

PRESERVICE TEACHERS' VIEW ON $y = x + 5$ AND $y = \pi x^2$ EXPRESSED THROUGH THE UTILIZATION OF CONCEPT MAPS: A STUDY OF THE CONCEPT OF FUNCTION

Örjan Hansson

Kristianstad University/Luleå University of Technology, Sweden

This paper considers a group of preservice teachers' construction of concept maps derived from $y=x+5$ and $y=\pi x^2$ with emphasis on their conceptual understanding of function. The two statements are perceived to represent a number of different concepts with indications of compartmentalized knowledge structures that might prevent the preservice teachers from building rich conceptual structures. The preservice teachers' view on the concept of function contrasts with a view where the function concept is a unifying concept in mathematics with a large network of relations to other concepts. Different properties and categorizations of functions are less frequently recognized. In the preservice teachers' response of drawing concept maps there are signs of metacognitive activity.

INTRODUCTION

A limited number of studies have been conducted using concept maps in mathematics education, with only a minor part accessing students' conceptual understanding of function (e.g., Doerr & Bowers, 1999; Grevholm, 2000a, 2000b; Hansson, 2004; Hansson & Grevholm, 2003; McGowen & Tall, 1999; Williams, 1998). Williams (1998) concludes that concept maps assess conceptual knowledge in studying maps drawn by calculus students and professors with PhDs in mathematics. McGowen and Tall (1999) give further support to this conclusion in studying concept maps drawn by students on different levels of achievement in their studies. Other studies in mathematics education (e.g. Laturno, 1994) give further credibility to concept maps as an assessment technique.

Functions are present within all areas of mathematics, and it is important for preservice teachers in becoming successful dealing with the concept of function to have a well-developed conceptual understanding of function, including the concept's relations to other concepts and significance in mathematics (Cooney & Wilson, 1993; Eisenberg, 1992; Even & Tirosh, 2002; Vollrath, 1994). Mathematical statements like $y=x+5$ and $y=\pi x^2$ can be perceived to represent a number of concepts on a variety of levels, related to the preservice teachers' future teaching as well as more advanced concepts in mathematics. It is possible to let the statements represent concepts such as a straight line, parabola, equation, formula, proportionality, function (if we assume that domain and codomain also are considered), and others. The two statements $y=x+5$ and $y=\pi x^2$ can in particular be recognized as real functions of a real variable in using a representation not uncommon for the concept of function (Eisenberg, 1991, 1992; Tall, 1996). The function concept can in this context be related to different

properties and classes of functions and recognized as a concept with a large network of relations to other concepts (Cooney & Wilson, 1993; Eisenberg, 1991, 1992).

The purpose of the current study is to examine preservice teachers' conceptual understanding of function in relation to $y=x+5$ and $y=\pi x^2$ through the utilization of concept maps.

THEORETICAL FRAMEWORK

In the chosen theoretical framework knowledge is considered as an individual construction built gradually. Knowledge is represented internally and described in terms of the way an individual's mental representation is structured (Goldin, 2002; Hiebert & Carpenter, 1992). Internal representations can be linked, metaphorically, forming dynamic networks of knowledge with different structure, especially in forms of webs and vertical hierarchies. Understanding grows as an individual's knowledge structures become larger and more organized, where existing knowledge influence relationships that is constructed. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. Hiebert and Carpenter (1992) describe the construction of larger and more organized networks of knowledge as learning with understanding. A similar notion is described by Ausubel (2000) as meaningful learning, as opposed to rote learning "only in rote learning does a simple arbitrary and nonsubstantive linkage occur with preexisting cognitive structure." (Ausubel, 2000, p. 3).

A central part of Ausubel's assimilation theory of meaningful learning is the idea that new meanings are acquired by the interaction of new, potentially meaningful ideas with what is previously learned. This interactional process results in a modification of both the potential meaning of new information and the meaning of the knowledge structure to which it becomes anchored. The process of assimilation results in progressive differentiation in the consequent refinement of meanings, and in enhanced potentially for providing anchorage for further learning.

The total cognitive structure that is associated with a concept in the mind of an individual is viewed in the way of Tall and Vinner (1981), as a concept image, including "all the mental pictures and associated properties and processes" (p. 152). When an individual meets an old concept in a new context it is the at that time evoked concept image – a portion of the concept image – with all the implicit assumptions abstracted from earlier contexts that respond to the task. In an individual's conceptual development Sfard (1991, 1992) suggests a process-object model. The formation of an operational conception, a process conception, precedes a more mature phase in the formation of a structural conception with focus on objects.

METHOD AND PROCEDURE

The current study is part of an ongoing study on preservice teachers' understanding of the function concept (Hansson, 2004). The preservice teachers that participate in the study are in the third year of a four and half year long teacher preparation program and specialize in mathematics and science, grades 4 to 9. During the sixth

term they enrolled in the final courses in mathematics of the program, and the study is conducted after calculus course where the concept of function is a central concept.

A group of twenty-five preservice teachers was given a presentation on concept maps during a lecture and showed examples of different types of concept maps such as non-hierarchical web-based maps and hierarchical maps, where the nodes represented concepts and the links were labeled in each case. (The hierarchical maps were constructed according to Novak & Gowin, 1984, and Novak, 1998). The maps that were displayed were often related to science education. Concept maps which were derived from mathematical concepts were largely avoided during the presentation, so as not to influence the contents of the maps which the students were to draw in the subsequent assignment.

After the introduction, the preservice students were each directed to draw concept maps based on the statements $y=x+5$ and $y=\pi x^2$, respectively. In the process of drawing maps with hierarchical structure they started to draw maps with a freely formatted structure. As a result, the preservice students constructed two maps for each mathematical statement; one freely formatted map and one hierarchical map. The preservice students were also asked to comment on their maps and their experiences of drawing the maps.

To draw concept maps based on $y=x+5$ and $y=\pi x^2$, respectively, differ from the use of concept maps derived from “function” (as in e.g. Doerr & Bowers, 1999; Grevholm, 2000b; McGowen & Tall, 1999; Williams, 1998) in that a concept is not explicitly stated. The statements can thus be perceived to represent a number of different concepts. The derived concept maps give the subject an opportunity to illustrate what concepts the statements are perceived to represent, and their different properties and relations.

All of the concept maps were analyzed. Each map was analyzed as an integrated unity in which the contents and structure of the map were noted. Furthermore, the manner in which the different sections of the map were related to each other was also studied. In particular, the manner in which the function concept was expressed on the maps was noted, its relationships to other concepts and properties that were assigned to functions were considered. The contents of the maps and the preservice students’ comments on the maps were also compiled in tables.

The presentation took about 30 minutes, and the group received 60 to 80 minutes depending on when the students decided they had completed the assigned task. Twenty-four students in the group submitted all of their maps.

RESULTS

The statements on the freely formatted maps are usually placed at the center of the map; the map has a web-like structure, but very few cross-links i.e. links connecting different parts of a map (e.g. those drawn by F3, F5 and F18). The maps are less detailed than the hierarchical maps, and often served as a draft of the information that later would be included in the hierarchical maps. The hierarchical maps tend to branch off into substructures which often have few cross-links (as in those drawn by

F2, F8 and F16), implying that the concepts which are perceived to be represented by the statement often develop separately from each other. Although the maps have a hierarchical form, the conceptual structure is less hierarchical; thus, more general and inclusive concepts tend to be mixed with more specific concepts (a completely hierarchical conceptual structure is however not possible, since the maps are derived from specific examples). Furthermore, it is clear that when the preservice teachers construct maps for $y=x+5$ and $y=\pi x^2$ at the same occasion, it influenced the maps' contents and structure (as indicate by the maps and comments of e.g. F1, M8, F16).

All of the preservice teachers in the group have included the function concept in their maps. None of the preservice teachers discuss domain or codomain in relation to the function concept. They often express the function concept as a "relation" (e.g. F1, F12, F16) or a "dependence" (e.g. F3, F10, F11) between the variables x and y , reflecting a process conception (Eisenberg, 1991; Sfard, 1992). The node which contains the function concept is usually connected to the statement from which the map is derived and often gives rise to an underlying structure with very few links to other sections of the map (e.g. F2, F9, F10) – even if there are students who integrate the function concept in their maps to a great extent (e.g. F1, M4, F14).

Maps that contain fewer cross-links highlight to a lesser degree the manner in which the different concepts – which are represented by the statement – are related to each other. For example, F2 has the nodes "straight line" and "function" but does not state that the straight line is a graph of the function. Similarly, M7 links "curve" to "minimum value", but does not link "minimum value" and "function". Whereas preservice teachers who largely link the function concept to other nodes on the map, also link it to concepts which give the impression that they do not fully understand the function concept. The preservice teachers experience difficulties in distinguishing relations between the function concept and the equation concept (similar to e.g., Grevholm, 2000b; Leikin, Chazan & Yerushalmy, 2001; Williams, 1998), where e.g. M8 place "function" as a sub-concept¹ to "equation" in his hierarchical maps, M2 links "inverse" to "equation" and M17 links "equation" to "has derivative".

Concerning classes of functions, about half of the preservice teachers did not tie the function concept to any special function class (e.g. F1, M6, F18). While nearly a third gave "second degree function" as a polynomial function and "function" for $y=\pi x^2$ and $y=x+5$, respectively, and thus link $y=\pi x^2$ but not $y=x+5$ to a function class (e.g. M5, F14, F16). Other preservice teachers state that $y=\pi x^2$ is a "second degree function", and that $y=x+5$ is a "first degree function" (M4 and M8) or a "linear function" (M2, F15, F17). It was noted that there were preservice teachers who incorrectly associated $y=\pi x^2$ with other function classes, e.g. F13 who link it to "exponential function". Otherwise, the preservice teachers do not mention any properties of functions, implying that they view $y=x+5$ and $y=\pi x^2$ as belonging to

¹ M8 links " $y=f(x)$ " to "function", the concept map implies that M8 interprets the notation " $y=f(x)$ " as an equation and thus views "function" as a sub-concept to "equation".

categories of functions that they met during their calculus course like even, continuous, increasing, differentiable etc. In those cases where the derivative is discussed (only a few of the students mention derivatives in their maps, e.g. M1, M4, F18), it is not viewed as a property which gives rise to a class of functions, but instead as a procedural skill in the form of a calculation process. For example, just as M2 states that the derivative determines the slope or F5 states that the derivative determines the minimum point. Derivatives are however more common in the maps derived from $y=\pi x^2$ (6 maps) than for maps derived from $y=x+5$ (4 maps).

Only a few preservice teachers establish links between teaching and learning in their maps (similar to Doerr & Bowers, 1999). F5 for example, states that “ $y=\pi x^2$ ” is a “second degree function” which is “difficult” and that “primary school students cannot manage it”. This is in contrast to “ $y=x+5$ ” which she labels a “simple function” which “is used in primary schools”. The preservice teachers also expressed similar opinions on degree of difficulty without establishing any connection to learning: F3 states that $y=x+5$ is a “simple function” and M6 states that “ $y=x+5$ ” is “easier” than “ $y=\pi x^2$ ” (on his map for $y=\pi x^2$).

There are elements of an “algorithmic nature” in the maps (in common with McGowen & Tall, 1999; Williams, 1998) in which e.g. F3 states that $y=\pi x^2$ has two roots $x = \pm\sqrt{y/\pi}$, F8 establishes a relation to the area of a rectangle “ $A=ab$ ” and a triangle “ $a=bh/2$ ” (in relation to $y=\pi x^2$), and F5 states that “the derivative” of $y=\pi x^2$ gives a “minimum point”. Moreover, the preservice teachers also raise matters which are trivial, to say the least, in their maps e.g. M1 states that $y=x+5$ is “easy to draw”, F2 states that x^2 is x “multiplied by itself” or F8 who writes the Greek letters “ ϵ , α , β ” in relation to π .

The preservice students’ written comments revealed clear perceptions that the maps promote metacognition and have a mediating role (Novak, 1998), e.g. M6 who states that “you see relations that you have not considered”, F3 who writes that “one really has to consider the meaning of what the different things mean and where they lead”, or F8 who says that one “learn from the map, facts are entered once again”. But there are also signs that the maps evoke concept images with conflicting pieces of information, e.g., M5 with the opinion that “It only causes confusion...”.

One preservice teacher’s maps

The two concept maps below are examples of how the preservice teachers might draw their maps. They illustrate a tendency for the maps in the study to have few cross-links – or as in this case a lack of cross-links – where different parts of a map are developed separately. Moreover, the two maps illustrate a common feature of the preservice teachers’ maps in containing graphical interpretations – although not always in relation to the concept of function in contrast to the maps below. They also show how preservice teachers less frequently recognize relations to different properties of functions. For instance, “ $k=1$ ” (in figure 1) might be related to an increasing function. Moreover, “min” and “min max” (in figure 2) are related to the concept of “parabola” rather than “function” or “2nd degree function”, respectively.

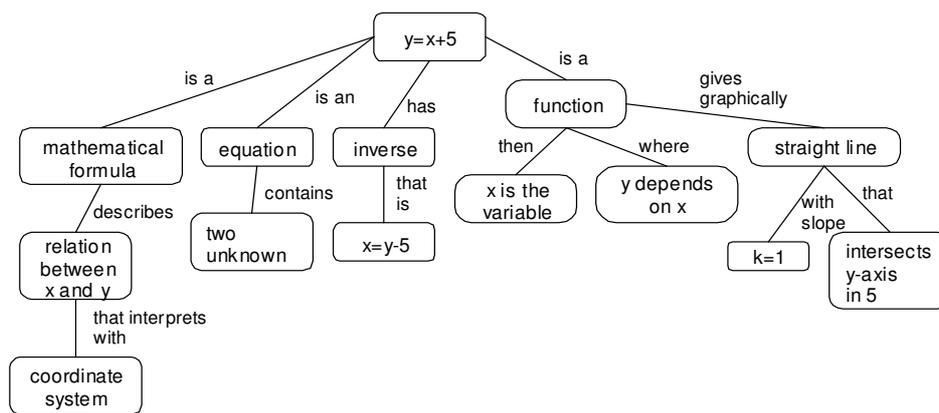


Figure 1. The hierarchical map of $y=x+5$ drawn by F16.

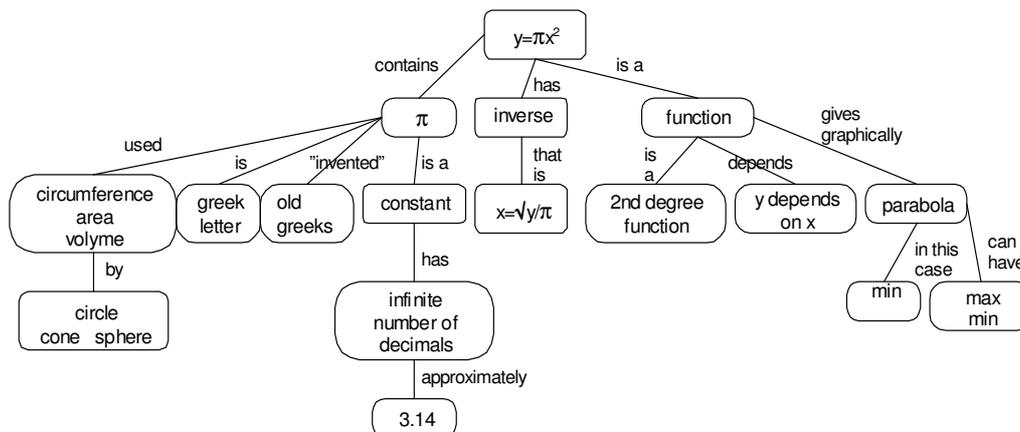


Figure 2. The hierarchical map of $y=\pi x^2$ drawn by F16.

The two maps also show individual characteristics as for example in the case of “inverse”, where the function concept seems to be a less meaningful concept. This indicates rote learning (Ausubel, 2000) in particular as $y=\pi x^2$ does not have an inverse (when considered as a real function of a real variable).

DISCUSSION AND CONCLUSIONS

There are clear indications that the function concept often is developed independently with few relations to other parts of the maps. This may be an expression of compartmentalized knowledge structures that prevent the preservice teachers from building rich conceptual structures that form a basis for meaningful learning (Ausubel, 2000) and learning with understanding (Hiebert & Carpenter, 1992), with consequences for the preservice teachers’ future teaching (Even & Tirosh, 2002; Vollrath, 1994). Moreover, the preservice teachers rarely relate to teaching and learning in their maps. This may be surprising, since their mathematics courses all contain parts related to mathematics education. Particularly as the characteristics of the two statements $y=x+5$ and $y=\pi x^2$ make them suitable for connection to different teaching scenarios which the preservice teachers will face as inservice teachers.

Preservice teachers often express the function concept as a dependency between the variables x and y – describing a process conception (Eisenberg, 1991; Sfard, 1991, 1992). In some cases, they give somewhat more elaborate explanations and state that an x gives one y . But none of the students discuss domain or codomain in association

with the function concept, and a more developed conceptual understanding in the form of an object with a set of properties is less frequent in the maps.

Elements that express procedural knowledge and skills of an algorithmic nature occur frequently (in agreement with results presented by e.g. Grevholm, 2000b; McGowen & Tall, 1999; Williams, 1998). The maps may also contain elements that are completely trivial at the expense of important concepts and relations between them, which indicates rote learning (Ausubel, 2000). Moreover, several misconceptions related to the concept of function can be identified on the maps.

The function concept's large network of relations to other concepts is frequently not part of the concept maps. This is usually a consequence of that the preservice teachers do not observe and relate to different properties of $y=x+5$ and $y=\pi x^2$ when they are regarded as functions. In those cases the preservice students give a concept, or some characteristic of a concept, they do usually not observe the relations to the concept of function. The perception of the function concept that is illustrated in the concept maps contrasts highly with the idea of functions playing a central and unifying role in mathematics (Cooney & Wilson, 1993; Eisenberg, 1991, 1992).

The concept maps seem to reveal a need for the preservice students to reflect upon the function concept's relevance in mathematics, its different properties and its network of relations to other concepts. The preservice teachers' comments on drawing the maps indicate that the process of drawing concept maps supports metacognitive activities (Novak, 1998). Such activities might promote the preservice teachers' conceptual understanding of the function concept and its significance in mathematics and require further research.

References

- Ausubel, D. P. (2000). *The acquisition and retention of knowledge: A cognitive view*. Dordrecht: Kluwer Academic.
- Cooney, T. J. & Wilson, M. R. (1993). Teachers' thinking about functions: Historical and research perspectives. In T. Romberg, E. Fennema & T. Carpenter (Eds.), *Integrating Research on the Graphical Representation of Function* (pp. 131-158). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Doerr, H. & Bowers, J. (1999). Revealing preservice teachers' thinking about functions through concept mapping. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of PME-NA* (Vol. 1, pp. 364-369). Columbus, OH: ERIC.
- Eisenberg, T. (1991). Functions and associated learning difficulties. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 140-152). Dordrecht: Kluwer Academic.
- Eisenberg, T. (1992). On the development of a sense for functions. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 153-174). Washington, DC: MAA.
- Even, R. & Tirosh, D. (2002). Teacher knowledge and understanding of students' mathematical learning. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 219-240). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Goldin, G. A. (2002). Representation in mathematical learning and problem solving. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 197-218). Mahwah, NJ: Lawrence Erlbaum Associates.

- Grevholm, B. (2000a). *Teacher education in transition. The case of Sweden*. (Paper presented at WGA7 at ICME9). Research report. Kristianstad: Kristianstad University.
- Grevholm, B. (2000b). *Research on student teachers' learning in mathematics and mathematics education*. (Regular lecture at ICME9). Research report. Kristianstad: Kristianstad University.
- Hansson, Ö. (2004). *Preservice teachers' view on the concept of function: A study including the utilization of concept maps* (LTU licentiate thesis series No. 2004:49). Luleå: Luleå University of Technology.
- Hansson, Ö. & Grevholm, B. (2003). Preservice teachers' conceptions about $y=x+5$: Do they see a function? In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th Conference of PME* (Vol. 3, pp. 25-32). Honolulu: University of Hawai'i.
- Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 65-97). New York: Macmillan.
- Laturno, J. (1994). The validity of concept maps as a research tool in remedial college mathematics. In D. Kirshner (Ed.), *Proceedings of the 16th Annual Meeting of the PME-NA* (Vol. 2, pp. 60-66). Baton Rouge, LA: Louisiana State University.
- Leikin, R., Chazan, D. & Yerushalmy, M. (2001). Understanding teachers' changing approaches to school algebra: Contributions of concept maps as part of clinical interviews. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of PME* (Vol. 3, pp. 289-296). Utrecht: Utrecht University, Freudenthal Institute.
- McGowen, M. & Tall, D. (1999). Concept maps & schematic diagrams as devices for documenting the growth of mathematical knowledge. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME* (Vol. 3, pp. 281-288). Haifa, Israel: Technion.
- Novak, J. D. (1998). *Learning creating, and using knowledge: Concept maps as facilitative tools in schools and corporations*. New York: Lawrence Erlbaum Associates.
- Novak, J. D. & Gowin, D. B. (1984). *Learning how to learn*. Cambridge: Cambridge University Press.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1-36.
- Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification - the case of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 59-84). Washington, DC: MAA.
- Tall, D. (1996). Functions and calculus. In A. J. Bishop (Ed.), *International handbook of mathematics education* (pp. 289-325). Dordrecht: Kluwer Academic.
- Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169.
- Vollrath, H. J. (1994). Reflections on mathematical concepts as starting point for didactical thinking. In R. Biehler, R. W. Scholtz, R. Strässer & B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline* (pp. 61-72). Dordrecht: Kluwer Academic.
- Williams, C. G. (1998). Using concept maps to assess conceptual knowledge of function. *Journal for Research in Mathematics Education*, 29(4), 414-421.