THE RELATIONSHIP BETWEEN FLEXIBILITY AND STUDENT PERFORMANCE ON OPEN NUMBER SENTENCES WITH INTEGERS

Lisa Lamb
San Diego State University
Lisa.Lamb@sdsu.edu

Jessica Bishop
University of Georgia
jpbishop@uga.edu

Randolph Philipp
San Diego State University
RPhilipp@mail.sdsu.edu

Ian Whitacre
Florida State University
iwhitacre@fsu.edu

Bonnie Schappelle
San Diego State University
bschappelle@projects.sdsu.edu

To better understand the role that ways of reasoning play in students' success on integer addition and subtraction problems, we examined the relationship between students' flexible use of ways of reasoning and their performance on integers open number sentences. Within groups of students in 3 participant groups—39 2nd and 4th graders who had negative numbers in their numerical domains, 40 7th graders, and 40 successful 11th graders—we found that flexibility and success were positively related. That is, the more flexibly students invoked ways of reasoning, the greater their success. These findings indicate that rather than searching for one particular model or approach to teaching integer operations, teachers should support multiple ways of reasoning and discuss features of problems that might make one way of reasoning more productive than another.

Keywords: Number Concepts and Operations, Cognition, Learning Trajectories (or Progressions)

Students' lack of success operating on negative numbers is well documented in the literature (Kloosterman, 2014; Vlassis, 2002). Understanding students' thinking about integer addition and subtraction is important for teachers to better support students' learning and to promote their transition from arithmetic to algebra (Peled & Carraher, 2008). We see our research on the flexibility of students' ways of reasoning about integers as connecting to the PMENA theme of Questioning Borders in our examining the role of implicit borders among instructional approaches to teaching integers. Existing literature tends to be focused on identifying a single best model or instructional practice for supporting students' understanding of integers, that is, promoting a *border* around a single practice. Our findings indicate that an opening of borders (or promoting more than one way of reasoning) may be more productive for students than a closed-border approach (or reliance on a single way to learn to operate with integers).

Theoretical Framework and Literature Review

We approach our research from a children's mathematical thinking perspective. We consider it important to see mathematics through children's eyes to better understand the sense that they make. This perspective is based on principles that children have existing knowledge and experiences they bring with them into the classroom and upon which they continue to build (e.g., Carpenter et al., 1999). We take this view because our ultimate goal in our research is to find ways to better support children's learning of mathematics, and instruction that builds on students' ideas benefits both teachers and students and supports rich instructional environments (Sowder, 2007; Wilson & Berne, 1999). Moreover, we hope that by studying the variety of ways students reason about integer arithmetic we can broaden existing instructional approaches so that more students can successfully engage with this important mathematical topic.

Literature Related to the Teaching and Learning of Integers

To provide background for this paper, we examine the research on computational fluency, students' understanding of integers, and integer instruction. Much of the research related to students'

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

understanding of integers is focused on secondary students' difficulties with integer arithmetic (Gallardo, 2002; Kloosterman, 2014; Vlassis, 2008). Students also have difficulties solving algebraic equations, simplifying algebraic expressions, and comparing quantities that include negative integers (Christou & Vosniadou, 2012; Vlassis, 2002). However, researchers have found children to be capable of reasoning about integers in relatively sophisticated ways, even in the lower elementary grades (Behrend & Mohs, 2006; Bishop et al., 2014; Bofferding, 2014). Despite these findings, few researchers have focused on students' ways of reasoning about integers. Instead, in the majority of integer-related studies, researchers developed and tested a variety of approaches to the teaching of integers, using various models, tools, and contexts (see Liebeck, 1990; Linchevski & Williams, 1999; Stephan & Akyuz, 2012). In most articles related to integers instruction, a single model or instructional approach, such as movement on a number line or zero pairs, was proposed. Although some models showed promise in terms of student achievement, no model is perfect, and each one has both affordances and limitations. The lack of compelling results across interventions led us to consider whether multiple approaches might be warranted; thus, we shifted to the literature on flexibility, described in the next section.

Literature Related to Flexibility

Although the research related to integers instruction has been focused on identifying one way to support students, the research on flexibility heightened our curiosity about the role that flexibility might play in relation to students' success in solving integers problems. Star and Newton defined *flexibility* as "knowledge of multiple [strategies] as well as the ability and tendency to selectively choose the most appropriate ones for a given problem and a particular problem-solving goal" (2009, p. 558). In recent years, researchers have identified flexibility as a critical characteristic of one who has attained deep procedural knowledge (Star & Newton, 2009) and as a characteristic shared by expert mathematicians (Dowker, 1992; Star & Newton, 2009). Additionally, many researchers have studied the influence or development of flexibility in different content domains, ranging from 2-digit-addition and -subtraction to estimation, multistep linear equations, and proportional reasoning (e.g., Berk, Taber, Gorowara, & Poetzl, 2009; Blöte, Van der Burg, & Klein, 2001; Dowker, 1992; Star & Newton, 2009).

Given middle-school students' well-documented struggles with integers, we questioned the emphasis in the literature on finding a single, best model to support students' learning. Instead, we sought to determine the degree to which flexible use of ways of reasoning influenced students' performance in solving integer problems. We asked the following research questions: How flexible are students in using ways of reasoning when completing integer open number sentences, and to what degree is students' flexible use of ways of reasoning related to their success in this activity? In reviewing the literature, we did not find any study that systematically sampled students at different grade levels to document the flexibility in their reasoning about integer addition and subtraction. We here provide a cross-grades view of flexibility in students' ways of reasoning about integer addition and subtraction and then relate flexibility to performance. The findings reported here advance the field's understanding of students' flexibility and performance on integer addition and subtraction problems. This study contributes to the efforts of the mathematics education research community to support instruction that enables students to successfully transition from arithmetic to algebra.

Methods

Background and Participants

This study is part of a larger project in which our goal was to understand K–12 students' conceptions of integers and integer arithmetic. In this paper, we focus on flexibility in students' ways of reasoning—an aspect of integer understanding rarely mentioned in the literature yet which we

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

believe supports students to develop deeper conceptions of integers as well as more efficient computational strategies. Data for this study include clinical interviews with 160 students from 11 schools (3 elementary, 3 middle, 1 K–8, and 4 high schools) in the Western United States. During the 2010–2011 school year, these students participated in individual interviews about integer addition and subtraction. We conducted clinical interviews with students in Grades 2, 4, 7, and 11 (40 children from each grade level). Students in Grades 2 and 4 had yet to receive school-based integer instruction; students in Grade 7 had completed integer instruction; students in Grade 11 were enrolled in a precalculus or calculus course and, because of their course taking, were deemed to be successful high school mathematics students. Because our focus is on understanding students' productive ways of reasoning, we restrict our findings in this paper to students who had some knowledge of negative numbers: *2nd/4th with negatives* (n = 39), those 2nd- and 4th-grade students who provided evidence of at least limited knowledge of negative numbers, *7th-grade students*, and *successful 11th-grade students*. The *2nd/4th with negatives* group included 13 Grade 2 students and 26 Grade 4 students who, on the basis of responses to tasks posed at the beginning of the interview, provided evidence of having at least some knowledge of negative numbers.

Clinical Interview

The videotaped 60–90-minute clinical interviews (Ginsburg, 1997) were conducted at the students' school sites. Although we sought to understand and follow the child's thinking during the interviews, the interviews were standardized: All children were posed the same set of 47 tasks, except for those students who did not have negative integers in their numeric domains. The interview had four categories of tasks: introductory questions, open number sentences (e.g., $-3 + \square = 6$ and $\square + 6 = 4$, with unknown location varying), contextualized problems, and comparison problems. Findings shared herein are from responses to the open number sentences.

Coding and Analysis

The interviews were coded at the problem level for both correctness and the underlying way of reasoning the child used. The Ways of Reasoning coding scheme was developed and refined iteratively over a period of 2 years. We identified five broad categories we call *Ways of Reasoning* and a total of 41 subcodes that provide more detail as to the child's specific strategy or strategies. The five ways of reasoning are order-based, analogy-based, computational, formal, and developmental (see Table 1 below for definitions). Each response to the 25 open number sentences was assigned a way-of-reasoning code (some responses involved more than one way of reasoning and, thus, received multiple codes). For example, some students completed $-3 + 6 = \square$ by counting up 6 units, by ones, from -3. This solution would be coded as an order-based way of reasoning because counting leverages the ordered and sequential nature of numbers. However, another student might complete the same sentence using analogy-based reasoning, explaining, "It's like I borrowed 3 dollars from my friend. It's like I owe him; that's minus 3. And I give him 3 from the 6 my mom gave me, and now I have 3." We would code this response as analogy-based reasoning because the student solved the problem by comparing negative integers to owing money. Of the 160 interviews 42 (or 26.25%) were double coded, and interrater agreement was 92% at the Ways of Reasoning level and 83% at the subcode level.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Table 1: Ways of Reasoning About Integer Addition and Subtraction

Ways of Reasoning	Definitions
Order-based	In this way of reasoning, one leverages the sequential and ordered nature of numbers to reason about a problem. Strategies include use of the number line with motion as well as counting forward or backward by 1s or another incrementing amount.
Analogy-based	This way of reasoning is characterized by relating numbers and, in particular, signed numbers, to another idea, concept, or object and reasoning about negative numbers on the basis of behaviors observed in this other concept. At times, signed numbers may be related to contexts (e.g., debt or digging holes). Analogy-based reasoning is often tied to ideas about cardinality and understanding a number as having magnitude.
Formal	In this way of reasoning, negative numbers are treated as formal objects that exist in a system and are subject to fundamental mathematical principles that govern behavior. Students may leverage the ideas of structural similarity, well-defined expressions, and fundamental mathematical principles.
Computational	In a computational way of reasoning, one uses a procedure, rule, fundamental mathematical principle, or calculation to arrive at an answer to a problem involving negative numbers either as part of the problem statement or as appearing in the solution set.
Developmental	This category of reasoning often reflects preliminary attempts to compute with signed numbers. An example of this category is a child's overgeneralization that addition always makes larger with the claim that a problem for which the sum is less than one of the addends (e.g., $6 + \square = 4$) has no answer. In this case, the domain of possible solutions appears to be restricted to natural numbers and the effect (or possible effect) of adding a negative number is not considered.

Measuring flexibility. *Flexibility* is a measure of the variety of ways of reasoning students use to solve integer-arithmetic tasks. Flexibility indicates whether a student primarily uses one way of reasoning or chooses different ways of reasoning depending on the affordances of the problem. We calculated a flexibility measure for each student by identifying the number of times each way of reasoning was used across all 25 open number sentences. Note that we did not include the developmental way of reasoning in this calculation because developmental approaches reflected preliminary attempts to compute and typically resulted in incorrect responses. We deemed a student to be proficient with a particular way of reasoning if that student used it three or more times during the interview (that is, used a given way of reasoning on at least 12% of the open number sentences). The number of ways of reasoning with which a student was proficient was our measure of flexibility. For example, if a student used order-based reasoning on 7 open number sentences, analogy-based reasoning on 2 open number sentences, formal reasoning on 3 open number sentences, and computational reasoning on 16 open number sentences, the student would be proficient with 3 ways of reasoning and receive a flexibility score of 3. We used this flexibility score to explore the relationship between flexibility and performance on the 25 open number sentences of students in the participant groups: $2^{nd}/4^{th}$ with, 7^{th}-grade students, and 11^{th}-grade students. In the next section, we report findings and provide case studies of three students to underpin the quantitative findings.

Findings

We found that flexibility is positively correlated with performance in our data within grades (r = .347, .523, .429 for 2/4, 7^{th}, and 11^{th}, respectively, two-tailed, all p-values < .05). In Table 2, we share frequency counts for flexibility scores of 0, 1, 2, 3 and 4 at each grade level. These frequency counts reflect the number of students who were proficient with 0, 1, 2, 3 or 4 ways of reasoning.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

More than half of the 2nd/4th graders with negatives used two ways of reasoning, and almost all (87%) used 1 or 2. The 7th graders had the greatest spread in flexibility; although almost half (45%) used 3 ways of reasoning, one fourth used 2 ways and one fourth used 4 ways of reasoning. Finally, more than half of the 11th graders (55%) used 3 ways of reasoning, and almost all (85%) used either 3 or 4. The findings that 11th graders were both the most flexible in their ways of reasoning and the most accurate indicate that particular ways of reasoning are not necessarily *replaced* by other, more sophisticated, ways of reasoning but, rather, that students who have access to and use multiple ways of reasoning are more successful than those who do not.

Table 2: Frequency Counts for Ways of Reasoning

Flexibility score	Grades 2/4 with negatives (n = 39)	Grade 7 (n = 40)	Grade 11 (n = 40)
0	2 (5%)	0 (0%)	0 (0%)
1	11 (28%)	3 (8%)	4 (10%)
2	23 (59%)	10 (25%)	2 (5%)
3	3 (8%)	18 (45%)	22 (55%)
4	0 (0%)	9 (23%)	12 (30%)

Cases

We provide cases of three 7th-grade students—Hannah, Sofia, and Maria (pseudonyms)—to exemplify the relationship between flexibility and accuracy. Hannah invoked order-based reasoning on more than 80% of all open number sentences, whereas Sofia primarily used a computational way of reasoning. Maria invoked every productive way of reasoning, each on one fourth to one half of the open number sentences.

The case of Hannah. Hannah completed about one third (32%) of the open number sentences correctly and used order-based reasoning almost exclusively. Although she could use this way of reasoning productively at times, more often than not her almost exclusive use of order-based reasoning hindered her success. Hannah tended to be able to productively use order by invoking the strategy *motion on a number line* for problems such as -3 + 6 = □ and -9 + □ = -4, for which the second addend (or, for subtraction problems, the subtrahend) is positive, correctly completing 73% of these types of number sentences. However, she also used the same order-based way of reasoning for almost every other problem and answered every one of them *incorrectly*. For example, on 6 – -2 = □, Hannah placed her pen at 6 on the number line and paused. She then moved her pen to the left 2 units and answered 4, explaining, "You go to positive 6 minus negative 2 equals 4." The interviewer asked for clarification and Hannah added, "I am just a little bit confused by this (points to the open number sentence), so I just looked at this (points to the number line). So then I just minused 2, and I got 4." The discussion continued.

Interviewer: Which part is a little confusing?
Hannah: Because it is a positive 6 minus a negative 2 (she points to the number line). I don't know if you go down here (moves her pen to the negative side of the number line), or, I don't know, but I think it's 4."
Interviewer: Was there something else (another answer) you were considering?
Hannah: Negative 4, but then I was like, "No. Because it [-4] is all the way down here (points to the negative side of the number line)."

Hannah appeared to implicitly recognize that the problem has a structure different from the other problems she had successfully solved, and she shared her confusion about how to adapt her use of motion on the number line when subtracting a negative number. Her second answer of -4 (also incorrect) appeared to reflect her attempt to account for subtracting -2 (rather than +2). Hannah's almost exclusive use of order-based reasoning limited her options for solving problems. Because she appeared to have no other ways of reasoning to support her completion of the open number sentences, she was often unsuccessful, as were most others who exclusively relied on one way of reasoning. In the next case we describe Sofia, who, like Hannah, tended to rely primarily on one way of reasoning and also had limited success. Unlike Hannah, however, Sofia used primarily computational reasoning.

The case of Sofia. Sofia was more successful than Hannah, inasmuch as she answered about two thirds (64%) of the problems correctly. She used primarily computational reasoning. For example, on the problem $6 + \square = 4$, Sofia correctly answered -2, using a rule. In particular, she used what is often referred to as the *different-signs rule*, such that when adding a negative number and a positive number, one finds the difference of the absolute values of the numbers and appends the sign of the number that has the larger absolute value.

Interviewer: So how come you can put -2?

Sofia: Because [the sum] can still become a 4 if you subtract from a positive and a negative. You can still get a number that is positive because this one (points to 6) is bigger, and then the answer has to be positive. If [the positive number] is a lower number, like if a negative number is higher (points to -2), and this one (points to 6) is like 3, [the sum] becomes a negative."

Sofia solved this problem by applying a rule: She subtracted 2 from 6—what she described as "subtract[ing] from a positive and a negative"—and the difference of 4 takes the sign of the number with the larger absolute value. Her focus on primarily computational approaches likely explains why she also applied the different-signs rule on a problem for which the rule was not applicable. For example, on the problem $-8 – 3 = \square$, Sofia incorrectly answered -5, explaining that she subtracted 3 from 8 and, because the 8 was negative, her answer should also be negative. "When [the absolute value of] a negative number is bigger than a 3, like, a positive number (points to 3), if this one (points again to 3) is lower [than the absolute value of -8], then the answer becomes a negative sign." For this problem, Sofia inappropriately invoked the different-signs rule. Similarly, she invoked a rule for multiplying negative numbers to complete $-5 + -1 = \square$. Sofia explained, "I think it is 6 because if you add both negatives, then it becomes a 6, but if you see the signs negative and a negative, it becomes a positive number, so I say it's a 6." When asked why two negatives become a positive, Sofia responded, "Because when the signs are the same, it becomes a positive. Well, my teacher said that if … it was two negatives, then it [the result] would become a positive."

Like Hannah's exclusive focus on order-based reasoning, Sofia's focus on computational reasoning appeared to hamper her success. Looking across both cases, we see that the particular way of reasoning upon which students focused was less important than the fact that each student appeared to have one way of reasoning on which she relied, and being limited to a single way appeared to negatively influence success.

The case of Maria. In contrast to Hannah and Sofia, Maria completed every open number sentence correctly and used all four productive ways of reasoning. For example, she used order-based reasoning for $-3 + 6 = \square$ by "jumping to" 0. Maria explained her answer of 3 saying, "Half of 6 is 3, so then that would bring it to the 0. And 3 more would bring it to the 3. And that would equal 6." Maria decomposed 6 into 3 and 3 so she could jump to 0 using a partial sum. Her strategy of using a decade number of 0 supported her to count more efficiently than counting by ones. Maria then used analogy-based reasoning on the problem $-5 + -1 = \square$, by comparing negative numbers to "bad guys." For this problem she correctly answered -6, saying, "Since negative numbers are like bad guys, 5 bad guys met up with one more bad guy, so there were 6 bad guys total." In contrast, she correctly completed the number sentence $6 – -2 = \square$ using computational reasoning, explaining, "I changed the signs so it was plus-plus, and 6 + 2 is 8." Finally, Maria used formal reasoning for $6 + \square = 4$.

She answered, "Negative 2. It's [the unknown] not going to be a positive because the sum is less than 6 and it [the operation] is addition. So it [the missing addend] has to be a negative number. So, -2." This response received a formal code because Maria used deductive reasoning (by noticing that if she added two addends and the sum was less than one of the addends, then the other addend must be a negative number) to determine that the sign of the number had to be negative.

Similar to others who used all the productive ways of reasoning, Maria not only flexibly invoked a wide range of strategies on the problems but also appeared to choose strategies that corresponded with the underlying structure of the problem, indicating that her (perhaps implicit) attention to the underlying structure evoked particular ways of reasoning. This attention appeared to be influential in her choice of strategy and, presumably, in her success.

Discussion

Our unique contribution to the research literature is documenting statistically significant and moderate-to-strong correlations within participant groups between flexibility and performance on integers open number sentences. Additionally, the cases of Hannah, Sofia, and Maria provide insight into how the degree of flexibility played out in relation to students' performance on the open number sentences. Although not causal, these findings indicate a need for students to have several ways of reasoning at their disposal. Further, the finding that 11th graders were both the most flexible in their ways of reasoning and the most accurate indicates that particular ways of reasoning are not necessarily *replaced* by other, more sophisticated, ways of reasoning but, rather, that a mark of expertise in operating with negative numbers is the flexible use of many ways of reasoning. These findings have implications for instruction, which we discuss below.

Implications and Contributions

In other work, we have found that many younger students used more than one productive way of reasoning about integer-related tasks prior to school-based instruction (Bishop et al., 2014). Thus, in any given classroom, teachers across grade levels may have different students approach the same problem in different ways. Teachers should be aware that they can leverage these different ways and that having multiple ways of reasoning appears to promote successful performance on these problems. That is, no single best model or way of reasoning that students successfully use across all problems exists, and attempting to teach students one all-encompassing way may have the unintended consequence of impeding students' success by limiting their flexibility. Rather than teaching any one particular strategy, teachers could instead discuss a variety of ways of reasoning and features of problems (signs of numbers, relative sizes of numbers in the problem, and operation) that might evoke one way of reasoning more than another.

Our findings align with the following famous quote: "I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail (Maslow, 1966, pp. 15–16). Those students who relied on a single tool (or way of reasoning) approached every problem with that tool, and they tended to be less successful than those who had a variety of tools from which to choose. This finding is particularly powerful in that the correlations between flexibility and accuracy held for every participant group—the more flexible students were, the more successful they were. Thus, the ways of reasoning may be thought of as a tool belt. We find that, in general, the more tools the better.

Acknowledgments

This material is based on work supported by the National Science Foundation (NSF) under grant number DRL-0918780. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

References

Behrend, J. L., & Mohs, L. C. (2006). From simple questions to powerful connections: A two-year conversation about negative numbers. *Teaching Children Mathematics*, *12*(5), 260–264.

Berk, D., Taber, S. B., Gorowara, C. C., & Poetzl, C. (2009). Developing prospective elementary teachers' flexibility in the domain of proportional reasoning. *Mathematical Thinking and Learning*, *11*(3), 113–135.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education*, *45*(1), 19–61.

Blöte, A. W., Van der Burg, E., & Klein, A. S. (2001). Students' flexibility in solving two-digit addition and subtraction problems: Instruction effects. *Journal of Educational Psychology*, *93*(3), 627–638.

Bofferding, L. (2014). Negative integer understanding: Characterizing first graders' mental models. *Journal for Research in Mathematics Education*, *45*(2), 194–245.

Carpenter T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Christou, K. P., & Vosniadou, S. (2012). What kinds of numbers do students assign to literal symbols? Aspects of the transition from arithmetic to algebra. *Mathematical Thinking and Learning*, *14*(1), 1–27.

Dowker, A. (1992). Computational estimation strategies of professional mathematicians. *Journal for Research in Mathematics Education*, *23*(1), 45–55.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). Mathematics instruction and teachers' beliefs: A longitudinal study of using children's thinking. *Journal for Research in Mathematics Education, 27*(4), 403–434.

Gallardo, A. (2002). The extension of the natural-number domain to the integers in the transition from arithmetic to algebra. *Educational Studies in Mathematics*, *49*(2), 171–192. doi:10.1023/A:1016210906658

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York: Cambridge University Press.

Kloosterman, P. (2014). How much do mathematics skills improve with age? Evidence from LTT NAEP. *School Science and Mathematics*, *114*(1), 19–29.

Liebeck, P. (1990). Scores and forfeits—An intuitive model for integer arithmetic. *Educational Studies in Mathematics*, *21*(3), 221–239. doi:10.1007/BF00305091

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in "filling" the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics*, *39*(1-3), 131-147.

Maslow, A. H. (1966). *The psychology of science: A reconnaissance*. New York: Harper & Row.

Peled, I., & Carraher, D. W. (2008). Signed numbers and algebraic thinking. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 303-328). New York: Routledge.

Sowder, J. T. (2007). The mathematics education and development of teachers. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 157–223). Charlotte, NC: Information Age.

Star, J. R., & Newton, K. J. (2009). The nature and development of experts' strategy flexibility for solving equations. *ZDM*, *41*(5), 557-567.

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education*, *43*(4), 428–464. doi:10.5951/jresematheduc.43.4.0428

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, *49*(3), 341–359. doi:10.1023/A:1020229023965

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. *Review of Research in Education*, *24*, 173–209.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.