

# STRONG AND WEAK METAPHORS FOR LIMITS

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*The metaphorical nature of first-year calculus students' reasoning about limit concepts is explored using an instrumentalist approach. Analysis of written and verbal language reveals that, while these students used motion terminology profusely when discussing limits, it was typically not intended to signify actual motion and did not play a significant role in their reasoning about limiting situations. In contrast, many of these students' employed other non-standard metaphors, involving for example collapsing dimensions, to explore these situations and to build their emerging understanding of limit concepts.*

## INTRODUCTION

Although limit concepts are foundational to the study of calculus, they have proven especially difficult for beginning calculus students to understand. Williams (1990, 1991) has revealed many ways in which students try to reason about limits using insufficient, intuitive ideas and metaphors for the concept including boundaries, motion, and approximation. Lakoff and Núñez (Lakoff & Núñez, 2000; Núñez, 2000) have attempted to show how intuitive ideas can form a more rigorous metaphorical basis for understanding limit concepts, but their work is neither based in student data nor intended as a theory of learning.

This paper presents results of a study investigating actual students' spontaneous reasoning about limit concepts and the aspects which determine whether that reasoning is helpful or a hindrance to developing a stronger understanding of limits.

### **Instrumentalism**

Most previous research on students' understanding and learning of limit concepts has focused on the structural aspects of their knowledge. While it is important to account for the content and internal structure of knowledge, equally significant are the functional ways in which those knowledge structures are applied against specific problems. In order to address this aspect of knowledge in the theoretical perspective of this study, we turn to John Dewey's "instrumentalism."

To understand a human activity, according to Dewey, it is necessary to examine the ways in which relevant tools are applied *technologically* against problematic aspects of situations (Hickman, 1990). In its modern use, the word "technology" typically refers to physical inventions rather than cognitive tools used in mental activity. Dewey argued, however, that such Cartesian lines between environment and organism and between mind and body are not so definite. The same principles that apply to human physical tool use also apply to productive mental activity. For Dewey, describing tool use as technological meant that it is active, testable, and productive. A cognitive tool is selected and applied in a dynamic process which actively engages the attention of the individual. It is used to perform tests upon the problem that gave rise to its selection, and reciprocally, the tool is itself tested against the problem and evaluated for appropriateness. Fortuitous interactions between aspects of the tool and problem are complex, reciprocal, and implicative, thus

effecting change in both. The artifacts of this dialectic are new meanings, which as they emerge, present situations that may themselves become the object of further inquiry.

In this process, an original idea becomes more “coherent” and “densely textured.” Since inquiry is situated and ongoing, one cannot separate knowledge from the context of its origins; it is bound to the unique circumstances and processes through which it was created, and truth is emergent, not located externally. Consequently, Dewey’s focus is on the process of inquiry rather than on transient pieces of knowledge. Meaning for a proposition, symbol, or metaphor is defined in terms of the object’s function in particular productive activity, just as it is for a physical tool such as a computer or hoe.

### Metaphors

Consistent with an instrumentalist approach, Max Black’s “interaction” theory of metaphorical attribution asserts that one must regard the two subjects of a metaphor as a complex, interacting system (Black, 1962, 1977). This requires two levels in which new and old meanings must be held active together: first, with respect to distinct meanings of the metaphorical subject with and without the context of the metaphor, and second, with respect to the extension of meaning imposed on the literal subject by the system. Strong metaphors, such as those that would be necessary for supporting creative thinking, force the relevant concepts involved to change in response to one another. The resulting perspective created is one that would not otherwise have existed, that is, strong metaphors are ontologically creative. In such metaphorical reasoning, one cannot simply apply an antecedently formed concept of the metaphor as-is; something new and actively responsive to the situation is required of all concepts involved. If pursued, the implications can support a degree of discovery that leads far beyond one’s original thoughts, providing the complexity and richness of background implications necessary for generating new ways of perceiving the world.

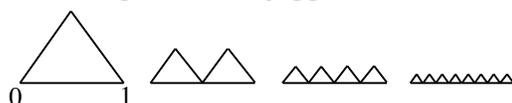
## METHODOLOGY

Students from a year-long introductory calculus sequence at a large southwestern university participated in interviews and submitted writing samples covering their attempts to make sense of problematic situations involving limit concepts. To observe functional aspects of students’ thought, data collection in this study was intended to encourage students’ technological application of their metaphors against challenging problems. Ten questions, roughly paraphrased below, were presented to the students. The first two were presented in clinical interviews with 9 students during which problematic aspects were called out by the interviewer for resolution by the subjects. Problems 3 and 4 were given to the entire class of 120 students as a short writing assignment, and the remaining six problems were offered as extra-credit writing assignments to the entire class with 25-35 students responding to each one. Follow-up interviews were conducted with an additional 11 students.

1. Explain the meaning of  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$ .

2. Let  $f(x) = x^2 + 1$ . Explain the meaning of  $\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$

3. Explain why  $0.\overline{9} = 1$
4. Explain why the derivative  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  gives the instantaneous rate of change of  $f$  at  $x$ .
5. Explain why L'Hospital's Rule works.
6. Explain how the solid obtained by revolving the graph of  $y = 1/x$  around the  $x$ -axis can have finite volume but infinite surface area.
7. Explain why the limit comparison test works.
8. Explain in what sense  $\sin x = 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$
9. Explain how the length of each jagged line can be while the limit has length 1.



10. Explain what it means for a function of two variables to be continuous.

Multiple rounds of open and axial coding (Strauss & Corbin, 1990) were used to identify emergent metaphorical themes in the language used by the students while confronting the problematic issues presented in these problems. Refined metaphorical categories were then analyzed for the following properties of instrumental use: support of implicative reasoning, commitment to the context of the metaphor, change in understanding of the problem, and change in meaning of the context.

## RESULTS

Eight major metaphorical contexts emerged from the data. Five contexts, involving collapse, approximation, closeness, infinity as a number, and physical limitation, possessed all four instrumental properties listed above, and were thus labeled “strong” metaphors. The other three contexts, involving motion, zooming, and arbitrary smallness, exhibited none of these instrumental properties and were thus labeled “weak” metaphors. A discussion of all eight contexts is beyond the scope of this paper (see Oehrtman, 2002 for details), so the following two sections present an example of one strong metaphor, collapse, and one weak metaphor, motion.

### **An Example of a Strong Metaphor for Limits: Collapsing Dimensions**

The collapse metaphor, while mathematically incorrect, did afford many students the ability to reason powerfully about the mathematics. In this context, students characterized a limiting situation by imagining a physical referent for the changing dependent quantity collapsing along one of its dimensions, yielding an object one or more dimensions smaller. A version of this metaphor, anecdotally familiar to most calculus teachers, is a fallacious justification of the fundamental theorem calculus. In this incorrect argument, students focus on the referent for the numerator of the difference quotient imagining, for example, a “final thin rectangle” of area underneath the curve. Ignoring the denominator,

they then argue that the limit as the width “becomes” zero causes that slice to “become” the one-dimensional height of the graph.

The collapse metaphor was observed in two main versions involving the definition of the derivative and volumes of unbounded solids of revolution. In both the interviews and written assignments about the definition of the derivative, approximately one third of the responses involved significant use of a collapse metaphor (3 of 9 students for Problem #2 and 36 of 98 students for problem #4). While describing the volume of a solid of revolution, nearly one sixth of the students used a collapse metaphor (5 of 31 students).

In the case of the definition of the derivative, students would describe a dynamic secant line through two points with the base and height of a right triangle as in a standard slope illustration. Moving these points closer together yields secant lines closer to the tangent, and the collapsed object is achieved when the two points are moved to the same location. The result is the tangent line at that point (see Figure 1). Some students characterized this as taking the slope at a single point while others reported thinking of the slope between two points at the same location.

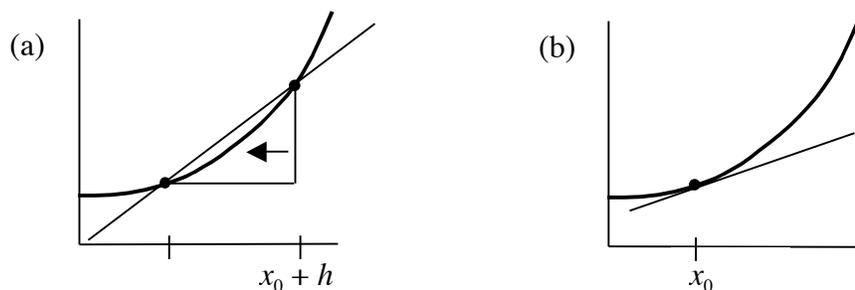


Figure 1: A secant line collapsing to a tangent. (a) Before collapse: a secant line between two points. (b) After collapse: a tangent line through “two points” at a single location.

Consider the following interview excerpt in which Amy wrestles with the role of  $h$  in the derivative definition and comes to the conclusion the two points become one:

As you take the limit, the value  $h$  is going to be getting continually smaller until it reaches zero at which point you'll be finding - the slope - of the line between  $[3,10]$  and  $[3,10]$ ...What you're doing is taking the limit of the slope - of what is - actually it's the slo-, it's not the slope of the tangent line, it's just what it ends up being, but you're taking the limit, you're taking the slope of two points. It only - and the limit is involved to allow you to eventually phase out the other point - and it just becomes to be, it would be just become the slope of the original po-, of the line at the original point... It involves taking  $x_2$  and - and making it gradually closer to  $x_1$  - until  $x_2$  is equal to  $x_1$ . Which - um - which you would also - you know  $y_2$  would be equal to  $y_1$ . And so - basically what you're doing is you're taking the slope of two points that are infinitely close together - so that they become the same point.

During the interviews, students were also asked to give an interpretation of the definition of the derivative for the position of a car as a function of time. The students all struggled with this new context, but while not ostensibly referring to their previous work, several who had already used the collapse metaphor gave a similar account for the new problem, imagining instantaneous speed to be an average taken over a time interval of no duration.

A different version of the collapse metaphor emerged while students attempted to explain how the volume of a solid of revolution could be finite (Problem #6). Here the dynamic object is a cross-sectional disk produced from revolving a point on the curve and varying in the dimension of its radius (Figure 2). The radius is imagined to decrease to zero at some definite point (possibly but not necessarily infinity) so that the two dimensions of the disk collapse to a point. Simultaneously imagining all of the collapsed “disks” beyond this point, one imagines the three dimensional solid “pinching off” to a one-dimensional line with no volume. Consider Karrie’s explanation for how the volume of a solid of revolution collapses in this manner:

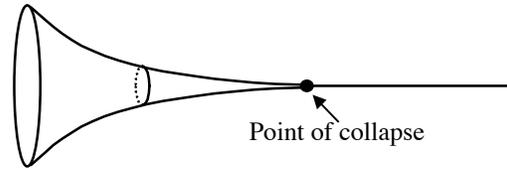


Figure 2: A collapsing solid of revolution.

The finite volume is not really finite in the same way that familiar containers such as bowls and ice cream cones are finite. The volume is the result of a line which stretches off into infinity into the  $x$  direction. Thus, we cannot actually imagine it pinching off and ending like an ice cream cone does. Rather, the radius of the disks in the volume gets so small as the  $x$  values get extremely large that at infinity the radius becomes zero in the same way that  $.9999 \rightarrow$  is actually exactly the same as 1. This progressively smaller disks actually add up to a finite amount. I imagine this "pinching off" as the two-dimensional volume (looking only at the disks, and taking two dimensions at a time) wrapping more and more closely around the one-dimensional line that is the  $x$ -axis, and then, at infinity, losing that radius entirely to zero and becoming one-dimensional, like the line. This is where volume ends, but surface area continues to exist in that single dimension

For Karrie, the collapse occurs “at infinity” but the object continues to exist beyond this where “volume ends, but surface area continues to exist in that single dimension.” This caused some concern for Karrie, and her subsequent explanation was full of hedges to soften her commitment to a complete idea of collapse.

Even though the collapse metaphor is mathematically incorrect, students like Karrie were able to use it to see valid connections between different types of limits (e.g.,  $0.\overline{9} = 1$  and a solid of revolution), between different contexts involving the same limits (e.g., the definition of the derivative and instantaneous velocity), and between different representations of limits (e.g., the “collapsed” tangent discussed above and slope via numerical approximations “collapsing” to an exact value). Making such connections enabled these students to organize their thoughts for further inquiry and to make substantial progress conceptualizing the meaning of limits in difficult contexts.

### **An Example of a Weak Metaphor for Limits: Motion**

Several researchers have found that a dynamic conceptualization of functions and variables is crucial to students’ understanding of key concepts in calculus such as limits (Monk, 1987, 1992; Tall, 1992; Thompson, 1994b). Unexpectedly, strong motion metaphors were nonexistent in the students’ responses in this study. While students frequently used words such as “approaching” or “tends to,” these utterances were not accompanied by any description of something actually moving. When asked specifically about their use of the word “approaches,” students almost always denied thinking of

motion and gave an alternate explanation. Motion for these students was something more “literal” as suggested here by Karen:

I guess with motion I think of - with motion I'm thinking force and work. I'm thinking of actual, like, locomotion. I don't necessarily think that that's what's happening when you're talking about a limit or talking about a number. I don't know that that's - I guess for me motion is a more literal term, like cars moving along the ground or I'm walking. That's more what I'm thinking than on the number line.

Only for Problem #10 about the continuity of functions of two variables were at least 10% of the students observed to discuss actual motion. In response to this question, 6 out of 25 students explicitly described something moving. Another 11 of the 25 respondents used motion language, but without applying it to an actual object. In the cases that something was imagined to be moving, that motion tended to be simply superimposed on another conceptual image that actually carried the structure and logic of their thinking. For example, all 6 of the motion references in responses to Problem #10 were to an object (an ant, a mouse, a moving truck, a baseball, the tip of a pencil, and a generic “you”) moving along the graph of the function. For both single- and two-variable cases, these students described the function as continuous if the object could move freely along the graph without having to traverse a jump or hole. In the following excerpt, a student describes continuity in terms of moving on the graph of a function of two variables.

A good example is the surface of a big wooden board. What does it mean for this to be continuous? Imagine a tiny mouse is on the board. If the board was continuous, the cute little mouse could venture all over the board without falling to its death. If the board wasn't continuous, maybe [it] contains a hole in the center.

Thus, the concepts about discontinuity for these students were presented as topological features of the surface (holes, cliffs, breaks, etc.). The addition of motion may add visual effect or drama, but not conceptual structure or functionality.

Whenever students used motion language, such as “approaches,” during the interviews, they were asked how they interpreted those terms. Of the 20 students interviewed, only eight ever agreed that they thought of motion when using a variation of the word “approaches.” Five of these students described the motion occurring on the graph of the function, one described motion along the  $x$ -axis, and two gave explanations in which it was impossible to tell what object was imagined to move. None of these students mentioned explicit motion other than during these exchanges initiated by the interviewer. Of the 12 students who denied imagining any type of motion, six explained that they thought of “approaches” as indicating closeness, five described picking points sequentially, and one student thought it meant that changing the value of input caused the output to change. Below are brief descriptions and examples of the responses from each of these categories.

Students’ descriptions of motion on the graph are exemplified by the previous excerpt involving a mouse running on the surface of a graph in which the reasoning is actually supported by static images such as breaks or holes. The single description of motion on the  $x$ -axis was not accompanied by corresponding motion on the  $y$ -axis. Instead she imagined moving to the point in question then “looking up” at the function value (or “the

hole” where the function should be.) Interestingly, she reported thinking this because the horizontal arrow in the limit notation indicated horizontal motion. Only one student explained that “approaching” meant that changing the input of a function caused a change in the output. In discussing the definition of the derivative, she described two points that “both approach the same limiting position” but denied that these points actually moved.

Half of the students (6 out of 12) who claimed to not think about motion when using words like “approaches” described a static closeness. For example, one used a metaphor of two train tracks meeting in the same place, with lengthy descriptions of “meeting up” in terms of being located in the same region in space. Karen, quoted earlier in this section drawing a distinction between “literal motion” and “approaching,” described the latter as meaning “close” in a very static sense:

I don't think that I necessarily picture motion, but picture that idea that you may have a value that your points are really close to that - so close that they - like in the first problem that they're almost that point but they're not quite that point, so I guess the way I think of approaches is that it's not necessarily moving from 3 to 2\_ to 2. You know, it's not moving, but it's the idea behind that it may not be - it may not be 2, but it's really close to 2

Finally, five students specifically explicated the term “approaches” as a process of sequentially selecting points closer to the point at which the limit was being evaluated. Here, Darlene describes this as picking numbers.

Interviewer: OK. The word “approaches” has a lot of - it sounds kind of like something is moving. Do you think of motion at all?

Darlene: No.

Interviewer: No?

Darlene: That’s just the way it’s always been explained to me.

Interviewer: OK. So, people have used that word before?

Darlene: Yeah. The book uses that word, too. [*laughs*] ... I don’t really think about it that way. I just, you know, pick numbers. [*points at several distinct points on the x-axis successively closer to 1*]...I'm not saying like a car approaches point a. I don't think of it as like that. I think of it as like, OK, I'm gonna take this value [*points at the x-axis near 1*]. The next time I'm going to take this value [*points at a spot closer to 1*], so it's approaching - approaching in intervals basically. I don't - yeah. I'm not thinking - that's what I'm thinking of. I'm not thinking of it like moving motion, like that. Like I take this interval - like I take a point, then I take this point, then I take this point, then it's approaching - yeah.

## CONCLUSIONS

Students in this study did not reason about limit concepts using motion metaphors. This is particularly surprising given the predominance of motion language used when talking about limits and abundant proclamations that intuitive, dynamic views of functions should help students understand limits. When these students did use motion language, their actual reasoning typically relied on a static graphical setting or, at most, the sequential selection of points. When they were asked to use limit concepts to think about

something new or approach a difficult problem, motion language tended to remain in the background and did not enter their descriptions as referring to anything actually moving.

Instead, other metaphors, such as collapsing dimensions, surfaced. This research found students using such metaphors as organizers of ideas and touchstones for reasoning. These metaphors became tools with which students were able to probe difficult problems, ask interesting questions, and develop further connections. They supported dynamic mental imagery that students were able to manipulate, extracting conclusions about the relevant mathematics. Although such reasoning was often technically incorrect, it remained a productive tool for the students' emerging understanding.

Such results suggest that research cannot fully uncover the nature of students' metaphors by examining only their surface language and responses to direct questions about their conceptualizations of the topic. Not only does this methodology miss the different structures that might appear in such problem solving contexts, but it also lacks the important characterizations of how conceptual tools are actually applied, of the questions the tools are used to ask and the resulting answers, and of the changes the tools undergo in the process. Students' reported structural organization of mathematical concepts does not account for their actual use of those ideas; research must look at richer data on their functional application of ideas in addition to their structure and logic.

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