A THEORETICAL MODEL OF ANALYSIS OF RATE PROBLEMS IN ALGEBRA

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During several years of experimentation, we accumulated a large collection of data on the analysis of problems generally encountered in algebra and on the reasoning developed by students at different levels when confronted with these problems. Analysis of this data led us to a new way of looking at differences among word problems, and to the development of a new theoretical approach to classify them with respect to their complexity and their structure. The aim of this paper is to extend this theoretical approach to the classification of algebra rate problems with respect to their complexity. An analysis of problems in textbooks from three different time periods allows us to circumscribe the respective fields of possibilities that students encountered with respect to rate problems.

INTRODUCTION

The failure of learning algebra and problem solving in secondary school is a problematic area that researchers know very well (Booth, 1984; Lochead & Mestre, 1988; Kaput & Sims-Knight, 1983; Clement, 1982; Bednarz & Janvier, 1992; Marchand & Bednarz, 1999, 2000). The search for new theoretical approaches to explain and improve the learning of algebra has produced new ways of looking at this problematic. For example, the rupture between the procedures used to solve arithmetic and algebraic word problems and the change of thinking involved in the passage from one to the other are a major point of consideration when formulating new strategies for teaching (Bednarz et al., 1992, 1996; Filloy & Rojano, 1989; Kieran & Chaloug, 1993; Lee & Wheeler, 1989; Chevallard, 1985, 1989; Puig & Cerdan, 1990). The same consideration of the mathematization process is needed within the algebraic domain (Lochead et al., 1988; Kaput et al., 1983; Clement, 1982; Sleeman, 1986). Those studies have provided new teaching approaches which take into account students' difficulties and their different modes of reasoning when facing an algebraic problem (Rojano, 1996; Sutherland & Rojano, 1993; Vergnaud, 1987; Rojano & Sutherland, 2001; Kieran et al., 1996; Landry, 1999; Bednarz, 2001). On the other hand, proposals for teaching strategies, if they are to provide an enlightened choice of problem situations for students, must provide a classification of different problem types and a better understanding of their complexity for students. Our past research enabled us to highlight different types of problems generally encountered in algebra and to analyze the complexity of those problems within a particular type (Bednarz et al., 1994). The investigation we present here is a natural extension of this previous research in that we are applying our theoretical model to a new class of problems: algebra rate problems (ARP).

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THEORETICAL ANALYSIS OF ARP

Algebra rate problems have a major role in building algebraic thinking; more precisely in the context of problem solving, and in the articulation between unknown and variable in the learning of algebra. These problems related to proportional reasoning, refer to relationships between non-homogeneous magnitudes, through intermediary of rates. These types of problems, where the concept of rate is present is important in making a connection between mathematics and the "real world", and mathematics and other sciences. For example, problems dealing with the concept of speed, debt, density, involve the mathematical concept of rate. How can we understand, from a theoretical perspective, students' difficulties when solving those problems? And how can we be in a better position to help students in their construction of algebraic thinking with respect to rate problems?

In algebra, analysis of problems and their complexity has usually focussed on their symbolic treatment. Indeed, a focus on equations guides the selection of problems for different school levels. The selection and ordering of problems that students must tackle, is guided by this classical approach. For example, equations with one unknown like $a x + b = c$ in Grade 8 are followed in Grade 9 by equations like $a x + b = c x + d$, and then in Grade 10 by systems of equations with two unknowns. The results we obtained from research with secondary school students (Grade 7 to Grade 11) challenge that classical approach because many problems that can be modeled by the same equation can be associated with very different levels of complexity. And some problems in Grade 10 and Grade 11 can be solved more easily than those in the earlier grades, not because of the strategies developed by the students but because of the low complexity involved in the problem (Bednarz & Janvier, 1994; Marchand & Bednarz, 1999).

There is little research related to the complexity of rate problems in algebra. For example, Presmeg & Balderas (2002) worked with rate problems to get a better understanding of graduate students' visualization rather than the complexity of word rate problems. Yerushalmy and Gilead (1999) developed a theoretical approach to classifying algebra rate problems and to exploring their complexity by looking at how algebra rate problems could be modeled in a functional approach. Their model is restricted to word problems describing constant-rate processes. This model seems to work well when dealing with continuous magnitudes but not when dealing with discrete ones. In their functional approach, they are interpreting data in a specific way that requires enhancing the proposed problem situation, and thus, modifying the original situation. As a consequence, their work does not provide an analysis of the real complexity of interpreting and mathematizing the problem from the student's perspective.

In previous research, we built a model to classify algebraic word problems (AWP) (Bednarz & Janvier, 1994). The approach developed by our team was in connection with relational calculus (Vergnaud, 1982) which involves the representation and solving of those problems (nature of the relations among data, linking relations...) focussing on the cognitive complexity of the task given to students. Our theoretical approach at that time

was dealing with AWP in connection with problems of unequal sharing. The aim of this paper is to extend this theoretical approach to classify ARP using the same methodology.

METHODOLOGY

Our grid was developed through the analysis of rate problems found in algebraic sections in past and present textbooks at different school levels. This first phase continued with an experimental approach, as a pilot study, with students of Grade 10, where ARP are usually tackled.

In a second phase, a deep analysis of textbooks over three different periods of time (related to three curriculum changes) mapped out the field of applications of the kinds of rate problems that students are facing in the various approaches that are in use (a functional approach, at present, versus an approach stressing problem solving and equations). Because of space limitations, this paper will focus on the theoretical approach rather than the experimental results.

Complexity in the Resolution of ARP

Our theoretical framework classifies algebra rate problems with respect to their complexity, taking into account their structure (underlying relational calculus), the kind of rate (if familiar, different levels of abstraction of the rate involved in the problem, for example, speed, debt, density, unitary cost price, etc.), and the formulation of the rate (formulation in terms of a relation between two magnitudes, for example, he/she traveled 75 km in one hour, versus the formulation 75 km per hour). We will restrict the analysis presented here to the analysis in terms of relational calculus. Our analysis has identified eight main categories. Let us consider several examples of different categories we have found. In what follows, we are using a notation to illustrate different kinds of relational calculus involved in certain types of problems.
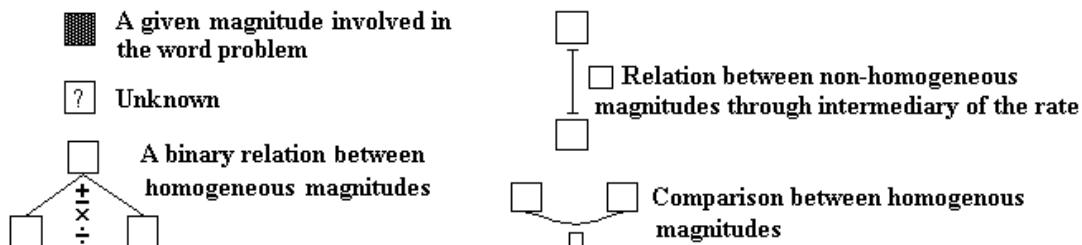
We would like to show the extent of the categories we found in our theoretical approach by presenting only a few examples; we are not being exhaustive.

## EXAMPLES USING THE GRID OF ANALYSIS FOR ARP

A word problem, first category

In this category we have word problems where the symbolization process is direct. These kinds of problems, that we found in almost all textbook sections in connection with rate problems, are classified for some authors as problems that promote transition from the students' arithmetical thinking to algebraic thinking (in connection with equations like $ax + b = cx + d$). Usually, in this category, the rate is a familiar one for students (of course, in this category we could increase the difficulty by creating word problems

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involving rates that are unfamiliar to students). In summary, these types of problems (see example below) are not complex for the students from a cognitive point of view.

> Judy and Carolyn are planning to go to Europe. The two friends are saving some money for t project. Judy has at this moment 500 dollars at the bank and she expects to deposit 20 dollars ev week. Carolyn has only 200 dollars but she wants to deposit 40 dollars every week. In how mu time will they have the same amount of money?

In our approach, we search the content of the problem to find a relation between non-homogeneous magnitudes (i.e., 20 Dollars every week). From there, we construct our diagram. That is,
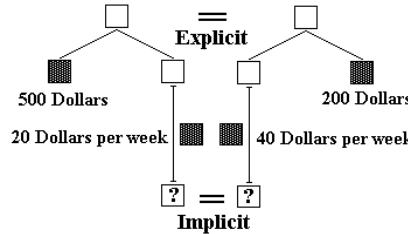

Example of category VI

In this category, the problems involve comparisons among homogeneous magnitudes. Depending on the type of relations, the complexity of the problems increases or diminishes. Let us give an example and at the same time compare our theoretical approach with that of Yerushalmy & Gilead (1999, pp. 187-188, see below).

Round trip 1. A biker traveled from town 1 to town 2 an average speed of 24 km/hour. Arriving at town 2, immediately turned back and traveled to town 1 at average speed of 18 km/hour. The whole trip too hours. How long was the trip in each direction?	
Round Trip 2. A biker traveled from town 1 to 2 at average speed of 24 km/hour. Arriving at town 2, immediately turned back and traveled to town 1 at average speed of 18 km/hour. The return trip was 1 h longer than going there. How long was the trip in ea direction?	
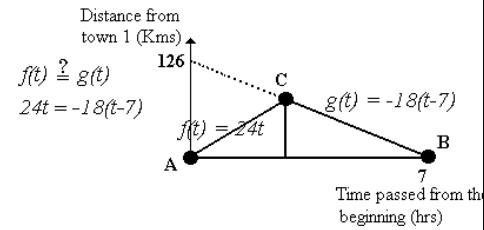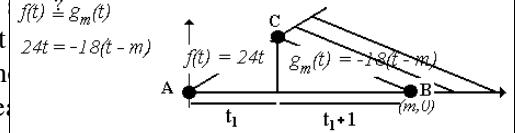
Their model is restricted to word problems describing constant-rate processes and the field of ARP is wider than that. For example, using the same schema we could find a large class of word problems referring to discrete versus continuous magnitudes and different rates. From our theoretical approach, we gain in extension. However, what is important is that we are keeping in our model the relations and organization of data at hand, which allows us to explicit the complexity of the relations involved and of their linking, to better see what is required by students' process of mathematization.

We agree with Yerushalmy & Gilead that a traditional solution of the problems "Round trip 1 and 2" suggests that they are similar, because the solution is related to a similar equation. Under their approach and ours, they are not. We classified them in the same

category but different subcategories (VIa and VIb). In one case (Round trip 1), we have a binary relation between homogenous magnitudes (t_1 and t_2, see our diagrams below), in the other case, we have a comparison between the two homogeneous magnitudes.

| Round trip 1.	Round Trip 2.
$d = d$ Implicit equality 24 Km/h 18 Km/h t_1 ? ? t_2 + 7 hrs	$d = d$ Implicit equality 24 Km/h 18 Km/h t_1 ? ? t_2 + $t_1 = t_2 + 1$
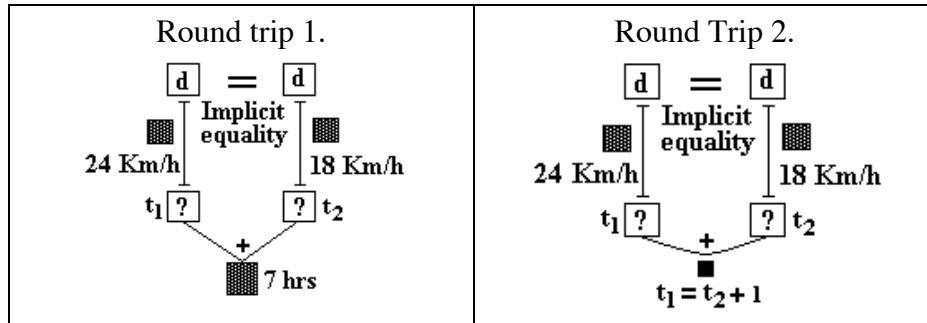
Example of category VII.

In this category, the problems involve relations among non-homogeneous magnitudes and comparisons among homogeneous magnitudes (rates). The rates are unknowns. These kinds of problems are very difficult for students.

| A biker augmenting his speed by 5 Km/h, gains 37 minutes and 30 seconds. If he diminishes his speed by 5 Km/h, he loses 50 minutes. What is his speed and how long is the trip? | Implicit   Implicit
 $d = ? = d$ |
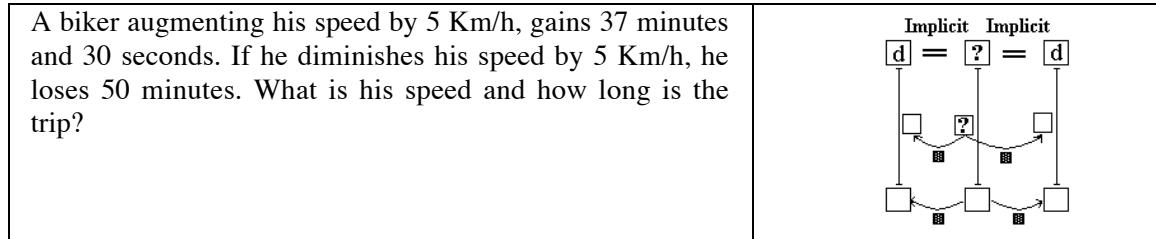
In summary, we have eight main categories with subcategories. In what follows we are briefly presenting the main categories.

I	II	III	IV

V	VI	VII	VIII
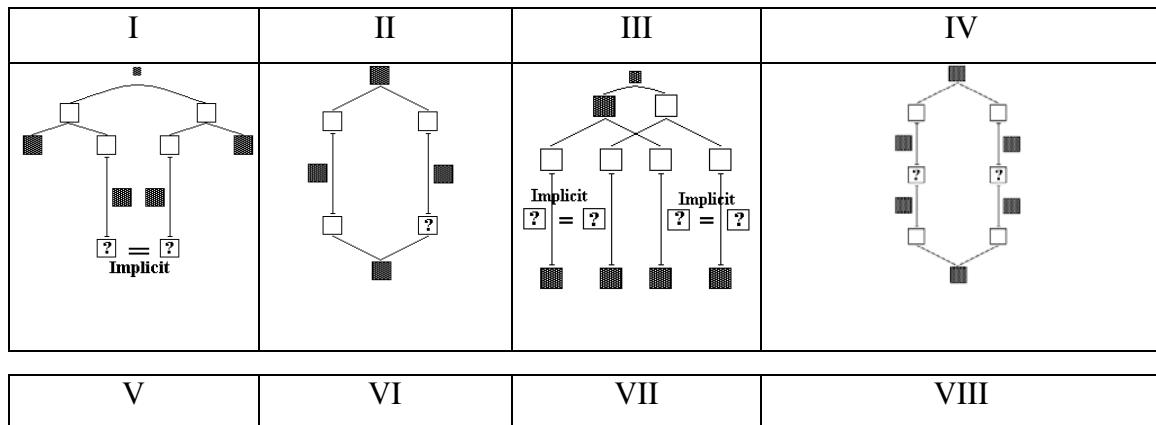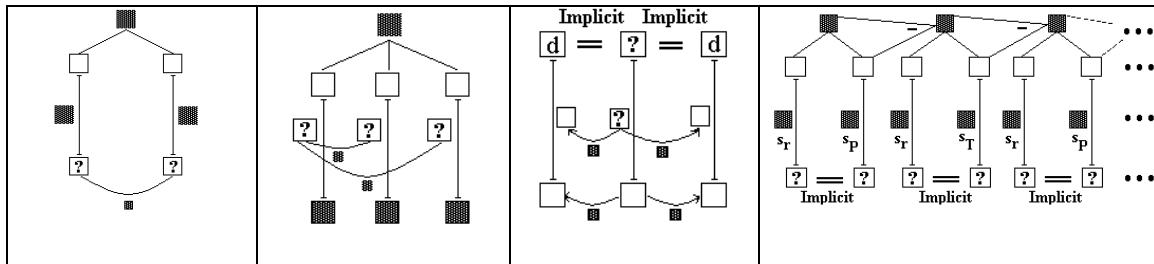
Analysis of the field of problems which students are faced with in ARP

A syllabus analysis shows that proportional reasoning is important for secondary students (Grade 7 and Grade 8); this is within a direct arithmetical approach where problems are dealing with non-homogenous magnitudes in connection with rate. Our analysis of problems shows however that authors of textbooks are not carefully taking into account the transition from arithmetic to algebra, a few rate problems being presented in the introduction of algebra (Marchand & Bednarz, 1999). These problems appear only in some textbooks in Grade 9 and frequently in Grade 10. The teaching process seems to create ruptures from one grade level to the next, because of the choice of problems presented to students. This leads us to the question: Must students confront this transition of thinking in connection with ARP on their own?

*The analysis of three different periods of time related to curriculum changes shows us that the field of problems the students must solve at the present time is relatively limited (see table below). The syllabuses we are considering in our analysis are the Quebec curriculum before 1980, between 1980 and 1993, and from 1994 to the present (they are marked in the table as *, **, ***). We classified the following textbooks according to the main categories in our grid.*

Textbook / Category	I	II	III	IV	V	VI	VII	VIII
Mathématiques d'aujourd'hui 2, (1979*)	1	5				11		1
BMS 4 (Module A), (1985**)	1	8	10	1		6		
Mathématiques Soleil, Sec. 4, (1986**)		13	10	4		8	2	
Mathophilie 436, (1997***)	5	4	17	3		2	1	
Scénarios 436, tome 2, (1997***)	4	2	1					

The selection, in connection with a functional approach in the Quebec curriculum (Grade 10), seems to be oriented towards relatively simple problems. Problems in categories VI to VIII are rare in the current curriculum textbooks we analyzed (see table and books marked with ***) and sometimes absent. In these books, the problems that are proposed to students do not permit a fluid evolution of the students' process of mathematization in relation to complex problems.

DISCUSSION

The theoretical approach presented above can be discussed in terms of intended use of the classification of ARP. We have developed a grid that can be used to predict the complexity of problems according to the structure of the relations involved. Our theoretical approach is a natural extension of the one used with AWP not involving proportional thinking.

We used the grid to analyze and compare textbooks with respect to the level of competence they are requiring of students. Using this grid, we found ruptures in the curriculum because it seems problems are classified in textbooks considering only the final equation or equations involved in solving the problem and not the type of homogeneous or non-homogeneous magnitudes and the type of relations between them. From our point of view, the curricular change to a functional approach has resulted in abandoning too soon the algebraic approach needed to build consistent algebraic thinking and in passing too quickly over the relationship between unknown and variable.

We believe that our theoretical approach can be considered as an example of organizing ARP with respect to their complexity. We conjecture that if teachers are confronted to such type of analysis of problems, they could get a better feel for the difficulties students can experience when solving ARP. The grid of analysis could provide instructors with a holistic approach because this technique takes into account the situated structure of the problem, shows the quantitative relationships involved and allows them to select and create problems in connection with the complexity of the algebraic tasks involved in students' algebraic processes.

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