Error Patterns with Fraction Calculations at Fourth Grade as a Function of Students' Mathematics Achievement Status

Robin F. Schumacher

Instructional Research Group

Amelia S. Malone

Vanderbilt University

in press @ *Elementary School Journal*

Accepted for publication February 2, 2017

This research was supported in part by Grant R324C100004 from the Institute of Education Sciences in the U.S. Department of Education to the University of Delaware, with a subcontract to Vanderbilt University and by Award Number R24HD075443 and by Core Grant HD15052 from the Eunice Kennedy Shriver National Institute of Child Health and Human Development Vanderbilt University. The content is solely the responsibility of the authors and does not necessarily represent the official views of the Institute of Education Sciences, the U.S. Department of Education, the Eunice Kennedy Shriver National Institute of Child Health and Human Development, or the National Institutes of Health.

Abstract

The goal of the present study was to describe fraction-calculation errors among 4th-grade students and determine whether error patterns differed as a function of problem type (addition vs. subtraction; like vs. unlike denominators), orientation (horizontal vs. vertical), or mathematics-achievement status (low- vs. average- vs. high-achieving). We specifically addressed whether mathematics-achievement status was related to students' tendency to operate with whole number bias. We extended this focus by comparing low-performing students' errors in two instructional settings that focused on two different types of fraction understandings: core instruction that focused on part-whole understanding vs. small-group tutoring that focused on magnitude understanding. Results showed students across the sample were more likely to operate with whole number bias on problems with unlike denominators. Students with low or average achievement (who only participated in core instruction) were more likely to operate with whole number bias than students with low achievement who participated in small-group tutoring. We suggest instruction should emphasize magnitude understanding to sufficiently increase fraction understanding for all students in the upper elementary grades.

Error Patterns in Fraction Calculation at Fourth Grade as a Function of Students' Mathematics Achievement Status

A solid foundation in fraction understanding is critical for success with higher-level mathematics topics, such as algebra, and for competing in the American workforce (Geary, Hoard, Nugent, & Bailey, 2012; Siegler et al., 2012). Fractions are typically introduced in third grade, and fourth grade is the first year of intensive focus on fractions. By the end of fourth grade, students are expected to be proficient with a range of fraction concepts, including adding and subtracting fractions with like and unlike denominators. However, fraction calculations pose substantial difficulty for many students, especially students at risk for mathematics learning difficulties (Bottge, Ma, Gassaway, Butler, & Toland, 2014; Mix, Levine, & Huttonlocher, 1999; Siegler & Pyke, 2013). These difficulties persist into high school and beyond (Calhoon, Emerson, Flores, & Houchins, 2007). Therefore, understanding common errors with fraction calculations in fourth grade can provide insight for how to design instruction to increase proficiency and ultimately decrease persistent failure.

This paper focuses on *fractions* presented with a numerator and denominator, i.e., $\frac{a}{b}$. We refer to *whole numbers* as positive integers. When the curriculum shifts to include fractions in the late elementary grades, many students struggle to consolidate whole numbers and fractions into a single numerical framework and tend to over-apply whole-number concepts to fractions (e.g., DeWolf & Vosniadou, 2011; Siegler, Thompson, & Schneider, 2011). As such, many students approach fraction learning with *whole number bias* (WNB), which is the overgeneralization of the one-to-one counting scheme to fraction problems (e.g., DeWolf & Vosniadou, 2011; Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004; Vamakoussi & Vosniadou, 2004; Vosniadou & Verschaffel, 2004). WNB, therefore, underpins fraction misunderstanding.

When students operate with WNB with fraction concepts, they fail to simultaneously attend to the numerator and denominator in a meaningful way. Pervasive WNB indicates poor understanding of foundational fraction concepts and indicates students are struggling to assimilate fraction properties into their general concept of “number”. Because proficiency with fractions is one of the strongest predictors of future success with algebraic concepts (e.g., Siegler et al., 2012; Wu, 2008), WNB needs to be prevented early (i.e., in the upper elementary grades), before students fall too far behind.

Not only do students tend to overgeneralize whole-number properties to fractions during initial fraction learning, but research suggests that they also tend to reciprocally apply fraction properties to whole numbers (e.g., when operating with mixed numbers) and often default to whichever representational understanding is more familiar (Mack, 1995). One example of operating with WNB is viewing the numerator and denominator in a fraction as two independent whole numbers. In doing so, students misapply whole-number logic and incorrectly assume that as the magnitude of the whole numbers that comprise the fraction increase, magnitude of the fraction also increases (Meert, Gregoire, & Noel, 2010a, 2010b; Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004). The relationship between the numerator and denominator determines magnitude for fractions, not the whole-number magnitudes that comprise it. Misinterpreting fraction magnitude can be detrimental to students’ ability to accurately solve fraction calculations.

Whole Number Bias (WNB) and Fraction Calculation Errors

Because conceptual and procedural knowledge are believed to develop iteratively (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001), students are unable to judge the accuracy of their fraction calculations without foundational understanding of fraction magnitude. For example, Siegler & Pyke (2013) found that individual differences in fraction calculations among sixth- and eighth-grade students correlated with their fraction-magnitude understanding. Fraction calculations involve a more tedious process than is required for whole-number calculation (e.g., Mix et al., 1999). Thus, students with poor magnitude understanding may operate with WNB to avoid multi-step procedures.

Consider the procedural differences to solve whole number vs. fraction addition problems. When adding 4 + 5, students find the answer by simply adding the numbers together. If a struggling student cannot immediately recall the answer from long-term memory, she can use a counting-up strategy, for example, and count up from 4 to reach 9. Even with this compensatory strategy, basic number combinations are essentially solved in a single step. By contrast, when adding $\frac{1}{4}+\frac{2}{4}$, students must consider the denominators, determine there is no need to write an equivalent fraction (i.e., they can add right away), understand that the denominator in the answer is 4, and then compute the answer, $\frac{3}{4}$. When the denominators are different, additional steps are required to write equivalent fraction(s) so the denominators are the same. For example, when adding $\frac{1}{4}+\frac{1}{2}$, the process includes the following steps: (a) find an equivalent fraction so the addends have the same denominator; (b) recognize 2 as a multiple of 4; (c) decide to rewrite $\frac{1}{2}$ as an equivalent fraction with 4 in the denominator; (d) multiply the numerator and denominator in $\frac{1}{2}$ times 2; (e) write revised addition problem as $\frac{1}{4}+\frac{2}{4}$; (f) determine 4 is the denominator in the answer; and (g) compute the answer, $\frac{3}{4}$. Understanding *why* the denominators must be the same to add and subtract fractions is an indicator of magnitude understanding. That is, you can only add or subtract the numerators if both fractions have the same size parts.

Proficiency with fraction calculations, therefore, requires students to effectively apply their magnitude knowledge of fractions (i.e., evaluating the denominators) and retrieve basic multiplication (i.e., computing equivalent fractions) and addition facts (i.e., solving the problem) to correctly to compute the answer. When students ignore fraction properties and rely on whole-number properties to solve fraction calculation problems, they demonstrate their misunderstanding of fraction magnitude. Operating under the assumption that conceptual and procedural understanding develop in iterative fashion (e.g., Rittle-Johnson & Koedinger, 2009), understanding magnitude—which is one aspect of conceptual understanding with fractions—has potential for boosting students' proficiency with fraction calculations (Siegler & Pyke, 2013).

The present study investigates students' fraction calculation errors to determine whether WNB is common among fourth-grade students and whether the tendency to operate with WNB differs by incoming mathematics achievement status. We summarize three relevant studies, which assessed fraction misunderstanding through analyzing error patterns. Two studies utilized a qualitative data analysis approach (e.g., Haseman, 1986; Newton, Willard, & Teufal, 2014) by characterizing errors and providing descriptions. One study (Bottge et al., 2014), used a quantitative data analysis approach. Results from across the three studies underscore the relationship of WNB to fraction calculations, especially for students who have disabilities or who are at risk for failure.

Haseman (1986) examined 70 case studies, including interview data, to analyze error patterns on fraction number line activities and computation problems. The author did not use quantitative methods to analyze error patterns. He explored students' misunderstanding of fractions by describing their problem-solving choices from written work and post interviews. He presented each student a variety of fraction representations and relied on error patterns as well as student explanations to inform theoretical conclusions regarding each student's mathematical thinking. Because his research question was theoretical in nature, he did not identify and describe specific errors independent of a student's verbal explanation of written work. Instead, he focused on whether students were consistent in their application and explanation for solving fraction problems. He concluded that errors were most often the result of WNB.

Newton et al. (2014) looked at fraction-calculation error patterns with addition, subtraction, multiplication, and division among 11 sixth-grade students with learning disabilities. During instruction, the classroom teacher explicitly linked fraction concepts to procedures in order to support relational understanding between the numerator and denominator, which is similar to the way we define magnitude understanding. One way their instruction focused on relational understanding was by linking adding unit fractions to multiplying a unit-fraction by a whole number (e.g., $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$ and $\frac{1}{5} \times 3$). Results from the study showed students systematically misapplied whole-number properties to addition and subtraction problems with unlike denominators. Although authors did not specifically address WNB, the error descriptions included a lack of attention to fraction properties that underscore magnitude understanding.

The approach Bottge et al. (2014) used in their fraction error analysis was most similar to the present study. They applied quantitative methodology to their error analysis and indexed common error patterns for fraction calculations among middle-school students with learning disabilities. They examined whether error patterns were remediated as a function of intervention vs. business-as-usual (BAU). They identified two error categories: those due to WNB (e.g., combining numerators together and denominators together, selecting a denominator from the two given, and adding together all numbers to find a total sum) or those related to the procedural algorithm (e.g., failing to rewrite an equivalent fraction when required). After coding for these two errors, Bottge et al. (2014) looked at how intervention or BAU affected students' error patterns from pre- to post-testing. Results showed error patterns at pretest were similar between the two groups, whereas posttest errors differed by treatment condition. Students who participated in intervention significantly reduced two of three errors that demonstrated WNB. Overall, errors that occurred for students in intervention were less frequent than those in BAU and students who participated in intervention significantly reduced their WNB errors, thus increasing overall performance on fraction calculations.

Our study extends the literature in three important ways. First, we extend the work from Newton et al. (2014) by looking at problem types and features in relation to error patterns. In our study, we analyze whether operation (addition vs. subtraction), like vs. unlike denominator, and problem orientation, (horizontal vs. vertical) affect errors. Second, we describe fraction-calculation errors among fourth-grade students across three achievement groups: low achieving (LA), average achieving (AA), and high achieving (HA). In Newton et al. (2014) and Bottge et al. (2014), students with or at-risk for disabilities were the focus. Because prior research has shown that fraction problems pose unique challenges to students who do not struggle with whole numbers (Hansen, Jordan, & Rodrigues, 2015; Jordan et al., 2013; Namkung & Fuchs, 2015; Vukovic et al., 2014), we address error patterns and misconceptions across achievement groups. By focusing on calculation errors during fourth grade (the first year of intensive focus on fractions) we look at important implications for designing curriculum to prevent the difficulties documented by others in the later grades (e.g., Bottge et al., 2014; Siegler & Pyke, 2013). Third, to determine whether WNB can be attenuated by instructional approaches, we include a fourth comparison group which is comprised of LA students who received small-group tutoring (LA-T) that primarily focused on magnitude understanding. LA, AA, and HA students received their fraction instruction in the core classroom. Unlike small-group tutoring, which had a strong emphasis on magnitude understanding, instruction in the classroom largely addressed part-whole understanding of fractions and procedures related to calculations (based on teacher report).

Research Questions and Hypotheses

The goal for the present study is to describe error patterns with fraction-calculation problems among fourth-grade students. We were specifically interested in errors that demonstrated WNB, although we coded for a range of errors. Our error analysis was guided by two research questions. Our first research question is exploratory, and includes an investigation of accuracy and WNB by problem type and orientation across the sample of students. Our second, and primary research question, addresses errors as a function of mathematics-achievement status. To understand the development of fraction misconceptions among LA students, we compare them to average- and high-achieving students. To gain a better understanding of possible ways to reduce students' misconceptions and frequency of WNB errors, we extend our focus by also assessing low-achieving students' error patterns who participated in supplemental tutoring (LA vs. LA-T) that focused on fraction-magnitude understanding.

Research Question 1: Does accuracy and WNB errors differ by problem type (i.e., addition vs. subtraction, like vs. unlike denominators) or orientation (vertical vs. horizontal) across the achievement groups? We expected greater accuracy on the following: (a) problems with like denominators because there are fewer steps required for solution and (b) addition problems because students tend to be more accurate for addition than subtraction (Kamii, Lewis, & Kirkland, 2001). We explore whether orientation (horizontal vs. vertical) affects accuracy and error patterns because findings from Raghubar et al. (2009) suggests problem orientation affects performance for students with disabilities. We hypothesize that WNB errors may be more common on horizontally-presented problems because students will simply add or subtract across numerators or denominators and vertically-presented problems may elicit more haphazard errors and reflect a lack of attention to the numerators and denominators altogether.

Research Question 2: Do accuracy and error patterns differ as a function of (a) incoming mathematics achievement and (b) instructional setting for LA students who received tutoring that emphasized magnitude understanding versus LA students who received fraction core mathematics instruction (i.e., in the general education classroom) that emphasizes part-whole understanding and procedures? We expect the HA group to be the most accurate, followed by AA then LA. We expect a higher frequency of WNB errors among LA students followed by AA students and then HA students. We categorized students as HA, AA, or LA in mathematics as indexed by a screener that included primarily whole-number problems. We chose this screener because the items spanned grades K-12 and provided an index of general mathematics achievement. We did not screen based on fraction achievement because incoming fourth graders have had limited experience with fractions.

We expected the LA-T students to be more accurate and have fewer WNB errors than LA students because small-group tutoring focused on magnitude understanding. By including the LA-T students in the analysis with the LA, AA, and HA groups, we are able to address whether a focus on magnitude increased accuracy and decreased WNB errors which tested the Siegler et al. (2011) hypothesis that increasing students' magnitude understanding affects conceptual understanding, thus boosting their procedural competence (Rittle-Johnson et al., 2001). (Note that the LA and LA-T students performed comparably at pretest, ES = .02, 95% CI [-0.23, 0.26]).

Method

Participants

Achievement Groups. The present study includes fourth-grade students who participated in a larger intervention study focused on fraction-magnitude understanding (Fuchs et al., 2013). In the larger study, students were screened at the beginning of fourth grade on a norm-referenced measure of mathematics, the *Wide Range Achievement Test - Arithmetic* (WRAT-4; Wilkinson, 2008) and students who scored below the 35th percentile were randomly assigned to tutoring or control conditions and effects of intervention were assessed (see Fuchs et al., 2013).

For our error analysis, we used a sub-sample of students who were screened on the WRAT-4 in the beginning of the larger study to comprise achievement groups. We used the 27th percentile for the cut point for LA students in the present study, which is lower than the cut point in the larger study. We made this decision for two reasons: to be more in line with the framework put forth by Geary (2013) that students at-risk perform below the 25th percentile,[1] and to ensure there was a substantive difference between the LA and AA groups.

LA, AA, and HA distinctions were made based on the WRAT-4 manual's qualitative performance descriptions corresponding to percentile rank standard score ranges (Table 3.1 in the WRAT-4 manual). In order to assess error types on an item-by-item basis, we were constrained to using categorical groups and therefore stratified students into three distinct achievement groups *a priori* based on their WRAT-4 percentile scores. However, because identifying achievement groups is not an exact procedure, especially for LA students (see Mazzocco & Myers, 2003), and there are differing schools of thought for how to delineate LA, AA, and HA students, we created score buffer zones among groups as an attempt to ensure a

[1] In the larger study, students below the 35th percentile were included so we could assess responsiveness to instruction for students with more vs. less severe risk for mathematics difficult (i.e., below 15th percentile and between the 16th and 34th percentiles; see Fuchs et al., 2013), which is not the purpose in the present study.

substantive (rather than arbitrary) difference among achievement groups. (Note that these buffer zones correspond to a 1-point standard score difference on the WRAT-4.) LA and LA-T students scored below the 27th percentile; AA students between the 43rd and 65th percentiles; and HA students scored at or above the 81st percentile.

We randomly sampled students from each of the categories to comprise the sample for the error analysis. Data were collected for 824 fourth-grade students from 53 classrooms in 13 schools as part of the larger study. Because only 51 students in the larger sample fell in the percentile range for HA, to create equal groups, we randomly selected 51 participants from each of the other three achievement strata to comprise our study sample (n=204). Three students had incomplete data, so our final sample included 201 students: 50 HA, 50 AA, 51 LA, and 50 LA-T. Note that as part of the larger study, LA-T students received additional and supplemental instruction on fraction-magnitude understanding. This was the impetus for using two groups of LA students so that we could also investigate how the instructional approach affected accuracy and errors. Fraction instruction for HA, AA, and LA students focused on part-whole understanding.

Instructional Context. Part of the second research question focuses on how instruction for low-performing students might change error patterns, therefore we describe some key differences between fraction instruction focused on magnitude (tutoring for LA-T students) vs. part-whole understanding (core instruction for LA, AA, and HA students). Per fourth-grade standards, students must be proficient with comparing two fractions, ordering three fractions, and placing fractions on number lines, all of which require students to determine magnitude. Core instruction (focused on part-whole understanding) emphasized fair shares and teaching understanding of parts of units through identifying fractions from set models (e.g., A picture of 4 apples is presented; 3 are red and 1 is green. Students write the fraction for the red apples.) or area models (e.g., A picture of a square divided into 6 equal parts. 2 of them are shaded. The student names the fraction for the shaded part of the square.). When teaching students to place fractions on a number line, for example, part-whole instruction emphasized dividing number lines into parts of a whole (fair shares) based on the denominator and then counting over the number of parts based on the numerator. To compare fractions, core instruction teachers reported using cross multiplying (i.e., multiplying the denominator of one fraction by the numerator of the other) to determine which fraction was greater. While this is not necessarily in line with part-whole instruction, it does not promote magnitude understanding and instead offers a "trick" to find the answer. By contrast, tutoring focused on magnitude understanding (i.e., LA-T students), which taught students to reason about the size of the numerator in relation to the denominator and to compare with benchmark numbers to evaluate magnitude. For example, when determining if $\frac{1}{3}$ or $\frac{3}{4}$ is greater, students are taught to use 0, $\frac{1}{2}$, and 1 as benchmark numbers rather than drawing area models or pictures of sets. Students are taught to compare $\frac{1}{3}$ to $\frac{1}{2}$, by considering the size of the denominators. They conclude $\frac{1}{3}$ is less than $\frac{1}{2}$ because thirds are smaller than halves. Then, students evaluate $\frac{3}{4}$ in comparison to the benchmark numbers $\frac{1}{2}$ and 1. Students conclude that $\frac{3}{4}$ is greater than $\frac{1}{2}$ because it is greater than $\frac{2}{4}$. They also conclude that $\frac{3}{4}$ less than1 because it is less than $\frac{4}{4}$. Therefore, $\frac{1}{3}$ is less than $\frac{3}{4}$. This instruction underscores the relationship of the numerator and denominator to determine whether the fraction is closer to 0 or less than or greater than 1. Reasoning in this way can be used when comparing two fractions, ordering three or more fractions, or placing fractions on a number line.

Measures

For the *WRAT-4-Arithmetic* (Wilkinson, 2008) screening measure, students complete calculation problems of increasing difficulty for 15 min. WRAT-4 fourth-grade problems almost entirely samples whole-number problems. Coefficient alpha on this sample was .84.

Although this study focuses on students' fraction computation, we also collected data on students' whole-number computation ability (two measures) and general fraction knowledge (one measure) to describe the sample's mathematics achievement. For all measures, students were tested in large groups.

Whole number computation. Both whole-number computation measures are timed fluency tasks. *Number Fact Fluency* (Fuchs, Hamlett, & Powell, 2003a) tests students' basic fact skills, and includes an addition and subtraction subtest. Each subtest includes 25 problems. For addition fact fluency, sums range from 0-18; for subtraction fact fluency, minuends range from 0-18. For each subtest, problems are presented horizontally on one page, and students have 1 min to write answers. Two staff members entered all responses independently into a computerized scoring program and any discrepancy was reconciled in joint effort for 100% agreement. Alpha on this sample was .87.

Procedural Calculations tests students' double-digit addition and subtraction skills, and includes an addition and subtraction subtest (Fuchs, Hamlett, & Powell, 2003b). Each subtest includes 20 problems, half of which require regrouping. Students have 3 minutes to write answers for each subtest. Two staff members independently entered all responses into a computerized scoring program and any discrepancy was reconciled for 100% agreement. Alpha on this sample was .91.

Fraction concepts. We administered 18 released fraction items from 1990-2009 *National Assessment of Education Progress* (NAEP) mathematics released items: the pool of items classified by NAEP as easy, medium, or hard from the fourth-grade assessment or as easy from the eighth-grade assessment (NAEP, 2015). Sixteen items assess magnitude understanding (half focus on part-whole understanding); one requires subtraction with like denominators; and one asks how many fourths make a whole. Students select an answer from four choices (11 problems); write an answer (3 problems); shade a portion of a fraction (1 problem); mark a number line (1 problem); write a short response (1 problem); or write numbers, shade fractions, and explain the answer (1 problem; multiple parts). The maximum score is 22. Alpha on this sample was .82.

All of these measures (except for the screener) were used for descriptive purposes. We used them to index the sample's performance on whole-number calculations and general fraction concepts, which are related to, yet not the primary focus of the present study.

Fraction computation. From the 2011 Fraction Battery (Schumacher, Namkung, & Fuchs, 2011), we included two measures of fraction calculation to be the focus of the error analysis study. *Fraction Addition* includes 10 problems: four addition problems with like denominators and six addition problems with unlike denominators. Five are presented vertically and five horizontally. *Fraction Subtraction* includes 10 problems: five subtraction problems with like denominators and five with unlike denominators; five are presented vertically and five horizontally. Before conducting the error analysis, problems were scored as correct or incorrect (without regard for answers being in simplest form). We used the total score across both tests, with a maximum score of 20. Alpha on this sample was .94. See Appendix A for test items.

Procedure

Students were tested on WRAT-4 in September of their fourth-grade year. For the larger study, LA students were randomly assigned to tutoring and BAU conditions. Across the district, teachers reported that they taught the fractions unit during the second nine weeks of school (October – December) with cumulative review through the remainder of the school year. Students assigned to small-group tutoring received their teachers' instruction in the general education classroom during the second nine weeks of school and participated in the 12-week small-group tutoring program that spanned those nine weeks and included three additional weeks (which coincided with cumulative review with their general classroom). Tutoring was typically scheduled for the last part of the math block; therefore, time spent on fractions was therefore similar across the four groups. Small-group tutoring focused on assessing fraction magnitude: comparing fraction, ordering fractions, and placing fractions on the number line. In March (after tutoring had taken place), students were assessed on the whole-number computation measures, the general fraction measure, and the fraction computation measures for which we applied error analysis.

Research staff administered all tests in the general education classroom setting (approximately 20 students per class). There were two testers present in every classroom: one tester ran all administration procedures by reading directions, keeping time (when specified), and pacing students on measures that did not have a finite time limit; the other tester circulated throughout the classroom to ensure students were working on correct sections, using their cover sheets, and focusing on their own work. Testers moved to the next activity on untimed tests when all but two students were finished.

Error Analysis Coding

We identified two WNB error types (Independent Whole Number and Combination) and one error type that reflected misapplication of fraction procedures (Incorrect Denominator Change). (Note that we capitalize the error codes for readability.) Note that we kept the two

WNB error subtypes separate in our analysis, unlike Bottge et al. (2014), who collapsed these error types into a single WNB category. We posit that these two errors represent fundamentally different aspects of WNB because Independent Whole Number errors account for fraction notation whereas Combination errors do not, although both ignore the relationship between the numerator and denominator.

Whole number bias errors. ***Independent Whole Number errors*** reflect a solution in which students operated on the numerators and denominators independently. We identified and coded three examples of this error. Example 1: students add or subtract across the numerator and denominator (e.g., $\frac{1}{2}+\frac{3}{8}=\frac{4}{10}$). Example 2: students add or subtract across the numerator and pick one of the denominators for the answer (e.g., $\frac{1}{2}+\frac{3}{8}=\frac{4}{10}$ or $\frac{4}{2}$). Example 3: students add or subtract across the numerator and apply a different operation (i.e., multiplication or division) to the denominator (e.g., $\frac{1}{2}+\frac{3}{8}=\frac{4}{4}$ or $\frac{4}{16}$). These three examples were all characterized by isolating the numerator and denominator in the calculation and operating on them independently. Note that we coded these three examples separately, but combined them to comprise the Independent Whole Number error for data analysis.

Combination errors reflect a solution in which students combined the numerals in the fraction calculation problem (rather than operating across numerator and denominators). We identified two examples. Example 1: students add the four numerals in the addition or subtraction problem and the answer is written as a whole number (e.g., $\frac{1}{2}+\frac{3}{8}=14$; i.e., $1+2+3+4+8=14$). Example 2: students add the numerator and denominator in the first fraction to become the numerator in the answer and add the numerator and denominator of the second fraction to become the denominator in the answer (e.g., $\frac{1}{2}+\frac{3}{8}=\frac{3}{11}$). When students solve problems in either of these ways, the meaning of the numerator and denominator and the requirement to treat them separately is ignored. Both Independent Whole Number and Combination errors demonstrate WNB. However, we consider the Independent Whole Number errors to demonstrate more advanced fraction understanding than the Combination errors because students operate on the numerators and denominators separately, which at least accounts for fraction notation, whereas Combination errors ignore fraction notation altogether.

Other errors. ***Incorrect Denominator Change errors*** demonstrate emerging fraction understanding as students attempted to apply correct fraction procedures but failed to complete them accurately. This error is only applicable to problems with unlike denominators and is categorized by a student's failed attempt to write an equivalent fraction for one of the operands to make the denominators the same (e.g., $\frac{1}{2} + \frac{3}{4} = \frac{6}{12} + \frac{3}{12} = \frac{9}{12}$–the student correctly changed $\frac{1}{2}$ to $\frac{6}{12}$ but incorrectly changed $\frac{3}{4}$ to $\frac{3}{12}$) This error is demonstrative of the difficult procedures required to solve fraction calculations, which could be related to poor multiplication skills or the multi-step process required to find a common denominator. This error does not demonstrate WNB; instead, this error demonstrates difficulty with the procedures required to solve fraction calculations.

We also coded for errors that did not fit into a particular category. These included: (a) not attempting the problem, (b) could not assess the error pattern, or (c) a calculation error (e.g., 1 + 4 = 4). Note that these errors represent careless or random errors, and do not indicate pervasive misunderstanding of fraction concepts.

Scoring. The scoring procedure occurred in three phases. In the first phase, the first author identified error types relevant to addition and subtraction of fractions by consulting the existing literature base and taking a quick review of the sample of tests. After identifying the three aforementioned error-type categories, both authors created a scoring rubric with 9 codes:

coded for Correct, Not Attempted, the three Independent Whole Number variations, the two Combination variations, Incorrect Denominator Change, and No Category. No Category errors did not follow a decipherable pattern. See Table 1 for our coding description, which includes examples and descriptions of each error and its variations. In the second phase, both authors independently scored the tests, applied the coding scheme, and assigned responses. The scorers hand-scored each test item. After hand scoring, we entered all data into separate excel spreadsheets. In the third and final scoring phase, we identified and reconciled all discrepant coded responses. The authors achieved 100% agreement.

Data Analysis

We first calculated frequency of correct answers and errors. To address the two research questions, we ran multiple cross-classified random effects models (CCREM) using the lme4 command in R (Baayen, Davidson, & Bates, 2008; Van den Noortgate, De Boeck, & Meulders, 2003) for each of the outcomes of interest. The data structure included two levels: Level 1 comprised item-by-item student responses and Level 2 crossed student by problem (see Figure 1). Because each student answered the same 20 problems, responses were cross-classified by student and problem at Level 2. We opted for a CCREM for our analysis of problem type, accuracy, and errors because variance within a student is lower than variance between students and CCREMs are advantageous for considering both student-specific and item-specific effects (Baayen, Davidson, & Bates, 2008). Including the cross-classified term accounts for the fact that a single student is more likely to operate with pervasive error across a test. Failing to include a cross-classified term has potential to over-inflate the variance within each parameter (e.g., Luo & Kwok, 2009). Note that to run an item-by-item CCREM necessitated division of groups into dummy codes, which is why we divided students into achievement groups *a priori* and included a buffer zone in an attempt to ensure groups were in fact different.

Research Question 1 includes two models (Accuracy and Independent Whole Number) for each of the problem types and orientation: addition vs. subtraction problems, like vs. unlike denominator problems, and vertical vs. horizontal orientation. Research Question 2 includes three models (Correct, Independent Whole Number, and Incorrect Denominator Change) across the LA, AA, HA, and LA-T groups.

To contextualize results for the CCREM parameters, we converted the coefficients to risk ratios (Viera, 2008). (Note that when the incidence of a particular error occurs greater than 10% of the time, the odds ratio tends to overinflate the relative risk. We therefore rely on risk ratios, rather than odds ratios.) Risk ratios contextualize frequency of correct answers and errors by achievement group. We use the HA group as the referent group because we conceptualized the HA group as the “gold standard” and wanted to compare achievement to the highest achieving group. Therefore, the calculated risk ratio indicates the likelihood of committing an error, relative to the HA group. Per Viera (2008), if the risk ratio equals 1, that group is just as likely to get an item correct or commit an error as the HA group; if the risk ratio is greater than 1, that group is more likely to get an item correct or commit an error as the HA group; if the risk ratio is less than 1, that group is *less* likely to get an item correct or commit an error as the HA group. Note that we also calculated risk ratios for the HA group, which compares the HA group to *all* other groups.

Results

See Table 2 for a summary of whole-number computation skill and general fraction understanding descriptors across achievement groups. HA students had greater fluency with whole-number calculations and better fraction understanding than AA and LA students.

Table 3 includes frequencies of errors by problem type (addition vs. subtraction; like vs. unlike denominators) and orientation (vertical vs. horizontal). WNB errors were the most pervasive errors across the sample, and of the two different types of WNB errors, Independent Whole Number errors occurred most frequently. Note that Combination errors (5% frequency with no difference among problem types or groups) and Incorrect Denominator Change errors (2% frequency with no difference among problem types or groups) occurred too infrequently to perform logistic regression (Stoltzfus, 2011).

Research Question 1

See Table 4 for CCREM parameters estimates on addition v. subtraction, like vs. unlike denominators, and vertical vs. horizontal orientation for accuracy and Independent Whole Number errors across the sample. Results indicated there were no differences for accuracy or Independent Whole Number errors on addition vs. subtraction or vertical vs. horizontal orientation. However, for problems with unlike denominators, accuracy was lower and Independent Whole Number errors occurred more frequently across the sample, $p < .001$. For like vs. unlike denominators, students across the sample were .48 times less likely to accurately answer unlike denominator problems and 6.58 times more likely to commit Independent Whole Number errors on these problems.

Research Question 2

Table 5 includes frequencies for accuracy and error type by achievement group. See Table 6 for CCREM parameter estimates comparing problem type across separate error type models. There were significant differences among groups for accuracy and Independent Whole Number errors, $p < .001$. Using the HA group as the referent group, AA students were .60 times less accurate, LA students were .51 times less accurate, and LA-T students were 1.22 times *more* accurate. For Independent Whole Number errors (again using HA as the referent group), AA students were 2.63 times more likely to apply Independent Whole Number errors, LA students were 2.73 times more likely to apply Independent Whole Number errors, and LA-T students were .14 times *less* likely to apply Independent Whole Number errors.

Discussion

Research Question 1 addressed whether accuracy and WNB errors differed by problem type (i.e., addition vs. subtraction, like vs. unlike denominators) and orientation (vertical vs. horizontal) across all students in the sample. Our findings indicate that operation (i.e., addition vs. subtraction), did not affect accuracy or frequency of errors, which was surprising given prior work with whole-number calculations that has shown that subtraction is more difficult than addition for whole numbers (e.g., Kamii et al., 2001). We anticipated this pattern to hold for fractions. Also surprising was the fact that students were just as likely to apply WNB errors on subtraction problems as addition problems. Consider this problem and its incorrect solution $\frac{1}{2} - \frac{2}{10} = \frac{1}{8}$. Students ignore the "small from big" whole-number concept to subtract across numerators (1 – 2) and denominators (2 – 10) independently. Instead, students reverse the subtrahend and minuend to avoid a negative number for the answer. Fourth-grade students do not typically know how to add/subtract negative numbers, so ignoring this principle of whole-number subtraction and operating on the numerator and denominator as independent whole numbers speaks to the fact that students likely do not recognize the bipartite structure of a fraction as a meaningful number and apply the only strategies they know how to apply (i.e., WNB), even though the strategy violates their known rules of subtraction. Three of the 10 items on the assessment included numerators/denominators that would result in a negative number if subtracted across from left to right and students who applied this error to these problems and subtracted from right to left and ignored the order of the fractions in the subtraction problem.

For orientation, horizontal vs. vertical, there were no difference for accuracy or errors. We expected students to apply Independent Whole Number errors more frequently on horizontal problems due to the presentation of the numerator and denominator lending themselves to adding or subtracting across on a horizontal problem. That is, it is conceivably more difficult spatially to keep track of adding the numerator and the denominator independently on vertical problems. In fraction addition and subtraction, students are required to evaluate whether denominators are the same in both fractions before they can perform the calculation. When fractions are presented vertically, the denominators and numerators are not next to one another, which may pose an increased challenge to students when determining if they need to rewrite one or both fractions in the problem before adding or subtracting. Because vertical problems present a more taxing spatial arrangement, we anticipated students might apply procedures more haphazardly, which was not the case. Raghubar et al. (2009) showed that students' reading status detrimentally affected performance on math problems with taxing spatial arrangements, which is one reason we wanted to explore error patterns with horizontal vs vertical problems. Even though we did not collect reading achievement data for the sample in this study, we assumed some students may have reading difficulties (Fletcher, 2005) and hypothesized that different response patterns might emerge between vertical and horizontal orientations. These data did not support this hypothesis.

We did, however, find that students were more accurate on problems with like denominators than with unlike denominators and Independent Whole Number errors were more common on problems with unlike denominators than those with like denominators. Given the conceptual foundation and multi-step procedures required for unlike denominator problems, this pattern supports the notion that students with poor conceptual understanding of fractions have a more difficult time judging the validity of their approach to solving the problem and also the accuracy of their answers (Rittle-Johnson et al., 2001). We posit that when students are presented with unlike denominators, they do not know which procedures to apply and choose to avoid the multi-step process of changing denominators and thus resort to familiar procedures that demonstrate WNB. For example, for the problem $\frac{2}{10} + \frac{1}{2}$, the correct answer is $\frac{7}{10}$. Adding across the numerators and denominators would result in $\frac{3}{12}$, which is less than one of the addends and violates the rule of adding two positive numbers together.

Research Question 2 examined whether accuracy and frequency of errors differed among HA, AA, and LA students and whether instructional context affected accuracy and error patterns between low-achieving students who received core classroom instruction based on part-whole understanding (LA) and low-achieving students who received supplementary small-group tutoring based on magnitude understanding (LA-T). (Note again that Combination and Incorrect Denominator Change errors occurred too infrequently to analyze with CCREM). We therefore center the discussion on the most pervasive WNB error across the sample – Independent Whole Number errors.

For students who received their mathematics instruction only in their core classroom (i.e., LA, AA, HA), mathematics-achievement status was related to the probability of applying Independent Whole Number errors. This error is characterized by students independently operating across numerators and denominators and occurred significantly more often for the LA group, followed by the AA group, and then the HA group, which was expected. Surprisingly, the AA and LA groups applied Independent Whole Number errors with similar frequency. Given that the AA students had greater incoming fraction understanding (as indexed by the NAEP, see

Table 2) than the LA group, we would have expected them to outperform LA students at the end of fourth grade on fraction calculations.

We also found that mathematics-achievement status was significantly related to accuracy for LA, AA, and HA students. The HA group was more accurate than the other two groups, and the LA group was least accurate. This pattern was expected. Accuracy for LA students was only marginally less than AA students, demonstrating again that the performance pattern for LA and AA students was similar, and significantly different from the HA students. The similar patterns for both Independent Whole Number errors and accurate responses between LA and AA students suggest that a measure of general mathematics achievement does not adequately predict performance in fractions, especially for students in the bottom two-thirds of the class. Therefore, before fractions are introduced into the curriculum, it appears that it is difficult to determine which students will struggle consolidating fraction properties with whole-number properties.

This finding corroborates the larger body of work that describes students' tendencies to operate with WNB when learning to solve fraction calculations (e.g., Bottge et al., 2014; DeWolf & Vosniadou, 2011; Mack 1995; Meert, Gregoire, & Noel, 2010a, 2010b; Mix et al., 1999; Stafylidou & Vosniadou, 2004) and prior work that has shown that fractions present challenges for a wide range of students (Ni & Zhou, 2005; Vamakoussi & Vosniadou, 2010) regardless of their whole number knowledge. Our findings support this by indicating that even students who are performing at grade level (AA) or higher (HA) in mathematics may still operate with WNB.

As mentioned, we focused on Independent Whole Number errors (rather than the Combination error which also demonstrates WNB) because it occurred most often across the sample and was applied by students across all three achievement groups who participated in core instruction. The three examples we documented for Independent Whole Numbers demonstrate how students incorrectly add and subtract fractions by viewing the numerators and denominators separately. Students ignore their relationship (and the quantity the fraction represents) and rely on whole-number properties to drive the procedure they apply. The misapplication of whole number properties in this way underscores the tediousness of fraction calculations as they require an understanding of the bipartite representation of fractions and the multiple steps required to solve them.

In terms of instructional setting and the nature of instruction, our results suggest that instruction focused on magnitude increases accuracy and attenuates applying Independent Whole Number errors. The LA-T group (who received small-group tutoring focused on magnitude understanding) applied Independent Whole Number errors significantly *less* often than *all* the other three groups. And similarly, the LA-T group was significantly *more* accurate on fraction calculations than *all* of the groups. Given the low-performance for LA-T students prior to tutoring, we did not expect their gains from tutoring focused on magnitude understanding to be over that of both the AA and HA groups. We expected LA-T students to be more accurate than LA students based on data from the larger study that assessed effects of intervention (see Fuchs et al., 2013), but were surprised that they were significantly more accurate than both the AA and HA groups. However, the LA-T group outperforming the AA group makes sense given the similar pattern that emerged between the LA and AA groups. Yet, the LA-T students outperformed the HA group, which was unforeseen.

In our opinion, the accuracy and error patterns for the four groups of students demonstrates the importance of instructional approach in addition to context. As a reminder, core instruction focused on part-whole understanding (LA, AA, HA) and supplemental tutoring focused on magnitude understanding (LA-T).[2] The performance of the LA-T students supports the Siegler et al. (2011) hypothesis that a strong focus on magnitude understanding boosts students understanding of fractions and allows students to judge the accuracy of their calculations (e.g., Rittle-Johnson et al., 2001). Our findings also echo prior work that suggests fraction instruction focused on magnitude understanding promotes a deeper understanding of fractions than instruction focused on part-whole understanding (Cramer, Post, & delMas, 2002; Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2016; Hecht, 2011; Siegler et al., 2011). We do, however, feel strongly that part-whole understanding is still important for learning fractions because it capitalizes on students' prior understandings of whole numbers and can therefore begin to bridge whole number and fraction concepts. In the tutoring program that the LA-T students participated in, the first week (out of 12) of instruction focused on bridging part-whole understanding to magnitude understanding (which was the focus for 9 out of the 12 weeks). During the magnitude focus, activities were centered around comparing two fractions, ordering three fractions, and placing fractions on the number line while reasoning about the size of fractions in relation to 0, 1, and ½ rather than focusing on fair shares.

Because of the high level of accuracy for which the LA-T students performed on fraction calculations, especially compared to AA and HA students, we recommend instruction focused on magnitude for students of all achievement levels in both core and tutoring contexts.[3] Magnitude understanding is a unifying property of all numbers (especially whole numbers and fractions) and drives proficiency when students evaluate quantities. When comparing or ordering fractions

[2] It is important to note that many of the kids in the LA-T group missed a portion of their core mathematics instruction by participating in tutoring, and therefore instructional time across the four groups was similar.

[3] It is important to note that fraction calculation was not a major emphasis in the tutoring program (Fuchs et al., 2013). That is, only three of the 36 days of tutoring included explicit instruction on fraction calculations.

or whole numbers and estimating fraction or whole number placement on the number line, magnitude understanding is critical (Siegler et al., 2011). Understanding fraction magnitude, as opposed to whole number magnitude, includes two fundamental fraction principles: (a) that there are an infinite number of fractions between any two given numbers on a number line and, (b) that any given point on a number line can be represented with an infinite number of fractions. To simultaneously attach meaning to whole number and fraction magnitude may minimize students' tendency to operate with WNB (Mack, 1995) when determining fraction magnitude or calculating with fractions.

In considering instructional context, many studies have focused on fraction intervention (rather than core) and have increased low-performing students' accuracy with fraction calculations, decreased errors that demonstrate WNB, and enhanced understanding of fraction concepts (Bottge et al., 2014; Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., 2016; Newton et al., 2014). In all of these studies, the focus was on intervention for students who perform below grade level or who are at risk for learning disabilities and did not include the full range of learners. In Asian countries, magnitude understanding typically drives fraction instruction, which could help to explain why they tend to outperform U.S. students in mathematics, and especially on rational number topics (Miura, Okamoto, Vlahovic-Stetic, Kim, & Han, 1999; Paik & Mix, 2003; Provasnik et al., 2012). By contrast, instruction in U.S. schools primarily focuses on the part-whole understanding of fractions (Mix et al., 1999). Even though part-whole understanding is foundational for magnitude understanding and important to overall fraction understanding, it is not sufficient (e.g., Charalambous & Pitta-Pantazi, 2007) for developing a deep understanding of fractions, as demonstrated by the performance for the three groups who only participated in core instruction focused on part-whole understanding.

The recommendations from the Fraction Practice Guide, *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade,* published by IES include centralizing the number line representation (Siegler et al., 2010) when introducing fractions in core instruction. However, this recommendation does not specify number line *estimation*, which forces students to confront fraction magnitude understanding. While number lines are critical for increasing magnitude understanding and for consolidating whole number and fraction properties, they do not generate these understandings on their own. Number lines can be divided into fair shares which illustrates part-whole understanding whereas number line estimation is the mechanism for developing magnitude understanding (Siegler & Pyke, 2013). Therefore, we propose not only centralizing the number line representation in core instruction, but also using it as a tool to support magnitude understanding by teaching students how to evaluate fraction magnitude through reasoning and then to estimate placement on the number line.

Limitations and Conclusion

One limitation of this study centers on the achievement group categorical structure, specifically the cut points we used on the screener, as they were chosen arbitrarily to capture three distinctive achievement groups. These cut points were not based on a widely known or accepted categorical structure, because there is not an agreed upon way to distinguish mathematics achievement groups. However, we feel that by including the buffer zones and constraining the range for each category, we can more confidently report that these students comprised LA, AA, and HA groups in fourth grade. A second limitation includes the categorization of error types. For one of the examples of the Independent Whole Number error (i.e., pick a denominator), occurrences were eligible only on problems with unlike denominators. When this strategy is applied to problems with like denominators, the correct answer results.

Despite this, we felt coding for this error was necessary since it reveals students' inability to concurrently operate on the numerator and denominator when denominators are not the same.

In conclusion, the findings of this study extend the literature on the difficulties students experience when the mathematics curriculum shifts to include fractions. Future research should continue to focus on instructional approaches that enhance magnitude understanding to improve students' fraction understanding and accuracy in solving calculations. Results from the present study support this, given the finding that LA-T students performed better than *all* students who primarily received part-whole instruction in their core mathematics program. We therefore propose that both core and supplemental instruction focus on magnitude understanding when introducing fractions in upper elementary school. We urge teachers to consider the increased difficulty of mathematics once fractions are introduced and encourage educators not to assume that a student with incoming average or high achievement with whole numbers will continue to perform at an adequate level with fractions.

References

Baayen, R. H., Davidson, D. J., & Bates, D. M. (2008). Mixed-effects modeling with crossed random effects for subjects and items. *Journal of Memory and Language, 59*(4), 390–412. doi:10.1016/j.jml.2007.12.005

Bottge, B. A., Ma, X., Gassaway, L., Butler, M., & Toland, M. D. (2014). Detecting and correcting fractions computation error patterns. *Exceptional Children*, *80*(2), 237–255. doi:10.1177/001440291408000207

Calhoon, M. B., Emerson, R. W., Flores, M., & Houchins, D. E. (2007). Computational fluency performance profile of high school students with mathematics disabilities. *Remedial and Special Education, 28*(5), 292–303. doi:10.1177/07419325070280050401

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics, 64*(3), 293–316. doi:10.1007/s10649-006-9036-2

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning for fourth- and fifth-grade students: A comparison of the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education, 33*(2), 111–144. doi:10.2307/749646

DeWolf, M., & Vosniadou, S. (2011). The whole number bias in fraction magnitude comparisons with adults. In L. Carlson, C. Hölscher, & T. Shipley (Eds.), *Proceedings of the 33rd Annual Conference of the Cognitive Science Society* (pp. 1751–1756). Austin, TX: Cognitive Science Society.

Fletcher, J. M. (2005). Predicting math outcomes: Reading predictors and comorbidity. *Journal of Learning Disabilities*, *38*(4), 308–312. doi:10.1177/00222194050380040501

Fuchs, L. S., Hamlett, C. L., & Powell, S. R. (2003a). *Number fact fluency.* Available from L.S. Fuchs, 228 Peabody, Vanderbilt University, Nashville, TN 37203.

Fuchs, L. S., Hamlett, C. L., & Powell, S. R. (2003b). *Procedural calculations*. Available from L.S. Fuchs, 228 Peabody, Vanderbilt University, Nashville, TN 37203.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., . . . Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology, 105*(3), 683–700. doi:10.1037/a0032446

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Malone, A. S., Wang, A., . . . Changas, P. (2016). Effects of intervention to improve at-risk fourth graders' performance on fraction understanding, calculations, and word problems. *The Elementary School Journal, 116*(4), p. 625–651. doi:10.1086/686303

Fuchs, L. S., Schumacher, R. F., Sterba, S. K., Long, J., Namkung, J., Malone, A., . . . Changas, P. (2014). Does working memory moderate the effects of fraction intervention? An aptitude-treatment interaction. *Journal of Educational Psychology, 106*(2), 499–514. doi:10.1037/a0034341

Geary, D. C. (2013). Learning disabilities in mathematics: Recent advances. In H. L. Swanson, K. Harris, & S. Graham (Eds.), *Handbook of learning disabilities* (2nd ed., pp. 239-255). New York: Guilford Press.

Geary, D. C., Hoard, M. K., Nugent, L., & Bailey, D. H. (2012). Mathematical cognition deficits in children with learning disabilities and persistent low achievement: A five-year prospective study. *Journal of Educational Psychology,104*(1), 206-223. doi:10.1037/a0025398

Hansen, N., Jordan, N. C., & Rodrigues, J. (2015). Identifying learning difficulties with fractions: A longitudinal study of student growth from third through sixth grade. *Contemporary Educational Psychology*. Advanced online publication. doi:10.1016/j.cedpsych.2015.11.002

Haseman, K. (1986). Analysis of fraction errors by a model of cognitive science. *European Journal of Psychology and Education*, *1*(2), 57–66. doi:10.1007/BF03172569

Hecht, S. A. (2011). *Effects of a conceptual knowledge intervention for seventh-graders with fraction difficulties*. Unpublished manuscript.

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Psychology*, *116*(1), 45–58. doi:10.1016/j.jecp.2013.02.001

Kamii, C., Lewis, B. A., & Kirkland, L. D. (2001). Fluency in subtraction compared to addition. *The Journal of Mathematics Behavior*, *20*(1), 33–42. doi:10.1016/S0732-3123(01)00060-8

Luo, W., & Kwok, O. M. (2009). The impacts of ignoring a crossed factor in analyzing cross-classified data. *Multivariate Behavioral Research, 44*(2), 182–212. doi:10.1080/00273170902794214

Mack, N. K. (1995). Confounding whole number and fraction concepts when building on information knowledge. *Journal for Research in Mathematics Education, 26*(5), 422–441. doi:10.2307/749431

Mazzocco, M. M., & Myers, G. F. (2003). Complexities in identifying and defining mathematics learning disability in the primary school-age years. *Annals of Dyslexia*, *53*(1), 218–253. doi:10.1007/s11881-003-0011-7

Meert, G., Gregoire, J., & Noel, M. P. (2010a). Comparing 5/7 and 2/9: Adults can do it by accessing the magnitude of the whole fractions. *Acta Psychologica, 135*(3), 284–292. doi:10.1016/j.actpsy.2010.07.014

Meert, G., Gregoire, M, & Noel, M. P. (2010b). Comparing the magnitude of two fractions with common components: Which representations are used by 10- and 12-year-olds? *Journal of Experimental Child Psychology, 107*(3), 244–259. doi:10.1016/j.jecp.2010.04.008

Miura, I. T., Okamoto, Y., Vlahovic-Stetic, V., Kim. C. C., & Han, J. H. (1999). Language supports for children's understanding of numerical fractions: Cross national comparisons. *Journal of Experimental Psychology*, *74*(4), 356–365. doi:10.1006/jecp.1999.2519

Mix, K. S., Levine, S. C., & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology*, *35*(1), 164–174. doi:10.1037//0012-1649.35.1.164

Namkung, J. M., & Fuchs, L. S. (2015). Cognitive predictors of calculations and number line estimation with whole numbers and fractions among at-risk students. *Journal of Educational Psychology, 108*(2), 214–228. doi:10.1037/edu0000055

National Assessment of Educational Progress [NAEP]. (2015). *NAEP item map: Mathematics* [Release questions from 1990-2009]. Washington DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics. Retrieved from http://nces.ed.gov/nationsreportcard/itemmaps/

Newton, K. J., Willard, C., & Teufel, C. (2014). An examination of the ways that students with learning disabilities solve fraction computation problems. *The Elementary School Journal, 115*(1), 1–21. doi:10.1086/676949

Ni, Y., & Zhou, Y. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27–52. doi:10.1207/s15326985ep4001_3

Paik, J. H., & Mix, K. S. (2003). U.S. and Korean children's comprehension of fraction names: A reexamination of cross-national differences. *Child Development, 74*(1), 144–154. doi:10.1111/1467-8624.t01-1-00526

Provasnik, S., Kastberg, D., Ferraro, D., Lemanski, N., Roey, S., & Jenkins, F. (2012). *Highlights from TIMSS 2011: Mathematics and science achievement of U.S. fourth- and eighth-grade students in an international context* (NCES 2013-009). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education: Washington, DC. Retrieved from http://files.eric.ed.gov/fulltext/ED537756.pdf

Raghubar, K., Cirino, P., Barnes, M., Ewing-Cobbs, L., Fletcher, J., & Fuchs, L. S., (2009). Errors in multi-digit arithmetic and behavioral inattention in children with math difficulties. *Journal of Learning Disabilities, 42*(4), 356–371. doi:10.1177/0022219409335211

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*(1), 175–189. doi:10.1037/0022-0663.91.1.175

Rittle-Johnson, B., & Koedinger, K. (2009). Iterating between lessons on concepts and procedures can improve mathematics knowledge. *British Journal of Educational Psychology, 79*(3), 483–500. doi:10.1348/000709908X398106

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346–362. doi:10.1037//0022-0663.93.2.346.

Schumacher, R. F., Namkung, J. M., & Fuchs, L. S. (2011). *2010 fraction battery*. Available from L.S. Fuchs, 228 Peabody, Vanderbilt University, Nashville, TN 37203.

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., . . . Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE 2010-4039). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from whatworks.ed.gov/publications/practiceguides

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworkth, K., Claessens, A., Engel, M., . . . Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*(7), 691–697. doi:10.1177/0956797612440101

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, *49*(10), 1994–2004. doi:10.1037/a0031200

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*(4), 273–296. doi:10.1016/j.cogpsych.2011.03.001

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503–518. doi:10.1016/j.learninstruc.2004.06.015

Stoltzfus, J. C. (2011). Logistic regression: A brief primer. *Academic Emergency Medicine, 18*(10), 1099–1104. doi: 10.1111/j.1553-2712.2011.01185.x

Vamakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14*(5), 453–467. doi:10.1016/j.learninstruc.2004.06.013

Vamakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*(2), 181–209. doi:10.1080/07370001003676603

Van den Noortgate, W., De Boeck, P., & Meulders, M. (2003). Cross-classified multilevel logistic models in psychometrics. *Journal of Educational and Behavioral Statistics, 28*(4), 369–386. doi:10.3102/10769986028004369

Viera, A. J. (2008). Odds ratios and risk ratios: What's the difference and why does it matter? *Southern Medical Journal, 101*(7), 730–734. doi:10.1097/SMJ.0b013e31817a7ee4

Vosniadou, S., & Verschaffel, L. (2004). Extending the conceptual change approach to mathematics learning and teaching. *Learning and Instruction, 14*(5), 445–451. doi:10.1016/j.learninstruc.2004.06.014

Vukovik, R. K., Fuchs, L. S., Geary, D. C., Jordan, N. C., Gersten, R., & Siegler, R. (2014). Sources of individual differences in children's understanding of fractions. *Child Development, 85*(4), 1461–1476. doi:10.1111/cdev.12218

Wilkinson, G. S. (2008). *Wide Range Achievement Test* (Rev. 4). Wilmington, DE: Wide Range.

Wu, H. (2008). *Fractions, decimals, and rational numbers*. Berkeley, CA: University of California. Retrieved from https://math.berkeley.edu/~wu/NMPfractions.pdf

Table 1

Description of Error Exmaples

Category	Description
Correct	Performed fraction algorithm correctly; reducing optional
Not Attempted	Student did not attempt problem
Independent Whole Number*	
Example 1	Perform operations across numerators and denominators separately; treats the numerators and denominators as independent whole numbers; e.g., $\frac{2}{10}+\frac{1}{2}=\frac{3}{12}$
Example 2	Picked denominator; treats the numerators as independent whole numbers, but computes the denominator using a different operation (multiplication or division); e.g., $\frac{2}{10}+\frac{1}{2}=\frac{3}{10}$ or $\frac{7}{10}-\frac{1}{2}=\frac{6}{10}$
Example 3	Computed denominator; treats the numerators as independent whole numbers, but computes the denominator using a different operation (multiplication or division); e.g., $\frac{2}{10}+\frac{1}{2}=\frac{3}{20}$ or $\frac{2}{10}+\frac{1}{2}=\frac{3}{5}$
Combination**	
Example 1	Combines denominators and numerators into a whole number; adds all (4) numbers together or observes no distinction between numerator and denominators; answer is whole number; e.g., $\frac{2}{10}+\frac{1}{2}=312$
Example 2	Combines denominators and numerators into a new fraction; makes top fraction the numerator and bottom fraction the denominator when set up vertically; e.g., $\frac{2}{10}+\frac{1}{2}=\frac{13}{3}$
Incorrect Denominator Change	Changes a fraction incorrectly in an attempt to make like denominators
No Category	An incorrect answer was written and procedures were unclear

Note: Independent Whole Number and Combination errors represent whole number bias errors.
*Independent Whole Number examples 1-3 were coded as distinct errors to increase accuracy of coding, but were collapsed for data analysis.
**Combination examples 1-3 were also collapsed for data analysis.

Table 2

Whole Number and General Fraction Understanding Descriptors Across Mathematics Achievement Groups

	Low-risk Sample				At-Risk Sample			
	High (*n* = 50)		Average (*n* = 50)		Low (*n* = 51)		Low + Tutoring (*n* = 50)	
	M	*(SD)*	*M*	*(SD)*	*M*	*(SD)*	*M*	*(SD)*
WRAT-4	32.76	(2.26)	28.72	(0.99)	23.67	(1.97)	23.68	(2.00)
Number Fact Fluency – Addition	21.42	(4.38)	17.65	(4.57)	13.35	(5.48)	14.04	(4.54)
Number Fact Fluency – Subtraction	19.10	(5.19)	12.18	(4.71)	8.69	(4.93)	8.66	(4.55)
Procedural Calculations – Addition	19.70	(0.65)	19.14	(2.31)	16.18	(4.83)	17.46	(3.45)
Procedural Calculations – Subtraction	18.20	(3.13)	15.84	(4.07)	10.10	(4.80)	9.34	(3.92)
NAEP	17.98	(2.83)	13.13	(3.49)	10.93	(3.33)	13.86	(2.79)

Table 3

Frequencies for Accuracy and Error Type by Problem Type and Orientation

		Error Category					
Item Category	Items[a]	Correct	IWN	COMB	IDC	NA	NC
Problem Type							
Addition	2,010	1160	598	37	21	130	64
Subtraction	2,010	1240	487	29	11	149	94
Like Denominator	1,809	1517	120	27	1	85	59
Unlike Denominator	2,211	883	965	39	31	194	99
Orientation							
Vertical	2,010	1121	548	65	23	164	89
Horizontal	2,010	1279	537	1	9	115	69

Note: IWN = Independent Whole Number; COMB = Combination; IDC = Incorrect Denominator Change; NA/NC = Not Attempted or No Category. IWN and COMB errors reflect WNB.

[a]Items reflects the number of problems answered per item category in the sample.

Table 4

CCREM Parameter Estimates Comparing Problem Type (Addition vs. Subtraction and Like vs. Unlike Denominators) and Orientation (Horizontal vs. Vertical) for Correct Answers and Independent Whole Number Errors

	Correct			Independent Whole Number		
Model/Estimate	β (*SE*)	*RR*	*p*	β (*SE*)	*RR*	*p*
Addition vs. Subtraction						
Intercept[a]	1.12 (.77)	0.94	.75	-3.61** (.88)	1.23	.03
Subtraction	.31 (1.03)	1.07	.81	-1.06 (1.15)	0.81	.01
Like vs. Unlike Denominators						
Intercept[b]	3.66** (.32)	2.10	.97	-7.09** (.52)	0.15	<.01
Unlike Denominators	-4.39** (.28)	0.48	.48	5.21** (.30)	6.58	.15
Horizontal vs. Vertical						
Intercept[c]	.79 (.76)	0.88	.69	-4.20** (.90)	1.02	.01
Horizontal	.93 (1.01)	1.14	.85	.12 (1.17)	0.98	.02

[a]The intercept represents addition as it was the referent type in the model.
[b]The intercept represents like denominators as it was the referent type in the model.
[c]The intercept represents vertical orientation as it was the referent orientation in the model.
**p* < .01; ***p* < .001.

Table 5

Frequencies for Accuracy and Error Type by Achievement Group

		Error Category					
Achievement Group	Items[a]	Correct	IWN	COMB	IDC	NA	NC
High	1,000	718	165	9	14	63	31
Average	1,000	429	435	0	4	78	54
Low	1,020	374	461	57	13	71	44
Low + Tutoring	1,000	879	24	0	1	67	29

Note. IWN = Independent Whole Number; COMB = Combination; IDC = Incorrect Denominator Change; NA= Not Attempted; NC = No Category. IWN and COMB errors reflect WNB.

[a]Items reflects the number of problems answered per item category in the sample.

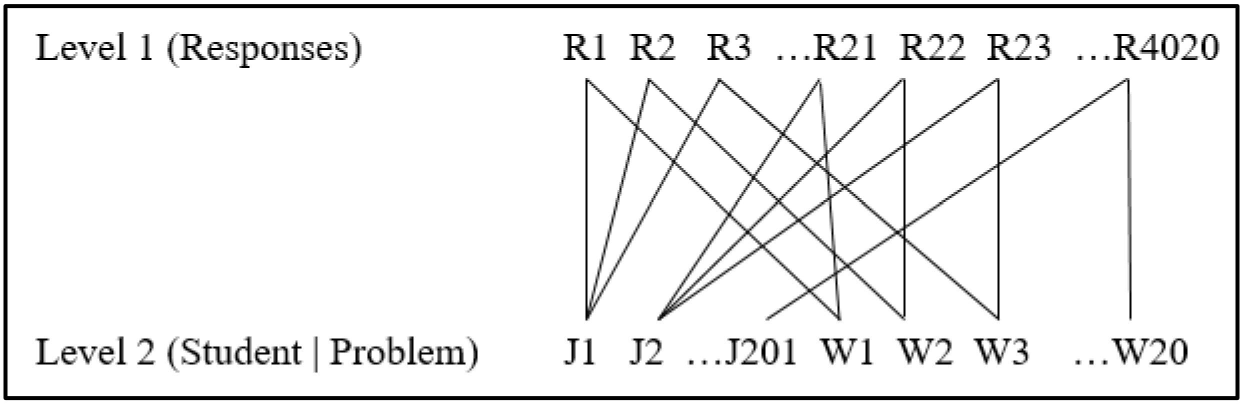
Table 6

CCREM Parameter Estimates Comparing Achievement Groups for Correct Answers and Independent Whole Number Errors

	Correct			Independent Whole Number		
Estimate	β (*SE*)	*RR*	*p*	β (*SE*)	*RR*	*p*
Intercept[a]	2.48** (.63)	1.29	.92	-5.62** (.83)	0.54	<.01
Average	-3.11** (.52)	0.60	.35	4.56** (.78)	2.64	.26
Low	-3.78** (.52)	0.51	.21	4.80** (.77)	2.74	.31
Low + Tutoring	2.13** (.55)	1.22	.99	-3.85** (1.16)	0.15	<.01

[a]The intercept represents the high-achieving group as they were the referent group in the model.

Figure 1. Cross-classified model.



Appendix A

Fraction Addition and Subtraction

Add.

A. $\frac{1}{8} + \frac{3}{8} =$

B. $\frac{2}{10} + \frac{1}{2} =$

C. $\frac{3}{4} + \frac{2}{4} =$

D. $\begin{array}{r} \frac{1}{2} \\ + \frac{4}{10} \\ \hline \end{array}$

E. $\begin{array}{r} \frac{1}{2} \\ + \frac{4}{12} \\ \hline \end{array}$

F. $\begin{array}{r} \frac{5}{8} \\ + \frac{1}{2} \\ \hline \end{array}$

G. $\frac{1}{2} + \frac{2}{4} =$

H. $\begin{array}{r} \frac{4}{6} \\ + \frac{3}{6} \\ \hline \end{array}$

I. $\begin{array}{r} \frac{5}{10} \\ + \frac{4}{10} \\ \hline \end{array}$

J. $\frac{1}{2} + \frac{5}{6} =$

Subtract.

A. $\frac{8}{12} - \frac{2}{12} =$

B. $\begin{array}{r} \frac{1}{2} \\ - \frac{2}{8} \\ \hline \end{array}$

C. $\frac{7}{10} - \frac{1}{2} =$

D. $\begin{array}{r} \frac{7}{8} \\ - \frac{2}{8} \\ \hline \end{array}$

E. $\begin{array}{r} \frac{1}{2} \\ - \frac{2}{6} \\ \hline \end{array}$

F. $\frac{6}{4} - \frac{2}{4} =$

G. $\frac{8}{12} - \frac{1}{2} =$

H. $\frac{7}{10} - \frac{3}{10} =$

I. $\begin{array}{r} \frac{9}{12} \\ - \frac{5}{12} \\ \hline \end{array}$

J. $\begin{array}{r} \frac{1}{2} \\ - \frac{3}{10} \\ \hline \end{array}$