OPPORTUNITIES TO POSE PROBLEMS USING DIGITAL TECHNOLOGY IN PROBLEM SOLVING ENVIRONMENTS

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This article reports and analyzes different types of problems that nine students in a Master's Program in Mathematics Education posed during a course on problem solving. What opportunities (affordances) can a dynamic geometry system (GeoGebra) offer to allow in-service and in-training teachers to formulate and solve problems, and what type of heuristics and strategies do they exhibit during this process? Results show that combining semi-structured problems with the use of GeoGebra can be useful in motivating and involving teachers in various episodes of problem formulation. In this context, important strategies included analyses of the variation in the attributes of figures using dynamic points and loci.

Keywords: Problem Solving, Technology, Geometry and Geometrical Spatial Thinking, Teacher Education-Preservice, Teacher Education-Inservice/Professional Development

1 Introduction

It is widely recognized today that formulating, or posing, problems is a central activity in the practice of professional mathematics and a fundamental component of mathematical thinking (Cai et al., 2013). In this regard, in the past two decades problem formulation and problem solving have been identified as central topics in mathematics education (Rosli et al., 2015). On this theme, Osana and Pelczer (2015, p. 470) commented that:

> A growing movement in mathematics education that placed problema solving at the center of school mathematics further contributed to researchers' focus on problem posing, particulary its role in teaching and learning. (p. 470).

In this perspective, and in educational contexts, mathematical activity is conceived as a form of thinking in which a community (teacher and students) formulates questions and new problems to give meaning to, and resolve, problematic situations. In this scenario, the community recognizes the importance of seeking different means of supporting their responses. Santos-Trigo, Reyes-Martínez and Ortega-Moreno (2015) observe that one objective of mathematical activity is to identify and contrast diverse approaches to representing, exploring, conjecturing, resolving and formulating new problems. In communities of this kind, the role of teachers is determinant for students' learning because they are responsible for choosing and presenting the tasks that will allow learners to develop their ability to formulate and resolve problems. However, some researchers recognize that, in general, in-service and in-training teachers experience serious difficulties when confronting the tasks involved in preparing and posing problems (Rosli et al., 2015; Lavy, 2015).

What role does the use of digital technologies play in learning communities that promote and value problem posing and problem solving? In mathematics education digital technologies can provide an effective way of developing mathematical knowledge and transforming teaching scenarios by orienting them towards the formulation and resolution of problems (Aguilar-Magallón & Reyes-Martínez, 2016). To date, however, little research has been conducted on the role of technology in designing and implementing tasks whose goal is to enhance the ability to formulate and resolve problems (Abramovich & Cho, 2015).

In light of the foregoing, the principal objective of this study consisted in analyzing how the systematic use of a Dynamic Geometry System (DGS) by in-service and in-training teachers can contribute to the processes of problem formulation and problem solving. Thus, our research is oriented by the following questions: what opportunities (affordances) can the *GeoGebra* Dynamic Geometry System offer current and future math teachers in relation to problem formulation and problem solving, and what kinds of heuristic resources and strategies are exhibited in this process?

2 Conceptual Framework

Posing Problems and the Use of Digital Technologies

The literature sustains that the process of posing problems centers on two fundamental activities: formulation and reformulation. The formulation consists in generating new problems based on certain information, situations or contexts, while the latter entails elaborating new problems by modifying the conditions and/or objectives of an earlier, given problem (Silver, 1994). Reformulation activity also occurs when a problem that is in the process of resolution is transformed or re-posed in order to simplify it (Silver, Mamona-Downs, Leung & Kenney, 1996). According to this characterization, formulation and reformulation activities may take place before (the formulation or understanding of a statement), during, or after (reformulation) the resolving problems (Silver, 1994).

Following these ideas, Stoyanova and Ellerton (1996) presented a typology of problems that specifies the following three categories: open, semi-structured, and structured, as a function of the formulation or reformulation activities they involve. In open problems, individuals must posit problems based on information presented in the form of figures, tables, numbers, etc. The statement of problems of this kind do not include any specific requirements or objectives. In semi-structured problems, individuals are required to generate and/or add conditions in order to reach a solution; that is, the statement of this type of problem contains only partial information or conditions. Structured problems, finally, stipulate both the objective and all the information and conditions necessary to resolve them. Thus, open problems entail primarily formulation activities, while structured problems involve reformulation activities. Semi-structured problems can propitiate both formulation and reformulation activities. Silver (1997) holds that open or semi-structured problems can be useful in propitiating episodes of problem formulation.

Santos-Trigo, Reyes-Martínez and Aguilar-Magallón (2015) underscores the importance of the systematic utilization of various digital tools in environments of problem formulation and problem solving. Here, the goal is to have individuals constantly identify and examine distinct types of relations, posit conjectures, determine and analyze patterns, employ different systems of representation, establish connections, apply distinct arguments, generalize and extend initial problems, communicate their results, and posit their own problems. Some research has focused on examining the processes involved in posing problems using specific digital tools, such as DGS (Leikin, 2015; Lavy, 2015). According to Lavy (2015), a DGS constitutes a cognitive visual support based on immediate interactions between the tool and its user that can facilitate the processes involved in posing problems. Imaoka, Shimomura and Kanno (2015) recommend that the design of problem formulation activities utilizing a DGS entail exploring variable attributes of such figures as areas, perimeters, lengths, and angles, among others. They further advise designing problems that can be represented and solved in distinct ways; that is, they underline the importance of posing problems that are not made trivial once a DGS is applied. Leikin (2015), finally, argues that an important strategy for designing activities related to posing problems consists in transforming structured problems into open or semi-structured ones; *i.e.*, eliminating the specific conditions or objectives of structured problems to encourage exploration and research with the aid of a DGS.

3 Methodology

Participants

The participants in this study were nine students enrolled in a Master's Program in Mathematics Education. The study design consisted of fourteen weekly sessions, each with duration of three and a half hours. The group included six in-service and three in-training math teachers, all of whom had formal academic training in the field of mathematics.

Design of Activities

A total of five activities were implemented during the study, taking into account the ideas proposed by Imaoka et al. (2015) and Leikin (2015); that is, we began with a series of structured problems related to the area of figures that were then transformed into semi-structured research topics. In this report, we analyze the results of one of those problems. That problem emerged when the structured problem used by Schoenfeld (1985) was transforming by modifying its conditions (Table 1).

Table 1: Transformation of a Structured Problem Into a Semi-Structured One

Structured Problem	***Semi-Structured Problem***
You are given a fixed triangle T with base B. Show that it is always possible to construct, whit straightedge and compass, a straight line that is parallel to B and divides triangle T into two parts with equal area. Can you similarly divide the triangle into five parts of equal area? Schoenfeld (1985, p. 16).	***P.1.*** *Given any triangle, divide it into two regions with the same area.*

Implementation of the Activity and Data Collection

The development of the activity can be characterized as including three phases: 1) individual or pair work; 2) plenary discussions; and, 3) on-line discussions. The first phase consisted in three (weekly) in person sessions of three hours each. They took place in a computer laboratory so that each student had access to a personal computer with internet. During the plenary discussions, participants presented their ideas or advances in resolving the activity to the whole group. The on-line discussions utilized a digital wall (*Padlet*) that allowed participants to continue the discussion outside and beyond the in-person sessions. Study data were collected by video taping the in person sessions, recording the participations in the digital wall, *GeoGebra* worksheets, individual written reports, and interviews.

4 Results

In this section, we discuss the resources, heuristics and strategies that were presented in participants' efforts to solve the problem (P.1). Special emphasis is placed on the episodes involving problem formulation propitiated by the use of *GeoGebra*.

Initial Solutions

In a first instance, participants solved the problem using two basic ideas: 1) bisecting the area of the triangle by means of a median (*i.e.*, dividing the base in two equal parts while maintaining the height); and, 2) bisecting the area by dividing the height into two equal parts but conserving the initial base. Thus, participants used both static and dynamic solution strategies. Some of the initial

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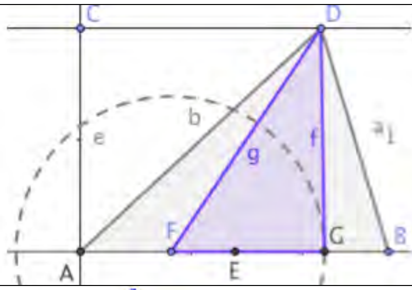
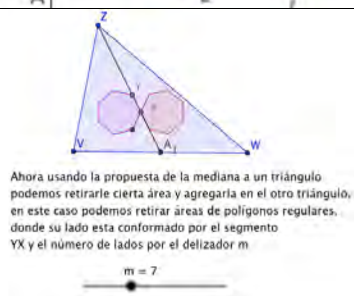
dynamic solutions are shown in Table 2. One essential aspect of these approaches was the search for diverse ways to identify regions with the same area.

Table 2: Some Initial Dynamic Solutions to the Problem

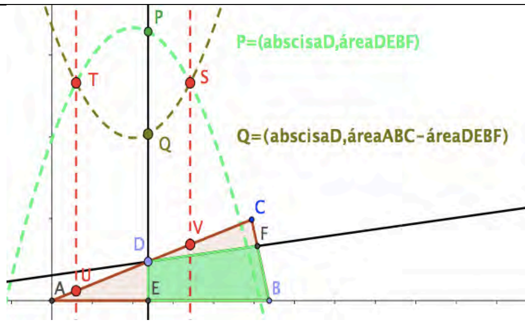
<table>
<tr><th>Solution</th><th>Resources and Strategies</th></tr>
<tr><td></td><td>Resources: circumference to transfer measurements, mobile point on the segment, mid-point, triangle.
Dynamic strategy: point F on AB. Construct a dynamic triangle, FGD, with the same height as triangle ABD, but a movable base, FG, of constant length equal to AE where
$AE = \frac{1}{2}AB$ (infinite solutions).</td></tr>
<tr><td></td><td>Resources: Median, regular polygon, mobile point on the segment.
Dynamic strategy: use a slider, “m”, to draw a regular polygon of “m” sides and reflect it by means of the median; add and subtract dynamic polygons of the same area on both sides of the median (infinite solutions).</td></tr>
<tr><td></td><td>Resources: parallel mean, mid-point, triangle, mobile point on the segment, Euclidean proposition 37.
Dynamic strategy: divide the height in two parts by the parallel mean, ED, and construct two triangles with the same base (equal to half of side AC) and mobile vertices, G and H, on the parallel mean (infinite solutions). The area of the green region is equal to that of the blue region.</td></tr>
</table>

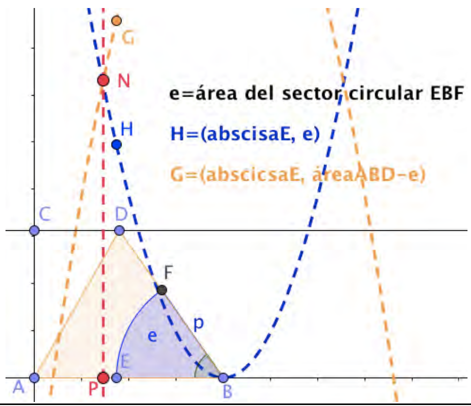
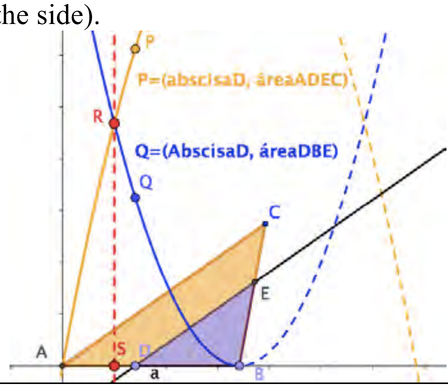
Problems Posed by Participants

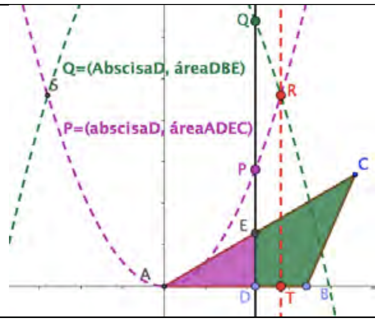
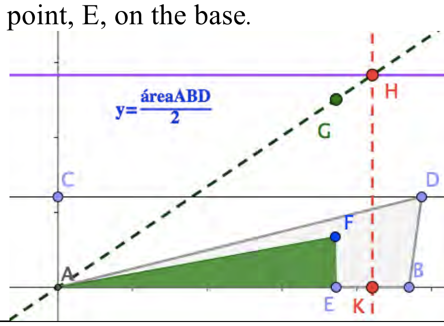
After presenting their initial solutions in a plenary discussion, participants proposed new ways to find regions with the same area as the given triangle, motivated by the dynamic exploration of elements inside the configuration (Table 3). For example, one participant suggested using a circular sector to divide the triangle in two sections of the same area. Another focused attention on a construction that involved a quadrilateral. All approaches were based on a graphic representation of the variation of the area of the figures involved (Table 3).

Table 3: Problems Posed with Exploration and Solution Strategies

Problem Posed	***Resources and Solution Strategies***
1.1. Divide the triangle with a quadrilateral whose sides are perpendicular to two sides of the triangle (variation of a point D on side AC).	***Exploration***. Constructing a family of quadrilaterals based on mobile point D and with sides perpendicular to those of the triangle.

<table>
<tr>
<td>
</td>
<td>Important question: where should point D be placed on side AC so that quadrilateral EDFB has the same area as the sum of the areas of triangles AED and DFC?
Solution. Construct the dynamic points
P = (x(D), areaDEBF) and
Q = (x(D), areaABC - areaDEBF).

The intersections T and S of the loci described by P and Q upon moving D determine solutions U and V.</td>
</tr>
<tr>
<td>1.2. Use a circular sector to divide the triangle (variation of a point E on side AB).

</td>
<td>Exploration. Constructing a family of circular sectors of variable area “e” from mobile point E.
Important question: where to place point E on side AB so that circular sector BEF has the same area as section AEFD?
Solution. Create the points
H = (x(E), e) and
G = (x(E), areaABD – e).
The intersection N of the loci described by H and G upon moving E determines solution P.</td>
</tr>
<tr>
<td>1.3. Division by means of a straight line parallel to one of the sides (variation of a point D on the side).

</td>
<td>Exploration. Constructing a family of triangles DBE with D mobile on AB and side DE parallel to side AC of the initial triangle ABC.
Important question: where to place point D on side AB so that quadrilateral EDAC and triangle DEB have the same area?
Solution. Create the points
P = (x(D), areaADEC) and
Q = (x(D), areaDBE). The intersection R of the loci described by P and Q upon moving D determines solution</td>
</tr>
<tr>
<td>1.4. Division by means of a straight line perpendicular to one of the sides (variation of a mobile point D on the side).</td>
<td>Exploration. Constructing a family of right-angled triangles ADE from mobile point D on side AB and with side DE perpendicular to side AB of the initial triangle ABC.
Important question: where to place point D on side AB so that quadrangle DBCE and triangle ADE have the same area?
Solution. Create the points
P = (x(D), areaADE) and
Q = (x(D), areaDBCE)</td>
</tr>
</table>

<table>
<tr>
<td>
</td>
<td>The intersection R of the loci
described by P and Q upon moving D determines solution T.</td>
</tr>
<tr>
<td>1.5. Bisect the area by means of a free mobile point, F, inside the initial triangle and another
mobile point, E, on the base.

</td>
<td>Exploration. Constructing a family of triangles AEF from mobile points F and E. Point F moves freely inside triangle ABD. Point E moves freely on side AB.
Important question: where to place points E and F such that triangle AEF has half the area of triangle ABD?
Solution. Create point G = (x(E), areaAEF)
The intersection H of the loci described by G (upon moving E) and the straight $y = \frac{áreaABD}{2}$ determine solution K.</td>
</tr>
<tr>
<td>1.6. Use any straight line to divide the triangle.

</td>
<td>Exploration. Constructing a family of triangles AFJ from mobile points J and E.
Important question: where to place points E and J so that triangle AJF has half the area of triangle ABD?
Solution. Create point G = (x(E), areaAFJ).
The intersection O of the locus described by G (upon moving E) and the straight line $y = \frac{áreaABD}{2}$ determine solution Q.</td>
</tr>
</table>

The exploration strategy applied in these problems consisted in constructing dynamic sections (quadrilaterals, circular sectors, triangles) inside the initial triangle and then visualizing the change in the area of those sections by dragging points until the section had half of the initial area. Visualization of the change in area was performed using dynamic points and their respective loci. Solutions were determined in terms of intersection points between those loci (Table 3).

New Problems Posed After Solving the Original Problem

The solutions reached by participants were reviewed in a plenary discussion. Those solutions involved using such loci as parabolas and hyperbolas. New problems emerged as a product of this discussion, and participants then attempted to resolve them by: i) determining the important elements geometrically (focus, directrix, axis of symmetry, vertex, etc.) of the conic sections utilized to resolve the problem; and, ii) finding the equations of those conic sections and obtaining a general algebraic solution of the problem. To find the important elements of the conics sections, participants had to review their geometric properties (in different on-line resources); for example, to find the focus of parabolas they used their reflexive property, while to find the equations, they generally used parametrization of the attributes of the triangle and extreme cases (Figure 1). Finally, participants pondered extending the initial problem by considering how to divide a triangle in three and more sections of the same area.

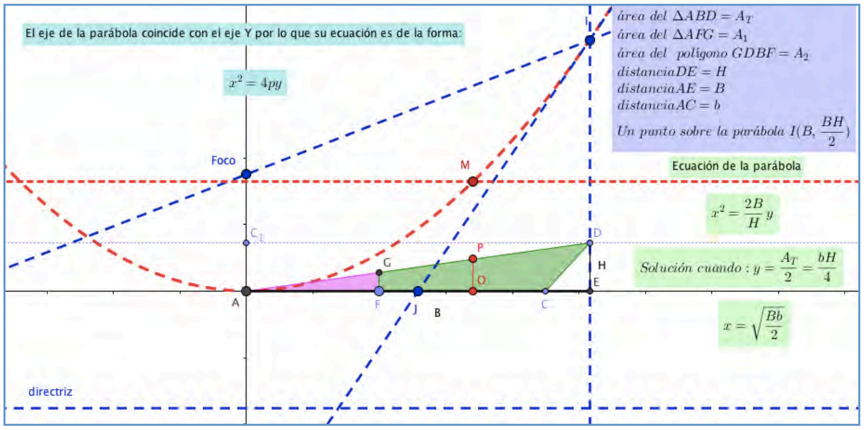


Figure 1. Parametrization and use of extreme cases to find the equation of the parabola used to resolve problem 1.4.

5 Discussion of Results

The results shown suggest that using the Dynamic Geometry System (*GeoGebra*) makes it possible to generate processes for posing problems by transforming a traditional (structured) problem into a semi-structured research problem. This transformation is achieved by *not making* certain objectives or conditions explicit. In the initial problem, *not making* the condition of dividing the triangle by means of a straight line parallel to one of the sides explicit proved to be determinant in leading the participants to formulate and resolve diverse problems.

Thanks to the ability to drag objects inside the dynamic configurations, participants were able to resolve the problem by applying dynamic approaches. These approaches allowed them to find infinite solutions that would be very difficult to visualize using traditional static tools like pencil and paper. Moreover, the dynamic exploration of the task motivated participants to pose a series of problems whose solution required analyzing variations in the areas of the figures. The use of dynamic points and their respective loci was crucial in this analysis. Later, another phase of posing problems emerged as participants explored the loci (conic sections) obtained to determine their important elements (focus, directrix, vertex, etcetera), their equations and, finally, algebraic solutions to the problems.

6 Conclusions

Any attempt to include the posing and resolution of problems in teaching and learning contexts in mathematics education depends, first and foremost, on the teacher(s) involved. In this regard, posing or formulating problems is important for teachers both in terms of their own training in the discipline and for their teaching practice. On the one hand, formulating problems allows both in-service and in-training teachers to develop their creativity and construct or strengthen their knowledge of mathematics. On the other, formulating problems is a fundamental pedagogical ability, because it is always necessary to formulate or reformulate problems as a function of students' needs, resources, ideas or errors. In this sense, it is necessary to address two key issues: 1) teacher training; and, 2) designing tasks that require formulating problems.

This study presented an example of the design and implementation of such a problem-formulation task. The use of a DGS was fundamental because it made it possible to transform a traditional problem in an activity that required exploration and research. We can conclude that the DGS can motivate processes of exploration and research that will eventually lead to the formulation and resolution of distinct problems. This idea could well become an essential element in the design of teacher training programs based on posing and resolving problems with the aid of digital technologies.

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