HOW SPATIAL REASONING AND NUMERICAL REASONING ARE RELATED IN GEOMETRIC MEASUREMENT

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The positive correlation between spatial ability and mathematical ability has been well-documented, but not well-understood. Examining student work in spatial situations that require numerical operations provides us with insight into this elusive connection. Drawing on student work with angle, length, volume, and area, we examine the ways in which students associate numerical operations with their spatial structurings of objects. We find that for students to correctly coordinate their spatial structurings and numerical operations, their solution methods must satisfy basic properties of measurement functions. We illustrate this claim by providing examples in which students successfully and unsuccessfully employ spatial-numerical linked structurings.

Keywords: Cognition, Geometry and Geometrical and Spatial Thinking, Technology

Numerous studies have found that spatial ability and mathematical ability are positively correlated (Mix et al., 2016). But specifying the exact nature of the connection between these abilities has been elusive, with much research in this area focused on understanding correlations between specific spatial skills (e.g., as measured by visualization and form perception tests) and mathematical performance (Mix et al., 2016). In this paper, we seek to precisely specify the spatial-mathematical connections in geometric measurement—a content area for which numerical and spatial reasoning must be properly coordinated. Indeed, de Hevia and Spelke claim that the human mind is "predisposed to treat number and space as related" (2010, p. 659). And researchers in mathematics education argue that understanding relationships between numerical and spatial reasoning is fundamental to developing a full understanding of geometric and measurement reasoning (Clements & Battista, 1992). However, although a great deal of research has investigated how students represent numbers on number lines (Gunderson, et al., 2012), in geometric measurement, numerous situations arise that are more complex than envisioning numbers on number lines. We have investigated numerous instances of these more complex situations, and in this paper, we analyze these situations to more fully understand the nature and properties of the connection between numerical and spatial reasoning in geometric measurement.

Theoretical Framework

Measurement Properties

For spatial reasoning and numerical reasoning to be properly connected in geometric measurement, certain basic properties of measurement functions must be followed, as described by Krantz: "When measuring some attribute of a class of objects or events, we associate numbers … with objects in such a way that properties of the attribute are faithfully represented as numerical properties" (1971, p. 1). That is, if *M* is the function that assigns measurement values to objects—*M(a)* is the measure of object *a*—then, consistent with Krantz et al. and basic axioms for geometric measurement (Moise, 1963), *M* satisfies the following properties:

1. If object *a* and object *b* are congruent, then $M(a) = M(b)$.
2. Object *a* is spatially larger than object *b* if and only if $M(a) > M(b)$.
3. If we join-without-intersection object *a* and object *b*, then $M(a\ \ join\ \ b) = M(a) + M(b)$.

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4. Given n copies of congruent and non-overlapping unit-measure objects $a_1 \ldots a_n$:

$$If \bigcup_{i=1}^{n} a_i \cong b, then\ nM(a_1) = M(b).$$

These properties justify the measurement iteration process in which we determine measure by iterating a unit measure to "cover" the object being measured with no gaps or overlaps. If, however, there are gaps in a unit-measure covering so that it is a proper subset of the object being measured, then Property 2 implies that the measure of the covering will be less than the measure of the object. If there are overlaps, then Property 3 is not satisfied, so we cannot count/add the unit measures to find the measure of the object.

Spatial-Numerical Linked Structuring

Beyond the basic measurement properties, linking spatial and numerical reasoning in geometric measurement requires use of what we call *spatial-numerical linked structuring* (SNLS). *Spatial structuring* is the mental act of constructing a spatial organization or form for an object or set of objects, imagined or real (Battista, 1999, 2007, 2008; Battista et al., 1998; Battista & Clements, 1996). *Numerical structuring* is the mental act of constructing an organization or form for a set of computations. A correct *spatial-numerical linked structuring* is a coordinated process in which a numerical measurement operation is performed along with a linked spatial structuring in a way that is consistent with the above measurement properties. Incorrect student enumeration is generally based on SNLS that violates at least one of the measurement properties. Note that each measurement property expresses a spatial-numerical linked structuring. For instance, putting one angle inside another to decide which is bigger spatially organizes the two angles with respect to each other. In this paper, we give examples of correct and incorrect spatial-numerical linked structuring.

Methods and Data Sources

The data we analyze comes from individual interviews and one-on-one teaching experiments with elementary and middle school students from several NSF-funded geometry projects awarded to the first author.

Sample Results and Discussion

To illustrate our results and analysis, we provide examples of spatial-numerical linked structuring for angle, length, volume and area. The examples (a) describe student actions, (b) discuss what students did, and (c) interpret what students did using the spatial-numerical linked structuring conceptual framework.

Spatial-Numerical Linked Structuring for Angles

For the computer-presented problem in Figure 1, KS employed several spatial-numerical linked structurings to find the angle that rotates the green point onto the red point.

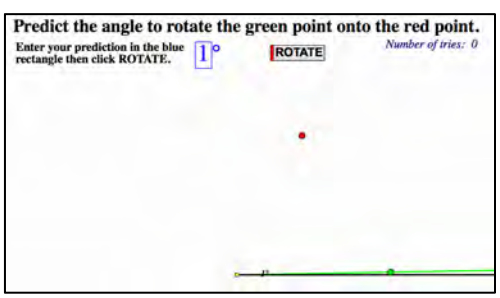

Figure 1.

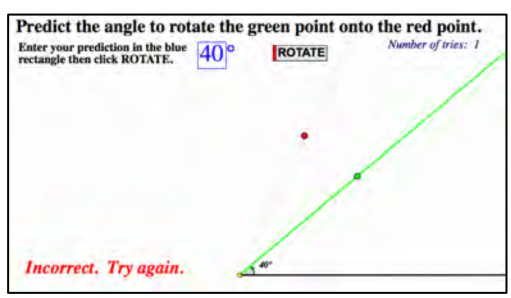

Figure 2.

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KS: I think it may be 40 [enters 40; green ray rotates to the 40° position Figure 2].
Int: So what are you thinking?
KS: So if this is 40 [angle in Figure 2], I may have to go up maybe *20 more*.
Int: Okay, why 20 more?
KS: Cause, if this was 40 [pointing at the interior of the green 40° angle], then half of it is this [pointing to the interior of the angle between 40° and the target angle; enters 60°; Figure 3].

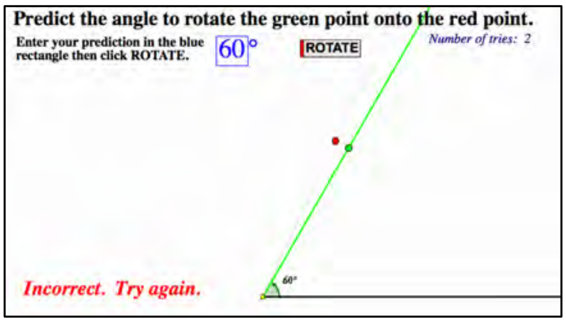


Figure 3.

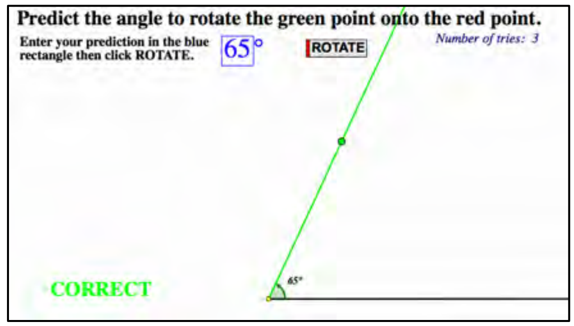
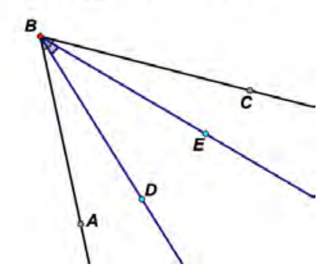


Figure 4.

Int: Very close, what are you thinking?
KS: Hum. So maybe with the other [computer page showing 5° iterations of a ray] it shows that they were really close together, so maybe it'd be 65 [enters 65°; Figure 4].

Discussion. KS used a sequence of spatial-numerical linked structurings (SNLSs) for solving this problem (see Table 1). After viewing the result of her first estimate, which is quite a bit off, KS reasoned that her original estimate was too small. This is an example of SNLS 1, in which KS recognized the smaller-than spatial relationship between the angle she made and the target angle. KS then, using SNLS 2, spatially compared the angle between the green ray and the black ray from her 40° estimate as half of the 40° angle. Then, using SNLS 3, she added 20° to 40° to produce a second estimate of 60°. Finally, in her third estimate, KS used SNLS 4 followed by SNLS 3 to recall a previously viewed 5° angle and add that to her estimate of 60°.

Table 1: Definitions of Types of Angle Spatial-Numerical Linked Structuring

SNLS 1. [Bigger Angle ⇔ Greater Measure; Property 2]: If Angle X is bigger than Angle Y, then the measure of Angle X is greater than the measure of Angle Y. Definition/Spatial Structuring: Angle X is spatially "bigger" than Angle Y if the angles have the same vertex and Angle Y fits in the interior of Angle X.	**∠ABC is bigger than ∠DBE.** 

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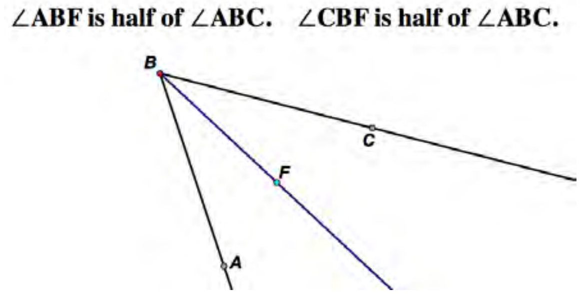
<table>
<tr>
<td>SNLS 2. [One-half angle ⇔ One-half measure] If Angle X is one-half of Angle Y, then the measure of Angle X equals one-half of the measure of Angle Y.

Definition: Angle X is one-half of Angle Y if the initial side of Angle X coincides with the initial side of Angle Y and the terminal side of Angle X is in the interior of Angle Y and is halfway toward the terminal side of Angle Y.</td>
<td>
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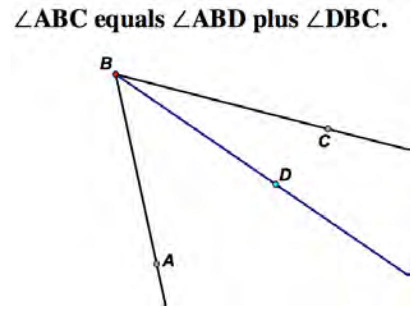
<td>SNLS 3. [Add angles ⇔ Add measures; Property 3] If Angle X equals Angle Y plus Angle Z, then the measure of Angle X equals the measure of Angle Y plus the measure of Angle Z.

Definition: If point D is in the interior of Angle ABC, then Angle ABC equals Angle ABD "plus" [joined with] Angle DBC.</td>
<td>
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<td colspan="2">SNLS 4. [Compare perceived angle to recalled angle; Property 1] The student compares a perceived angle to the recalled visual image of a previously seen angle, and says that the two angles are congruent so their measures are equal.</td>
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Spatial-Numerical Linked Structuring for Length

To examine the way students use spatial-numerical linked structuring with length, we consider a student's work in a computer golf game (Figure 5). Students "putt" a ball by entering a distance and angle. When students click the *PUTT* button, the ball travels to the right the entered distance, then arcs around counterclockwise as it sweeps out the entered angle, which is a multiple of 5° (Figure 6). Students receive visual feedback on each of their estimates until they determine a correct putt angle and distance. SJ is doing the problem in Figure 5.

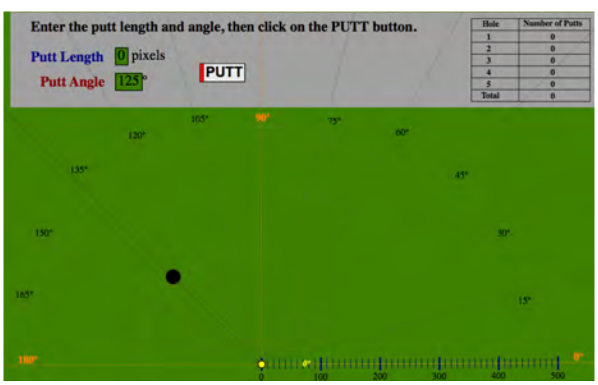

Figure 5.

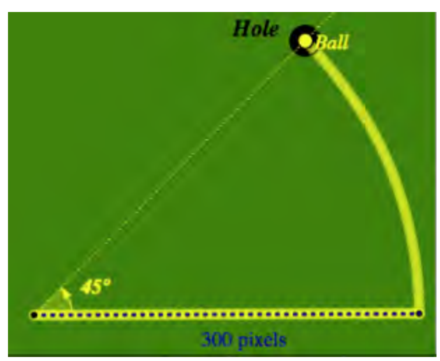

Figure 6.

Example 1:

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SJ: [Pointing along hash marks 0-140 on the number line with the cursor] These lines are the pixels right?

Int: Yep. So this [pointing with a finger] is 100 pixels. That's 200 [pointing]. So they might be counting by, what do you think, in those little ones [points to hash marks between 100 and 200 on the number line]?

SJ: 25s?

Int: So let's see. If this is 100 [pointing to 100]. That'd be 125 [pointing to 110], 150 [pointing to 120], 175 [pointing to 130], 200 [pointing to 140].

SJ: Aw, never mind.

Int: So what do you think?

SJ: 10, 15, [points along hash marks 110 to 190 on the number line] 45. No [goes back to 110 on the number line]. *Oh, tens!*

Discussion: SJ understood that each hash mark represents the same amount of space (Measurement Property 1), but she could not immediately determine the correct numerical value for the distance associated with the space between each hash mark. When she estimated 25 as the distance, the Interviewer iterated by 25 starting at the landmark for 100 so that SJ recognized that the numerical value of 25 for each hash mark was too large. After choosing a smaller value of 5 and realizing it was too small, she correctly concluded that 10 was the distance between each hash mark. This is an example of a student using the measurement properties and the iteration spatial-numerical linked structuring to develop an understanding of the coordinate system inscription embedded in the game. The mistakes she made with the 25 and 5 estimates for hash-mark values seem to arise from not coordinating the number iterations of the hash-mark interval with the beginning and end values of the 100-to-200 interval. In essence, she violated Measurement Property 4.

Example 2:

Figure 7.

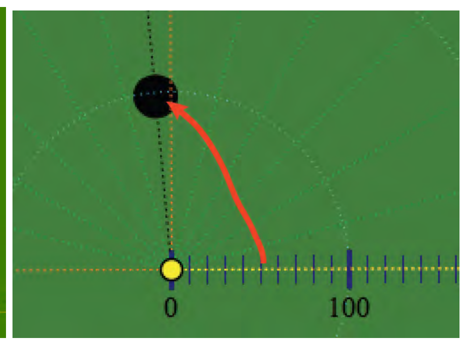


Figure 8.

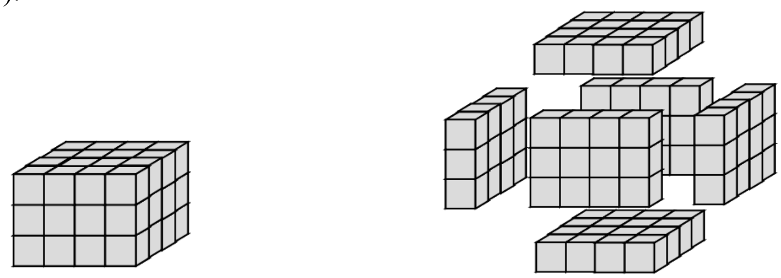
SJ: [For the problem in Figure 7] Okay. This one is probably going to be 50 [points the cursor at 50 on the number line]. Because like 10, 20, 30, 40, 50 [counting on the 10-50 hash marks with the cursor]. Here's the 50 [moves from 50 towards the hole; Figure 8]. Maybe even 60.

Discussion: In this example, SJ's spatial structuring of the rotation path of the ball is incorrect. Because of this incorrect spatial structuring, her numerical choice for the length of the putt was incorrect—her spatial structuring violated Measurement Property 2. Importantly, note that SJ does not understand the meaning of the distance-arc inscriptions for the embedded coordinate system. Because of her incorrect structuring of a point rotation, she does not recognize that every point on a distance arc is the same distance from the origin as the reference measurement on the number line.

Similar to many elementary students using rectangular coordinates (Battista, 2007; Sarama et al., 2003), SJ does not conceptualize the spatial-structural metric properties of the coordinate system. In order to accurately recall spatial relations, students must abstract not pictures but mental models that have encoded spatial properties of objects (e.g., Hegarty & Kozhevnikov, 1999).

Spatial-Numerical Linked Structuring for Volume

In the complex context of enumerating unit cubes in rectangular boxes, students must link their numeric structuring to their spatial structuring. For instance, consider the following example (Battista, 2004, 2012).

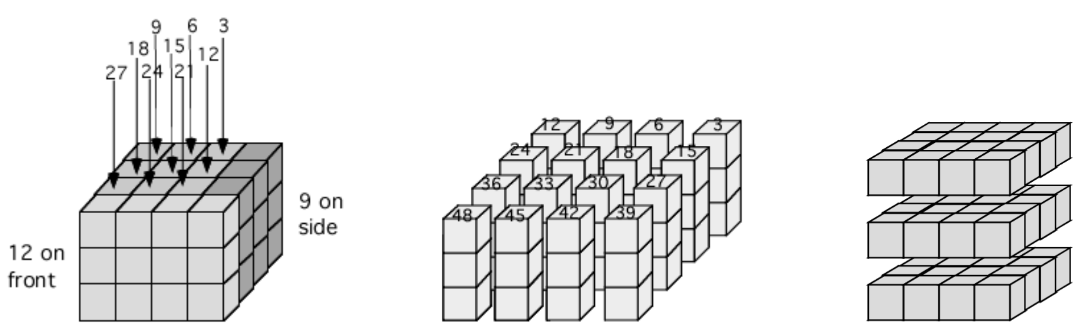


a b

Figure 9.

For the building shown in 9a, Fred counted based on the spatial structuring shown in 9b. He said that there are 12 cubes on the front, then immediately said there must be 12 on the back; he counted 16 on the top, and immediately said there must be 16 on the bottom; finally, he counted 12 cubes on the right side, then immediately said there must be 12 on the left side. He then added these numbers. Fred's numerical structuring of *12 + 12 + 16 + 16 + 12 + 12* corresponded to his spatial structuring of *(front + back) + (top + bottom) + (right side + left side)*. So his spatial structuring of the building into composite units of cubes violated Measurement Property 3—the cubes that he double-counted occupied the same space.

Below we see two alternative SNLSs for the same cube building. On the left, the spatial structuring of *front + what's left on right side + (9 columns of 3)* corresponds to the numerical structuring of *12 + 9 + (repeat 9 times counting 3 cubes in a column)*. In Figure 10b, we see a column spatial structuring that a student numerically structured as *3, 6, 9...45, 48*. Another spatial structuring is horizontal layers (Figure 10c) which students variously structure numerically as *16 + 16 + 16*, *3 x 16*, or *skip counting 16, 32 48*.



a b c

Figure 10.

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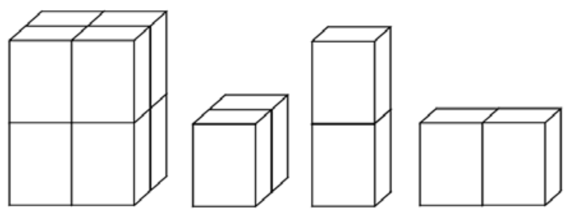
Note that, unlike the first SNLS in Figure 9b, the last three SNLSs produce correct answers. Given that there are multiple correct SNLSs for this cube building enumeration task, part of SNLS reasoning is consideration of enumeration efficiency. The SNLS in Figure 10a is correct but too cumbersome to be efficient and too unwieldy for large arrays. The SNLS in Figure 10b could be conceptualized in terms of 3 cubes in each column times 4 columns in a horizontal row times 4 horizontal rows, leading to the standard volume formula, as could the layer structuring SNLS (Figure 10c). So part of SNLS reasoning is metacognitive consideration of enumeration efficiency. Furthermore, using SNLS reasoning to make sense of the volume formula illustrates how SNLS reasoning can be used for generalization, not just enumeration.

SNLS reasoning as sense making for volume. The next example further illustrates how SNLS reasoning can be used to make sense of geometric measurement problems that deal with generalizations rather than enumeration. Consider the following problem (Battista, 2012). *The dimensions of a box are 3 cm by 2 cm by 4 cm. Give the dimensions of a box that has twice the volume.* The most common error that students make on this problem is to multiply all three dimensions by 2. SNLS reasoning can help students understand why the numerical structuring of *multiply all the dimensions by 2* is incorrect and what correct numerical structurings are possible. For instance, Figure 11a shows that doubling all the dimensions of a *3 cm by 2 cm by 4 cm box* gives 8 times the original box volume, whereas Figures 11b, c, d show that doubling any one of the dimensions of the box doubles its volume.

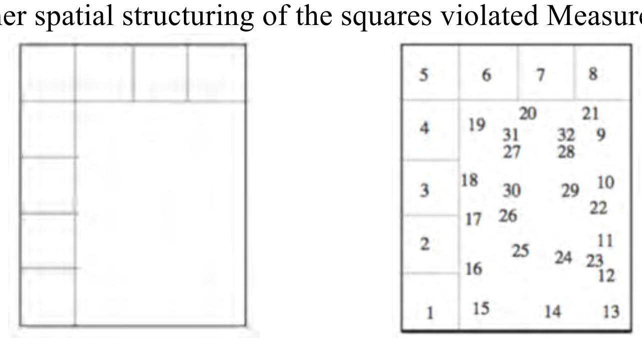


a b c d

Figure 11.

Spatial-Numerical Linked Structuring for Area

The ability to mentally construct an accurate spatial structure for rectangular arrays is a critical reasoning process for students determining area. But this process is surprisingly difficult for students to construct (Battista, et al., 1998). For example, student CS was asked to determine the number of squares required to completely cover the inside of the rectangle in Figure 12a (Battista, et al., 1998). CS counted in a non-random way as shown in Figure 12b. She counted the pre-drawn squares first, then she counted 9, 10, 11, 12, 13 down the right side and an equivalent number (15, 16, 17, 18, 19) up; overall counting in a clockwise spiral (Battista, et al., 1998). Because of the overlapping positions of CS' squares, her spatial structuring of the squares violated Measurement Property 3.



a b

Figure 12.

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Significance

In addition to helping us untangle the complicated nature of students' coordination of spatial and numerical reasoning, this research helps us decompose the basic mental processes that students use in geometric measurement. It therefore helps us understand, for one content area, more precisely how spatial reasoning is related to numerical reasoning in geometry, which in turn helps us start penetrating why spatial reasoning has been found to be related to mathematical reasoning in so many correlational studies.

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