

## Preservice Secondary Mathematics Teachers' Knowledge of Students

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### Abstract

The aim of this paper is to present the nature of preservice secondary mathematics teachers' knowledge of students as emerged from a study investigating the development of their pedagogical content knowledge in a methods course and its associated field experience. Six preservice teachers participated in the study and the data were collected in the forms of observations, interviews and written documents. Knowledge of students is defined as teachers' knowledge of what mathematical concepts are difficult for students to grasp, which concepts students typically have misconceptions about, possible sources of students' errors, and how to eliminate those difficulties and misconceptions. The findings revealed that preservice teachers had difficulty in both identifying the source of students' misconceptions, and errors and generating effective ways different than telling the rules or procedures to eliminate such misconceptions. Furthermore, preservice teachers' knowledge of students was intertwined with their knowledge of subject matter and knowledge of pedagogy. They neither had strong conceptual knowledge of mathematics nor rich repertoire of teaching strategies. Therefore, they frequently failed to recognize what conceptual knowledge the students were lacking and inclined to address students' errors by telling how to carry out the procedure or apply the rule to solve the given problem correctly.

**Keywords:** *Knowledge of students; pedagogical content knowledge; mathematics; preservice teachers*

### Introduction

Preservice secondary mathematics teachers deal with different aspects of learning, teaching, and curricular issues in their teacher education programs. Teacher education programs provide several content, general pedagogy, and content-specific methods courses to support the development of professional knowledge for teaching. In these courses, preservice teachers are expected to construct and improve different knowledge domains for effective teaching.

Unquestionably, having strong subject matter knowledge is essential to becoming a teacher but it is not sufficient for effective teaching (Ball & Bass, 2000; Borko & Putnam, 1996). Teachers should know how to teach a particular mathematical concept to particular students, how to represent a particular mathematical idea, how to respond to students' questions, and what curriculum materials and tasks to use to engage students in a new topic. Shulman (1986) used the term *pedagogical content knowledge* to name a special knowledge base that involves interweaving such various knowledge and skills. He stated that pedagogical content knowledge (PCK) includes teachers' knowledge of representations, analogies, examples, and demonstrations to make a subject matter comprehensible to students. It

involves knowledge of specific topics that might be easy or difficult for students and possible conceptions or misconceptions that student might have related to the topic.

Although many scholars agree upon the existence of PCK as a distinct knowledge domain (Brown & Borko, 1992), there are different views about what constitutes it (e.g., Gess-Newsome, 1999; Grossman, 1990; Hill, Ball, & Schilling, 2008; Marks, 1990). Because PCK is perceived as knowledge of *how to teach* a particular subject matter (An, Kulm, & Wu, 2004), knowledge of subject matter and knowledge of pedagogy is not enough to achieve effective teaching practices without knowing the students, curriculum, educational goals, and instructional materials. In most studies, knowledge of subject matter, knowledge of pedagogy, knowledge of students, and knowledge of curriculum are accepted to be the components of PCK (e.g., An, Kulm, & Wu, 2004; Marks, 1990; Morine-Dersheimer & Kent, 1999). Teachers need to know personal and intellectual characteristics of a particular group of students, and their conceptions and misconceptions about a particular topic that will be taught. Then teachers should tailor their lesson in a way that address students' needs and their difficulties in understanding the subject matter and eliminate their misconceptions effectively. They also need to know the arrangement of the topics within a particular grade level and between grade levels, and how to use curriculum materials to achieve the learning goals identified in the written curriculum. Therefore, not only knowledge of subject matter and knowledge of pedagogy but also knowledge of students and knowledge of curriculum are essential components of PCK (Ball, Thames, & Phelps, 2008; Park & Oliver, 2008).

Pedagogical content knowledge is assumed to be developed as teachers gain more experience in teaching because it is directly related to act of teaching (Borko & Putnam, 1996). However, studies of preservice mathematics teachers' knowledge and skills related to teaching have revealed that methods courses and field experiences are likely to contribute to the development of PCK to some extent (Ball, 1991; Ebby, 2000; Graeber, 1999; Grossman, 1990; Tirosh, 2000; van der Valk & Broekman, 1999; van Driel, de Jong, & Verloop, 2002). Although there is no widely accepted standardized instrument specifically developed to measure teachers' PCK or the development of their PCK, researchers could learn about the nature of teachers' PCK by using different methods such as classroom observations, structured interviews, questionnaires, and journals (e.g., An, Kulm, & Wu, 2004; Even & Tirosh, 1995; Foss & Kleinsasser, 1996; Grossman, 1990; Marks, 1990). In other cases, workshops for inservice teachers could be designed with an intention of raising their awareness about the level of their PCK and improving their PCK through various practice (e.g., Barnett, 1991; Clermont, Krajcik, & Borko, 1993; Hill & Ball, 2004; van Driel, Verloop, & de Vos, 1998) or a methods course for mathematics teachers could be designed in a way that preservice teachers would have various opportunities such as analyzing students' error, developing a task, and microteaching to improve their PCK (e.g., Ball, 1988; Ebby, 2000; Graeber, 1999; Kinach, 2002; Tirosh, 2000). Therefore, I aimed to investigate what components of preservice secondary mathematics teachers' PCK developed in a secondary mathematics methods course and its associated field experiences. However, in this paper, I will discuss the findings about the nature of one of the components, namely knowledge of students and how it was influenced by the other components of PCK. Because of the space limitation, I will only discuss the findings obtained from interview data.

### **Knowledge of Students**

Knowledge of students is generally defined as knowing about the characteristics of a certain group of students and establishing a classroom environment and planning instruction accordingly to meet the needs of these students (Fennema & Franke, 1992). Shulman (1987) stated that teachers should know their subject matter thoroughly and be aware of the process of learning in order to understand what a student understands and what is difficult for them to grasp. Then, they need to develop a

repertoire of effective ways of teaching a particular subject, assessing students' understanding, and addressing their difficulties. Furthermore, An, Kulm, and Wu (2004) identified four aspects of PCK of students' thinking. These aspects are 1) building on student ideas in mathematics, 2) addressing students' misconceptions, 3) engaging students in mathematics learning, and 4) promoting student thinking about mathematics. They noted that teachers need to relate students' prior knowledge with new knowledge through various representations, examples, and manipulatives and focus on students' conceptual understanding rather than procedures or rules. Teachers also need to identify students' misconceptions correctly and eliminate such misconceptions by probing questions or using appropriate tasks.

In fact, teachers not only need to be able to help students when mistakes arise but also need to craft their lesson plans to either avoid or deliberately elicit common student errors. Moreover, teachers need to be able to determine the source of students' difficulties and errors in order to correct them effectively. For instance, a student's difficulty in solving a geometry problem might not necessarily be due to not knowing the geometric concept but may be due to a lack of arithmetic or algebraic skills.

The studies on teachers' knowledge of students have shown that beginning teachers lack knowledge of students' mathematical thinking (Fennema & Franke, 1992; Morris, Hiebert, & Spitzer, 2009; van Dooren, Verschaffel, & Onghena, 2002). They do not know much about what problems students may encounter when learning a specific topic. Moreover, they do not have a rich repertoire of strategies for presenting the material in a way that facilitates students' understanding or for eliminating students' misconceptions effectively.

Furthermore, teachers' own knowledge influences their efforts to help students learn (e.g., Ball & McDiarmid, 1990; Even & Tirosh, 1995; Grossman, 1990; Morris, Hiebert, & Spitzer, 2009; van Dooren, Verschaffel, & Onghena, 2002). Teaching is not just delivering procedural information but helping students improve their conceptual understanding. For instance, Even and Tirosh (1995) examined teachers' presentations of certain content in terms of their knowledge of subject matter and students. Their study was premised on the idea that to generate appropriate representations and explanations for a concept, teachers should not only know the facts, rules, and procedures but also know why they are true. For instance, one participant knew that 4 divided by 0 is undefined but did not know why. Therefore, this participant would tell students that it is one of the mathematical axioms that should be memorized. Additionally, Even and Tirosh noted that the preservice teachers were unable to address students' misconceptions effectively. Given two cases of incorrect solutions for 4 divided by 0 (e.g.,  $4 \div 0 = 0$  and  $4 \div 0 = 4$ ), they preferred to suggest their own answers rather than attempting to understand the students' reasoning. Thus, Even and Tirosh concluded that teachers' knowledge of subject matter and students' thinking had a strong influence on their pedagogical decisions.

### **Theoretical Framework**

Based on the literature about teacher knowledge, I accepted that PCK includes knowledge of subject matter, knowledge of pedagogy, knowledge of students, and knowledge of curriculum. Furthermore, I adopted Shulman's (1986, 1987) ideas about PCK and defined it as the ways of knowing how to represent a topic effectively to promote students' understanding and learning and being able to diagnose and eliminate students' misconceptions and difficulties about that topic.

In my definition of PCK, knowledge of subject matter refers to knowledge of mathematical facts and concepts and the relationships among them. I define strong mathematical knowledge as knowing how mathematical concepts are related and why the mathematical procedures work. Subject matter

knowledge also influences teachers' instruction and students' learning (e.g., Ball, 1990; Ball & Bass, 2000; Borko & Putnam, 1996; Ma, 1999; Thompson, 1992). Therefore, subject matter knowledge includes being able to relate a particular mathematical concept with others and explain or justify the reasons behind the mathematical procedures explicitly to promote students' understanding.

Knowledge of pedagogy covers knowledge of planning and organization of a lesson and teaching strategies. Teachers who have strong pedagogical knowledge have rich repertoires of teaching activities and are able to choose tasks, examples, representations, and teaching strategies that are appropriate for their students (Borko & Putnam, 1996). In addition, they know how to facilitate classroom discourse and manage time for classroom activities effectively.

Knowledge of students refers to knowing students' common difficulties, errors, and misconceptions. Teachers who possess a strong knowledge base in this domain know what mathematical concepts are difficult for students to grasp, which concepts students typically have misconceptions about, possible sources of students' errors, and how to eliminate those difficulties and misconceptions (An, Kulm, & Wu, 2004; Even & Tirosh, 1995; Tirosh, 2000).

Finally, knowledge of curriculum includes knowledge of learning goals for different grade levels and knowledge of instructional materials. Teachers with strong knowledge in this area know the state's or national standards for teaching mathematics identified for different grade levels and plan their teaching activities accordingly (Grossman, 1990; Marks, 1990). They choose appropriate materials (e.g., textbooks, technology, and manipulatives) to meet the goals of the curriculum and use them effectively.

### **Methodology**

This study was designed to investigate the nature of PCK developed in a methods course and its associated field experience in a group of preservice secondary mathematics teachers. I observed the methods course and its associated field experience course in fall 2008 at a large public university in the southeastern part of the United States. I wanted to understand the variety and the extent of the issues discussed in these courses and how preservice teachers could benefit from those discussions and field experiences. I decided to conduct a qualitative study because I was "concerned with process rather than simply with outcomes or products" (Bogdan & Biklen, 1998, p. 6).

I used multiple sources for collecting data, including interviews, observations, a questionnaire, and written documents. I was a participant-observer in all class sessions in both classes and took field notes. I conducted three interviews with each participant throughout the semester and collected all artifacts distributed in the courses and looked at the students' assignments to gain a better understanding of the course topics and students' thoughts and reflections about those topics. The methods course and its associated field experiences were not designed with an intention of developing preservice teachers' PCK. Therefore, at the beginning of the semester I interviewed the instructor of each course to learn about their goals for the course. Then, I attempted to triangulate all data to reduce the risk of the biases and the limitations of a specific data source (Bogdan & Biklen, 1998; Cohen, Manion, & Morrison, 2007).

### **Participant Selection**

From the 29 preservice teachers taking both courses, I chose 6 representative students as my participants based on a questionnaire administered at the beginning of the semester. The questionnaire consisted of 13 items; 8 of them were multiple-choice, 1 was Likert-type and 4 were

short-answer question. Through multiple choice and Likert-type items I aimed to learn how preservice teachers *perceive* their level of knowledge for each component of PCK. Short-answer type questions were context-specific and were similar to the questions that I would ask during the interviews. Therefore, they not only helped me learn more about my participants but also decide probing questions that I could ask during the interviews.

The questionnaire items were written to address the components of PCK that I identified in my theoretical framework. Each multiple-choice item was aligned to one knowledge type. For instance, Items 1 and 6 were aligned with knowledge of subject matter, Items 2 and 5 were aligned with knowledge of pedagogy, Items 3 and 7 were aligned with knowledge of curriculum, and Items 4 and 8 were aligned with knowledge of students. The short-answer questions involved multiple knowledge types. For instance, Item 10 entailed knowledge of subject matter, pedagogy, and students. The alignment of each questionnaire item with aspects of PCK was discussed with two faculties from the mathematics education department and reached an agreement on all items. The questionnaire with alignment and the rubric for the items are illustrated in Appendix.

Because I wanted the participants to be a representative group of preservice teachers taking the both courses, I assigned points to each questionnaire item to categorize preservice teachers in terms of their *perceived* knowledge level of PCK as *having* low, medium or high level of PCK and then choose two preservice teachers from each category. Such categorization not allowed me to work with a representative group of preservice teachers taking the both courses but also learn about whether their perceptions about their knowledge level of PCK had changed by the end of the semester. For short-answer type questions I discussed the ratings for each answer with a peer and we had .90 inter-rater reliability (Cohen, Manion, & Morrison, 2007) on the scores. In cases where we disagreed on a rating, we discussed what points to assign those answers and agreed on the final scores.

The total scores ranged between 29 and 43 (out of a total possible of 52 points). Because the categorization was mostly based on preservice teachers' perceptions about themselves, I did not specify the PCK levels in terms of scores. Instead, I ranked all scores from the smallest to highest and divided them into three groups having the same size. Therefore, 10 students with scores between 29 and 35 were categorized as perceiving themselves having a low level of knowledge; the next 10 students with scores between 36 and 38 were categorized as perceiving themselves having a medium level of knowledge; and the last 9 students with scores between 39 and 43 were categorized as perceiving themselves having a high level of knowledge. Then, I asked two volunteers from each group to be the participants of this study.

Based on the analysis of questionnaire data, 2 male and 4 female students agreed being the participants of the study. Laura and Linda (pseudonyms) were categorized as perceiving themselves having a low level of PCK with overall scores of 29 and 34, respectively. Laura was 21 years old, White, and a senior. Linda was 21 years old, White, and a senior. Monica and Mandy (pseudonyms) were categorized as perceiving themselves having a medium level of PCK with overall scores of 36 and 37, respectively. Monica was 20 years old, African American, and a senior; she was pursuing a double major in mathematics and mathematics education. Mandy was 34 years old, White, and a senior. Henry and Harris (pseudonyms) were categorized as perceiving themselves having a high level of PCK with overall scores of 42 and 43, respectively. Henry was 26 years old, White, and a graduate student. Harris was 22 years old, White, and a senior. The choice of pseudonyms of the participants was purposeful such that the initial letter of the pseudonym represents the participant's perceived level of PCK (L for low, M for medium, and H for high).

## Data collection

In the methods course the preservice teachers usually worked in groups to discuss given tasks, and then they shared their ideas with the rest of the class. I took extensive notes about their performance on the given tasks and what the 6 participants said during whole class discussions. Furthermore, I collected any artifacts (e.g., handouts, and multimedia presentations) discussed in the class in order to make inferences about the goals of that particular lesson and make a list of major topics discussed in the methods course and the field experience course. In the field experience course, the preservice teachers were required to write field reports during their time in schools. I examined all assignments and field reports completed by the participants to gain a better understanding of their experiences in the methods course and in the field.

I conducted three interviews with each participant. The first interview was held during the third week of the semester, the second one was held during the eighth week of the semester just after their second field experience, and the third one was held during the last week of the semester. At the beginning of the interviews, I asked them to reflect on the issues discussed in the methods and the field experience courses and how they contributed to each aspect of their PCK. Then I gave them some content-specific questions to understand the nature of their PCK. I also wanted them to reflect on their field experiences. During the last interview, I gave them a shortened version of questionnaire including multiple-choice and Likert-type items to see how they perceived their knowledge levels at the end of the semester. Furthermore, I asked them to make an overall evaluation of the methods and field experience course in terms of their gains from these courses.

## Data analysis

I used the PCK framework developed for this study to analyze the interview transcripts, field notes, and students' written work. I read through each students' work, transcripts, and daily field notes to get familiar with the content. I read each transcript to code each participant's answers in terms of the type of knowledge demonstrated in the questions, and then I compared the answers to similar types of questions to determine the similarities and differences between the explanations and also to detect any change, if there was, in their knowledge level of that particular knowledge domain. I discussed my decisions about each participant's responses to the interview questions with a faculty from the mathematics education department and we agreed on almost all of them.

The preservice teachers' answers to given mathematical problems and the validity of their explanations were counted as the indicators of their knowledge of subject matter. When their answers and explanations were mathematically valid, I categorized their responses as 1) procedural without reasoning (e.g., flipping the inequality sign when multiplying or dividing both sides of the inequality by a negative integer because it is *the rule*), 2) procedural with reasoning (e.g., using the *FOIL method* when multiplying binomials because *FOIL method* is based on the distributive property), and 3) conceptual (e.g., in Cartesian coordinate system, if a system of equations has no solution it means there is no common point satisfies the both equations, that is, the lines represented by those equations are parallel.) When their answers or explanations were mathematically invalid I noted them as the indicator of deficiencies in their knowledge of subject matter.

The variety and the reasonableness of preservice teachers' choice of teaching activities, tasks, examples, and representations and comprehensiveness of their lesson plans were accepted as their pedagogical knowledge. For instance, using the example of "finding the number of all possible arrangements of five different books on a shelf" is valid to explain permutation concept but the

example of "finding all possible two-letter words from the word BOOT" is not valid to explain combination concept.

The preservice teachers' repertoire of students' possible difficulties and misconceptions in mathematics and their ability to identify and eliminate such difficulties and misconceptions was coded as their knowledge of students. I gave some tasks such as error analysis to the preservice teachers and I categorized their responses in terms of their ability to identify all possible sources of difficulties or errors and their ability to suggest various ways to eliminate such errors. Therefore, they either 1a) diagnosed all possible difficulties or misconceptions correctly, or 1b) diagnosed some of the possible difficulties or misconceptions (in the case of there were more than one) correctly, or 1c) could not diagnose the possible difficulties or misconceptions. Then, they either 2a) suggested telling the rules and procedures to solve the given problem correctly, or 2a) suggested a reasonable way different than telling the rules or procedures to eliminate them. Finally, the preservice teachers' ability to identify a reasonable order of mathematical concepts to be taught in a semester, to differentiate learning goals for different grade levels, and to choose appropriate instructional materials such as textbooks, technology, and manipulatives to meet those goals were identified as their curriculum knowledge. For instance, linear equations are placed before quadratic equations in a typical secondary mathematics curriculum. Therefore, given a list of topics (including linear and quadratic equations) to be taught in a semester, linear equations should precede quadratic equations. Furthermore, a teacher may prefer to discuss the similarities and differences between linear functions and quadratic functions through the graphs of each type of functions by using graphing calculator or similar computer applets.

### **Findings**

In this study, knowledge of students is defined as teachers' knowledge of students' common difficulties and errors in different contexts and teachers' ability to diagnose and eliminate them. The preservice teachers' knowledge of students' common difficulties and errors is limited by their classroom observations during their field experiences. They noted that they did not know much about them. To understand the nature of how they would address and eliminate students' errors and misconceptions, I gave some content-specific cases to them during the interviews. I gave some student work involving errors and asked them how to address those errors and I also asked them how they could help students who are struggling with understanding some mathematical concepts. When given examples of students' errors and asked how to address them, the preservice teachers tended to repeat how to carry out the procedures or explain how to apply a rule or mathematical fact to solve the problem. That is, their responses mostly fell into categories of "diagnosed some of the possible difficulties or misconceptions correctly" and "suggested telling the rules and procedures to solve the given problem correctly." They had limited repertoire of teaching strategies to help students understand mathematics. Although, in some cases, the preservice teachers noted that they would first ask students to explain their solutions to help students assess their own understanding and realize their mistakes, they usually preferred to tell how to solve the given problem rather than using various visual aids such as tables, schemas, computer applets to help students solve the problem. Moreover, when they explained the solution of the given problem they rarely mentioned the reasoning underlying the procedures. That is, in terms of their knowledge of subject matter, their explanations mostly fell into category of "procedural without reasoning."

The most salient finding about the nature of preservice teachers' knowledge of student was the weakness in analyzing the reasons behind students' errors or difficulties which was emerged as a result of the nature of their subject matter knowledge. The preservice teachers usually came up with a reason, which was apparent and procedural. However, they did not state how flaws in students' conceptual understanding would likely lead to failure in generating a correct solution. For example,

during the first interview, I asked them how they could help a student who was having difficulty in multiplying binomials. Most of them said they would explain the procedure for using the “FOIL method” to multiply binomials. FOIL is a mnemonic used for multiplying the terms of two binomials in an order such that first terms, outer terms, innner terms, and last terms are multiplied and then simplified to find the result of the multiplication. The preservice teachers did not attempt to justify the reasoning behind the procedure, but two of them indicated that they were applying the distributive law when multiplying binomials. They assumed that applying the distributive law after separating the terms would help students understand the multiplication of the binomials. However, the students might not understand why the distributive law works and just try to memorize the procedure. The preservice teachers did not consider that students might know how to apply the distributive law but fail to multiply variables or negative integers correctly. For instance, students might think that  $2x \cdot 5x = 10x$  or  $-2(x-3) = -2x - 6$ . Laura and Henry did point out that students might struggle with multiplying variables and adding similar terms, but they did not explain how they would clarify those issues for the students.

In another task, I asked the preservice teachers how to help a student who simplified a rational expression inappropriately by using “canceling” as shown in Figure 1. All of them started by saying they would explain the procedure of simplifying rational expressions.

*Simplifying rational expressions*

Look at the student work given below. How can you explain to the student that his or her solution is incorrect?

$$\frac{2x^3y^2 - 6xy}{3xy^2 - x^3y^3} = \frac{\cancel{2x^3}^2 - \cancel{6xy}}{\cancel{3xy}^2 - \cancel{x^3y^3}} = \frac{2 - 6y}{3 - y^3}$$

Figure 1. The Simplifying Rational Expressions Task

Mandy and Henry were unsure how to clarify the student’s misconception. Mandy said that she would tell the student that the numerator and denominator are a unit, and therefore she cannot randomly cancel out the terms. She stated that the rules for multiplication of exponents are different from the rules for addition; however, she did not give examples of such rules or explicitly relate them to this task. She suggested using the idea of a *complex conjugate* to get rid of the denominator, but then she realized that she could not use a complex conjugate in the context of *real numbers*. Although she was aware of that the student’s solution was incorrect, she could not recognize that the numerator and denominator should be written in factored form before simplifying the terms. Hence, she failed to generate an effective way to approach the student’s misconception and help her to understand how to simplify rational expressions.

Similarly, Henry said he would tell the student that a term cannot be simplified when it is associated with another term through addition or subtraction. However, he did not explain what he would do to clarify such misconception. Instead, he said that explaining why the solution is incorrect is harder than solving the problem.

In contrast, some participants mentioned that they would show the student how to factor the given expressions and then simplify them. Laura and Linda said they would explain how to factor the numerator and denominator and then cancel out common terms. Laura would tell the student that “when we want to cancel out we need to remember that we are taking away every term in our

numerator and every term in our denominator.” Then she would show how to factor the numerator and denominator and then simplify them. She also said, “Being able to explain is tricky.” She noted that she would emphasize the idea of factoring and try to make sure that the student understood it. Similarly Linda would show how to factor the terms step by step, first working on the  $x$  terms and then the  $y$  terms. She said that she did not know whether there is an easier way to explain it.

Although Laura and Linda, explained how to factor, this might not be convincing for the student because it does not include a rationale for why it is necessary to find common terms in the numerator and denominator and then cancel them. They did not clarify the reasoning behind writing the numerator and the denominator in factored form rather than leaving them as they are. Furthermore, Linda used the term “taking away” to explain how to simplify the common terms in the numerator and denominator. Because “taking away” is used to indicate subtraction operation students may confuse about whether simplification refers to division or subtraction.

Harris also would explain how to factor the numerator and the denominator. However, first, he would try to convince the student that his or her reasoning was invalid by rewriting the given expression as the sum of two fractions, that is,  $\frac{a}{c+d} + \frac{b}{c+d}$  and then applying the student’s method to the fractions such that for each fraction, he would simplify the single term in the numerator with one of the terms in the denominator. Thus, he would show that the answer obtained in this way was different from the student’s answer in the example. While Harris’ explanation would help the student realize her mistake, it would not necessarily help her to understand *why* she needs to factor the expressions.

During the second interview I showed preservice teachers student work where the student found the solution of the equation  $2x^4 - 18x^2 = 0$  to be  $\pm 3$  by taking  $18x^2$  to the other side of equation and then dividing both sides by  $2x^2$  (see Figure 2). I asked them how they could explain that the solution is invalid.

*Solving polynomial equations:*

Look at the student work given below. How can you convince your student that his/her answer is invalid?

$$2x^4 - 18x^2 = 0$$

$$2x^4 = 18x^2$$

$$\frac{2x^4}{2x^2} = \frac{18x^2}{2x^2}$$

$$x^2 = 9$$

$$x = \mp 3$$

Figure 2. The Solving Polynomial Equations Task

With the exception of Henry, the preservice teachers did not recognize the student’s error. They stated that they would tell the student that factoring is a better way to solve that equation because it will help you find all of the solutions, including zero. For instance, Monica said “you just have to remind them that there are other ways of solving the problem, and this is one way she didn’t

necessarily get every solution.” It was evident that she did not notice the student’s error and therefore did not recognize that her explanation would not help the student understand why her method was incorrect. Henry also said he would explain how to factor the given equation; however, he would first tell the student that when dividing by  $x^2$  she needs to make sure that  $x$  is not zero. Thus, he was able to identify and clarify the student’s confusion about why her method did not work. The preservice teachers’ approaches to this problem revealed that they were unable to recognize the gap in students’ understanding of solving polynomial equations. Instead, they merely focused on the procedural steps and suggested another method that they were sure would yield all solutions.

During the third interview, I gave an example of student work in which the student forgot to change the direction of the inequality sign when dividing both sides of the inequality by a negative number (see Figure 3). With the exception of Linda, the preservice teachers failed to remember the reason behind this procedure. They noted that there exists a mathematical explanation for it, but they were unable to recall it.

*Solving inequalities*

Look at each of the student work given below. How can you explain to the student that his or her solution is incorrect?

$$-2x + 5 \leq x - 1$$

$$-2x - x \leq -1 - 5$$

$$-3x \leq -6$$

$$x \leq 2$$

Figure 3. The Solving Inequalities Task

Linda explained that if a number is less than a negative number, then it is itself a negative number. Therefore,  $-3x$  has to be a negative number. Then she used the fact that the product of two numbers is negative if and only if one of the numbers is negative and the other is positive. Thus,  $x$  would be a positive number. Henry attempted to explain it by using the idea of solving systems of inequalities. He suggested setting up  $y = -3x$  and  $y = -6$  to investigate the common solution as if they were inequalities. However his reasoning was vague because he did not identify the inequalities clearly. Based on his explanations, I concluded that he assumed that  $y \leq -6$ , but it was not clear whether he thought  $y \leq -3x$  or  $y \geq -3x$  because he did not solve the problem completely. To obtain the answer as “ $x$  is greater than or equal to 2” he probably considered the latter inequality, but he did not state it explicitly.

On the other hand, when preservice teachers had a deeper understanding of a particular topic, they attempted to justify the reasoning behind mathematical procedures and facts by using visual or concrete representations or by making connections with other concepts. For instance, during the first interview, I asked the preservice teachers how they could help a student who was confused about getting  $2 = 0$  as the solution of a system of linear equations, namely  $2x - y = -1$  and  $2y = 4x$  (see Figure 4).

*Solving systems of linear equations*

Assume that one of your students got confused when he or she found  $2 = 0$  as the result of the solution of a system of linear equations. How do you explain to him or her the meaning of this result?

Sample student work:

$$\begin{array}{l}
 2x - y = -1 \rightarrow \\
 2y = 4x \rightarrow \\
 2 = 4x - 4x \\
 2 = 0
 \end{array}
 \quad
 \begin{array}{l}
 2x + 1 = y \\
 2y = 4x
 \end{array}
 \rightarrow
 \begin{array}{l}
 2 - (2x + 1) = 4x \\
 4x + 2 = 4x
 \end{array}$$

Figure 4. The Solving Systems of Linear Equations Task

Although Henry and Mandy did not recognize that the solution  $2 = 0$  meant that there was no solution of the system or that the lines did not have a point of intersection, the others did recognize and suggest sketching the graphs of each to show that they are parallel. Henry thought that "it means you divided by zero or did some kind of illegal maneuver." He suggested writing the equations in the slope-intercept form to find the wrong step, but he did not explain further how it would help him to detect the error. Likewise, Mandy said "Whenever you get something like  $2 = 0$  or  $7 = 3$ , somewhere along the line here you didn't follow the mathematical rule." She rewrote the second equation as  $y = 2x$  but did not continue working on this question. Mandy failed to realize that the lines have the same slope and are therefore parallel, even though she wrote the equations of the lines in slope-intercept form. It is unclear whether she did not know that the slopes of lines provide information about the relationship between (i.e., parallel lines have the same slope) or whether she was simply unable to recall and apply this knowledge at the time of the interview. However, neither preservice teacher was able to reason about the task by thinking about what a solution to a system of linear equations represents (a point of intersection of the lines). Neither one suggested using visual aids such as graphs to investigate the given case and help students understand the context better; rather these participants said they would explain the procedural steps for solving the system of equations to students.

In contrast, the other participants said they would graph the lines to show students that they would not intersect. Linda noted that getting such an answer would indicate that there is no  $x$  value that satisfies both equations for any  $y$  value. Then she said, "Graphing it would be the easiest way because...if you give them a picture they can understand a lot better." Linda said she would graph the equations to support her explanations and foster students' understanding.

Laura stated that she would ask the student to check the calculations first. If the student got the same answer, then she would tell her that "this  $x$  in the first equation is probably not equal to this  $x$  in the second equation." Then, she would graph both equations to show that the graphs would not intersect. She suggested using graph paper or a graphing calculator to sketch the graphs. She would also talk about parallel lines because "when lines do not intersect that means they have the same slope and further they are parallel." Thus, her reason for graphing the equations was twofold: to address the student's difficulty in understanding systems of linear functions and to make connections with other concepts such as parallelism and slope.

Harris also said he would suggest checking the answer for accuracy and then he would talk about what it means to get no solution as the result of systems of linear equations. He would relate that discussion to the idea of independent lines, and then he would graph the lines to show that getting  $2 = 0$  means that there is no solution and the lines are independent; that is, they are not intersecting. It was evident that he would graph the lines to support his explanations and help students understand the given case better.

Monica said she would prefer to talk about all possible cases of the solution of systems of linear equations. She would rewrite the given equations in the slope-intercept form and then graph them to show that the graphs are not intersecting. Then she would give examples of the other two cases and graph them to show how the solution of the system relates to the graphs of the lines on the coordinate plane. It seemed that Monica's goal was to put this particular example in a larger context by providing examples of each case: A unique solution means the lines intersect, no solution means the lines are parallel, and infinitely many solutions means the lines coincide. By approaching the problem in this manner, Monica was trying to help the student make sense of systems of linear equations more generally rather than just in the given case.

### Discussion

The interview data revealed that the preservice teachers' knowledge of students was intertwined with their knowledge of subject matter and pedagogy such that they sometimes had difficulty in identifying the source of students' difficulties and errors correctly, and in finding effective ways to eliminate them. The preservice teachers thought that students fail in mathematics because they do not know the procedures or rules to be applied or they apply them incorrectly. Therefore, they were inclined to address students' errors by repeating how to carry out the procedures or explaining how to apply a rule. Such approach of the preservice teachers could be counted as an indicator of the weakness of their repertoire of appropriate examples, representations, and teaching strategies could be used when teaching mathematics, that is, it was the indicator of the weakness in their knowledge of pedagogy.

Although there are a number of more conceptual approaches to address students' difficulties and errors, the preservice teachers did not mention during the interviews. For instance, in the case of multiplying binomials, a teacher could work with small numbers to show how the distributive law works. For instance, one could create a simple word problem to show that  $3 \cdot 7 = 3 \cdot (2 + 5) = 3 \cdot 2 + 3 \cdot 5$ . Similarly, it is possible to use an area model to explain the multiplication of binomials in the form of  $ax + b$ . Given two binomials  $ax + b$  and  $cx + d$ , draw a rectangle having these binomials as the dimensions and then construct four small rectangles with dimensions  $(ax) \times (cx)$ ,  $(ax) \times d$ ,  $(cx) \times b$ , and  $b \times d$ . The sum of the areas of all of the rectangles gives the area of the original rectangle, which is a visual illustration of the multiplication of binomials. Also, using algebra tiles would allow students to find the area of a rectangle as the sum of partial areas in a manner similar to the area model just described. The teacher could also use more conceptual approach to help students even if the distributive property is not the cause of the problem but lack of prior knowledge such as operations with variable expressions.

In the case of simplifying variable expressions, the preservice teachers might use particular numerical examples to show that the student's reasoning was invalid. For example, if the 2s are canceled in  $\frac{2+4}{5-2}$ , the answer is  $\frac{4}{5}$ , but the correct answer is 2. The order of operations could be used to explain this task as well, noting that when the numerator or denominator of a fraction involves more than one term, they are assumed to be inside parentheses. Because the division operation does not

precede parentheses, simplification cannot be applied randomly over the single terms. In addition, the idea of equivalent fractions and simplification could be applied in this situation. For instance, showing

that  $\frac{6}{8} = \frac{2 \cdot 3}{2 \cdot 4} = \frac{3}{4}$  and then extending the analogy to examples with variables would show how

these concepts are related to the given problem. Furthermore, the preservice teachers said they would explain to students how to factor the numerator and denominator before canceling out the common terms. They noted that the student failed to simplify the given expression because she did not know how to factor variable expressions. However, another reason underlying the error might be weakness in the student's knowledge of exponents and operations with them. Although Monica stated

that she would review the properties of exponents, such as showing that  $x^3 = x \cdot x \cdot x$  or  $\frac{x^3}{x^2} = x$ ,

she did not state explicitly how she would relate these properties to the idea of simplifying the terms or writing the expressions in factored form. Therefore, not only the weakness in preservice teachers' knowledge of pedagogy might be the cause of incomplete responses but also their knowledge of subject matter. For the simplifying variable expressions tasks, the preservice teachers could not recognize all possible sources of the student's error. Thus, they did not suggest alternative ways of helping the student.

Similarly, in the case of solving polynomial equations the preservice teachers could not recognize the student's error. They confused with the student's answer because her solution was seemingly correct but they knew that zero is also in the solution set of the given equation. Although they realized that something had to be wrong with student's solution they preferred to explain the solution in their minds, that is, factoring the equation first and then solving for  $x$ . Such an attempt not only revealed deficiencies in preservice teachers' knowledge of subject matter but also nature of such knowledge, which is procedural. The preservice teachers came up with two *methods* to solve polynomial equations: either factorize the equation or simplify. They thought that both methods have to yield the same answers. However, it was not the fact because they overlooked a special case that one of the values of the unknown was zero. Although some of them recalled the fact that the degree of a polynomial function determines how many roots the function would have, they could not justify this fact to address the student's error more effectively. They preferred to tell the student that she might check the accuracy of her answer by using this *rule*. Another example of the preservice teachers' procedural knowledge of mathematics was "solving inequalities task." Except one participant, the preservice teachers did not explain why the inequality sign should be flipped when multiplying or dividing both sides of inequality by a negative number. Seemingly, they just memorized it as a mathematical rule and did not reason why it works. On the other hand, in the case of solving systems of linear equations the preservice teachers attempted to use representations to explain the underlying concept. Except two of the participants, the preservice teachers had solid understanding of solving systems of equations and they suggested using the geometric meaning of such solution by graphing the given linear equations.

Briefly, the examples discussed here and above revealed that the preservice teachers' knowledge of subject matter and pedagogy had an impact on their knowledge of students. If they knew the concept in depth, then they were able to detect the flaws in students' understanding. If they had rich repertoire of teaching strategies, representations and examples then they could address students' errors and misconceptions effectively.

### **Conclusion and Implications**

The aim of this paper was to present the findings about preservice teachers' knowledge of students as emerged from a study designed to investigate the development of preservice teachers' PCK in a methods course and its associated field experiences. The findings support the earlier studies on teachers' knowledge of students (e.g., Ball, Thames, & Phelps, 2008; Even & Tirosh, 1995; Kagan, 1992) that the preservice teachers lacked knowledge of students' mathematical thinking. They neither knew much about what problems students might encounter when learning a specific topic nor how to help students overcome their difficulties and correct their misconceptions.

To improve preservice teachers' knowledge of students, they should be given opportunities to work with individual students to develop their repertoire of students' misconceptions and also improve their ability to help address students' difficulties effectively. Graeber (1999) suggested that preservice teachers should be given different examples of students' misconceptions and asked to analyze students' thinking and generate a way of eliminating such misconceptions in the methods course to improve their knowledge of students' thinking. Although the preservice teachers in this study were given such examples a few times during the methods course, it seemed that the number of those activities should be increased to help preservice teachers improve their knowledge of students. Furthermore, the preservice teachers should be given opportunities to work with individual students or a group of students to experience how to help students understand mathematics. Thus, they could improve their repertoire of different ways of addressing students' difficulties and misconceptions such that they may need to use representations, manipulatives, or real-life examples rather than merely telling of the rules or procedures.

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## Appendix

### Questionnaire<sup>15</sup>

Instruction: For each of the following items choose the response that best fits you.

1. At the end of my degree program I will have taken enough content courses to be an effective mathematics teacher in grades 6-12. (*KSM*)
  - a. Agree
  - b. Somewhat agree
  - c. Disagree
  
2. At the end of my degree program I will have taken enough courses about teaching mathematics to be an effective mathematics teacher in grades 6-12. (*KP*)
  - a. Agree
  - b. Somewhat agree
  - c. Disagree
  
3. I know what mathematics content is to be addressed in each year of the 6-12 mathematics curriculum. (*KC*)
  - a. Agree
  - b. Somewhat agree
  - c. Disagree
  
4. I know possible difficulties or misconceptions that students might have in mathematics in grades 6-12. (*KS*)
  - a. Agree
  - b. Somewhat agree
  - c. Disagree
  
5. I have a sufficient repertoire of strategies for teaching mathematics. (*KP*)
  - a. Agree
  - b. Somewhat agree
  - c. Disagree
  
6. I know how mathematical concepts are related. (*KSM*)
  - a. Agree
  - b. Somewhat agree
  - c. Disagree
  
7. I know how to integrate technology in mathematics lessons. (*KC*)
  - a. Agree
  - b. Somewhat agree
  - c. Disagree
  
8. I know how to diagnose and eliminate students' mathematical difficulties and misconceptions. (*KS*)
  - a. Agree
  - b. Somewhat agree
  - c. Disagree

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<sup>15</sup> Alignment of the questions are given in the parentheses with abbreviations. KSM: Knowledge of subject-matter, KP: Knowledge of pedagogy, KS: Knowledge of students, KC: Knowledge of curriculum

9. Read the definitions of the following Knowledge Bases:

*Knowledge of subject-matter:* To know mathematical concepts, facts, and procedures, the reasons underlying mathematical procedures and the relationships between mathematical concepts.

*Knowledge of pedagogy:* To know how to plan a lesson and use different teaching strategies.

*Knowledge of students:* To know possible difficulties, errors, and misconceptions that students might have in mathematics lessons.

*Knowledge of curriculum:* To know learning goals for different grade levels and how to use different instructional materials (e.g., textbook, technology, manipulatives) in mathematics lessons.

How do you perceive your knowledge level in each knowledge base identified above? Use the following scale: 1-not adequate 2-adequate 3-competent 4-very good

Knowledge of subject-matter: .....

Knowledge of pedagogy: .....

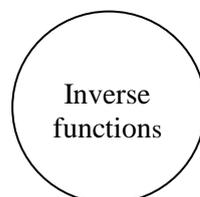
Knowledge of students: .....

Knowledge of curriculum: .....

10. Look at the student work given below. How can you explain to the student that his or her solution is incorrect? (*KSM, KP, KS*)

$$\sqrt{9x^2 + 25y^4} = 3x + 5y^2$$

11. Assume that you will introduce "inverse functions". Make a concept map for inverse functions showing which mathematical concepts or facts relate to inverse of functions. (*KSM, KC*)



12. If you were introducing how to factor trinomials, which of the following trinomials would you use first? Explain your reasoning. (*KSM, KP, KS*)

$$2x^2 + 5x - 3, \quad x^2 + 5x + 6, \quad 2x^2 - 6x - 20$$

13. Assume that you will teach the following topics in a semester. In which order would you teach them to build on students' existing knowledge? Explain your reasoning. (*KSM, KC*)

Polynomials, trigonometry, factorization, quadratic equations

### Rubric

**Scale for Items 1 through 8.** Disagree: 1 pt., Somewhat Agree: 2 pts., Agree: 3 pts.

**Scale for Item 9.** Not Adequate: 1 pt., Adequate: 2 pts., Competent: 3 pts., Very Good: 4 pts.

**Scale for Items 10 through 13.** 0: No answer, 1: Vague answers or answers without explanations, 2: Answers without justifications or answers with minor mathematical errors, 3: Valid explanations or justification.

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