Development of ideas in a GeoGebra – aided mathematics instruction

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With GeoGebra introduced into mathematics instruction the teaching/learning process is not improved in terms of speed and quality only. Mathematical concepts, rules and procedures must be adjusted to the new environment. On the other hand, characteristics of the computer and the educational software in use must be thoroughly examined and a successful teacher has to use their advantages and avoid their flaws in order to maximize the efficiency of the instruction process. This paper deals with development of ideas when the concept of ellipse is being built with GeoGebra. We shift from dynamic applets that show the ellipse characteristics to applets that are built by students and teacher together. The old applets were perceived by students as dynamic representations of the ellipse and its characteristics. They offered a very low level of freedom for students to explore the material on their own. This meant the students were expected only to receive and learn information about the ellipse. On the contrary, the new applets were designed in a way that the genesis of the ellipse and its relations to other objects were clearly visible. Besides, the process of building new applets enables students to discover independently the ellipse characteristics. Thus, the students become active participants in the process of the knowledge building. In other words, we use GeoGebra to move from static exploration to dynamic exploration. This process could easily be accomplished in Dynamic and Interactive Mathematics Learning Environments (DIMLE). All the information and procedures meant to be learnt by students were presented within the "*I – you – we*" concept that was very easy to implement in such an environment.

Introduction

Modern mathematics instruction relies on computer and educational software usage. GeoGebra is a software that combines DGS (Dynamic Geometry System) features which enables us to create Dynamic and Interactive Mathematics Learning Environments (DIMLE) – a term coined by Martinovic and Karadag (2012). That is why it is ideal to carry out plane analytic geometry in a computer aided instruction. Thanks to its features, simple usage and availability, teachers and students are enthusiastic to use GeoGebra in analytic geometry instruction. Since 2008, we have been using GeoGebra as a means to organize and carry out

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analytic geometry instruction in mechanical and gymnasium classes of "Nikola Tesla" Engineering School in Leposavić, Serbia. Though initial experiences were positive, we learned that introducing GeoGebra alone into a classroom does not mean better classes necessary. Better classes and results require a meaningful usage of GeoGebra and subject material adaptation to it as well as teachers' devotion to build creative applets and encourage students to work creatively in such an environment. In the paper we present a gradual development of ideas in building dynamic applets we used to describe, explore and "discover" characteristics of the ellipse.

Initial ideas

Plane analytic geometry is taught in third grades of Serbian secondary schools. That means the students learning the plane analytic geometry are of age 17. The majority of the material is taught according to the workbooks (Bogoslavov, 1995; Kečkić, 1990). In majority of our schools, the instruction process still relies on blackboard/chalk and notebook/pencil usage. Having in mind all technologies teachers and students have at their hand, the way we used to teach our students can be seen as disadvantageous.

Usually, we define the ellipse as follows:

The ellipse is a set of points M in a plane such that the sum of distances from point M to two steady points F_1 and F_2 is constant.

Each of points F_1 and F_2 is called a focus of the ellipse. According to the definition, we draw a sketch and derive the equation of the ellipse. All the sketches and calculations needed are done on the blackboard. These can consume significant amounts of time. They can also draw student's attention off the subject material taught. Besides, the sketches are static and students can see only a single stage of ellipse construction at any time.

On the other hand, GeoGebra usage to teach and learn analytic geometry seems to be the easiest way to start digitalization of mathematics instruction. However, our experience suggests that one should pay attention not only to the subject material chosen for such an instruction, but also to the way the new technology will be implemented into the instruction process. To clarify this, we will show the way we taught and explored ellipse.

Using GeoGebra, we built a dynamic worksheet *ellipseconstruction.ggb* that showed the characteristic of the points that form the ellipse. Both, foci and the point *M* mentioned in the definition are shown in this applet. Sum of the distances from point *M* to the focus F_1 and from *M* to F_2 is constant regardless of the position of the point. The values of both distances and their sum are visible in the applet. Students were told to turn on the trace of the point M and experiment with its positions. Thus they come visually to the shape of the ellipse. Then, we derived the ellipse equation out of the applet. However, the way we derived the equation was not much different than the way we did it without GeoGebra applets. Later, we built another applet – *ellipse.ggb* wherefrom semi minor and semi major axis and the foci of the ellipse can be determined as well as their impact on shape and position of the ellipse. The details of the applets are shown in figures 1 and 2 respectively.
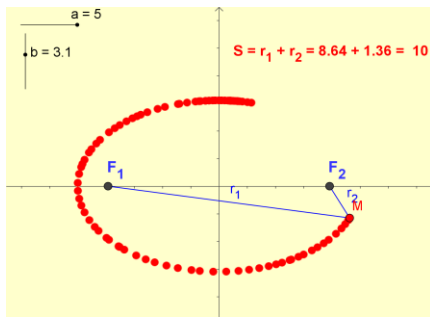
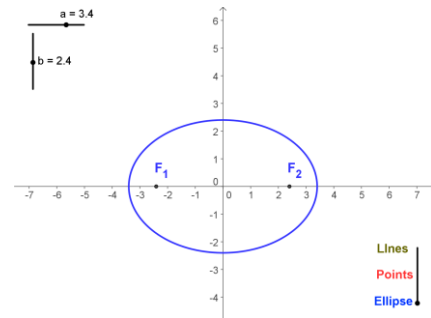**Figure 1:** A detail of the applet representing the ellipse definition

Figure 2: The second applet was used to explore the ellipse properties

Thus, we brought dynamics into mathematics classes and eased understanding the subject material. We solved mathematical problems in two ways: on paper and with computer.

Students have done tests on analytic geometry and results they achieved were compared with results of the same tests done by students that learned analytic geometry without GeoGebra and computers. Results Ljajko, Mihajlović Pavličić (2010) suggest that the students that used computers and the applets showed better results in problem solving and homework but they lacked deeper understanding of mathematical concepts. This resulted in difficulties at building and understanding of more complex concepts and determining connections among them.

Here we demonstrate the way we used *ellipse.ggb* applet in solving the following problem:

Example 1:

Prove that point $M = \left(-4, \dfrac{12}{5}\right)$ *lies on the ellipse* $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$, *determine equations of lines and length of segments* $F_1 M$ *and* $F_2 M$ *as well as the angle between them.*

First of all, we set the sliders *a* and *b* in the applet at values *a = 5, b = 4* wherewith we obtain the given ellipse. Then we enter coordinates of the point *M*. The figure looks like the point *M* lies on the ellipse but it can be inaccurate. That is why we need a more accurate tool to check whether a point belongs to a set or not. For that reason we use the option *Relation between two objects* to bring a clear decision, figure 3. At the next stage, we set straight lines through points M and F_1 and M and F_2 respectively. We read their equations in algebra window, figure 4.
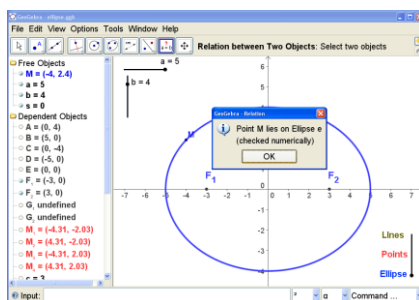

Figure 3: Using a dynamic worksheet to determine whether a point lies on the ellipse or not
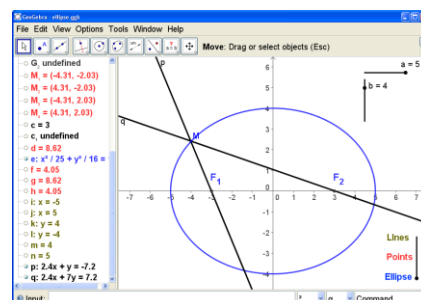

Figure 4: Lines through the point M and the foci of the ellipse

We determine the lengths of segments F_1M and F_2M using option *Distance or Length* which enables us to read them directly from algebra window, figure 5. Finally, by using the *Angle* feature, we determine the value of the angle between the lines, figure 6.
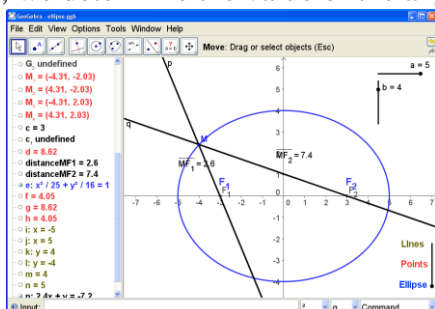

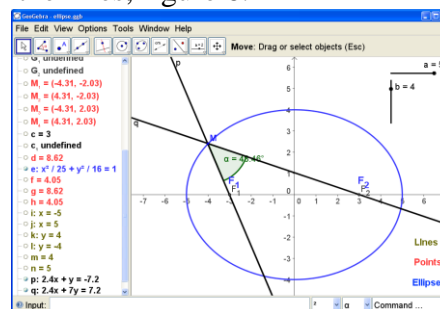**Figure 5**: Measuring the distance between two points done in the applet

Figure 6: The Angle feature used to determine the angle between two lines

Advantages and flaws

Introducing GeoGebra into mathematics instruction is not an easy task for teachers, but it's appropriate usage improves the instruction quality and helps students achieve better results in learning mathematics. Dynamic applets we used to present the ellipse and its characteristics improved our mathematics instruction with regard in several aspects:

- We spent less time drawing sketches and doing calculations; thus, more time could be used for learning and exploring characteristics of the ellipse.
- The applets were dynamic which means students had a possibility of re-examining them if something was not clear.

Anyway, the first applet, *ellipseconstruction.ggb* had quite limited possibilities concerning development of students' creativity. We used it only to present basic features and elements of the ellipse. On the other hand, the second one, *ellipse.ggb*, enables us examine most of the characteristics of the ellipse in several ways. Students used it when solving problems and, while doing so, they learnt mathematical concepts related to the ellipse. Both applets improved the instruction process by enabling us consume less time on drawing sketches. The second applet reduced time needed for calculations. All these also caused students have better results at tests.

In the classroom, we noticed a flaw in the first applet. In fact, the way we built it determined the only way we could use it – it could be no more than a dynamic sketch of the ellipse. It made us perform the instruction in a manner that was very similar to the "old fashioned way" – still, most of the instruction was done by using blackboard/chalk technology. We also used to explain the concepts to the students and expected them to understand the concepts.

Shift of ideas

During the first year of GeoGebra usage we made a plain mistake instructing analytic geometry in the "old fashioned way" but still having dynamic applets at hand. After a few advices of Croatian colleagues, we started changing our approach. We built a new applet – *equtionellipse.ggb* where we described the basic feature of points that make an ellipse. To do so, we used the "pins – and – string – method". Students examine the applet on their own or in cooperation with the teacher and that was planned to be the way they "discover" the equation of the ellipse.

First, they experiment with different ratios of segments r_1 and r_2 in the applet with a sole condition that their sum is constant (and equal to the length of the string - *2a*). This can be done by moving point *T* along the slider at the top of the applet. On that occasion, they observe circles with variable radiuses (their values are exactly r_1 and r_2) and their intersection points. While doing so, the students can notice that the intersection points of the circles have the same feature described in the definition and take positions that, taken together, resemble the ellipse. This intuitive conclusion can be proved. For instance, one can switch the traces of the intersection points on and then move the point *T*. The traces of the intersection points lie on the ellipse. If we want a continuous line that describes trajectories of the intersection points we can do it using option *Locus: Locus [T_1, T]* and *Locus [T_2, T]*. In this way we come, at least visually, to an ellipse.

Finally, using *Ellipse* tool, we can construct the ellipse with foci at F_1 and F_2 that passes through point T_1. At the following step, we turn the students' attention to points where the ellipse intersects coordinate axes. At the beginning, it is sufficient to notice that $x-$coordinate of the intersection point of the ellipse with the $x-$axis equals *a*, which is $semi-major$ axis of the ellipse. Knowing that, and half of the distance between the foci, $c = \dfrac{1}{2}|F_1 F_2|$, we can define $semi-minor$ axis *b*. One can adjust properties of the ellipse (right click on the ellipse: *properties – show label – value* and *properties – algebra – equation* $\dfrac{(x-m)^2}{a^2} + \dfrac{(y-n)^2}{b^2} = 1$) in the way that it's equation depends on $semi-major$ and $semi-minor$ axis. Details of the applet are shown in the figures 7 and 8.
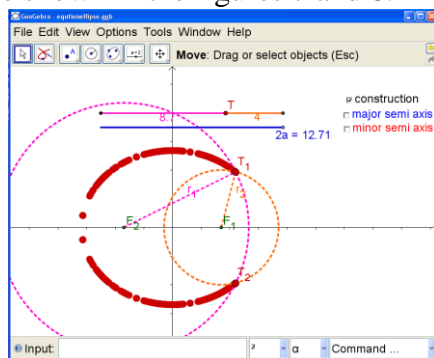

Figure 7: Trajectories of the points described in the pins – and – string method „resemble" the ellipse
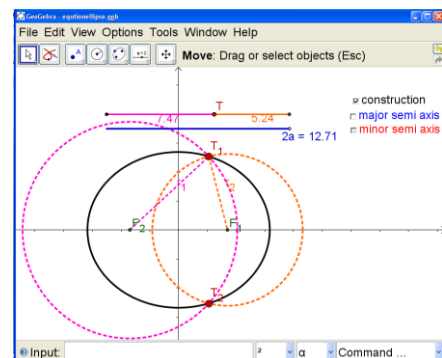

Figure 8: Using the new applet, students come to the equation of the ellipse

Ways of concept – building

The amount and the quality of the knowledge incorporated into students' cognitive structure depends on the way the subject is represented to students and the ways the teacher, students, the subject material and the applets are interrelated in the instruction process. Since we already showed in detail the ways students used the applets, we are going to describe here the ways the knowledge was built.

Having in mind technical constraints we are facing (a computer is being used by two students), we decided to implement "*I – you – we*" concept described in Ulm, (2002). It consists of three stages:

(1) "*I*" – students explore the dynamic worksheets individually. They come to certain conclusions about the material explored. Those conclusions are not necessary correct, but they reflect the ways students understood the material and its representation in the applet. Their conclusions also reveal points where students are more likely to make mistakes in incorporating the new knowledge.

(2) "*You*" – work in pairs and/or small groups. Students communicate among themselves, compare results they have come to, correct possible misunderstandings and synchronize the ways they understand the concept taught.

(3) "*We*" – in the class as a whole, the teacher compares the results different groups have come to, accents the correct ones and turns students' attention to incorrect conclusions and results.

Since most of the modern software have different user forums and internet pages for cooperation and user communication, students can find thousands of users with interest in similar fields. There, they can refine their understanding of the concepts they learnt in the class. This means that the third phase of the "*I – you – we*" concept can easily be done in much larger groups than a class.

Effects of the new applets usage

After introducing the new applets we realized that our goal – shifting the students' position from a passive recipient to an independent explorer – was achieved. Students have to be far more active in order to use the new applet meaningfully. It is not just a mere presentation of properties and elements of an ellipse, but a dynamic process that represents the genesis of the ellipse. In this applet, properties and elements of the ellipse have to be discovered in an exploring activity and not simply told to students. Such a way of learning requires students to take active part in it which leads to acquisition of knowledge that is of higher quality and lasts longer. This makes better conditions to apply the *ellipse.ggb* applet into problem solving and induces better understanding of the ellipse features.

 With the use of the new – built applet we made a shift towards an environment that encourages students' creative activity. New technology introduced into the classroom in this way is not just a sheer replacement of the blackboard/chalk. It offers students an active exploration of the subject material and a possibility to bring conclusions independently. The role of teacher is to direct and monitor students' work. Mathematical concepts and procedures learnt in this way are longer lasting and better incorporated into students' cognitive structure which makes them easier to apply. Such an instruction does not work in one direction only. Here, students are not only passive recipients of information meant to be remembered but active explorers and creators.

The experiences we earned while teaching/learning ellipse in the two different ways helped us identify fruitful ways of creating dynamic applets and organizing instruction in a computer environment. In other words, we have built gradually a Dynamic and Interactive Mathematics Learning Environment. This means that a mere introduction of new technologies into the classroom is not sufficient to achieve better results. To achieve better results in building DIMLE, teachers have to pay attention on several things:

- A careful selection of the subject material and appropriate software would give us better chances to achieve the goal – building a creative DIMLE – an environment that encourages students to explore the material on their own.

- Teachers should create dynamic worksheets – applets that represent the most important features of the material taught.
- The dynamic worksheets have to be built in such a way to allow students to explore them independently and "discover" mathematics with a minimal help of teachers.

Still, we identified a significant danger that can be easily overseen while using digital technologies in mathematics instruction. Namely, all the efforts described here can be futile if one neglects symbolic approach to mathematics instruction. For this reason, we should insist on usage of educational software that allows a symbolic approach to mathematics as well as precise and fast calculations and dynamic sketches.

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