On Doing Mathematics: Why We Should Not Encourage 'Feeling,' 'Believing,' or 'Interpreting' Mathematics.

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Abstract

On Doing Mathematics: Why We Should **Not** Encourage Feeling, Believing, or Interpreting Mathematics.

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P. R. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of the methods employed in teaching students to do mathematics with a modified Moore method (MMM). It states, "I see, I forget; I hear, I remember; I do, I understand." In this paper we build upon the suggestions made in, On the Nature of Mathematical Thought and Inquiry: A Prelusive Suggestion (2004, ERIC Document ED502336) and attempt to explore why the differences between reading, seeing, hearing, witnessing, and doing give rise to the contrast between and betwixt feeling, believing, interpreting, opining, and knowing.

We refine in this paper the philosophical position proposed in the 2004 paper and accentuate how reading, seeing, or hearing do not lead to understanding whilst feeling or believing do not lead to truth. We submit that 'interpreting' gives the impression Math is as imprecise as Psychology and is rooted in relativism (the 'eye of the beholder') rather than certain conditional truth deduced from axioms. We posit that feeling, believing, & "interpreting mathematical phenomena," are actually harmful to authentic meaningful mathematical learning.

1. Introduction, Background, and the Modified Moore Method.

This paper is one of a sequence of papers ([146], [147], [148], [150], [151], [152], [153], [154], [145]) the author has written over the last decade discussing inquiry-based learning (IBL) and the modified Moore method (M^3) that he employs whilst teaching mathematics, directing research, and doing mathematics himself. We seek in this paper to expand on the thoughts previously presented and expand upon the basic philosophical points made in his paper about mathematical thought [149]. So, we shall argue that reading, seeing, or hearing do *not* lead to understanding whilst feeling or believing do not lead to truth. We submit that 'interpreting' gives the impression Math is an imprecise relativistic endeavour devoid of epistemological certainty rather than certain conditional truth deduced from axioms. We posit that feeling, believing, & "interpreting mathematical incidents," are actually harmful to authentic meaningful mathematical learning.

What prompted this paper (and the accompanying talk) is: twenty-five years of teaching and 45 years of schooling; articles read in *Journal of Mathematical Behavior*, *Pi Delta Kappan*, *Journal for Research in Mathematics Education*, *Mathematical Thinking and Learning*, etc. that seem to indicate mathematics is a relativistic endeavour; and the use of inquiry-based learning (IBL) to teach (based on the Moore method[1]).

Many of the articles (especially by Boaler, Taylor, and Zevenbergen) seem to argue for a disconnected incidental schema (usually termed phenomenological, hermeneutical, or constructivistic schema) that is anti-objectivist.[2] The works are converse to the author's experience and seem to be of the type that is best described as 'sounds great (on paper) but does not work.' So, the prompt for writing this paper is a practical, pragmatic prompt for an *a forteriori* argument.

We submit that in order to learn one must *do* **not** simply witness. The demand for *doing mathematics* is a hallmark of the Moore method, a modified Moore method (M^3), or most (if not all) inquiry-based learning (IBL) methods. In a Moore method class the individual is supreme and there is a focus on competition between students. Moore, himself, was highly competitive and felt that the competition among the students was a healthy motivator; the competition among students rarely depreciated into a negative motivator; and, most often it formed an *esprit d' corps* where the students vie for primacy in the class. Whyburn notes that Moore's beliefs "gives one the feeling that mathematics is more than just a way to make a living; it is a way of life, an orderly fashion in which you want to consider all things."[3] The Moore method assists in students' developing an internal

[1] The Moore method due to R. L. Moore, H. S. Wall, and H. J. Ettlinger at the University of Texas – see [140]; [121]; [78]; [72]; [52]; [44]; [220]; and, [155].

[2] Such being based upon the works of Freire, Giroux, Noddings, etc.

[3] Whyburn, page 354.

locus of control; confidence; perspective; and, a sense of curiosity.

Distilled to its most basic components it seems generally agreed that a Moore method, M^3, or IBL course has the following characteristics: the instructor is a facilitator, guide, or mentor; there is limited to no use of books (instructor notes are handed out throughout the semester or quarter); there is no collaboration before student presentations (in many Moore method classes this is expanded to absolutely not 'group work' or consultation between students); and, there is a competitive atmosphere in the class - students compete to get to the board to present work (but not for the grade - the grade is determined by each individual's work: quality, quantity of produced work and quality of questions and comments in the class). The focus on IBL, the Moore method or a modified Moore method is on the student – student created arguments, proofs, examples, counter-examples, etc. In the author's M^3 almost everything is in the form of a claim (universal): to prove or disprove or a construction of an example or counter-example (existential).

We have argued in [149] that 'positive scepticism' is a central tenet to the nature of mathematical thought and is particularly present in any work a student does when directed to do research in a Moore method manner. 'Positive scepticism' is meant to mean demanding objectivity; viewing a topic with a healthy dose of doubt; remaining open to being wrong; and, not arguing from an *a* priori perception. IBL, M^3, and the Moore method are interlinked are based on Moore's Socratic philosophy of education - - the student must master material by *doing*; not simply discussing, reading, or seeing it and that *authentic* mathematical inquiry relies on inquiry though 'positive scepticism' (or the principle of *epoikodomitikos skeptikistisis*).[4]

Under the author's modified Moore method (M^3), students occasionally are allowed books for review of pre-requisite material or for 'applications' (which are not of much interest to the author) We opine we should not be afraid to direct students to books or use books ourselves; but, we should train our students to use them wisely (for background (pre-college) material mostly) and sparingly. We should not be inflexible with regard to the use of a computer algebra system (CAS) (such as *Maple* or *Mathematica*) on *rare* occasions to allow the student to investigate computational problems (in Probability and Statistics or Number Theory, for example) where or when such might be helpful to understand material or form a conjecture.

[4]Moore was a Socratic. If one employs an IBL, M^3, or classic Moore method philosophy of education, one is a Socratic. However, we opine Sophistry is ascendant in the 21^{st} century university. In the competition between Sophistry and Socraticism, Socraticism is authentic, preferred, and correct. Sophistry prevails in many a mathematics classroom because it is easier for the instructor–no arguments with students, parents, or administrators; complaints of things being 'hard' are almost non-existent if one employs sophistry and the instructor does not have to "think as hard;" it is easier for the student–he does not have to "think as hard" (or think at all), she can "feel good," it can have its self-esteem 'boosted;' it is easier for the institution– standardisation can be employed (which seems to be a goal at many institutions); students retained ('retention' and 'assessment' seem to be a 21^{st}, 'buzz words'), graduation rates increase, and accrediting agencies are mollified.

We should not be fearful of directing students to a CAS or use a CAS ourselves; but, we should encourage our students to use them infrequently and wisely. We must be very careful with a CAS for it can become a crutch quickly and there are many examples of students who can push buttons, copy and paste syntax, but not understand why they are pushing the buttons, what is actually the case or not, and are very convinced that crunching 10 quintillion examples proves, for example, a claim let us say a universal claim in $\mathbb{R}$.[5] Understanding mathematics – really understanding it – is not something that is learnt through reading other people's work or watching a master teacher demonstrate his or her great skill at doing a proof (no errors, elegant, and compleat). The student learns by getting his 'hands dirty' just as an apprentice plumber gets his hands dirty taking a sink apart, fixing it, and then putting it back together. He must explore the parts of the sink, learn how the parts interact, take things apart, put the part or the whole system together again, make mistakes (leave a part out so the sink leaks or puts the parts together incorrectly so the pipes do not deliver water to the sink), learn from the triumphs and mistakes (**but most especially we learn from our mistakes**). Doing mathematics not only takes skill but takes initiative, imagination, creativity, patience, perseverance, and hard work. The nuts-and-bolts of a skill can be taught; but, initiative, discipline, imagination, patience, creativity, perseverance, and hard work can not be taught – such must be nurtured, encouraged, suggested, and cultivated.

2. What makes Math different from many other Arts or Sciences

We note that the mathematics exercise is centred on reasoning– whether the aspect of the mathematics is applied, computational, statistical, or theoretical; the retention of facts is not as important to Mathematics as say English or History whilst the end product of something marketable is not a core consideration in Mathematics as it is in Communication Design, Software Development, Building Science, Architecture, or Engineering. This is not to say these arts or science are inferior or lacking – it is merely noted to highlight that reasoning plays so important a role a student in Mathematics could get every problem wrong on a test and earn a 'A' whilst not so in other areas; for a research mathematician the process of how an idea is developed and proven is more important (to many) than the result (e.g.: the Four Colour Theorem); and we opine that there are unanswerable questions within the realm of any given system (the simplest of which we call axioms since we assume them) but such atomic principles might not be needed or interesting in other arts or sciences.

Some principles, ideas, or facts relegated to memory *are* necessarily utile and essential for learning mathematics and the more advanced the material the more understanding of previous material is needed. But in mathematics

[5] Sadly, some faculty think that such is 'better' than a proof.

it is not recalling discrete facts (like Euler created the Gamma function) that is needed but *an authentic understanding of the idea* (like Euler's Gamma function's definition, proof of its iterative property, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, etc.) as opposed to facts in History, English, Spanish, Geology, Psychology, etc. Hence, students' education in mathematics is fiducially centred on the encouragement of individuals thinking: conjecturing, analysing, arguing, critiquing, and proving or disproving claims whilst being able to discern when an argument is valid or invalid.

Indeed we opine that the unique component of mathematics which sets it apart from other disciplines in academia is the demand for proof - - the demand for a succinct argument from a logical foundation for the veracity of a claim such that the argument is constituted wholly within a finite assemblage of sentences which force the conclusion of the claim to necessarily follow from a compilation of premises and previously proven results founded upon a consistent collection of axioms. Mathematicians demand logic from the foundation of the process of invention and discovery as opposed to faith that can be used in, say, Theology. Principles, programmes, and theories are scaffolded –that is built from previous or are constructed anew if an area of mathematics is discovered that denies a previously held 'truth' (an example being spaces which have non-integer Hausdorff dimension as opposed to large inductive and small inductive dimension of a space). Scaffolding is not needed, for example, to read *A Tale of Two Cities* beyond basic English language competence. Sometimes in literature a tome owes much of its imagery or flavour to a previous work but such is not *necessary* as Analysis needing Set Theory and Logic to inform its works and clarify the veracity of a claim within Analysis. We demand axiomatics – we need terms defined unambiguously, consistently, and precisely. We demand to know from whence an idea came; such is not needed in many other areas of the academy. We claim that mathematics obeys the demands of 'positive scepticism' (*epoikodomitikos skeptikistisis*), by and large, since it demands objectivity; to do mathematics well requires viewing a topic with a healthy dose of doubt; to do mathematics well requires remaining open to being wrong; and, to do mathematics well requires not arguing from an *a priori* perception. What binds and supports the mathematical endeavour is a search for authenticity, a search for pragmatic truth, and a search for what is apposite within the constraints of the demand for justification. The means of solution are matter – not the ends – the progression of deriving an answer, the creation of the application, or the method of generalisation so as not to be fallacious.

3. The Nature of Mathematical Thought and Inquiry.

These procedures demand more than mere speculative ideas; they demand reasoned and sanguine justification. Hence, the nature of the process of the inquiry that justification must be supplied, analysed, and critiqued is the essence of the nature of mathematical enterprise: knowledge and inquiry are inseparable and as such must be actively pursued, refined, and engaged.

The mathematics we are discussing with is not concerned with ontological questions. Such questions might make a wonderful topic for a paper; but, such are beyond the scope of this paper. We are discussing the epistemological and axiological nature of mathematical thought and inquiry.

We put forward that mathematicians seek epistemological understanding. Mathematics is a science of thinking, one can not *do* mathematics with no thought – mathematics is an epistemological endeavour so how can one do mathematics without thought? Mathematics is not for witnessing or being a spectator to an event (though some *seem* to think such is the case with lectures, books, and calculators).

Mathematical practitioners are concerned with the nature of truth – conditioned from an axioms system or induced from physics, chemistry, biology, etc. and then formalised for analysis. Mathematics is *not* an unorganised, relative, or subjective collection of activities that each individual creates such that each 'relative math' is equi-valid with all others.

Mathematics is an abstract, a logical realm, with clarity of concepts and certitude in its conclusions – conclusions that are predicated by axioms systems and previously proven results. As such these truths are not *self-evident* but *agreed to* assumptions by society and are relative to the axioms system that the human race has collectively agreed. There are *not* billions of different mathematical systems of equal validity and sanguinity constructed by different perceptions.

We put forward that mathematicians seek axiological verisimilitude. We suggest that it is the case that mathematicians also accept and acknowledge the axiological value of completing a proof; creating an argument; and, establishing a theory. It also seems to be the case that mathematicians try to mold aesthetically pleasing arguments. We value parsiminous elucidations. We enjoy creating or reading novel ways to go about proving or disproving a claim. We find elegant arguments on par with a fine work of art, symphony, or book. This is an axiological position for it is a value-judgement that inquiry into the nature of mathematics is positive. It is further an axiological position for it posits that there is an ethic involved in the art of mathematics: the ethic of *epoikodomitikos skeptikistisis*. We must do in order to transcend from rudimentary to more refined epistemological and axiological understanding of mathematics.

The axiological position forwarded in this paper is in direct opposition to the position taken by D'Ambrosio, Powell, Frankenstein, Joseph, Gerdes, Anderson, and Martin (to name but a few) (see [6]; [48]; [5]; [74]; [75]; [76]; [77]; [84]; [122]; [174]; [175]; [51]; [85]; and [141] for a few examples) who build upon the works of Giroux, Noddings, Friere, etc. and expand the notion of alternate realities or relative ethics in mathematics. Our axiological position could not be farther from these authors' positions for they adopt a position which minimises *epoikodomitikos skeptikistisis* and "an ethic based on a set of values intrinsic to mathematics, such as rigor [sic], precision, resilience, and others of the same kind."[6] They adopt a more expansive definition of mathematical inquiry and 'knowledge gathering' to go well beyond any positive scepticism. To wit, D'Ambrosio states "I am also concerned with an ethic of respect, solidarity, and cooperation. In fact, we know that so much is involved in the acquisition of knowledge. Knowledge results from the complexity of sensorial, [sic] intuitive, emotional, and rational components. Is this incompatible with mathematics? If not, how do they relate?"[7] It begs the question, how can one gather mathematical understanding from emotional experience? What is happy math or sad math? What does solidarity have to do with Linear Algebra, Real Analysis, or Probability Theory?

We argued about the nature of mathematical thought and its philosophical roots in [149]; whilst herein we posit that there is mathematical truth that is knowable and that mathematical inquiry is a reason for an activity; not knowledge attainment. Much of the literature with which we take issue imagines mathematics as a sequence of facts to be acquired; a sequence of processes to me memorised; or, a set of rules to be followed (though the authors do not use such terms but when one reads the works it becomes clear that mathematics to many of them is a 'thing' to procure and not a way of thought and understanding in and of itself). Furthermore, many of the authors draw broadly across the academic landscape merging ways of learning from kindergarten through graduate schools as if they are similar; we opine there is not much authentic meaningful mathematical learning occurring in today's universities, schools, or academies because of this mistaken view of learning. If one is to learn mathematics in a meaningful and authentic manner one must be taught through a programme that encourages thought and individual effort so that understanding comes to each student rather than being 'taught' with a goal of 'knowledge acquisition.'

4. The Proposed Taxonomy: Feeling, Believing, Interpreting, Opining, and Knowing.

To know a thing means that it is an incontrovertible fact under an agreed to axiom system. The thing has been proven. To opine a thing is so means that there is a body of evidence that is open to review and critique such that a strong (hopefully) case is made that given the state of our understanding and the evidence that the thing is so (it is more likely to be than not to if there were a well defined probability sample space that adjoins the thing so the probability of the thing happening can be rigourously quantified). To believe a thing is so means that there is some evidence that is open to review and critique such that a case is made that given the state of our understanding and the evidence that the thing seems

[6]Ubiratan D'Ambrosio, "Foreword." In *Ethnomathematics*, Powell, A. and Frankenstein, M. (Eds.), Albany, NY: State University of New York Press, 1997.

[7]Ibid.

to be so; to believe a thing can be relative – that is to say one belief does not disallow another. To feel a thing is so means whatever one wishes to feel; that person can feel as that person wishes.[8] Let us consider these four points along with the concept of interpreting mathematics in the order of feeling, believing, interpreting, opining, and knowing.

4.1. **Feeling.**

Many professors have encountered the feelings of students oft when grades are discussed, "I feel that . . . (fill - in - the - blank[9])" Perhaps due, in part, to the 1970s to the present educational theories (we term 'edu-babble') pop-psychology and advertising agency inspired feelings are more important than deeds much of the educational system in the United States is infected with the concept that feelings are of paramount importance. Read any Noddings, Boaler, Taylor, Powell, Fennema, etc. and one finds repeatedly the call for not only acknowledgement of feelings but using such as the basis for the academy. Literally one finds educational researchers such as Delphit, Ferraro, and others using feelings as a 'justification' for knowing something is or is not! Further the insistence that concepts such as self-esteem, communality, or humanism are, if we are correct that what was written by said authors was meant to mean what was written, more important in education than classic academic subjects. With feelings comes indignation, entitlement, demand, offence, righteousness, insult, umbrage, etc.– there can be no meaningful objective inquiry just disjointed disconnected incidental relativistic blather.

Some anecdotes which serve to illustrate said are, for example, in a freshman Statistics course the author taught when asked for an explanation for why a student (call him Mr. X) said that nine-tenths of 120 is 100 the student said, "I feel it is." There was no irony nor sarcasm in his voice– he meant what he said and said what he meant. When other students pointed out that he was wrong, offence and indignation followed and for Mr. X the topic was no longer of importance nor something to be discussed. In a senior Probability Theory course where an oral quiz was administered to each individual student in the class the author asked for an explanation for how to determine that for $X, Y \sim Dirichlet(x, y, \alpha = 4, \beta = 3, \gamma = 1)$ the probability that $X + Y < \frac{1}{2} \vee Y > \frac{1}{4}$ a student (call her Ms. Y) wrote some things on the board and offered a solution which in no way had a connection to the question. However she stated, "I feel this is the answer." Pressed repeatedly to explain why, the response remained, "I feel this is the answer" (for at least it seemed over five times with the question restated in different terms and with suggestions meant to hint at a possible path for her to figure out the solution). Indeed when the author suggested drawing the graph of the region of integration in the plane the student did not respond and acted

[8] The use of wishes as the active verb in indicative of the verisimilitude of the feeling.

[9] I deserve an 'A' for I . . .

as if such a suggestion was 'out of left field' and was novel.

The misuse of the term, 'feel,' may also contribute to this conundrum. In the vernacular students oft now respond to a question of understanding of comprehension with the response, "I feel you." Consider the question, "so does it make sense that if U is a well defined universe and $A \subseteq B$ then it is the case that $B^c \subseteq A^c$?" On more than one occasion the slang response, "I feel you," was proffered. How does a feeling indicate grasping of a concept or comprehension? Under what logical system could such, "I feel you," create any certification of recognition, attachment of significance of an idea, or comprehension?

The misapplication of psychology to education by educational theorists may be a partial explanation of how such has occurred or perhaps some educators sincerely *believe* feelings have a logical connection to knowing; but, there is scant (if any) evidence to even entertain such a possibility within science or mathematics.

4.2. **Believing.**

There are some philosophers who argue that in order to know 'X' one must believe 'X.' We have argued [149] such is not the case. We contend that the classical philosophical position that person M knows that thing p is true if and only if 1) M believes p; 2) p is true; and, 3) M is justified in believing that p is true is not necessary. We have argued in [149] that person M knows that thing p is true if and only if 1) p is true and 2) M is justified in opining that p is true. That p is true implies that there is something that can be known apart from the individual M. That M is justified in opining that p is true requires a method of argument from the justification, requires that the justification be understandable, and that there was an accepted schemata employed for providing said justification.

We hold that belief is not a necessary condition for obtaining mathematical truth for it seems that belief is a consequent rather than an antecedent for knowing something and might not be needed even after obtaining knowledge. For example, the wonderful example of Gabriel's Horn illustrates this well:
Let $U = \mathbb{R}$ for one dimension and $V = \mathbb{R} \times \mathbb{R}$ for two dimensions. Consider R to be the region bounded by $y = 0, x = 1$, $y = \frac{1}{x}$, to the right of $x = 1$. The area of R does not exist (is infinite) because $\int_1^\infty \left(\frac{1}{x}\right) dx$ does not exist. Yet, the volume of the resulting object, T, obtained by rotating R about the x-axis does exist since $\int_1^\infty \left(\pi \cdot \frac{1}{x^2}\right) dx$ exists. Hence the region R has no area but the region T (based on R) has volume π^3. Another way to explain it is that the region R has infinite area but the region T has finite volume. The student need not believe a result in order to deduce it or know it.

Another interesting example of belief doing harm to the learning of mathematics is with the class example of the real number $0.\overline{9}$. There are a plethora of students (and faculty who believe $0.\overline{9} < 1$, and when confronted with a proof that $0.\overline{9} = 1$ oft attempt to try to parse the language in order indirectly reject the Axiom of Completeness of $\mathbb{R}$. If one wishes to study a system that is similar to $\mathbb{R}$ but is not $\mathbb{R}$ (only use the axioms of order and fields for $\mathbb{R}$) that is fine but it is not Real Analysis! That is what so many fail to comprehend. They confuse $\mathbb{I}$ with $\mathbb{C}$; know not of the density of $\mathbb{Q}$ in $\mathbb{R}$ and the density of $\mathbb{I}$ in $\mathbb{R}$; etc. but believe they 'know' much about the structure and nature of $\mathbb{R}$. Misapplication of language and of concepts run amok to aid in retaining the false belief $0.\overline{9} \neq 1$. So, such people contend $0.\overline{9}$ is 'infinitesimally' close to one, 'approaches' one, is 'almost but not quite' close to one,or other such rubbish.

4.3. **Interpreting.**

To interpret has multiple meanings, *Oxford English Dictionary* reference, amongst which include "(1) to give one's own interpretation of (as in a musical composition, a landscape, etc.); rendition;" or "(2) to expound the meaning of; to render clear and explicit." The former definition of the term is what we will discuss in this paper for the latter definition we hold to be formulated also under opining and as such should be encouraged. Hence, if a person wishes to encourage interpreting to mean opining then we agree such is within the realm of behaviour or methods that should be encouraged in the mathematics classroom. However, when using the term in the second form above as one would interpret the "Star Spangled Banner" with different voice inflection or words than the F. S. Key poem or traditional music or a company would interpret "The Nutcracker" with a different setting, costumes, etc. than Tchaikovsky's original – we would state such is problematic and should not be a part of the mathematics educational experience.

Similar to believing is interpreting in the secondary sense from above – the cliché is 'interpreting results' or 'interpreting the math' – which is one of the new fads in mathematics education in the 21^{st} century. It seems to have come from educational constructivism and statistics education. 'Interpreters of the math' seem to follow a relativism paradigm such that the mathematical endeavour (a process, solution, theorem, counterexample, etc.) is open to interpretation; hence, is different depending on the interpreter. So, in a trivial set theoretic discussion let us say that $U = \mathbb{R}$ is the universe; 'interpreters of math' hold $\emptyset$ and $\{\}$ are the same; $\dfrac{1}{0} = \infty$ (which allows that $\dfrac{1}{\infty} = 0$ for them); $\infty - \infty = 0$; and so on. A faculty member stated that such nonsense is acceptable since the course taught was remedial; students are given college credit though the material is really remedial; the course taught was for non-majors (mathematics majors); or, the students taught would never be math majors. When is teaching something erroneously (and realising it is incorrect), wrong, false, or just-plain-wrong assisting students? How is that engaging in meaningful inquiry? How does that fulfill a professor's fiduciary responsibility to the university, state, nation, or to humanity?

Let us turn our attention back to of mathematics is with the classic example of the real number $0.\overline{9}$. Shockingly (to the author of this paper) some students (and some faculty) 'interpret' $0.\overline{9}$ getting closer and closer to 1; another student *still* insists that $0.\overline{9}$ is the closest real to 1 that is not 1 but is less than 1. The student who believes that $x = 0.\overline{9}$ is the closest real to 1 such that $x < 1$ refuses to accept a proof $0.\overline{9} = 1$. It is his right; but, he is wrong and with such a stubborn insistence on belief trumping truth is have a very difficult time in his mathematics course (needless to say). The failure to understand that a real number (including $0.\overline{9}$, π, $\sqrt{2}$, 4, etc.) is a point on the line and does not move, shift, or change. We have been in more than one didactic discussion about said. A faculty member (no longer at our university) stated that he had not taken Real Analysis and did not suffer from it; yet he taught Calculus, Differential Equations, and other courses and held the 'interpretation' previously noted along with 'interpreting' violations of theorems are not a problem so long as the calculations 'work' (agree with a solution manual), one can teach any course (even if one has no experience in the area or with the material) so long as one stays 'a day or two ahead.'

An interesting example of interpretation doing harm to a student's understanding of mathematics is with a discussion on $\displaystyle\int_2^\infty \frac{1}{x^2}dx$. The aforementioned faculty member who 'interprets' math and claims it is fine so long as calculations 'work' stated (paraphrasing):

(1) $\displaystyle\int_{1/2}^\infty \frac{1}{x^2}dx = \frac{-1}{x}\Big|_{1/2}^{\infty} = \frac{-1}{\infty} - \frac{-1}{1/2} = 2;$
(2) dividing by something so 'big' is 'essentially' zero so the previous is correct;
(3) $\displaystyle\frac{1}{\infty}$ is 'essentially' zero;
(4) when one puts $\displaystyle\int_{1/2}^\infty \frac{1}{x^2}dx$ into a calculator one gets 2 so all else does not matter (move on);
(5) in *Mathematica* $\displaystyle\int_{1/2}^\infty \frac{1}{x^2}dx$ 'is' 2 so arguing about how somebody gets an answer does not matter;
(6) OK, so theoretically one has to use L'Hôpital's rule for $\displaystyle\int_{1/2}^\infty \frac{1}{x^2}dx$ but that is just a technicality (or put another way he stated he is an 'applied guy')

Contrived but nonetheless elucidating is another example of the dance of 'interpretation of mathematical phenomena' that this author finds most amusing. Consider the question in arithmetic:

Let $U = \mathbb{R}$ Reduce fully $\dfrac{1\,6}{6\,4}$.

$$\frac{1\cancel{6}}{\cancel{6}4} =$$
$$\frac{1}{4}$$

Is such not a 'different' way of 'knowing;' a matter of 'interpretation;' and though not generalisable it still ends with ('accomplishes') the desired result? Some would argue the need for generalisability but not all mathematically valid arguments or techniques to solve something are generalisable (example: proving $\emptyset \subseteq A$ where A is a set given a well defined universe U).

When we consider Newtonian and Liebnitzian differential calculus the fact that one has a well defined smooth curve, C, over domain D lying in $\mathbb{R}^2$ and the slope of the tangent line to C at a point of D where the difference quotient is well defined is $\frac{dy}{dx}$ is not a different interpretation of calculus than given the well defined function $p : \mathbb{R} \longrightarrow \mathbb{R}$ such that $p(t)$ is a position at time t; so, $p'(t)$ is the velocity of the particle at time t when the difference quotient is well defined for the original function. Likewise Riemann, Riemann-Darboux, Riemann-Stieljes, and Lesbegue integrals are not different interpretations of the calculus but are different mathematical objects or constructs. In a vernacular sense in mathematics education there seems to be a misconstruing of different mathematical theories, different mathematical objects, and different mathematical processes with the idea of different views of reality and different interpretations of said (quite like in a literary sense). We contend that faculty need to understand that such a meaning of interpretation is incorrect (or debate why such is or is not openly, fervently, and honestly) for in so doing comprehend what makes such so – therefore enabling the professoriate to assist student learning (if we do not learn how can we direct others to learn?).

4.4. **Opining.**

To opine is something that we hold should be encouraged. When one considers the term, one finds that to opine means, *Oxford English Dictionary* reference, to "to hold an opinion; to hold as one's own opinion; to think; to suppose," or "to express an opinion; to say that one thinks (so and so)." The expression of the opinion must be supported by an argument and cannot include 'interpretation of a thing' (such as in art, music, literature, dance, etc.).

We should work diligently before the requirements of proof and full rigour (and beyond) to encourage students to opine about whatever is being discussed and studied. Everything under discussion should be open to critique – authentic inquiry – and every discussion should demand justification, support, explanation, and verisimilitude from faculty and students. The should not be one thing not open to discussion (including axioms – why do we assume such, which are agreed to and by whom, why, what happens without such, etc.).

To encourage a student to opine is to encourage him first to conjecture, hypothesise, and formulate ideas (an opinion). The formulation of the opinion is a key ingredient in understanding for knowledge cannot be obtained by cracking open a head and pouring in knowledge not can it be obtained by connecting a person to a data base and downloading facts.

4.5. **Knowing.**

To know something is the ultimate goal of any epistemological endeavour. So, we should try ourselves to know all we can and not 'short cut' or 'short change' the process of knowing (which is difficult to do for patience is rare); in the mathematics educational experience there is so much opportunity to know something that we hold academics should be encouraging student inquiry at every opportunity that arises. When we reflect on the definition of the word, 'know,' we find that the *Oxford English Dictionary* [168] offers as a reference many different uses of the term but they all seem to centre around sagacity; to have personal knowledge of a thing (to have 'figured it out' in the vernacular); to ascertain; to become well versed in something; to be skilled and versed; and, to understand. How does one understand that which he has not experienced? How does one comprehend a thing of which one has heard but not seen? How does one grasp a thing of which one has seen but not done?

Let $U = \mathbb{N}$. Assume the laws of logic, axioms of set theory (ZFC), Peano axioms, and axioms of the reals.
Construct a proof of $\nexists$ a prime $p \ni p$ is the greatest prime. Does not a person who can do this have a greater, better, firmer, and enhanced comprehension of the laws of logic, axioms of set theory (ZFC), Peano axioms, and the axioms of the reals than one who cannot do such? Is it not the case that a person who can do this also have a more refined, better, and superior understanding of the nature of the naturals and the properties of primes?

It is possible; but, unlikely that a person could prove such without understanding such. It is also possible and more likely than the previous case that there are those who could memorise a proof of such without understanding said. Such persons do exist and they exist as faculty and students; however, they are not authentic academics and will oft find themselves in terrible conundrums where their lack of knowing something becomes notable and leads to problems.

Let $U = \mathbb{R}$. Assume the laws of logic, axioms of set theory (ZFC), and axioms of the reals. Construct a proof of $x, y \in \mathbb{R} \ni x \geq y \; \forall \varepsilon > 0$, $x \leq \varepsilon + y \Longrightarrow x = y$. This problem possibly yields more difficulty for most than the previous question (in fact there are many who would attempt a counterexample to this claim). A person needs to know the laws of logic, axioms of set theory (ZFC), and the axioms of the reals along with some basic topology of $\mathbb{R}$. The person who can do this clearly has a modicum of understand of $\mathbb{R}$. A person who will attempt this; who opines it is true but cannot compleat the proof; and, who finds the question a challenge (positive exercise worthy of discussion) outshines the person who 'Googles it,' looks it up in a book, or asks an 'expert' for the answer.

We claim that a person who attempts to do a proof but fails to compleat a proof is a better student, academic, faculty member, researcher, or teacher than a person who searches the internet for a proof, looks up a proof in a book, pushes some buttons on a calculator for an example or two, or repeats clichéd phrases about the idea but offers no insight.

5. To Opine or To Know

Why is it accepted that for a person to become a great athlete he must practice for years; but, such seems not to be the case for some areas of academics? Raw talent is not enough for a Michael Jordan in basketball, a Roger Federer in tennis, an Albert Pujhols in baseball, or a Michael Phelps in swimming – all had to practice relentlessly and after becoming great had to practice in order to maintain their position as great in their sport lest others rise to challenge them and possibly pass them (which eventually does occur for anyone). By the same token such is accepted that for a person to become a great artist, musician, sculptor, writer, etc. – he must practice or train for years and must continue after becoming great so as not to become stale, hackneyed, or mediocre.

There are many creative athletes, musicians, writers, and entertainers who are not great but are capable and productive; such persons, we hold, have to practice *even more* than a great athlete, musician, etc. in order to continue to be competent or proficient. Should we not be encouraging academic pursuits in at least a similar manner? Should not it be the case that a better student, researcher, or teacher practices continually his craft and attempts to learn *more* than what is known presently? Thus, we should be encouraging thinking for one's self, understanding, creating, opining, conjecturing, critiquing, and knowing rather than pushing buttons for internet searches or on a calculator; or, memorising clichéd phrases about the idea (or whole proofs someone else has written on a board in a class or in a book). We should be accentuating authentic learning, authentic thought, and an internal locus of control so that students acclimate toward *wanting to do* something instead of waiting for someone else to do it for them, witnessing something, hearing about something, or reading about something.

Let us agree we mean that person A understands thing B if and only if he is 1) able to comprehend it; to apprehend the meaning of or import of, 2) to be expert with or at by practice, 3) to apprehend clearly the character or nature of a thing, 4) to have knowledge of to know or to learn by information received, 5) to be capable of judging with knowledge, or, 6) the faculty of comprehending or reasoning. Such a definition complements Bloom's taxonomy and focuses the discussion on the idea of thinking. A person can only comprehend when that person is thinking so thinking is an antecedent to understanding – no thought, no understanding. To encourage students to seek to be able to opine or to know should be the goal of every learning experience created by a faculty member – each exercise, lecture, assignment, quiz, test, or project should have tasks which create the opportunity for a student to creatively think – meaning to opine or to know. To know requires the highest form of mathematical reasoning - proof whilst to opine requires much thought and reflection. To opine one needs to be able to image, to create, to ponder, reflect, and to wonder. Such is not required to interpret, believe, or feel.

We submit that in a classroom the goal of assisting students in opining or knowing is supported by creating an atmosphere or culture which embodies a **pronounced, overt, and clear celebration** of authentic intellectual effort by encouraging attempts a student makes in trying to solve a problem, create a proof, argue a point, forge an example, produce a counter-arguments, or construct a counterexample. A professor should inspire inquiry and comprehension; in order that such is encouraged, there should be a celebration when one makes a mistake when seeking to explain, understand, opine, or prove. Indeed, we put forward that a professor should be willing to make mistakes (oft consciously) to model the act of learning from mistakes; encourage students to identify mistakes; and, demonstrate that making a mistake is a natural part of the act of inquiry. Ideally we should be influencing the discussion in a classroom to point toward clear, succinct, and detached argumentation. We should discourage disconnected incidental schema to 'get a solution.' Such method goes against prevailing norms and archetypes - the critical theory, radical constructivist, and post-modernist paradigm present in predominant popular educational theories.

In order to encourage students to opine or know and not sit idly by and copy perfect notes from the board we should seek to introduce classical Aristotelian logic into the mathematics curriculum in the elementary or secondary curriculum. We should consider also introducing more of the axiomatic method into the secondary curriculum because the use of axiomatics, of encouraging opining, proving, or disproving creates a condition which also encourages honing an internal locus of control. If a student can see that he, himself, can have an impact on his learning; that it is not a parent, sibling, friend, or teacher who gets that individual to learn—footnoteThey may assist the student to learn – especially the parent – but they cannot *make* that individual learn.; and, that once he truly learns and understands such is not a flimsy perch on which the idea falls and is forgotten (as is the case with most if not all of that which one is told).

An illustrative example in the mathematics curriculum of encouraging opining which leads to later knowing is in basic Calculus I. Classical American pedagogy begins a discussion of calculus with the naïve discussion of the idea of limit. It is introduced graphically, algebraically, and definitionally. We discuss the intuitive idea of a left limit, a right limit, and the need for both to be real and to be the same. Students are encouraged (required) to justify answers to $\lim\limits_{x \longrightarrow 2} \dfrac{x^2 - 4}{x - 2}$ beyond writing a number down (hopefully). They do not prove said and this method of leads to some erroneous ideas (like x moves or for some distance in the x-direction there is a distance in the y-direction, etc. which is opposite the actual meaning) that have to be clarified through discussion, exemplars, and questions.

Another example is from set Theory: the use of either Euler or Venn Diagrammes. In fact any graphical representation of a concept used in order to better understand concepts, construct examples or counter-examples, investigate ideas, etc. is a fine example of what we mean by encouraging opining because such leads often to confidence and an internal locus of control because the individual produces said rather than being given said or watching said (on a computer screen or calculator).

Opining helps one choose whether to prove or disprove a claim, encourages experimentation, evaluation, reflection, and pausing before tackling a claim. We submit such a pause is an important part of the mathematician's process toward discovery, invention, and truth; therefore, the professoriate should train students of mathematics to do such (which is especially difficult with 21^{st} century diversion such as cell phones, the internet, etc. Opining leads toward understanding which should inevitably lead toward rigour because to opine one must offer a sanguine argument as to why something is posited and it demands reason to the scaffold on which the argument rests.

6. Summary & Conclusion

If one subscribes to the radical relativist position of feeling, believing, and interpreting as the core activity in mathematics (or anything in the academy) as forwarded by Taylor, Noddings, Powell, Buerk, D'Ambrosio, et al. then the Calculus, Analysis, Topology, Algebra, etc. melt away into an epistemological purgatory on par with Anthropology, Sociology, etc. or worse Alchemy, Sorcery, Magic, etc. To exile mathematical thought to such a realm is not only a mistake but a crime for it does an injustice to the subject, tosses aside the marvellous discoveries and inventions of the past, and teaches nothing to the students of today and tomorrow.

So, we proposed a taxonomy of increasing credibility, reliability, and realistic verisimilitude that sets feeling as inferior, rises to believing, then moves to interpreting, is succeeded by opining, and is crowned by knowing. We have a mathematics which has universals and existentials; application and theory; areas of agreement and areas of new discoveries and different systems; whilst there is a plethora of principles yet to be discovered, created, or invented. But these aspects of mathematical thought and inquiry are quite different than the social sciences, arts, humanities, etc. which hinge on subjectivity, interpretation, and plausibility along with appeals to the masses for approval (belief or emotive acceptance) rather than. Mathematical truth, thought, and ideas appeal to logic - - 'cold,' 'hard' deduction and objectivity. Mathematics cannot be slanted to a political position - - there is not a marxist, capitalist, or anarchist mathematics. There is not a female mathematics and male mathematics at war with one another. We do not have a mathematics for Catholics, a mathematics for Jewish people, one for atheists and another for Buddhists. A Hindu person studies the same mathematics as a marxist, capitalist, or anarchist. We do not have a Muslim mathematics - there is not an interpretive licence to conclude something based on subjective *a priori* biases, beliefs, or feelings. To contend otherwise is to appeal to naïvety, to posit that which is not supportable by logic, or to play semantic or rhetorical games.

The philosophy of mathematics is unalterably and steadfastly tied to the notion of being correct, of bounding error (when error exists), and of being able to note when we are wrong. The foundation of positive scepticism and objectivism is fundamental to mathematical thought and inquiry. We can understand; but, must also get it right.

We can deduce analytic truths. We can *do* mathematics and can explain *why.* Phenomenology, hermeneutics, radical constructivism, and their ilk see mathematics primarily as a social construct, as a product of culture, subject to interpretation, personal whims, politics, and emotion. The position put forward by Taylor, Gustein, Joseph, or the Feel Mathematics Institute that mathematics is constrained by the fashions of the social group performing it or by the needs of the society financing it is seemingly without merit given the permanence of mathematics.

What binds and supports mathematics is a search for truth, a search for what works, and a search for what is applicable within the constraints of the demand for justification, clarification ,and proof. It is not the ends, but the means which matter the most - - the process at deriving an answer, the progression to the application, and the method of generalisation. These procedures demand more than mere speculative ideas; they demand reasoned and sanguine justification. Furthermore, we put forward that 'positive scepticism' is meant to mean there is an explicit or implicit demand for objectivity; a topic should be viewed with a healthy dose of doubt; the mathematician must remain open to being wrong; and, the mathematician must not argue from an *a priori* perception.

For a student of mathematics it is critical that he be encouraged to make mistakes, to take risks, and to opine. We put forward that **one learns from one's mistakes not from one's successes!** A key point for students is that after one assumes an axiom system, definitions are pliable; but, methods of proof and the fundamental methods of reasoning are not. Erroneous or fallacious arguments are not 'another way of knowing' but are simply wrong and can be corrected. Therefore we must seek to encourage students to understand said and strive to opine or to know. We (each and every human) can learn from the errors *so long as we understand that the errors are errors* and there is not a requirement that a human be perfect. Hence, the nature of the process of the inquiry (where justification must be supplied), the analysis, or the critique is **the essence of the nature of mathematical enterprise: knowledge and inquiry are inseparable and as such must be actively pursued, refined, and engaged.** The demand that one opines or knows is also an axiological position for it is a value-judgement that inquiry into the nature of mathematics is positive; executable, justifiable, and knowable.

P. R. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of the methods employed in teaching students to do mathematics with the fusion method and provides incite into the foundation of the philosophy of positive scepticism. It states, "I see, I forget; I hear, I remember; I do, I understand." It is in that spirit that a core point of the argument presented in the paper is that the strength of an argument cannot be dismissed with 'pop' cultural drivel that centres on 'emotional mathematics,"belief mathematics,' or 'interpretive mathematics.' This paper proposes a philosophical position that deviates from the disconnected incidental schema (usually termed phenomenological, hermeneutical, or constructivist schema) and the interpretive schema that the nature of mathematical thought is one that is centred on constructive scepticism which acknowledges conditional truth **can be** deduced, recognises the pragmatic need for models and approximation, and suggests that such is based on the experience of doing rather than witnessing.

We further hold to the position that inquiry-based learning (IBL), the Moore method, or a modified Moore method leads to an authentic understanding of mathematics: encouragement of thought, encouragement of deliberation, encouragement of contemplation, and encouragement of a healthy dose of scepticism so that one does not wander too far into a position of subservience, 'give-me-the-answer'-ism, or a position of arrogance, 'know-it-all'-ism. I (the author of this paper) become more convinced each day that R. L. Moore was right - in the competition between Sophistry and Socraticism, Socraticism is correct, authentic, and should be preferred. Unfortunately, Sophistry is ascendant in the 21^{st} century from elementary through post-graduate study. It prevails in many a classroom because:
1) it is easier for the instructor–no arguments with students, parents, or administrators; complaints of things being 'hard' are almost non-existent if one employs sophistry and the instructor does not have to "think as hard;"
2) it is easier for the student–he does not have to "think as hard" (or think at all), she can "feel good," it can have its self-esteem 'boosted;'
3) it is easier for the institution– standardisation can be employed (which seems to be a goal at many institutions); students retained ('retention' and 'assessment' seem to be a 21^{st} 'buzz words'), graduation rates increase,[10] and accrediting agencies are mollified (such as the Middle States Association of Colleges and Schools ('Middle States'), the Southern Association of Colleges

[10]Not because of higher achievement; therefore, a sophist success.

and Schools ('SACS'), or National Council for the Accreditation of Teacher Education ('NCATE')); and 4) there are no errors, no wrong methods, no deductive principles of logic to learn for everything is beautiful, everything is wonderful, everything is happy, and everything is relative (indeed all is tosh in such a scheme).

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