US-China Education Review B 3 (2011) 430-437
Earlier title: US-China Education Review, ISSN 1548-6613

Using Multiple Representations to Make and Verify Conjectures*

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This article reports on the results of research, the objective of which was to document and analyze the manner in which students relate different representations when solving problems. A total of 20 students attending their first year of university studies took part in the study. In order to design the problem, the underlying information in each representation was deemed to be the starting point of different inferences and different cognitive processes. The findings obtained make it possible to assert that the underlying information in each representation is not visible to all students and that a problem can foster handling of different representations, the making and verifying of various conjectures, as well as the transfer of knowledge acquired in previous courses.

Keywords: cognitive processes, relating representations, conjectures

Introduction

A consensus seems to exist among researchers in mathematics education, with respect to the importance of relating different representations of a mathematical concept in order to attain the solution to a problem. For example, when students solve a problem that involves the function concept, they draw a graph or build a table of values to represent it. From the graph or the table, it is possible to obtain the information relevant to the function, which can then be employed to arrive at the solution of the problem.

Some researchers (Parnafes & Disessa, 2004) have expressed themselves along such lines. On the one hand, they indicated that student reflections are tied to the representation and context that they are using. They further stated that each representation either highlights or hides aspects of a concept, and that when the students use several representations, they develop a more flexible understanding of the concept (p. 251). While on the other hand, they stated that the relationship among different representations also provides information regarding the cognitive processes of the students in the problem-solving process. The underlying information in each representation is the point of departure of different inferences and, consequently, of different cognitive processes (p. 252). The latter idea is the starting point of the research reported here, which was undertaken with students attending first year of university. The objective of the research is to document and analyze the manner, in which the students relate different representations while solving a problem.

Theoretical Framework

Since the relationship that exists among representations is the core of the research reported here, the

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authors of this article believe it is advisable to clarify that in this document the term "to relate representations" is used in the sense of Goldin and Kaput (1996). These authors pointed out that, "A person relates representations when he is able to integrate his cognitive structures in such a way that given an external representation, the individual is able to predict or identify its counterpart" (p. 416). The relationship among representations is considered as an important element in stimulating reflection in students (Parnafes & Disessa, 2004; Goldin & Kaput, 1996; Duval, 1988). It is possible to use different representations in two different ways: (1) using of representations in order to achieve greater understanding of a concept; and (2) using of a representation in order to promote cognitive processes, such as abstraction and generalization (Parnafes & Disessa, 2004).

Using of Representations in Learning a Mathematical Concept

Related with the use of representations in comprehension of a concept, each representation underlies information concerning the concept (Duval, 1988; 1993). For example, it is possible to regard the case of a function represented by a line, in which the relevant elements are the direction of the slope of the line, the angles that the line forms with the axis and the position of the line with respect to the origin of the vertical axis; each visual variable is associated with a meaningful unit in the algebraic expression. The direction of the line's slope is related to the sign of coefficient "x" in the equation "$y = ax + b$"; the angle formed by the line's intersection with the axis, to the absolute value of parameter a; and the position of the line with the intersection of the vertical axis, to the value b. In this sense (Duval, 1993), the adequate knowledge of a concept is considered as the invariant of multiple semiotic representations of the concept and it is accomplished when a student skillfully handles changes between representations.

Using of Representations to Make Students' Cognitive Processes Visible

Each representation provides specific information and promotes certain cognitive processes. For instance, some authors (Parnafes & Disessa, 2004, p. 252) have documented the work of students within a computer-aided learning environment, in which two representations were used. The first is a representation that simulates the movement of two turtles, while the second is a number representation made up of two lists of values, one of positions and the other of speeds. The two representations characterize the same structure—movement of two objects, yet each one is the starting point of different cognitive processes. Based on their data analysis, the authors identified two types of reasoning. In the one type, the students arrive at the solution by assigning a value to each variable in the problem, ensuring that all of the restrictions are complied with. In the other, the students create a mental model of the movement of the turtles and analyze those images in order to infer qualitative descriptions from those images.

Digital Technologies and Representations

One can also identify studies (Goldin & Kaput, 1996) that refer to the role of digital technologies and their contributions to having students relate representations. When digital tools provide resources for an individual to relate representations, the foregoing is achieved when based on an action in an external representation and the subject is able to predict and identify the results in another representation.

Each representation is associated with specific information (Parnafes & Disessa, 2004). The problem reported in the latter document was redesigned, so that in addition to using verbal and iconic representations, the students also build a table of values. The use of a spreadsheet (Excel) can be very useful in this type of activity due to three characteristics of these spreadsheets that may help students to make generalizations and gain a better understanding of the variables (Wilson, Ainley, & Bills, 2004) as follows: (1) analysis of the calculations; (2) use of notation; and (3) the possibility of feedback.

Since the students had used a spreadsheet in three previous activities, which were minimal technical problems related to use of the software and such problems are not cause for thought in this document. Therefore, we would not like to detract from the importance of the role of digital technologies in students' reflections.

Subjects, Methods and Procedures

The research was carried out with ten pairs of first-year university students. The activity reported on here was undertaken during two sessions of one hour and one half each. The sessions were videotaped and the students submitted their work both in writing and electronic files.

The Problem: The Triangular Stack of Boxes

The problem consists of determining how the quantities in a sequence of steps change and identifying that from the very first step a relationship among the quantities is maintained.

The idea is to build a triangular stack of boxes, as shown in Figure 1.

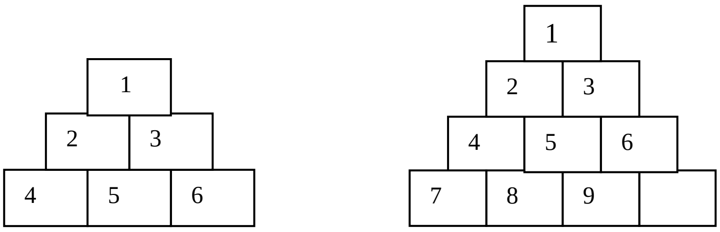


Figure 1. Source: Bassein (1993, p. 7).

The problem was presented to the students in the following way:

(1) Determining the number of boxes needed to form a stack of a given height: In this problem, the boxes are rectangular and with all the same size. The height of the stack will be measured in terms of the number of levels in the stack. For example, a stack made up of three levels will have a height of three;

(2) Beginning your explorations by drawing stacks made up of different levels;

(3) Writing down in a table the number of levels and boxes needed in each drawing.

Two representations are referred to in the problem wording: verbal and iconic. The instruction provided under clause (3) suggests to the students that they should make drawings and write down the values represented by different triangular stacks. The clause was included in order to guide the work of the students toward using a numerical representation and having them prepare a table of values that included the variables of levels and boxes. This was done re-broaching the ideas of some authors who indicated that tables of values contribute to facilitating identification by students of patterns and regularities among the series that make up the table, and that they are the basis of particular cognitive processes (Parnafes & Disessa, 2004).

The table of values suggested in clause (3) was also included with the intention of orienting the work of the students toward using an algebraic representation. It is important to mention at this point that to obtain a mathematical model makes it possible to determine the number of boxes needed to build a stack of a given height; it is indeed advisable to work with an algebraic representation.

The students worked on the problem during two sessions. A detailed analysis of their work was subsequently undertaken, which provided information on the manner in which they related the variables presented in the problem and the manner in which they developed a mathematical model to represent the relation between them. The actions carried out by the students were followed very closely, paying special attention to their handling of representations.

The questions outlined follow, which arose from the analysis of the data, are the focal point of this document:

(1) What information do the students obtain from each representation that they use while solving the problem?

(2) In what manner do the students relate different representations?

Data Analysis

Upon analyzing the data, evidence that enabled us to respond to the questions was identified. The manners of relating representations arise and are exemplified with the work of three students: Carlos, Luis Miguel and Leonardo, the work of who is representative of the work carried out by the remainder of the group. The data used come from different sources. Brief episodes of the videotaped sessions have been taken up again, hence, the actions and gestures are explained in parenthesis and pauses are indicated by the use of ellipsis.

First Manner: Carlos Related Verbal, Iconic, Numerical and Algebraic Representations

Carlos read the wording of the problem (verbal representation), drew stacks containing different levels (iconic representation) and carried out a numerical exploration (numerical representation). The instruction to draw stacks with different levels was a contributing factor in his being able to identify a relationship between the height of the stack and the number of boxes in the base level, as is shown in the following episode:

> Carlos: "… so, analyzing the problem you gave us, the base is equal to the height sought; so we have… if the base is two, we add two plus one, then, if we have a height of seven, the base is seven".

The drawings done together with the numerical exploration also contributed to Carlos' ability to identify that the total number of boxes needed to create a stack of a given height was equal to the sum of the number of boxes in the base, plus the number of boxes in a stack with one level less, as he explained to the group:

> Carlos: "… if its height is seven, its base is seven. So finding the total number of boxes is seven, plus six, plus five, plus four, plus three, plus two, plus one".

The explanation provided by Carlos attested to his having associated the total number of boxes with an expression that enabled him to calculate the sum of the natural numbers from 1 to *n*. Carlos was already familiar with this expression.

> Carlos: "I don't know if you remember, but this formula was given to us by our Physics professor when he asked us what the sum of one though one hundred was and that was when he gave us the formula, (writes the formula on the whiteboard)" (see Figure 2).

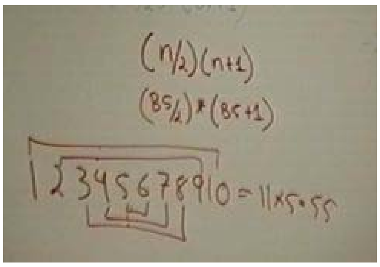


Figure 2. Carlos writes a formula on the whiteboard.

> Carlos: "So to find the total number of boxes, the base is seven, plus six, plus five, plus four, plus three, plus two, plus one, in other words, all of its previous numbers, and thus, we have the formula, *n* is the number of levels".

With a few examples, Carlos checked that the formula worked to calculate the number of boxes needed to create a stack of a given height. It is important to point out here that it was not simply a matter of his having memorized a formula; the explanation he gave his classmates and what he wrote down on the whiteboard confirmed that he had established a relationship between the numerical representation of the sum of the natural numbers from 1 to *n* and the algebraic representation that corresponds to that sum, as seen in Figure 2. What is more, Carlos was able to transfer the information he obtained from working with the verbal, iconic and numerical representations to a new context.

Second Manner: Luis Miguel Related Verbal, Iconic and Numerical Representations

Luis Miguel began his exploration on a sheet of paper, in which he represented particular cases. It is noteworthy to mention that unlike Carlos, Luis Miguel counted the number of boxes needed to obtain a stack of a given height. Apparently, he did not consider the number of boxes needed to create a stack with one level less. Figure 3 depicts Luis Miguel's drawings, submitted in his written report, where he had one stack made up of six levels with twenty-one boxes, one stack made up of seven levels with twenty-eight boxes.

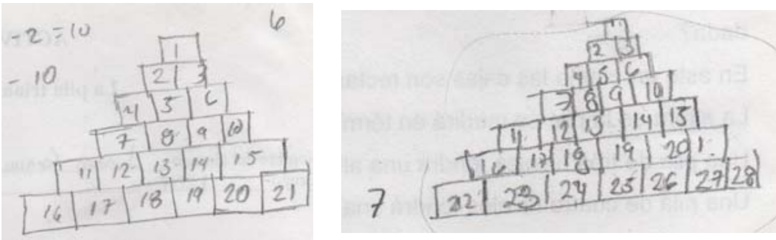


Figure 3. Luis Miguel draws particular cases.

In the episode included below, Luis Miguel explained to his classmates how he went from the wording of the problem (verbal representation) to the drawings of stacks containing different levels (iconic representation), from which he obtained the information needed to build a table of values (numerical representation).

> Luis Miguel: I created a table in Excel (inaudible) and by levels I started with zero…
>
> Researcher: You worked directly in Excel?
>
> Luis Miguel: No (inaudible) first on a sheet of paper (points with his hand to the shape of a triangular stack).

In Figure 4, Luis Miguel explained the manner in which he developed a table of values on the spreadsheet. As can be seen during his explanation, the student used Excel syntax.

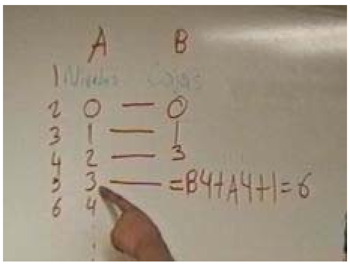


Figure 4. Luis Miguel explains his work with the spreadsheet.

The episode outlined below includes a transcription of Luis Miguel's participation and represents evidence that the numerical representation contributed to his identification of a relationship among the values in the table. The relation enabled him to determine the number of boxes needed to build a stack of a given height. Nonetheless the operation that he proposed, B4 + A4 + 1, which corresponded to adding the number of boxes needed to create a two-level stack, plus the number of boxes in the base level of that stack, plus one, made it possible to infer that Luis Miguel did not identify the same relation as Carlos did. Carlos pointed out that if its height was seven, its base was seven.

> Luis Miguel: For one level it's one box: for two levels, 3 boxes... And what I did was: add the two previous ones, plus 1; for example, for 3…, (He writes the formula = B4 + A4 + 1, and using it he calculates the number of boxes needed to create a three-level stack).

> Luis Miguel: So I would be left with three plus two is five, plus one is six. And that's the way I would fill it in up to 50.

In order to encourage Luis Miguel to reflect upon the fact that the expression he have come up with only worked in particular cases, the researcher intervened as follows:

> Researcher: And what about if you have a stack made up of 75 levels?

> Luis Miguel: I would have to make the list all the way down to 75.

> Luis Miguel: The formula depends on the whole list (Moves his hand upward to indicate the previous numbers in the table).

> Luis Miguel found a relation based on his analysis of particular cases, but he was unable to find a general expression to determine the number of boxes needed to build a stack of a given height.

Third Manner: Leonardo Related Verbal, Iconic, Numerical and Algebraic Representations

Leonardo initially worked with the verbal representation, then changed to the iconic representation and drew stacks with different heights. From his analysis of the figures, he obtained information that he used to build a table in Excel. Table 1 shows the table he built. Column 1 indicates the number of levels, while column 2 indicates the number of boxes.

Table 1

Excel Table Built by Leonardo

Number of levels	Number of boxes
1	1
2	3
3	6

The numerical exploration undertaken by Leonardo in the table facilitated his ability to identify relations between number of levels and number of boxes variables. The extract provided below portrays the analysis he carried out and the explanation he gave to his classmates.

> Leonardo: The way I solved it is that you have levels and numbers of boxes: for one level, one box, for two levels, three boxes, and then for three levels, six boxes. And here, what I found was that this quantity of boxes (Points to the value three in the number of boxes column) is equal to the sum of the previous level (Points to the value one in the number of boxes column), plus the number of levels that you want to have (Points to value two in the number of levels column).

The work done subsequently by Leonardo was divided into two sections. In each section, one could identify a relation between the number of boxes and number of levels variables.

First Relation Established by Leonardo

In the extract provided follow, Leonardo proposed an expression to obtain the number of boxes needed to create a stack containing *n* number of levels.

> Leonardo: I found that the number of boxes is equal to the number of levels that you want to have, plus the number of boxes that you had before, and that gave the result. So, I applied the formula to calculate them, equal to…

In Figure 5, he has drawn a table on the whiteboard and uses Excel syntax cell B2 + A3 and it gave the authors a result of six. So then, the authors just copy and paste (Uses Excel-related language, copy and paste).

Leonardo identified that the number of levels in each stack coincided with the number of boxes in the base; he then used that relation to determine the total number of boxes in each stack.

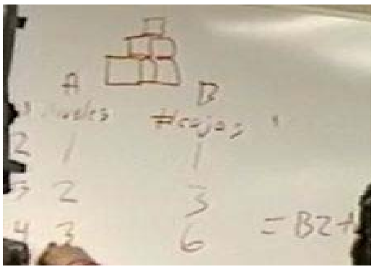


Figure 5. Table built by Leonardo.

Second Relation Established by Leonardo

Leonardo obtained the total number of boxes by adding the number of boxes needed to create a stack with one level less, plus the number of levels from the next stack. In the episode transcribed follow, Leonardo explained that in order to obtain the total number of boxes, it was necessary to build a table with a consecutive number of levels.

> Leonardo: The consecutive continued to be maintained (the column with the number of levels), but when there was a later one, from 2 to 7 for example, then it didn't give the result, it didn't work. I agreed that this result was correct for consecutive stacks that increased one by one.

> Leonardo: So I began to investigate what relation existed between those two columns (the number of levels and number of boxes columns), and thought that this result was a function of this number, with something that could be a sum with something or the product of something, and so I ended up dividing this column (the number of boxes column) by this one (the number of levels column) and found that this was the result (Added an additional column, and writes down the result of the divisions).

> Leonardo: I divided B1 by A1 (Divides the number of boxes by the number of levels) which gives me one, then one point five, then two, then two point five, and so on and so forth, point five more each time.

> Leonardo wanted to determine a formula that related the number of levels (column A) with the results of quotients (column C), so as to obtain the number of boxes without having to depend on previous results.

> Leonardo: And then I began to investigate what relationship existed between column A and column C. I knew that the values in column C times the values in column A give the values in column B.

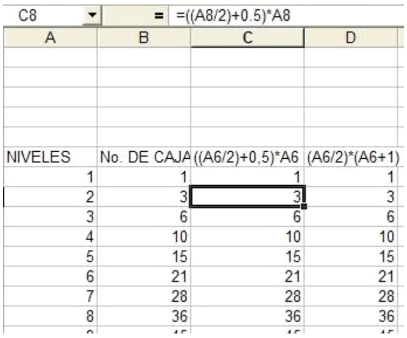
> Leonardo: I cannot depend on these results (the values in column B) because that is the result I want to arrive at. So, what I did was to see what the relationship between column A and column C was… And I found that it was 3; number of levels divided by 2 plus 0.5 (then he does calculations for the remaining values in the table).

Leonardo: So to be able to arrive at that result, I inputted in the cell the same number as the number of levels, say A1 over 2 plus 0.5, all of this times A1, and that way it no longer depends on the result of B.

Then, he calculated a result for a 10-level stack as follows:

$$\left(\frac{10}{2}+0.5\right)\times(10)=55$$

In Figure 6, one can identify the operations that Leonardo carried out in Excel.



C8 = =((A8/2)+0.5)*A8

A	B	C	D
NIVELES	No. DE CAJA	((A6/2)+0,5)*A6	(A6/2)*(A6+1)
1	1	1	1
2	3	3	3
3	6	6	6
4	10	10	10
5	15	15	15
6	21	21	21
7	28	28	28
8	36	36	36

Figure 6. Table built by Leonardo in Excel.

Conclusions

From our analysis of the data, we were able to find evidence that makes it possible to respond to the questions raised in this document. Each representation provides specific information concerning the problem. The underlying information in each representation is not visible to all of the students, as can be seen in the work of Luis Miguel and Leonardo. The problem put the students' triggers usage of different representations, the making and verification of conjectures as well as the transferring of knowledge acquired in previous courses, as can be seen in the work done by Carlos.

Use of the numerical representation, which was suggested in the wording of the problem, was a contributing factor in enabling the students to identify regularities and relations among the variables number of boxes and number of levels in the manner pointed out by Parnafes and Disessa (2004). The language used by Carlos and Leonardo provided proof that they kept the context of the problem in mind at all times. The written reports submitted by Luis Miguel as well as his participation, enabled us to infer that he did not relate the numerical representation with the wording of the problem and the figures of the stacks, even when he proposed an expression to calculate the number of boxes needed to build a stack.

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