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# Reflective Abstraction and Mathematics Education: The Genetic Decomposition of the Chain Rule—Work in Progress

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Students have experienced difficulty in understanding and using the chain rule. This study aims at assisting the students to understand and apply the chain rule and thus inform the author's teaching for future learning of students. A questionnaire will be designed to explore the conceptual understanding of the concept of the chain rule by first year university of technology students using APOS (action-processes-objects-schema) which proposes in the form of the genetic decomposition a set of mental constructions that the students might make in order to learn the concept of the chain rule in calculus and accessing it when needed. This instrument will be used to collect data on how students learn derivatives of trigonometric functions in calculus, using the chain rule. This will be with a view to clarify their understanding of the composition of functions, derivative and the chain rule. The study consists of two phases, both using a qualitative approach. A structured way to describe an individual student's understanding of the chain rule is developed and applied to analyzing the evolution of the understanding for each of 30 first year students. Other ways to collect data include tests, written exercises and classroom observations. The purpose of the questionnaire will be to establish the correlation between the students' ability to deal with composition of functions and using the chain rule successfully. Students ($n$ = 10) will then be interviewed based on their written responses to elicit their thinking involved when answering. The analysis of written responses and interviews should establish whether the instrument provided substantial information for identification of certain mental constructions that the researches proposed to consider.

*Keywords:* APOS (action-processes-objects-schema), genetic decomposition, chain rule and composition of functions

## Introduction

The chain rule is an underlying concept in many applications of calculus: implicit differentiation, solving related rate problems and solving differential equations. The rule states that if $g(x)$ is a function differentiable at $c$ and $f$ is a function differentiable at $g(c)$, then the composite function $fog$ given by $(fog)(x) = f(g(x))$ is differentiable at $c$ and so that it is $(fog)'(c) = f'(g(c).g'(c)$. Cottrill (1999) asserted that conventional wisdom holds that students' conception of the chain rule (as with other rules) is that of the symbol manipulation. This conception appears to be a straight-forward manipulation of symbols which can easily be applied to problem situations. However, this application of symbol manipulation carries a heavy

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requirement for the function to be given by an expression, fostering students' tendencies towards instrumental understanding, where they are unable to apply the chain rule.

This study aims at assisting the students to understand and apply the chain rule, and thus, inform the author's teaching for future learning of students. This study is guided by one research question: How do students construct various structures to recognize and apply the chain rule in the context of calculus?

This is with the view of clarifying:

(1) The students' understanding of function composition;

(2) Their understanding of the derivative;

(3) The students' difficulties in explaining the chain rule;

(4) Students' schema alignment with the genetic decomposition of the chain rule;

(5) The triad stage of schema development with respect to the chain rule that is the students' operating;

(6) Whether students see the reverse application of the chain rule in the substitution technique for integration.

## Literature Review

The processes used by students to build their knowledge of the chain rule in calculus are of interest to this study. Clark et al. (1997) who studied students' understanding of the chain rule and its applications concluded that the difficulties with the chain rule for a large number of students could be attributed to students' difficulties in dealing with composition and decomposition of functions. This hypothesis was confirmed by Cottrill (1999) in his study of correlation between a student's understanding of composition of functions and understanding of the chain rule in which the understanding of the composition of functions was the key to understanding the chain rule.

Reviewed literature has, therefore, addressed functions, their properties, how students understand functions, composition of functions, rules for differentiation and misconceptions about the chain rule. It is evident from the above discussion that many well-known functions have simple expressions for their derivatives, while composite functions require the use of the chain rule for differentiation. Functions with fairly complicated expressions have explicit formulas for derivatives. It was the development of formulas and rules, such as the chain rule that enables mathematicians to calculate derivative that motivated the use of the name calculus for this mathematical discipline.

## Theoretical Framework

The framework for this research consists of theoretical analysis, data collection and analysis and then design and implementation of instruction. The theoretical analysis will result in genetic composition of the chain rule. This will constitute a set of mental constructs which might describe how the chain rule can develop in the mind of an individual. Dubinsky (1991b) proposed that reflective abstraction could be a powerful tool in the study of advanced mathematical thinking, could provide a theoretical basis that supported and contributed to the understanding of how students think and could suggest explanations of the difficulty experienced by students with mathematical concepts, including the chain rule.

He further suggested that usually, it became necessary that the genetic decomposition in the original theoretical analysis was revised as a result of data. He, therefore, believed that the incorporation of the triad concept of Piaget and Garcia (1989) would lead to a better understanding of the construction of schema.

Previous studies conducted on the genetic decomposition of the chain rule (Clark et al., 1997), where they report on students' understanding of the chain rule; Cottrill (1999) studied on the chain rule and its relation to composition of functions; and Baker, Cooley, and Trigueros (in press) discussed about the relationships between the graph of a function and properties of its first and second derivatives, revealing that the understanding of schemas as described in reflective abstractions was not adequate to provide a satisfactory explanation of the data.

The introduction of the triad then helped to elaborate a deeper understanding of schemas and better explanations of the data. The theory on reflective abstraction and the triad suggested by Piaget and Garcia (1983) was important for higher mathematics, as they were useful to explain children's logical thinking. In extension of this theory, Dubinsky (1991) isolated some essential features of reflective abstractions, reorganized and reconstructed them, and formed a coherent theory of mathematical knowledge and its construction, APOS (actions-processes-objects-schema). The author has decided to adopt the APOS approach (Dubinsky, 1991a), based on its intuitive appeal, as there has been little empirical research done before to document the use of it on students' conception of various mathematical concepts in the African continent.

This approach, through which this study is conducted, begins with a statement of an overall perspective of what it means to learn and know something in mathematics as prescribed by Asiala, Brown, Devries, Dubinsky, Mathews, and Thomas (2004, p. 7) that,

> An individual's mathematical knowledge is his/her tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing and reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations.

They further believed that the understanding of a mathematical concept began with manipulating previously constructed mental or physical objects to form actions that were then interiorized to form processes which are then encapsulated to form objects. They said that these objects could be de-encapsulated back to the processes from which they are formed, which would be finally organized in schemas.

## Methodological Framework

The study consists of two phases, both by using a qualitative approach. A structured way to describe an individual student's understanding of chain rule is developed and applied to analyzing the evolution of that understanding for each of the 15 first year students. The methodology is a multiple case study. Interviews, including both task-based and open-ended questions, will be the primary instruments for collecting data on each student's understanding of the chain rule. Other ways used to collect data include tests, written exercises and classroom observations. The case study approach allows the researcher to select the examples that illustrate the points that he/she wishes to make (Cohen, Marion, & Morrison, 2000). They further asserted that the analysis of the individual student's answers would consist of a construction of taxonomies, resulting from the various observation sessions of each student's work. Since the main aim of this study is to analyze students' mathematical thinking in the context of the chain rule, an interpretive paradigm is used. More specifically, the researcher examines students' attempts to answer the tasks given in class, their tests and exercises with regard to their understanding of functions, composition of functions and the chain rule.

**Participants**

The subjects for this study are first year civil engineering students (197) at Mangosuthu University of Technology who have been taught more than half of calculus concepts like the limits, the rate of change of a function, finding the derivatives of polynomials and algebraic together with trigonometric functions and also the use of product and quotient rules in calculus. Some of these students have been through a foundation course including calculus for a period of six months at the university, while others had good matric symbols and were registered for first year without going through any foundation course. A small sample of 30 volunteering students is selected because: (1) Interpretive case studies depend on descriptive foundation; and (2) This type of research takes a lot of time.

**Data Sources**

Data for this analysis include the results from the pilot study. It also includes the analysis of students' performances in the tasks given regarding their understanding of the function, derivative, composition of functions and the chain rule. Interviews are also conducted to analyze students' responses in written tasks and class discussions. After the analysis of the tasks in phase 1, there will be video recording of a number of sequential lessons on the chain rule. During this period, the interaction between the sequence of lessons and students' activities will be compared. Audio-recording is also employed, whilst interacting with the students and when providing guidance to individual students. Students work individually.

**Data Collection Procedures: The Pilot Study**

Here, a pilot study is conducted. It involves collection of data via questionnaires which are administered to 30 previous semester students of known ability, willing to participate in the study. These are students who have already written an examination on calculus at first year and passed it. This is done to check for errors, validity and reliability in the instrument. The validity of an instrument determines whether an instrument measures what it is supposed to do. The content validity will be given prior attention in the instrument. Reliability attempts to answer the questions: Does the instrument give consistent results? (de Vos, 2002). The questions will be testing the understanding of: (1) definitions of functions using graphical methods; (2) definition of function using domain and range; (3) composition problem to determine understanding of the "o" notation; (4) decomposition of a composed function; (5) determining derivative; and (6) application of the chain rule.

These participants are then interviewed to explain their responses to the questionnaire and the results of the pilot study will be included in this study. Data then will be coded.

**Data Analysis**

This aspect of the study is based on APOS. Different students are expected to perceive the chain rule differently. Various activities on the use of the chain rule in differential calculus are designed and given to all groups. The aim of these activities is to draw from the explanations that are given by the students on how they arrived at particular solution, but not on whether the students' answers are correct or wrong. For example, in answering problem (2) $f(x) = \sin^2 2x$, a student might differentiate correctly, show a minor error by dropping (+) sign or not putting a bracket where it is due, indicate an error with derivative of a trig function, and apply the chain rule indiscriminately or attempt to avoid the chain rule by expanding or rearranging the terms or may show no evidence of considering the chain rule at all. All these categories of answers are looked at and coded differently.

## Research Framework

The framework used in this research follows the Framework for Research and Curriculum Development in Undergraduate Mathematics Education as proposed by Asiala et al. (2004). This framework utilizes qualitative methods for research and consists of three components: (1) theoretical analysis of the concept; (2) design and implementation of instruction; and (3) data collection and analysis.

APOS is used for the theoretical analysis of the chain rule, that is, a description of some specific mental constructions that a student might make in order to develop his/her understanding of the chain rule. APOS proposes, in the form of a genetic decomposition, a set of mental constructions that a student might make in order to learn the concept of the chain rule and access it when needed. Instruction is designed to help students make the mental constructions and relate them to the mathematical concept of the chain rule. The instructional strategies are used to get the students to reflect on their work through working on activities, class discussions and exercises. The activities help students to relate and reflect on properties and relationships in which the chain rule is used. When they have learnt the chain rule, the students are assigned class-work, homework and assignment to be done in the form of a tutorial on the chain rule containing many standard exercises, designed to make the necessary mental constructions proposed by the genetic decomposition.

Observation and assessment follow the instructional treatment and allow the researcher to gather and analyze data. Data gathered is used to report on the performance of students on the mathematical tasks related to the chain rule. This data is represented in mathematical terms rather than in terms of what mental constructions might or might not have been made. Also, the results of the data analysis may lend support to or lead to revisions or changes in the initial genetic decomposition, and thus, directing the formulation of a new genetic decomposition.

## Initial Genetic Decomposition

The genetic decomposition of the concept of the chain rule given here is used to guide the author’s teaching instruction in class and also guide the construction of the interview tasks. The chain rule schema develops through the levels of the triad, intra, inter and trans. At the first level, the intra-level, the student has a collection of rules for finding derivatives of functions in various situations, but has no recognition of the relationships between them. This collection may include some special cases of the chain rule, and perhaps even the general formula which is perceived as a separate rule rather than a generalization of the others. The inter-level is characterized by the student’s ability to begin to (mentally) collect all different cases and recognize that these are related. At this stage, the collection of elements in the chain rule schema is being formed and the collection is called a pre-schema. At the trans-level, a student has constructed the underlying structure of the chain rule. He/she links the composition and decomposition of functions to differentiation and recognizes various forms of the chain rule as linked in the sense that they follow from the same general rule through function composition. It is only at this stage of development that the underlying structure of the chain rule schema is constructed through reflection on relationships between various objects from previous stages. The elements in the schema must go beyond being described essentially by a list, to being described by a single rule (Clark et al., 1997).

For a student to have his or her function schema:

(1) He/she will have developed a process or object conception of a function;

(2) He/she developed a process or object conception of a composition of functions.

For a derivative schema:

(1) He/she will have developed a process conception of differentiation;

(2) The student will then use the previously constructed schemas of functions, composition of functions and derivative to define the chain rule. In this process, the student has to recognize a given function as the composition of two functions, take their derivatives separately and multiply them;

(3) The student recognizes and applies the chain rule to specific situations.

Even though most activities are written under ordinary test conditions with little interaction for the pilot study, a five-point rubric based on guidelines adopted from Carlson (1998) is used for coding. For example, finding $\frac{dy}{dx}$ for $y = \sin^2(4x^2 + e^{\sqrt{2x-\cos e}})$ requires a student to recognize the composition of functions and the sequence guided by the chain rule to find the derivative of the above: a 1 for no evidence of considering the chain rule; a 2 for attempting to avoid the chain rule by expanding or rearranging the terms; a 3 for leaving out brackets where appropriate after finding the derivative; a 4 for minor error such as dropping a minus sign or putting one where unnecessary; and a 5 for a well-presented computation of the derivative showing correct use of the chain rule.

Data analysis is based on the initial genetic decomposition. The development of the chain rule schema is then described by explaining the observations from the data in terms of action, process and object conceptions based on the theoretical framework (Asiala, Brown, Devries, Dubinsky, Matthews, & Thomas, 1996). Students who are in the intra-stage of the chain rule schema development will be those who see the various rules for differentiation as not related. They would be able to solve some of the problems by simply applying rules which have been memorized and in some cases not remembered correctly. These are students who would be skilled at algebraic manipulations, easily able to assimilate rules and procedures in a cognitive structure that consists of a list of unconnected actions, processes and objects to produce correct answers.

Students in the inter-stage will show the evidence of having collected some or all the differentiation and integration rules in a group and perhaps provide the general statement of the chain rule without yet constructing the underlying structure of the relationships. That student would tackle the above mentioned tasks, by applying the power rule, not sure that he/she is using the chain rule. This student during interviews, and further questioning would explain the connection between his general statement of the chain rule and its applicability. Lastly a student who displays coherence of the understanding of a collection of derivative rules and the understanding of composition of functions as a schema will have moved to the trans-stage of development. He/she will be able to reflect on the explicit structure of the chain rule and be capable of operating on the mental constructions which make up his collection. Without stating the chain rule, this student will be able to use it proficiently. This student should at this stage be able to link function composition and decomposition with the differentiation and the integration and be able to link the two.

## Conclusions

This study is intended to address some of the difficulties students have related to the concepts of function, its composition and inversion. There are some evidence presented to support that the understanding of composition of functions is key to understanding the chain rule, in an American study (Clark et al., 1997).

The APOS paradigm, by using selected activities, will bring the students to the point of being better understand the chain rule and inform the teaching strategies for this concept. This theory has been scarcely explored mainly by researchers using an American context (Cottrill, Dubinsky, Nichols, Schwingendorf, & Vidakovic, 1996). Useful testing out of features of theoretical framework in the South African context was done to university students in real analysis (Brijlall & Maharaj, 2008) and continuity (Maharaj, Brijlall, & Govender, 2008). No such investigation has been done extensively in a South African context, hence, this study will inform us on how our students compare with students from the first world countries.

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