HEURISTICS OF TWELFTH GRADERS BUILDING ISOMORPHISMS*

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This report analyzes the discursive interactions of four students to understand what heuristic methods they develop as well as how and why they build isomorphisms to resolve a combinatorial problem set in a non-Euclidian context. The findings suggest that results of their heuristic actions lead them to build isomorphisms that in turn allow them to justify a conjecture of theirs, using transitivity.

This research report focuses on four twelfth graders who, in an extended, self-structured problem-solving session, build heuristics and isomorphisms. The study arises from our general research program into the development of mathematical ideas by individual students as they work collaboratively in a small group. Specifically, this investigation connects to our inquiry into students' discursive practices (Powell & Maher, 2002) and how through their discursive practices they structure their own investigation and build mathematical practices and ideas appropriate for their problem task. The data is part of an ongoing, fifteen-year longitudinal, cross-sectional research project of the Robert B. Davis Institute for Learning, directed by Maher, that has been conducted in public elementary and secondary schools in a suburban, working-class, and immigrant town of New Jersey. Overall, our longitudinal study aims to contribute basic scientific understanding of cognitive behaviors as well as pedagogical conditions for which mathematics learning occurs as a process of sense making.

The participants in the present study are four students, Brian, Jeff, Michael, and Romina, in their senior year of high school who, from their entry into first grade have participated in mathematical activities of our longitudinal study. Over the years, these students have engaged tasks from several strands of mathematics, including algebra, combinatorics, probability, and calculus both in the context of classroom investigations as well as in after school settings (Maher, 2002). In this study, during an after-school problem-solving session, the students collaborate on a culminating task—The Taxicab Problem—of the research strand on combinatorics:

> A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.
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> What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

Accompanying this problem statement, the participants have a map, actually, a 6 x 6 rectangular grid on which the left, uppermost intersection point represents the taxi stand. The three passengers are positioned at different intersections as blue, red, and green dots,

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respectively, while their respective distances from the taxi stand are one unit east and four units south, four units east and three units south, and five units east and five units south.

Besides the new, non-Euclidean geometric setting, this task has an underlying mathematical structure and encompasses concepts that resonate with those of other problems the participants have worked on in the longitudinal study. They, therefore, potentially revisit and deepen mathematical ideas they have already built as well build new ideas. Their implicit task was to formulate and test conjectures. We explicitly announced that they were to explain and justify conclusions. After they worked on the problem for about an hour and a half, we listened as they presented their resolution and asked questions to follow movements in their discourse toward further justification of their solution. Their resolution goes beyond the problem task to generalize it and to propose isomorphic propositions. A research objective is to inquire into and track how the participants develop their resolution of the problem task. Expressly for this report, we explore the following two questions: What heuristics do the participants employ? How and why do they build an isomorphism between the problem task and other problems on which they have worked?

THEORETICAL FRAMEWORK

Our theoretical perspective involves notions concerning the development of representations, models of the growth of understanding, and ideas about the generation of meaning references for which are detailed in Maher (2002). Here we build on this perspective and incorporate into it specific criteria for noticing, within the fine details of discourse, propositions that lead toward building isomorphisms. We explore a conceptual category about the contents of mathematical experience as proposed by Gattegno (1987). He theorizes how the human capacity to be aware of something and attach importance to can beget different sciences. For him, the instrument of knowing that allows scientists to be cognizant of the content of their awareness is "a dialogue of one's mind with one's self" (p. 6). Different sciences develop from the repeatable findings that stem from dialogues of minds with themselves specializing, for instance, on different human senses and on specific ways of knowing. He discusses a special "conquest of the mind at work on itself":

> mathematics…is the clearest of the dialogues of the mind with itself. [It] is created by mathematicians conversing first with themselves and with one another.…Based on the awareness that relations can be perceived as easily as objects, the dynamics linking different kinds of relationships were extracted by the minds of mathematicians and considered per se. (pp. 13-14).

From Gattegno's view on the psychological and dialogic development of mathematics, three notions of the contents of human experiences upon which the discipline is built can be identified: objects, relations among objects, and dynamics linking different relations. As the data from this study show, an additional category concerns heuristics. It pertains to methods of responding to questions raised in dialogues of the mind with the self. Extending Gattegno's categories, in mathematics, the content of experiences, whether internal or external to the self, can be *objects, relations among objects, and dynamics linking different relations*, and *heuristics*.

The notion of dynamics linking different relations provides guidance for identifying isomorphic propositions. Powell (1995) gives an example of this notion. It concerns the correspondence one can perceive between the two processes of (a) raising 2 to consecutively increasing integral powers and (b) the multiplicative process of doubling. In each process, we have objects (2 and the implicit objects $\ldots \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8\ldots$) and a relation (raising to a power and doubling). In the study, we employ this notion to identify and then analyze the propositions of participants that contribute to building of isomorphisms.

METHOD

Our data sources consist of the problem task; a video record of about 100 minutes of the activity of the four participants from the perspective of two video cameras; a transcript of the videotapes combined to produce a fuller, more accurate verbatim record of the research session; the participants' inscriptions; and researcher field notes. The participants' inscriptions are scanned and saved as picture documents. The video recordings are digitized, compressed, and stored on five compact disks as MPEG1 files. The transcript is a textual rendering of verbal interactions, specifically, turn exchanges among the participants and between them and researchers and in all consists of 1,619 turns at talk. Our analytic method employs a sequence of phases, informed by grounded theory (Charmaz & Mitchell, 2001), ethnography and microanalysis (Erickson, 1992), and approaches for analyzing video data (Pirie, 1996). Specifically, our method of data analysis involves the following nine non-linear, interacting phases: (1) attentively viewing the videotapes several times without intentionally imposing a specific analytic lens; (2) describing consecutive time intervals; (3) identifying critical events; (4) transcribing the video record; (5) inductive and deductive synchronous coding of transcript, videotape, and inscriptions; (6) writing analytical memoranda; (7) categorizing codes, identifying properties, and dimensionalizing properties within categories; (8) constructing a storyline; and (9) composing a narrative. (For an elaboration and examples of these phases, see Powell, Francisco, & Maher, 2001).

RESULTS

The problem-solving session lasted for approximately 1 hour and 40 minutes. Analysis of the video data reveals that, without assistance from the researchers, the participants through their conversational exchanges structure their own investigation. Further analyses of their discourse and inscriptions reveal that they use their time to understand and plan how to resolve the problem task; develop problem-solving strategies and overcome heuristic hurdles; hypothesize and create combinatorial algorithms; build explanations and justifications of their ideas; challenge each other to clarify their explanations and justifications as well as accept challenges of the same from researchers; and formulate isomorphisms, focusing on the one between the Taxicab Problem and the Towers Problem.[1]

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[1] The Towers Problem is to build towers (for example, with Unifix cubes) of particular heights when selecting from a certain number of colors. From grades 3 to 10, the participants have worked on versions of this problem with varied conditions.

During the session, the participants develop and employ sixteen heuristics. The following are their different heuristics with indication of when ([hours:minutes:seconds]) from the start of the session they initially implement each one: (1) counting routes from the taxi stand to a pick-up point while outlining without drawing the routes [0:02:30]; (2) traveling on the grid lines only east and south [0:04:41]; (3) parceling out different mini-tasks among group members as well as collecting and recording the data they generate [0:05:59]; (4) counting routes to a pick-up point while drawing the routes on the same sub-grid [0:06:15]; (5) attending to dynamical links among objects and relations between two systems [0:07:31]; (6) attending to numeric patterns in generated data [0:12:10]; (7) doing easier sub-problems; counting routes from the taxi stand to nearby intersection points while outlining without drawing the routes [0:14:53]; (8) counting routes from the taxi stand to nearby intersection points while drawing the routes on the same sub-grid [0:16:12]; (9) parceling out the same mini-task, each counting routes to intersection points nearby the taxi stand, drawing them on the same sub-grid, as well as recording and comparing resulting data [0:16:50]; (10) planning to count systematically points nearby the taxi stand and anticipating that a numerical pattern will emerge [0:22:12]; (11) talking aloud how one is finding all shortest routes an intersection point [0:24:23]; (12) drawing each route between the taxi stand and an intersection point on a separate sub-grid [0:24:23]; (13) finding opposite routes to each drawn route to ensure that all possible routes are found [0:25:00]; (14) parceling out mini-tasks to compare the data generated from different combinatorial algorithms [0:39:08]; (15) building isomorphisms among the Taxicab, Tower, and Pizza Problems, using Pascal's triangle as an iconic representation upon which to build the isomorphisms [1:02:37]; and (16) processing their findings with researchers to see *where they lead themselves* through their presentation of their ideas [1:04:34]. Some participants supplant some heuristics with others, and some heuristics once initiated remain active strategies for some participants.

With these heuristics, the participants generalize the problem task and propose isomorphic propositions. They notice relational connections between this problem task and others on which they have worked. They develop combinatorial algorithms with which they generate reliable data from which to perceive numerical patterns. Based on these patterns, they conjecture that the underlying mathematical structure is Pascal's triangle of binomial coefficients. To convince themselves of the veracity of their conjecture, they build an isomorphism between the problem task and the Towers Problem. They know from previous work on block-tower tasks that Pascal's triangle underlies their mathematical structure. In what follows, we further focus our analysis on the isomorphism they build by identifying their discursive propositions about dynamical links that establish one-to-one correspondences between, on the one hand, objects and relations or actions in one system and, on the other hand, objects and relations of another system in such a way that an action on objects of one system maps to an analogous action on the corresponding objects in the other system.

A prerequisite to articulating a proposition that indicates an isomorphism is to attend to particular features of objects and relations among the objects within each system to determine whether dynamical links can be formulated between the systems. Early in the session, the participants manifest embryonic thinking about an isomorphism.

Romina wonders aloud: "can't we do towers on this" (turn 159).[2] Her public query catalyzes a negotiatory interlocution among Michael, Jeff, and her. Jeff, responding immediately to Romina, says, "that's what I'm saying," (turn 160) and invites her to think with him about the dyadic choice ("there or there" turn 162) that one has at intersections of the taxicab grid. Furthermore, he wonders whether one can find the number of shortest routes to a pick-up point by adding up the different choices one encounters in route to the point (turn 162). Romina proposes that since the length of a shortest route to the red pick-up point is 10, then "ten could be like the number of blocks we have in the tower" (turn 169). Romina's query concerning the application of towers to the present problem task prompts Michael's engagement with the idea, as well. As if advising his colleagues and himself, he reacts in part by saying, "think of the possibilities of doing this and then doing that" (turn 180). While uttering these words, he points at an intersection; from that intersection gestures first downward ("doing this"), returns the to point, and then motions rightward ("doing that"). Similar to Jeff's words and gestures, Michael's actions also acknowledge cognitively and corporally the binomial aspect of the problem task. He, Jeff, and Romina have put into circulation the prospect of as well as insights for building an isomorphism between the Taxicab and Towers Problems.

The prospect and work of building such an isomorphism reemerges several more times in the participants' interlocution. With each reemergence, the participants further elaborate their insights and advance more isomorphic propositions. Eventually the building of isomorphisms dominates their conversational exchanges. Approximately thirty-five minutes after Romina first broached the possibility of relating attributes of the Towers Problem to the problem at hand, the participants reengage with the idea. Romina speculates that between the two problems one can relate "like lines over" to "like the color" and then "the lines down" to the "number of blocks"(turn 738). What is essential here is Romina's apparent awareness that each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem.

Romina uses this insight later in the session. She transfers the data that she and her colleagues have generated from a transparency of a 1-centimeter grid to plain paper. Their data are equivalent to binomial coefficients. She identifies one unit of horizontal distance with one Unifix cube of color *A* and one unit of vertical distance with one Unifix cube of color *B*:

> Like doesn't the two- there's- that I mean, that's one- that means it's one of *A* color, one of *B* color [pointing to the 2 in Pascal's triangle]. Here's one- it's either one- either way you go. It's one of across and one down [pointing to a number on the transparency grid and motions with her pen to go across and down]. And for three that means there's two *A* color and one *B* color [pointing to a 3 in Pascal's triangle], so here it's two across, one down or the other way [tracing across and down on the transparency grid] you can get three is two down [pointing to the grid]. (turn 1210)

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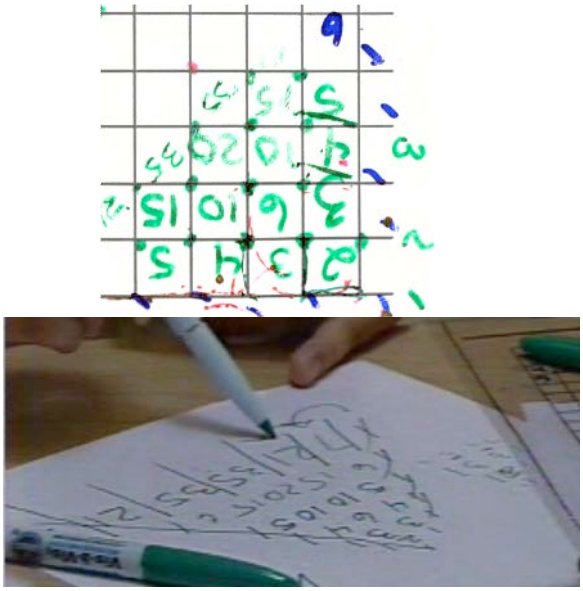
[2] For Romina and other participants in the longitudinal study, this comment is pregnant with mathematical and heuristic meaning derived from their constructed, shared experiences with tasks and inscriptions in the combinatorial and probability strands of the study (see, for instance, Kiczek, 2000; Martino, 1992; Muter, 1999).

Furthering the building of their isomorphism, Michael offers another propositional foundation. Pointing at their data on the transparency grid and referring to its diagonals as rows, he notes that each row of the data refers to the number of shortest routes to particular points of a particular length. For instance, pointing the array—1 4 6 4 1—of their transparency, he observes that each number refers to an intersection point whose "shortest route is four" (turn 1203). Moreover, he remarks that one could name a diagonal by, for example, "six" since "everything [each intersection point] in the row [diagonal] has shortest route of six"(turn 1205). In terms of an isomorphism, Michael's observation points in two different directions: (1) it relates diagonals of information in their data to rows of numbers in Pascal's triangle and (2) it notes that intersection points whose shortest routes have the same length can have different numbers of shortest routes.



A B

Figure 1. Participant's data arrays (from their perspective): (A) In green, empirical data of shortest routes between the taxi stand and nearby intersection points. Jeff wrote the ones in blue to augment the appearance of the numerical array as Pascal's triangle. From the participant perspective, to the left of Jeff's numbers, Romina wrote in green the numbers 1, 2, and 3 to indicate the row numbers of the triangular array. (B) The first five rows contain empirical data; the remaining two rows contain assumed data values based on the addition rule for Pascal's triangle.

Later in responding to a researcher's question, the participants develop a proposition that relates how they know that a particular intersection in the taxicab grid corresponds to a number in Pascal's triangle. They focus their attention on their inscriptions, A and B, in Figure 1. Michael and Romina discuss correspondences between the two inscriptions. Referring to a point on their grid that is five units east and two units south, Romina associates the length of its shortest route, which is seven, to a row of her Pascal's triangle by counting down seven rows and saying, "five of one thing and two of another thing"(turn 1313). Michael inquires about her meaning for "five and two" (turn 1314). Both Romina and Brian respond, "five across and two down"(turns 1317 and 1318). She

then associates the combinatorial numbers in the seventh row of her Pascal's triangle to the idea of "five of one thing and two of another thing," specifying that, left to right from her perspective, the first 21 represents two of one color, while the second 21 "is five of one color" (turn 1320), presuming the same color. Using this special case, Romina hints at a general proposition for an isomorphism between the Taxicab and Towers Problems.

The above presents evidence that students work to build an isomorphism during the course of the problem-solving session. The content of the phases include the following with indication of when from the start of the session each occurs: (1) there exists a relationship between the Towers and Taxicab Problems, [0:07:37]; (2) Similar to the Towers Problem, the Taxicab Problem has a dyadic choice or binomial aspect, [0:07:39 and 0:08:55]; (3) The length of a shortest route to an intersection point corresponds to the height of a tower, [0:08:15]; (4) Each of the two different directions of travel in the Taxicab Problem needs to be associated with different objects in the Towers Problem, [0:44:26]; (5) Rebuild the meaning of 2 to the n in the environment of the Towers Problem, [0:08:26 and 0:44:51]; (6) Identify one unit of horizontal distance with one Unifix cube of color A and one unit of vertical distance with one Unifix cube of color B, [1:14:59]; (7) A row "diagonal" of their data contains the number of shortest routes for intersection points whose shortest distance from the taxi stand is n, [1:16:00]; (8) Intersection points whose shortest routes have the same length can have different numbers of shortest routes, [1:16:37]; (9) A tower 3-high with 2 of one color and 1 of another color, to routes to a point 2 down and 1 across, [1:18:40]; and (10) Intersection point five units east and two south from the taxi stand corresponds to five of one thing and two of another thing and, therefore, go the seventh row of Pascal's triangle and the second and fifth entries of the triangle to find the number of shortest routes from the taxi stand to the intersection point five units east and two south from the taxi stand, [1:22:40].

DISCUSSION

The forgoing has presented the mathematical processes and strategies that participants employ as they resolve the problem task. Through their various heuristic actions, among other consequences, the participants generate data that they consider reliable. Reflecting on numerical patterns in their data, they conjecture that Pascal's triangle is the underlying mathematical structure of the problem task. How do they justify this conjecture? The data suggest that to justify their conjecture is the reason why the participants build an isomorphism between the problem task and the Towers Problem. Furthermore, to understand how they build their isomorphism, we have focused analytic consideration on one of their heuristics: *attending to dynamical links among objects and relations between two systems*. By doing so, we have identified the locus of how they build an isomorphism. We observe that early in the problem-solving session by attending to dynamical links three participants—Romina, Jeff, and Michael—articulate awareness of object and relational connections between their current problem task and a former one, the Towers Problem. Later, upon noticing that their array of data resembles Pascal's triangle and conjecturing so, the participants embark on building an isomorphism between the Towers Problem and the Taxicab Problem as an approach to justifying their conjecture since from previous experience they know that Pascal's triangle underlies the mathematical structure of the Towers Problem. In this sense, their strategy can be

interpreted as justifying their conjecture by transitivity: (a) Pascal's triangle is equivalent to Towers and (b) Towers is equivalent to Taxicab; therefore implying that (c) Pascal's triangle is equivalent to Taxicab. They know (a) is true and embark on demonstrating (b) to justify and conclude (c).

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