DEVELOPING AND CONNECTING CALCULUS STUDENTS' NOTIONS OF RATE-OF-CHANGE AND ACCUMULATION: THE FUNDAMENTAL THEORM OF CALCULUS

Marilyn P. Carlson
Arizona State University

Joni Persson
Arizona State University

Nanci Smith
Arizona State University

An overview of the conceptual underpinnings, reasoning abilities and notational issues related to learning the Fundamental Theorem of Calculus is provided. Using this theoretical framework, curricular materials were developed to promote these understandings and reasoning abilities in students. Results from a study that investigated the effectiveness of these materials on first semester calculus students' understandings of the FTC revealed significant advances in their understandings of accumulation and the FTC. Some specific difficulties that were observed in select students provided insights for further refinement of the theoretical framework and for revision of the FTC activities.

INTRODUCTION AND BACKGROUND

The Fundamental Theorem of Calculus has been described as one of the intellectual hallmarks in the development of the calculus (Boyer, 1959). However, studies have documented that most first semester calculus students do not emerge from the course with an understanding of this concept; nor do they appear to be developing the foundational reasoning abilities needed to understand and use the FTC in applied settings (Bezuidenhout & Olivier, 2000; Kaput, 1994). Student difficulties with the Fundamental Theorem of Calculus have been attributed primarily to their impoverished view of function (Carlson, 1998; Thompson, 1994) and rate-of-change (Thompson, 1994). However, little research is available articulating what is involved in knowing and learning this concept. The purpose of this paper is to provide additional clarity about the understandings and reasoning abilities involved in learning and using the FTC. It also reports the results of a study that investigated the effectiveness of curricular materials for first semester calculus students that were developed using this framework as a guide.

Reasoning about and with the Fundamental Theorem of Calculus involves mental actions of coordinating the accumulation of rate-of-change with the accumulation of the independent variable of the function. The accumulating quantity can be imagined to be made of infinitesimal accruals in the quantities, which when thought of multiplicatively, make up the accruals in the accumulating quantity. Both a process view of function and *covariational reasoning* have been shown to be foundational for coordinating these accumulations (Thompson, 1994).

Covariational reasoning refers to the coordination of an image of two varying quantities, while attending to how they change in relation to each other (Carlson, Jacobs, Coe, Larsen and Hsu, 2002). A more detailed characterization of covariational reasoning has been articulated in a Covariation Framework that characterizes covariational reasoning in terms of the mental images that support the mental actions of coordinating: i) changes in one variable with change in the other variable; ii) the direction of change in one variable with changes in the other variable; iii) the amount of change of one variable with changes in the other variable; iv) the average rate-of-change of a function with changes of the independent variable; and v) the instantaneous rate-of-change of the function with continuous changes in the input variable. The mental image that supports all five mental actions has been classified as Level V covariational reasoning. As this body of literature suggests, it seems reasonable that students should develop both a process view of function and Level V covariational reasoning abilities prior to their study of the Fundamental Theorem of Calculus.

Thompson (1994) has described the Fundamental Theorem of Calculus as a means of expressing the relationship between the accumulation of a quantity and the rate-of-change of the accumulation. He advocates that an understanding of the FTC involves coordinating images of respective accruals in relation to the total accumulation. According to Thompson, this is the idea that motivated Newton's development of the Fundamental Theorem. Newton first determined the average rate-of-change of an area and determined that the total area could be computed by multiplying the rate-of-change by the accumulation of the independent variable. This led to his observation that the rate of change of the accumulated quantity is equal to the immediate accrual. This line of thinking emphasizes the importance of understanding that the accrual is a multiplicative relationship and that the total accumulation is made of infinitesimal (multiplicatively composed) accruals of the quantities (e.g., accruals of lines compose area and accruals of area compose volume). It is these understandings that enable the relationship expressed by $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$ to be appreciated and understood. A more careful articulation of the reasoning abilities and understandings related to accumulation and the Fundamental Theorem of Calculus are provided in the theoretical framework for this study.

THEORETICAL PERSPECTIVE:

The FTC Framework

This framework contains four dimensions that describe the foundational reasoning abilities and understandings of the Fundamental Theorem of Calculus.

Part A: Foundational understandings and reasoning abilities

(FR1) Ability to view a function as an entity that accepts input and produces output.

(FR2) Ability to coordinate the instantaneous rate-of-change of a function with continuous changes in the input variable (Level V covariational reasoning).

(FU1) Understanding that the average change of a function (on an interval) = the average rate-of- change (multiplied by) the amount of change in the independent variable.

(FU3) Understanding that the multiplicative relationship that represents the accrual of change on an interval can be represented by area.

Part B: Covariational reasoning with accumulating quantities.

The Mental actions of the Fundamental Theorem of Calculus (The function refers to the rate-of-change function, *f*).

(MA1) Coordinating the accumulation of discrete changes in a function's input variable with the accumulation of the average rate-of-change of the function on fixed intervals of the function's domain.

(MA2) Coordinating the accumulation of smaller and smaller intervals of a function's input variable with the accumulation of the average rate-of-change on each interval.

(MA3) Coordinating the accumulation of a function's input variable with the accumulation of instantaneous rate-of-change of the function from some fixed starting value to some specified value.

Part C: Notational aspects of accumulation

Notation	Meaning
$F(x) = \int f(x)dx$	i) The antiderivative of f is F ii) f is the function that describes the rate-of-change of F.
$F(x) = \int_a^x f(t)dt$	i) The value of $F(x)$ represents the accumulated area under the curve of f from a to x; ii) The value of $F(x)$ represents the total change in F from a to x.

Part D: The statements and relationships of the FTC

$\int_a^b f(x)dx = F(b) - F(a)$	i) The accumulated area under the curve of f from a to b is equal to the total change in F from a to b.
$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$	i) The instantaneous rate-of-change of the accrual function at x is equal to the value of the rate-of-change function at x

METHODS

The subjects were 24 beginning calculus students enrolled in the same section of first semester calculus at a large university in the United States. A Pre-Calculus Concept Assessment Instrument (focused primarily on assessing the reasoning abilities and understandings described in Part A of the FTC Framework) was administered to the students at the beginning of the semester, and a post-instruction written assessment instrument was administered at the end of the course. The mean score and number of correct responses for each item were compiled. Four students who were somewhat representative of the diverse understandings of the class (based on their performance on the Pre-Calculus Concept Assessment Instrument) were invited to participate in eight (~75 minute) clinical interviews. The interviews were conducted in pairs and were designed to gain information about students' ability to understand and reason using the major concepts of the course (covariation, limit, derivative, accumulation, the FTC). During each interview the students were asked to complete a collection of thought revealing tasks (Lesh, 2002) that paralleled the conceptual focus of instruction during the previous two weeks of the course. Each pair verbalized their thinking while responding to the written problems and questions. The role of the interviewer was to promote discussion among the pair of interviewees and to gain insight into the understandings and reasoning of the individuals. Digital videos of the sessions were transcribed, coded and analyzed, using the FTC framework.

THE COURSE

The text for the course was *Calculus Early Transcendentals* (Stuart, 1999). However, about half of the instruction was delivered using the *Conceptual Calculus Modules* currently under development by the first author. Each module contains a collection of in-class and take-home activities designed to promote the development of students' conceptual connections and reasoning abilities relative to the central concept of the module. Carefully sequenced prompts and tasks (situated in context whenever possible) were included to promote students' articulation of their thinking.

The *Precalculus Concept Assessment* instrument was administered to the students at the beginning of the semester. Instruction during the first two weeks of the semester included a strong focus on the foundational reasoning and understandings described in Part A of the FTC framework. Post-instruction assessment of these understandings suggested that most students emerged from this instruction with these reasoning abilities and understandings. Instruction leading up to the FTC module included a balanced focus on concept development, acquisition of notational understanding, facts and procedures, and the development of students' mathematical practices and problem solving behaviors. Students were expected to be regular participants in the classroom. Whole class discussion, group work and lecture were the primary modes of instruction.

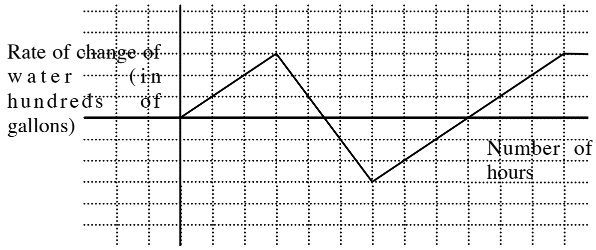
RESULTS

Select data from administering a post-instruction written assessment of students' understandings related to accumulation and the FTC are reported. The presentation of results provides a statement of the item, the number of students who provided a correct response (out of the 24 who completed the course), and the mean score (out of 3) on each part of each item.

The collection of responses on Item 1 suggests that the beginning calculus students in this study were proficient in applying covariational reasoning with accumulation tasks. Over 70% of the students completing the course provided a completely correct response to parts d, e and f (Item 1), suggesting proficiency in coordinating the accumulation of a function's input variable with the accumulation of instantaneous rate-of-change of the function from some fixed starting value to some specified value (MA3). Over 90% of these same students also provided a correct response to the prompts that assessed students' understanding of the notational aspects of the FTC (parts b and c).

Item 1: The Water Problem

Let f represent the rate at which the amount of water in Phoenix's water tank changed in (100's of gallons per hour) in a 12 hour period from 6 am to 6 pm last Saturday (Assume that the tank was empty at 6 am (t=0)). Use the graph of f, given below, to answer the following.



	Number Correct (out of 24)	Mean Score (out of 3)
a. How much water was in the tank at noon?	21	2.8
b. What is the meaning of $g(x) = \int_0^x f(t)dt$	24	3.0
c. What is the value of $g(9)$?	22	2.7
d. During what intervals of time was the water level decreasing?	22	2.7
e. At what time was the tank the fullest?	17	2.3
f. Using the graph of f given above, construct a rough sketch of the graph of g and explain how the graphs are related.	17	2.4

Student responses on Item 2 also suggest that this collection of students possessed both a strong understanding of notational aspects of the FTC (parts a, b) and proficiency in applying covariational reasoning with accumulation tasks (parts f, g, and h). Responses on parts c, e and i indicate moderate proficiency in understanding the statement of the FTC, with only about 60% of these students providing correct responses on this collection of questions.

Item 2: The Circle Problem

Consider a circle that expands in size from $r = 0$ to $r = x$. Let A be a function that represents the accumulation of the rate-of-change of the circle as it increases in size from $r = 0$ to $r = x$.

	Number Correct (out of 24)	Mean Score (out of 3)
a. Define $A(x)$ as an accumulation function.	18	2.3
b. Construct a circle and illustrate what $\int_2^4 2\pi r\, dr$ represents.	22	2.8
c. Describe what $\frac{d}{dx}\int_a^x 2\pi r\, dr$ represents relative to the circle.	14	1.8
d. Construct the graph of f (the rate of change of the area of a circle), on the axes on the left and the graph of A (as defined above) on the graph on the right. Label your axes.	20	2.6
e. Explain how the two graphs are related.	16	2.2
f. Construct the graph of A. Estimate the area under the graph of A from $r = 1$ to $r = 5$ using eight approximating rectangles and	19	2.5

right endpoints		
g. Given that n represents the number of subdivisions on the interval from $r = 1$ to $r = 5$, explain what is involved in letting the $\lim_{n\to\infty}$ for this interval.	20	2.6
h.What is the result of this evaluation?	19	2.7
i. What does $A'(x) = f(x)$ mean in the context of this situation?	16	2.1

Results for item 3 reveal that most of these students recognized this question as an application of the FTC. They were also successful in translating the situation to symbols (Part D of the FTC framework). However, most students had difficulty recognizing that they needed to sum the distance traveled in both the positive and negative directions.

Item 3: The Distance Problem

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in feet per second). Find the distance traveled during the time period from $t = 1$ to $t = 4$. Show Work!

Number Correct (out of 24): 12 Mean Score (out of 10): 7.2

Number of students who set up the integral correctly: 23

Common Error: Computed position from start instead of total distance traveled.

The collection of student-responses on these items suggests that most of the students completing the course emerged with proficiency in using and understanding notational aspects of the FTC (Item 1, parts b and c; Item 2, parts a, b). These results also suggest that these students were able to apply covariational reasoning with accumulation tasks. Their understanding of the statements and relationships of the FTC tasks were weaker, with only a little over half of the students completing the course providing correct responses to the collection of questions assessing this ability.

Interview Results

The four interview subjects were Lisa, Harold, Chad and Katie. Lisa received a C in the course, Harold and Chad received B's, and Katie received an A. Analysis of the four interviews revealed that: i) all four students were able to apply covariational reasoning with accumulation tasks; ii) all four students possessed a strong understanding of most of the notational aspects of the FTC; and iii) Chad, Katie and Harold possessed a strong understanding of the statements and relationships of the FTC. Select interview excerpts from the interviews in which these four students explained the reasoning they used to respond to the above items follow.

When responding to Item 1, Chad demonstrated that he was able to coordinate the accumulation of time and accumulation of rate (MA3). He also demonstrated that he understood the role of the input variable to g (i.e., determining the upper limit of the integral); however, further probing suggests that he had some confusion about the role of x and the rationale for labeling the independent variable of f using another variable.

I: So what did you notice about the relationship?

Chad: One figure is always twice the area of the other.

I: Explain the meaning of $g(x)$ (see item 1c above).

Chad: I see g as giving the amount of area under the graph of f.

I: What does the input variable x represent?

Chad: This tells you how far out on the right on the graph of f you want to go (student sweeps his hand across the graph.)

I: Can you explain this in the context of the question.

Chad: Um…since f is the rate of water flowing into the tank and g is the integral of f from 0 to x, when you find $g(x)$ you are finding how much water came in or went out of the tank from the starting time, up until the time that you want….that is the time x (MA3).

I: So, how do you think about evaluating $g(9)$?

Chad: I see that as finding the time that passes from 0 to 9 and thinking about how much area gets added under the curve as I move along. I see that water is coming into the tank, first at an increasing rate, then at a decreasing rate. Then after 4_ hours, water starts to go out of the tank (MA3). As you add up the area under the curve you see that the same amount of water comes in between 0 and 4 _ that goes out between time 4_ and 9….so, the result is that there is no water in the tank after 9 hours have passed.

I: How are g and f related?

Chad: The derivative of g gives the graph of f. What I don't get is why t is the variable that is used in f. I never really understood this on some of the other problems we did either.

The interview responses to item 2 (parts d and e) revealed that three of the interview subjects (Harold, Chad and Katie) held a strong understanding of the statement of the FTC for the circle problem. These three students were proficient in constructing the graphs of f and A. They were also able to provide a clear articulation of how the two graphs are related. Harold set up a table that computed the accumulation of the area under the graph of f from 0 to various values of x. He continued to explain that he viewed the accumulation of the area from 0 to specific values of the input to f as producing a value that provided the total area. He went on to explain that he also viewed the accumulation of area as the output of A. Later in the interview, he expressed that he viewed the accumulation of rate-of-change of the circle as adding up "infinitely many infinitesimally small" circumferences. When probed to explain how to use the graphs to compute $\int_2^5 2\pi r\, dr$, he responded that it could be computed in several ways. He continued by subtracting the two areas under the graph of f; he then drew a picture of the circle and shaded the area represented by this definite integral. He went on to explain that what he was actually finding was $A(5) - A(2)$ and expressed that this value was just the difference in the heights of the graph of A between $r = 5$ and $r = 2$.

The interview with Lisa revealed some weaknesses in her understanding of the statements and relationships of the FTC. More specifically, she was unable to articulate what $\frac{d}{dx}\int_a^x 2\pi r\, dr = 2\pi x$ expressed about the relationship between accumulation and accrual.

Her response suggested that she did not view $\int_a^x 2\pi r\, dr$ as a representation of the accrual of $2\pi r$ from some specific value a, to some specified value for x. Her utterances suggested that she did not view this as an object that she was able to differentiate.

CONCLUSIONS AND DISCUSSION

The quantitative and qualitative data suggest that most of the first semester calculus students in this study completed the course with a strong understanding of notational aspects of accumulation. They also demonstrated an ability to coordinate the accumulation of a function's input variable with the accumulation of instantaneous rate-of-change, from some fixed starting value to some specified value, for various contextualized situations. Although some weaknesses were observed in some students' understandings of the statements and relationships expressed in the FTC, the performance of this collection of students relative to the attributes of accumulation and the FTC expressed in this framework were relatively good, especially if one compares this with what has been reported of secondary teachers and graduate students (Thompson, 1994).

The framework for this study served as a useful tool for analyzing students' reasoning abilities and understandings relative to both conceptual and notational aspects of the FTC. The results of this study suggest that further refinement of Part D of the framework is needed. In particular, the weaknesses that were observed suggest that the framework needs to include a more careful articulation of the mental actions involved understanding and applying the statements and relationships expressed by the Fundamental Theorem of Calculus. This refinement should also lead to the development of additional ideas for curricular tasks and prompts to better assist students in developing these understandings and reasoning abilities.

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