Braithwaite, D. W., Pyke, A. A., & Siegler, R. S. (in press). A computational model of fraction arithmetic. *Psychological Review.* (Anticipated publication date: 2017).

A Computational Model of Fraction Arithmetic

David W. Braithwaite and Aryn A. Pyke

Carnegie Mellon University

Robert S. Siegler

Carnegie Mellon University and Beijing Normal University

Author Note:

David W. Braithwaite, Department of Psychology, Carnegie Mellon University; Aryn A. Pyke, Department of Psychology, Carnegie Mellon University; Robert S. Siegler, Department of Psychology, Carnegie Mellon University and The Siegler Center for Innovative Learning (SCIL), Advanced Technology Center, Beijing Normal University. Correspondence concerning this article should be addressed to David Braithwaite or Robert Siegler, Department of Psychology, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213. baixiwei@gmail.com.

This work was supported in part by the Institute of Education Sciences, U.S. Department of Education, through Grants R305A150262, R324C100004:84.324C, Subaward 23149, and R305B100001 to Carnegie Mellon University and the Advanced Technology Center and Siegler Center for Innovative Learning, Beijing Normal University. Preliminary results from Studies 1 and 3 were presented on posters at the 2015 Institute of Education Sciences Principal Investigators meeting and the 38th Annual Conference of the Cognitive Science Society (2016); studies 2, 4, and 5 have not been presented before. The authors would like to thank John Anderson and Christian Lebiere for helpful comments. Python code for the simulations reported here is included in the Supplemental Materials.

Abstract

Many children fail to master fraction arithmetic even after years of instruction, a failure that hinders their learning of more advanced mathematics as well as their occupational success. To test hypotheses about why children have so many difficulties in this area, we created a computational model of fraction arithmetic learning and presented it with the problems from a widely used textbook series. The simulation generated many phenomena of children's fraction arithmetic performance through a small number of common learning mechanisms operating on a biased input set. The biases were not unique to this textbook series – they were present in two other textbook series as well – nor were the phenomena unique to a particular sample of children – they were present in another sample as well. Among other phenomena, the model predicted the high difficulty of fraction division, variable strategy use by individual children and on individual problems, relative frequencies of different types of strategy errors on different types of problems, and variable effects of denominator equality on the four arithmetic operations. The model also generated non-intuitive predictions regarding the relative difficulties of several types of problems and the potential effectiveness of a novel instructional approach. Perhaps the most general lesson of the findings is that the statistical distribution of problems that learners encounter can influence mathematics learning in powerful and non-intuitive ways.

Keywords: numerical cognition; fractions; production systems; textbook analysis; fraction arithmetic

A Computational Model of Fraction Arithmetic

Mathematical knowledge is important for both academic and occupational success (Davidson, 2012; McCloskey, 2007; Murnane, Willett, & Levy, 1995; Sadler & Tai, 2007). The foundations of later success are laid early. Mathematics achievement at age 7 predicts socio-economic status (SES) at age 42, even after controlling for the child's general intelligence, reading ability, and birth SES (Ritchie & Bates, 2013).

Fractions are among the most important topics encountered in mathematics education. One reason is that understanding them is critical to success in more advanced mathematics (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014). Consistent with this perspective, fifth graders' fractions knowledge uniquely predicts their knowledge of algebra and overall mathematics achievement in tenth grade, even after controlling for their parents' income and education and their own whole number arithmetic knowledge, IQ, working memory, reading achievement, race, and gender (Siegler et al., 2012). Fractions knowledge is also required in a wide range of occupations: a recent, large-scale survey of American white collar and blue collar employees and service workers found that 68% said that they employed fractions and decimals during their work (Handel, 2016). Reflecting this importance, fractions are a major topic of instruction in fourth, fifth, and sixth grade classrooms (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000).

Despite the importance of fractions and the prolonged fraction instruction students receive, many children (and adults) struggle to understand them (Stigler, Givvin, & Thompson, 2010). Illustratively, Algebra I teachers in the United States who were presented 15 topics viewed as prerequisites for learning algebra and asked about their students' preparedness in them rated knowledge of fractions and decimals second lowest (with the amorphous category “story problems” being the lowest-rated topic; Hoffer, Venkataraman, Hedberg, & Shagle, 2007).

Fraction arithmetic poses a particular challenge for learners. In contrast to whole number arithmetic, which most children eventually master at least for single-digit numbers, many learners never reach proficiency in fraction arithmetic (Gabriel et al., 2013; Lortie-Forgues, Tian, & Siegler, 2015; Stigler et al., 2010). Further, the ability of children in the United States to solve fraction arithmetic problems lags far behind that of children in countries with high math achievement, such as China (Torbeyns, Schneider, Xin, & Siegler, 2015). The fact that fraction arithmetic was part of more than half of the formulas provided on the reference sheets for recent (2014) Advanced Placement chemistry and physics exams further illustrates the far-reaching importance of fraction arithmetic (Lortie-Forgues & Siegler, in press).

A recent review identified two classes of difficulties children face in learning fraction arithmetic (Lortie-Forgues et al., 2015). One class includes difficulties relating to the specific fraction arithmetic procedures that children are expected to learn, such as the sheer number of procedures; the opacity of why the procedures make sense; the complex, partially overlapping relations among procedures for different arithmetic operations; and the complex relations between fraction and whole number arithmetic procedures. The other class includes difficulties relating to the context in which the procedures are learned: the clarity of exposition by teachers and textbooks, the types and number of practice problems that children are presented, and the prior mathematical knowledge that children bring to the task of learning fraction arithmetic.

To examine the ramifications of both sources of difficulty, and to test a number of specific hypotheses about how learning of fraction arithmetic occurs, we created a formal computational model of how children acquire, or fail to acquire, fraction arithmetic proficiency.

Such computational models are useful because they make explicit the mechanisms that link theoretical accounts to observed behavior and test whether the mechanisms actually generate the phenomena they purport to explain. Computational models also can be used to simulate the results of future experiments, thereby generating empirical predictions that allow further tests of the underlying theory. Moreover, such models can identify potentially effective instructional interventions.

We named the present model FARRA (Fraction Arithmetic Reflects Rules and Associations). FARRA predicts accuracy, patterns of strategy use, and specific errors that children make on all four fraction arithmetic operations on varied types of problems. Simultaneously examining performance on all four operations seems particularly important for models of fraction arithmetic, because confusions among procedures that are appropriate for different fraction arithmetic operations appear central to children's difficulties learning fraction arithmetic. The frequency and range of such confusions are only evident when all four arithmetic operations are considered together.

FARRA reflects four main hypotheses. The central hypothesis is that poor fraction arithmetic performance reflects well-documented learning mechanisms operating on biased input in ways that generate weak learning of confusable procedures. Overgeneralization of these confusable procedures seems to be an especially large source of difficulties.

A second, related hypothesis is that the relative difficulties of different types of fraction arithmetic problems largely reflect imbalances in the distribution of practice problems that children receive. In particular, children encounter the most challenging types of problems the least often, making these problems more difficult than they otherwise would be. A general implication of this hypothesis is that detailed analyses of the problems that children are presented can yield valuable insights about learning in specific domains.

A third hypothesis that FARRA instantiates is that people use statistical associations between problem features and solution procedures to guide their choices of procedures. Such statistical learning is beneficial in many situations, but it can be harmful in mathematics learning, where the normative criteria for applying procedures depend on explicit rules rather than statistical associations. In particular, if the practice problems that children receive are biased, the children's choices of strategies will reflect the biases.

A fourth hypothesis underlying FARRA is that conceptual knowledge plays little role in most children's learning of fraction arithmetic. In some cases, children have relevant conceptual knowledge, but ignore it when doing fraction arithmetic. For example, most sixth and eighth graders know that the sum of two positive fractions must be larger than either addend (Siegler & Lortie-Forgues, 2015), but they often violate this principle when solving fraction addition problems, for example by stating "2/4" as the answer to 1/2+1/2. Other times, children lack relevant conceptual knowledge. For example, most sixth graders, eighth graders, and pre-service teachers incorrectly believe that the product of two fractions with magnitudes between 0 and 1 is larger than either multiplicand (Siegler & Lortie-Forgues, 2015). These findings do not imply that children have no conceptual knowledge of fraction arithmetic, but they do suggest that for many children, procedural knowledge develops and is represented in isolation from whatever conceptual knowledge they have.

This last hypothesis suggests that numerous aspects of fraction arithmetic performance can be explained by considering procedural knowledge alone. FARRA tests this view by examining whether a model devoid of conceptual knowledge can generate and explain the main phenomena of children's fraction arithmetic that have been documented.

Prior research has emphasized the role of conceptual knowledge in fraction arithmetic primarily from the perspective that children fail to learn fraction arithmetic because of a systematic misunderstanding (Byrnes & Wasik, 1991; Fischbein, Deri, Nello, & Marino, 1985). For example, Gelman (1991) argued that children use the conceptual framework of whole numbers to think about fractions and therefore think of a fraction as two separate whole numbers, a phenomenon known as "whole number bias" (Gabriel et al., 2013; Ni & Zhou, 2005). Whole number bias leads to errors in which fraction arithmetic problems are treated as two separate whole number arithmetic problems, as in 1/2+1/2 = (1+1)/(2+2) = 2/4 (Byrnes & Wasik, 1991; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Mack, 1995).

However, whole number bias does not explain the many errors that occur in cases where the above strategy yields a correct answer, including all fraction multiplication problems. More generally, poor conceptual understanding may contribute to children's poor fraction arithmetic performance, but it does not explain the full range of errors that they make, why some errors are more frequent than others, or why the frequencies of different errors vary systematically on different types of problems. In the present research, we address these questions by comparing to children's behavior the output of a model in which fraction arithmetic is totally unconstrained by conceptual knowledge.

Major Fraction Arithmetic Phenomena

Eight well-established empirical phenomena that any comprehensive model of fraction arithmetic would need to explain are summarized in Table 1. We first present the phenomena and evidence regarding them, and then describe our model and how it explains each one. To provide a common frame of reference, we cite data from Siegler and Pyke (2013) to illustrate each phenomenon and also reference other studies that have documented the same phenomena.

===================== Table 1 about here =====================

Low Overall Accuracy

Numerous studies have documented poor fraction arithmetic performance among fourth to eighth grade children in the United States and Western Europe (Byrnes & Wasik, 1991; Fuchs et al., 2013, 2014; Hecht, 1998; Hecht & Vagi, 2010; Jordan et al., 2013; Newton, Willard, & Teufel, 2014; Siegler & Pyke, 2013; Siegler, Thompson, & Schneider, 2011; Torbeyns et al., 2015). Performance does improve with age, but slowly and to a low asymptotic level. For example, in one representative study in which U.S. sixth and eighth graders solved a set of fraction arithmetic problems including all four arithmetic operations, equal and unequal denominators, and operands (numbers in the problem) with numerators and denominators of five or less, average accuracy was 46% in sixth grade and 57% in eighth grade (Siegler & Pyke, 2013). Performance remains poor among high school students (G. Brown & Quinn, 2006), community college students (Richland, Stigler, & Holyoak, 2012), and even pre-service teachers (Newton, 2008).

Especially Low Accuracy on Division Problems

Fraction division problems are consistently more difficult for children than problems involving addition, subtraction, or multiplication (Siegler & Pyke, 2013; Siegler et al., 2011). Inaccurate fraction division performance may in part reflect the fact that the standard procedure for fraction division that is taught in the United States – that is, to invert the second operand and then to multiply the numerators and denominators separately – is more complex than the procedures taught for other operations. For example, only for the standard division procedure do learners need to invert an operand and decide which one to invert. The difficulty also may reflect the paucity of fraction division problems in at least some U.S. math textbooks. Son and Senk (2010) found 250 fraction multiplication problems but only 54 fraction division problems in the fifth and sixth grade textbooks and workbooks in a widely used U.S. textbook series, Everyday Mathematics (2002). In this article, we found similarly few fraction division problems in three other textbook series: enVisionMATH, (Charles et al., 2012), GO MATH (Dixon, Adams, Larson, & Leiva, 2012), and a newer edition of Everyday Mathematics (University of Chicago School Mathematics Project, 2015c).

Variable Responses Within Individual Problems

In addition to low accuracy, children exhibit striking variability in the mix of correct answers and errors they generate on a given problem (Hecht, 1998; Newton et al., 2014; Siegler & Pyke, 2013; Siegler et al., 2011). For example, on the problem 2/3+3/5, Siegler and Pyke (2013) observed 22 distinct answers from the 59 children who answered incorrectly.

Variable Strategy Use by Individual Children

Substantial variability is present within children as well as problems (Siegler & Pyke, 2013; Siegler et al., 2011).Thus, 60% of children in Siegler and Pyke (2013) used different strategies on at least one pair of virtually identical problems (e.g., 3/5×1/5 and 3/5×4/5). This variability usually involved a correct strategy on one but not both highly similar items.

Greater Frequency of Strategy Errors Than Execution Errors

Fraction arithmetic errors can be divided into strategy errors and execution errors. Strategy errors result from using an incorrect procedure for the given arithmetic operation, for example separately adding the two numerators and the two denominators on a fraction addition problem. Execution errors result from flawed execution of procedures that are correct for the operation. They include whole number arithmetic errors (e.g., 6×9=48), failure to change numerators as well as denominators when establishing common denominators, and inverting the wrong operand in fraction division (G. Brown & Quinn, 2006; Byrnes & Wasik, 1991; Hecht, 1998; Newton et al., 2014; Siegler & Pyke, 2013; Siegler et al., 2011).

Strategy errors account for a larger proportion of fraction arithmetic errors than do execution errors (G. Brown & Quinn, 2006; Byrnes & Wasik, 1991; Gabriel et al., 2012, 2013; Hecht, 1998; Newton et al., 2014; Siegler & Pyke, 2013; Siegler et al., 2011). For example, in Siegler and Pyke (2013), strategy errors occurred on 46% of trials and accounted for 91% of errors, whereas execution errors occurred on 5% of trials and accounted for 9% of errors.

The line between strategy and execution errors is not always clear; for example, inverting the wrong operand in fraction division might be viewed as an incorrect strategy rather than incorrect execution of a correct strategy. In the studies reported below, we adopt a relatively conservative definition of strategy errors, in which potentially ambiguous cases such as the example just mentioned are classified as execution errors rather than as strategy errors. As will be seen, strategy errors are more common than execution errors even when such a conservative definition of strategy errors is used.

The Most Common Strategy Errors Are Wrong Fraction Operation Errors and Independent Whole Numbers Errors

The most common flawed strategies are wrong fraction operation and independent whole numbers approaches (Siegler & Pyke, 2013; Siegler et al., 2011). Wrong fraction operation strategies involve using a procedure that is correct for one fraction operation on a problem that calls for a different fraction operation. For example, a correct strategy for adding fractions with equal denominators is to apply the operation to the numerators while maintaining the common denominator, as in 3/5+1/5=(3+1)/5=4/5. Applying this strategy to a division problem, as in 3/5÷1/5=(3÷1)/5=3/5, constitutes a wrong fraction operation error. Similarly, a correct strategy for dividing fractions is to invert the denominator fraction and then to multiply the numerators and denominators of the two operands separately, as in 3/5÷1/4=3/5×4/1=(3×4)/(5×1)=12/5. Applying an overgeneralized version of this strategy to a multiplication problem, as in 3/5×1/4=3/5×4/1=(3×4)/(5×1)=12/5, constitutes a wrong fraction operation error.

Independent whole numbers strategies involve applying specified arithmetic operations separately to the numerators and denominators of the operands to obtain the numerator and denominator of the answer. This strategy is correct for multiplication, but yields errors in addition and subtraction, as in 3/5+1/4=(3+1)/(5+4)=4/9.

Both types of flawed strategies are common. In Siegler and Pyke (2013), wrong fraction operation strategies accounted for 51% of errors, and independent whole numbers strategies accounted for 27% of errors.

Equal Denominators Increase Addition and Subtraction Accuracy but Decrease Multiplication Accuracy

Equal denominators are associated with higher accuracy for addition and subtraction, but lower accuracy for multiplication (Gabriel et al., 2013; Siegler & Pyke, 2013; Siegler et al., 2011). For example, in Siegler and Pyke (2013), fraction addition and subtraction accuracy was 80% for equal denominator problems and 55% for unequal denominator problems, whereas fraction multiplication accuracy was 37% for equal denominator problems and 58% for unequal denominator problems.

The Most Frequent Type of Error on Each Operation Varies With Denominator Equality

The most common errors on each arithmetic operation also vary with whether denominators are equal. On addition and subtraction problems, children more often commit independent whole numbers errors on unequal than on equal denominator items (Gabriel et al., 2013; Newton et al., 2014; Siegler & Pyke, 2013; Siegler et al., 2011). Thus, 3/5+1/4 elicits the incorrect answer 4/9 more often than 3/5+1/5 elicits the incorrect answers 4/10 or 2/5 (Siegler & Pyke, 2013). On multiplication problems, by contrast, children commit wrong fraction operation errors more often on equal than on unequal denominator items (Gabriel et al., 2013; Newton et al., 2014; Siegler et al., 2011). Thus, 3/5×4/5 elicits the incorrect response 12/5 more often than 3/5×1/4 elicits the incorrect responses 60/20 or 3/1.

The Model

Production Rules

Within FARRA, as within other production system models, rules are modular components of knowledge used to perform tasks. Each rule is a condition-action pair, in which the condition specifies when the rule may be fired, and the action specifies what happens when it fires. Rules are selected and fired iteratively until an answer is obtained.

When solving a problem, FARRA maintains a representation of the problem state, including the numerators, denominators, and arithmetic operation in the problem, as well as results of intermediate calculations. FARRA also maintains a list of goals that it is trying to achieve. Together, the problem state and goals determine which rules' conditions are met. Firing rules causes changes to the problem state and/or goals, thus affecting which rule fires next.

FARRA includes two types of rules—strategy rules and execution rules—each of which can be further divided into correct rules and mal-rules. Correct rules were devised by formalizing the fraction arithmetic procedures presented in a widely-used textbook series (e.g., Charles et al., 2012). No substantial differences were found between these procedures and those presented in two other textbook series (Dixon et al., 2012; UCSMP, 2015). Mal-rules were devised based on our review of common fraction arithmetic errors, as described in the previous section. We first describe similarities and then differences among the different types of rules.

Strategy rules. FARRA begins each problem-solving episode with a single goal: find the answer. We refer to the relatively few rules that can fire when this is the only goal as *strategy rules*. A complete list of FARRA's strategy rules is shown in Table 2.

===================== Table 2 about here ======================

Correct strategy rules. Rules 1-4 describe normatively correct strategies. Rules 1 and 2 describe the standard correct procedures for fraction addition and subtraction, collectively termed "Correct Add/Sub." Rule 1 describes the version of this strategy for equal denominator addition and subtraction problems, whereas Rule 2 describes the version for unequal denominator problems, the latter rule requiring conversion of the operands to a common denominator. Rule 3 describes the standard correct procedure for fraction multiplication, termed "Correct Mult," and Rule 4 describes the standard correct procedure for fraction division, termed "Correct Div."

Strategy mal-rules. Consistent with cognitive science research on mathematics problem-solving (J. S. Brown & VanLehn, 1980; Ohlsson, 2016; Payne & Squibb, 1990; Sleeman, 1984; Young & O'Shea, 1981), we assume that most fraction arithmetic errors result from *mal-rules*, that is, rules that incorporate relatively small deviations from normatively correct procedures.

All but one of FARRA's strategy mal-rules (Rules 5-8 in Table 2) reflect overgeneralization of Rules 1-4. Overgeneralization was implemented by deleting the arithmetic operation from the condition side of correct rules. Thus, the strategies described by these rules are called “Op-deleted Add/Sub,” “Op-deleted Mult,” and “Op-deleted Div.” These rules lead FARRA to overgeneralize Add/Sub, Mult, and Div rules to arithmetic operations other than the operation specified in the problem.

One likely reason why such mal-rules are used frequently is the blocked presentation of problems of a given arithmetic operation (e.g., fraction addition) in classroom instruction and textbooks. The blocked presentation may lead learners to ignore the arithmetic operation beyond the first problem or two in a set. Indeed, given the blocked presentation of particular operations in textbook chapters, many children may not attend to the operations of any problems in a set; why encode the operation if this is (e.g.) the fraction multiplication chapter?

The one mal-rule that does not reflect overgeneralization of correct fraction arithmetic procedures instead reflects overgeneralization to fraction multiplication of the cross-multiplication procedure for comparing fractions. In the fraction comparison context, cross multiplication involves multiplying the numerator of one operand by the denominator of the other, multiplying the remaining numerator by the remaining denominator, and choosing as larger the fraction that contributed the numerator to the larger product. In the fraction multiplication context, the cross-multiply mal-rule (Table 2, rule 9) establishes the goal of performing similar cross multiplications (Newton, 2008). This mal-rule yields the same answers as the Op-deleted Div strategy, so trials on which the two strategies are used are grouped together in analyses of FARRA’s accuracy.

Execution rules. Firing any strategy rule creates new goals that permit *execution rules* to fire. As shown in Table 3, some execution rules implement strategies correctly (Rules 10-15); other execution rules produce the most frequent execution errors that children make (Rules 16-

18). (Several execution rules of little theoretical interest are omitted from the table for purposes of brevity; a complete list of FARRA's rules appears in the Supplemental Materials, Part A.)

===================== Table 3 about here =====================

Correct execution rules. Once a strategy is chosen, rule 10 sets up a whole number arithmetic problem relevant to computing the numerator, and Rules 11-15 serve similar functions for computing the denominator. Other execution rules, presented in Supplemental Materials, Part A, solve the whole number arithmetic problems needed to specify the numerator and denominator in intermediate calculations and in the answer.

Execution mal-rules. Incorrect variants of correct execution rules are called execution mal-rules. The main such rules in FARRA reflect *incomplete execution*, that is, failure to perform one or more parts of the correct procedure. Table 3 lists three common execution mal-rules: leaving numerators unchanged when converting operands to a common denominator (rule 16); changing the operation from division to multiplication but not inverting either operand when dividing (rule 17); and inverting a random operand instead of the second operand when dividing (rule 18). All execution mal-rules are shown in the Supplemental Materials, Part A.

Another mechanism, the assumption of *closure of whole number arithmetic*, generated execution mal-rules that reflected the belief that whole number arithmetic problems must yield whole number answers. This belief may arise because it is consistent with the input children receive prior to learning fraction arithmetic. Whole number addition and multiplication always yield whole number answers. Whole number division problems with fractional or decimal answers, and whole number subtraction problems with negative answers, do violate the assumption of closure, but they are not recommended by the Common Core State Standards for instruction before fifth or sixth grade and were not introduced before then in any of the textbook series we examined. The mal-rules reflecting belief in closure of whole number arithmetic were a) when subtracting whole numbers, always subtract the smaller from the larger number; and b) when dividing whole numbers, always divide the larger by the smaller number and ignore any remainders. When these rules were used, they led FARRA to avoid generating answers that included a fraction, decimal, or negative in the numerator or denominator.

Rule Selection and Learning

FARRA often must choose among multiple rules whose conditions are met, such as between a correct strategy rule and overgeneralized versions of other rules. We assume stochastic rule selection combined with a reinforcement learning mechanism, in which using a rule increases the likelihood of using that rule in the future to a greater extent on problems with similar features if its use led to a correct answer. Equations 1-3 describe these mechanisms formally.

$$\eta_j = \sum_i w_{ij} x_i \tag{1}$$

$$p_j = e^{\gamma\eta_j} \Big/ \sum_k e^{\gamma\eta_k} \tag{2}$$

$$\Delta w_{ij} = \begin{cases} e & \text{if the answer was correct} \\ (1-d)\cdot e & \text{if the answer was incorrect} \end{cases} \tag{3}$$

At each step in a problem-solving episode, FARRA first determines which rules' conditions are met. The activation of each of these candidate rules is calculated according to Equation 1, which states that the activation η_j of rule *j* is the sum of the problem's values x_i (0 or 1) on features *i*, weighted by the associative weights w_{ij} connecting the features to rule *j*. FARRA then selects from among the candidate rules using probabilities specified by Equation 2, which states that the probability p_j of selecting rule *j* from among candidate rules *k* is a softmax function (a generalization of the logistic function) of the rules' activations. Noise in this decision rule is governed by γ, a free parameter of the model.

The associative weights w_{ij} are initialized to 0, with the result that all candidate rules initially have an equal probability of being selected. When FARRA is in learning mode, it receives feedback about whether its answer was correct after the answer is advanced. Then, for each step in the problem-solving episode, for each problem feature *i* that was present at that step, and for whichever rule *j* was fired on that step, the weights w_{ij} are adjusted according to Equation 3. The equation contains two free parameters: the learning rate *e* and the error discount *d*, both constrained to be positive numbers. The learning rate governs the degree of reinforcement after the model obtains a correct answer. The error discount determines how much less positive reinforcement is received after the model generates an incorrect answer.

In principle, the error discount can permit decrements in the associative weights of rules that yielded incorrect answers. However, in Studies 1-4 of the present article, *d* was set to values that precluded such decrements in associative weights. This constraint follows several previous models of learning whole number mathematics (Shrager & Siegler, 1998; Siegler & Araya, 2005; Siegler & Shipley, 1995), in which even when use of a rule yields an incorrect answer, the probability of using that rule slightly increases. In Study 5, we tested the effects of this constraint on FARRA's learning.

Equations 1 and 3 presuppose a set of problem features. Within FARRA, three features were encoded. One feature indicated the arithmetic operation. Another indicated whether the operands' denominators were equal. The third feature indicated whether both operands were fractions, one operand was a mixed number and the other a fraction or mixed number, or one operand was a whole number and the other a fraction or mixed number.

Input to the Model

A major assumption of the present research is that the problems that children encounter shape their learning. To understand the characteristics of the problems that children encounter, we extracted problem sets from three commercial textbook series: Pearson Education's enVisionMATH (Charles et al., 2012), Houghton Mifflin Harcourt's GO MATH! (Dixon et al., 2012), and McGraw Hill's Everyday Mathematics (UCSMP, 2015a, 2015b, 2015c). These series were selected to include one representative from each of the three largest publishers of primary and middle school mathematics textbooks.

The problem sets included all fraction arithmetic problems from the textbooks that had two operands, at least one of which was a fraction or mixed number; were in symbolic form (i.e., not story problems); and required an exact numerical answer (i.e., not an estimate). Such problems accounted for the large majority of fraction arithmetic practice problems in the three series, and in three other textbook series analyzed by Cady, Hodges, and Collins (2015). The number of problems meeting the above criteria was 659 in enVisionMATH, 807 in GO MATH!, and 464 in Everyday Mathematics.

The three problem sets were very similar in several respects. First, as shown in Table 4, division was less common than the other three arithmetic operations in all three problem sets.

===================== Table 4 about here =====================

Second, equal denominator multiplication and division problems were almost completely absent, accounting for no more than 2% of the total in any of the sets (Table 4). Thus, the vast majority of equal denominator problems involved addition or subtraction. By contrast, the largest number of unequal denominator problems involved multiplication, and the second largest involved division.

A third similarity among the three textbook series involved the relative frequency of problems with different types of operands (Table 5). Problems were classified according to whether their operands included only fractions, only mixed numbers, or one whole number and one fraction or mixed number. (Items of the last type were classified as unequal denominator problems in Table 4.) In all three sets, among problems in which one operand was a whole number and the other was a fraction or mixed number, addition and subtraction were almost completely absent, accounting for no more than 2% of the total in any of the sets. Thus, the vast majority of these problems involved multiplication or division. By contrast, among problems whose operands included only fractions and/or mixed numbers, the majority involved addition or subtraction.

===================== Table 5 about here =====================

The problems in the three textbook series did differ in a few respects. One difference was that addition and subtraction were more common than multiplication in the enVisionMATH set, whereas in the GO MATH! and Everyday Mathematics sets, multiplication was the most common operation. Also, some series include considerably more practice problems than others did. Nonetheless, the similarities were far more striking than the differences.

The fact that three textbook series published by three different high selling textbook companies (Broussard, 2014; Noonoo, 2012) shared very similar distributions of problems suggests that these distributions are likely to be representative of the fraction arithmetic practice problems encountered by children in the United States. The enVisionMATH set was arbitrarily selected as the primary learning set for training FARRA in the simulations reported below.

Additional simulations conducted using GO MATH! problems as the learning set yielded similar results, as described in Study 2.

Empirical Predictions

We now describe how and why FARRA exhibits each of the empirical phenomena listed in Table 1 when it is trained on problems like those in the textbook series described above.

Low Overall Accuracy. Several factors work toward the model resembling children in being relatively inaccurate at fraction arithmetic. One is that FARRA only learns to distinguish between correct rules and mal-rules via trial and error. Practice increases the model's accuracy by reinforcing correct rules more than mal-rules, but the learning is slow.

Another factor that contributes to low accuracy over a protracted period of learning is that practice at solving problems involving a given fraction arithmetic operation increases the likelihood not only of correctly using the appropriate strategy on problems involving that operation but also of incorrectly generalizing the strategy to problems involving other operations. The model's rule set includes two types of rules for each solution strategy: strategy rules that are specific to the fraction arithmetic operation for which the strategy is appropriate, and mal-rules representing overgeneralizations of the strategy to all fraction arithmetic operations. These two types of rules are identical, except that the mal-rules do not specify the arithmetic operation, and thus can be applied to operations where they are inappropriate. Because the overgeneralized mal-rules yield correct answers on problems involving the arithmetic operation for which the strategy is appropriate, these rules are reinforced fairly often, thereby increasing their use on other types of problems as well.

Especially low accuracy on division problems. FARRA predicts especially low accuracy for division. Two reasons are primacy and frequency effects. Fraction division problems are presented later and less frequently than problems involving other arithmetic operations. By the time FARRA encounters division problems, the Add/Sub and Mult strategies, including the overgeneralized mal-rule versions of those strategies, already have been repeatedly reinforced. Competition from those strategies reduces the frequency of FARRA selecting the correct fraction division strategy.

Even if the model correctly chooses the Div strategy to solve a fraction division problem (rules 4 or 8, Table 2), there are more opportunities for execution errors, such as inverting the wrong operand, not inverting either operand, or randomly inverting operands, than with addition, subtraction or multiplication. The many opportunities for execution errors also slow FARRA's learning of the correct division strategy. Because FARRA often obtains incorrect answers when using the Div strategy, the model reinforces that strategy less than other correct strategies.

Variable responses within individual problems. On a single problem, FARRA sometimes generates correct responses; sometimes commits strategy errors, which lead to different responses depending on the particular strategy that is chosen; and sometimes produces execution errors, which generate different responses depending on the execution error. For example, on 3/5+1/4, the model might generate the correct response 17/20; might make the strategy error of choosing the mal-rule version of multiplication and answer 4/9 (i.e., (3+1)/(5+4)), might make the execution error of converting denominators to common form but leaving the numerators unchanged, yielding 4/20 (i.e., (3+1)/(5×4)), etc.

Variable strategy use by individual children. FARRA predicts high variability in individual children's strategy use for the same reason as it does within individual problems. Indeed, it predicts considerable variability even when the same child does the same or highly similar problems more than once.

Greater frequency of strategy errors than execution errors. FARRA predicts that strategy errors should be more frequent than execution errors because the rule set offers more opportunities for strategy errors. FARRA implements three high-level strategies: Add/Sub, Mult, and Div/Cross-Multiply (Table 2). It can use any of these strategies on any problem, but two of the three would be incorrect. Thus, FARRA always has at least two different ways to commit a strategy error. By contrast, if a correct strategy rule is chosen, the model usually has fewer ways to commit execution errors. For example, if the model is presented the problem 3/5+1/4 and correctly chooses the strategy of converting to a common denominator and then adding the numerators, there is only one possible execution error within the model: leaving the numerators unchanged when converting operands to a common denominator (rule 16, Table 3). These characteristics do not guarantee that strategy errors will be more common – a single execution error could be more frequent than several types of strategy errors combined – but it usually leads to that outcome.

The most common strategy errors are wrong fraction operation errors and independent whole numbers errors. All of FARRA's strategy errors stem from overgeneralization. Nearly all of these errors are wrong fraction operation errors or independent whole numbers errors[1]. Overgeneralization of the standard correct strategies for fraction addition/subtraction and division (rules 5, 6, and 8, Table 2) result in wrong fraction operation errors. Overgeneralization of the standard correct strategy for fraction multiplication (rule 7, Table 2) results in independent whole number errors.

Equal denominators increase addition and subtraction accuracy but decrease multiplication accuracy. Two mechanisms produce this effect. One involves effects of frequency of presenting different types of problems. Associations between a rule and the features of the problem on which the rule is used strengthen each time the rule is used. If a given problem feature is frequently paired with a given arithmetic operation, the model strengthens the association between the two, thus increasing the likelihood of using that strategy when that feature is present subsequently. Thus, FARRA uses problem features as cues for choosing strategies, even if those features are irrelevant to the formal criteria for using each strategy.

In the input to FARRA, addition and subtraction problems with equal denominators are more frequent than ones with unequal denominators (Table 4). This is one factor that leads to FARRA using the correct strategy and obtaining the correct answer more often for equal denominator addition and subtraction problems than for unequal denominator ones. By contrast, multiplication problems with unequal denominators are far more common than ones with equal denominators (Table 4). This leads to the model using the correct strategy and obtaining the correct answer more often on multiplication problems with unequal than equal denominators.

A second mechanism that influences effects of denominator equality involves the likelihood of FARRA committing execution errors. When FARRA selects the correct rule for equal denominator addition and subtraction problems, no execution errors are possible[2]. In contrast, when FARRA selects the correct strategy for unequal denominator addition and subtraction problems, it can err by leaving the numerators unchanged when converting operands to a common denominator (rule 16, Table 3). Thus, FARRA should commit more execution errors on addition and subtraction problems with unequal than equal denominators.

The most frequent type of error on each operation varies with denominator equality. The same factors that cause FARRA to associate the correct strategy rules for addition and subtraction more strongly with equal denominator problems than with unequal denominator ones also cause FARRA to associate the overgeneralized versions of these rules more strongly with equal denominator problems. Wrong fraction operation errors resulting from overgeneralization of addition/subtraction strategies should be more common on equal denominator multiplication and division problems than on unequal denominator ones. For the same reason, FARRA should commit independent whole numbers errors more often on unequal denominator addition, subtraction, and division problems than on equal denominator ones.

Study 1: A Simulation of Children's Fraction Arithmetic

The goal of Study 1 was to evaluate how well FARRA's performance matches children's behavior. The model was first trained on the 659 problems from the fourth, fifth, and sixth grade books of a widely used mathematics textbook series. After this training, the model was tested on all 16 fraction arithmetic problems from Siegler and Pyke (2013). FARRA's performance on this test set was compared to children's performance on the same problems in Siegler and Pyke (2013), with the analysis focusing on the eight aspects of children's performance summarized in Table 1.

Method

Problems. The 659 fraction arithmetic problems presented in the fourth, fifth, and sixth grade volumes of enVisionMATH were FARRA's *learning set*. Items were presented once each to the model, in the order they appeared in the textbooks. We used enVisionMATH as the source of our learning set because it is widely used in schools, presented the median number of problems among the three textbook series we examined, and includes books for the grades when fraction arithmetic is usually taught: fourth, fifth, and sixth grade.

The fraction arithmetic problems presented to children in Siegler and Pyke (2013) served as the *test set* in Study 1. This test set included 16 problems, four problems for each arithmetic operation, two with equal and two with unequal denominators. The four addition problems were

3/5+1/5, 4/5+3/5, 2/3+3/5, and 3/5+1/4. The problems for the other three arithmetic operations involved the same operand pairs but different operations.

Procedure. One thousand simulated students were created by generating instances of FARRA with parameter values drawn randomly from uniform distributions over the following ranges: [0.001, 0.02) for the learning rate parameter *e*, (0.8, 1.0] for the error discount parameter *d*, and [0.5, 1.5) for the decision noise parameter γ. Sixty four additional simulations were run using a wider range of parameter values – in particular, with *e* set to values from 0.005 to 0.05, *d* to values from 0.0 to 1.0, and γ to values from 0.5 to 5.0. These simulations generally yielded similar results to those reported below, with exceptions noted in the text. Results of the additional simulations are described in more detail in the Supplemental Materials, Part C.

Because children would be unlikely to use strategies not yet taught in class, the strategy rules corresponding to each arithmetic operation were excluded from the model until the first time the textbook presented a problem involving that operation. For example, rules 4 and 8, which describe the correct and mal-rule versions of the Div strategy, were not introduced until the first division problem was presented. The one exception involved rule 7, the Op-deleted Mult strategy, which involves applying the specified arithmetic operation separately to the numerators and denominators of the operands. This rule was included from the beginning of training, because it is a straightforward application of previously taught whole number arithmetic and because previous findings indicated that many children use this strategy well before it has been taught in the context of fraction multiplication (Byrnes & Wasik, 1991; Ni & Zhou, 2005). Thus, the rule could be thought of as “Whole number/Op-deleted Mult.”

After receiving the learning set problems, each simulated student received the test set problems. No changes in associative strength occurred during the test set.

Coding. Both FARRA's and children's responses to the test set problems were classified as correct if they were numerically equal to the correct answer, regardless of the procedure used. Answers containing decimals were also counted as correct if they met this standard. Responses were further classified into three groups based on the strategy that generated them: Add/Sub, Mult, or Div/Cross-Multiply (Table 2). Siegler and Pyke (2013) had classified verbal strategy reports after each trial into 31 categories. Because the present modeling efforts changed our perspective on the processes that generated the verbally reported strategies, we re-categorized children's strategies into the same three categories used to classify FARRA's responses. Any response that did not clearly belong to one of the three groups was classified as Other/None.

Results and Discussion

Our description of FARRA's performance is organized around the eight phenomena in Table 1.

Low overall accuracy. Average accuracy in the simulation was 52%, equal to the 52% accuracy exhibited by children in Siegler and Pyke (2013). The equality of the percentages of correct answers suggested that the combination of parameter values within the model was reasonable. In additional simulations conducted over a broader range of the model's free parameters, accuracy ranged from 42% to 65% (Supplemental Materials, Part C).

Especially low accuracy on division problems. As predicted, and like children, the simulation performed worse on division than on other arithmetic operations (Figure 1). This low accuracy reflects the relatively high rate of execution errors when using the Div strategy, as well as frequency and primacy effects favoring the other three arithmetic operations over division.

===================== Figure 1 about here =====================

Variable responses within individual problems. Like children, FARRA generated a range of different responses for each fraction arithmetic problem. Table 6 illustrates this phenomenon by showing the most common answers generated by FARRA and by children on four problems in the test set, one for each operation. On each of these problems, at least four distinct answers were generated by at least 2.0% of children in the experiment. Nearly all of these answers were also generated by the simulation, with frequencies approximating those in the children's data in most cases. For example, the most common error was identical for FARRA and the children on all four problems.

===================== Table 6 about here ======================

To assess the strength of the relation between the frequencies of answers generated by the model and by children, we correlated the frequency in FARRA's and children's data of each problem-answer pair that appeared in either dataset. Over the entire set of problem-answer pairs (N=391), answer frequencies between the experimental and model datasets correlated r=.958, $p < .001$. The strength of this correlation partially reflected the frequency of correct answers being relatively high in both the children's and the simulation's data. However, the correlation remained strong when only errors were considered (N=354), r=.878, $p < .001$. Thus, children's and FARRA's frequencies of different answers, both correct and incorrect, were closely related.

In general, FARRA generated few responses that were not also advanced by children. Children, on the other hand, occasionally advanced responses that were never generated by FARRA. The following three were the most common. First, children made several whole number arithmetic errors that FARRA never made, such as incorrectly retrieving arithmetic facts (e.g., 3/5÷1/5 = (3÷1)/5 = 1/5, 5% of trials) or incorrectly handling the remainder in whole number division (e.g., 3/5÷1/4 = 12/20÷5/20 = (12÷5)/20 = (2 remainder 2)/20 = 2 2/20, 4% of trials).

Second, children sometimes used both an incorrect strategy and an incorrect arithmetic operation when executing that strategy, as in $2/3 \times 3/5 = 10/15 \times 9/15 = (10+9)/15 = 19/15$ (4% of trials). Finally, children occasionally used decimals to calculate the answer, as in $3/5 \div 1/4 = 0.6 \div 0.25 = 2.4$ (a correct response, advanced on 3% of trials).

Variable strategy use by individual children. FARRA, like children, generated highly variable strategies. Nearly all runs of the simulation (99%) used different strategies on at least one of the eight pairs of virtually identical problems in the test set, such as 3/5+1/5 and 4/5+3/5.

Greater frequency of strategy errors than execution errors. Errors generated using strategies other than the standard correct strategy for a given problem were classified as strategy errors, whereas errors generated by executing the standard correct strategy incorrectly were classified as execution errors. As predicted, strategy errors were more common than execution errors for all four arithmetic operations (Table 7).

===================== Table 7 about here ======================

The most common strategy errors are wrong fraction operation errors and independent whole numbers errors. As with children, nearly all of FARRA's strategy errors (93%) were wrong fraction operation or independent whole numbers errors. Wrong fraction operation errors accounted for 64% of strategy errors; independent whole number errors accounted for 29%. The remaining 7% of FARRA's strategy errors involved use of the cross-multiplication approach.

To test whether the model correctly predicted when each type of strategy error would be most frequent, we classified errors according to the strategy and the type of problem on which it was employed. In both the children's and FARRA's data, wrong fraction operation errors that involved overgeneralization of the Add/Sub strategy to multiplication and division problems were more common than overgeneralizations of the Mult and Div strategies to addition and subtraction problems (Table 7). In contrast, independent whole number errors were most common on addition and subtraction problems.

Equal denominators increase addition and subtraction accuracy but decrease multiplication accuracy. As predicted, and like children, FARRA demonstrated higher accuracy for addition/subtraction problems involving operands with equal rather than unequal denominators (Figure 2). For multiplication problems, the reverse was true: accuracy was higher for problems involving operands with unequal denominators (Figure 2).

===================== Figure 2 about here ======================

Considering the paucity of equal denominator division problems in the learning set, one might expect lower accuracy on such problems than on unequal denominator ones. However, neither children nor FARRA displayed such a tendency. The reason seems to be that the paucity of equal denominator division problems was offset by use of a non-standard strategy that consistently yielded correct performance on some such problems. This strategy, which was used on roughly 20% of trials by both FARRA and children, involves applying the operation specified in the problem separately to numerators and denominators (rule 6, Table 3). For division, this meant dividing the numerator of one operand by the other and dividing the denominator of one operand by the other, as in “3/5÷1/5=(3÷1)/(5÷5)=3/1.

On the one problem in the test set where dividing numerators and dividing denominators both yielded whole number answers (3/5÷1/5), performance when using this strategy was accurate for both FARRA (100% correct) and children (69% correct). In contrast, on the other three division problems, where dividing numerators and denominators did not yield whole number answers, this strategy was inaccurate for both FARRA (35% correct) and children (2%

correct). The reason for such answers was that the assumption of closure of whole number arithmetic led FARRA and, we believe, children to round or truncate non-whole-number numerators and denominators in their answers, thus yielding errors. However, because this non-standard strategy generated correct answers on one of the two equal denominator division problems, performance on such problems was more accurate than it otherwise would have been.

The most frequent type of error on each operation varies with denominator equality. FARRA committed strategy errors resulting from overgeneralization of the addition/subtraction strategy more often on equal than on unequal denominator multiplication and division problems (Table 8). In contrast, FARRA committed strategy errors resulting from overgeneralization of the multiplication strategy more often on unequal than on equal denominator addition and subtraction problems (Table 8). Children showed the same pattern. Again, the phenomenon appeared to stem from children, like FARRA, learning the statistical relations between denominator equality and arithmetic operation in the input problems.

===================== Table 8 about here ======================

In summary, after practice on a learning set drawn from a widely used mathematics textbook series, FARRA generated test performance similar to that of middle-school students on all eight of the well-documented phenomena listed in Table 1: low overall accuracy; especially low accuracy for division; high response variability both within problems and within children; high frequency of strategy errors relative to execution errors; the most common errors being independent whole number and wrong fraction operation errors; and denominator equality exercising different effects on different arithmetic operations for both accuracy and specific errors. Thus, the model captures a number of prominent aspects of fraction arithmetic.

Study 2: Testing the Model Against Different Data and Problem Sets

We wanted to test the generality of the model to different children and different input problems to the model to ensure that FARRA's accurate simulation of children's performance did not depend on idiosyncratic characteristics of either the children's performance or the learning set employed in Study 1. Therefore, in Study 2, FARRA's fraction arithmetic performance after training on either the enVisionMATH learning set, used in Study 1, or an alternate learning set drawn from the GO MATH! textbook series was compared to that of the children in Siegler et al. (2011) rather than those in Siegler and Pyke (2013).

Method

Problems. Two learning sets were used to train FARRA. One was the learning set used in Study 1, consisting of 659 problems from the fourth fifth, and sixth grade books of the enVisionMATH textbook series. The other set consisted of 807 problems from the fourth, fifth, and sixth grade books of the GO MATH! textbook series. The distributional characteristics of the two sets are described in Tables 4 and 5.

The fraction arithmetic problems presented to children in Siegler et al. (2011) served as the test set. It included 8 problems, two problems for each arithmetic operation, one with equal and one with unequal denominators. The two addition problems were 3/5+2/5 and 3/5+1/2. The problems for the other three arithmetic operations involved the same operand pairs but different operations.

Procedure. One thousand runs of the model were conducted using each learning set, following the same procedure as in Study 1. The model runs conducted using the two learning sets will be referred to as the enVisionMATH simulation and the GO MATH! simulation.

Coding. Children in Siegler et al. (2011) reported the strategy they used to solve each problem immediately after they answered it. To compare children's strategy use to FARRA's, these verbal strategy reports were classified into the same categories as in Study 1: Add/Sub, Mult, Div/Cross-Multiply, or Other/None.

Results and Discussion

As in Study 1, our description of FARRA's performance is organized around the eight phenomena in Table 1.

Low overall accuracy. Average accuracy was 52% in the enVisionMATH simulation and 54% in the GO MATH! simulation, both reasonably close to the 46% accuracy exhibited by children in Siegler et al. (2011).

Especially low accuracy on division problems. Accuracy on division problems was lower than on any other operation in Siegler et al. (2011) and in the simulations using each textbook series as input problems (Figure 3).

===================== Figure 3 about here =====================

Unlike children in Siegler and Pyke (2013), children in Siegler et al. (2011) were more accurate on multiplication than on addition or subtraction. A similar difference appeared between the enVisionMATH simulation, which was more accurate on addition and subtraction than on multiplication, and the GO MATH! simulation, which was more accurate on multiplication than on addition or subtraction (Figure 3). This difference likely reflects the fact that the enVisionMATH learning set contained more addition and subtraction than multiplication problems, whereas the GO MATH! learning set showed the opposite trend (Table 4). The different patterns of accuracy in Siegler and Pyke (2013) and Siegler et al. (2011) might similarly reflect different distributional characteristics of children's practice problems.

Variable responses within individual problems. Given either input set**,** FARRA generated a similar range of responses for each problem to those generated by children in Siegler et al. (2011). For example, FARRA generated both the most common correct response and the most common incorrect response advanced by children for the four problems shown in Table 9, as well as the second most common one for three of these (the exception being 3/5×2/5, for which FARRA did not generate the second most common incorrect response, 6/10).

===================== Table 9 about here ======================

To assess the correspondence of different response frequencies in FARRA's and children's data, we correlated the frequency in each simulation and in children's data of each problem-answer pair that appeared in either one. For both simulations, this analysis involved 115 problem-answer pairs, including 102 pairs in which the answer was incorrect. The correlation between frequencies in the enVisionMATH simulation and children's data was r=.792, $p < .001$, or r=.635, $p < .001$ when only incorrect answers were included. The correlation between frequencies in the GO MATH! simulation and children's data was r=.820, $p < .001$, or r=.661, $p < .001$ when only incorrect answers were included.

Variable strategy use by individual children. FARRA, like children, generated highly variable strategies. Half (50%) of children used different strategies on the two problems for at least one arithmetic operation, such as 3/5+2/5 and 3/5+1/2. The same was true on the majority of simulation runs (96% in the enVisionMATH simulation and 94% in the GO MATH! simulation).

Greater frequency of strategy errors than execution errors. Errors were classified as strategy errors or execution errors in the same way as in Study 1. Just as in Siegler and Pyke (2013), strategy errors in Siegler et al. (2011) accounted for the majority (87%) of all errors.

Similarly, strategy errors accounted for the majority of all errors in both the enVisionMATH simulation (85%) and the GO MATH! simulation (85%).

The most common strategy errors are wrong fraction operation errors and independent whole numbers errors. Wrong fraction operations errors resulting from overgeneralization of the Add/Sub strategy, and independent whole numbers errors resulting from overgeneralization of the Mult strategy, together accounted for the great majority of strategy errors in children's data and both simulations. However, in children's data, errors due to overgeneralization of the Mult strategy were more common than those due to overgeneralization of the Add/Sub strategy (28% vs. 8% of trials). In the simulation data, the reverse was true: errors due to overgeneralization of the Mult strategy were *less* common than those due to overgeneralization of the Add/Sub strategy in both the enVisionMATH simulation (12% vs. 17% of trials) and the GO MATH! simulation (14% vs. 15% of trials). The reason for these deviations from the children's data in Siegler and Pyke (2013) and from the simulation results in response to both the GO Math! and enVisionMATH practice sets remain to be determined.

Equal denominators increase addition and subtraction accuracy but decrease multiplication accuracy. As in Siegler and Pyke (2013), children in Siegler et al. (2011) were more accurate on equal than on unequal denominator addition and subtraction problems, but they were more accurate on unequal than on equal denominator multiplication problems (Figure 4A). The same pattern appeared in both the enVisionMATH and GO MATH! simulations (Figures 4B and 4C). The GO MATH! simulation, though not the enVisionMATH simulation, also correctly predicted virtually no difference in accuracy between equal and unequal denominator division problems.

===================== Figure 4 about here =====================

The most frequent type of error on each operation varies with denominator equality. Children in Siegler et al. (2011), like those in Siegler and Pyke (2013), committed strategy errors involving use of the Mult strategy more often on unequal than on equal denominator addition and subtraction problems, but committed strategy errors involving use of the Add/Sub strategy more often on unequal than on equal denominator multiplication problems. Both the enVisionMATH and GO MATH! simulations showed the same pattern (Table 10).

===================== Table 10 about here ======================

In summary, FARRA's performance resembled that of children in Siegler et al. (2011), as it resembled that of children in Siegler and Pyke 2013), regardless of which textbook series it was trained on. This outcome increased our confidence that the model captures general phenomena and mechanisms of fraction arithmetic learning, rather than idiosyncrasies of a particular textbook's practice problems or of a particular sample of children.

A useful model should not only simulate previously observed phenomena: It also should generate novel, testable predictions. We examined FARRA's ability to do this in Studies 3, 4, and 5.

Study 3: Learning Associations Between Problem Features and Solution Strategies

As described in the Introduction, the more often a problem feature is paired with an arithmetic operation, the more strongly FARRA associates that feature with the correct strategy for that operation. In Studies 1 and 2, this property of the model, together with the frequencies of equal and unequal denominator problems for each arithmetic operation in the learning set, led FARRA to generate the interactions between arithmetic operation and denominator equality that had been observed in children.

The success of this account implies that children and FARRA may also learn other relations in the problems they encounter. One such relation is that between type of operand – fraction, mixed number, or whole number – and arithmetic operation. In each of the three textbook series described in Table 5, problems with only fraction or mixed number operands almost always involved addition or subtraction, whereas problems with one whole number operand and one fraction or mixed number operand usually involved multiplication.

We were unable to find any data on people's performance on these types of arithmetic problems. However, on the basis of FARRA's success in accounting for children's fraction arithmetic performance, we predicted that the degree to which FARRA associates the correct strategy for each arithmetic operation with problems having each type of operation and operand should mirror the frequency with which each arithmetic operation is matched with each operand type in the learning set. Thus, we predicted that FARRA would use the correct strategy more often on addition and subtraction problems when they involve only fraction and mixed number operands, but would use the correct strategy more often on multiplication problems when one operand is a whole number.

Method

Problems. The enVisionMATH learning set used in Studies 1 and 2 was also used as the learning set in Study 3. Table 5 indicates the frequencies in the learning set of problems classified in the way relevant to Study 3.

The 40 test set problems included the 16 fraction test items from Study 1 and, for each of the four arithmetic operations, two problems with a mixed number and a fraction with equal denominators, two problems with a mixed number and a fraction with unequal denominators, and two problems with a fraction and a whole number. For example, the equal denominator addition problems with a mixed number and a fraction were 2 3/5+1/5 and 3 4/5+3/5; the unequal denominator addition problems with a mixed number and a fraction were 2 3/5+1/4 and 3 1/3+3/5; and the addition problems with a whole number and a fraction were 2+1/4 and 3+1/5. The subtraction, multiplication, and division problems involved the same pairs of operands as the addition problems.

Procedure. As in Studies 1 and 2, 1,000 simulated participants were created, run on the learning set with feedback on each item, and run on the test set with no feedback. FARRA first encoded the relevant features (types of operands, denominator equality, and arithmetic operation). Then the model converted whole number operands to fractions with a denominator of 1 (e.g., 3 became 3/1), and converted mixed number operands to improper fractions with denominators equal to those of their fractional part (e.g., 2 3/5 became 13/5).

Results and Discussion

For addition and subtraction problems, FARRA used the Add/Sub strategy more often when both operands were fractions or mixed numbers than when one operand was a whole number. For multiplication problems, FARRA used the Mult strategy more often when one operand was a whole number than when both operands were fractions or mixed numbers (Figure 5). These were the patterns predicted from the frequencies of problem presentation in the textbook series. The prediction that children will show the same pattern remains to be tested.

===================== Figure 5 about here =====================

Study 4: Effects of Differentiated Feedback on Strategy Learning

Simulation models allow tests of whether intuitions about the workings of learning mechanisms are correct. In Study 4, FARRA provided such a test of the way in which the timing of feedback affects mathematics learning, as well as a test of the usefulness of the theoretical distinction between strategy errors and execution errors.

FARRA's learning algorithm relies on a single feedback signal (correct or incorrect) received at the end of each problem-solving episode. This corresponds to the usual situation in classrooms and homework assignments. However, the model's inability to differentially assign credit to different steps in a problem-solving episode might inhibit learning in at least two ways.

First, execution errors can reduce reinforcement of correct strategies. For example, for an unequal denominator addition or subtraction problem, FARRA can correctly select the Add/Sub strategy, but execute it incorrectly by not revising the numerators when converting to a common denominator. Similarly, for a division problem, FARRA can correctly select the Div strategy, but execute it incorrectly by inverting the wrong operand. In both cases, the strategy choice was correct, but it would not receive strong reinforcement because the answer was wrong.

Execution errors also can lead to reinforcement of incorrect strategies. For example, on a multiplication problem, FARRA can select the Op-deleted Add/Sub strategy, and also execute this strategy incorrectly by committing the same error as in the addition example above. These two errors offset each other, yielding a correct answer, as in $3/5\times1/4=(3/(5\times4)\times1/(4\times5)=3/20\times1/20=(3\times1)/20=3/20$. The incorrect strategy choice would be reinforced as if it were correct, because the final answer is correct.

These considerations suggest that FARRA would learn more effectively if provided *differentiated feedback*, that is, separate feedback on strategy selection and execution. In Study 4, we tested this prediction by running five simulations. In the control simulation, no differentiated feedback was provided. In the other four simulations, differentiated feedback was provided for one of four problem categories: addition/subtraction with equal denominators, addition/subtraction with unequal denominators, multiplication, or division. Differentiated feedback was expected to improve test performance for the problems in which the differentiated feedback was given, relative to performance on that category in the control simulation.

Method

Problems. To afford an unbiased comparison between simulations, the enVisionMATH learning set used in Studies 1-3 was modified to eliminate frequency and primacy effects favoring addition and subtraction. Frequency effects were eliminated by equalizing the frequencies in the learning set of the four aforementioned problem categories. The most frequent problem category in the textbooks, addition/subtraction with equal denominators, included 218 problems. We therefore added items to the other three categories of problems by randomly selecting existing problems from the set and duplicating them until the total number of problems in each category equaled 218. Primacy effects were eliminated by randomizing problem order separately for each simulated participant. The test set was always the same as in Study 1.

Procedure. Simulated participants were run on the learning set and then on the test set problems. This was done as in Studies 1-3, except as noted below.

Simulations were run under five conditions. In the control condition, 1,000 simulated participants received standard undifferentiated feedback for all training trials. In each of the other four conditions, 1,000 simulated participants received differentiated feedback on the problems in the designated category (addition/subtraction with equal denominators, addition/subtraction with unequal denominators, multiplication, or division) and standard feedback on the other types of problems. For example, in the differentiated multiplication condition, differentiated feedback was provided on multiplication items but not on the other three types of problems.

When receiving undifferentiated feedback, the model adjusted its associative weights for both strategy and execution rules after completing each training problem. Correctness of the final answer determined the amount of adjustment of both. In contrast, differentiated feedback involved two rounds of adjustments. The first round was presented immediately after the model selected a strategy; this information led to the model adjusting the associative weights for the strategy it selected according to its appropriateness for that problem. Then, regardless of which strategy had been selected, goals corresponding to those of the correct strategy were assigned, and the model continued to run. After generating an answer, the model adjusted the associative weights for all execution rules that had been used, with the amount of adjustment determined by the correctness of the final answer.

Results and Discussion

Because our predictions concerned effects of differentiated feedback on strategy use, our analysis focused on the proportion of test trials on which the model employed the standard correct strategy for each problem category. Relative to the control condition, differentiated feedback yielded increased use of standard correct strategies for three of the four categories: addition/subtraction with unequal denominators, multiplication, and division (Table 11). The improvement was largest for division: use of the correct Div strategy on division problems increased from 44% in the control condition to 62% when differentiated feedback was provided. The greater effect on division was expected, due to its greater opportunities for execution errors, which would lead to correct strategy choices not being reinforced in the standard condition.

===================== Table 11 about here =====================

Addition/subtraction with equal denominators deviated from the general pattern in that differentiated feedback on such problems yielded no improvement in accuracy relative to the control condition (the 1% difference in Table 11 appeared due to random variation among simulation runs). For such problems, the feedback presented to FARRA was always identical in differentiated and undifferentiated conditions, because there were no execution mal-rules for them. Under the other conditions, however, differentiated feedback led to greater learning. Results of Experiment 4 thus illustrated the potential instructional usefulness of distinguishing between strategy errors and execution errors and providing separate feedback on them.

An important caveat is that differentiated feedback on a given problem type not only increased correct strategy use on that problem type, but also decreased correct strategy use on other problem types. For example, relative to the control condition, differentiated feedback on unequal denominator addition and subtraction led to 4% less correct strategy use on multiplication problems (47% vs. 43%) and 2% less correct strategy use on division problems (44% vs. 42%). The reason is that, as noted earlier, conditions that increase reinforcement of a given strategy in FARRA tend to increase not only correct use but also overgeneralization of that strategy. Nevertheless, each differentiated feedback condition led to an increase in correct strategy usage across all problem types (Table 12), suggesting that the positive effects of differentiated feedback outweigh its potential negative effects.

Study 5: Effects of Input Set Characteristics on Learning

In Study 5, we varied three characteristics of the simulation to examine their effects on FARRA's learning. First, the number of learning set problems was varied, to test if providing more practice would increase learning. Second, the distribution of different types of learning set problems was balanced, to test whether providing a greater proportion of difficult problems would improve learning. Third, adjustments to associative strengths following incorrect answers were varied, to test whether reducing, rather than slightly increasing, the strengths of rules following incorrect performance would improve learning. The effects of all combinations of these three variables on FARRA's performance were examined.

Method

Problems. Four learning sets were created by modifying the enVisionMATH learning set used in Studies 1-4. The four sets represented the possible combinations of number of problems (659 or 2400) and distribution of problems (textbook or balanced). Sets with 659 problems and the textbook distribution included the same items as the original enVisionMATH learning set. Sets with 2400 problems and the textbook distribution were created by randomly duplicating problems of each type in the enVisionMATH learning set to increase the total number to 2400 while maintaining the proportion of problems belonging to each combination of arithmetic operation and denominator equality/inequality. Sets with a balanced distribution of problems were created by randomly duplicating problems to equalize the proportions belonging to each combination of operation and denominator equality/inequality, either maintaining the total number of problems at 659 or increasing the number to 2400. Within these constraints, a unique learning set was generated for each simulated participant.

Procedure. One thousand simulated participants were run in each of eight conditions, representing all combinations of the four types of learning set described in the previous paragraph with two types of reinforcement: positive only or positive and negative. In the positive reinforcement only conditions, the error discount parameter had the same values as in Studies 1-4. In the positive and negative reinforcement conditions, incorrect answers during the learning set led to decrements in associative weights involving the rules that fired on the trial. Amount of decrement following incorrect answers was arbitrarily set at half the amount of the increment following a correct answer. In all other respects, the procedure was the same as in Studies 1-4.

Results and Discussion

Each of the three modifications to FARRA's training led to improved accuracy on the test problems (Figure 6). Averaged over the other variables, accuracy improved from 55% when there were 659 problems in the learning set to 68% when there were 2400 such problems, from 60% when the learning set had its original distribution to 63% when it had a balanced distribution, and from 55% when only positive reinforcement was allowed to 68% when both positive and negative reinforcement were allowed. The simulation was most accurate, 80% correct, when all three improvements were implemented simultaneously.

===================== Figure 6 about here =====================

General Discussion

In this concluding section, we discuss implications of our findings for learning of fraction arithmetic, for learning of other aspects of mathematics, and for real world learning in general.

Implications for Fraction Arithmetic Learning.

The role of input problems. Perhaps the most striking findings to emerge from the present research concern the distribution of fraction arithmetic problems in textbooks, the effects of this distribution on FARRA's learning, and its apparent importance in children's learning. Sets of training problems drawn from three widely adopted textbook series included very few equal denominator multiplication or division problems over the period in which fraction arithmetic is primarily taught, fourth, fifth, and sixth grades. In contrast, addition and subtraction problems with equal denominators were very common.

FARRA's performance and the closely similar performance of middle school students suggest that the lack of equal denominator multiplication problems hampers children's learning. The higher frequency of equal denominator addition and subtraction problems relative to equal denominator multiplication problems caused FARRA to make specific types of overgeneralization errors, in which standard rules for solving addition and subtraction problems with equal denominators were overgeneralized to multiplication, leading to frequent errors of the form 3/5×4/5=12/5. These simulation results explain previous findings with children that equal denominator multiplication problems are more difficult than unequal denominator ones, a surprising finding given that the standard multiplication algorithm is identical for equal and unequal denominator problems (Gabriel et al., 2013; Newton, 2008; Newton et al., 2014; Siegler & Pyke, 2013; Siegler et al., 2011). Presentation of such imbalanced sets of practice problems in textbooks seems like an unforced error, one that could be corrected easily by increasing the number of equal denominator multiplication and division problems.

Other aspects of the distribution of textbook problems also appeared to influence fraction arithmetic learning. There were more addition and subtraction problems with equal than with unequal denominators, despite it being inherently more difficult to correctly execute appropriate strategies on addition and subtraction problems with unequal denominators. The difficulty is likely exacerbated by textbooks presenting unequal-denominator problems more often for multiplication than for addition or subtraction. This imbalance leads FARRA to overgeneralize the multiplication rule to addition and subtraction problems with unequal denominators. Children do the same, probably for the same reason.

A similar argument applies to division problems. The model-based analysis indicated that correctly executing appropriate strategies on division problems is inherently more difficult than doing so on multiplication problems. Again, the distribution of problems seems to exacerbate this difficulty, in that children and the model encounter relatively few division problems.

These results did not appear to be due to any idiosyncrasy of the particular textbook series from which the learning set problems were drawn. The three textbook series that we examined had highly similar distributional patterns. Less detailed prior analyses of two textbook series, Saxon Math and Everyday Mathematics, also revealed that division problems were less frequent than multiplication problems in them, though those analyses did not examine denominator equality and thus did not reveal the striking interaction between arithmetic operation and denominator equality that was documented here (Lortie-Forgues et al., 2015; Son & Senk, 2010). Interestingly, an analysis of a textbook series from Korea, a country with very high math achievement, showed the opposite pattern: considerably more fraction division than fraction multiplication problems (Son & Senk, 2010). Given the inherent difficulty of correctly executing the standard correct division algorithm, the more frequent presentation of division problems in the Korean textbook makes sense.

We recognize that assessing textbook problems is an imperfect means for assessing the input that children receive. Most teachers do not present all problems in a textbook, different teachers use different textbooks, and many teachers present problems from sources other than textbooks, such as websites. However, the distribution of problems drawn from enVisionMATH and GO MATH! seemed to provide a reasonable approximation to the problems that children receive, especially given the similarity of the distribution to that in the other textbook series we examined, Everyday Mathematics. The assessment of textbook input proved very useful in accounting for numerous aspects of children's performance, which again suggests that the textbook problems were a reasonable approximation to the problems children receive. This conclusion dovetails with results of a recent study that successfully predicted aspects of children's conceptual understanding of rational numbers based on properties of rational number story problems found in textbooks (Rapp, Dewolf, & Holyoak, 2014).

The role of overgeneralization. Another contribution of the present study to understanding why fraction arithmetic is so difficult was to provide a parsimonious account for a wide range of fraction arithmetic errors. In FARRA, a single error-generating mechanism – overgeneralization – accounted for most errors. Other theoretical accounts have provided specialized explanations for some of these errors. Most prominently, independent whole number errors, such as 3/5+1/4=(3+1)/(5+4)=4/9, have been attributed to "whole number bias" – a tendency to view a fraction as two separate numbers (numerator and denominator) rather than as a single number (e.g., Gabriel et al., 2013; Gelman, 1991; Ni & Zhou, 2005).

However, whole number bias does not predict or explain wrong fraction operation errors, which involve overgeneralization of strategies that are appropriate for fraction operations other than the one in the problem. Such errors appear to be at least as common as independent whole numbers errors (Bailey et al., 2015; Ni & Zhou, 2005; Siegler & Pyke, 2013). In FARRA, overgeneralization accounts for both types of errors, and thus appears to be a more powerful explanation of fraction arithmetic errors than whole number bias. This does not imply that whole number bias does not exist or that it does not lead to errors. Rather, lack of conceptual understanding of fraction arithmetic appears to open the door to overgeneralizations, which generate both independent whole number errors and wrong fraction operation errors.

A practical implication of our emphasis on overgeneralization as a source of fraction arithmetic errors is that children should receive explicit instruction and practice in selecting which solution strategy to use on the eight types of fraction arithmetic problems identified here (four operations with equal and unequal denominators). As the results of Study 4 suggest, the difficulty of correctly executing fraction arithmetic strategies complicates children's task of learning which strategy to use. Children likely would benefit from practicing fraction arithmetic strategy selection alone, without the added burden of correctly executing the strategies.

A related instructional approach would be to present fraction arithmetic problems involving different operations in interleaved sequences after operations have been introduced separately. Mathematics textbooks in the United States typically present problems involving arithmetic operations in separate blocks (e.g. Charles et al., 2012; Dixon et al., 2012). Blocked study promotes development of fluency in the execution of each strategy, but not skill in selecting which strategy to use. Such blocked presentation of each operation in textbooks is a likely contributor to the high frequency of overgeneralizations of strategies to fraction arithmetic operations to which they do not apply. The most common strategy mal-rules in FARRA were all of the same form – identical to correct rules except for not specifying the operation.

This analysis points to a previously unsuspected source of difficulty in fraction arithmetic learning. Children might learn to ignore the operation in the usual blocked presentation of fraction arithmetic problems because, after the first problem in a set, seeing the operation is uninformative. Even before that first problem in the set, children might not attend to the operation if they know that they are in (e.g.) a multiplication unit.

By contrast, once a given fraction arithmetic procedure has been learned, interleaving problems involving different arithmetic operations with equal and unequal denominators would encourage children to identify the conditions under which each strategy is appropriate and thus to reduce overgeneralization. Consistent with this conclusion, interleaved practice has been found to lead to improved learning in fraction arithmetic (Patel, Liu, & Koedinger, 2016) and other areas of mathematics (Rohrer & Taylor, 2007; Taylor & Rohrer, 2010).

The role of conceptual knowledge. Many investigators have cited lack of conceptual knowledge as a key difficulty in learning fraction arithmetic (e.g., Byrnes & Wasik, 1991; Fischbein, Deri, Nello, & Marino, 1985; Gelman, 1991; Lortie-Forgues et al., 2015). Children's frequent implausible errors, such as claiming that 1/2+1/2=2/4, reflect this lack of conceptual understanding, or at least lack of application of whatever conceptual knowledge children have to solving fraction arithmetic problems. Such lack of conceptual knowledge is implicit in FARRA, insofar as the model does not constrain fraction arithmetic strategy choices but instead relies on trial and error to learn which procedures to use under which circumstances—a slow and imperfect process. FARRA's success in simulating children's performance suggests that many children do the same.

A question raised by the model is how some children are able to master fraction arithmetic procedures without huge numbers of practice problems. Even with very large input sets, balanced presentation of different types of problems, and negative reinforcement in response to incorrect answers, FARRA only generated 80% correct answers. One possibility is that the learning of students who master fraction arithmetic to very high levels is constrained by their conceptual understanding of the operations. Conceptual knowledge of fraction arithmetic procedures includes knowing the rationales and principles underlying the procedures (Crooks & Alibali, 2014; Prather & Alibali, 2008), which could allow children to check their solutions for plausibility and reject procedures that yield implausible answers, such as the previously mentioned 1/2+1/2=2/4. Examining learning processes of children who do master fraction arithmetic seems a valuable focus for future research.

General Implications for Mathematics Learning

FARRA's explanatory power comes largely from domain-general mechanisms operating on domain-specific input to the model. In particular, FARRA’s reinforcement learning mechanism, together with the biased distribution of fraction arithmetic problems found in the textbooks and therefore in FARRA's learning set, allowed the model to generate performance much like that of children.

The same pattern of general learning mechanisms operating on biased input to produce non-intuitive aspects of performance may also explain other aspects of mathematics learning. One illustration comes from a study of preschoolers’ addition, in which parents of preschoolers were asked to present addition problems to their children as they would at home (Siegler & Shrager, 1984). The distribution of problems that parents presented paralleled several aspects of preschoolers’ performance that were otherwise difficult to explain. Especially striking, parents presented “+1” problems (e.g., 4+1) far more often than “1+” problems (e.g., 1+4), and children were much more accurate on +1 than on 1+ problems. Neither pattern reflected children counting-on from the larger addend; none of the preschoolers in the study used that strategy. Similarly, parents presented tie problems (e.g., 2+2) much more often than non-tie problems, and their children were much more accurate on tie than on non-tie problems.

A study of second graders’ learning of whole number multiplication (Siegler, 1988) yielded similar findings. Both of the textbooks that were examined presented tie problems more often than non-tie problems, and children displayed superior performance on tie problems. Inclusion of frequency of problems in textbooks in a computer simulation of multiplication learning allowed the simulation to capture this phenomenon.

A related implication of FARRA for mathematics learning in general is that when statistical associations exist between problem features and solution strategies, children use the features as cues for strategy selection even when the features are formally irrelevant. This implication dovetails with previous empirical findings regarding people's reliance on formally irrelevant features to guide problem solving in other mathematical domains (Braithwaite & Goldstone, 2015; Braithwaite, Goldstone, van der Maas, & Landy, 2016; Chang, Koedinger, & Lovett, 2003; Landy & Goldstone, 2010). For example, university students who were trained to compare algebraic fractions using two distinct procedures subsequently used each procedure more often on problems similar to those with which the procedure had been paired during training, even though both procedures were valid for all problems (Ben-Zeev & Star, 2001).

These studies, like the present one, suggest that attention to statistical properties of problem sets should be an important element in designing mathematics curricula. Spurious correlations in problem sets can bias strategy selection in unfortunate directions. Instructional designers should avoid building such spurious correlations into textbooks and other materials and should provide students greater experience with problems that have been found to elicit inaccurate performance.

A third general implication for mathematics learning concerns relations between strategy selection and strategy execution. In FARRA, strategies whose correct execution is difficult receive less reinforcement during learning, and therefore are less likely to be selected subsequently, compared to strategies that are easy to execute correctly. This observation suggests that instruction that increases learners' ability to execute correctly the component steps of a correct procedure should make them not only more likely to obtain correct results when they use that procedure but also more likely to choose the procedure over other alternatives.

Implications for Learning in General

It would be difficult to find a theory of learning that did not predict that learning is shaped by the frequency of experiences. Despite this broad recognition of the importance of input, most cognitive research proceeds without assessment of the problem environment. There are of course exceptions, but only a small minority of studies assess input in the domain being examined. This is understandable, because in many domains, the relevant input is unknown or difficult to assess. However, the importance of the specifics of input in the present research, and its value for understanding other domains in which it has been assessed, suggest that our grasp of real-world learning and cognition would benefit from assessing input when possible.

One well-known example of the benefits that can be gained by assessing input involves language development. Hart and Risley (1995, 2003) conducted monthly, hour-long observations of family interactions in homes of varied socio-economic status from the time children were 7-month-olds to the time they were 3-year-olds. They found that children from professional families heard more than 2100 words per hour, whereas those from families on welfare heard only 600 words per hour. Children's vocabularies showed large differences that paralleled these differences in linguistic input, and individual differences in vocabulary at age three were strongly predictive of vocabulary differences at age nine. Hearing relatively rare words at mealtimes has been found to be especially predictive of children's later vocabulary (Snow & Beals, 2006).

Technology has made possible assessment of some environments that previously were impossible to assess. One example involves head mounted cameras (head cams), which have been used to assess infants' visual environments, that is, the objects in the environment at which infants look. These studies have revealed non-intuitive findings such as that infants look at obstacles in their paths more often when crawling than when walking (Kretch, Franchak, &

Adolph, 2014), that the first two years of life see a shift in visual input from dense face input to dense hand input (Fausey, Jayaraman, & Smith, 2016), and that infants see people acting as causal agents three times as often as they see them engaging in self-propelled motion (Cicchino, Aslin, & Rakison, 2011). None of these findings were self-evident; for example, beginning walkers might have been expected to look at obstacles more often than crawlers do, because stepping on obstacles when walking produces harder falls.

Assessments of broader environments have also proved useful in predicting concurrent and subsequent intellectual outcomes. The HOME (Home Observation for Measurement of the Environment; Bradley, 1994) examines numerous aspects of early environments, including whether children have books of their own, whether mothers frequently read stories to their children, and whether children have play equipment appropriate for their age. HOME scores of 2-year-olds have been found to predict IQ and school achievement of the children at age 11 years (Olson, Bates, & Kaskie, 1992). When HOME scores are relatively stable over time, IQ scores also tend to be stable; when HOME scores change substantially, IQ scores tend to change in the same direction (Bradley, 1989). Influences of the home environment tend to be especially large for children from low-income families (Tucker-Drob et al., 2011; Turkheimer et al., 2003).

Environmental assessments can reveal surprising commonalities across different topics. Consider, for example, the previously mentioned areas of mathematics where inputs on specific problems were examined: preschoolers’ whole number addition, elementary school students’ whole number multiplication, and elementary and middle-school students’ fraction arithmetic with all four operations. In all cases, input was disproportionately frequent on the most accurately performed problems: +1 and tie problems in preschoolers’ addition, tie problems in elementary school students' multiplication, and equal denominator addition and subtraction problems and unequal number multiplication problems in middle school students' fraction arithmetic.

Two non-mutually-exclusive explanations seem plausible. One is *the causal explanation*: children might do well on frequently presented problems precisely because they are frequently presented. The other account might be labeled *the self-medication explanation*: parents and textbooks might disproportionately present easy problems so that children feel good about themselves; so that teachers feel good about themselves and their students; so that parents feel good about their children, the children's teachers, and their schools; and so that decision makers in school districts feel good about their teachers and their textbook adoption choices. Whatever the reason, creating instructional materials that provide substantial experience with challenging as well as less challenging problems holds promise for improving learning in many areas.

Methodological Implications

Newell (1973) advocated as an experimental strategy "to accept a single complex task and do all of it." Doing so, he argued, would force researchers to generate mutually compatible explanations for many phenomena – the various phenomena associated with the task – rather than studying each phenomenon in isolation from the others.

FARRA represents an attempt to apply this recommendation to the complex task of fraction arithmetic. Accounting for a wide range of fraction arithmetic phenomena required us to specify numerous cognitive mechanisms, including knowledge representation, problem representation, reinforcement learning, strategy selection, and generalization. Further, our assumptions about each of these mechanisms were constrained by our assumptions about the others. This approach, we believe, led to a more credible theory – and one of broader relevance – than would have been achieved by focusing on a more limited task, such as fraction addition alone, or on a single phenomenon, such as whole number bias errors. The present study thus points to the utility of providing a unified account of multiple, apparently disparate phenomena within the context of a complex task.

Conclusions

FARRA is a formal theoretical account of fraction arithmetic learning that explains many aspects of children's performance. It demonstrated that a small number of domain-general learning mechanisms, operating on the distribution of input problems in textbooks, can explain a wide range of phenomena in fraction arithmetic. The model also yielded novel predictions regarding student performance on specific categories of problems and regarding the effectiveness of potential instructional interventions. It also yielded implications regarding learning of fraction arithmetic, of mathematics more broadly, and of learning in general. Research is needed to test these predictions and implications and to examine in a variety of domains the explanatory power of models that combine domain-general learning mechanisms with empirically assessed, domain-specific input.

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Endnotes

1. Errors involving the Cross-Multiply strategy (rule 9, Table 2) are an exception. These errors result from overgeneralization of a procedure for fraction comparison, rather for fraction arithmetic, and thus do not meet the definition of wrong fraction operation errors.

2. An exception is that the correct strategy for subtraction with equal denominators can be executed incorrectly in cases where the correct answer would be a negative number. In these cases, the model can reverse the order of operands in order to guarantee a positive number as the answer (Supplemental Materials, Part A, Table A2, rule 33), yielding errors such as 1/5–3/5=(3–1)/5=2/5 instead of -2/5. However, no problems whose correct answers were negative numbers were included in any simulation in the present study.

Figure Captions

Figure 1. Percent correct by arithmetic operation and dataset (Study 1). “Children” denotes data from Siegler and Pyke (2013). Here and throughout, error bars indicate standard errors.

Figure 2. Percent correct by arithmetic operation and denominator equality (Study 1). (A) Children’s data from Siegler and Pyke (2013), (B) Simulation data. Here and throughout, “denoms” denotes “denominators.”

Figure 3. Percent correct by arithmetic operation and dataset (Study 2). “Children” denotes data from Siegler et al. (2011).

Figure 4. Percent correct by arithmetic operation and denominator equality (Study 2). (A) Children’s data from Siegler et al. (2011), (B) Simulation data generated with enVisionMATH learning set, (C) Simulation data generated with GO MATH! learning set.

Figure 5. Percent of test trials on which the simulation used the standard correct strategy on each arithmetic operation and operand type (Study 3).

Figure 6. Percent correct by number and distribution of problems in the learning set (Study 5). Number of problems is indicated by “N=659” or “N=2400.” (A) Positive only reinforcement. (B) Positive and negative reinforcement.

Table 1. Major empirical phenomena of fraction arithmetic.

Number	Phenomenon
1	Low overall accuracy
2	Especially low accuracy on division problems
3	Variable responses within individual problems
4	Variable strategy use by individual children
5	Greater frequency of strategy errors than execution errors
6	The most common strategy errors are wrong fraction operation errors and independent whole numbers errors
7	Equal denominators increase addition and subtraction accuracy but decrease multiplication accuracy
8	The most frequent type of error on each operation varies with denominator equality

Table 2. FARRA's strategy rules. Mal-rules are marked with *. In all strategy rules, the condition includes the goal of finding the answer; we omitted this feature below for purposes of brevity.

Rule	Strategy Name	Condition	Action
1	Correct Add/Sub	Operation is + or – and denominators of operands are equal	Create goals to set denominator of answer equal to common denominator, and set numerator of answer equal to result of performing given operation on numerators
2	Correct Add/Sub	Operation is + or – and denominators of operands are unequal	Create goals to convert operands to a common denominator, then set denominator of answer equal to common denominator, and set numerator of answer equal to result of performing given operation on numerators
3	Correct Mult	Operation is ×	Create goals to set denominator and numerator of answer equal to results of performing given operation on operand denominators and numerators respectively
4	Correct Div	Operation is ÷	Create goal to execute Div strategy (see Table 3, Rule 15)
5*	Op-deleted Add/Sub	Operand denominators are equal	Same as Rule 1
6*	Op-deleted Add/Sub	Operand denominators are unequal	Same as Rule 2
7*	Op-deleted Mult	None	Same as Rule 3
8*	Op-deleted Div	None	Same as Rule 4
9*	Cross-Multiply	Operation is ×	Create goals to set numerator and denominator of answer equal to results of performing given operation on operand numerators and denominators cross-wise

Table

Table 3. FARRA's main execution rules (for a complete list, see Supplementary Materials, Part A). Mal-rules are marked with *.

Rule	Rule Name	Condition	Action
10	Operate-Numerators	Goal exists to set answer numerator to result of performing given operation on numerators	Create goal to solve appropriate whole number arithmetic problem
11	Operate-Denominators	Goal exists to set answer denominator to result of performing given operation on denominators	Create goal to solve appropriate whole number arithmetic problem
12	Maintain-Common-Denominator	Goal exists to set answer denominator to common denominator of operands	Set answer denominator equal to common denominator of operands
13	Convert-Common-Denom-By-LCD	Goal exists to convert operands to common denominator	Create goals to convert both operands using the LCD procedure
14	Convert-Common-Denom-By-Mult-Denoms	Goal exists to convert operands to common denominator	Create goals to multiply numerator and denominator of each operand by denominator of other operand
15	Execute-Invert-Operate	Goal exists to execute Div strategy	Create goals to invert 2nd operand, change ÷ (if present) to ×, then set answer denominator and numerator to results of performing given operation on operand denominators and numerators respectively
16*	Convert-Denoms-Only-By-Mult-Denoms	Goal exists to convert operands to common denominator	Create goals to multiply the denominator of each operand by the denominator of the other operand
17*	Execute-Invert-Operate-Forget-Invert	Goal exists to execute Div strategy	Create goals to change ÷ (if present) to ×, then set answer denominator and numerator to results of performing given operation on operand denominators and numerators respectively
18*	Execute-Invert-Operate-Random-Invert	Goal exists to execute Div strategy	Create goals to invert one operand, change ÷ (if present) to ×, then set answer denominator and numerator to results of performing given operation on operand denominators and numerators respectively

Table 4. Percent of problems belonging to each combination of arithmetic operation and equality of denominators of operands in the enVisionMATH, GO MATH!, and Everyday Mathematics problem sets (problem counts are given in the Supplemental Materials, Part B). Cells with exceptionally low values are bolded. Here and throughout, “add” refers to “addition,” “sub” to “subtraction,” “mult” to “multiplication,” and “div” to “division.”

	Arithmetic Operation			
	Add	Sub	Mult	Div
	enVisionMATH			
Equal denominators	17	16	**1**	**2**
Unequal denominators	11	12	24	17
	GO MATH!			
Equal denominators	8	10	**1**	**1**
Unequal denominators	12	14	33	21
	Everyday Mathematics			
Equal denominators	11	13	**0**	**2**
Unequal denominators	18	8	30	19

Table 5. Percent of problems belonging to each combination of operand type and arithmetic operation in the enVisionMATH, GO MATH!, and Everyday Mathematics problem sets (problem counts are given in the Supplemental Materials, Part B). Cells with exceptionally low values are bolded.

	Arithmetic Operation			
Type of Operands	Add	Sub	Mult	Div
	enVisionMATH			
Two fractions, two mixed numbers, or a fraction and a mixed number	28	27	10	7
A whole number and a fraction, or a whole and a mixed number	**0**	**1**	14	11
	GO MATH!			
Two fractions, two mixed numbers, or a fraction and a mixed number	20	22	13	6
A whole number and a fraction, or a whole and a mixed number	**0**	**2**	21	15
	Everyday Mathematics			
Two fractions, two mixed numbers, or a fraction and a mixed number	29	19	16	11
A whole number and a fraction, or a whole and a mixed number	**0**	**2**	14	9

Table 6. The most common answers on four fraction arithmetic problems in children's data from Siegler and Pyke (2013) and the simulation data (Study 1). For each problem, all answers that were advanced on more than 2.0% of trials in the children's data, the simulated data, or both are shown. Answers are ordered by decreasing frequency in the children's data. Correct answers are bolded.

		Frequency (% of responses)	
Problem	Answer	Children	Simulation
2/3+3/5	**19/15**	**50.8**	**43.1**
	5/8	23.3	34.3
	5/15	5.0	12.0
	1/3	2.5	0.6
	6/7	1.7	2.6
	7/6	0.0	7.4
3/5–1/4	**7/20**	**54.2**	**45.0**
	2 or 2/1	20.0	30.8
	2/5	3.3	0.0
	1/10	2.5	0.9
	2/20	2.5	10.9
	1/4	1.7	4.9
	-1/4	0.0	5.4
4/5×3/5	**12/25**	**40.0**	**38.7**
	12/5	36.7	40.9
	15/20	4.2	5.5
	20/15	3.3	9.6
3/5÷1/5	3/5	37.5	48.5
	3 or 3/1	**28.3**	**21.3**
	15/5	7.5	15.1
	1/5	5.0	0.0
	5/15	3.3	5.5
	1/3	2.5	0.8
	3/25	2.5	9.1

Table 7. Percent of trials on which different types of errors occurred, by arithmetic operation and dataset (Study 1). "Children" denotes data from Siegler and Pyke (2013).

	Children			Simulation		
Error Type	Add/Sub	Mult	Div	Add/Sub	Mult	Div
Execution error	8	3	19	6	0	16
Strategy error	25	50	57	31	49	53
Add/Sub strategy	-	33	33	-	29	40
Mult strategy	20	-	12	17	-	13
Div strategy/Cross-Multiply	2	9	-	13	20	-
Other/None	2	7	12	-	-	-

Table 8. Percent of trials on which different types of strategy errors occurred, by arithmetic operation, equality or inequality of denominators, and dataset (Study 1). “Children” denotes data from Siegler and Pyke (2013).

	Add/Sub		Mult		Div	
	Equal denoms	Unequal denoms	Equal denoms	Unequal denoms	Equal denoms	Unequal denoms
			Children			
Add/sub strategy	-	-	41	25	40	26
Mult strategy	14	26	-	-	10	15
Div strategy/Cross-Multiply	2	2	8	10	-	-
Other/None	1	4	8	6	8	15
			Simulation			
Add/sub strategy	-	-	40	18	49	31
Mult strategy	10	24	-	-	4	22
Div strategy/Cross-Multiply	9	18	19	20	-	-
Other/None	-	-	-	-	-	-

Table 9. The most common answers on four fraction arithmetic problems in children's data from Siegler et al. (2011) and the enVisionMATH and GO MATH! simulations (Study 2). For each problem, all answers that were advanced on more than 2.0% of trials in the children's data or either simulation are shown. Answers are ordered by decreasing frequency in the children's data. Correct answers are bolded.

		Frequency (% of responses)		
Problem	Answer	Children	enVisionMATH Simulation	GO MATH! Simulation
3/5+1/2	4/7	41.7	33.6	31.9
	11/10	**37.5**	**42.9**	**47.6**
	4/10	4.2	11.2	11.6
	6/5	2.1	3.2	2.0
	1/2	2.1	0.0	0.0
	11/12	2.1	0.0	0.0
	19/10	2.1	0.0	0.0
	2/3	2.1	0.0	0.0
	4/6	2.1	0.0	0.0
	4/8	2.1	0.0	0.0
	5/5	2.1	0.0	0.0
	5/6	0.0	7.4	5.7
3/5–1/2	2/3	43.8	30.4	29.3
	1/10	**41.7**	**45.6**	**50.7**
	2/10	4.2	11.1	10.3
	1/4	2.1	9.0	7.0
	4 or 4/1	2.1	3.2	1.9
	4/3	4.2	0.0	0.0
	4/4	2.1	0.0	0.0
3/5×2/5	**6/25**	**52.1**	**40.3**	**58.4**
	6/5	16.7	39.7	25.3
	6/10	10.4	0.0	0.0
	2/3	4.2	1.1	0.7
	1/2	4.2	0.0	0.0

	10/15	2.1	6.9	7.1
	10/20	2.1	0.0	0.0
	150/25	2.1	0.0	0.0
	5/10	2.1	0.0	0.0
	5/25	2.1	0.0	0.0
	5/5	2.1	0.0	0.0
	15/10	0.0	11.0	7.2
3/5÷2/5	1/5	22.9	23.7	20.1
	1/1	20.8	9.1	10.6
	3/2	**16.7**	**1.6**	**1.6**
	15/10	**8.3**	**16.5**	**16.9**
	1/2	4.2	0.0	0.0
	15 or 15/1	4.2	0.0	0.0
	3 or 3/1	4.2	0.0	0.0
	6/25	2.1	10.3	13.0
	2/3	2.1	0.6	0.6
	0/5	2.1	0.0	0.0
	2/5	2.1	0.0	0.0
	4/10	2.1	0.0	0.0
	5/10	2.1	0.0	0.0
	5/2	2.1	0.0	0.0
	6/5	2.1	0.0	0.0
	65/4	2.1	0.0	0.0
	1.5/5	0.0	25.6	19.7
	10/15	0.0	4.8	4.9

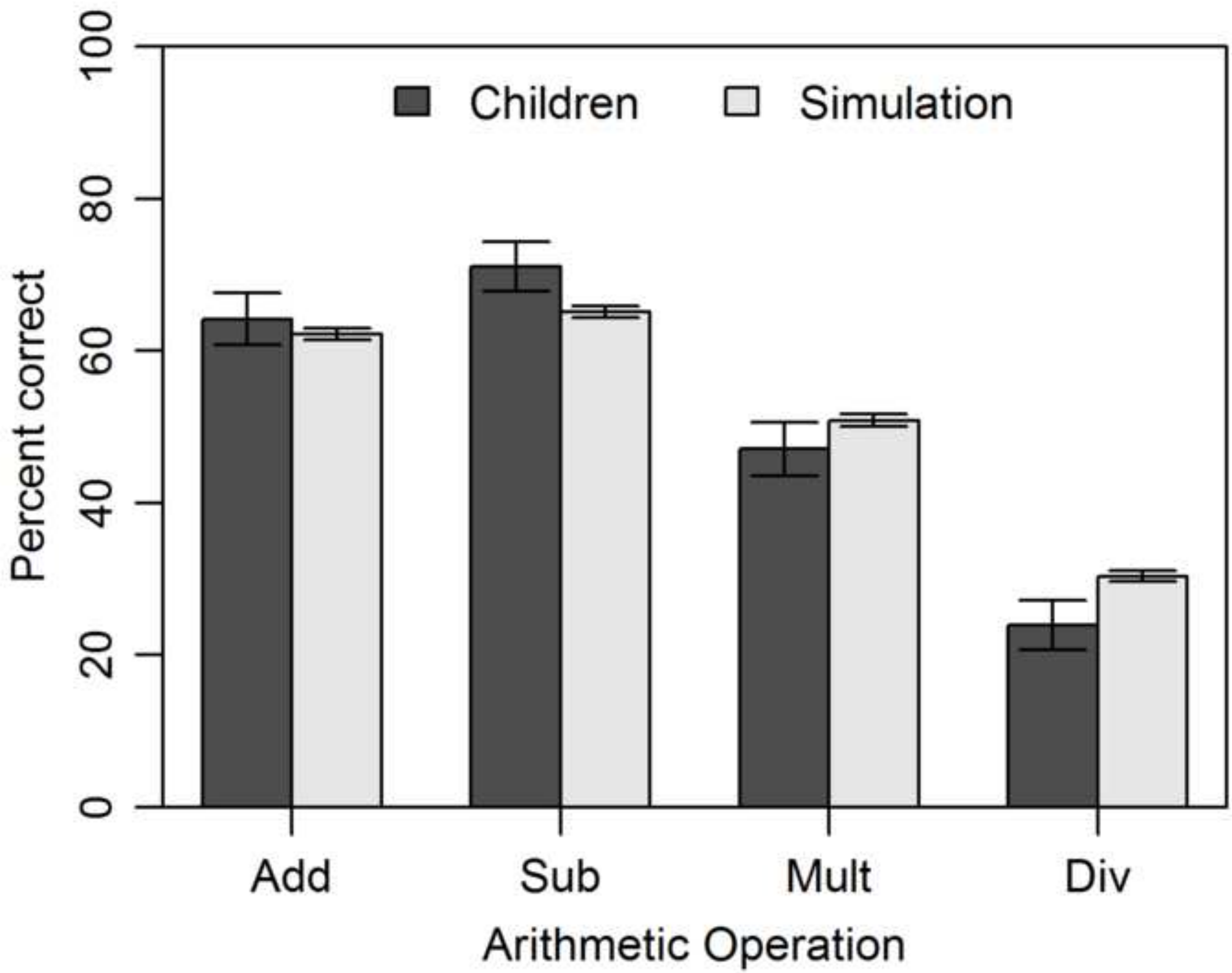
Table 10. Percent of trials on which different types of strategy errors occurred, by arithmetic operation, equality or inequality of denominators, and dataset (Study 2). "Children" denotes data from Siegler et al. (2011).

	Add/Sub		Mult		Div	
	Equal denoms	Unequal denoms	Equal denoms	Unequal denoms	Equal denoms	Unequal denoms
			Children			
Add/sub strategy	-	-	21	8	17	15
Mult strategy	31	48	-	-	33	35
Div strategy/Cross-Multiply	2	2	6	6	-	-
Other/None	2	5	4	8	21	21
			enVisionMATH Simulation			
Add/sub strategy	-	-	40	16	49	30
Mult strategy	10	26	-	-	9	14
Div strategy/Cross-Multiply	10	18	20	20	-	-
Other/None	-	-	-	-	-	-
			GO MATH! Simulation			
Add/sub strategy	-	-	25	20	40	39
Mult strategy	15	27	-	-	11	17
Div strategy/Cross-Multiply	12	12	16	12	-	-
Other/None	-	-	-	-	-	-

Table 11. Percent of test trials on which the standard correct strategy was used (Study 4). Bolded in each row are the percent correct in the control and differentiated feedback conditions on the type of problem specified in the "Test Problem Category" column.

	Type of Differentiated Feedback				
Test Problem Category	None (Control)	Add/Sub Equal Denoms	Add/Sub Unequal Denoms	Mult	Div
Add/Sub with Equal Denoms	**74**	**75**	75	76	74
Add/Sub with Unequal Denoms	**52**	52	**65**	50	45
Multiplication	**47**	48	43	**54**	39
Division	**44**	44	42	43	**62**

Figure


Children
Simulation
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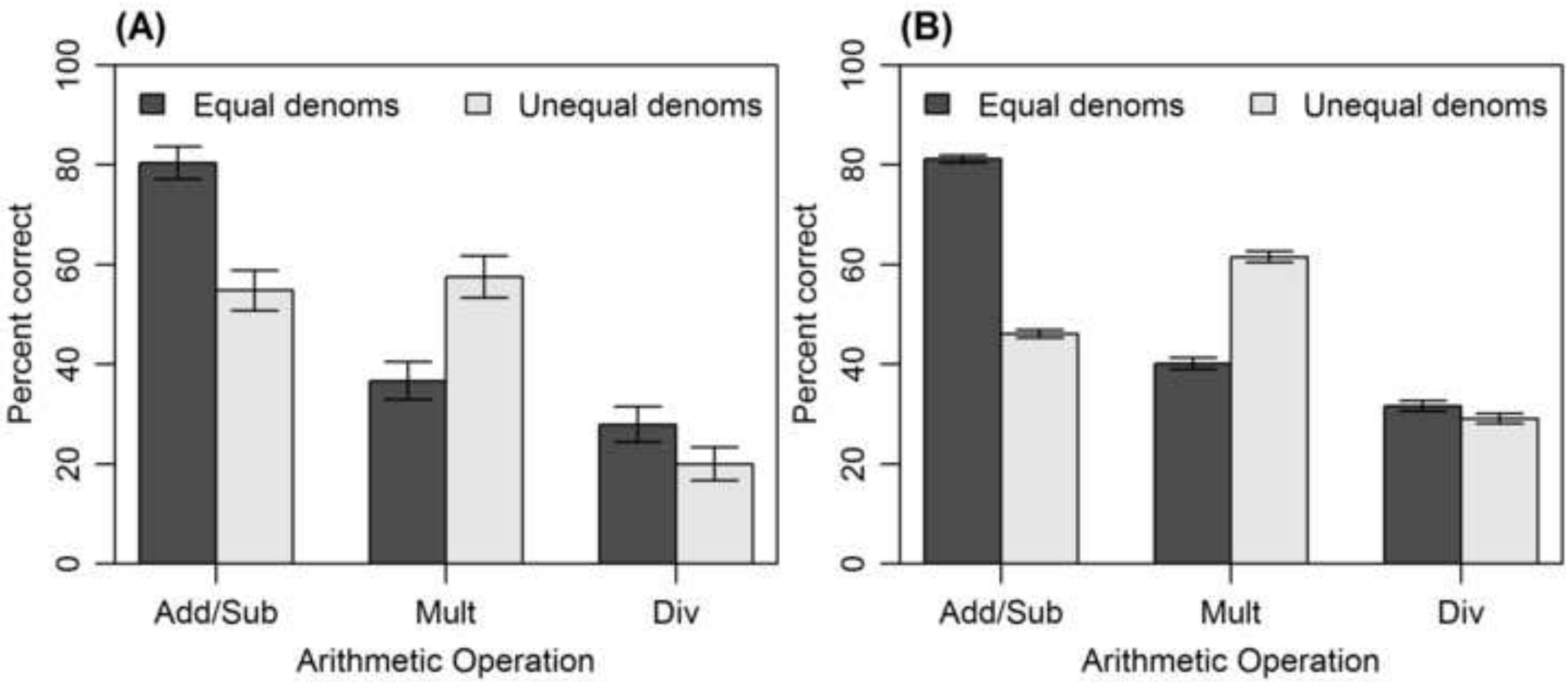
80
60
40
20
0
Percent correct
Add
Sub
Mult
Div
Arithmetic Operation

Figure



(A)
100
80
60
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Percent correct
Equal denoms
Unequal denoms
Add/Sub
Mult
Div
Arithmetic Operation
(B)
100
80
60
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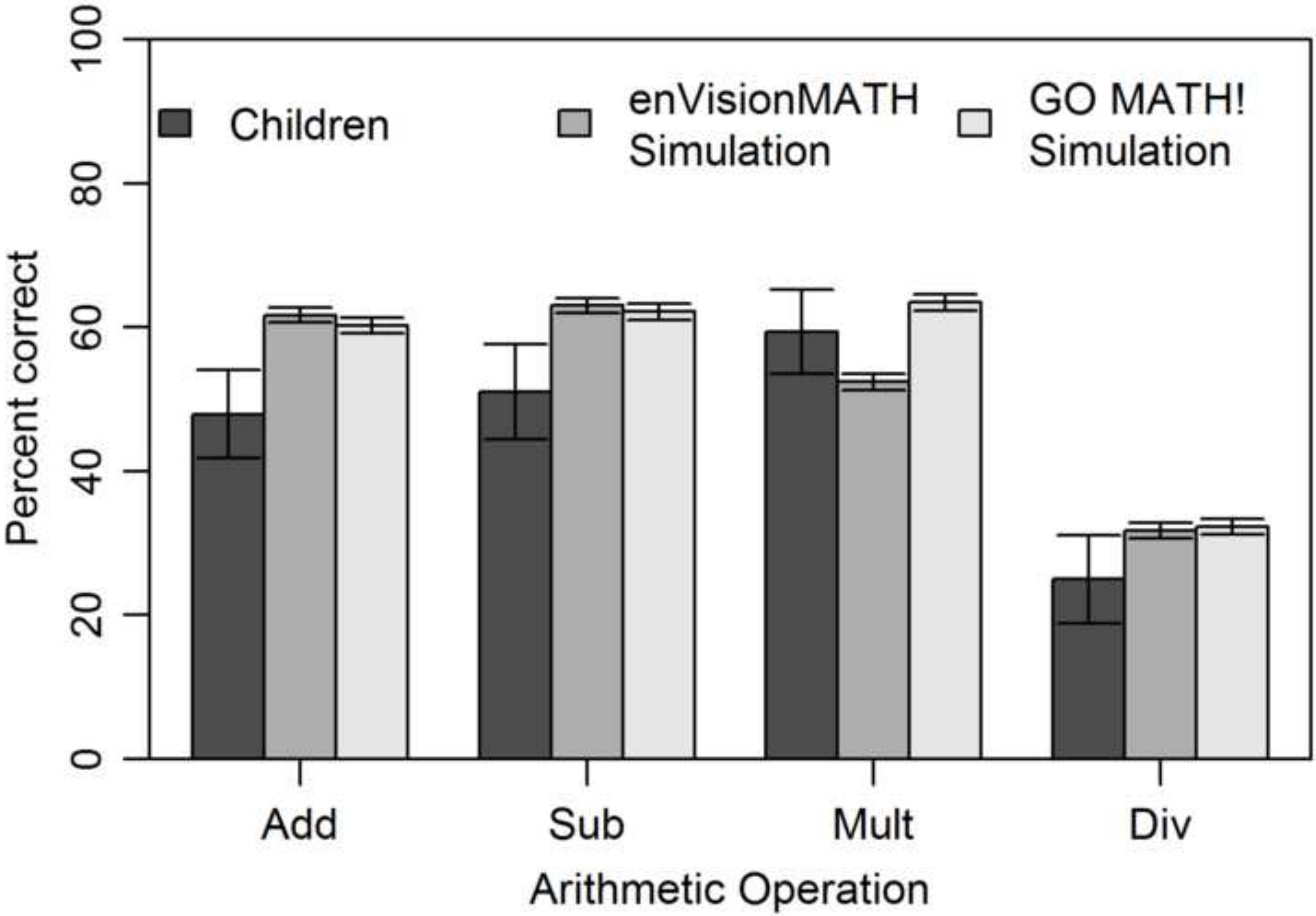
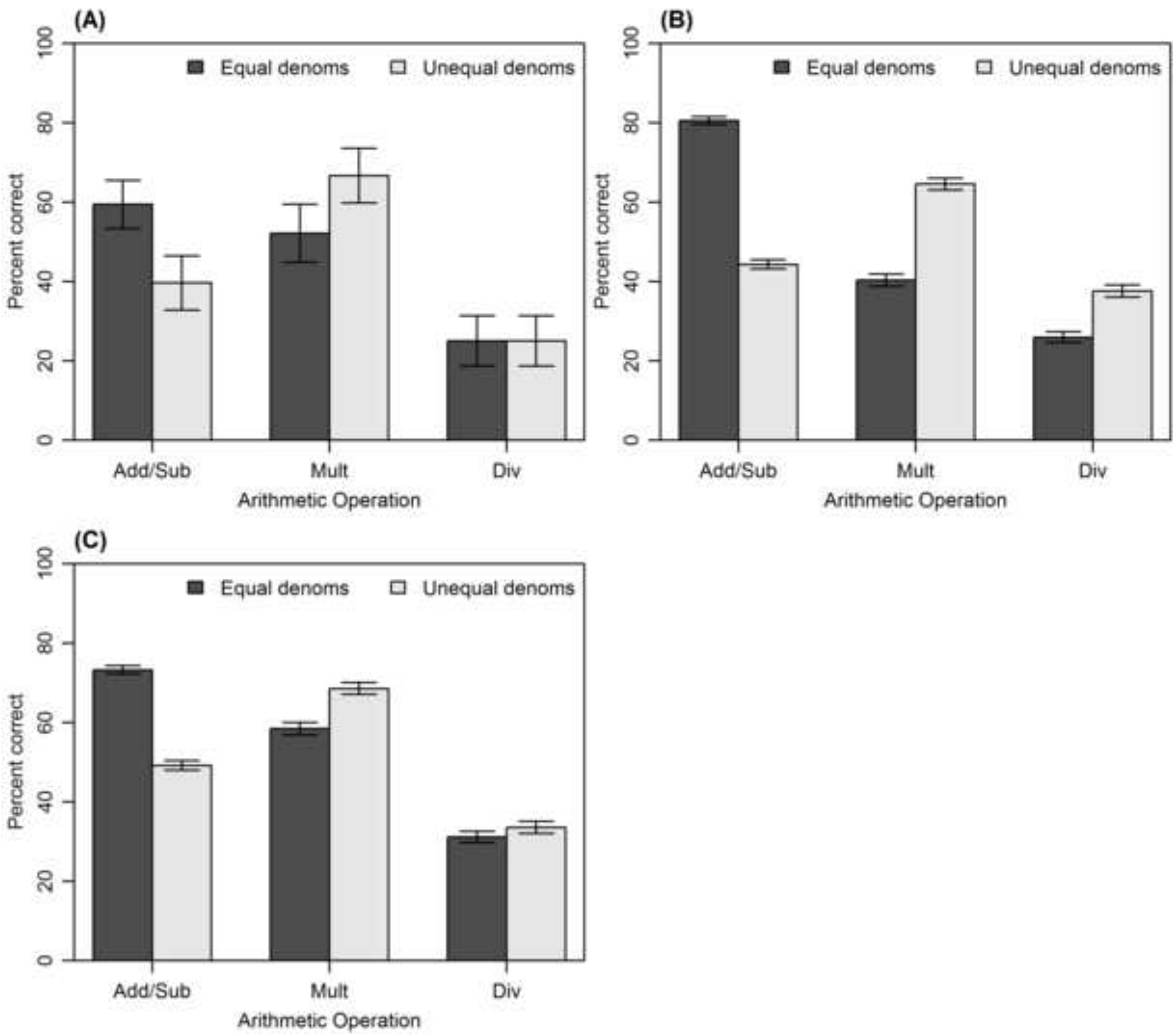
0
Percent correct
Equal denoms
Unequal denoms
Add/Sub
Mult
Div
Arithmetic Operation

Figure

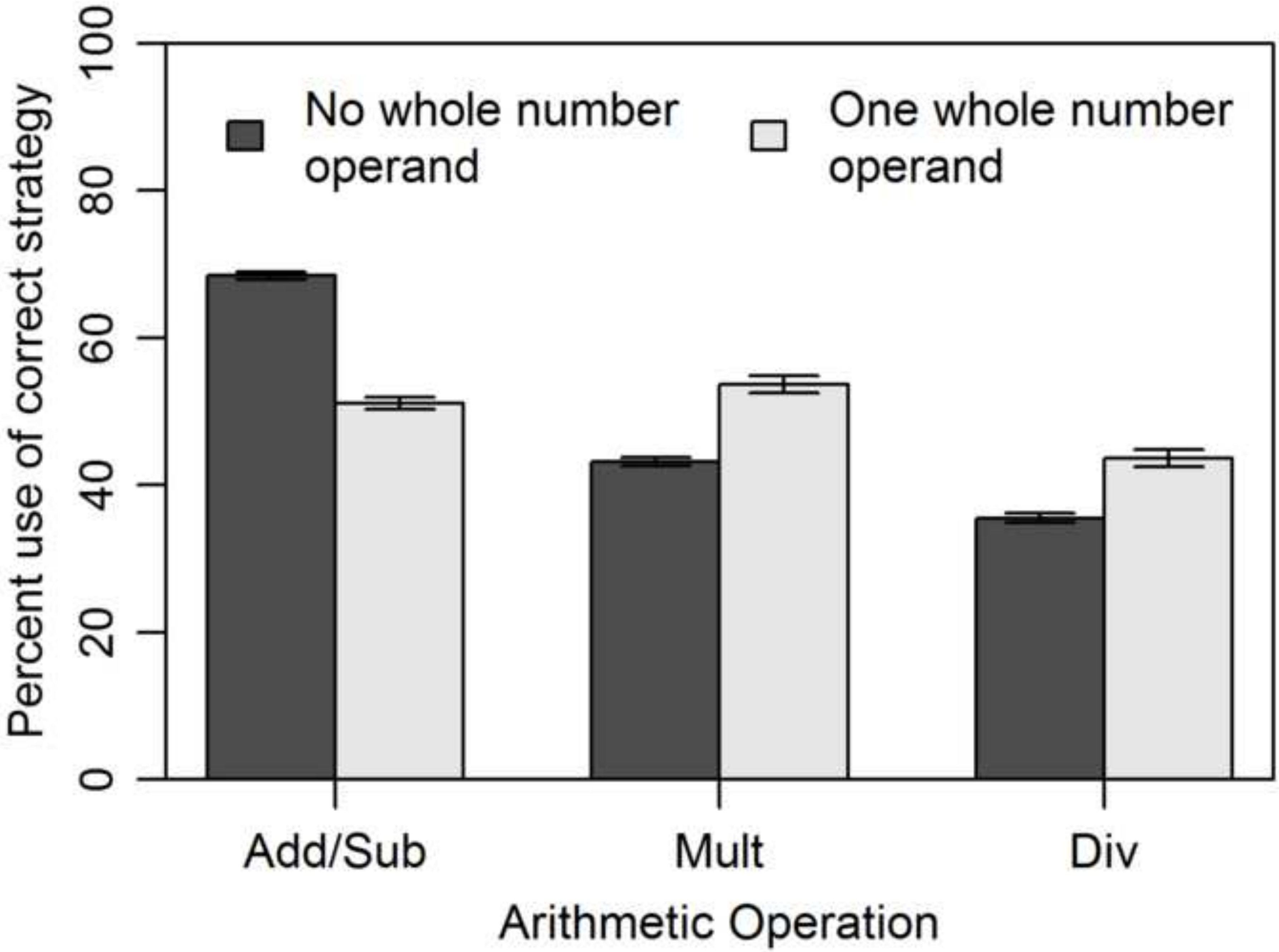


100
80
60
40
20
0
Percent correct
Children
enVisionMATH Simulation
GO MATH! Simulation
Add
Sub
Mult
Div
Arithmetic Operation

Figure

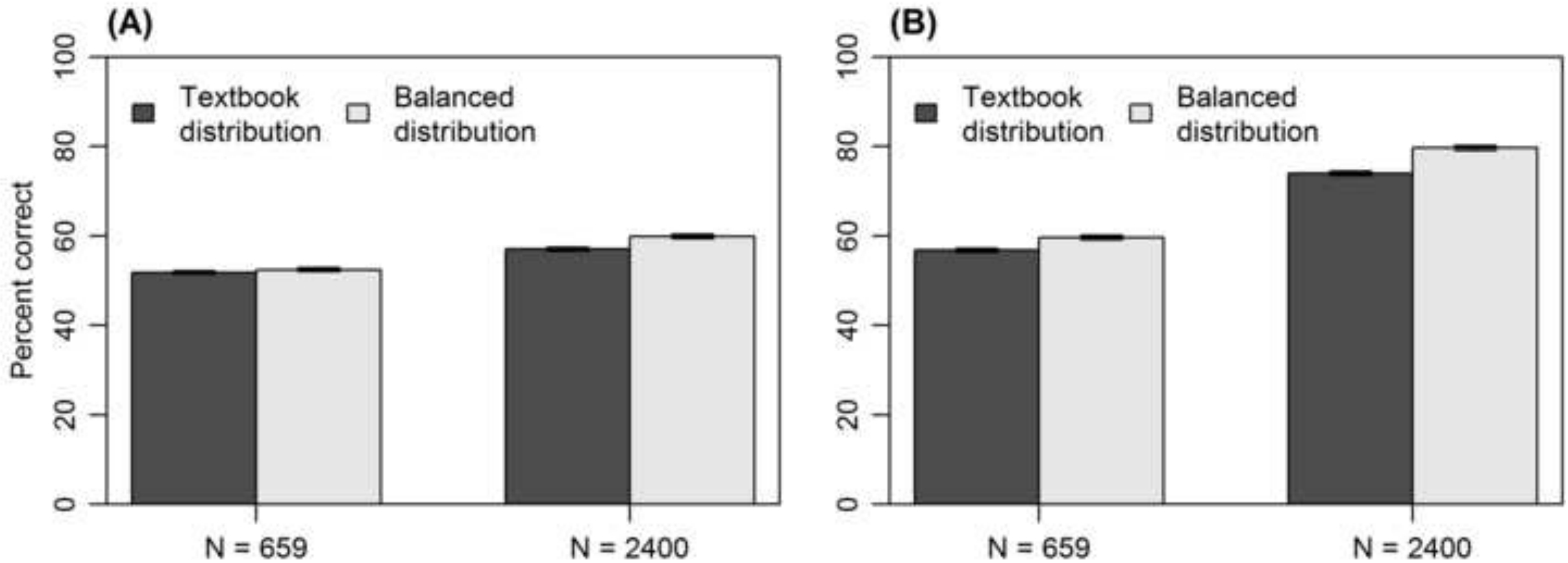

(A)
Equal denoms
Unequal denoms
Percent correct
0
20
40
60
80
100
Add/Sub
Mult
Div
Arithmetic Operation
(B)
Equal denoms
Unequal denoms
Percent correct
Add/Sub
Mult
Div
Arithmetic Operation
(C)
Equal denoms
Unequal denoms
Percent correct
Add/Sub
Mult
Div
Arithmetic Operation

Figure



100
80
60
40
20
0
Percent use of correct strategy
No whole number operand
One whole number operand
Add/Sub
Mult
Div
Arithmetic Operation

Figure



(A)
Textbook distribution
Balanced distribution
Percent correct
0
20
40
60
80
100
N = 659
N = 2400
(B)
Textbook distribution
Balanced distribution
0
20
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100
N = 659
N = 2400