RESEARCH + PRACTICE BRIEF

Interactive STEM

Interactive Technologies in STEM Teaching and Learning

Mathematics in the Early Grades: Operations & Algebraic Thinking

In the domain of *Operations & Algebraic Thinking*, Common Core State Standards indicate that in kindergarten, first grade, and second grade, children should demonstrate and expand their ability to understand, represent, and solve problems using the operations of addition and subtraction, laying the foundation for operations using multiplication (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). In this brief we describe the key concepts and skills that are involved in meeting the standards in this domain and provide a sample learning trajectory that illustrates how they develop over time and with experience.

Addition and Subtraction

The operation of addition can be understood as adding items to a set or joining two sets; subtraction can be understood as taking away items, or taking apart a set into two or more smaller sets. Children have some understanding of these operations even before they make their knowledge about addition and subtraction explicit (Clements & Sarama, 2014; Cross, Woods, & Schweingruber, 2009). For example, infants have a very early sense of appropriate results for addition and subtraction situations (Kobayashi, Hiraki, Mugitani, & Hasegawa, 2004; Wynn, 1992), and as children become proficient at counting they are essentially learning to perform an addition operation for which each count adds 1 to the previous number (Clements & Sarama, 2014; Wu, 2011). Children are first able to perform these operations using very small numbers and with the aid of manipulatives that allow them to act out the addition or subtraction situation (Clements & Sarama, 2014; Cross et al. 2009).

Addition can be more formally defined using its *commutative* ($a + b = b + a$) and *associative* properties [$(a + b) + c = a + (b + c)$]. Children may not recognize these names or call them by these names, but incorporating their meaning into children's understanding of addition is critical for developing strategies to perform this operation (Clements & Sarama, 2014). For example, children can count 2 green bears and 3 red bears to get 5 bears in all. They can then count 3 red bears and 2 green bears to get the same total number of 5 bears. That is, children can change the order in which they count two groups of items but still arrive at the same sum.

Subtraction is formally defined as the inverse of addition and can be thought of as an unknown addend in situations where the total and remaining amounts are known. For example, the subtraction problem $5 – 2 = ?$ can be reframed as $2 + ? = 5$, or "What can be added to 2 to make 5?"

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The commutative and associative laws of addition do *not* apply to subtraction, where the order of numbers and sequence of operations do matter.

Key Vocabulary

Expression – math phrase with numbers, unknowns, and/or operation symbols

An expression is a mathematical phrase that can include numbers, representations of unknown or changing quantities (such as variables), and symbols for operations including addition, subtraction, multiplication, and division. An expression can represent a known quantity (such as 5) or a relationship between known quantities (such as $5 - 2$) and/or unknown quantities (such as $2 + a$). Expressions that include one or more unknown quantities may change total value if the value(s) of the unknown quantities change.

Five examples of expressions: 3 $4 + 1$ $2 + 4 \bullet ?$ $5 - x$ $3(a - 4)$.

Equation – math sentence with an "=" sign

An equation is a mathematical sentence (also called a "statement") with an "=" (equals) sign. An equation represents an equality relationship between two expressions, one expression on the left side of the equals sign and the other expression on the right side of the equals sign. The expressions can include known quantities (represented by numbers) and/or unknown quantities (possibly represented by variables, a box, or a question mark). For an equation to be true, the two expressions are always equivalent (have the same total value) to each other, even if values for the unknown quantities change.

Here are four example equations: $2 + 4 = 6$ $3 + ? = 9$ $a - 5 = 10$ $10 + \square = 5 + \square + 5$

Commutative law of addition – for real numbers a and b, $a + b = b + a$

The commutative law or property is an ordering property; given two numbers, the sum of the numbers is the same regardless of which number is added to the other. Children can apply the commutative property to help make addition problems more efficient or accurate to compute mentally, such as when counting up from a number by changing 4 + 23 (starting at 4 then counting 5, 6, 7, 8,… all the way to 27) to the problem 23 + 4 (starting at 23 and counting "24, 25, 26, 27").

Associative law of addition – for real numbers a, b, and c, $(a + b) + c = a + (b + c)$

The associative law or property of addition is a grouping property; for three numbers taken in order, the sum of all three numbers is the same whether the first and second numbers are added first or the second and third numbers are added first. The associative law of addition allows a mental addition strategy that simplifies some computations, such as starting 4 + 4 + 6 by adding the second and third addends: $4 + (4 + 6) \rightarrow 4 + 10 = 14$.

Addition and Subtraction Strategies

Counting: Children's early use of addition and subtraction operations relies on subitizing, counting skills, and understanding of cardinality. The earliest strategies involve direct modeling of the situation in which manipulatives are used to act out the problem and then counted to find the solution. The strategies below are examples Clements and Sarama (2014) describe and illustrate:

» Counting all (when presented with two small sets of items, a child counts both sets separately, then counts the set of combined objects or images [a third count] to determine the cardinality of the combined set; example: *When adding 3 + 4, counts "1, 2, 3" and "1, 2, 3, 4," then counts "1, 2, 3, 4, 5, 6, 7."*)

» Counting on (when presented with two small sets of items, a child counts the first set, then "counts on" the items in the second set by continuing to count from the last item of the first set; example: *When adding 3 + 4, counts "1, 2, 3" then "4, 5, 6, 7."*)

» Counting backward (when presented with a set of items and a subtraction situation, counting down from the total as items are removed from the set; the item(s) remaining represent the difference; example: *When subtracting 5 – 3, counts "5, 4, 3… 2."*

Composing and Decomposing Number: With support, children can learn to apply their knowledge of part-whole relationships to addition and subtraction situations. They can put together two parts to make a whole (composition), take apart a whole into two or more parts (decomposition), and eventually move fluidly between different combinations of parts and whole to find a sum or difference (a missing addend). The following are example strategies that build on children's ability to compose and decompose numbers (Fosnot & Dolk, 2001):

- Compensation (when a child increases one part by 1 or more and decreases another part by an equal amount to compensate; children can use this strategy to arrive at combinations of numbers that are more familiar or easier for them to perform the operation; example: *When adding 2 + 4, adds 1 to 2 and compensates by subtracting 1 from 4, to change the problem to 3 + 3*)

- Working with 5s and 10s (when a child uses the familiar structures of 5 and 10 to make addition or subtraction problems easier; examples: *8 + 6 = 5 + 3 + 5 + 1 = 5 + 5 + 3 + 1 = 10 + 4 = 14* and *8 + 6 = 10 + 4 = 14*)

- Doubles with addition or subtraction (after the doubled amounts of different numbers become familiar, children can use these "doubles" numbers in combination with counting or addition to compose different numbers; examples: *4 + 7 = 4 + 4 + 3 = 8 + 3 = 11 and 6 + 7 = 6 + 6 + 1 = 12 + 1 = 13)*

Memorization: Many of the strategies described above rely on children's fluency with different combinations of numbers, which are sometimes referred to as addition or subtraction "facts." Although it is critical that children develop automatic recall for certain combinations of numbers (for example, doubles and the sum of combinations of single-digit numbers), this should be achieved through numerous opportunities to think about and describe number relationships rather than rote memorization or drilling to master facts (Clements & Sarama, 2014; Fosnot & Dolk, 2001).

Grouping: When presented with a set of items, children may create an addition situation using both their knowledge of part-whole relationships and counting skills. By arranging the items into groups of equal number, they can then add the number used to form each group based on the number of groups they were able to form. This lays the groundwork for both multiplication and unitizing (working with units other than one) (Van de Walle, Lovin, Karp, & Bay-Williams, 2013). Example: Given a set of 34 items, a child creates three separate groups of 10 items and a group of 4 items. The child adds 10 + 10 + 10 + 4 to find the sum 34.

Addition and Subtraction Situations

Cross et al. (2009) outline three general scenarios where addition and subtraction operations can be applied:

1. Change plus/minus: Start with a number, apply a change by either adding or taking away, and get a result.
2. Put together/take apart: When given two (or more) parts, figure out the whole; when given the whole, break into two (or more) parts.
3. Comparison: Compare two numbers and find how many more or less.

Word Problems: The scenarios described above can be played out in a virtually endless variety of written or verbal stories that put addition and subtraction in context. These situations or contextualized problems can help children develop their understanding of addition and subtraction (Van de Walle et al., 2013). Children's success on word problems depends in part on their knowledge of the number words and understanding of cardinality, but also on the difficulty of the problem (Carpenter, Fennema, Franke, Levi, & Empson, 2015; Cross et al., 2009; Van de Walle et al., 2013). At least initially, children will have more difficulty solving problems that include the following characteristics:

- Larger numbers (vs. numbers less than 5)

- Unknown *start* or *change* numbers (compared to an unknown *result*) in a change plus/minus situation

- Situations that are difficult to act out or model

- Unfamiliar contexts

- Wording that makes the actions, quantities, and/or chronology unclear

There is some disagreement about whether it is useful to have children learn and identify the different types of addition and subtraction situations. Although Clements and Sarama (2014) argue for its importance, Carpenter et al. (2015) caution against teaching children to identify these problem types by name and using overly rigid keyword strategies (for example, that "more" indicates an addition situation) that can lead to confusion or misrepresentation of the addition or subtraction situation. However, what is clear is that children should have ample opportunities to practice and describe each of these problem types, so they can learn to be attuned to the critical aspects of the contextualized situation that help them represent and solve the addition or subtraction problem.

Sample Learning Trajectory

The following table contains samples from one of Clements and Sarama's (2009) learning trajectories.[1] Note that the ages are approximate and heavily dependent on experience. Complete versions of this and other trajectories, which include the full age range and sample instructional tasks, are available in *Learning and Teaching Early Math: The Learning Trajectories Approach* (2014; see Resources).

	Developmental Progression	Description
Age 4	Small Number +/-	Finds sums for joining problems up to 3 + 2 by "counting all" with objects.
Age 4-5	Find Result +/-	Finds sums for joining… and part–part–whole… problems by direct modeling, counting all, with objects. Solves takeaway problems by separating with objects.
	Make it *N*	Adds on objects… without needing to count from 1. Does not (necessarily) represent how many were added.
	Find Change +/-	Finds the missing addend (5 + _ = 7) by adding on objects. Compares by matching in simple situations.
Age 5-6	Counting Strategies +/-	Finds sums for joining… and part–part–whole… problems with finger patterns and/or by counting on.
Age 6	Part-Whole +/-	Has initial part–whole understanding. Solves problem types using flexible strategies (may use some known combinations) Sometimes can do "start unknown"… but only by trial and error.
Age 6-7	Numbers in Numbers +/-	Recognizes when a number is part of a whole and can keep the part and whole in mind simultaneously; solves "start unknown" … problems with counting strategies.
	Deriver +/-	Uses flexible strategies and derived combinations (e.g., "7 + 7 is 14, so 7 + 8 is 15) to solve all types of problems. Can simultaneously think of 3 numbers within a sum, and can move part of a number to another, aware of the increase in one and the decrease in another. May solve simple cases of multi-digit addition (sometimes subtraction) by counting by tens and/or ones.
	Problem Solver +/-	Solves all types of problems, with flexible strategies and known combinations. Multidigit may be solved by incrementing tens and ones by counting

Representing Addition and Subtraction

As children's strategies for performing addition and subtraction operations become more sophisticated—moving from counting-based strategies to those that rely on a fluid knowledge of number relationships—their methods for representing these situations also evolve with instruction, support, and opportunities to practice. Across all methods, especially as children move from more informal, self-made, or self-selected representations to those that are more conventional, for the representations to become meaningful and useful children must have supported opportunities to map their knowledge and ideas onto the representational tools (Van de Walle et al., 2013).

Manipulatives: Young children are able to solve addition and subtraction problems with the use of physical objects that they can use to act out the problem situation. Initially, these might be the actual objects involved in the problem situation (e.g., marbles, toys); later, other manipulatives such as fingers or blocks can be used to either directly model the situation or act as a support for a child's chosen strategy. Research suggests that use of manipulatives can help children build confidence in their addition and subtraction skills and use of manipulatives should be supported until they have developed successful strategies for solving particular types of problems (Clements & Sarama, 2014). It is important to note that manipulatives can also be virtual. There is evidence that virtual manipulatives can help children assign meaning to math symbols (e.g., "+" or "–") (Van de Walle et al., 2013) and can support the development of mental strategies (Clements & Sarama, 2014).

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Drawings: Children can also directly model or represent addition and subtraction situations using drawings or diagrams that they create on paper or a touch-screen device. These representations can be literal (such as drawings of the actual objects involved in the problem situation), abstract (such as dots or slashes), and/or diagrams that relate the different quantities featured in the problem (Clements & Sarama, 2014; Fuson & Abrahamson, 2009; Van de Walle et al., 2013).

Expressions and Equations: As children develop concepts, strategies, and language related to addition and subtraction, it is important to provide and scaffold opportunities to map their knowledge in these areas onto symbols and syntax that can then be used to represent addition and subtraction operations using expressions and equations (see "Key Vocabulary"). They may initially find the "+," "–," and "=" symbols confusing (Carpenter et al., 2015; Cross et al., 2009; Van de Walle et al., 2013) and may create and use their own symbols (possibly arrows or other marks for operations and relationships between quantities) in their early use of expressions (Carpenter et al., 2015).

Language: Regardless of how children represent an addition or subtraction situation and no matter which strategies they use to arrive at a solution, it is important that they have opportunities to verbally represent the problem situation, describe their thinking in their own words, and be supported in building an appropriate mathematics vocabulary (Clements & Sarama, 2014; Van de Walle et al., 2013).

Resources

Children's Mathematics: Cognitively Guided Instruction (2nd ed.)
Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B., 2015
Chapter 2: Addition and Subtraction: Problem Types
Chapter 3: Addition and Subtraction: Children's Solution Types

Learning and Teaching Early Math: The Learning Trajectories Approach (2nd ed.)
Clements, D., & Sarama, J., 2014
Chapter 5: Arithmetic: Early Addition and Subtraction

Early Childhood Mathematics Education Research: Learning Trajectories for Young Children
Sarama, J., & Clements, D., 2009
Part II: Number and Quantitative Thinking

Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity
National Research Council, 2009
Chapter 5: The Teaching and Learning Paths for Number, Relations, and Operations

Teaching Student-Centered Mathematics: Developmentally Appropriate Instruction for Grades Pre-K–2 (2nd ed.)
Van de Walle, J., Lovin, L.H., Karp, K. S., & Bay-Williams, J. M., 2013
Chapter 9: Developing Meanings for the Operations

References

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics: Cognitively guided instruction* (2nd ed.). Portsmouth, NH: Heinemann.

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach. (Studies in mathematical thinking and learning series)* (2nd ed.). New York, NY: Routledge.

Cross, C. T., Woods, T. A., & Schweingruber, H. (2009). *Mathematics learning in early childhood.* Washington, DC: National Academies Press.

Fosnot, C., & Dolk, M. (2001). *Young mathematicians at work: Constructing number sense, addition and subtraction*. Portsmouth, NH: Heinemann.

Fuson, K. C., & Abrahamson, D. (2009). *Word problem types, numerical situation drawings, and a conceptual-phase model to implement an algebraic approach to problem-solving in elementary classrooms.* Chicago, IL: Northwestern University.

Kobayashi, T., Hiraki, K., Mugitani, R., & Hasegawa, T. (2004). Baby arithmetic: One object plus one tone. *Cognition, 91*, B23–B34.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards for mathematics: Counting & cardinality. Retrieved from http://www.corestandards.org/Math/Content/OA/

Van de Walle, J., Lovin, L. H., Karp, K. S., & Bay-Williams, J. M. (2013). *Teaching student-centered mathematics: Developmentally appropriate instruction for grades pre-K–2.* (2nd ed.). New York, NY: Pearson.

Wu, H. H. (2011). *Understanding numbers in elementary school mathematics*. Providence, RI: American Mathematical Society.

Wynn, K. (1992). Addition and subtraction by human infants. *Nature, 358*, 749–750.