DEPARTMENT OF THE INTERIOR
BUREAU OF EDUCATION.

BULLETIN, 1920, No. 1

THE PROBLEM OF MATHEMATICS IN SECONDARY EDUCATION

A REPORT OF THE COMMISSION ON THE REORGANIZATION OF SECONDARY EDUCATION, APPOINTED BY THE NATIONAL EDUCATION ASSOCIATION

WASHINGTON
GOVERNMENT PRINTING OFFICE
1920

REPORTS OF THE COMMISSION ON THE REORGANIZATION OF SECONDARY EDUCATION.

The following reports of the commission have been issued as bulletins of the United States Bureau of Education and may be procured from the Superintendent of Documents, Government Printing Office, Washington, D. C., at the prices stated. Remittance should be made in coin or money order. Other reports of the commission are in preparation.

1915, No. 23. The Teaching of Community Civics. 10 cents.
1916, No. 28. The Social Studies in Secondary Education. 10 cents.
1917, No. 2. Reorganization of English in Secondary Schools. 20 cents.
1917, No. 49. Music in Secondary Schools. 5 cents.
1917, No. 50. Physical Education in Secondary Schools. 5 cents.
1917, No. 51. Moral Values in Secondary Education. 5 cents.
1918, No. 19. Vocational Guidance in Secondary Schools. 5 cents.
1918, No. 35. Cardinal Principles of Secondary Education. 5 cents.
1919, No. 55. Business Education in Secondary Schools. 10 cents.
1920, No. 1. The Problem of Mathematics in Secondary Education. 5 cents.

COMMITTEE ON THE PROBLEM OF MATHEMATICS IN SECONDARY EDUCATION.

William H. Kilpatrick, *chairman*, professor of education, Teachers College, Columbia University, New York, N. Y.
Fred R. Hunter, superintendent of schools, Oakland, Calif.
Franklin W. Johnson, principal University High School, University of Chicago, Chicago, Ill.
J. H. Minnick, assistant professor of educational methods, University of Pennsylvania, Philadelphia, Pa.
Raleigh Shorling, Lincoln School, 646 Park Avenue, New York, N. Y.
J. C. Stone, head of department of mathematics, State Normal School, Montclair, N. J.
Milo H. Stuart, principal Technical High School, Indianapolis, Ind.
J. H. Withers, superintendent of schools, St. Louis, Mo.

THE REVIEWING COMMITTEE OF THE COMMISSION ON THE REORGANIZATION OF SECONDARY EDUCATION.

(The Reviewing Committee consists of 26 members, of whom 16 are chairmen of committees and 10 are members at large.)

Chairman of the Commission and of the Reviewing Committee:
Clarence D. Kingsley, State high-school supervisor, Boston, Mass.

Members at large:
Hon. P. P. Claxton, United States Commissioner of Education, Washington, D. C.
Thomas H. Briggs, associate professor of education, Teachers College, Columbia University, New York, N. Y.

Members at large—Continued.

Alexander Inglis, assistant professor of education, in charge of secondary education, Harvard University, Cambridge, Mass.
Henry Neumann, Ethical Culture School, New York, N. Y.
William Orr, senior educational secretary, international Y. M. C. A. committee, 347 Madison Avenue, New York, N. Y.
William B. Owen, principal Chicago Normal College, Chicago, Ill.
Edward O. Sisson, president University of Montana, Missoula, Mont.
Joseph S. Stewart, professor of secondary education, University of Georgia, Athens, Ga.
Milo H. Stuart, principal Technical High School, Indianapolis, Ind.
H. L. Terry, State high-school supervisor, Madison, Wis.

Chairmen of Committees:

Administration of Secondary Education—Charles H. Johnston, professor of secondary education, University of Illinois, Urbana, Ill.[1]
Agriculture—A. V. Storm, professor of agricultural education, University of Minnesota, St. Paul, Minn.
Art Education—Royal B. Farnum, president, Mechanics Institute, Rochester, N. Y.
Articulation of High School and College—Clarence D. Kingsley, State high-school supervisor, Boston, Mass.
Business Education—Cheesman A. Herrick, president, Girard College, Philadelphia, Pa.
Classical Languages—Walter Eugene Foster, Stuyvesant High School, New York, N. Y.
English—James Fleming Hosic, Chicago Normal College, Chicago, Ill.
Household Arts—Mrs. Henrietta Calvin, United States Bureau of Education, Washington, D. C.
Industrial Arts—Wilson H. Henderson, extension division, University of Wisconsin, Milwaukee, Wis. (now Major, Sanitary Corps, War Department, U. S. A.).
Mathematics—William Heard Kilpatrick, associate professor of education, Teachers College, Columbia University, New York, N. Y.
Modern Languages—Edward Manley, Englewood High School, Chicago, Ill.
Music—Will Earhart, director of music, Pittsburgh, Pa.
Physical Education—James H. McCurdy, director of normal courses of physical education, International Y. M. C. A. College, Springfield, Mass.
Sciences—Otis W. Caldwell, director, Lincoln School, and professor of education, Teachers College, Columbia University, New York, N. Y.
Social Studies—Thomas Jesse Jones, United States Bureau of Education, Washington, D. C.
Vocational Guidance—Frank M. Leavitt, associate superintendent of schools, Pittsburgh, Pa.

[1] Deceased Sept. 4, 1917.

CONTENTS.

LETTER OF TRANSMITTAL.

DEPARTMENT OF THE INTERIOR,
BUREAU OF EDUCATION,
Washington, October 11, 1919.

SIR: One of the committees of the Commission on the Reorganization of Secondary Education, appointed by the National Education Association, and several of whose reports this bureau has already published in the form of bulletins, undertook the study of mathematics in the high schools. As stated by this committee in the introduction to this report and by the chairman of the commission in the preface, the committee found itself unable to make final recommendations in regard to the reconstruction of the courses of study in this subject in the high schools. The committee has, therefore, confined its work to a preliminary report, presenting an analysis of the subject, and raising certain fundamental questions which must be answered before the reconstruction desired can be undertaken intelligently and with any certainty of satisfactory success.

I am transmitting this preliminary report for publication as a bulletin of the Bureau of Education, in order that in this form it may be accessible to students of education, teachers of mathematics, and directors of mathematics teaching in high schools. It is expected that it will give rise to such discussion and experimenting as will enable other committees to carry forward the work of the point of definite reconstruction of courses of study in this subject for the several classes of high-school pupils.

Respectfully submitted.

P. P. CLAXTON,
Commissioner.

The SECRETARY OF THE INTERIOR.

PREFACE.

The Commission on the Reorganization of Secondary Education finds itself confronted with problems of great difficulty in recommending a reorganization of the mathematical studies of the secondary school. Antecedent to new courses, there should be an agreement among psychologists and educators such as has not yet been reached. It seems, therefore, that the best service that the commission can at this time render is to present an analysis of the situation. This report, therefore, is submitted primarily for the purpose of stimulating discussion. It is hoped that the practical suggestions will also serve to direct experimentation in planning new courses for secondary school students of the various types here recognized.

CLARENCE D. KINGSLEY,
Chairman of the commission.

THE PROBLEM OF MATHEMATICS IN SECONDARY EDUCATION.

I. INTRODUCTION.

Few subjects taught in the secondary school elicit more contradictory statements of view than does mathematics. What should be taught, how much of it, to whom, how, and why, are matters of disagreement. There is every variety of position. A conservative group would keep substantially unchanged the customary content and division into courses, and find the hope of improvement in a more adequate preparation of teachers. To this limited reform an increasing number object, with little agreement, however, among themselves. Amid the conflict of opinions the committee on the problem of mathematics in secondary education believes that a reconsideration of the whole question is desirable.

To present the finished details of a working plan would have been most gratifying to the committee, but this has been judged impossible. The situation seems to force the limitation. To carry weight, such a detailed plan would have to be based upon a wider range of experiment than in fact exists. Only recently has there been serious effort to consider the problem of the proper content and arrangement of the courses in secondary mathematics. The pertinent experiments available for study do not as yet present a variety of type and testing sufficient to establish the necessary conclusions. Within the time allotment available to the committee there seemed then only the choice between no report and an admittedly preliminary report. The committee has chosen the latter alternative, and proposes to lay before the American educational public (1) some of the considerations that demand a fresh study of the problems involved, (2) some of the factors that bear upon the solution of the problem, and (3) certain tentative suggestions for experimentation to develop new and better courses.[1] It is but fair to say that few of the specific suggestions made are in fact new, many being already somewhere actually in practice.

II. THE DEMAND FOR AN INQUIRY.

An inquiry into the advisability of reorganizing and reconstituting secondary mathematics is demanded from a variety of considerations.

[1] It is gratifying to note that the Mathematical Association of America is pushing a program of study and experimentation along lines quite similar to those here discussed.

1. It is being insisted as never before that each subject and each item in the subject justify itself; or, negatively, that no subject or item be retained in any curriculum unless its value, viewed in relation to other topics and to time involved, can be made reasonably probable. No longer should the force of tradition shield any subject from this scrutiny. A better insight into the conditions of social welfare, and the many changes among these conditions, alike make inherently probable a different emphasis upon materials in the curriculum, if not a different selection of actual subject matter. This calls for a review and revaluation, in particular, of all our older studies, mathematics not least.

2. Moreover, a growing science of education has come to place appreciably different values upon certain psychological factors involved, chief among which is that relating to "mental discipline." No one inclusive formulation of the older position can be asserted, yet on the whole there was acceptance of the "faculty" psychology with an uncritical belief in the possibility of a good-for-all training of the several "faculties." To the extremist of this school the "faculty of reasoning," for example, could be trained on any material where reasoning was involved (the more evident the reasoning, the better the training), and any facility of reasoning gained in that particular activity, could, it was thought, be accordingly directed at will with little loss of effectiveness to any other situation where good reasoning was desired. In probably no study did this older doctrine of "mental discipline" find larger scope than in mathematics, in arithmetic to an appreciable extent, more in algebra, most of all in geometry.

With the scientific scrutiny of the conditions under which "transfer" of training takes place, the inquiry grows continually more insistent as to whether our mathematical courses should continue unchanged, now that so much of their older justification has been modified. Possibly both purpose and content need to be changed.

3. Yet another demand for reconstruction is found in the now generally accepted belief that not all high-school pupils should take the same studies. The fact of marked individual differences has been scientifically established. The principles of adaptation to such individual differences, that is, to individual needs and capacities, is now widely accepted in the high schools of America. The exception calls for scrutiny. Traditionally, algebra and geometry have been required for graduation. Is this necessary or advisable? In this growing practice of differentiation and adaptation we have then a third reason for at least reconsidering the customary mathematics courses.

4. A demand for reconsideration well worthy of our attention is found in the insistent question whether a content chosen to furnish

preparation for further but remote study does necessarily or even probably include the wisest selection of knowledge useful for those who do not reach that advanced stage of study. Whether all should learn first the more assuredly useful topics, or whether alternative courses should be offered, are proper subjects of inquiry. In either event we find in this consideration a fourth reason for studying anew the offerings of our high-school mathematics.

5. A fifth reason for reconsideration is found in the problem of method. Educators are studying now with new zeal the proper presentation of subject matter in all school work. Should not this study extend to secondary mathematics? Have we arranged the subject matter of that field in the best form for appropriation? Might it even be possible that mathematics should be reorganized in a way to run across customary lines of division? Or might this be true of some parts of mathematics for some groups of pupils and not be true of all? The proper answers to such questions are not at once evident, but certainly there is enough point in the inquiry to add a fifth reason for our proposed investigation.

III. ANALYSIS OF THE SITUATION.

1. *The problem of presentation.*—Far-reaching differences of method carry with them widely different organizations of subject matter, especially in introductory courses. From this consideration, at least, there are certain advantages in discussing as the first factor in the situation the problem of presentation.

The traditional school method has been that based upon the "logical" arrangement of subject matter. Thus our fathers studied English grammar before they took up composition, the "science" being "logically" anterior to the "art." The science, in this case grammar, began with a definition of itself and the analysis of the subject into its four principal divisions. Then came the definitions of the "parts of speech." It was a long—and generally dreary—road before the pupil could see any bearing of what he learned upon anything else. At length, after toilsome memorizing, there appeared within the subject itself a new variety of mental gymnastics which called forth from some a certain show of activity. In the end the survivors caught some glimpse of what it had all been about. But when they took up the "art" of composition, the "science" proved of small assistance. Somehow the "art" had to be learned as if it alone faced the actual demand.

From an implicit reliance upon this "logical" arrangement there has come a revolt, not yet universal, but still unmistakably at hand. The demand has now become insistent that in arranging subject matter for learning, consideration be given, not to "logic" as formerly conceived, but to economy in learning and effective con-

trol of subject matter. This reversal of method, coupled with a distrust of the theory of discipline, has thus not only reduced grammar to a small fraction of its former self, but has, besides, greatly rearranged and rewritten the study.

Keeping before us the demand for economy in learning and effective control of subject matter, what can we say about method? How does learning in fact take place? (1) Repetition is a factor in learning known to all. (2) An inclusive "set" which shall predispose the attention, focus available inner resources, and secure repetition is a necessary condition less commonly considered. (3) The effect of accompanying satisfaction to foster habit formation is a third factor to be noted.[1] These three factors are necessary, then, to adequate consideration of the problem of method. It accords with these considerations and with undisputed observation that, other things being equal, any item is more readily learned if its bearing and need are definitely recognized. The felt need predisposes attention, calls into play accessory mental resources, and in proportion to its strength secures the necessary repetition. As the need is met, satisfaction ensues. All factors thus cooperate to fix in place the new item of knowledge. The element of felt need thus secures not only the learning of the new item, but it has at the same time called into play the allied intellectual resources so that new and old are welded together in effective organization with reference to the need which originally motivated the process.

Lest some should fear that by need is here meant a mere "bread and butter demand," the committee hastens to say that it is psychologic and not economic need which acts as the factor in learning. Economic need may indeed be felt; and, if so, may then serve to influence learning; but there is nothing in the foregoing argument to deny that a purely "theoretic" interest might not be as potent as any other to bring about the learning and organization of subject matter.

To speak of the bearing and need of any new material is to imply the presence and functioning of already existent purposes and interests. From this consideration thus related to the foregoing the committee believes that, speaking generally, introductory mathematics—ordinarily conceived as separate courses in algebra, geometry, and trigonometry—should be given in connection with the solving of problems and the executing of projects in fields where the pupils already have both knowledge and interest. This would make the study of mathematics more nearly approximate a laboratory course, in which individual differences could be considered and the effective devices of supervised study be utilized. The minimum of

[1] The behaviorist psychologist by definition rejects the subjective connotation of "satisfaction." If we had access to the actual psychology involved, possibly the difference of statement would in effect disappear.

the course might well in this way be cared for in the recitation period, reserving the outside work rather for allied projects and problems in which individual interests and capacities were prominent factors.

The significant element in this conception is the utilization of ideas and interests already present with the pupils as a milieu within which the mathematical conception or process to be taught finds a natural setting, and from which a need to use the conception or process can as a consequence be easily developed. Where this state of affairs exists, the bearing and felt need utilize the laws of learning as was discussed above, and the mathematical knowledge or skill is fixed in a manner distinctly economical as regards both present effort and future applicability.

As was stated at the outset, this suggested procedure reaches beyond the questions of economy of learning and application—controlling though these here are—to the question of content. The procedure here contemplated makes definite demand for an appropriate introductory content. To work along this line there must be made a selection of conceptions and processes which can serve the pupils as instruments to the attainment of the ends set before them in the projects or problems upon which they are at work. This instrumental character becomes then the essential factor in any introductory course. It is these instrumental needs and not "logical" interconnectedness which must give unity to such a course. A content thus instrumentally selected will, on the one hand, be free of the old formal puzzles, the complex instances, the verbal problems which in the past have wasted so much time and destroyed so much potential interest; and will, on the other, run across the divisions heretofore separating algebra, geometry, and trigonometry.

A distinct advantage in the procedure here suggested is the better promise it holds out of meeting in one introductory course the needs of both those who will go on to advanced study in mathematical lines and those who will not. Where the basis of selection and procedure is instrumental, all can begin together. The future specializers in mathematics will as the course proceeds take increasing interest in the mathematical relationships involved and will stress this aspect in their individual problems and projects. Those whose tastes and aptitudes lead them elsewhere will in the meanwhile have had the opportunity to learn in practical situations some of the mathematical concepts and processes which they will later use in their own chosen fields. Their individual projects in the course can serve well as connecting links between the mathematics taught and their later field of vocational application.

After the introductory course has been completed, and differentiation has begun, the same principles still hold, though in the different

fields. Those who have chosen to continue the study of mathematics as such will find their problems or projects within the field of mathematics itself, quite likely examining anew in the light of wider acquaintance assumptions freely made in the earlier period. Euclid's system of axioms and postulates might here receive its first careful consideration. Those who had elected to prepare for engineering and the like might continue to find their mathematics in connection with problems or projects devoted now particularly to a preliminary engineering content. Conceptions usually reserved for college analytics and calculus—if not indeed already used in the introductory course—can well have a place here. Their rich instrumental character will justify their presence, even if they lack somewhat in relationship to a fully developed logical system.

2. *The several needs for mathematics*—Among the multiplicity of specific occasions for using mathematics and among the various types of subject matter, there are certain possible groupings which promise aid in the determination of the mathematical courses.

Without implying the possibility always of sharp differentiation, we may distinguish in the realm of mathematical knowledge (i) those items the immediate use of which involve a minimum of thinking, as, for example, adding a column of figures, and (ii) those items which are primarily used as notions or concepts in the furtherance of thinking. It is clear that the distinction here is of the way in which the knowledge is used and not of the knowledge itself; for any item of knowledge might at one time serve one function and at another time the other. It would still remain true, however, that certain groups of people might have characteristically different needs along the two lines. Under the first head we should include the mechanic's use of a formula, the surveyor's use of his tables, the statistician's finding of the quartile. The man in the street would call this the "practical" use of mathematics. Under the other head we should include the intelligent reader's use of mathematical language by which he would understand an account of Kepler's three famous laws. Some may wish to call this the "cultural" use of mathematics. The term "interpretative" might, however, more exactly express the differentiating idea.

We may next ask whether there are differentiable groups among high-school pupils whose probable destinations or activities determine within reasonable limits the extent and type of their future mathematical needs. In a democracy like ours, questions of probable destination are of course very difficult. There must be no caste-like perpetuation of economic and cultural differences; and definite effort must be made to keep wide open the door of further study for those who may later change their minds. But differentiating choices are in fact made; and in view of the wealth of

offerings on the one hand and of individual differences on the other, such choices must be made. Properly safeguarded by an intelligent effort to adopt social demands to individual taste and aptitude, these choices should work to the advantage both of the individual and of the group. The committee considers that four groups of users of mathematics may be distinguished:

(*a*) The "general readers," who will find their use of mathematics beyond arithmetic confined largely to the interpretative function described above.

(*b*) Those whose work in certain trades will make limited, but still specific, demand for the "practical" use of mathematics.

(*c*) Those whose practical work as engineers or as students of certain sciences requires considerable knowledge of mathematics.

(*d*) Those who specialize in the study of mathematics with a view either to research or to teaching or to the mere satisfaction of extended study in the subject.

It is at once evident that these groups are not sharply marked off from each other; and that the needs of the first group are shared by the others. It is, moreover, true that the "general readers" represent a wide range of interest. The committee has taken all these things into account, and still believes that the division here made will prove of substantial utility in arranging the offerings of high-school mathematics.

3. *Comparative values.*—Out of the conflict of topics for a place in the program there emerges one general principle, already suggested in these pages, which is being increasingly accepted for guidance by students of education. In briefest negative terms: *No item shall be retained for any specific group of pupils unless, in relation to other items and to time involved, its (probable) value can be shown.* So stated the principle seems a truism, but properly applied it proves a grim pruning hook to the dead limbs of tradition. A final method of ascertaining such comparative values remains to be worked out; but the feasibility of a reasonable application of the principle will hardly be denied. In accordance with this, many topics once common have been dropped from the curriculum and more are marked to go. Thus our better practice has ceased to include the Euclidian method of finding the H. C. F., because the knowledge of this method is nowhere serviceable in life; and in secondary algebra itself little if anything else depends on it. Indeed, the H. C. F. itself might well go, as it is used almost exclusively in simplifying fractions made for the purpose.

In a full discussion, many terms of the statement would need consideration. What constitutes value is probably the point where most questioning would arise. The committee takes this term in its broadest sense, specifically denying restriction to a "bread and but-

ter" basis or other mere material utility, though affirming that remunerative employment is normally a worthy part of the worthy life. What the statement then in fact demands is (i) that the value of the topic be not a mere assumption—a positive case must be made out; and (ii) that the value of the topic so shown be sufficiently great in relation to other topics and to the element of cost (as regards time, labor, money outlay, etc.) to warrant its inclusion in the curriculum.

This principle of exclusion seems especially applicable to those items which now remain merely as a heritage from the past and to those which have been introduced mainly to round out the subject or where the unity of the subject matter has been found in the content itself and not in the relation of the content to the needs of the pupil.

In offering such a principle for guidance, the committee considers that it is merely stating explicitly what has been implicitly assumed in all such controversies. The committee none the less believes that conscious insistence on the point is necessary in order to disclose whatever indefensible elements may be in our present program of studies.

4. "*Formal discipline.*"—A full discussion of this topic, of course, is impossible within the limits of this paper. Such a discussion is, moreover, for our purpose unnecessary, because we shall wish to use only the most general conclusion, in which there is substantial concurrence. We can thus avoid the niceties of elaboration, about which agreement has not yet been reached. The older doctrine assumed uncritically a very high degree of what we now call "general transfer" of training. Modern investigation, to speak generally, restricts very considerably the amount of transfer which may reasonably be expected, and inquires strictly into the conditions of transfer. Under the older doctrine it was a sufficient justification for the requiring of any subject that pupils should gain through it increased ability in the use of any important "faculty," because the increase in ability was naively assumed to mean an increase in the equally naively assumed faculty itself and would accordingly be effective wherever the faculty was used. As pupils show such an increase of ability in one or more "faculties" by the simple fact of learning any new subject, the convenience of this older doctrine for curriculum defense is evident. When this old psychological doctrine was first called in question by scientific measurement, the idea gained popular currency that all transfer was denied. No such claim has serious support. The psychologists, however, have so far found it difficult to agree upon any final situation as to the amount of transfer which in any particular situation may be a priori expected. All agree, none the less, in greatly reducing the old claim both as to the amount and

as to the generality of conditions under which transfer may be expected. In accordance with these considerations the committee has not used the factor of "formal discipline" in determining the content of the mathematical courses to be recommended in this report.

5. *The needs of the several groups.*—With these several principles and factors before us, we are now ready to consider more fully the needs of the several groups of users as distinguished above. We are particularly concerned to ask whether or not their respective group needs are compatible with one introductory course to be taken in common; and if yes, when the differentiation from such a common course should begin.

(1) *The "general readers."*—This group will need to use in "practical" fashion but little of mathematics other than ordinary arithmetic. As general readers, however, they will still require a certain acquaintance with mathematical language and concepts. Just what terms, symbols, and concepts would meet the requirements of this group will have to be determined by extensive inductive studies. Assuming, however, ordinary arithmetic and mensuration, some items can be at once named as fairly certain to be included: How to interpret and evaluate a simple literal formula; the meaning and use of an algebraic equation of one unknown; the notion and use of negative numbers in such simple cases as temperature, latitude, and stock fluctuations; the simpler conception of space relations (inductively obtained); the notion of function (the dependence of one quantity upon another); the graph as a means of interpreting statistical information, with such terms as average and median.

(2) *The group preparing for certain trades.*—Under this head the committee would group those whose use of "practical" mathematics is, while generally quite definite, still relatively small—such, for example, as machinists, plumbers, sheet-metal workers, and the like. The general run of the need here contemplated can be gathered from the requirements laid down for machinists in one of the more recent vocational surveys—simple equations, use of formulas, measurement of angles, measurements of areas and volumes, square root, making and reading of graphs, solution of right triangles, geometry of the circle. Much practice would of course be necessary to make even this small amount of mathematics function adequately.

It is at once evident that if no more algebra is needed than formulas, simple equations and the graph, and no more geometry than is here suggested, then the ordinary high-school courses in these subjects are but ill-adapted to the needs of such pupils. It would seem to follow that this group of pupils has no need to follow courses in mathematics other than (i) arithmetic, (ii) the "interpretative"

(introductory) mathematics discussed above, and (iii) the special applications of these to the specific subject matter of their several specializations.

This group might then well study in common with the preceding until the completion of the work there laid out. The presentation of this common course along the lines previously laid down (p. 11) would well harmonize the somewhat diverse interests of the two groups. What little additional content and whatever practice in specialized application this second group might need could then be given either in a parallel or in a succeeding course (or courses) especially devised for that purpose.

(8) *The group preparing for engineering.*—This group will consist mostly of boys intending to study in engineering schools. In contrast with the two preceding groups, appreciably more mathematics is here needed. In contrast with the following group, there are here specific aims external to mathematics itself which define and limit the mathematical knowledge and skill needed. Although recognizing that the individual teacher will require a certain leeway as regards content in getting his class effectively to work at any topic, we may still profitably ask as to the minimum content fixed for this group by its peculiar needs.

The minimum mathematical content suitable for the use of this group can probably best be secured by working simultaneously along two lines: First, to ascertain inductively what mathematics the engineer needs (including experiment to find out what part of this can best be taught in the secondary school); second, to criticize the existing courses to see what they lack and what they include that is useless for this group. It is much to be hoped that necessary inductive studies and experiments along the first line of procedure may be vigorously pushed. The second in important respects waits for the first, but it is possible from certain inherent considerations at once to exclude some matter now customarily taught.

Taking the customary high-school mathematics as a basis for comparison, we find at least three principles of criteria for exclusion from the present offerings: (*a*) Exclude all those items which are not themselves to be directly used in practical situations or which are not reasonably necessary to the intelligent mastery or use of such "practical" items; (*b*) exclude all involved and complicated instances of otherwise useful topics or applications which do not serve to clarify the main point under consideration; (*c*) exclude all such proofs and discussions as do not in fact help the pupil to an intelligent use of the topic. It is probably correct to say that these exclusions relate to material introduced from considerations of theory rather than of intelligent practical mastery; from considerations of

the pleasure that theorizers (teachers mostly) get from the study of mathematics rather than from a conscious purpose to give that familiarity and grasp which the future practical man will need.

Under the head of (*a*) topics excluded as not needed in this group the committee would mention such as the H. C. F. and the L. C. M.; operations with literal coefficients (except for a few formulas); radical equations; the theory of exponents, except the simplest operations with fractional and negative exponents (these to be retained to give meaning to logarithms and the slide rule); operations with imaginaries; cube root; proportion as a separate topic (the simple equation suffices for the progressions).

Among (*b*) excluded complex applications might be mentioned the following: All lengthy exercises in multiplication and division; factoring beyond the simplest instances of the four forms (i) $ax+ay$, (ii) a^2-b^2, (iii) $a^2+2ab+b^2$, (iv) $x^2+(a+b)x+ab$; all but the simplest fractional forms (the more complicated are in fact given to illustrate factoring); all radicals beyond $\sqrt{ab}$ and $\sqrt{a+b}$; simultaneous equations of more than two unknowns; simultaneous quadratics (except possibly a quadratic and a linear); the clock, hare and hounds, and courier problems and the like; the extended formal demonstrative geometry of our ordinary schools; most trigonometry beyond the use of sine, cosine, and tangent in triangle work.

(*c*) Proofs excluded or deferred are mostly cared for in (*a*) and (*b*). The chief instances in the past (too often yet remaining for the "specializers") have been the distinction between negative quantities and negative numbers, the (supposedly) rigorous generalizing of $a^m \times a^n = a^{m+n}$, the proof of too evident propositions in geometry, the incommensurable cases in geometry, the general proof of $\sin(x+y)$.

It may be mentioned in this connection that teachers of mathematics from arithmetic onward only too frequently deceive themselves as to the place that the presentation of a rigorously logical proof plays in bringing conviction. The worth of a sense of logical cogency can hardly be overestimated; but we who teach not infrequently overreach ourselves in our zeal for it. The teacher of introductory mathematics can well take lessons from the laboratory, where careful measurement repeated under many different conditions will bring a conviction often otherwise unknown to the pupil who is not gifted in abstract thinking. Probably in most instances an inductively reached conviction is the best provocative of an appetite for a yet more thoroughgoing proof.

Everything so far points to one common introductory course. With this group as with the preceding, the point of differentiation would seem to come at the end of the interpretative course first discussed for the "general readers." Whether this third (engineering)

group should proceed further in common with the fourth group (of specializers), we later consider further in common with the fourth group.

(4) *The group of specializers.*—This group will include those pupils, both boys and girls, who "like" mathematics. While these best of all could continue to work with the present offerings, the considerations urged under the discussion on presentation suffice, in the committee's opinion, to demand even for this group a far-reaching reorganization of practically all of secondary mathematics.

Since we are here planning for those who specialize in mathematics, we are not called upon—after meeting the interpretative need—to consider any external demands upon mathematics, but only such a selection and arrangement within the subject itself as best furthers the mathematical activity. Hitherto the arrangement within the course has been made, as we saw in the discussion on presentation, in answer to considerations rather of "logical" organization than of psychological experiencing and growth. The results have not been satisfactory. Algebra, geometry, and, to a lesser degree, trigonometry have been treated as separate logical entities, with consequent loss to the pupil of both interest and power. The committee thinks that the selection and organization should be made in the light of experiment as to which conceptions do in fact prove successively most strategic in the pupils' continued approach to mathematical power. The result would probably take a form somewhat analogous to the "general science" course which is now being worked out in that field.

That this group should take its introductory work in common with the others has perhaps been sufficiently implied. The intelligent choice of a specialty could hardly precede the actual experiencing of taste and aptitude. How far beyond the common introductory course this group should go in company with the third (the preliminary engineering) is not easy to say. In all but the largest schools administrative considerations will probably keep the two together in whatever work is offered. Where numbers and funds suffice, differentiation may well begin immediately upon the completion of the common introductory course, according to considerations already laid down. In that case the preliminary engineering group would get their mathematics more in terms of engineering content and situations; those specializing in mathematics would get theirs more directly in terms of "pure" mathematics. The contents of such courses could well differ considerably.

6. *Selecting mathematical ability.*—From the point of view both of society and its needs and of the individual and his satisfactions, it is highly desirable that ability, or the lack thereof, be disclosed in order that intelligent choice may be made. Mathematical ability as

expressed in mathematical achievement and application is a most powerful agency in advancing civilization. In order that society may profit by its available stock of mathematical ability, there is urgent need of some process that shall disclose this ability. Analogous considerations demand that the individual learn by a less costly process than occupational trial what degree of probable success he may expect from an occupation in which mathematical ability is an important factor. We hope much from further psychological study in this field of disclosing specific abilities, but as matters now stand the opportunity in the high school for trial of mathematical success and liking is at least one important part in the disclosing of mathematical ability. This factor must be taken into account in arranging the introductory work in mathematics.

IV. SUGGESTIONS AS TO COURSES.

Each valid consideration in the foregoing discussion should have its effect in the resulting determination of the mathematics courses. Considerations of presentation demanded that we give up the "logical" arrangement of subject matter, especially for introductory work, and find instead an organization based upon the successful attack of projects and problems in connection with which the pupils already have both knowledge and potential interest. Four groups of pupils judged by probable destination showed four types of mathematical needs: (i) The "general readers," whose needs lie largely in the "interpretative" function of mathematics; (ii) those who, expecting to enter trades, would have a small but still definite need for "practical" mathematics; (iii) those who, as prospective engineers, would need a considerable body of content determined by the demands of engineering study and practice; (iv) those specializing in mathematics who would wish a content determined by the satisfactions inherent in the activity and by the demands of further study. From considerations of comparative values nothing should enter into the curriculum except as it can show probable value in relation to other topics and to time involved. "Formal discipline" was not considered by the committee in determining the content of courses to be recommended. Care should be given that at an early stage mathematical taste and ability may be disclosed so as to allow appropriate choice of school work and occupational preparation. It seemed clear that a new introductory course should be offered which all the students should, normally, take in common. College entrance considerations, except as inherently cared for above, are deliberately disregarded.

With these demands before us, can an appropriate school procedure be devised and feasibly operated?

The task certainly is great. Nothing short of extended study and experimentation can meet the situation. The committee makes the following tentative suggestions as possible lines along which research and trial might prove profitable.

1. *The work of the junior high school.*—It seems to the committee that the work of grades 7, 8, and 9 should, in addition to whatever review of previous arithmetic may be necessary, include—

A. A body of processes and conceptions commonly called arithmetic, where the study, however, is of social activities—trade or otherwise—which need mathematics, rather than of mathematical topics artificially "motivated" by social relationships. As a constituent part of these processes the committee would include any use of algebra or intuitive geometry within easy reach of the pupils which can prove its worth by actual service in common life outside of the school.

B. A body of mathematical symbols, concepts, information, and processes—commonly thought of as belonging to algebra and geometry or beyond—which the intelligent general reader of high school or college standing will need in order to meet the demands of his social and intellectual life. As a part of this content, it seems safe to suggest the ordinary algebraic symbols, the use of the formula, the simple equation, and the (statistical) graph.

C. The opportunity for at least a preliminary testing of mathematical taste and aptitude.[1]

D. Such additional content—relatively small in amount—as may be needed to make effective the teaching of the foregoing.

The appropriate contents of parts A and B can be fixed only by a carefully made inductive study of the demands as they actually exist; the contents of parts C and D, only after extended experimentation. The contents of B, C, and D, respectively, the committee judges to be in the descending order of size and importance. Pending the scientific determination of these several contents, the committee feels that a wide diversity of offerings is to be welcomed as a sign of healthy variation likely to promote progress.

Just what course groupings of the foregoing should be made must likewise be for some time a matter of experimentation. Some will wish to consider the whole three years' work as one unity, the various items being presented in such connections among themselves and with the situations of application as good teaching may suggest. Others will wish to give A in grade 7, and devote grades 8 and 9 to an extended treatment of B, C, D. Still others will give two years to A, probably reducing the weekly time allowance, and in the ninth grade concentrate on B, C, D.

[1] We have grounds for hoping that psychological tests may prove of material assistance in this connection.

Especial attention is called to the course to be made up of B, C, and D (whether extending through one year or two). This is the common introductory course referred to many times in these pages. It is assumed that as a rule all pupils would take it (or at least begin on it), and that no further mathematics would be customarily required for college entrance.[1]

2. *Trade mathematics.*—For the groups of small but definite "practical" use the committee judges that the foregoing will commonly suffice so far as concerns specific mathematical content. In some of the trade curriculums, however, it will be necessary to provide a specific course (or courses) in which sufficient practice in the trade application can be found. The more directly such courses can be connected with the work of application the better.

3. *Preliminary engineering.*—For the preliminary engineering group there are, after the common introductory course, several possibilities. One would be to have this group work as heretofore with the "specializers" (see 4 below). This is perhaps less desirable, but will probably continue for some time to come as the more usual procedure, especially in the secondary school of not more than moderate size and income. Another possibility would be to construct a course, specifically for this group from a careful study of the specific demands of their future work (see pages 18 and 19 above). Such a content could then be given according to the principles of presentation discussed earlier (see p. 11). Here again experimentation will be necessary to develop an effective organization and procedure. Such experimentation may be expected to show a wide range of variation—at the one extreme an approximation to the old formal "logical"; at the other, an effort to make all mathematics teaching purely incidental to other work. And again, a wide diversity is at the first a healthy indication.

4. *For the specializers.*—Where numbers and income warrant, there should be elective work during the grades 10, 11, and 12 for those specializing in mathematics. There is need, as previously stated, to reorganize the customary offerings for these years in such a way as to displace a presentation based on classification for a presentation based on experimentally determined conditions in growth, in interest, and power.

Such a reorganization will naturally run across the lines of division heretofore maintained, and will probably anticipate certain conceptions and procedures confined now to analytics and calculus. Where the preliminary engineering group is included with this

[1] Any pupil would of course be permitted to offer for entrance credit any other mathematics he had elected in his secondary school. It is, moreover, probable that certain college courses open to freshmen would specify as a necessary prerequisite an amount of mathematics greater than that here included in this "general readers'" course.

group it may prove advisable to utilize to a considerable extent problems and projects from the natural sciences to give a certain desirable concreteness of thinking. This may well result in benefit to all concerned. With progress in the work should come, however, for the group of specializers an increased interest in and desire for mathematics on its own account. Such a reorganization as is suggested above would probably reduce in appreciable degree the quantity of formal demonstrative geometry, a result that the committee anticipates with equanimity. It seems probable that by suitable experimentation a new course can be worked out which will prove more alluring to the pupil and at the same time furnish a better introduction to the further study of the subject. Various efforts tending to corroborate this belief have already been made both in this country and abroad.

It may be asked whether all secondary schools should try to make full offerings of the courses here suggested. The committee thinks not. It will expect that work substantially equivalent to that here suggested for the grades 7, 8, and 9 will everywhere be offered; that the trade courses will naturally be restricted to trade curriculums; but that the elective work for the senior high school may be restricted in small schools where the income is not large. It seems probable that the relative reduction attending elective mathematics in the college will extend itself similarly to the secondary school. The committee in conclusion deprecates the continued disposition on the part of some colleges unwisely to dictate the contents of courses in secondary schools. It feels that such a usurpation of power operates to prevent the secondary school from making the most intelligent adaptation of its work to the needs of its pupils.