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JAIMIE DUPLASS/THINKSTOCK (BOY); TOLGA TEZCAN (BUBBLES)

Relational Thinking: What's the Difference?

Instructional activities designed to encourage relational thinking in primary-grades classrooms can give students advantages when they reason about subtraction.

Ian Whitacre, Robert C. Schoen, Zachary Champagne, and Andrea Goddard

How much is 41 – 39? How about 100 – 3? Which of those computations was easier for you to do? It so happens that first graders are much more likely to solve 100 – 3 correctly than 41 – 39. Likewise, second graders are much more likely to solve 100 – 3 correctly than 201 – 199. Our data (Schoen et al. 2016) suggest that the latter problems are more difficult for students to solve correctly, because many students' understanding of subtraction is limited by thinking about the operation only as take-away or by using a default procedure, such as the standard subtraction algorithm in the United States. In this article, we argue the importance of students learning to reason flexibly about subtraction. We highlight a useful but often-ignored way of reasoning, and we offer suggestions for teaching about subtraction.

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In a 2014 study (see **fig. 1**), we found that of the majority of second graders who attempted to use the standard U.S. subtraction algorithm to compute 201 – 199, only 29 percent were successful (see **table 1**). Some students obtained such answers as 198 as a result of "subtracting up" errors and, unfortunately, failed to notice the unreasonableness of such a large answer. Furthermore, even when these approaches were successful, the correct answer could have been obtained much more easily by counting forward from 199 to 201 or counting backward from 201 to 199.

We found similar results in first graders' attempts to solve 41 – 39 = ___. Many students struggled with this problem, despite being able to correctly solve problems like 100 – 3 = ___. For many adults, by contrast, problems like 201 – 199 = ___ and 41 – 39 = ___ can easily be solved mentally. A characteristic that can make problems like these easy is that the numbers are close together. However, this characteristic is only useful if one thinks about subtraction as asking such questions as "How far apart are 41 and 39?" or "What would I have to add to 39 to get 41?" If, instead, one asks, "How much is left if I take 39 away from 41?" one has much work to do to find the answer and many potential pitfalls along the way.

Take-away subtraction

We found that students in both grade levels relied heavily on the take-away meaning for subtraction. For problems such as 100 – 3 = ___, thinking in terms of take-away served many students well. One popular strategy was to start from 100 and count down (99, 98, 97), often using fingers. For problems like 100 – 3, this way of reasoning is advantageous, because the subtrahend is small; the student must take away only 3. In 201 – 199, by contrast, the difference is small, but the subtrahend is large. In the latter situations, thinking about subtraction as take-away can be highly inefficient, whereas thinking of the difference as the distance between the given numbers is advantageous.

It is not a new idea that that there are limitations to the take-away meaning of subtraction. Gibb (1954) pointed this out more than sixty years ago in a paper that discussed different types of subtraction word problems and various strategies that children use to solve them. Her points are still valid and relevant today. When initially learning about the operation, students in the United States learn to associate subtraction with taking away, or removing, objects from a set. This meaning for subtraction is consistent with separate-result-unknown problems, such as this:

> Connie had 13 marbles. She gave 5 to Juan. How many marbles does Connie have left? (Carpenter et al. 1999)

Many teachers value the take-away meaning, especially when it comes to introducing subtraction initially (e.g., Maples 1959; Page 1994). In fact, teachers often orally read the subtraction symbol as "take away" (e.g., reading 13 – 5 as "Thirteen take away five"). The take-away meaning is certainly important and useful, especially when students are first learning about subtraction. However, equating subtraction with take-away is problematic, because many

FIGURE 1

In the spring of 2014, 614 first and second graders were interviewed to determine how they solved a variety of math problems. Below are a few of the many problems. **Table 1** shows the percentages of correct answers on each of them.

Students had access to base-ten blocks, snap cubes, paper, and markers. They were also free to solve problems mentally or to use their fingers to help them. Students were presented with each problem on a sheet of paper and asked to write the number that would make the equation correct. First graders were asked to solve 100 – 3 = ___ and 41 – 39 = ___.

Second graders were asked to solve 100 – 3 = ___ and 201 – 199 = ___.

TABLE 1

Of *all* second graders in the study who were posed 201 – 199, 26% got the correct answer. Of those ***who attempted to use the standard subtraction algorithm*** (56% of the total number of students), 29% found the correct answer.

Total numbers of students who correctly solved selected subtraction problems

Level	Subtraction problem		
	100 – 3 = __	41 – 39 = __	201 – 199 = __
Grade 1 (*n* = 333)	213 (63%)	50 (15%)	—
Grade 2 (*n* = 281)	231 (82%)	—	72 (26%)

situations involving differences do not fit neatly with a take-away interpretation (Fuson 1986). For example, in a join-change-unknown problem, such as "Connie has 5 marbles. How many marbles does she need to have 13 altogether?" (Carpenter et al. 1999), the answer of 8 marbles can be obtained by computing 13 – 5; yet, nothing in the story is being taken away. If children are told to use subtraction to solve these kinds of problems—and yet learn that the sole meaning of subtraction is take-away—they may experience mathematics as not making sense.

Differences as distances

The point we wish to highlight is how powerful flexible reasoning about subtraction can be for students. Beyond the take-away meaning, many situations exist in which reasoning about differences as distances between numbers is helpful. Students who solve 201 – 199 by counting forward from 199 to 201 seem to be answering a different question than those who create a set of 201 objects and remove 199 objects from it. In the case of these numbers, "How far is 199 from 201?" seems to be a simpler question to answer than "How much is left if you take 199 away from 201?" because 199 and 201 are close to each other.

Thinking about differences as distances between numbers (e.g., on a number line) can help students make important connections among the ideas of addition and subtraction, counting forward and backward, and even linear measurement. It can also help students to sensibly relate different types of story problems to the subtraction operation, as well as to make otherwise difficult computations like 201 – 199 much easier. The distance meaning arises in integer arithmetic, where it can provide a sensible interpretation of such expressions as 3 – –5. Furthermore, the distance interpretation is used in algebraic and geometric applications in secondary mathematics and beyond, such as using subtraction to determine the horizontal or vertical distance between two points in a plane. Thus, reasoning about differences in terms of distance is good preparation for the transition from arithmetic to algebra (Matsuura and Xu 2014).

Relational thinking

The Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) call for students to solve addition and subtraction problems that involve a variety of situations (1.OA.A.1, 2.OA.A.1). For students to reason effectively about different subtraction computations, flexible thinking about subtraction is important. Also important is the habit of pausing to consider the numbers in the problem and what is being asked before choosing an approach. Some students notice that 39 and 41 are very close to each other, and they choose a strategy that takes advantage of that feature (1.OA.B.4). Others interpret the problem as asking them to take a set of 39 away from a set of 41; they plow ahead with a default procedure, whether or not it is efficient to do so. In other words, some students use relational thinking to make problems like 41 – 39 easy; others miss these opportunities.

Relational thinking involves a mindful application of place value and the properties of number, operations, and equality in solving mathematics problems (Jacobs et al. 2007). Curriculum standards and textbooks encourage students to learn different strategies for solving problems, but relational thinking involves more than just using strategies; it involves making strategic decisions. A student with a disposition toward relational thinking has a habit of thinking before acting. We believe that students who approach 100 – 3 as "100 take away 3" and then shift their thinking to view 41 – 39 as the distance between 41 and 39 are demonstrating relational thinking.

XUA/THINKSTOCK (GIRL); TOLGA TEZCAN (BUBBLE)

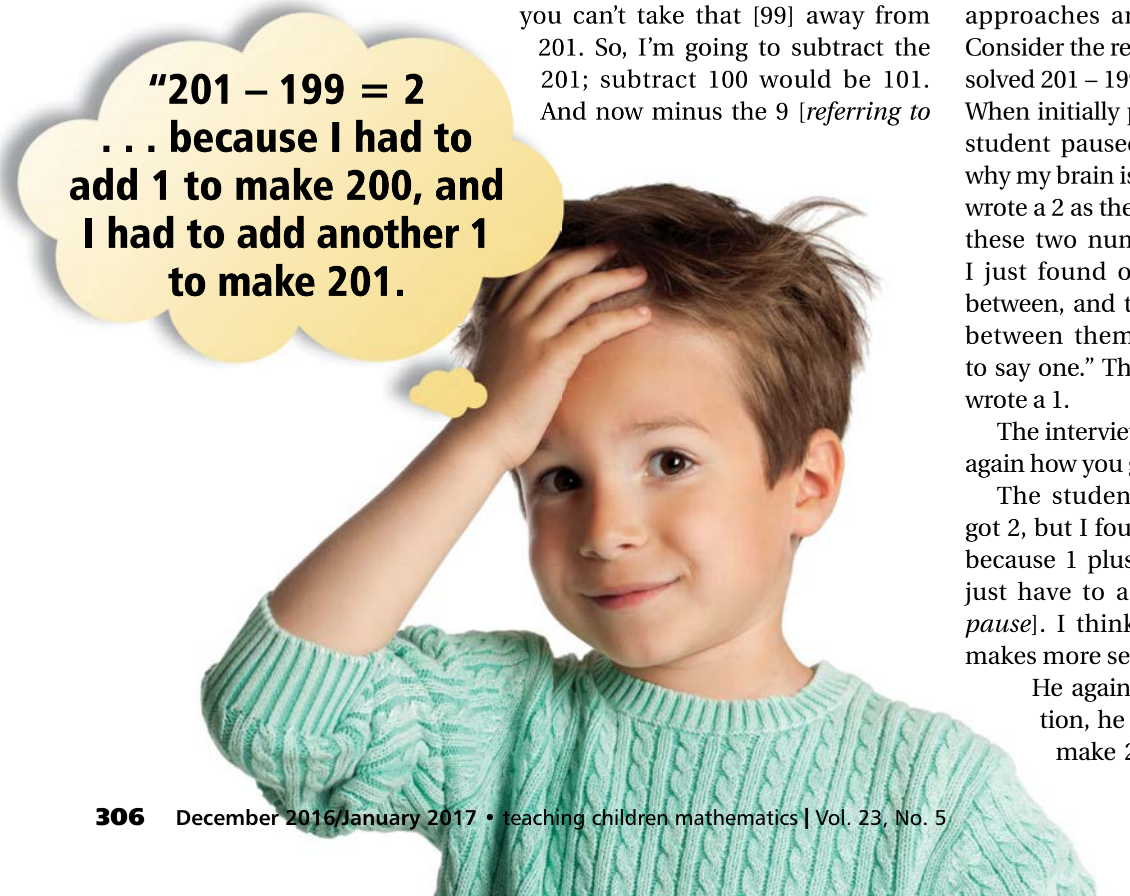
Children's reasoning about subtraction computations

We present three examples from interviews with first and second graders solving subtraction problems. These episodes are not selected to be the three most typical kinds of responses. Rather, they are intended to represent a range of responses and to illustrate important points concerning relational thinking in reasoning about subtraction.

Young children tend to focus on the take-away meaning

The following quote from a student interview highlights the complex process involved in thinking about a problem such as 201 – 199 = ___ using take-away subtraction. A second grader had previously solved 100 – 3 = ___ by counting back from 100 (99, 98, 97) and answered 97. His use of take-away subtraction to solve 100 – 3 worked very well, because it required counting back only three. When this student was presented with 201 – 199, he again used a take-away model, chosing an incremental approach based on place value to find his answer:

> So, you can't take away the 99; so, I'm going to subtract the hundreds first—because you can't take that [99] away from 201. So, I'm going to subtract the 201; subtract 100 would be 101. And now minus the 9 [*referring to the nine tens*] from the 101 would be 91, 81, 71 [*pausing*], 61, 51 [*pausing*], 41, 31, 21, 10. So, I have 10 left, minus 9 [*holding up ten fingers and counting back*]. It will go 9, 8, 7, 6, 5, 4, 3, 2, 1. The answer is 1.



Even when working with such large numbers as 201 and 199, this student was able to use an advanced strategy involving incrementing and counting backwards. However, the process of taking away 199 was cumbersome, and the student happened to make a mistake when he mentally took 10 away from 21 and got 10, rather than 11. This small mistake resulted in an incorrect solution. We certainly believe that this student could have arrived at the correct answer with his method. In fact, he came very close. It is not the case that take-away reasoning dooms students to get incorrect answers. It is just that taking away 199 by incrementing requires a lot of working memory and many steps in the process, and those steps create opportunities for errors.

Reasoning about differences as distances

Although reasoning about subtraction in terms of the distance between two numbers can be advantageous, we acknowledge that such approaches are also not immune to errors. Consider the reasoning of another student, who solved 201 – 199 = ___ by counting up from 199. When initially presented with the problem, the student paused and then said, "I don't know why my brain isn't thinking right now." Then, he wrote a 2 as the answer. He explained, "Because these two numbers are really close together, I just found out what were the numbers in between, and there were only two numbers in between them—wait—I mean one. I meant to say one." The student crossed out the 2 and wrote a 1.

The interviewer asked, "OK, so explain to me again how you got one."

The student explained, "The same way I got 2, but I found out that it's really actually 1, because 1 plus 199 would be 200, so I would just have to add 1 more to get to 201 [*long pause*]. I think I'm going to make it 2. That makes more sense."

He again wrote a 2. In his final explanation, he said, "Because I had to add 1 to make 200, and I had to add another 1

ICONOGENIC/THINKSTOCK (BOY); TOLGA TEZCAN (BUBBLE) to make 201, and that's how I got 2." This student struggled with deciding whether it was the one whole number between 199 and 201 or the count from 199 to 201 that would give the correct solution. This is not a trivial issue, and it is one that can be troubling even for students who are proficient with subtraction. Thus, encouraging students to reason about distances between numbers is only one important aspect of a comprehensive approach to teaching subtraction. Just as with take-away approaches, challenging conceptual issues must be tackled.

Opportunities to bridge from take-away to distance

When a first grader was posed the problem 41 – 39 = ___, she decided to use snap cubes. She connected them end-to-end until she had a very long chain of 41 cubes, counting them as she went. Just before the chain was complete, she reached 39. She then added 2 more cubes for a total of 41, and said, "That's forty-one." Then she said, "Take away thirty-nine." She seemed prepared to remove 39 cubes from the chain by counting them individually, but then she suddenly exclaimed, "Oh, that's easy!" She removed the last 2 cubes that she had added to the chain, and said, "That's thirty-nine. It equals two! Yay! That's easy."

In her sudden insight regarding this problem, this student noticed something. She may have thought about taking 39 cubes away from the chain or taking 2 cubes away, or perhaps she thought about rewinding her actions to the moment before she had added the last 2 cubes to the chain. In any case, she split the chain into two parts with lengths 39 and 2, and she could see that the answer was 2. The proximity of 39 to 41 created an opportunity for this student to do something special. Rather than laboriously removing and counting 39 cubes to discover how many were left, she broke the chain at the meeting point of the 39th cube and 40th cube, making it quick and easy to see the solution of 2.

The purpose of these interviews was to investigate students' thinking, not to influence it. However, moments like this suggest opportunities that teachers could capitalize on in classroom settings.

We have suggested many ways that the ideas in this article can be applied in the classroom, and additional suggestions are available in the **more4U online appendix**. The most important point is the goal: for students to develop the habit of making thoughtful, strategic decisions about how to approach subtraction problems, supporting those decisions with multiple ways of reasoning.

The importance of flexibility

After interviewing hundreds of first- and second-grade students and observing them as they solved mathematics problems, we conclude that learning to reason flexibly about subtraction is important for students. Subtraction is more complicated than addition (Fuson 1984), and subtraction can be thought about in different ways (Selter et al. 2012). The take-away meaning is important and plays a central role in early subtraction instruction. However, children who think about subtraction only in terms of take-away are limited in their flexibility and thus tend to rely on inefficient and error-prone approaches. Some problems lend themselves to thinking of subtraction as take-away; others lend themselves to reasoning about the difference as a distance. Promoting these different ways of reasoning when opportunities arise is important to assist students in becoming accurate, efficient, and flexible with computation.

Common Core Connections

1.OA.A.1
1.OA.B.4
2.OA.A.1

REFERENCES

Carpenter, Thomas P., Elizabeth Fennema, Megan Loef Franke, Linda Levi, and Susan B. Empson. 1999. *Children's Mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics (CCSSM). Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Fuson, Karen C. 1984. "More Complexities in Subtraction." *Journal for Research in Mathematics Education* 15 (May): 214–25.

———. 1986. “Teaching Children to Subtract by Counting Up.” *Journal for Research in Mathematics Education* 17 (May): 172–89.

Gibbs, E. Glenadine. 1954. “Take-Away Is Not Enough!” *The Arithmetic Teacher* 1 (April): 7–10.

Jacobs, Victoria R., Megan Loef Franke, Thomas P. Carpenter, Linda Levi, and Dan Battey. 2007. “Professional Development Focused on Children’s Algebraic Reasoning in Elementary School.” *Journal for Research in Mathematics Education* 38 (May): 258–88.

Maples, Eva Jean. 1959. “Developing the “Take-Away” Idea.” *The Arithmetic Teacher* 6 (April): 161.

Matsuura, Ryota, and Yu Yan Xu. 2014. “Mathematical Explorations: Find the Distance: No Formula Necessary.” *Mathematics Teaching in the Middle School* 19 (May): 564–70.

Page, Anita. 1994. “Helping Students Understand Subtraction.” *Teaching Children Mathematics* 1 (November): 140–43.

Schoen, Robert C., M. LaVenia, Zachary Champagne, and K. Farina. 2016 Mathematics Performance and Cognition (MPAC) Interview: Measuring First- and Second-Grade Student Achievement in Number, Operations, and Equality in Spring 2014. (Report No. 2016-01.) Tallahassee, FL: Florida Center for Research in Science, Technology, Engineering, and Mathematics.

Selter, Christoph, Susanne Prediger, Marcus Nührenbörger, and Stephan Hußman. 2012. “Taking Away and Determining the Difference—A Longitudinal Perspective on Two Models of Subtraction and the Inverse Relation to Addition.” *Educational Studies in Mathematics* 79 (3): 389–408.

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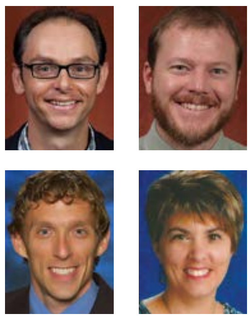

Ian Whitacre, iwhitacre@fsu.edu, is an assistant professor in the School of Teacher Education at Florida State University. **Robert C. Schoen, rschoen@lsi.fsu.edu**, is the associate director of the Florida Center for Research in Science, Technology, Engineering, and Mathematics (FCR-STEM) at Florida State University. **Zachary Champagne, zchampagne@lsi.fsu.edu**, is an assistant in research at FCR-STEM. **Andrea Goddard, goddarda@leonschools.net**, is the elementary school mathematics curriculum developer for Leon County Schools. The four authors are interested in using what they learn about children’s mathematical thinking to offer practical recommendations to help improve mathematical learning experiences for all students.

Last chat of 2016

On the second Wednesday of each month, *TCM* hosts a lively discussion with authors and *TCM* readers about an important topic in our field.
On **December 14, 2016,** at **9:00 p.m. ET**, we will discuss “Relational Thinking: What’s the Difference?” by Whitacre, Schoen, Champagne, and Goddard.
Follow along using #TCMchat.

Unable to participate in the live chat? Follow us onTwitter@TCM_at_NCTM and watch for a link to the recap.

more4U

Go to **http://www.nctm.org** for an appendix of suggestions to encourage flexible subtraction reasoning. This online content is a members-only benefit.

Encouraging flexible subtraction reasoning

We offer suggestions that apply to classroom instruction related to subtraction. We present these in the form of things for teachers to notice, to use, and to encourage. We have tried or observed some of these in classrooms. Others are inspired by opportunities that we noticed in interviews.

Notice

- Notice the different situations in which you use subtraction and the ways that you reason about subtraction. (SMP 8: *Look for and express regularly in repeated reasoning*.)
- Notice the situations that your students associate with subtraction. (MTP7: *Elicit and use evidence of student thinking*.)
- Notice the language that you and your students use when talking about subtraction. (SMP 6: *Attend to precision*.)

Use

- Use the terms *minus* or *subtract* when reading the subtraction symbol, and refrain from reading the symbol as "*take away*" (Fuson 1986). (SMP 6: *Attend to precision*.)
- Use tasks such as 41 – 39 = ❑. Write these horizontally, rather than vertically, to encourage students to use nonstandard strategies. (MTP5: *Pose purposeful questions*.)
- Use story problems with Compare situations (Carpenter et al. 1999; CCSSI 2010) to emphasize thinking about differences as distances between numbers. (SMP 7: *Look for and make use of structure*.)
- Use think-alouds to model how you choose an approach to a subtraction problem in order to help students learn to articulate their reasoning about subtraction. (MTP4: *Facilitate meaningful mathematical discourse*.)
- Use a number line posted on the wall of your classroom when discussing subtraction problems and strategies. Emphasize distances between numbers when this idea relates to a problem or strategy being discussed. (MTP3: *Use and connect mathematical representations*.)
- Use true/false equations, such as 72 – 68 = 74 – 70, to support students in thinking about how adjusting both the minuend and the subtrahend by the same amount does not change the distance between the two values. (MTP2: *Implement tasks that promote reasoning and problem solving*.)

Encourage

- Encourage students to use mental math and to reason flexibly about subtraction. (MTP6: *Build procedural fluency from conceptual understanding*.)
- Encourage students to discuss and make sense of one another's strategies. (SMP 3: *Construct viable arguments and critique the reasoning of others*. MTP4: *Facilitate meaningful mathematical discourse*.)
- Encourage students to ask specific questions, such as, "How were you thinking about subtraction?" or "Which meaning of subtraction were you using?" (MTP4: *Facilitate meaningful mathematical discourse*.)
- Encourage students to notice when a strategy is most helpful, depending on the numbers. (SMP 7: *Look for and make use of structure*.)
- Encourage students to describe what they are trying to find out when solving a subtraction problem. (SMP 6: *Attend to precision*.)

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