CONTEXTUALIZED MATHEMATICS PROBLEMS AND TRANSFER OF KNOWLEDGE: ESTABLISHING PROBLEM SPACES AND BOUNDARIES

Rebecca McGraw
University of Arizona
rmcgraw@email.arizona.edu

Cody L. Patterson
University of Texas at San Antonio
cody.patterson@utsa.edu

In this study, we examine how inservice secondary mathematics teachers working together on a contextualized problem negotiate issues arising from the ill-structured nature of the problem such as what assumptions one may make, what real-world considerations should be taken into account, and what constitutes a satisfactory solution. We conceptualize the process of negotiating these questions as the construction of a "problem space," characterized by the boundary between considerations deemed relevant or essential to the problem and ones thought to be beyond the scope of the problem. We use data from group discussions of the problem to consider ways in which problem spaces are co-constructed by learners, instructors, and problem authors and how these problem spaces evolve over time. We conclude by discussing implications of these findings for the design and implementation of contextualized mathematics problems.

Keywords: Problem Solving, Modeling, Classroom Discourse, Affect, Emotion, Beliefs, and Attitudes

Background

The focus of this paper is on teacher work on one particular mathematics problem. The problem is contextualized and open-ended and involves aspects of both problem-solving and mathematical modeling. Lesh and Zawojewski (2007) define *problem solving* as

> the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing and revising mathematical interpretations – and sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics. (p. 782)

The problem we discuss in this paper requires integration of multiple mathematical concepts, as well as interpretation, modification, and revision of ideas within and outside of mathematics. With respect to mathematical modeling, when learners work on a problem involving a real-world context, part of the problem solving process may involve the construction of *mathematical models*, or systems of objects, relationships, and rules that can explain or predict the behavior of other systems (Doerr & English, 2003). Although we do not claim that the problem discussed in this paper is a modeling problem *per se*, participants engage in aspects of the modeling process (*e.g.*, developing a model and interpreting solutions) as they solve the problem. The problem used in this study is contextualized and ill-structured, and requires that the learner find and use information from the real world.

Our focus in this paper is on the negotiation of problem spaces. We defined a "problem space" as the collection of mathematical ideas and classroom and real-world issues and resources that learners take up and use as part of their solution process. These ideas, issues, and resources become visible as *the boundaries of the problem spaces* are constructed and explicitly negotiated. For example, while working on the problem of designing an enclosure with the greatest possible area given a fixed perimeter, a learner may decide (by themselves or by asking a teacher) that they only need to consider rectangular shapes. This decision about the problem boundary leads to a problem space that includes rectangles but not other shapes. By investigating the development of problem boundaries, we hope to better understand the ways in which problem spaces are created and how they evolve, as well as surface implicit assumptions about problem spaces and boundaries that we may take into our own mathematics teaching.

In our research, we assume that learners' problem-solving work is situated within particular mathematics classroom contexts, with associated norms and expectations that will influence the negotiation of the problem space, as will learners' previous experiences in mathematics classrooms. We assume that learners' beliefs about mathematics, and their mathematical and school-learner identities, will influence how problem spaces/boundaries are established, as will power and authority relationships among learners and between teachers and learners. Lastly, we assume that the establishment of problem spaces is an ongoing negotiation that takes place among learners, teachers, and "animated others" such as problem authors or representatives of the real world (e.g., people in a town, a business owner, etc.). Within this framework, we address the following questions:

4. How do mathematics teacher learners, engaged in an ill-structured contextualized problem, negotiate the problem space?
5. What boundaries do the teachers establish and how are they determined? How do the boundaries evolve throughout the problem-solving process?

Method of Study

Context and Problem Design

In Summer 2015, the authors taught an 80-hour mathematics content focused professional development (PD) course to 33 middle and high school mathematics teachers from three school districts in the Southwestern United States. Teachers spent most of their time during the PD working in small groups on problem sets and activities meant to highlight key ideas in middle grades and secondary mathematics.

One of these was the "Quantitative Reasoning Cards" activity, in which participants work in groups of four on a sequence of problems involving real-world contexts. Each problem consists of a statement and several pieces of information. For the problem analyzed in this study, both the statement and the information are on a single card given to the group member designated as the leader for the task. The text on the card is shown below:

> The town of Squareville (population 25,600) relies on a nearby lake for drinking water. The water has been tainted due to an industrial accident. The lake can be cleaned, but it will take about two weeks to do so. In the meanwhile, the state plans to use trucks to send clean water to Squareville from a town 23 miles away. How many trucks will the state need?
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> In order to proceed to the next task, you (the person holding this card) must give a referee a convincing argument answering this question.

Figure 1. The Water Shortage Problem.

The leader may share the information on the card with other group members and help guide the discussion, but they are not permitted to write anything down nor look up any additional information. Other group members may write down their thoughts and mathematical work, but are not allowed to see nor touch the leader's card. Once the group reaches a consensus solution, the leader must explain it to one of the PD instructors (designated as the "referee"), who may then ask follow-up questions of the other group members. The design of the problem is meant to foster interdependence among group

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members (Cohen & Lotan, 2014); because the group leader cannot perform calculations, and other members have no direct access to the information on the card, group members must communicate about their overall problem-solving strategy as well as the details of the solution so that the leader can clearly describe the group's work to the referee.

The Water Shortage Problem, designed by one of the authors of this paper, requires participants to answer a practical question (how many trucks are needed to deliver water to a town) by analyzing rates of water consumption and delivery rather than absolute amounts. The information card intentionally leaves some essential questions unanswered, such as how much water a truck can carry, and how much water each person will need. The purpose of providing incomplete information is to stimulate discussion among participants about what quantities are relevant to the problem's solution, and to encourage participants to seek information from sources external to the activity.

The problem is designed to elicit thinking from participants about how to estimate quantities whose values cannot be determined exactly. For example, if a disaster preparedness website recommends that each person receive 2 to 4 gallons of water per day, should one assume that each person will receive 2 gallons, 4 gallons, or some amount between these two extremes? We have found in our own implementations of this and similar problems, given a range of possible estimates for a quantity, participants will often select an estimate at the middle of the range, even when a lower or upper bound might be more useful for the situation at hand.

The problem is also designed so that solutions that do not contain rate thinking (*e.g.*, thinking only about how many gallons total are needed for 2 weeks, rather than thinking about gallons per day) will likely lead to unreasonably large answers. This problem feature is intended to spur learners to reconsider their solutions and seek ways to decrease the number of trucks needed. For this to occur, participants must expand the problem space to include consideration of whether a given number of trucks is practically feasible; while a request for ten trucks is likely to be honored by an emergency management agency, a request for five thousand will almost surely be rebuffed.

Participants, Data Collection, and Analysis

During the problem implementation, we captured video and audio recordings of two groups of teachers working on the Water Shortage Problem. Each group consisted of four inservice secondary mathematics teachers. Group 1 consisted of three female middle school teachers and one male high school teacher; Group 2 consisted of one female high school teacher, one female middle school teacher, and two male middle school teachers. Group 1 spent 22 minutes on the problem, and Group 2 spent 30 minutes.

After the conclusion of the professional development course, the two researchers viewed both videos independently and made note of instances in which participants and instructors appeared to question or negotiate the boundaries of the problem. For each such instance, we attempted to identify factors in the group discussion, the instructor's comments, or the design of the task that may have influenced the group's decision about how to define the problem space. We repeatedly met together to compare analyses and come to consensus on any discrepancies. We report results of this initial work here; however, we intend to continue to refine our analysis process as we attempt to apply it to the data we have collected (video and audio) for small groups working on other contextualized problems.

Results

In both groups that participated in the study, the group leader read the task, and the group worked gradually toward a consensus solution, making assumptions about the situation described, making preliminary estimates, and refining these estimates to produce a reasonable and practically feasible solution. Along the way, each group confronted questions about which elements of the real-world situation should be taken into account and which considerations lay beyond their co-constructed

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boundaries. In this section, we analyze each group's negotiation of the problem space and observe how this space evolved over the duration of the group's work on the problems. All names used below are pseudonyms.

Shifting Responsibility for Boundary-Setting: The Case of Group 1

Vicki, the leader of Group 1, introduced the problem by reading her card aloud to her teammates Tina, Kenny, and Nalda. Shortly after reading the card, Vicki questioned whether the group was allowed to consider information not on the card. The question of how much discretion the group has in negotiating problem conditions and goals occurred again later, as Vicki asked whether the question was about "efficiency" or about how many trucks the state should send. Upon asking this, Vicki said, "I don't know how far we're allowed to take this," suggesting that authority for determining problem boundaries lay at least partially outside of the group itself. We hypothesize that many teachers' prior experiences with contextual problems (as teachers or learners) may consist mainly of problems for which the boundaries are largely pre-determined by the problem statement, or as structured by the teacher.

Table 1: Interactions Influencing the Problem Space for Group 1

Interaction	Action/response	Possible causes of interaction
Vicki: Are we allowed to extrapolate outside of what is on the card? We would need to know how much a truck could carry, average family size…	Tina begins to look up information on phone.	Contextual problems encountered in school often provide the information that is needed; no more, no less. In this setting the group must negotiate the boundaries of the problem space.
Nalda: Are we looking for realistic solutions to this? Because the state isn't going to pay for that many a day… each truck can make four trips…	Nalda's teammates assert that they are counting truckloads, not distinct trucks.	Nalda believes that in this case, issues of realism should at least be considered. Nalda uses the pronoun "we," while Vicki uses the pronoun "they."
Kenny: What if they don't have tankers, they have an average water truck?	Group considers both scenarios and produces an estimate for each.	The problem is ambiguous on the issue of which type of truck the state will use. The group does not have the resources to resolve this ambiguity, but is willing to manage it as a condition of the problem.
Tina: So did we answer the question? Vicki: I feel like we would need more parameters though to be able to really integrate the 23 miles. Nalda: I feel like they give us the 23 miles for us to estimate how many trucks.	Group begins to consider multiple trips per truck.	Task design: Tina cannot look at the card with the question on it. Nalda pushes the group not to set aside the mileage information. She reframes the problem so as to put group members inside the real-world situation.
Kenny: How long is a tanker? Nalda: I don't know. … Kenny: Where are they storing this? Nalda: Well, water towers. Kenny: I'm just thinking/ Nalda: /Half an hour to fill, half an hour to get there…	Tina pulls up a picture of a tanker on her phone and shows the group. Nalda redirects the group's attention to the calculation of the number of trucks.	Nalda seems to view Kenny's queries as outside of the problem's boundaries.

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The group ultimately developed estimates for the number of trucks needed in two different scenarios: if the state sends large tanker trucks, and if it sends smaller water trucks. The group's initial approach assumed that each truck would make only one trip per day, and that each resident of the town would receive 90 gallons of water per day. This led to an estimate of 221 tankers per day. At this point Nalda raised the concern that sending 221 tankers per day would not be realistic, and suggested a model in which each tanker makes several trips per day. The group initially dismissed this suggestion, claiming that the problem was to estimate the number of truckloads, not tankers. However, by pointing out that the group had not used the information provided about the distance between the towns, which we interpret as an appeal to an external source (i.e., the problem author) in order to determine a problem space boundary, Nalda later persuaded her teammates to consider the possibility of allowing each truck to make several trips per day, and count the number of trucks rather than the number of truckloads of water. The group eventually produced an estimate of 56 tanker trucks.

Table 1 outlines some instances in which the problem space was negotiated, explicitly or implicitly, by members of Group 1. We note here that, for Group 1, interpreting the implicit intentions of the problem author appears to be a central part of their effort to negotiate the problem boundaries, and thus the problem space. At the same time, the group also attended to whether a particular approach or solution was realistic. In the data, we found multiple examples of this push-pull between school mathematics norms for contextualized problems (*e.g.*, figuring out what the problem author intends) and the desire to find a realistic solution. Importantly, we note that attention to realism may itself relate back to expectations about how we do mathematics in school when faced with contextualized problems for which some information is not given.

The Instructor's Role in Expanding the Problem Space: The Case of Group 2

Vince introduced the problem to his teammates Tobias, Darla, and Violet by summarizing the information on his card rather than reading it verbatim. The group immediately began searching the internet for information relevant to the problem and found that a water truck can carry 5000 gallons, and that the average American uses between 80 and 100 gallons of water per day. Based on this information, they obtained an initial estimate of 6450 trucks, which Violet deemed to be "excessive."

Spurred in part by the infeasibility of this estimate, the group then began to identify ways they could significantly decrease this estimate. Tobias suggested researching the minimum amount of water a person needs each day; based on his research, the group accepted a much lower estimate of 5 gallons per person per day. The group thus arrived at a more modest estimate of 358 trucks, still reflecting the implicit assumption that each truck will make only one run over the two-week period. The group presented this solution to Nancy, one of the PD facilitators. Nancy stated that the state did not have 358 trucks to spare, and that the group should try to determine the *minimum* number of trucks needed. After she left the group, Violet pointed out that the question did not ask for the *least* possible number of trucks, and Tobias claimed that Nancy had changed the question.

After this exchange, Tobias suggested considering how many trucks are needed per day (rather than for the entire two-week period); this brought the group's estimate from 358 down to 26. The group then gradually developed a plan in which six trucks take turns dropping water off at Squareville; at any given time, one truck is in Squareville dropping water, one truck is in the nearby town collecting water, and four other trucks are in transit between the two towns. Vince presented this solution to the other PD instructor, who endorsed it as an acceptable solution.

As the group worked on the problem, the problem space grew to encompass considerations of how much water a person needs during an emergency, and how much water an "average" truck can hold. However, only after Nancy visited the group and encouraged members to develop a more feasible solution did they seriously consider the possibility of having trucks perform multiple runs on

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the same day. This consideration entered the problem space at least in part due to Nancy's intervention. Table 2 below shows some instances in which Group 2 interacted to define the problem space and its boundaries, and our interpretations of possible causes of the interactions.

Table 2: Interactions Influencing the Problem Space for Group 2

Interaction	Action/response	Possible causes of interaction
Violet: Do we know how much water the trucks hold? Or how much each person needs? Vince: No.	Tobias starts to look for information on the internet using his tablet.	The problem cannot be solved without information that is not on the card.
Violet: Are they telling the people to limit the water? Because I feel like that would be beneficial.	Tobias determines that on average, a person uses between 80 and 100 gallons per day. The group doesn't pursue Violet's idea yet.	The group seems to feel that limiting water is beyond the boundaries of the problem.
Violet: It's 80-100 gallons per day, so do we just want to use 90?	Group calculated 90 x 14 x 25,600 = 32,256,000 gallons. Divided this by 5000 to obtain 6451 trucks.	Using the midpoint of a range as an estimator is possibly related to prior experience with school math problems; in this case, it may actually be worthwhile to use the lower end of the range in order to minimize the number of trucks needed.
Violet: 6451 trucks, that seems really excessive. Tobias: Let's see how much a person *needs* in a day. Violet: We don't know what limitations have been set for this town.	Group discusses different uses of water and eventually settles on 5 gallons per person per day, leading to an estimate of 358 trucks.	Initially, Violet seems to view the issue of water rationing as outside the boundaries of the problem. Eventually, the group shifts the boundaries to encompass this question.
Nancy: Yeah, well Circleville's also having a water issue, and I just don't have 358 trucks, so what's the minimum number I need? … Nancy: So think just a little more about how many trucks you *need*. Like what's the minimum number I can give you?	Group turns to the question of how long it takes for a truck to complete one cycle of loading, driving to town, unloading, and driving back.	Nancy observes that the group has not incorporated the possibility of trucks making multiple deliveries per day into the problem space; she uses the impracticality of a request for 358 trucks to encourage the group to reconsider the problem boundaries they have constructed.
Darla: How long does it take to unload a water truck? Violet: And how do you decide who gets water first? Are we figuring out the least? Is that the question? The least number? It just said how many trucks need to be sent, it didn't say least! Vince and Tobias: She changed it.	Group estimates how much time is needed for a delivery cycle and how many cycles are needed per day, and eventually decides upon 6 trucks.	Violet, Tobias, and Vince indicate their belief that they have been asked to enter a different problem space.

The push-pull between school mathematics and associated expectations of problem authors on the one hand and concerns for realism played out somewhat differently in this group. The group seemed initially unconcerned with limiting water, an aspect of the situation that would most certainly come into play in the real world. Yet, an unrealistic number of trucks did spur the group to reconsider water consumption, and this then became a part of the problem space. At this point, Violet raised another issue related to negotiating the problem boundaries, namely that the group did not know what limitations on water use had been put in place for the town in question. The group expressed frustration with the ambiguous nature of the problem space after the instructor questioned whether 358 was realistic.

Discussion

We offer this report as an initial analysis of the construct of "problem space" as it applies to contextual problems in mathematics. We make no claim that our findings generalize across all classes of mathematics problems and all groups of learners. The negotiation of a problem space may look markedly different in the context of a more closed-ended task, and may also vary according to the age and mathematical background of learners. One may argue that in this particular study the participants' shared familiarity with rate reasoning allowed them to devote additional attention to considering boundary issues such as which quantities in the problem should influence the problem's solution and which should not. Further study, with different types of tasks and with different populations of learners, is needed for a better understanding of how problem spaces develop in different settings.

The expectations that learners have of teachers, problems, and genres of mathematical tasks are central to the establishment of problem boundaries and spaces within them. If learners are accustomed to tasks in which all relevant and necessary information is explicitly provided, they may initially hesitate to consider external sources of information when presented with an open-ended problem. This may lead to learners attempting to work within a problem space that is too narrow to provide the intellectual resources necessary to construct a solution. At the same time, learners may make decisions to expand the problem space when faced with a problem that does not explicitly provide all the resources necessary for its solution. However, the boundaries defining the problem space cannot expand endlessly; learners must, at some point, accept that the situation they are attempting to analyze contains some details that are inaccessible to them and therefore cannot be modeled mathematically.

Our analysis of the negotiation of problem boundaries has implications for the practice of designing open-ended problems. In analyzing the groups' work on the Water Shortage Problem, we found that the problem worked as intended in at least one respect: both groups originally obtained infeasibly large estimates for the number of trucks needed, and thus were encouraged (without external feedback) to revise their assumptions. Both groups decided that the problem space should include some consideration of whether the solution obtained was fiscally responsible. Additionally, both groups decided to include some analysis of whether the solution obtained was physically feasible; for example, Group 2 developed a scheme in which six trucks rotate in and out of Squareville in succession, dropping off water as they arrive. Thinking about the problem at this level of detail helped the group develop confidence that a solution with six trucks was feasible and would deliver enough water. We posit that open-ended problems that contain supports for the development of detailed models and that encourage winnowing out unreasonable solutions may support learners in expanding problem spaces to include practical considerations.

Our analysis also has implications for the orchestration of open-ended problems. Both groups had questions that one could easily imagine asking of a teacher; for example, Vicki might have wanted to ask a PD instructor whether it was permissible to consider information outside of the problem

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statement. However, in the absence of instructor guidance, the group quickly decided that information from the real world lay squarely within the boundaries of the problem, since the information on the card was inadequate. Because problem spaces evolve over time, a teacher implementing an open-ended problem may wish to take an observer role initially and allow the problem space to develop according to the explicit and implicit demands of the problem.

We conclude this report by highlighting two ways in which the problem space of the Water Shortage Problem may communicate with the broader space of students' real-world experience. Since the time of the creation of this problem, serious water crises have occurred in places such as Flint, Michigan and Corpus Christi, Texas. In subsequent implementations of the Water Shortage Problem, the authors have noticed that teachers who have experienced water crises such as these sometimes interact differently with the problem; they are more knowledgeable about how water is actually distributed during a water crisis, and more attentive to logistical issues such as how a town should time and manage water collection. We offer this as an example of learners' real-world experiences interacting with the problem space. As an example of the problem space talking back to the broader world in which the learners live, consider the following comment from Violet: "If this [90 gallons per day] is what I use on a regular basis and this [4-6 gallons] is what I use in a disaster… like… I feel like *this* is the disaster!" Seeing the disparity between everyday water usage in the U.S. and recommended water usage during an emergency may heighten learners' awareness of the possibility of scaling back water consumption and using natural resources at a more sustainable rate.

Endnote

Alphabetical listing of author names is intended to indicate equal contributions to the paper.

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