REDUCING ABSTRACTION: THE CASE OF ELEMENTARY MATHEMATICS

Orit Hazzan, Technion – Israel Institute of Technology

Rina Zazkis, Simon Fraser University

There is a growing interest in the mathematics education community in the notion of abstraction and its significance in the learning of mathematics. "Reducing abstraction" is a theoretical framework that examines learners' behavior in terms of coping with abstraction level. This article extends the scope of applicability of this framework from advanced to elementary mathematics.

The notion of abstraction in mathematics and in mathematical learning has recently received a lot of attention within the mathematics education research community. The significance of this topic, as well as the magnitude of community interest was highlighted at the 2002 PME Research Forum #1. The purpose of this research forum was to discuss and critically compare three theories of abstraction, all aimed at providing a means for the description of the processes involved in the emergence of new mathematical mental structures. The forum was geared towards formulating an integrated theoretical framework that may serve to explain a vast collection of observations on mathematical thinking.

This article examines the notion of abstraction from the perspective of "reducing abstraction" – a mental activity of coping with abstraction. The theoretical framework of reducing abstraction (Hazzan, 1999) is usually associated with advanced mathematical thinking. Here we use it to describe and explain the mathematical thinking of preservice elementary school teachers on topics of elementary mathematics. Our contribution is twofold: (a) we provide a different perspective on the notion of abstraction in the learning of mathematics, and (b) we expand the scope of abstraction theories by focusing on elementary mathematics.

THEORIES OF ABSTRACTION IN MATHEMATICAL LEARNING

Abstraction is a complex concept that has many faces. As such, in a general context it has attracted the attention of many psychologists and educators (e.g., Beth and Piaget, 1966). In the more particular context of mathematics education research, abstraction has been discussed from a variety of viewpoints (cf. Tall, 1991; Noss and Hoyles, 1996; Frorer, Hazzan and Manes, 1997). There is no consensus with respect to a unique meaning for abstraction; however, there is an agreement that the notion of abstraction can be examined from different perspectives, that certain types of concepts are more abstract than others, and that the ability to abstract is an important skill for a meaningful engagement with mathematics.

The aforementioned research forum was assembled in an attempt to explore the variety of interpretations and the multi-faceted nature of abstraction. The theme of reducing abstraction builds on this variety, focusing on learner's mental activities. Similarly to other theories of abstraction, the theme of reducing abstraction, we believe, has "the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice." (Dreyfus and Gray, 2002, p.113)

THE THEME OF REDUCING ABSTRACTION

The theme of reducing abstraction (Hazzan, 1999) was originally developed to explain students' conception of abstract algebra. Abstract algebra is the first undergraduate mathematical course in which students "must go beyond learning 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes (problems)." (Dubinsky, Dautermann, Leron and Zazkis, 1994, p. 268). Indeed, it is in the abstract algebra course that students are asked, for the first time, to deal with concepts that are introduced abstractly. That is, concepts are defined and presented by their properties and by an examination of "what facts can be determined just from [the properties] alone." (Dubinsky & Leron, 1994, p. 42). This new mathematical style of presentation leads students to adopt mental strategies which enable them to cope with the new approach as well as with a new kind of mathematical objects. The theme of reducing abstraction emerged from an attempt to explain students' ways of thinking about abstract algebra concepts. The following description is largely based on Hazzan (1999).

The theme of reducing abstraction is based on three different interpretations of *levels of abstraction* discussed in literature: (a) abstraction level as the quality of the relationships between the object of thought and the thinking person, (b) abstraction level as reflection of the process-object duality, and (c) abstraction level as the degree of complexity of the concept of thought. It is important to note that these interpretations of abstraction are neither mutually exclusive nor exhaustive. What follows is a brief description of each of the above three interpretations.

(a) The interpretation of *abstraction level as the quality of the relationships between the object of thought and the thinking person* stems from Wilensky's (1991) assertion that whether something is abstract or concrete (or on the continuum between those two poles) is not an inherent property of the thing, "but rather a *property of a person's relationship to an object*" (p. 198). In other words, for each concept and for each person we may observe a different level of abstraction that reflects previous experiential connection between the two. The closer a person is to an object and the more connections he/she has formed to it, the more concrete (and the less abstract) he/she feels about it. Since new knowledge is constructed based on existing knowledge, unknown (hence abstract) objects and structures are constructed based on existing mental structures. Based on this perspective, some students' mental processes can be attributed to their tendency to make an unfamiliar idea more familiar or, in other words, to make the abstract more concrete. This is consistent with Hershkowitz, Schwarz and Dreyfus (2001) perspective that emphasizes the learner's role in the abstraction processes. They claim that "abstraction depends on the personal history of the solver". (p. 197). Specifically, based on Davidov's theory (1972/1990) they claim that "when a new structure is constructed, it already exists in a rudimentary form, and it develops through other structures that the learner has already constructed". (p. 219). Accordingly, abstraction is defined as "an activity of vertically reorganizing previously constructed mathematics into a

new mathematical structure". Vertical mathematization is "an activity in which mathematical elements are put together, structured, organized, developed etc. into other elements, often in more abstract or formal form than the originals." (Hershkowitz, Parzysz and van Dermolen, 1996 in Hershkowitz et al., 2001, p. 203).

(b) The interpretation of *abstraction level as reflection of the process-object duality* is based on the process-object duality, suggested by several theories of concept development in mathematics education (Beth & Piaget, 1966; Dubinsky, 1991; Sfard, 1991, 1992; Thompson, 1985). Some of these theories, such as the APOS (action, process, object and scheme) theory, suggest a more elaborate hierarchy (cf. Dubinsky, 1991). However, for our current discussion it is sufficient to focus on the process-object duality. Theories that discuss this duality distinguish between a *process conception* and an *object conception* of mathematical notions, and, despite the differences, agree that when a mathematical concept is learned, its conception as a process precedes – and is less abstract than – its conception as an object (Sfard, 1991, p. 10). Thus, process conception of a mathematical concept can be interpreted as being on a lower (that is, reduced) level of abstraction than its conception as an object.

(c) The third interpretation of *abstraction level* examines abstraction *by the degree of complexity of the mathematical concept of thought*. For example, a set of elements is a more compound mathematical entity than any particular element in the set. It does not imply automatically, of course, that it should be more difficult to think in terms of compound objects. The working assumption here is that the more compound an entity is, the more abstract it is. In this respect, this interpretation of abstraction focuses on how students reduce abstraction level by replacing a set with one of its elements, thereby working with a less compound object. As it turns out, this is a handy tool when one is required to deal with compound objects that haven't yet been fully constructed in one's mind.

The theme of reducing abstraction has been used for explaining students' conception in different areas of advanced mathematics and in computing science. It was utilized to analyze learners' work in abstract algebra (Hazzan, 1999), differential equations (Raychaudhuri, 2001), data structures (Aharoni, 1999) and computability (Hazzan, in press). These analyses illustrate that a wide range of cognitive phenomena can be explained by one theoretical framework. Here we expand the applicability of the framework by examining reducing abstraction in the area of elementary mathematics.

REDUCING ABSTRACTION IN ELEMENTARY MATHEMATICS

Examples in this section are taken from the work of preservice elementary school teachers in the "Principles of Mathematics for Teachers" course at Simon Fraser University (Canada), which is a core course for certification at the elementary level. Our aim here is to describe teachers' tendencies through the lens of reducing abstraction, rather than to report frequency of occurrence.

(a) Relationships between the object of thought and the learner

This interpretation for abstraction is illustrated by the preservice teachers' tendencies to retreat to the familiar base 10 when asked to solve problems in terms of other bases.

Int: We're in base five now. Can you add 12 and 14 (read: one-two and one-four) in base 5?

Sue: 12 (read: one-two) in base five is what? 7, yea, 5,6,7 and 14 (one-four) would be 9. So together this is 16.

Int: Is this in base 5?

Sue: Oh - no. I have to put this back into base 5. So 10 is 5, and we go 11, 12 (read: one-one, one-two, etc), 13, 14, 20… So I see, 20 is 10, and 30 will be 15 so 16 is 31, three-one base 5.

Different bases are often used in courses for elementary school teachers to reinforce the common algorithms for multi-digit addition and subtraction and to create appreciation for the meaning of "carrying" and "borrowing", rather than to perform these operations automatically following learned rules. However, as the above excerpt illustrates, Sue successfully avoids addition in base 5 by converting back to base 10, performing the operation in base 10 and then calculating the result in base 5. Her solution can be interpreted as reducing abstraction from the unfamiliar base-5-addition to the familiar base-10-addition via conversion, which she achieved by counting and matching.

(b) Process-object duality

This interpretation for abstraction is illustrated by preservice teachers' working with the concept of divisibility.

Int: Consider the following number $3^3 \cdot 5^2 \cdot 7$. We'll talk about it a bit, so let's call it M. Is M divisible by 7, what do you think?

Mia: OK, I'll have to solve for M… [pause] Yes, it does.

Int: Would you please explain, what were you doing with your calculator?

Mia: I solved and this, this is 1575, and divided by 7 gives 225. Like it gives no decimal so 7 goes into it.

The tendency of students to calculate rather than attend to the structure of the number as represented in its prime decomposition has been discussed in Zazkis & Campbell (1996). It has been reported that even students who are able to conclude divisibility of M by 7 or 5 based on its structure, tend to calculation when prime non-factors (such as 11) or composite factors (such as 15 or 63) are in question. These students reduce the level of abstraction by considering the process of divisibility, that is, attaining the whole number result in division, rather than the object of divisibility, which indicates a property of whole numbers and is independent of the specific implementation of division.

(c) Degree of complexity of mathematical concepts

The following excerpt is taken from Zazkis & Campbell (1996).

Int: Do you think there is a number between 12358 and 12368 that is divisible by 7?

Nicole: I'll have to try them all, to divide them all, to make sure. Can I use my calculator?

Int: Yes, you may, but in a minute. Before you do the divisions, what is your guess?

Nicole: I really don't know. If it were 3 or 9, I could sum up the digits. But for 7 we didn't have anything like that. So I will have to divide them all.

Nicole exhibits a common tendency – she wishes to find a number divisible by 7 between the two given numbers in order to claim its existence. The task invites her to consider the interval of ten numbers; however, Nicole prefers to consider and check for divisibility of each number separately. In doing so she is considering a particular object, a number, rather than a more complex object, a set or interval of numbers. Therefore, the abstraction level is reduced: a property of a set of elements is being examined one by one, rather than a property of the set as a whole.

(d) Multifaceted examination from the perspective of reducing abstraction

As mentioned earlier, the classification of ways in which learner's reduce abstraction is neither exhaustive nor mutually exclusive. Consider for example the following problem:

> *A length of 3 cm on a scale model corresponds to a length of 10 meters in a park. A lake in the park has an area of 3600 square meters. What is the area of the lake in the model?*

In her solution, Brenda assigned the dimensions 90□40 to the lake, converted each length separately and then calculated the area of lake in the model. Some of her classmates considered the lake to be a 36□100 rectangle or a 60□60 square. For most students, the random assignments of units and even the random restriction of the lake shape to either a square or a rectangle, still led students to a correct answer. However, no one could explain why the final calculation of the area was not influenced by the choice of shape and measurements.

The task in this example was geared at testing students' abilities to perform the conversion of square units. Regression to the units of length can be interpreted as reducing abstraction in several ways: In accordance with section (c), the assignment of units of lengths provides learners with a lesser degree of complexity. That is, it provides an opportunity to deal with one particular object rather than with *any* object of a given area. In accordance with section (a), the measures of lengths could have been perceived as more familiar, and therefore less abstract, than the measures of area. In accordance with section (b), the calculation of area can be interpreted as students' conception of area as a process, rather than as an object that assigns a measure to a shape.

Noss & Hoyles (1996) state that "[t]here is more than one kind of abstraction." (p. 49). Consequently, as the above examples illustrate, there is more than one way to reduce the level of abstraction and more than one way to describe a learner's activity in terms of reducing abstraction level.

CONCLUSION

Schoenfeld (1998) proposed four major criteria for judging theories and models that embody them: descriptive power, explanatory power, predictive power and scope of applicability. The theory of reducing abstraction meets each of these standards. It provides a lens for describing, explaining and predicting students' encounters with a wide variety of topics and concepts. This article has extended the scope of its applicability from the content domains of advanced mathematics and computing science to elementary mathematics. We conclude by inviting the readers to examine their own observations of learners' mathematical encounters through a lens of reducing abstraction.

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