GENERALISING THE CONTEXT AND GENERALISING THE CALCULATION

Janet Ainley, Kirsty Wilson and Liz Bills

Mathematics Education Research Centre, Institute of Education

University of Warwick, UK

The Purposeful Algebraic Activity Project[1] *is concerned with the development of algebraic activity in pupils in the early years of secondary schooling. Here we report on the different ways in which we have observed children articulating generalisations during semi-structured interviews which are being used to gain snap-shots of this development. We describe how generalising the* context *appears to be distinct from generalizing the* calculation*, and discuss the implications of this for the design and implementation of our teaching programme.*

BACKGROUND

The importance of generalising as an algebraic activity is widely recognised within research on the learning and teaching of algebra. Kieran (1996) describes the first of three categories of algebraic activities as *generational activities* which involve

> *the generating of expressions and equations… for example equations involving an unknown that represent quantitative problem situations, expressions of generality from geometric patterns or numerical sequences and expressions of the rules governing numerical relationships.* (p. 272)

However there seems to have been relatively little research which looks in detail at what it is that novices actually do when they are encouraged to generalise, and how this differs from what more experienced mathematicians might do, or at what kinds of pedagogic tasks support the construction of meaning for generalising.

Mason *et al* (1985) describe three important stages in the process of generalising a pattern or relationship as *seeing*, *saying* and *recording*; that is, seeing or recognising the pattern, verbalising a description of it, and making a written recording. Several researchers have looked at the last part of this sequence, and particularly at the difficulties which students encounter in producing formal written expressions of generalizations (for example Macgregor and Stacey, 1993), and at the role of intermediate or idiosyncratic notations in the moving towards such formal expressions (for example, Bednarz, 2001, Ainley, 1999).

Radford (2001) offers a detailed linguistic analysis of pupils working on the problem of generalising a matchstick pattern, and identifies three levels of generalisation related to such geometric-numeric patterns. These are *factual* generalisations, which generalise numerical actions and enable students to tackle particular cases*, contextual* generalisations, which are performed on conceptual, spatio-temporal objects, and *symbolic* (algebraic) generalisations, which deal with algebraic objects which are

[1] The Purposeful Algebraic Activity Project is funded by the Economic and Social Research Council.

unsituated and atemporal. Whilst giving a detailed account of the linguistic indicators of the shifts between these levels of generalisation, Radford offers a description of the complexity experienced by students in constructing meanings for algebraic symbols, which he suggests are initially haunted by ‘the phantom of the students’ actions’ (p. 88).

Within the Purposeful Algebraic Activity project we are studying the development of pupils’ constructions of meanings for algebra during the early years of secondary schooling. We have based our data collection around the notion of *algebraic activity,* using the broad definition given by Meira (forthcoming), which implies that

> *The student who is unable to fully and competently use algebra as intended by experts may be engaged in algebraic activity …[when] he or she is producing meaning for algebra.*

The main strand of the project is the development of pedagogic tasks, based on the use of spreadsheets, designed using the principles of *purpose* and *utility* proposed by Ainley and Pratt (2002) (Ainley, Bills and Wilson, forthcoming). These tasks are being used in the introductory stages of teaching algebra to pupils in the first year of secondary schooling. The tasks are designed to support a range of algebraic activity, focussing on *generational* and *meta-level* activity (as described by Kieran, 1996) and making explicit links to *transformational* activity, which traditionally receives emphasis in the school curriculum.

Semi-structured interviews are being used to collect snap-shots of the development of pupils’ algebraic activity. The interviews aim to explore different aspects of algebraic activity, and are to be repeated, with essentially the same set of questions, at regular intervals. Analysis of the first phase of interviews has given us some insight into the ways in which pupils with little experience of algebra generalise numerical relationships which differ in emphasis from, but may complement, Radford’s (2001) categorisation.

DATA COLLECTION

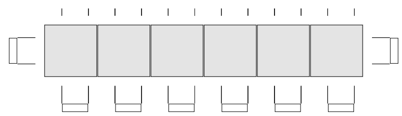
Two cohorts of pupils have been interviewed in the first phase of the project; cohort A at the end of their first year at secondary school (mostly aged 12), during which they had had some explicit teaching in algebra, and cohort B at the beginning of their first year (mostly aged 11). Twelve pairs of pupils from two schools were interviewed in cohort A, and fourteen pairs in cohort B. The pupils were selected by their teachers from those who were willing to take part, with the aim of making compatible pairs of similar ability. The pairs were distributed across the perceived ability range in the year group, and contained a balance of boys and girls.

The interviews were conducted by a researcher (the second named author), who presented the questions in written form, and also read them aloud. All interviews were video taped, and audio taped for transcription. Copies of any pupil writing were collected, and added to the transcripts, which were also annotated to include non-verbal behaviour observed in the video tapes. For the purpose of the analysis presented here, the contributions of the two individuals during the interview have generally been amalgamated, since our concern in this paper is less with the particular pupils than with the ways in which they are expressing their thinking.

WAYS OF GENERALISING

This paper focuses on responses to the question shown in Figure 1, which comes about two thirds of the way through the interview.

Tables and chairs are arranged like this in the school dining room:



Can you explain how to work out how many chairs are needed for each long table? Can you write it down?

Figure 1: The ‘tables and chairs’ question

This question was chosen so that we could see whether pupils were able to articulate a generalised relationship containing a variable (in Mason et al’s term *seeing and saying*). They were encouraged to write down (*recording*) so that we could see whether or not they would make use of an algebra-like notation. During the interviews, the researcher encouraged the children to articulate and share ideas, and, if they seemed stuck, sometimes suggested that they try particular numerical examples (e.g. *what would happen if there were 20 tables?*). The interviewer also prompted the pupils to try to write down an answer, but did not insist on this if they seemed reluctant to do so for any reason. Often the pupils preferred to say aloud what they would write.

Cohort A

All of the pairs in cohort A (those who had some experience of algebra), showed some evidence of algebraic activity, in that they were able to articulate a general relationship between the numbers of tables and chairs. Here are some examples of their statements.

1. *Two on each table except for the ends, which is three.*
2. *For each end one you would need three and … for each other table you would need two*
3. *For every table there’s two chairs plus the other two that are on the end*
4. *You would do, two for each table, two chairs for each table … and then you would need … two more for the sides*
5. *Well, every table you get, you need to double it and then … add ‘em on … the number of chairs must be double and add two.*
6. *Just however many tables double that, and then, plus two for the ends*
7. *So it could be times how many tables there are, times two … plus two*
8. *You take the tables and you times it by two and then plus two*

These examples vary in a number of ways. Clearly some are expressed more fluently and confidently than others. Statements 1 and 2 seem to see the arrangement in terms of two kinds of table (the ‘ends’ which have three chairs, and the ‘middles’ which have two) while the remaining statements seem see each table as having two chairs, and treat the chairs at the ends as extras. Although both of these are reasonable ways of describing the arrangement, the second is much easier to translate into a symbolic expression.

This range of statements could be analysed in a number of ways, but our initial analysis focused on a distinction between two different things which are being generalised: the context and the calculation. The first four examples appear to be descriptions of the *context;* the way in which the tables and chairs are arranged. We might see them as general instructions for how to place the furniture. In contrast, examples 5 – 8 describe the *calculation* required to find out how many chairs are needed. These statements are distinguished by the use of terms for operations, such as *double, times, plus, add*. They also include phrases that indicate a sense of a variable involved in the calculation: *every table you get, however many tables, how many tables, the* [number of] *tables*.

Within cohort A, four pairs of pupils gave general descriptions of the arrangements of furniture (the context) but did not describe the calculation, four gave general descriptions of the context and then of the calculation, and four gave descriptions of the calculation as their first response. We now look in more detail at these three groups of responses.

<u>Generalising the context only</u>

The four pairs who gave a general description of the arrangement of furniture, but did not describe the calculation, had been identified by their teachers as middle to low ability in mathematics. Three of these pairs initially gave a description which treated the end and middle tables separately (e.g. *if it's like a corner table you'd need three…but if it was one of the middle ones you'd only need two*). The calculation which arises from this description is more complex than the 'two chairs for each tables and two for the ends' image, and this may have been a factor which inhibited these pairs from being able to express a calculation. The researcher encouraged each of these pairs to try to write something down, but no pair in this group wrote or articulated a symbolic rule.

<u>Generalising the context then a calculation</u>

The four pairs who gave a description of the arrangement of furniture, and then later described a calculation, had been identified by their teachers as middle or high ability. They were generally more confident in approaching this question than those pairs who did not move beyond describing arrangements of furniture. In some cases the initial description of the arrangement of furniture was a dynamic one, focusing on what would happen as additional tables were added. For example, Pair 7 said:

> *'cause you just use this one again, it just goes to the end of the next table, but you need two more of those to put on to the side of it.*

As their discussion continued, Pair 7 described the calculation as follows:

> *so it could be times how many tables there are, times two … plus two*

Pairs 6, 7 and 8 managed to articulate algebra-like expressions for the calculation, though these were not always written down.

> Pair 6: *They could represent table as t and you could do two, I mean t two, no you could write*
> $\boxed{T_2 + 2}$

Pair 7: *But it could be, how many tables times by, n is tables, well, t is tables, times by two … t times two plus two could equal the chairs*

Pair 8: *So it would be like, um, c equals two t plus two*

Pair 5 gave a clear description of the calculation (*The number of chairs must be double and add two*), but then became confused when they decided that their first attempt at a written response $2\times x+2$ required some brackets.

<u>Calculation as first response</u>

Not surprisingly, the pairs who gave a description of the calculation as their first response were ones who had been identified as middle or high ability by their teachers. These pairs generally approached the question confidently, and with relatively little discussion. They were usually able to move easily to a written version of the calculation expressed in algebra-like notation, even though they sometimes struggled with syntax.

Pair 9: *double the number of tables add two,... t, you could, if you were doing it in algebra* $\boxed{T2+2}$

$\boxed{2x+2}$

Pair 10: *So we could try double it plus two? … So shall we say ...x, two x…plus two*

Pair 12: *However many tables, it would be n tables, times two… plus two*

In summary, this analysis suggests that although all the pairs in cohort A were able to generalise the relationship between the numbers of tables and chairs, those pairs who were successful in articulating a version of the rule in a recognisably algebra-like notation had also given a general description of the calculation necessary to work out the number of chairs. This led us to conjecture that being able to generalize the *context* was not sufficient to enable pupils to express the relationship in algebra-like notation, and that being able to generalise the *calculation* required was a significant ‘bridge’ which supported pupils in constructing meaning for a symbolic expression of the relationship.

Cohort B

To test our conjecture about the significance of a verbal description of the calculation as a pre-cursor of constructing meaning for an algebra-like expression, we looked at the transcripts from the interviews with cohort B. This group of pupils had just begun secondary school, and had had no real teaching of algebra prior to the interviews. Not surprisingly, fewer of the pairs in this cohort were able to use algebraic notation with any confidence, but nevertheless most showed evidence of algebraic activity which could be categorised in similar ways to that of cohort A.

However, three of the fourteen pairs were unable to give a general description of the relationship between tables and chairs. Their attempts included methods of counting the chairs, or specific numerical examples, though one pair felt the task was not possible ‘unless it tells you how many tables you actually want’.

<u>Generalising the context then a calculation</u>

Four pairs initially gave generalised descriptions using the ‘ends and middles’ approach.

Pair 5: *You need three for the end tables and two for the rest of the tables*

The interviewer's normal prompt to explore pupils' thinking further was to ask if they could write something down. In response to this kind of prompt, these four pairs gave generalisations of calculations, but were not confident about writing these: for example,

Pair 5: *You could put end tables times three chairs and then middle tables times two chairs.*

Expressing algebra-like rules

The remaining seven pairs of pupils all managed to articulate a rule for calculating the number of chairs which was recognisably algebra-like, using a variety of non-standard notations. Only one pair produced a rule in 'standard' notation. One pair gave a description of the calculation as their first response, while the remaining six moved from general descriptions of the context to generalising a calculation, and then to a symbolic version of the rule (see examples in Figure 2.) Although their responses are less confident than those of cohort A, they display explicit attempts to work with a variable.

	Generalisation of context	Generalisation of calculation	Symbolic rule
Pair 9	*You have two on each end and then there's two on each side of the table*	*You could times how many tables there are by two and then add the extra two on the …(ends)*	How many tables x2 +2
Pair 10	*For one table you'd need two chairs or for an end one three*	*You'd like, you'd double it then add the two on the end*	A numbAS x2 +2 Originally they wrote '15' then crossed this out and wrote 'a number', saying "it's not necessarily gonna be fifteen"
Pair 12	*It starts off with three for the first one and then its carry on adding two and then when it gets to the end it's add three again.*	*Times two that amount … add two*	*Just write times the amount of squares* [i.e. tables] *by two then add two*

Figure 2: examples of responses from cohort B showing algebra-like rules.

The overall pattern from cohort B appears to support our conjecture. Those pairs who were able to articulate an algebra-like version of the rule had previously generalised the calculation required; generalizing the context did not seem to be sufficient to support pupils in moving to a symbolic version of the rule.

DISCUSSION

The Purposeful Algebraic Activity Project centers around a teaching programme in which we have designed spreadsheet-based tasks to support generational activity (Ainley, Bills & Wilson (forthcoming)). Typically, such tasks involve pupils in generalising relationships and expressing these as spreadsheet formulae in order to use the power of the technology to solve problems. The spreadsheet environment provides a purposeful context for the use of an algebra-like notation, but we see a key element of such tasks as the construction of meaning for such notation through the activity of generalising.

The analysis of the ‘tables and chairs’ question has provided an important insight into the ways in which such relationships may be generalised, and suggests that if pupils’ activity is focused only on generalising within the context (as in describing arrangements of the furniture), and does not move to generalising the required calculations, then an important link in the construction of meaning for the symbolic notation may be lost.

Although our study is similar to that used by Radford (2001) as the basis for his categorisation of levels of generalising, in terms of the type of problem and the age of the pupils, there are some differing features which make a direct comparison difficult. These differences lie in the questions on which the pupils were working. Although both questions are based on geometric-numeric patterns, in our study the pattern had been contextualised in an ‘everyday’ setting (tables and chairs), whilst Radford’s question is based on an arrangement of matchsticks. In designing our interview question we had deliberately set what could have been presented as an abstract spatial design in an ‘everyday’ context, in which the expression of a generalised relationship could be used in a purposeful way, which parallels the pedagogic tasks in our teaching programme. In Radford’s study, there is no contextual setting to which pupils might refer, and indeed in the extracts he quotes pupils refer only to the numerical (rather than the spatial) patterns.

Furthermore, in Radford’s study, the question is broken down into stages as pupils are asked to find the number of matchsticks used in the fifth, and then the twenty-fifth, term in the pattern sequence before being asked to generalise for any term. We had made a choice to present the problem with a single image rather than as a sequence in order to discourage term-by-term approaches which obscure the need for algebraic generalisation. These two features (problem context and the staging of the question) appear to have produced different kinds of responses from pupils. Although there seems to be much in common between our analysis and that of Radford, particularly in terms of the shift in the level of abstraction of the objects involved in the generalisation, the different features of the question seem to have altered the process of generalising for these relatively inexperienced pupils. Whilst we feel that the overall level of success pupils achieved in making some kind of generalisation of the relationship between the numbers of tables and chairs indicates that both the problem context and the one-stage question were effective in supporting the constructing meanings for generalising, we need to consider carefully what effect they may have on the construction of meaning for symbolic notation.

We see this as significant in relation to the design of pedagogic tasks. One way in which we might read pupils’ responses to the tables and chairs problem is that they focus too much on the context and fail to successfully ‘negotiate the boundary between the ‘mathematical’ and the ‘real’’ in the design of the question (Cooper and Dunne, 2000). They see the problems as essentially about arranging furniture, and do not appreciate the pedagogic intention to express a relationship or calculation. We did, indeed, find that some children became distracted by discussing other possible arrangements of the tables and chairs, thinking about how the chair had to moved as other tables were added, or asking how many children needed to sit down for lunch. Adding an element to the task which signals clearly the need to describe a calculation (such as ‘Could you tell the caretaker how to work out how many chairs she should get out of the storeroom?) might signal the pedagogic emphasis of the task to pupils in a meaningful way.

We also see the distinction we have observed as potentially significant in helping to focus teachers' interventions. We are working closely with a group of teachers in the development of the teaching programme, discussing teaching approaches as well as the content of the tasks. If teachers become aware of the different ways in which pupils may be generalising, this offers the opportunity for targeted interventions to support pupils' construction of meanings for formal notation by asking them to articulate and generalise their calculations.

References

Ainley, J. (1999). Doing Algebra Type Stuff: Emergent Algebra in the Primary School. In O. Zaslavsky (Ed.) *Proceedings of PME23,* Haifa: Technion, Vol. 2, 9-16

Ainley, J., Bills, L. and Wilson, K. (forthcoming) Designing Tasks for Purposeful Algebra. In *Proceedings of the 3rd Conference of the European Society for Research in Mathematics Education.*

Ainley, J. and Pratt, D. (2002). Purpose and Utility in Pedagogic Task Design. In A. Cockburn and E. Nardi (Eds.) *Proceedings of PME26*, Norwich, UK, Vol. 2 17-24.

Bednarz, N. (2001). A problem-solving approach to algebra: Accounting for reasonings and notations developed by students. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra*, Melbourne, The University of Melbourne, Vol. 1 69-78.

Cooper, C. & Dunne, M. (2000). Assessing Children's Mathematical Knowledge, Buckingham, Open University Press

Kieran, C. (1996). The Changing Face of School Algebra. In C. Alsina, J. M. Alvares, B. Hodgson, C. Laborde & A. Pérez (Eds.) *Proceedings of the Eighth International Congress on Mathematics Education: Selected Lectures*, Seville: S. A. E. M. 'Thales',. 271-290

Macgregor, M. and Stacey, K. (1993). Seeing a Pattern and Writing a Rule. In I Hirabayashi, N. Nohda, K. Shigematsu and F.L. Lin (Eds) *Proceedings of PME17,* Japan, Vol 1, 181-188.

Mason, J., Pimm D., Graham, A. and Gower, N. (1985). *Routes to/Roots of Algebra*, Milton Keynes: Open University

Meira, L. (forthcoming) Students' Early Algebraic Activity: sense making and the productions of meanings in mathematics. In J. Kaput (Ed.) *Employing children's natural powers to build algebraic reasoning in the context of elementary mathematics*. Hillsdale: Lawrence Earlbaum.

Radford, L. (2001). Factual, Contexual and Symbolic Generalizations in Algebra. In M. van den Heuvel-Panhuizen (Ed.) *Proceedings of PME25,* Utrecht, Freudenthal Institute, Vol. 4, 81-88.