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Research Reports

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STUDENTS' USE OF ICT TOOLS - CHOICES AND REASONS

Anne Berit Fuglestad

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In a development and research project over a period of three years, with classes 8 to 10 in school, the aim was to develop students' competence to use computer tools. The aim was that the students should be able to judge what ICT tools are appropriate to use for a specific task. Seven classes with six teachers took part in the project. In a working period in the final part of the project the students work were observed, and afterwards they were given a questionnaire about their experiences in the work, what tools they chose to use and why. The results indicate that about half of the students could make reasonable choices and give appropriate reasons for their choice.

BACKGROUND

The project is based on a social constructivist view of learning, aiming that students will be stimulated to develop their mathematical concepts and understanding of computer tools for solving mathematical problems. This is in accordance with the mathematics plan in the Norwegian curriculum guidelines, L97, which state that the students should "know about the use of IT and learn to judge which aids are most appropriate in a particular situation" (KUF, 1999). The plan expresses a constructivist view of learning and the need to emphasise conversation and reflection. The use of meaningful situations and realistic problems as starting points will motivate the students according to L97. Similar ideas are expressed in other plans and documents, e.g. Principles and Standards for School Mathematics (NCTM, 2000).

A main aim in the project was in line with these ideas to develop the students' competence to use computer tools, and experiment with mathematical connections in an open co-operative environment.The main question for this research was to explore to what extent the students can utilise the ICT tools and make reasonable choices of tools for a specific mathematical task or problem and give reasons for their choices?

In this context I think of ICT tools as open and flexible software, not made for specific topics or limited to teach specific tasks. ICT tools are computer software that makes it possible for the user to plan and decide what to do. Such tools can be used for a wide variety of problems and can provide learning situations to explore and experiment with mathematical connections, and provide new ways of approaching the tasks. I think this is in accordance with seeing ICT tools as cognitive tools that influences the way we develop mathematical concepts and connections, and are reorganizer rather than as amplifiers (Dörfler, 1993; Pea, 1987). Amplifier implies doing the same as before, without changing the basic structure, methods and approaches, whereas "re-organising occurs when learners' interaction with technology as a new semiotic system, qualitatively transforms their thinking" as expressed by Goos et al. (2003).

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Based on this view of learning and the potential of computer as mediation tool to provide rich mathematical learning situations we chose software that are open and flexible, provides a learning environment for exploration and investigation, integrate different representations, and stimulate reflection (Hershkowitz et al., 2002). In our project, a spreadsheet (Excel), graph plotter (Grafbox) and dynamic geometry, Cabri, were the main tools together with use of Internet for collecting data and information.

A THREE YEAR DEVELOPMENT OF COMPETENCE

The project lasted three years starting with students in year 8 finishing in year 10 when they ended compulsory school. Seven classes with six teachers from three schools, named S, F and L, took part in the project. All the teachers had some experience of using computers, but the teacher of one class in school S had more experience with ICT in mathematics classes and he was also my co-project leader.

In order to develop the students' competence with ICT tools we planned regular use in the classes and build an open learning environment with options for students choices of tasks, tools and methods. The project leaders provided some material and ideas for use in the classes, both existing material and new. As the project developed some teachers also prepared some material in computer files and shared with the rest of the group. An important issue at the project meeting was to report and discuss experiences from the classes, further ideas and features of the ICT tools.

The teachers were in charge of what tasks and ideas they implemented in their classes. This gives less control from the project, but we think this makes the work realistic and sustainable. The teachers reported in a simple form each term what ICT tools and tasks they worked on in their classes.

STAGES OF COMPETENCE

The development of students ICT competence involve gaining facility with the software and understanding of the connection to mathematical concepts and models that can be expressed with the tools. This competence has to be build over time as students use the tools starting from simple tasks, develop insight and move on to more complicated models using suitable tasks (Fuglestad, 2004):

1. Basic knowledge of the software tools. The students can utilise the functionalities of the software to solve simple tasks prepared for it, when they are told what software to use. For example to make formulas in a spreadsheet when given the main outline or to use a graph plotter to plot a function when the formula is given.
2. Develop simple models. The students can make the layout of text, numbers and formulas to plan a model for a spreadsheet. For a graph plotter they can judge what functions to draw, use different scales on axis, zoom in or out. Be able to use dynamic geometry to make constructions that can resist dragging, i.e. the figure do not fall apart when parts of it, points or line segments, are moved.
3. Judge the use of tools for a given problem. The students are able to think of different ways and means for solving a problem, which software is most appropriate to use or when other methods are better.

I think the stages do not describe levels of competence, although to have some basic knowledge of the software is necessary for stage 2. Stage 2 I think cover several levels of competence. Models on a spreadsheet can vary from very simple to large complicated models which utilises complicated combinations of functions. Stage 3, to be able to judge for example if a spreadsheet is suitable, can perhaps be achieved without mastering very complicated models. In the same way other software can be used on different levels and depth of insight.

Development of mathematical competence is involved at all stages. In order to prepare their own models there is understanding of variables, expressing functional connections with formulae in Excel or Grafbox or analyse geometrical connections and properties and use corresponding menu tools to make drag resistant figures in Cabri. In order to develop this competence the tasks should be developed to give experience with different and typical models (Fuglestad, 2004).

METHODOLOGY

The project was a combination of development and research, with main focus on development of students' competence the first two years, and further development and research during the last year. The work involved project meetings, development and discussion of teaching ideas and teachers implementing ICT use in their classes. I visited the classes to support the teachers, observe and get known to the classes.

Data were collected during all the three years, but more focussed the last year with a close observation period in the last term. Observations and interviews with students were audio and video recorded. The teachers reported their activities in the classes on a simple report form and experiences from work in the classes were discussed in the project meetings. Students work in computer files and partly on paper were collected.

During a few weeks, approximately 8 – 10 hours of work the students worked on a collection of 12 tasks in a small booklet. The tasks were of different levels and degree of openness; some had clear questions and some presented just an open situation with information and the students had to set their own tasks. The tasks were designed to give variation in ICT use, but taken into account that the experiences were different in the classes. A draft version was discussed with the teachers in the previous project meeting. The students chose what tasks they worked on, what order and what tools to use and they could chose ICT tools, mental calculation a calculator, paper and pencil or a combination. They could work alone or in pairs and discuss their solutions or get some help from the teachers.

The activities when students worked on the booklet was observed and partly recorded. The students filled in a log-sheet telling what tasks they solved, what tools they used and who they cooperated with and other comments to the task.

About a week later the students were given a questionnaire to answer individually, via Internet, connected to the work on the tasks, what tools they chose and why, and some questions about attitudes to their work with ICT tools. In the last part of the

questionnaire four new tasks were presented and the students had to read the tasks and comment on what tools can be used, what they would use themselves for and why. The questionnaire had a combination of boxes to tick and open answers where they were asked to write their comments.

CLASSROOM OBSERVATIONS

A task in the booklet was about prices for a bus tour for the class using different bus companies:

> "The students at Lie school are going on a tour with their three teachers. They ask for a price offer from three bus companies. Ryenruta gives this offer: 1100 kr fixed charges and 5.50 kr per km, Lie tour service offers: 9 kr per km and Hagenruta offer a fixed price: 4000 kr for tours up to l 800 km. The students judge three possible tours: to Kristiansand Zoo, about 200 km, Tusenfryd 140 km or Hunderfossen 320 km"

Since no question was included, the students had to judge what to do, how to compare prices and what tools to use.

I observed two boys working alongside each other and discussing how to start. One boy named Tim, was very clear he wanted to use Grafbox since he had used this before; he enjoyed this software and thought he could handle it well. The other one, Hans, wanted to use Excel and was just as sure about it. They worked in parallel, each on their own solution on computes close to each other and could see both screens. They discussed in between if they got the same answers, and how to compare their solutions.

Tim talked a lot, telling what he did:

> See when you drive 800 km how much you earn on it. This is crazy. Here there is something variable *x*, I have to look see, 1100 times five. But can you write five comma five times five point five, or is it a point?

He found something is wrong, and look into the formula he made, then just check with me if the decimal sign is comma or a point in Grafbox (comma is used in Norwegian). He asked me a few questions, when I (ABF) observed, I did not answer directly but asked new questions:

ABF: I do no quite remember, but you can just try a point and see. Software are different.

TIM: Yes, eh it is.

ABF: five comma five times five comma five (expression he wrote in Grafbox)

TIM: Yes five comma five crowns per kilometre.

ABF: Yes.

TIM: And then times *x* because it is uncertain what number of kilometre.

ABF: Yes.

TIM: Times, will this be correct? I will try. But you, this can not be .. they are not both here.

We looked at the graph window and found the axis adjusted automatically, and made one graph invisible due to the scale. I helped him to change scale and we could see both graphs he had made so far, only the one did not look reasonable.

ABF: Look at this.

TIM: Yes, ok, se now, this "change graph" (referring to the menu in Grafbox) take this. This is strange it is nought to eight.

ABF: What have you written in the box for the formulae, can you have a closer look at it?

TIM: Yes I wrote one thousand one hundred.

ABF: Yes.

TIM: Perhaps that is why (looking at his expression).

ABF: one thousand one hundred, what is that

TIM: This is the … the fixed price , yes.

ABF: and so?

TIM: I times it.

ABF: Why ?

TIM: Of course it is plus, eh plus.

ABF: Why is it plus, you think?

TIM: Yes, because you take plus the money they charge to drive.

I found in the way he worked he had a good understanding of the software, and how to enter the expression of a function. But he only got confused when the scales changed automatically. He made a mistake using multiplication instead of addition in the formulae for the second graph he made. But he discovered there was a problem and changed the formulae accordingly and then worked on the other price offers.

I moved away for a while and observed at a distance. I could see other students moving over to Tim and Hans, asked what they did and they discussed the solution. From the audio recording I can hear they discussed mainly the task. They seemed to be concentrated on the task even if it was close to the end of the lesson and got fairly noisy in the classroom. Tim also invented his own price models, one very unusual for a bus company, making the tour cheaper for longer journey, and made a graph accordingly. Here he demonstrated that he managed very well the linear functions.

Hans made a model on the spreadsheet, calculating the prices just for the specific distances and prices given and did not make a more general comparison. The two students looked at each others solutions, and discussed what they had done, how to find the connection between the solution and if they gave the same answer. Later when the teacher arrived Hans explained his solution correctly.

What choices did the students make in the questionnaire? The work of Tim and Hans is reflected in the answers they gave in the questionnaire. Tim chose this task about the bus tour as the task he liked to work on, and commented: It was because this I have done this before, in year 8. I felt I got the grip on it.

He chose Grafbox but commented he also solved the task using Excel, and why: Because in this program you can also get an overview on task 10 that is about what pays off different to use things. But on the task I think curve plotting program is the best. And on the question what did you learn from this task: I learned that before you hire a bus you have to check and compare several companies.

Hans chose another task as the one he liked best. The task was to compare prices for a new car and an old car (veteran), where the new got cheaper by 15% and the veteran car more valuable, 10% price raise per year. Starting prices were 190 000 NOK and 15 000 NOK and the task was to find out how long time it will last before they have the same price. His choices were similar to Tim's. He chose both Grafbox and Excel as tools and commented why he liked it: We managed to do the task in Grafbox, vi managed the formulae after a good deal of discussion and some guidance from the teacher. And why he chose these tools: In a spreadsheet we wrote formulae and got the value in number for 5 years. But in Grafbox we made curves, using formulae that we wrote and then we found a point where the cars were of the same value.

The answer from Tim and Hans to the new tasks in the questionnaire had similar references to the tools: Hans wrote for three of the tasks, that all can be solved with a spreadsheet: Because with a spreadsheet you can write in formulae and find exact answer! Tim wrote for one of these*:* Here you can get an overview of how much to pay depending on how many rounds you drive. And he gave a similar answer to another.

For the last task, finding the larges rectangular sports arena inside a triangular area, Hans wrote; because then you can construct the task and with the help of the menus in Cabri can you solve the task! And Tim wrote: You can find both area and measure of the arena. These answers correspond with the use of Cabri to find the solution.

The two boys' choices of tools and comments to why corresponds with their work in the class. The answers to the new tasks are short, but relevant. These questions came late in the questionnaire and that could explain the short answers.

RESULTS FROM QUESTIONNAIRE – CHOICE OF TOOLS

In the questionnaire the students were asked to choose a task they liked to work on, why they liked it, and tell what tools they used and explain why. The choices were among the ICT tools, paper and pencil or calculator, and several tools could be chosen. The same question was asked for a task they did not like. They were also presented with four new tasks that they had to read and think about, but were not requested to solve. For those questions they were asked what tool could be used, and what tool they would prefer to use and why. In the context of a short survey we can not expect extensive writing, so answers were in most cases short. But some students wrote more, and judged by the teachers in a project meeting, the students gave fairly good answers and they were quite pleased with the amount of writing. There were also several open fields for writing comments.

I looked at the students' choice of tools and the reasons they gave. Many students gave acceptable or good comments. Not all students wrote their comments, the frequency varied from 88 on tasks they did not like to 139 of the 163 students in the project.

About 18% of the students that answered the questions why they chose the specific tools gave good reasons with indication of features in the tools and some very relevant where they also indicated solution methods. Varying on different questions, from 46 % to 60% of the answers given was acceptable and coherent with the tools chosen, but did not give much information. They were for example: this is most appropriate for the task, this is straightforward, it is easy. Other answers were I like this tool or neutral answers, not related to the tools.

The students comments to why they liked a specific task revealed that 39 students answered it was because it was challenging, not too easy and not too much of the same again and again. Examples: This task was quite demanding, but we found ideas how to solve it. It was sometimes a little difficult but very fun when we managed! (Butterpack task) It was fun, but it could have been more difficult (Planning for a sale in a kiosk) But some also liked easy tasks.

SUMMARY AND CONCLUSION

Observations in the classes confirm that the use of computers with the tools in an open setting had a mediating role for students' construction of knowledge. The use of an open learning environment and possibilities for students to make their choices seemed to stimulate this. The students discussed what they see on the screen and asked questions to their peers. Their understanding developed as they worked, and observation of some cases revealed that what they learned on one task, they recognized they could use on a similar task later.

An aim in the project was to develop the students' competence to make reasonable choices of tools. From the analysis of answers to the questionnaire I found that about 18 % gave good reasons for their choices with reference to features of the tools. For about 46 – 60 % of the students' answers was less informative, telling they used the tool because it was "the best choice for this task", "it is easy with a spreadsheet", and similar. For some answers, perhaps a geometrical figure or a table of numbers triggered the answer. These answers can be judged as superficial but the choice of a relevant tool itself has to be counted, and in this setting we can only expect short answers. In most cases their choices are relevant and give a good starting point for the task. I think the depth of choices will increase with more knowledge of the tools.

Another question is whether the students can solve the task with the tools they chose, e.g., if they can make a spreadsheet model when they have chosen this, and tell it is the best choice for this. In order to answer this in full, more analysis of observations is needed. But from the example of Tim and Hans given in this paper, and other cases in the data, indicate that the answers given are closely related to the experiences students had in their work on similar tools and are reasonable.

What level of competence with the tools is necessary to make appropriate choices of tools for a given task? The observations indicate that students can make reflections and choices even if they have limited facility with the tool. Important competence to develop for this is the understanding of how mathematical connections can be represented with the tool. Judging what tools to choose requires, planning without solving, thinking on a meta level, and reorganising the approach to tasks (Dörfler, 1993; Goos et al., 2003). I think this question needs further research.

From answers to what hey liked, many students answered they like challenges and many answers indicate they liked tasks that they master, either challenging or easy.

I think implications from this is to give student challenges and variation in tasks, further development of an open learning environment with computer tools as mediators, and options for students to co-operate, discuss and set their own tasks. Further research is needed to gain deeper insight into the connection between students' knowledge of the ICT tools and their competence to choose tools.

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INTERACTION OF MODALITIES IN CABRI: A CASE STUDY

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In this paper we consider an experiment in which 15 years old students explore a phenomenon of covariance, by using Cabri for drawing geometric figures, measuring, and sketching graphs. In this way they collect different types of information. A first research question is how students deal with them in situations of exploration, in particular, which role they ascribe to the numerical data. Another research question is how students approach formal aspects. The activity on which the experiment is based is a telling example of how Cabri enlarges the scope of the exploration, in a way that could not be possible in paper & pencil environment.

INTRODUCTION AND BACKGROUND

It is widely recognized that the manipulation of figures in Cabri enhances the ability of producing conjectures, see (Arzarello, 2002, De Villiers, 1997, Mariotti, 2001). But this is not all: Cabri gives access to measurement of lengths, areas etc., provides numerical data which evolve in a dynamic way, according to the changing of the corresponding geometric figure, and sketches analytical representations (diagrams) of the geometric situation. These aspects enlarge the scope of action of Cabri, since they give students the chance of studying a mathematical phenomenon from different points of view and enhance the production of conjectures in different domains. The other side of the coin is that Bruner's three modes of representation (1966) - enactive, iconic, symbolic - are available *at the same time*. Thus the students need to manage different types of information and to transfer the information from one mode to another. This situation may also imply to switch back and forth from an operational to a structural conception, see (Sfard, 1991). The following research questions arise:

- In which way students relate geometric figures to Cartesian diagrams, and to measures?
- Which is the role ascribed to numerical data, e.g. students consider measures as an instrument for exploration of geometric situations?
- Are there moments in which the students feel the need of resorting to formulas and formalisation and, if any, which are the stimuli and the circumstances?

The mentioned studies will be the background of our attempt to answer our research questions. We will also use the representational worlds, which are related to Bruner's modes, but seem to better apply to mathematical learning, as discussed in (Tall, 2003): embodied (related to Bruner's enactive and iconic mode), symbolic-proceptual

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and formal-axiomatic (both related to Bruner's symbolic mode). Furthermore we will take into account that Tall et al. (2001) described the path from the environment to the axiomatic mathematics as happening through two ways of operating (perception and action), which converge to reflection, and afterwards to theory. Answering our research questions means reflecting on how this path may actually develop in Cabri environment.

THE SETTING OF OUR EXPERIMENT

The students involved in the experiment are 15 years old; they attend a secondary school with a strong scientific orientation. Thanks to the innovative spirit of their teacher, they are 'frequent users' of the mathematics laboratory. Here the mathematics laboratory has to be intended not simply as a physical place, but as a 'place for learning', in which mathematical activity is carried out by different modalities (computer and other artefacts, work in groups, communication). The aim is to give students the *chance* and the *time* to construct the meaning of mathematical objects evolving from the personal meaning to the institutional meaning. In contrast with traditional mathematics teaching style, which evokes 'Fast food for mind', 'Slow food for mind' could be the motto of such a laboratory.

The students involved in the experiment have a good experience in using Cabri, working in group and exploring open situations. When using Cabri, they are accustomed to use the geometric and the analytic representations. At the moment of our experiment, they already had a first approach to calculus; in particular, they were introduced to functions, namely they knew the representation of linear and quadratic functions. During the experiment the students worked in small groups (three students for each group), each group working at one computer. The teacher sometimes intervened in the groups' activities with prompts and suggestions.

We focus on the activity of a group of three students (Nicolas, Alessio, Andrea), who, according to the teacher, are "enough good to work on the problem, but not so brilliant to solve it in a couple of minutes". They were allowed to write notes, which were collected. They were video-taped during the activity. An external observer took notes and interacted with them.

Our analysis is based on the video, the protocols, the field notes taken by the observer.

The exercise

The following statement was given to the students:

> In a circle a chord AB is given. Take a point C on the circle. What may we say about the area of the triangle ABC? Is it possible to have information on how the area of the triangle varies?

The students were accustomed to these kind of exercises and knew that the expected answer to the first question is of the type "It varies", while the answer to the second is

the description of how it varies. The Cabri figure was already on the screen of their computer (see Fig.1).

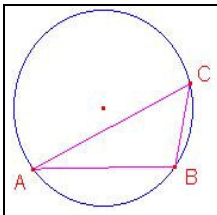

Fig.1. The figure on the screen

First of all, we observe that the exercise encompasses a situation of covariance: the area varies according to the position of the point *C* on the circle. Dragging with Cabri shows that the area varies according to the position of *C* (answer to the first question). The variation of the area may be studied with the help of a diagram done with Cabri. This implies the choice of variables. When working in Cabri it is natural to use measures (of segments, areas, angles, …): we will see that the students chose to study the relationship involving side *AC* (or *BC*) and the area, and the relationship between the height and the area. These choices encompass different kinds of dependence and, consequently, different diagrams (see below the discussion of the students' production): for example, the relation between the height and the area is functional; the relation between sides and area is not.

In the following, we report our microanalysis of the three students' behaviour when facing this problem. Their whole process of exploration is split into its main steps, to which we refer as 'episodes'. These episodes are presented together with our comments.

EPISODES FROM THE STUDENTS' EXPLORATION

First episode: getting on touch with the problem

The first approach to the problem is the usual one with Cabri, that is to manipulate the figure, first by dragging and after by animation. The students observe the changes in the figure without describing the phenomenon. They also measure the sides and the area, by using one of the functions of Cabri. Measuring is a necessary stage toward the analytical representation of the situation, which is implied by the second question in the statement. We point out that there is no reflection on these numerical data, the students are only collecting all the available data.

This is the phase in which the students "get in touch with the problem": they are reflecting, but not yet producing ideas. In this step the communication between mates is confined to the exchange of technical instructions ("move the mouse", "measure",…). Here Cabri seems to foster a permanence in the embodied mode of representation, see (Tall, 2003), with prevalence of the enactive mode.

Second episode: working on the analytical representation

Alessio proposes to produce an analytical representation. This is the crucial point, since this decision entails to choose two entities to represent on the axis. The first natural choice of variable is the area. The other variable may be the side *BC*, or the side *AC*, or the height of the triangle (which is not yet drawn on the figure and whose measure, consequently, is not on the screen). Andrea proposes to choose as variables

the area and the side *AC*. Alessio proposes to choose as variables the area and the sum of the two sides *AC* and *BC*. We guess that Alessio tries to take into account *as much data as possible*, namely *all* the things that vary in the figure: the area and the two sides (the third side is the fixed chord). This fact brings to the fore that for the moment the choice of variables comes from the figure, where the height is not present, and not from a more formal reflection on the area based on the formula (base*height)/2.

The group agrees to Andrea's proposal. At first, the students observe the point moving across the Cartesian plane. After, they produce the trace of the movement of the point and they start the actual reflection on the diagram (see fig.2). Andrea observes that "it rises until it reaches a maximum value". The students pay attention to particular points of the diagram (such as maximum and minimum) and link them to the geometrical configuration. We realise there are characters of the graph which are more evident and telling to the students, such as the existence of a maximum, respect to the trend of the diagram. This behaviour encompasses a 'causal' interpretation of the graph: when the point *C* is there, the point on the graph is…

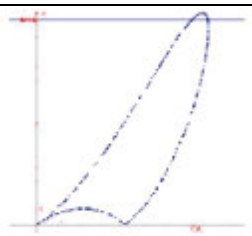

Fig.2: The side-area relation

We point out that the students' mode is still enactive, since they do not go beyond what the physical action of dragging shows.

The successive step is to reflect on the presence of two pieces of curve (see fig. 2). The students work back and forth from the diagram to the geometric figure and they understand there are two pieces of graph because of the movement of the point *C* along two arcs of the circle. The figure is helping to understand the graph, that is there is interplay between the two ways of representing the situation in Cabri.

We also observe that they did not previously try to guess the shape of the diagram, by means of the study of the geometric phenomenon. This is in line with their conception of the graph in Cabri as an instrument to better explore the situation, rather than the representation of the result of an exploration. The approach to the problem is operational.

Third episode: producing and testing a conjecture

Alessio proposes his conjecture about the maximum value of area: he thinks the maximum area is obtained when the point *C* is in the intersection of the circle and the perpendicular bisector of the chord *AB* (that is to say the triangle is isosceles). This conjecture comes from the perception of the figure supported by the check on numerical values. He constructs the perpendicular bisector and drags the point *C* to the intersection in order to test his conjecture. The transcript shows that the conjecture is tested also through numerical values.

Observer: Where did you see that the area is the maximum?

Alessio: We saw it both in the graph and in the numerical values.

We have an example of dragging test modality, see (Arzarello et al., 2002), performed also with the numbers. We note that, once the dragging confirms the conjecture, the students don't feel the need of a formal justification, that is to say they don't look for the reason why the maximum area is reached in that configuration. They act operationally.

Forth episode: searching for a justification

The teacher gives some prompts addressed to enlarge the scope of the students' work. He suggests to reflect on the numbers that are on the screen and to explain why the conjecture holds, rather than just testing it.

We report a short excerpt from the transcription of the video:

Andrea: The teacher asked why the triangle isosceles is the triangle of maximum area.

Alessio: I was thinking that… since the area of the triangle is b*h/2 [he writes down on the paper the formula], if the base is fixed, the area depends from the height. In this case, if we have the maximum height we have the maximum area. […] How can we say that it is the maximum height?

Nicolas: The triangle isosceles is the triangle of maximum area and the triangle of maximum height [among the triangles that may be inscribed in a circle].

We observe that the intervention of the teacher pushes the students to reflect on the reasons, which are behind their results. In this way they pass from the exploration on graph/figure to the reflection on the formula of area. This reflection brings to the fore a hidden element, which was not yet present on the figure: the height. As a consequence, Alessio proposes to do a height-area diagram. In this way they start a new exploration, with a new element that comes from theoretical reflection. We point out that this new element does not change their way of looking at the problem. This moment could have marked the passage from an embodied/operational way of looking at the problem to a symbolic/proceptual one, but this did not happen. They do not exploit the formula, but immediately they are *attracted* again to the exploration with Cabri. As we'll see in the fifth episode, the teacher intervenes with the aim of making the students to exploit the insight coming from the algebraic formula and to produce a more abstract thinking.

Fifth episode: Foreseeing the new diagram

The teacher encourages the group to describe the kind of diagram they expect to obtain.

Teacher: Before doing the graph, try to imagine how it can be.

Alessio: When the point *C* is next *A* or *B*, the height is shorter and then the area is smaller.

Teacher: OK, but do you remember the previous graph? The new graph will be the same?

The three students reflect on the figure and agree the diagram will be "the same as the previous one" (see Fig. 2). They are led to this conclusion by a rough reflection on the manipulation of the figure (compare Andrea's sentence): they do not see any difference between the phenomenon of the area increasing or decreasing according to the variation of the height, and the phenomenon previously observed (area increasing or decreasing according to the variation of one side). Thus, they conclude the new diagram will be the same. Here we have that the structural view of the diagram is very poor; there is not the concept of the kind of variation, but only the naive idea of growing, diminishing, maximum, minimum. The students are still acting in the embodied mode.

The formula of area, which was reminded before, does not provide further elements of reflection. The students don't reflect on the formula to anticipate the diagram. We may say in this case they lack of anticipatory thinking, but we are aware that the functions of Cabri, by making direct and immediate the result of transformations and decisions, may hinder the role of anticipatory thinking. What seems to us interesting is that, even when they are forced to anticipate by the teacher, they rely on figure (manipulation of the figure through Cabri) rather than reflecting on the formula.

Sixth episode: Introducing numbers

The teacher suggests to resort to numbers (measures) as a further information to study the nature of the graph. The aim of the teacher is to make the students aware of the difference between the two relationships. The key concept is the way of changing of numbers. But, as we'll see in the following transcript, this peculiarity is not evident to the students. Andrea's attention is caught mainly by the covariance.

Teacher: Let's look also at the numbers, how are they varying?

Andrea: The rate of change of the numbers … this means that if it [the height] increases or decreases more and more… here it [the area] increases more and more … and decreases more and more.

Seventh episode: the surprising diagram

At this point the students have at disposal four poles of interest, that is four representations of the problem: geometric (the dynamic figure), analytic (the diagram), numerical (the measures) and algebraic (the formula of area). Up to now they have worked mainly between the geometric and analytic poles. The lack of meaningful interaction of the four poles is proved by the fact they are very surprised when Cabri gives the actual diagram, see fig. 3. Andrea claims "It is a

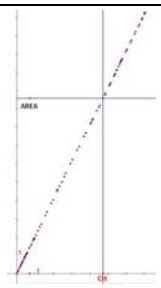

Fig. 3 The height-area relation

linear function", Alessio looks at his notes where he wrote the formula of area (see the forth episode), but nobody makes explicit that the linearity was already present in the formula.

Once more, we observe they don't rely on formula as a source of understanding.

Eighth episode: looking at numbers … again

At this point, the teacher intervenes in order to foster the interplay between modes of representation. The intervention of the teacher leads the students to explain the surprising diagram they obtained: the diagram is quite different from the previous one, even if the geometric situation seemed analogous (variation of the area according to the variation of the side/height). He reminds that the students had also reflected on measures to check this conjecture, see the sixth episode.

Teacher: What did the numbers suggest?

[…]

Teacher: There seems to be a contradiction between numerical and graphical aspects. From the numbers we expected a different thing [graph]…

Andrea: At first I looked at the height, now I'm trying to look at the values of the two sides, in order to see if…because I think we should see something. We saw that the variation of the height and the area gives a graph, the variation of the numbers [the two sides] should be different. It should be different because the graph is different. […] Here… the variation of the area follows the variation of the height. The area decreases and the height decreases, while for the sides… the area decreases and the side still increases. When the area varies, height and side vary in a different way. Area and height should vary in the same way: when the height increases the area increases, for the side it is not the same thing.

The numbers play a key role in changing the way of looking at the situation. The students grasp that in the covariance it is important not only what is varying, but also the way in which it varies. We see here the teacher's key role in scaffolding the students' understanding.

What seems to be still a source of problems is the 'formal' justification of the linear dependence. (of course, we do not focus on the lack of formal reflection on the non-linear situation of the relation side BC-area). The students have at disposal the formula and are aware (from the diagram) that it is a linear dependence. What they lack is a semantic control of the manipulation of the formula.

CONCLUSIONS

To answer our initial questions we summarise our comments to the episodes. The aim of the activity was to lead the students from the exploration of the problem to the construction of concepts (covariance, linear dependence). This path should have implied to pass from an operational view of the problem to a structural view based on the symbolic modes (numeric/algebraic and logic/verbal). Through Cabri four poles

of interest, that is four modes of representation, are available: geometric (the dynamic figure), analytic (the diagram), numerical (the measures) and algebraic (the formula of area). The students deal easily with geometric and analytic poles, but their mode of working is enactive/iconic so that we may say that their way of approaching the problem is operational. Numbers (here measures) are used spontaneously mainly to support dragging exploration, but the students need the help of the teacher to make numbers become source of insight for the problem.

Our experiment offers an instantiation of the way the path from the embodied stage to formal mathematics may happen in a rich dynamic geometric environment. The purpose of the exercise was not to reach the formal-axiomatic representational world; anyway, we note that even the symbolic-proceptual world is hindered to students: the dynamic figure and the diagram may act as a burden in the effort to go on.

From our experiment we may draw some didactical implications. Our exercise stresses all the potentialities of dynamic geometric environments. Without Cabri the students would have missed the exploration of the role of dependent and independent variables that emerged in our experiment and would have not had the deep insight into linear dependence; really through Cabri they had the chance to face all the complexity of the phenomenon. On the other hand, to deal with the richness of information turns out to be more complex and requires the flexibility for switching from one pole to another.

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THE DUALITY OF ZERO IN THE TRANSITION FROM ARITHMETIC TO ALGEBRA [1]

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This article shows that the recognition of the dualities in equality (operator–equivalent) of the minus sign (unary – binary) and the zero (nullity – totality) during the transitional process from arithmetic to algebra by 12 – 13 year-old students constitutes a possible way to achieve the extension of the natural number domain to the integers.

LITERATURE AND THEORETICAL FRAMEWORK

Since the nineteen seventies, researches such as those carried out by Freudenthal (1973), Glaeser (1981), Bell (1982), Janvier (1985), Fischbein (1987), Resnick (1989), Vergnaud (1989) among others, have shown that students presented extreme difficulties related to the conceptualization and operating with negative numbers in the pre-algebraic and algebraic scope. The study of this topic, baseline for the mathematical education, continues to be in effect to date.

The results of the first steps of a research project in process that intends to go deeper in the problematic with negative numbers, through the study of zero, are reported in this article. Questions as the following lead this project in the transition of arithmetic to algebra:

a) How does zero contribute to the extension of the numerical domain of natural numbers to integers?
b) Do students consider zero a number?
c) Are they aware of the dual nature of zero?
d) Do they understand the addition, subtraction, multiplication and division by zero?
e) Does a historical-epistemological analysis of zero as a number would contribute to the understanding of the conflicts presented by students nowadays?
f) Which cognitive hanges are provoked in students by the teaching of integers through technological environments?

The theoretical foundation of the initial stage of the project is based on the ruling ideas of the following authors: Piaget (1933) expressed that when showing the child

[1] Project sponsored by CONACYT. Research Project: 44632. "Abstraction Processes and Communication Patterns in Mathematics and Science Classrooms with Technological Learning Environments: Theoretical – Experimental Study with 10 – 16 year-old students.

representations of the world, a "system of tendencies" are discovered in him, and of which the child had not have conscience, therefore, he had not been able to express them explicitly. In this same direction, Filloy (1999) explained that there are tendencies due to the cognitive structures of the subject that appear in every individual development state that give preference to several mechanisms to proceed, several ways to code and decode mathematical messages. These "cognitive tendencies" can be observed both in the classroom and during clinical interviews. In what is reported in this article, two cognitive tendencies identified by Filloy with numbers 2 and 8 were clearly brought up and which are described as follows:

Conferring intermediate senses. This tendency appeared in the solution of additions and subtractions of integers when assigning multiple senses to negative numbers that were corresponded to with the acceptance levels reported by Gallardo (2002). She showed that it is in the transitional process from arithmetic to algebra that the analysis of a student's construction of negative numbers becomes meaningful. During this stage students face equations and problems having negative numbers as coefficients, constants or solutions. She found that five levels of acceptance of negative numbers were abstracted from an empirical study with 35 pupils of 12–13 years old. These levels are the following: Subtrahend where the notion of number is subordinated to magnitude (in $a–b$, a is always greater than b where a and b are natural numbers); Signed number where a plus or a minus sign is associated with the quantity and no additional meaning of the term is necessary; Relative number where the idea of opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry becomes evident in the continuous domain; Isolated number that of the result of an operation or as the solution to a problem or equation; Formal negative number a mathematical notion of negative number, within an enlarged concept of number embracing both positive and negative numbers (today's integers). This level is usually not reached by 12–13 years old students.

The presence of inhibitory mechanisms. The non-recognition of the subtraction of a greater number from a smaller number arose. This tendency of avoidance obstructed the concept of general number[2] in open sentences[3]. Likewise, the presence of negative solutions provoked the inhibition of syntactical rules which had already been mastered.

THE BEGINNING

The research began with a preliminary phase where The Algebraic Blocks Model [MB] was used, a teaching model for negative numbers, where zero expresses its dual feature, that is to say, *as a null element:* $a = a + 0 = a + 0 + 0 = \ldots$ and as *contained*

[2] General number is a symbol that represents an undetermined entity that can assume any value. Ursini & Trigueros, 2001.

[3] The open sentences belong to the type: $ax \pm bx \pm c =$, and the equations are in the form of: $ax \pm b = c$ and $ax \pm b = cx \pm d$, where a, b, c, d, are natural numbers.

by endless couples of opposites: a + (– a) = 0; where a is a natural number. The zero duality, *nullity – totality* contributed to the extension of the numerical domain in the tasks posed in this preliminary phase of the study.

The MB shows graphical representations with fixed arrangements of numbers, numerical operations and algebraic expressions. The black blocks represent positive terms, uncolored blocks represent negative terms, for example:

■ ■ ■ represents +3; ▯ ▯ ▯ ■ ■ represents (–3x) + (+2)

In this model, the count of positive numbers is extended to the negative numbers. Thus, the positive numbers lack their exceptional feature, that of being the only numbers with which you can count. There will be another type of numbers with which you can count with the peculiarity that when facing another positive with the same value, they void each other.

The adding action is represented by joining or linking the corresponding blocks, that is, colored and uncolored blocks. If in this link simultaneously appear blocks from both types, they are matched becoming void elements (main principle of the Model). As an example of the teaching carried out in the model, let us consider the addition 3x + (–2x) =.

These numbers are represented in a diagrammatic way: ▮▮▮ + ▯▯

They are added, that is, they are put together: ▮▮▮ ▯▯ In this case, zeros are formed. The result is a rectangle with color that represents *x.:* ▮

The action of subtracting means to take out the elements that constitute the subtrahend from the minuend, marking the eliminated blocks with a cross. The alternate representation of the number: $a = a + 0 = a + 0 + 0 = \ldots$ becomes relevant for the subtraction when the subtrahend is greater than the minuend. For example, 2 – 3 =, is chosen. The subtraction is represented as follows:

■ ■ – ■ ■ ■ It is observed that you can not take 3 from 2, then we continue to add one zero to number 2.

The alternative representation of the minuend 2 + 0: ■ ■ ■ □ is obtained. Now, the subtraction can be made by marking with a cross the squares that have been taken out: ⊠ ⊠ ⊠ □ An uncolored square is obtained, which represents –1.

In this preliminary phase, this MB was used as a resource which exhibited the different cognitive tendencies of the students when they face new mathematical concepts and operations (Hernández, 2004). Questionnaires and individual clinical

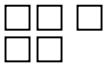
interviews were carried out to 16 students of 8th grade level who had already been taught with MB on the following topics:

1) To identify positive and negative numbers represented in MB.
2) To solve through arithmetical - algebraic language addition and subtraction operations represented in MB.
3) To solve additions and subtractions expressed in arithmetic – algebraic language.
4) To simplify open sentences.
5) To replace any numerical values in algebraic expressions.
6) To solve linear equations.

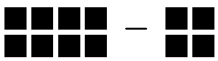
In this article, only the results on the performance of the best student Pamela (**P**), who was able to find a possible path towards the numerical extension during her performance in the interview, are reported. She had obtained the greater number of correct answers and been competent in the use of MB before her interview.

The most representative dialogues of the interview are shown as follows. What was expressed by **P** is written between quotation marks "… ". The interviewer (**E**) interpretations are written in brackets […]. The six topics listed above were approached.

In topic 1, **P** correctly represents algebraic expressions and numbers relating black color with positive numbers and no color with negative numbers. In the solution of operations represented in (Topic 2), the ambiguity between the negative number and the subtraction operation arose. Four cases are displayed as follows:

<u>1st Case</u>. Before addition:  add  , she writes: **–5 + 7=.**

Affirmation: *"It is as if I have seven minus five equals two"* [She reads the expression right to left and transforms the addition into one subtraction. Note that the equivalence: (–5) + (+7) = 7 – 5 is correct. However, the first member represents an addition of signed numbers and the second one a subtraction of natural numbers. In fact, **P** expresses the cognitive tendency that consists in giving the sense of subtrahend to the negative number -5].

<u>2nd Case</u>. Before subtraction:  . She writes: + 8 – + 4.

She observes the previous expression and affirms: *"It is the same as eight minus four, because this sign* [+8 – + 4] *is always minus, because minus by plus is minus"* [She considers the – + signs as a unique sign – , since she recurs to the rule of signs (–) (+) = – . This rule belonging to the multiplicative domain was learned without understanding it and she mechanically applies it in the additive mechanism. This fact leads her to consider a subtraction of natural numbers and not a subtraction extended to positive and negative numbers.)

<u>3rd Case</u>. Before representation: She writes:–8–(+7) = –1.

She observes the previous expression and says: *"It is as if I had minus 8 plus 7, equals minus 1"*. [**E** asks her: Why did not you make the subtraction?] She answered: *"I can not subtract because 7 is greater than negative 8. I made an addition"*. She writes: – 8 + 7 = –1. [**E** asks her again: Why is – 8 + 7 equals –1?] She responds: *"Because* –8 is equal to –1 *and –7, both forming the minus 8,* and she writes: – 8 + 7 = –1 –7 + 7 = –1. [In Fact, she learned through MB to disarrange numbers and to "make zeros". In this case, she transfers what is learned to the syntax and she forms zero: –7 + 7 = 0.]

The most outstanding issue on the 3rd case is that the written expression: –8 + 7; incorrectly considered by **P** equal to – 8 – (+7), has allowed us to discover that she accepts the addition of signed numbers in the case of: – *8* + 7 = –1. This acceptance was possible since she introduced the zero as the couple of opposites: – 7 + 7. The duality of zero has contributed to the extension of the addition beyond natural numbers. The above leads to the affirmation that she recognizes signed numbers, since she says: *"negative eight"*; relative numbers, since she writes: *"– 7 + 7"*, as well as the isolated number: "*–1"*, all of them intermediate senses of the negative and, consequently, manifestations of the cognitive tendency 2.

<u>4th Case</u>. Before representation:  She writes: 10– (– 4)=.

Explains: "*This sign* [10 – (– 4) =] *is from the operation and the other sign* [10 – (– 4) =] *belongs to the number"*. She adds: *"When there are two signs like these* [10 – (– 4)=], *it is as if we add"*. [It is observed that she distinguishes the sign of the number (unary) and the sing of the operation (binary), that is, she recognized signs but not the negative number – 4 per se. This is why, once she detaches from the model and is before the syntactical expression: 10 – (– 4) =, she applies the law of signs (–) (–) = +, disappearing the subtraction and, consequently, the possibility to extend the numerical domain of natural numbers to integers.]

Now we report the most relevant part of **P**'s performance when solving operations syntactically, that is, without the explicit presence of MB (topic 3).

Before the expression: (+ 8) – (+10) =. She writes: + 8 – 10 = 2. And expresses: *"No! It is minus two"* and corrects the result: +8 –10 = – 2. She explains: *"Because it is as if we had a subtraction, ten minus eight and the result is two, but since this number* [+8–10 = – 2] *is negative and greater than eight, the result is negative."* [Note that **P** considered a subtraction of natural numbers, orally expressed as: "*ten minus eight"* and, simultaneously, related the same minus sign to number ten in the written sentence: +8 – 10 = – 2.

> In fact, she read the same expression twice, one from right to left for the subtraction *"ten minus eight"* and another from left to right where she recovers number 10 as a negative number, that is, as –10.
>
> This *"incorrect duality of a unique minus sign"* can be interpreted as a progress, although incipient towards the extension of the numerical domain since she was able to subtract a greater number in absolute value from a minor number in absolute value through her "personal conception" of the minus sign duality.]

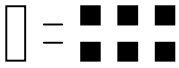
She interprets the unique minus sign [(+8) (+10) = considering they have a dual nature, that is to say, she warns that the sign is linked to the number and also considers it as the sign of the operation and believes that this fact can occur simultaneously]. Although incorrect, this double meaning related to one sign allows her to subtract a greater number from a minor number, situation she did not accept in the 3rd case. Regarding the simplification of algebraic expressions (topic 4), **P** does not express any difficulties. In the replacement of numerical values in open sentences (topic 5), the inhibiting tendency regarding the subtraction operation appears once again, as it can be observed in the following dialogue corresponding to item x – 8 =.

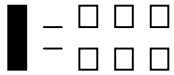
> ↓
>
> She writes: 3. And affirms: *"Here* [x – 8 =] *it would be three minus eight,…no!...it can not be done. I am going to choose another number since I can not subtract eight from three."* [Note that tendency 8 brought up in open sentences provokes an obstruction in the general number concept.]

It is important to point out that in the algebraic field, particularly before the expression x – 8 =, **P** presents an avoidance even with positive 3 since it is a number smaller than 8 that would allow the operation 3 – 8 = and she expresses: *"It can not be done."* During the replacement process, she does not extend the subtraction to the integers, despite she had done it in the numerical field in the case: 8 – 10 = –2 (topic 3). Therefore, when not accepting that *x* can take any values, **P** expresses that she does not recognize the variable as a general number in x – 8 = .

In the solution of equations (topic 6), **P** always uses the same procedure of adding or subtracting the additive inverse in both members of the equation. This has been the usual method she was taught to solve equations in MB and she transfers it to the algebraic language without difficulties. This fact shows that **P** recognizes the equality in its dual feature, that is, not only as an arithmetical equality that means carrying out one operation and obtaining a result, but also as an equivalent relationship between both members of the equality (Kieran, 1980). However, before the equations: 6 – x = 12, she is wrong as it can be verified as follows:

6 – x = 12 She adds number 6 to both members of the equation:

6 + 6 – x = 12 + 6. She affirms: "*I better solve with squares.*" [She uses MB and finally reaches the arrangement:  And explains: "*I am changing the color of the squares.*"

And shows the following arrangement:  **P** is quiet a few minutes… (question):"*Then x = –6?*" E: Why do you doubt?

She affirms: "*Because x is now positive, but six is still negative.*"

[**P** has difficulty to recognize a negative number as solution of the equation, when in some cases she had already recognized negative isolated numbers as results of operations (topic 2; 3rd case and topic 3), but not as a possible value for the unknown].

It is important to point out that the fact of considering the equality as one equivalence of expressions is what previously allowed **P** to solve equations which solutions were always positive. However, before an expression with a negative solution (x = – 6), P decodifies x as positive forgetting the equivalence of equality, fact recognized in the preceding tasks. The negative value unbalanced a knowledge that seemed to be consolidated in her.

FINAL DISCUSSION

From the analysis and interpretation of the interview, it can be concluded that during the transition of the arithmetic to algebra, the identification of the cognitive tendencies of **P** showed difficulties, but also possibilities of success leading to understanding the numerical extension. Therefore, if **P** only warns the subtrahend level and, besides, is not able to subtract a greater number from a minor number, or if she applies the laws of signs (–) (–) = +, (–) (+) = –, she will not be able to reach integers. But if **P** recognized the levels of the signed number, relative and isolated, as well as the extension of the subtraction operation (through the duality of the minus sign: unary – binary); besides, she can add signed numbers (through the duality of zero: nullity – totality) and knows a general method for the solution of equations (through the duality of equality: operator symbol – equivalence relation). **P** is before the possibility to accept other numbers different to the natural ones. We can affirm that these first results partially answer the three questions of the research: a), b) and c) posed in the introduction of this article, since zero contributed to the extension of the numerical domain through MB. **P** was able to carry out operations with zero, fact that proves that she considers it a number. Likewise, in her written productions she used, although without conscience of it, the three dualities above mentioned that she transferred from MB to the arithmetic – algebraic language. Now, we can neither affirm that these facts are completely stable in **P** nor that we know if it is about a transitory knowledge only linked to the teaching model she learned. She showed a good performance in "the elementary algebra of positive numbers". However, when

negatives arose, persistent obstacles were present (inhibition of the subtraction operation, obstruction of the general number, non-acceptance of the negative solution) that prove the need of carrying out the theoretical analysis at a deeper level, as the one posed in the outgoing project mentioned at the beginning of this article.

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CONFLICTS IN OFFSHORE LEARNING ENVIRONMENTS OF A UNIVERSITY PREPARATORY MATHEMATICS COURSE

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Many university programs are offered offshore to students whose first language and culture is not the same as the program's creator and where teachers' language and culture may not be the same as the students or the creators. This study investigated potential conflicts in a Hong Kong setting in a mathematics foundation classroom using an ethnographic approach and analysed the data using Valsiner's Zone theory. It is important to investigate the issues that arise from implementing such a program so that the best interests of both the student and the university are served while providing adequate support for the teachers.

INTRODUCTION

Australian universities are currently concerned with principles of provision regarding offshore partnerships so that 'universities should ensure that these services of the partner are of the same standard as offered by the university itself' (AVCC, 2002, p. 36). But what is same standard? We must not, as the UNESCO Assistant Director of Education warned (in Gribble & Ziguras, 2003, p. 212) 'dissociate education from the social, cultural and political origins of a country'. In Hong Kong universities, Pratt and colleagues found teaching traits deeply rooted in the cultural values and social norms where teachers guide students systematically, so 'the effective Chinese teachers are expected to adapt to their audience, guiding them step-by-step through content' (1999, p.253). This particular trait appears to be in conflict with expectations of western universities where lecturers and curricula aim to foster student independence.

But differences in cultural values extend beyond teaching, as Hofstede (1991) believes culture is a fundamental phenomenon that affects 'the way we live, are brought up, manage and are managed, and die; but also the theories we are able to develop to explain our own practices' (p. 170).

Many foundation or preparatory programs at university aim to address some of the differences especially in relation to students' knowledge and skills in particular subject areas (including mathematics), academic English, and reorientation to study at a western university. However to implement an offshore program successfully, staff also need to directly address cultural difference not only at the classroom level but also at government, management, curriculum, teacher and parent levels.

To investigate the issues and conflicts that arise in implementing an offshore program in detail, the author undertook an ethnographic study in Hong Kong in a four-course tertiary preparatory program from which themes or identifiable phenomena could

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emerge. These phenomena were then analysed using sociocultural theories. This paper will detail some of the results in terms of the mathematics course.

THEORETICAL FRAMEWORK

To investigate the multiple perspectives in such an international setting a suitable framework incorporating sociocultural practices was needed which encompassed many of the themes which emerged from the study. Zone Theory (Valsiner, 1997) was used as it accounted for the 'dynamic interactionism' in the classroom and could identify cultural conflicts between all stakeholders. It also lends itself well to qualitative research (Pressick-Kilborn & Walker, 1999) and has been used in mathematics education (Galbraith & Goos, 2004). Within the Zone framework, other theories were used to help explain particular phenomena. The theory of didactic contracts by Brousseau (1997) provided avenues to investigate further conflicts between the teacher and the student; work by Watkins and Biggs (1996) provided insights into cross-cultural perspectives on learner preferences; and Seah's (2002) work highlighted cross-cultural differences in values and beliefs in the classroom from a teachers perspective based in part on Hofstede's (1991) work on cultural dimensions.

Valsiner's theoretical framework includes three Zones: Zone of Free Movement (ZFM); Zone of Promoted Action (ZPA) and Zone of Proximal Development (ZPD). These zones constitute an interdependent system that can account for the complex relationships between the many constraints in any teaching and learning environment, actions specifically promoted, and the changing perceptions of the teacher.

Zone of Proximal Development is the difference between a learner's 'actual development level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers' (Vygotsky, 1978 in Valsiner, 1997, p. 152). In this present setting for example, if teachers or curricula promote mathematics tasks in English that are too far away from students' present capabilities, then these tasks are unlikely to be assimilated unless sufficient scaffolding is provided.

According to Valsiner (1997), the Zone of Free Movement represents environmental constraints that limit freedom of action and thought based on a person's relationship with the structure of a given environmental setting. This relationship is socially constructed by others (teachers, administrators, the curriculum writers) and their cultural meaning system. The ZFM provides the framework for their activities and emotions as these are controlled in 'culturally expected ways in different social settings' (p. 189). These areas can be 'on or off limits' and are time and context dependent. What is off limits in one environment may be within limits in another. In a school, while there are some explicit changes in zones from year to year and from class to class, some ongoing cultural norms are assumed and some constraints that have been internalised by the person as they develop may become part of the deep value system of the person, which may be resistant to change.

However if there is a change in any of the teacher/student/curriculum cultures then the ZFM may change dramatically, and there could be conflicting beliefs/values, class rules, different expectations etc. Thus the constraints on learning, both inside and outside the classroom, have to be identified and articulated. In this context both students and teachers ZFM are considered. For teachers/students in the current setting, elements of the ZFM include: their students/teachers, whose perceived abilities and behaviours may constrain teaching/students actions; their expertise in the language; past didactical contracts (Brousseau, 1997); curriculum and assessment requirements which set the choice of topics, teaching methods, and the time available to teach required content; outside constraints such as other jobs (both student and teacher), cultural environment, parents and government policies (e.g., bilingual education policy, visa requirements); and resources (e.g., teaching materials).

While the ZFM suggests which teaching, student or administration actions are possible, the Zone of Promoted Action (ZPA) symbolizes the efforts of a teacher, the curriculum or others, to promote particular actions. For example the curriculum may promote a particular mathematics course or teaching approach. However the ZPA is not binding, thus students may not want to learn any mathematics. It must also be in the ZPD, so having a low listening and speaking English score in a class which promotes an advanced communicative approach, may result in students' inability to participate or learn. It is also important that the ZPA be within the ZFM, and is also consistent with their ZPD; For example the actions promoted by the curriculum must be within the teacher's reach if a teacher is to embrace a different approach to teaching.

This particular ZPA/ZFM system working specifically in the classroom between the student and the teacher is reflected in the didactic contract concept (Brousseau, 1997). Agreements and expectations between the HK teacher and HK students about what is happening in the classroom may not be fully understood by curriculum designers. This contract reflects cultural assumptions about the work of teachers and students.

Education is generally seen as heavily influenced by culture (Eckermann, 1994) and hence values of that culture, but it also creates it own cultural practices so it is not surprising that when two educational systems meet, conflicts occur. Hofstede (1991) considers each culture as uniquely defined along five cultural dimensions, the first four of which emerged from western research and the fifth from Asian culture research. Table 1 compares Hofstede's dimensions in Hong Kong and Australia. While Hofstede's work is the benchmark for discussion on the implications of differences in national cultures, it does have its critics (e.g., McSweeney, 2003/4). The importance here is simply to highlight the large cultural distance Hofstede found in most dimensions particularly in individualism and long-term orientation. These dimensions permeate all aspects of society and at the heart of these are values. Values are 'broad tendencies to prefer certain states of affairs over others [and] many values

remain unconscious to those who hold them... they can only be inferred from the way people act under various circumstances' (Hofstede, 1991, p. 8).

Table 1: Hofstede's cultural dimension indices (1-100) for Hong Kong and Australia

Index	HK	Aus	Brief description
Power distance	68	36	The extent to which the less powerful expect and accept that power is distributed unequally
Individualism	25	90	Ties between individuals are loose and everyone is expected to look after oneself and family
Masculinity	57	61	Societies where social gender roles are distinct
Uncertainty avoidance	29	51	Extent to which members feel threatened by uncertain or unknown situations
Long term orientation	96	31	Includes persistence, ordering relationships by status & observing this order, thrift, sense of shame, (short term: personal steadiness, protection of face, respect for tradition & reciprocation of greetings favours)

These deep values are usually programmed in our mind by the age of 10, mainly by family and reinforced by teachers and classmates in early school life, and have a direct influence on learning behaviour. However, there are also manifestations of values in practice and many practices are learned through socialization in the various education systems and work. Surface differences in values can be highlighted by orientation programs, information booklets etc and often resolved through adaptation or compromise, but deep differences are more difficult to recognize and address.

To bring many of the value and cultural differences into sharper focus in HK, a range of data collection techniques were used. These provided information not only on teacher values and learner orientations and strategies, but a range of areas where conflicts could arise in this offshore setting. Valsiner, Brousseau, Hofstede's theories were then used to attempt to clarify and explain the conflicts.

Research design and method

Most of the data collection took place over a one-week period in Hong Kong in a preparatory program of 23 students and three teachers. James (not his real name) was the mathematics teacher and it is his case that will be highlighted here. He is a 34 year old from Hong Kong who had studied a Bachelor degree from a Canadian university. After teaching Microsoft courses part time, he obtained a teaching certificate. He is currently studying an MBA with an Australian university.

There were eight lesson observations (researcher observer) in all four courses (partially audio recorded); pre and post-lesson semi structured interviews of teachers; teacher values questionnaire (Seah, 2002); teacher marking of student written work;

student questionnaires (Biggs Student Process Questionnaire (SPQ, 1987) and background); SPQ questionnaire given to HK teachers and Australian moderators on their perceptions of students answers to the SPQ; other documents as necessary (Education Department approval forms; contracts etc); and two informal interviews with administrators of the program There was one focus group of students but will not be used because of ethical issues. Parent interviews were also sought but only one parent was informally interviewed. A follow up email questionnaire/interview to teachers was undertaken to validate teachers' comments and provide further information to saturate the categories (Creswell, 1998, p. 56).

After quantitative analysis of the SPQ data, the rest of the data were then analyzed following Creswell (1998) analysis and interpretation (pp. 191-195). Zone Theory assisted in the extraction and identification of the potential cultural conflicts from the phenomena and Hofstede was used to tentatively explain these conflicts.

RESULTS

Several broad categories emerged from the data analysis: those related to teaching, students, curriculum, outside issues and the use of English. An example of some of these is provided below:

Teaching issues (style): James has been teaching this course for a number of semesters and understands he is to teach English immersed in a mathematics problem-solving environment. From his teaching it is evident he has recognized differences between what students are used to and what will be expected. He writes: 'Students stick to their teaching culture of their home country. I have to pay a lot of effort to adjust their learning style before they get to university'

Student issues (learner orientation): Comparing James and the students' SPQ, there were a number of items with significant differences. For example for the item: 'I restrict my study to what is specifically set as I think it is unnecessary to do anything extra' students generally disagreed with the statement (2.39 out of 5); while James thought the students would strongly agree with the statement (5 out of 5)

Chinese/English issues: The language issues raised were quite prevalent in most of the data sets. Comments were made about students' level of expertise. James: 'English is an international language that the students must learn irrespective of culture…but their English standard may not be good compared to the rest of the world…' A follow-up questionnaire asked how much they and the students used English, he said the students used Cantonese frequently: 90%, with each other; 70% with the teacher; and 60% in front of the class. This was also noted in the classroom observations. The issue of English in mathematics was specifically mentioned by James: 'Students are handling maths calculations quite well but they are feeling uneasy to express maths in terms of English writing/speaking. They tend to avoid elaboration on the maths answer'. English also emerged as an issue with administration and parents as they imagined a more immersion approach to teaching.

Using Zone Theory

By moving all the issues from the categories into the Zones of Free Movement, Proximal Development and Promoted Action, possible conflicts may emerge when the issues are too far from the current zones of the stakeholders. The following exemplifies one set of issues.

When ZFM = ZPA, a strict set of rules is set up by the teacher. Valsiner used an example of rote learning by the child who was requested to repeat exactly the material the teacher provided and not allowed to do anything else as an illustration of this (1997:196). The following is a description of an observed part of a mathematics lesson. In the set curriculum, the students self-pace through a series of mathematics modules with quizzes at the end of each module. Students sit for a quiz when they feel they are ready (with some negotiation with the teacher). Hence in a typical class, students may be doing exercises, reading, helping or being helped or completing quizzes. In HK the format was rather different. The following is an excerpt from observation notes and transcripts. The mathematics in the two of the modules studied (4A and 4B) were on basic statistics (class from 9 – 11 am):

9.12 At the beginning of the lesson the teacher sets questions from the book for the students to do [so not self-paced].

James: 'Today we do practical exercises as normal and test in the second hour. Before we do 4B are there any questions in 4A?… good….I will go into a group and help you…so please go' (claps hands).

9.15 Students then work by themselves doing exercises, sometimes speaking to the person nearby. Teacher walks around room explaining where necessary. Some of the students have actually successfully completed these tasks in a previous semester but since the curriculum asks them to do these quizzes, the teacher has set these tasks for all the class. Some students appear to be ready to do the tests, so are doing little…..

10.20 James: 'Remove everything from the desk except the calculator, pencils and student id, your HK id is OK as well'.

The teacher then distributes several versions of the quizzes randomly.

James: 'Don't turn over the page until all students get a copy'…

The students then do the test together and finish altogether…some students have finished early, but have little choice but to sit and wait to the end of class.

11.05 Observer to James after class: 'If they finish early they look over the test. Why not get them to do the test when they are ready to leave when they think they are finished? They could even do the next test if they give you advanced notice so they have the option to finish the modules early' [as practised in Australia].

James: ‘Students prefer it this way. Parents prefer it this way. If they leave early parents will want to know why they are not in class. They are paying for this. Students prefer to do it all at the same time. What would they do? As it is not a university where would they go?…another class at 11.30’

Here the ZPA has been set up by the teacher from previous environments and are influenced by the perceived ZPA of the parents. Hence the ZFM of the students are narrowly defined by the ZPA. Students start and finish at the same time and have to do these tasks even if they know it already. To the curriculum designers, it appears the restriction of the ZFM impedes the aim of independent learning, but for the teacher this needs to be sacrificed for more important considerations, which may be part of more deeply held cultural values. In the questionnaire James did not think the outlooks he promoted or responses to cultural value conflicts that arise, are influenced by the society and culture he is teaching in, but rather, are guided by parents concerns and his personal values as he stated: ‘my teaching style is driven by my character’. He did, however, acknowledge that he had to adjust his ‘teaching style to include more group discussion (evident in class observations) and self-paced activities rather than purely lecture style in order to build up their self-learning ability’. The conflict may be due to the interaction of Hofstede’s power difference and uncertainty avoidance. While there may be a high power difference between the teachers and the HK administration, a low power difference is expected between the teachers and the university. Moreover the weaker uncertainty avoidance of the teachers in HK compared to the Australians may mean the HK teachers will change things and not inform the Australian teachers who want the communication with HK teachers. This resulting conflict needs to be resolved to maximize course success.

CONCLUSION

The aim of this paper was not to highlight all conflicts that exist (although the whole study does this) but to investigate the usefulness of the sociocultural framework to examine conflicts. The ethnographic approach to the study established the themes and categories of concern. Once established, Zone Theory helped to unravel each phenomenon with Hofstede’s work giving possible insights into reasons for conflict.

The emerging themes of teaching, student, curriculum issues, outside influences and use of English, would not be surprising to anyone working in the field. However the way these are investigated through Zone Theory and cultural dimensions may provide a framework to approach different cultural settings. In taking such an approach, I realize the generalizations that may occur, especially in Hofstede’s approach, and the basically etic approach to the research i.e. the theory and instruments used originate from one culture and used on another, may skew the data. It is hoped in the future a more collaborative approach can be taken. I also realise that some questionnaires may have been more appropriate than others. For example in the future a combination of the Biggs and other questionnaires may give more insight

into the students and teacher perceptions. A version of the Hofstede questionnaire may give more insight into the cultural values and perceptions of these values.

The value of the study is for the stakeholders themselves. It allowed the teachers in Australia and HK to become more aware of the similarities and differences; brought forward issues that may be addressed in future curriculum changes, contract documents and ongoing communication; and provide a framework for future studies.

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THE DIVERSE LEARNING NEEDS OF YOUNG CHILDREN WHO WERE SELECTED FOR AN INTERVENTION PROGRAM

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This paper examines the learning needs of 35 Grade 1 children and 60 Grade 2 children who were identified as vulnerable in their number learning on the basis of a clinical interview and reference to a set of research-based growth points. All children were selected for an intervention program that aimed to accelerate learning in four number domains. A key finding was that the children had diverse learning needs. Therefore it seems that intervention programs must not be formularised, but be flexible enough in structure and instructional design to cater for diversity.

INTRODUCTION

Improving the literacy and numeracy outcomes for Australian students is a key concern of the Australian Commonwealth Government, and for this purpose a National Literacy and Numeracy Plan has been developed (Department of Education Training and Youth Affairs, 2000). The National Plan recognises that "there is a need to provide effective assistance to students who need extra support, as part of ensuring that all students gain a level of numeracy essential for successful participation in schooling, in work, and in everyday life (p. 6)," and calls for the "provision of intervention for those students identified as being at risk of not making sufficient progress" (p. 7). Prevention and intervention in early childhood is viewed widely in the community as important for increasing the opportunities of children at risk of poor learning outcomes, and for ensuring the educational success and general wellbeing of young people (Doig, McCrae, & Rowe, 2003; McCain & Mustard, 1999). However, effective intervention requires specialist knowledge about the instructional needs of vulnerable students and how best to cater for these needs.

This paper explores the instructional needs of 35 Grade 1 children and 60 Grade 2 children who were selected for the *Extending Mathematician Understanding (EMU)* intervention program (Gervasoni, 2004). Of particular interest is whether these children form a group with similar instructional needs, and whether there are any patterns in the domains or combinations of domains for which children were vulnerable. The findings have potential for informing the structure and instructional design of mathematics intervention programs.

VULNERABILITY AND RESPONDING TO LEARNING NEEDS

The research reported in this paper was based on the assumption that it is important for school communities to identify children who, as emerging school mathematicians and after one year at school, have not thrived in the school environment, and to provide these children with the type of learning opportunities and experiences that will enable them to thrive and extend their mathematical understanding. Further, the

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perspective that underpinned this research was that those children who have not thrived, have not yet received the type of experiences and opportunities necessary for them to construct the mathematical understandings needed to successfully engage with the school mathematics curriculum, or to make sense of the standard mathematics curriculum. As a result, these children are vulnerable and possibly at risk of poor learning outcomes. The term *vulnerable* is widely used in population studies (e.g., Hart, Brinkman, & Blackmore, 2003), and refers to children whose environments include risk factors that may lead to poor developmental outcomes. The challenge remains for teachers and school communities to create learning environments and design mathematics instruction that enables vulnerable children's mathematics learning to flourish.

A common theme expressed by researchers in the field of mathematics learning difficulties is the need for instruction and mathematics learning experiences to closely match children's individual learning needs (e.g., Ginsburg, 1997; Wright, Martland, & Stafford, 2000). Rivera (1997) believes that instruction is a critical variable in effective programming for children with mathematics learning difficulties, and that instruction must be tailored to address individual needs, modified accordingly, and evaluated to ensure that learning is occurring.

Ginsburg (1997) articulated a process for responding to children's learning needs that used Vygotsky's zone of proximal development (Vygotsky, 1978). Ginsburg's process requires that the teacher first analyses children's current mathematical understandings and identifies their learning potential within the zone of proximal development. For this purpose, the notion of a framework of growth points or stages of development is important for helping teachers to identify children's zones of proximal development in mathematics, and thus identify or create appropriate learning opportunities. This approach is aligned also with the instructional principles advocated by Wright et al. (2000) for the *Mathematics Recovery* program. They believe that instruction for low-achieving children should be closely aligned to children's initial and ongoing assessment, and should be at the 'cutting edge' of each child's knowledge (Wright et al., 2000). The *Early Numeracy Research Project* (ENRP) research team also advocated this approach (Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002). Indeed, a feature of the ENRP was the use of a mathematics assessment interview and associated framework of growth points that enabled teachers to identify children's current mathematical knowledge, and locate children's zones of proximal development. Importantly, the assessment process also enabled those children who were vulnerable in aspects of learning mathematics to be identified.

IDENTIFYING CHILDREN WHO ARE VULNERABLE

In order to identify children who are vulnerable, often a line is drawn across a distribution of test scores, and children 'below' the line are deemed at risk and are recommended for a specialised program (Ginsburg, 1997; Woodward & Baxter,

1997). The decision about where to 'draw the line' is arbitrary, but in this research was made on the basis of *on the way* growth points (Gervasoni, 2004). The *on the way* growth points relate to a framework of growth points in nine domains that were research based and drew on what was known about the course of children's mathematical knowledge formation (Clarke, McDonough, & Sullivan, 2002; Gervasoni, 2003). The *on the way* growth points indicate children who have constructed the mathematical knowledge that underpins the initial mathematics curriculum in a particular domain and grade level, and who are likely to continue to learn successfully. Not yet reaching the *on the way* growth point in a particular domain is an indicator that children may be vulnerable in that domain and may benefit from opportunities to help them reach the *on the way* growth point as quickly as possible. Otherwise, they may not benefit from all classroom mathematics learning experiences because they do not have the conceptual knowledge that underpins these experiences.

The development of appropriate *on the way* growth points for Grade 1 and 2 children was guided by three data sources: (1) the ENRP growth point distributions for 1497 Grade 1 and 1538 Grade 2 children from 34 trial schools in March 2000. These schools included Government, Catholic and Independent schools from across Victoria that were widely representative of the Victorian population; (2) the Victorian Curriculum and Standards Framework II (Board of Studies, 2000) for Grade 1 and Grade 2; and (3) the opinions of ENRP Grade 1 and Grade 2 classroom teachers (Gervasoni, 2004). The analyses and synthesis of these data resulted in the following *on the way growth* points being established for Grade 1 children in Counting, Place Value, Addition and Subtraction, and Multiplication and Division respectively:

- counting collections of at least 20 items (Growth Point 2);
- reading, writing, ordering and interpreting one digit numbers (Growth Point 1);
- counting-all in addition and subtraction situations (Growth Point 1); and
- counting group items as ones in multiplication and division tasks (Growth Point 1).

The *on the way* growth points established for Grade 2 children were:

- counting forwards and backwards beyond 109 from any number (Growth Point 3);
- reading, writing, ordering and interpreting two digit numbers (Growth Point 2);
- counting-on in addition and subtraction situations (Growth Point 2); and
- using group structures to solve multiplication and division tasks (Growth Point 2).

In order to determine the growth points reached by each child, their mathematics knowledge was assessed and analysed at the beginning and end of the year (March and November) using the ENRP assessment interview (Clarke, McDonough, & Sullivan, 2002), and their growth points were entered into an SPSS database (SPSS Inc., 2002). The interviews used a clinical interview approach (Ginsburg, 1997), took between 30-40 minutes for each child and were conducted by the classroom teacher. Procedures were developed and implemented to maximise consistency in interview administration across all schools (Rowley et al., 2001) in order to enhance the

validity and reliability of the assessment data collected, and to ensure that the data were as consistent and as free from bias as possible. Procedures were also implemented to enhance the consistency with which the growth points were assigned (Rowley et al., 2001). This meant that a large database of reliable and valid growth point data (Rowley & Horne, 2000) that was representative of the Victorian population was available for analysis.

For the purposes of this research, any children who were vulnerable in each domain were identified using a process developed by Gervasoni (2004). This involved examining children's growth point profiles to identify those who had not reached the *on the way* growth points in each domain. ENRP school communities were then invited to select children for participation in an intervention program, and to train teachers to implement this program in their school.

The question of interest for this paper is whether or not the children who were selected for an intervention program formed a group with similar instructional needs. Such information would be useful for preparing suitable experiences and instruction for intervention programs, and associated professional learning programs for teachers. The profiles of participating children were therefore analysed in order to identify whether there were any patterns in the domains or combinations of domains for which participating children were vulnerable.

IDENTIFYING THE INSTRUCTIONAL NEEDS OF CHILDREN

In 2000, 22 schools participating in the ENRP chose to implement mathematics intervention programs for their Grade 1 and Grade 2 children who were identified as vulnerable in number learning. Overall, 576 of the 1497 ENRP Grade 1 children (38%), and 659 of the 1538 ENRP Grade 2 children (43%) were identified as vulnerable in at least one number domain (Gervasoni, 2004). Of these, 35 Grade 1 children and 60 Grade 2 children from the 22 schools were selected by their communities for participation in the *Extending Mathematical Understanding* (EMU) intervention program. The number of children who were vulnerable in each domain is shown in Table 1.

Table 1: Number of Grade 1 and Grade 2 Children Vulnerable in Each Domain.

Domain	Grade 1 (n=35)	Grade 2 (n=60)
Counting	20 (57%)	48 (80%)
Place Value	14 (40%)	55 (92%)
Addition and Subtraction	15 (43%)	44 (73%)
Multiplication and Division	23 (66%)	32 (53%)

These data highlight some important issues. First, it is clear that the participating children were not vulnerable in all domains. This may surprise some teachers. Second, a much higher percentage of Grade 2s than Grade 1s were vulnerable in

Counting, Place Value and Addition and Subtraction. This suggests that Grade 2 children were more likely than Grade 1s to be vulnerable in multiple domains. To investigate this issue, the number of domains in which children were vulnerable was calculated (see Table 2).

Table 2: Number of Domains for Which Children Were Vulnerable.

Domain	Grade 1 (n=35)	Grade 2 (n=60)
Vulnerable in 1 domain	14 (40%)	4 (7%)
Vulnerable in 2 domains	11 (31%)	16 (27%)
Vulnerable in 3 domains	5 (14%)	17 (28%)
Vulnerable in 4 domains	5 (14%)	23 (38%)

The findings show that Grade 2 children were more likely to be vulnerable in 3 or 4 domains than were the Grade 1s. This comparison highlights the possibility that as children who are vulnerable in learning mathematics progress through the school, the difficulties they experience become more complex. Thus, it seems important to identify and intervene with vulnerable children as early as possible in their schooling.

Figure 1 provides a diagrammatic representation of the intersecting domains for which Grade 1 children were vulnerable, and the number of children who were vulnerable in each intersecting domain. The intersections between Counting and Addition and Subtraction and Place Value and Multiplication and Division are shown with additional circles.

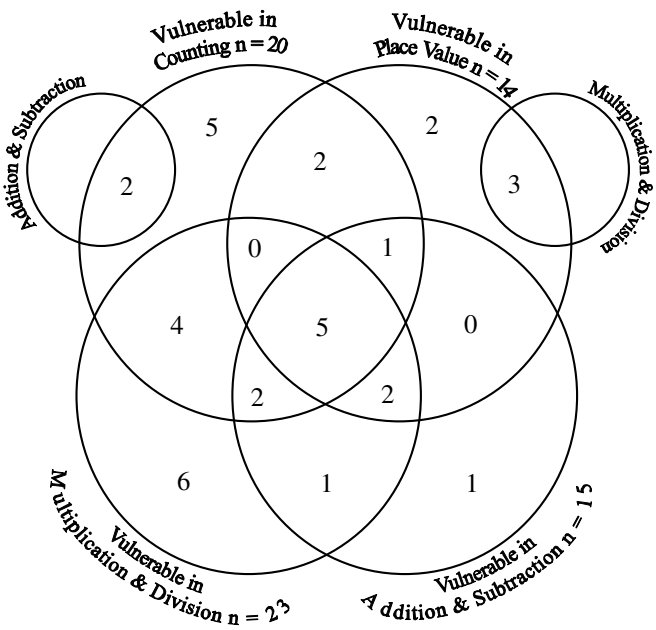

Figure 1: Diagrammatic representation of the intersecting domains for which Grade 1 children were vulnerable (*n*=35).

This diagram highlights two important issues. First, the diversity of domains and combinations of domains in which children were vulnerable is striking. There was a spread of vulnerability across all domains, and there were not any combinations of domains that were common for the children who were vulnerable. Thus, there was no single 'formula' for describing these Grade 1 children who were vulnerable in learning school mathematics, or for describing the broad instructional needs of this diverse group of students. Second, most Grade 1s were vulnerable in only one or two domains, but these domains varied.

Figure 2 shows the intersecting domains for which Grade 2 children were vulnerable, and the numbers of vulnerable children. The findings suggest that the learning needs of the Grade 2 group were also quite diverse. Most striking is that almost all children were vulnerable in Place Value, and also the large number of students who were vulnerable in 3 or 4 domains.

In summary, these data suggest that children who are vulnerable in aspects of number learning have diverse learning needs. It is also clear that Grade 2 children were more likely to be vulnerable in multiple domains than were Grade 1 children.

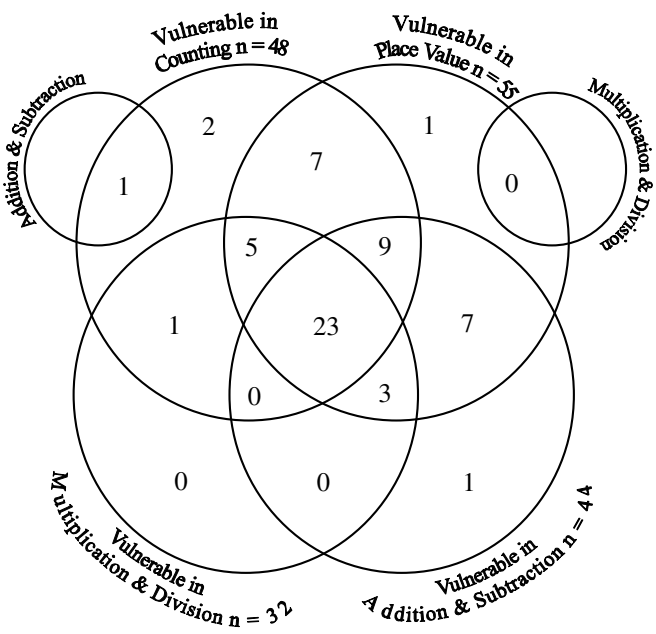

Figure 2: Diagrammatic representation of the intersecting domains for which Grade 2 children were vulnerable. (n=60).

DISCUSSION AND IMPLICATIONS

The findings presented in the previous section indicate that children who participated in an intervention program have diverse learning needs, and are vulnerable in a range and combination of domains. Indeed, there was no single formula that described the instructional needs of Grades 1 and 2 children who were selected for an intervention program, and there were no patterns in the domains or in any combinations of domains for which children were vulnerable. Vulnerability was widely distributed across all four domains and combinations of domains in both grade levels. However, it was found that most, but not all, Grade 2s were vulnerable in Place Value.

These findings have several implications for the structure and design of intervention programs. First, the diverse learning needs of children call for customised instructional responses from teachers. This supports the approach advocated by other researchers in the field of mathematics learning difficulties (e.g., Ginsburg, 1997; Rivera, 1997; Wright et al., 2000). It is likely that teachers will need to make individual decisions about the instructional approach for each child because there is no ‘formula’ that will meet all children’s instructional needs. This does not mean that separate intervention programs are needed for individual children, but rather that teachers need to know how to customise activities and instruction so that they may focus each child’s attention on salient features of their experiences so that they notice the aspects that lead to the construction of mathematical knowledge. For this to happen, it is optimal for group sizes to be three or less, and for teachers to be aware of each child’s current mathematical knowledge and associated zones of proximal development. This requires frequent assessment and knowledge of the ‘pathways’ of children’s learning.

The diversity of children's mathematical knowledge across the four domains also suggests that knowledge in any one domain is not necessarily prerequisite for knowledge construction in another domain. For example, some teachers may assume that children need to be *on the way* in Counting before they are ready for learning opportunities in Addition and Subtraction. On the contrary, the findings presented in Figures 1 and 2 indicate that some children who are not *on the way* in Counting are already *on the way* in Addition and Subtraction, and this pattern is maintained for the other domains also. This finding has implications for the way in which the mathematics curriculum is introduced to children. It seems likely that children will benefit from learning opportunities in all four number domains, provided in tandem with one another, and that learning opportunities in one domain should not be delayed until a level of mathematical knowledge is constructed in another domain.

In summary, the implications of these findings for intervention programs are:

- Intervention programs need to be flexible in structure in order to meet the diverse learning needs of each participating child;
- Intervention teachers need to provide instruction and feedback that is customised for the particular learning needs of each child, and based on knowledge of children's current mathematical knowledge. Further, teachers need to draw children's attention to the salient features of activities and learning experiences to facilitate the construction of knowledge and understanding;
- Intervention programs need to focus on all number domains in tandem. It is not appropriate to wait until children reach a certain level of knowledge in one domain before another domain is introduced; and
- Teachers need opportunities to gain professional knowledge about how to effectively customise learning experiences for children.

CONCLUSION

The findings presented in this paper highlight the diversity of domains and combinations of domains in which Grade 1 and Grade 2 children who participated in a mathematics intervention program were vulnerable, and the associated challenge teachers face to tailor instruction to meet each child's learning needs. Clearly, intervention programs need to be flexible enough to cater for diversity. This research also found that Grade 2 children were more likely than Grade 1s to be vulnerable in 3 or 4 number domains. An implication of this finding is that it is important to provide intervention programs for children who are vulnerable as early in their schooling as possible, before their difficulties increase in complexity. Meeting the diverse learning needs of children is a challenge, and requires teachers who are knowledgeable about how to effectively identify each child's learning needs and customise instruction accordingly. Assisting teachers to gain this knowledge is an important pursuit for school systems wanting to improve numeracy outcomes for students.

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USING CONTRADICTIONS IN A TEACHING AND LEARNING DEVELOPMENT PROJECT

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Yrjö Engeström (2001) asserts that contradictions in an activity system create potential for change and development. In this report a large research project aimed at developing mathematics teaching and learning is described and it is shown how Activity Theory can be applied to support understanding and progress. The project is producing evidence of contradictions that it is hoped will have a role in achieving its goals. We use accounts from the early stages of field work to outline some of the contradictions that are being encountered.

INTRODUCTION

Our[1] purpose here is to report on the early phase of a long term research and development project that aims at exploring the development of communities of inquiry comprising university didacticians[2] and school teachers. The developmental purpose of the project is for teachers and didacticians to work alongside each other in developing the quality of students' experience and learning of mathematics. In the first part of the paper we offer a brief description of the project using the framework proposed in Jaworski (2003) and then offer an analysis of the project in terms of Activity Theory (AT). In particular we will draw attention to the elements of artifacts, rules, community and division of labour that are identified within AT as mediating tools and context for the development of knowledge within an activity system (e.g., Engeström, 1999). In the second part of the paper we discuss the existence of contradictions or tensions (Engeström, 2001) within an activity system and their role in the development process. We then offer some evidence of contradictions and tensions within our own project.

THE LCM PROJECT

The project, Learning Communities in Mathematics (LCM)[3] was conceived by the Mathematics Education Research Group at Agder University College (hereafter referred to as the 'College' and is planned to run, initially, for four years. It began in January 2004, although preparation extended for fourteen months prior to this. The essence of the project is to develop communities of teachers and didacticians, in which we learn together through inquiry. 'The communities both support the inquiry and grow through the inquiry' (Jaworski, 2004a, p. 26). The concept of *community of inquiry* has been used and explained by Gordon Wells (1999), Ed Elbers and Leen Streefland (2000) and Elbers (2003); who, in turn have related it to other eminent scholars. We believe that we are giving the concept a particular interpretation: for example in the positioning of didacticians and teachers as 'co-learners' and in the use

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 41-48. Melbourne: PME.

of inquiry as a tool to achieve 'inquiry as a way of being' (Jaworski, 2004a), we hope we will bring new insights and understanding to teaching and learning development.

Groups of teachers from eight schools (at least three teachers from each school) have agreed to join the project; a necessary condition is that the school principal supports the activities and goals of the project within the school. The schools include elementary, middle and upper secondary schools, thus involving students from 6 to 19 years of age. Initially the project entails regular workshops held at the College, meetings between teachers (occasionally including didacticians) within schools, and school visits by didacticians. A principle aim is to design and study classroom activity that is inquiry-based. Inquiry is seen as a design, implementation and reflection process in which teachers should be central (Jaworski, 2004b). As the project develops it is intended to produce video recordings of classroom activity to study the outcomes of design, and as a developmental tool. The data collected will also enable the exploration of the process of developing 'communities of inquiry' (e.g. Wells, 1999), within and between schools and the College, and provide evidence of changing classroom practices. The project was first introduced to PME in Jaworski (2004a); here we provide a very brief analysis of the project using the framework proposed in Jaworski (2003).

Knowledge and learning

The project is concerned with teachers' and didacticians' learning more about processes of teaching and learning, about teacher and teaching development and the development of communities of inquiry. It is hoped that the knowledge generated will be of value to the international community concerned with mathematics teaching development. LCM also aims to provide a supportive context within which teachers can reflect upon and develop their own knowledge of teaching and learning and classroom practices, and improve their mathematical learning environment for pupils.

Inquiry and reflection

It is intended that all participants within LCM will engage reflectively in inquiry into their own practices. All participants within the project are researchers, inquirers and generators of new knowledge within the context of their own practices and activity.

Insider and outsider

Although LCM was conceived by researchers within the College it is a fundamental aim of the project to engage teachers as co-learners. The didacticians are simultaneously *outsider* researchers (Carr & Kemmis, 1986), as we seek to explore developments in school, and *insider* researchers as we monitor, explore and evaluate our own activities and progress in developing inquiry communities. It is intended that teachers will also become researchers/inquirers into their own practice, and thus they too will become insider researchers; also that teachers and didacticians will work closely together as co-learners in their respective practices. The development of

teachers as researchers/inquirers is a principle aim of the project (Jaworski, 2004a, 2004b).

Individual and community

As the title of the project suggests, at its heart lies the notion of 'community' where all participants can share individual experience and knowledge resources. It is intended that the diversity of knowledge (Wenger, 1998) available will be recognised and valued by all members of the community and that community knowledge will develop through joint activity. As individuals, teachers and didacticians each contribute to the knowledge shared by the community and grow within the community. Each can or will draw upon community knowledge as they operate individually within their own classrooms and spheres of activity.

LEARNING COMMUNITIES IN MATHEMATICS - AN ACTIVITY THEORY MODEL

Activity Theory is widely used as both principle of explanation and object of study (Engeström, 1999) in the context of teaching and learning development projects (e.g. Engeström, Y. Engeström R., & Suntio, 2002, Karaağaç & Threlfall, 2004). Engeström (1999) develops Vygotsky's picture of a 'complex mediated acted act' to present a 'structure of an emerging activity system' that 'explicate[s] the societal and collaborative nature' of the activity (p. 30). It is this model (fig. 1.) of an activity system that we use here to analyse the LCM project.

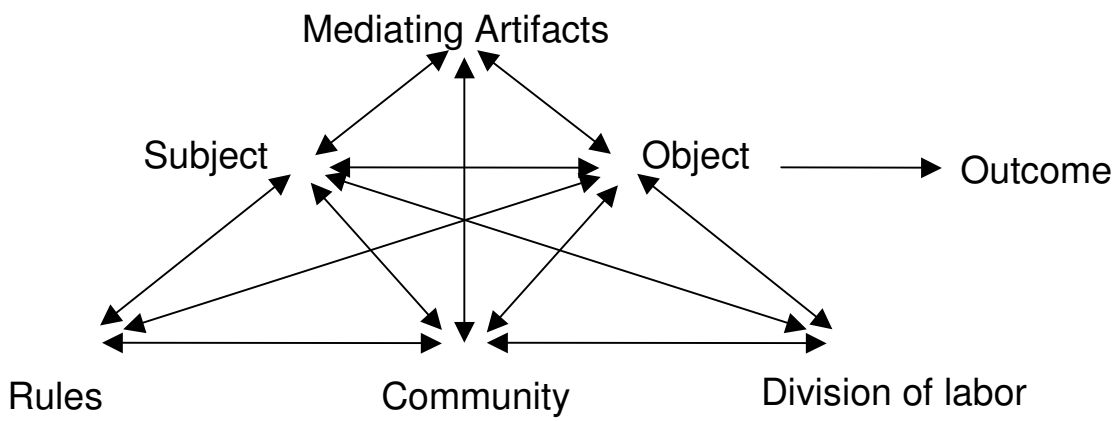

Fig. 1. The structure of a human activity system (based on Engeström, 1999, p. 31)

The LCM project exists at a number of levels at which the above model could be used as an analytic tool. It could be applied at the level of the individual participant (teacher or didactician) within the project, or at classroom and school levels, or applied to the project as a whole. At each level the 'subject', that is the 'acting person or persons' would be defined differently as would the object, or goals, of their activity. At the present phase of the project it is possible to identify communities at an institutional level, which is the focus in this report. We can consider the subject as the individual teacher working on, as object, her/his own classroom practices to

stimulate inquiry as a means of teaching and learning (the outcome), or it may be a group of teachers within a school working collaboratively on producing new teaching materials to use with their classes, or it could be the team of didacticians planning the agenda for a workshop. The range of possibilities that can be analysed with the activity theory model is, indeed, very broad. What should be noted is that in the model 'subject' does not mean 'research subject', in LCM we are co-learners and in this respect we are each of us participants of joint inquiry.

It is possible to identify the community as comprising the group of collaborating individuals, didacticians or teachers within each school, or the college, or indeed the wider professional or 'interested' communities (parents, employers, curriculum authorities, etc.) within which the teachers or didacticians are placed. Each community will have different characteristics and identity which will be, in part, evident in the rules (leadership, accountability, conventions for decision making, behaviour and speaking, etc.) and the division of labour (planning, preparation, reviewing events, etc.) but also in its values, goals and shared history. Community, rules and division of labour provide the socio-historical context for the activity.

'Mediating Artifacts' elsewhere referred to as 'Tools and Signs' (Engeström, 2001), are those objects and events that are used by the acting subject to achieve the desired goal. Thus, within the project community as a whole we are using workshops, school meetings, video recordings and relevant literature to facilitate the communication of ideas and support the development of approaches to teaching and learning. At an individual level, teachers may take ideas introduced in the workshops and produce materials suitable for use with their classes, these ideas and materials would be part of the collection of mediating artifacts that teachers use in their work, and tools that will enable them to review and revise their approach to their work. Language and discourse also play a crucial role in mediating activity and these also provide an important means of gaining insight into the nature of activity in individual classrooms, school meetings and meetings of didacticians.

CONTRADICTIONS AS SOURCES OF CHANGE

Engeström writes of 'the central role of contradictions within an activity system as sources of change and development' (2001, p. 137). He is careful to note that these contradictions (and/or tensions) within the activity system are not the same as problems or conflicts. The contradictions may be perceived by the acting 'subject' (as in the case reported by Karaağaç and Threlfall, 2004) or 'latent', that is evident to the outsider but not to the acting 'subject' as in Engeström et al. (2002). Given the potential of contradictions and tensions as agents of change, one focus of our research in this phase of the project has been the identification of contradictions within participants' activity in the LCM project.

For example, here, briefly, is a crucial tension that we experience in our own practice. At this stage of the project our evidence comes mainly from accounts recorded by didacticians, and as we record observations of events in school inevitably these

emerge as *outsider* accounts. This is in stark contradiction of the project goal, which is to position didacticians and teachers as co-learners. Here the tension lies within the 'division of labor' where we experience a demarcation between insider and outsider yet seek to establish co-learning partnership. We return to this point later. The concern of didacticians to engage teachers as co-learners and the intent of didacticians as insider researchers within their own practice are both described in detail in Cestari, Daland, Eriksen and Jaworski (2005).

Our discussion here focuses on our interpretation of observations made in meetings and classroom visits and subsequent discussions. An issue that is of concern to us, as to all researchers, is to ensure that our evidence base is the outcome of *systematic* inquiry. Our data is accumulating through a combination of structured, organised research events, such as the completion of questionnaires and tests, and unstructured but, none the less systematically documented meetings at the College and schools, and classroom visits that have a variety of purposes, such as discussing expectations and purposes, familiarisation of the context or simply courtesy calls. The accumulating collection of data is already large and varied. We make claims for our inquiry being 'systematic' through the approach taken in the examination and analysis of data in the light of current theory and the thorough testing of ideas against the rich variety of data available. At this early stage, some of our evidence has the *appearance* of being anecdotal – taking the form of stories. Such story-accounts are rooted in our field notes and in audio and video recordings of meetings and workshops; and as such they transcend the substance of anecdotes. They are recorded systematically, and the notes and interpretations produced are tested by other members of the team who have participated in the events recorded. The episodes that we record illuminate theory and can be used to guide the development of our research activity. We offer one such story.

School A is an upper secondary school (for students aged 16 to 19). To teach at this level the teachers require at least a Masters degree. The team of teachers in LCM is well qualified and highly experienced. Shortly after the series of workshops commenced, a team of didacticians visited the school to meet with the teachers in LCM and the school Principal. The purpose of this meeting was to explore the teachers' goals, constraints and opportunities within the project. The teachers articulated their goals for joining in the research as, first, 'to be part of a project led by the College' (didactician's field notes 041013), thus strengthening ties between the two institutions. Second, they expressed a desire 'to improve their approaches to teaching mathematics' (ibid.). These goals are located within the whole project 'community' and teachers individual 'mediating artifacts' respectively. The latter goal was later in the meeting contradicted, implicitly, by other statements made by the team of teachers. As the discussion moved on to consider approaches to a specific topic in mathematics the teachers talked with some satisfaction about their current approaches to teaching the topic (recorded in didactician's notes and reflections). In a later meeting, comprising the LCM team of teachers and a different team of

didacticians (one didactician in common to both visits), consideration was given to the results of tests conducted in the school as part of a longitudinal study that contributes to the project. The results included some surprisingly weak responses in basic arithmetic (e.g., adding fractions, identifying the correct operation for a word problem, etc.). The reaction from the teachers (as recorded in didactician's field notes) was: – (1) *Are the teachers in the lower schools being told these results*? Which could possibly be interpreted as, it's a problem that the teachers in School A face but it is the task of teachers in other schools to resolve. (2) *Will these test results be the focus for a workshop*? Which could possibly be interpreted as, it is up to the didacticians to provide a context for the discussion of the problem. (3) *The curriculum planners need to be made aware of these results*. This could possibly be interpreted as seeing a resolution in a redesign of the curriculum. With each statement the teachers appeared to distance themselves from a responsibility to support their own students in the development of basic arithmetic skills, and possibly to reform their own teaching approaches. That is, we can perceive contradictions between the stated desire to be part of the project community yet an apparent distancing of themselves from sharing the responsibilities of the community as a whole. We accept that the interpretations suggested are also indicative of a set of beliefs held by didacticians. An aim of the project is to develop sufficient trust between didacticians and teachers so that these interpretations can be shared without any implication of criticism being intended or perceived (as highlighted in Cestari et al. 2005).

An additional point is worth noting from the account of the discussion of the longitudinal test results. Within school A, students are separated into classes of higher and lower achieving students and despite the fact that the outcome from the test was disappointing overall, it was the teachers of the lower achieving students who showed an interest in what they might do themselves to address the issues. A possible interpretation here is that syllabus and assessment requirements create contradictions and tensions within rules, community and mediating artifacts. Assessments can be used, especially with the higher attaining classes, as instruments to motivate students' engagement, they can also be seen as conferring status within the school community. Success in examinations is also a 'passport' to further education and employment opportunities. In this account, we believe it is possible to find evidence of both perceived and latent contradictions. The teachers of the higher attaining classes (especially) feel a pressure from a demanding syllabus and a high stakes assessment regime. Although the teachers have met to discuss how some of the ideas that have been shared in the workshops (relating to the use of inquiry processes) can be implemented in their classes, their overriding concern is with the syllabus which they must address and the preparation of their students for their examinations. Furthermore, moving away from the practice that has proved acceptable for years entails a high risk, both with the students' performance and routine demonstration of teachers' competence. Here we see tensions between the project goal in terms of the development of inquiry approaches and the constraints of the curriculum, and between teachers current tried, tested and trusted practices and the uncertain state of

'inquiry as a way of being'. There is also, we believe, a latent tension between the teachers expressed desire to develop their teaching and other expressions that indicate that they believe they are performing well and that developments in practice need to occur elsewhere.

As noted above, we are conscious of contradictions within our own (didacticians') activity and we want to draw attention to these because we are also 'insider researchers' exploring our own activity. Our (team of didacticians) concern is to develop inquiry as a way of being for ourselves and the teachers and to begin this we have, so far, organised three workshops with teachers. A central feature of each workshop has been working together in small groups on mathematical tasks that we believe will open up possibilities to think together in inquiry ways, and to use inquiry as a tool to promote inquiry as a way of being. In the process, we expect these tasks to inspire and challenge teachers at a variety of levels. Teachers can also re-package tasks for use with their own classes if they wish to do so. The feedback received following the workshops is that teachers have enjoyed them and found them of value. However, the 'contradiction' here lies in the mediating artifacts, the tasks, in that many teachers are seeing the tasks as the purpose of the project rather than the inquiry processes which the tasks help to illuminate. In other words some teachers are seeing the project as about 'inquiries' or 'inquiry tasks' (in some contexts these might be described as 'investigations') rather than the processes they model as a way of being. Didacticians thus face new challenges in design of activity to promote inquiry.

Another contradiction that we (team of didacticians) experience arises because of the position that we have been required to take to ensure the project becomes established. This has inevitably initiated the development of a community in which the team of didacticians are *perceived* as taking a leadership, and possibly authority role. Our (didacticians') intention is to work in partnership as co-learners with the teachers and although there will be a necessary 'division of labour' we do not want there to be an imbalance of power. It is the expression of what "we want" that captures the dilemma here.

CONCLUSION

The application of an Activity Theory model and the identification of tensions within and between the nodes of the model are providing a means for understanding our own practices as researchers and interpreting the engagement of teachers and didacticians as co-learners. Casting contradictions and tensions in the role of sources for development provides access to areas of the project to which developmental effort needs to be made. The project will progress as we (didacticians and teachers) develop trust within and between our communities, reflect on the contradictions, and find the means to bring those which are latent to the surface, and resolution to those which are perceived.

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1 Unless otherwise indicated the personal pronouns 'our', 'we' refer to the authors of this paper.

2 Didacticians are people who have responsibility to theorise learning and teaching and consider relationships between theory and practice. In the project we refer to the university educators as 'didacticians' in order to recognise that both teachers and didacticians are educators and both can engage in research.

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A SOCIOCULTURAL ANALYSIS OF LEARNING TO TEACH

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This paper addresses the question of how teachers learn from experience during their pre-service course and early years of teaching. It outlines a theoretical framework that may help us better understand how teachers' professional identities emerge in practice. The framework adapts Vygotsky's Zone of Proximal Development, and Valsiner's Zone of Free Movement and Zone of Promoted Action, to the field of teacher education. The framework is used to analyse the pre-service and initial professional experiences of a novice secondary mathematics teacher in integrating computer and graphics calculator technologies into his classroom practice.

A challenge for mathematics teacher education is to understand how teachers learn from their experiences in different contexts – especially when their own schooling, university pre-service program and practicum sessions, and initial professional experiences can produce conflicting images of mathematics teaching. This challenge is sometimes associated with the perceived gap between the decontextualised knowledge provided by university-based teacher education and the practical realities of classroom teaching. As a result, novice teachers can find it difficult to implement innovative approaches they may have experienced during their pre-service program when they enter the more conservative setting of the school (Loughran, Mitchell, Neale & Toussaint, 2001). Clearly, a coherent theory of teacher learning is needed to account for the influence of these varied experiences.

Rather than appealing to cognitive theories that treat learning as an internal mental process, some researchers have begun to draw on situative or sociocultural perspectives in proposing that teachers' learning is better understood as increasing participation in socially organised practices that develop their professional identities (Ensor, 2001; Lerman, 2001; Peressini, Borko, Romagnano, Knuth & Willis, 2004). Identity can be said to emerge in practice, but identity also affects the ways in which a teacher interprets and analyses problems of practice. In the process of making instructional decisions and reconciling competing priorities, teachers construct their professional identities as individuals-acting-in-context.

The purpose of this paper is to outline a sociocultural framework for studying how teachers learn from experience in complex social settings, and how this shapes their professional identities. A case study from a three year longitudinal project is presented to demonstrate how the framework can guide analysis of pre-service and initial professional experiences of secondary school mathematics teachers.

THEORETICAL FRAMEWORK

Studies of teacher socialisation from a functionalist perspective typically identify influences such as the beliefs that students bring to the pre-service course from their

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 49-56. Melbourne: PME.

own schooling, and the classroom practices they observe and experience as novice teachers (Brown & Borko, 1992). Such approaches view teachers as being passively moulded by external forces to fit the existing culture of schools – thus producing the common explanation for why many beginning teachers give up their innovative ideas in the struggle to survive and conform to institutional norms of traditional practices. However, an alternative, sociocultural, perspective proposes that any examination of teachers' learning and socialisation needs to consider the "person-in-practice-in-person" (Lerman, 2000, p. 28), a unit of analysis that allows us to shift our analytical focus between the individual and the social.

The theoretical framework explored in this paper adopts a neo-Vygotskian approach, extending the concept of the Zone of Proximal Development (ZPD) to incorporate the social setting and the goals and actions of the participants. Vygotsky (1978) defined the ZPD as the distance between a child's independent problem solving capability and the higher level of performance that can be achieved with expert guidance. In a teacher education context, the ZPD can be thought of as a symbolic space where the novice teacher's pedagogical knowledge and skills are developing under the guidance of more experienced people. However, this gap between present and potential ability is not the only factor influencing development. Valsiner (1997) proposed two further zones to account for development in the context of children's relationships with the physical environment and other human beings: the Zone of Free Movement (ZFM), representing environmental constraints that limit freedom of action and thought; and the Zone of Promoted Action (ZPA), a set of activities offered by adults and oriented towards promotion of new skills.

Blanton, Westbrook and Carter (2001) have employed Valsiner's zone theory to examine the development of novice mathematics teachers. Their approach involved analysing patterns of classroom discourse to uncover contradictions between the ZFM organised by the pre-service teacher (what did the teacher *allow*?) and the ZPA she established for her students (what did the teacher *promote*?). The focus is on *students'* learning: the ZFM represents the classroom and the ZPA the activities offered by the teacher. The research reported here extends this study by applying Valsiner's ideas to *teachers'* learning by considering their social and institutional contexts and how these environments enable or constrain teaching actions.

For pre-service or beginning teachers, elements of the Zone of Free Movement might include their students (behaviour, motivation, perceived abilities), curriculum and assessment requirements, and the availability of teaching resources. While the ZFM suggests which teaching actions are *possible*, the Zone of Promoted Action (ZPA) represents the efforts of a university-based teacher educator, school-based supervising teacher, or more experienced teaching colleague to *promote* particular teaching skills or approaches. It is important that the ZPA be within the novice teacher's ZFM, and is also consistent with their ZPD; that is, the actions promoted must be within the novice's reach if development of their identity as a teacher is to occur. This is represented diagrammatically in Figure 1.

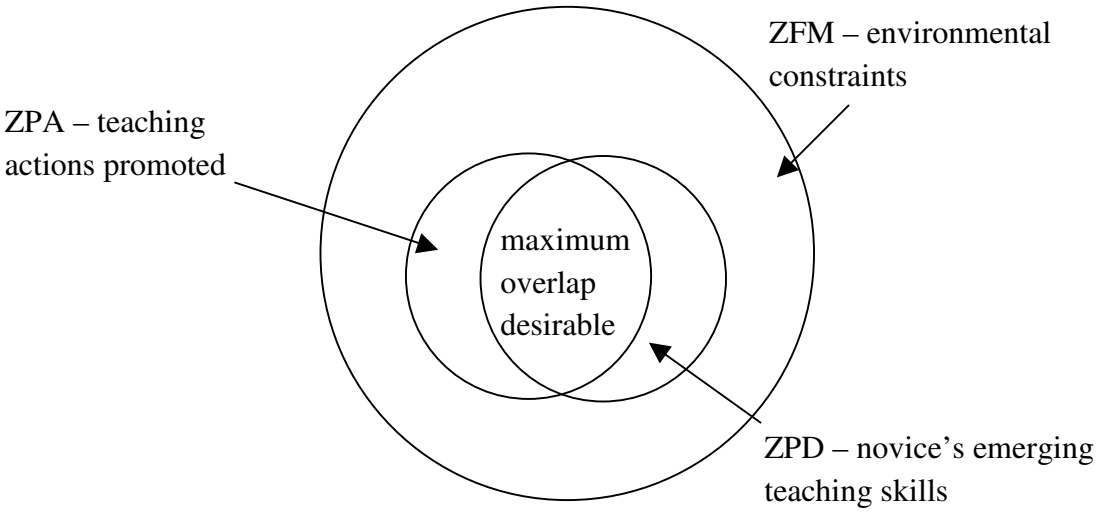

Figure 1: Relationships between the ZFM, ZPA and ZPD for novice teachers

Additionally, pre-service teachers develop under the influence of two ZPAs – one provided by their university program, the other by their supervising teacher(s) during the practicum – which do not necessarily coincide. These three zones constitute a system that can account for the dynamic relationships between opportunities and constraints of the teaching environment, the teaching actions specifically promoted, and the development of the novice teacher's pedagogical identity.

BACKGROUND TO THE STUDY

This research study is investigating the transition from pre-service to beginning teaching of secondary school mathematics. A major aim is to identify factors that influence how beginning teachers who have graduated from a technology-rich pre-service program integrate computer and graphics calculator technologies into their practice. This focus on relationships between technology and pedagogy in pre-service education and the early years of teaching provides rich opportunities to analyse teachers' learning and development in terms of identity formation. How do beginning teachers justify and enact decisions about using technology in their classrooms? How do they negotiate potential contradictions between their own knowledge and beliefs about the role of technology in mathematics education and the knowledge and beliefs of their colleagues? How do they interpret aspects of their teaching environments that support or inhibit their use of technology?

Questions such as these, when framed within a sociocultural perspective, may help us re-interpret and extend existing research findings on mathematics teachers' use of technology. This research has identified a range of factors influencing uptake and implementation, including: skill and previous experience in using technology; time and opportunities to learn; access to hardware and software; availability of appropriate teaching materials; technical support; support from colleagues; curriculum and assessment requirements; knowledge of how to integrate technology

into mathematics teaching; and beliefs about mathematics and how it is learned (Fine & Fleener, 1994; Manoucherhri, 1999). In terms of the theoretical framework outlined earlier, these different types of knowledge and experience represent elements of a teacher's ZPD, ZFM, and ZPA. However, previous research has not necessarily considered possible relationships between the setting, actions and beliefs, and how these relationships might change over time or across school contexts.

METHOD

This longitudinal study involves three successive cohorts of pre-service secondary mathematics teachers. In the final year of their pre-service program participants complete an integrated mathematics methods course, and two 7 week blocks of supervised practice teaching in schools. I design and teach the methods course so that students experience regular and intensive use of graphics calculators, computer software, and Internet applications (see Goos, in press). Thus the course offers a teaching repertoire, or ZPA, that emphasises technology as a pedagogical resource.

Case studies of individual pre-service teachers were conducted to capture developmental snapshots of experience during the second block of practice teaching (August) and again towards the end of their first and second years of full-time teaching (October/November). Six cases were selected in each year of the study to sample a range of different practicum school settings, including technology-rich and technology-poor government and independent schools in capital city and regional locations. These participants were chosen because of the interest and skills they demonstrated in using technology resources in mathematics teaching. Because they were eager to use technology, it was anticipated that their experiences in schools could provide insights into how they dealt with obstacles or took advantage of opportunities in incorporating technology into their pedagogical repertoire.

I visited these teachers in their schools at the times described above. The school visits involved lesson observations, collection of teaching materials and audio-taped interviews. Two types of interviews sought information on factors shaping the formation of beginning teachers' professional identities. A Post-lesson Interview was carried out immediately after the observed lesson to assist teachers to reflect on pedagogical beliefs that influenced lesson goals and methods. A more general Technology Interview was also conducted to discover what opportunities participants had to use technology in mathematics lessons, their perceptions of constraints and opportunities affecting their use of technology, and their views on the influence of technology on mathematics curricula, learning, teaching and assessment. (A full description of interview methods is provided in Goos, in press.)

All interviews were transcribed to facilitate analysis. Participants' responses to the interview questions were categorised as representing elements of their ZPDs, ZFMs, and ZPAs. As the zones themselves are abstractions, this analytical process focused on the particular circumstances under which zones were "filled in" with specific people, actions, places, and meanings.

CASE STUDY ANALYSIS

An analysis of one case study is presented below. This participant (Adam) completed the pre-service course in 2003 and entered his first year of teaching in 2004.

Adam's Experience as a Pre-Service Teacher

Adam's practicum placement was in a large suburban school where he was assigned to teach a range of junior and senior secondary mathematics classes under the supervision of the Head of the Mathematics Department. The school had recently received funding from the State government to establish a Centre of Excellence in Mathematics, Science and Technology, and had used this money to refurbish classrooms in the Mathematics Department and buy resources such as graphics calculators, data logging equipment, and software. Every mathematics classroom was equipped with twelve computers, a ceiling mounted data projector, and a TV monitor for projecting graphics calculator screen output. All students in the final two years of secondary school (Grades 11 and 12) had continuous personal access to a TI-83 PLUS graphics calculator via the school's hire scheme, and there were also sufficient class sets of these calculators for use by other classes (Grades 8-10). Some of these changes had been made in response to new senior mathematics syllabuses that now mandated the use of computers or graphics calculators in teaching and assessment programs. Thus the school and curriculum environment offered a Zone of Free Movement that afforded the integration of technology into mathematics teaching.

Adam had previously worked as a software designer and was a very confident user of computers and the Internet. Although he had not used a graphics calculator before starting the pre-service course, he quickly became familiar with its capabilities and took every opportunity to incorporate this and other technologies into his mathematics lessons, with the encouragement of his Supervising Teacher. For example, in the lesson I observed he used the graphics calculator's *ProbSim* program to introduce Grade 8 (13 year old) students to ideas about chance and frequency distributions via a dice rolling simulation where the outcomes were displayed as a histogram. In theoretical terms, then, the supervisory ZPA was consistent with that offered by the university course and also with the ZPD that defined the direction in which Adam wished to develop as a teacher. However, when interviewed Adam wondered how he could prevent students from becoming dependent on the technology – they might simply "punch it into their calculator and get an answer straight away" – and this might rob them of some important learning experiences. He acknowledged that his concern probably stemmed from the fact that he had only ever used technology in his teaching, or observed its use by other teachers, as a tool for saving time in plotting graphs and performing complicated calculations, or for checking work done first by hand, and he speculated that teaching styles would need to change to incorporate technology in new ways that enhanced students' learning of mathematical concepts. Nevertheless it seemed that Adam's professional identity was emerging in a context similar to the "ideal" situation depicted in Figure 1.

Adam's Experience as a Beginning Teacher

After graduation Adam was employed by the same school where he had completed his practicum. As the school environment (ZFM) and mathematics teaching staff (ZPA) were unchanged, one might predict that Adam would have experienced a seamless transition from pre-service to beginning teacher; yet I found that this was not the case when I visited him there towards the end of his first year of teaching.

I observed him teach a Grade 11 class about the effects of the constants a, b, and c on the graph of the absolute value function $y = a|x + b| + c$. The students first predicted what the graph of $y = |x|$ would look like, and then used their graphics calculators to investigate how the shape of the graph changed with different values of a, b, and c. Although Adam clearly had specific goals in mind, the lesson was driven by the students' questions and conjectures rather than a predetermined step-by-step plan. For example, at the start of the lesson one student noticed that the graph of $y = |x|$ involved a reflection in the y-axis and she asked how to "mirror" this graph in the x-axis. Immediately another student suggested graphing $y = -|x|$, and the teacher followed this lead by encouraging the class to investigate the shape of the graph of $y = a|x|$ and propose a general statement about their findings.

After the lesson Adam explained that he had developed a much more flexible teaching approach:

> I had a rough plan and we kind of went all over the place because we found different things, but I think that's better anyway. Because the kids are getting excited by it and they're using their calculators to help them learn.

Instead of viewing technology purely as a tool for performing tasks that would otherwise, or sometimes also, be done by hand, Adam now maintained that the role of technology was "to help you [i.e., students] get smarter" by giving students access to different kinds of tasks that build mathematical understanding, especially tasks that involve modelling real world situations. Here he claimed to have been influenced by the university pre-service course and the highly experienced mathematics teacher who was the Director of the school's Centre of Excellence project described earlier.

These observations suggest that Adam's potential for development – the ZPD representing his beliefs about teaching, learning and the role of technology, and his knowledge and expertise in using technology – had expanded since the practicum, and that his potential would be promoted with the assistance (ZPA) of his colleagues in the Mathematics Department. But this was the case for only *some* colleagues. Many of the other mathematics teachers were unenthusiastic about using technology and favoured teaching approaches that Adam claimed were based on their faulty belief that learning is linear rather than richly connected:

> You do an example from a textbook, start at Question 1(a) and then off you go. And if you didn't get it – it's because you're dumb, it's not because I didn't explain it in a way that reached you.

Because he disagreed with this approach, Adam deliberately ignored the worksheet provided for the lesson by the teacher who coordinated this subject. The worksheet led students through a sequence of exercises where they were to construct tables of values, plot graphs by hand, and answer questions about the effects of each constant in turn. Only then was it suggested that students might use their graphics calculators to check their work. Conflicting pedagogical beliefs were a source of friction in the staffroom, and this was often played out in arguments where the teacher in question accused Adam of not teaching in the "right" way and not preparing students properly for their examinations. Adam realised that as a pre-service teacher he had not noticed the "politics of teaching" because he had the luxury of focusing on a small number of classes and his relationship with a single supervising teacher. He now found himself in a more complex situation that required him to defend his instructional decisions while negotiating a harmonious relationship with several colleagues who did not share his beliefs about learning. Adam explained that he was willing:

> …to stand up and say "This is how I am comfortable teaching". I just walk away now because we've had it over and over and the kids are responding to the way I'm teaching them. So I'm going to keep going that way. But I've made the adjustment that at the end of the topic we'll say "Right, now to get assessed on this you need to do these steps".

In terms of the theoretical framework outlined earlier, Adam has interpreted his technology-rich ZFM as *affording* his preferred teaching approach and he has decided to pay attention only to those aspects of the Mathematics Department's ZPA that are consistent with his own beliefs and goals (his ZPD) and also with the ZPA offered by the university pre-service course. At the same time he has recognised the need to explain more clearly to his students how they should set out their work in order to gain credit for correct solutions to examination problems. One could say that his professional identity has developed to the extent that he is now able to reconcile his pedagogical beliefs (a part of his ZPD) with externally imposed assessment requirements (an element of his ZFM).

CONCLUDING COMMENTS

Valsiner's zone theory, when applied as illustrated in this paper, may help us analyse relationships between teachers' pedagogical beliefs, the teaching repertoire offered by their pre-service course, and their practicum and initial professional experiences, in order to understand how their identities might emerge as users of technology. The analysis presented here has examined how these relationships can change over time within the same institutional setting. In 2005 Adam has been transferred to a different school with very limited technology resources, so it is likely that his interpretation of his new context – the school's ZFM – will be crucial to the continued development of his professional identity, his sense of "being" as a teacher.

Although the three zone framework has been used here as an analytical tool, it could also support teachers' learning in several ways. First, it could help pre-service teachers to analyse their practicum experiences (ZFM), the pedagogical models these

offer (ZPA), and how these experiences reinforce or contradict the knowledge gained in the university program (university ZPA). Second, in the early years of teaching it could be used to create induction and mentoring programs that promote the sense of individual agency Adam displayed within the boundaries of the school environment (ZPD within ZFM). Finally, the framework could assist in the design of professional development for more experienced teachers (ZPA to stretch ZPD).

Acknowledgement

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THREE CASE STUDIES ON THE ROLE OF MEMORISING IN LEARNING AND TEACHING MATHEMATICS

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We extend an earlier study on the role of memorising in learning science with three case studies on mathematicians' experiences as learners of mathematics and their views on memorising in students' learning. Each report reflects a different view of memorising — from a minimal role in learning mathematics, to a 'stepping stone' role to memorising as a key learning strategy. The mathematicians' views about memorising in their students' learning appears consistent with their reports on memorising in their own learning. Our data leads to the conjecture that personal experiences of memorising underpin attitudes to memorising in teaching mathematics and that further investigation is warranted on this little researched aspect of "coming to know" mathematics.

INTRODUCTION

How did academic mathematicians learn mathematics when they were students and how do they perceive their students learn mathematics? We investigate this question with a focus on perceptions of memorising in learning mathematics. Our discussion draws on and extends our study on the role of memorising in learning physics, physiology, mathematics and statistics (Cooper, Frommer, Gordon & Nicholas, 2002). In that study we conducted interviews with 16 academics from the respective discipline areas. A phenomenographic analysis of the interviews revealed three categories of conceptions of the role of memorising in learning science: memorising plays a minimal role, memorising serves as a stepping stone to learning and memorising is a key strategy in learning.

A set of phenomenographic categories represents a collective awareness rather than an individual awareness expressed by Marton and Booth as "a description on the collective level, and in that sense individual voices are not heard" (Marton and Booth, 1997, p.114). Thus, in phenomenography, individuals are seen as the "bearers of fragments" (Marton and Booth, 1997, p.114) of differing ways of experiencing a phenomenon. In contrast to the "stripped" categories of description of memorising in learning science presented in our earlier paper we now take a deeper look at the views held by three mathematicians, whose conceptions were included in and contributed to the construction of the three major categories outlined above.

University teachers' earlier experiences of teaching and their perceptions of the teaching context have been shown to be significant dimensions of their current teaching approaches (Prosser & Trigwell, 1999). Moreover, there are relationships between the way teachers approach their teaching and students approach their

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 57-64. Melbourne: PME.

learning (Trigwell, Prosser & Waterhouse, 1999). Our framework brings to the foreground mathematicians' experiences both as learners and teachers of mathematics. We concur with Akerland (2004) that "a focus on academics' experience of teaching separated from their larger experience of being a teacher may encourage over simplification of the phenomenon of university teaching, in particular in terms of academics' underlying intentions when teaching".

Our theoretical framework draws on and amplifies Burton's theories on "coming to know mathematics" (Burton, 2001, 2003). Burton (2003, p. 13) proposes that "coming to know mathematics is a product of people and societies" and is heterogeneous, connected with experiences and with aesthetic and other feelings, and interdependent on networks of practice. We conjecture that mathematicians' experiences of coming to know mathematics at university are important to how they view the process of learning mathematics and that their beliefs and ideas impact on teaching. Hence mathematicians' experiences of memorising in their own learning of mathematics and their perceptions of the role of memorising in their students' learning are important to pedagogy. These are areas in which there is a dearth of data. Further, Burton (2001) proposes that mathematicians at university have considerable impact on the continuing development of the discipline and the enculturation of the next generation of mathematicians and mathematics educators.

> Mathematicians in universities have great power to influence what is learned and how it is learned in the discipline. They do this, in part, through the respect which society accords their discipline, and consequently those within it. ... But these mathematicians are also responsible for teaching the next generation of teachers and, consequently, for contributing to the definition and history of the practices which are seen to be appropriate to the communities that they touch (Burton, 2001, pp. 589-590).

We provide insights on the perceptions and experiences of three mathematicians as indicators of the connectedness of learners, learning and teaching mathematics at university and to highlight a need for further research in memorising to help develop theories on coming to know mathematics.

METHOD

We propose and answer the following research questions.

- How did the mathematicians learn mathematics and view memorising in their own learning?
- What were the mathematicians' perceptions of student learning and memorising?

Our three participants were experienced mathematicians and mathematics educators working in metropolitan universities in Australia and South Africa. The interviews were semi-structured and consisted of seed questions such as: *What role did memorising play, in your own learning of mathematics? How do you think students learn mathematics*? *Do you feel there is a relationship between memorising and*

understanding? Follow up questions depended on the responses and aimed at an in-depth understanding of the participants' experiences and perceptions. Interviews were audio-taped and transcribed.

In our initial phenomenographic analysis (Cooper et al, 2002) we focussed on the construction of broad categories from the data to form an outcome set, ignoring personal and individual perceptions. Our re-analysis of the interviews for the current investigation had a different purpose — to listen to and represent the individual reports of our participants. We examined independently and extensively the mathematicians' responses as they related to the two research questions. We compared, discussed and integrated our individual analyses and reviewed the interviews in ongoing cycles. We present case studies to illustrate views that best fit with each of the three categories describing memorising. However the view of each participant presented is more than an expansion and delineation of a category – it is an individual construction as interpreted by us, the researchers. This methodology can be considered as ethnographic. All three participants are well known to us as fellow mathematicians and mathematics educators, and in one case, also a fellow postgraduate student of mathematics. Our interpretations are cognisant of the environments in which our participants work, the resources available to them and the constraints which organise and shape their teaching.

RESULTS

Case Study 1: Peter

For Peter memorising was not a conscious strategy in his learning and therefore played a minimal role if any. His major strategy for making mathematics meaningful was writing out his lecture notes so he could understand them.

> I wrote out everything as much as I could of what he (the lecturer) said and what he wrote. And then I tried to write it in a way that I could understand it. I know that it was the effort of writing it out that helped me understand it.

Peter described his learning as including doing many exercises, essential to gain proficiency and understanding, and reported that learning mathematics was an effortful business. He said: "I did many exercises, so when I finished each course I knew how to do the problems and I had an idea of what the lecturer was saying I think". He continued: "I was lucky, because I didn't have any tutorials, so I had to grapple with stuff".

When asked for his perceptions of how his students learned mathematics, this mathematician talked for the most part about a large group of students who he thought were memorising without understanding.

> I think many of them memorise. ... I'm looking at one student now who really shouldn't be in second year. She has serious algebraic difficulties, and I know she is hanging on my every word and thinks all she has to do is repeat and she'll be right.

He considered this strategy had little value in learning mathematics as it was the "shallowest thing which was active" and "it is not sufficient just to memorise facts without any understanding of them".

He was of the view that the majority of students arrived from school with "an incorrect image of mathematics in their heads". Their approach to learning mathematics consisted of learning formulae and techniques and passing the examinations was a matter of applying those. However, he considered that these students in first year were "serious about learning" and they were "quite able to grow" even though they struggled to understand what the lectures were about. He believed that by third year this seriousness about learning had disappeared and students expected everything to be given to them easily, using tutorials as the principal venue for their own learning.

> They don't understand what the lectures are about, they think there are hoops, that they have to jump through these hoops, and all they've got to do is do that, and they pass and go on to the next thing.

These students, he felt, were able to pass examinations with a minimal understanding of the mathematics course by learning techniques and memorising rules. He thought that the examinations were "testing a very basic manipulative ability to follow the rules that we have given".

On the other hand, Peter acknowledged that there was another group of students who were serious about learning mathematics, and that "the student who gets a rudimentary understanding of mathematics I think does a great deal of learning".

Case Study 2: Mary

Reflecting on her learning Mary recounted that memorising fulfilled a role as a "stepping stone". She explained that by memorising she could ensure that an idea was captured with the necessary completeness, precision and conciseness. This allowed her to use an idea with confidence and certainty enabling her to progress. Memorising was interlinked with personal understanding and this was a dynamic, incomplete process.

> Memorising would never be like the lecturer wrote it, like 1, 2 and 3. I'll write it as 1, 2, and 3 because it did make sense to her (the lecturer) as 1, 2 and 3 but now it makes sense to me as well as 1, 2, and 3 so I'll learn it as 1, 2 and 3.

Mary reported that as a student had learned mathematics by making a line by line examination of a section of work trying to understand each and every facet. When asked how she knew she understood, she replied as follows.

> A feeling that the statements were validated. Yes a feeling that each statement was valid and followed a sequence, that there were no gaps. If there were gaps I would put in those gaps for myself. And those gaps later you would take them out again as your understanding became more sophisticated so your need to fill those gaps became less profound, you could take them out.

This mathematician perceived memorising as playing an important part in all students' learning. She identified two distinct groups of students characterised by a qualitative difference in their memorising. She considered that the majority were the weaker students who used memorising as a "crutch" for their learning mathematics — for them memorising was employed when there was no underlying understanding.

> Many students when they don't understand will learn by rote, learn by heart, learn to reproduce a piece of work according to how they've been given it. But it is quite clear when you ask a leading question they have no understanding to accompany that learning.

She perceived that for these students the goal was to reproduce what they were given.

Mary distinguished these students from the second group of students who, like her, memorise "only after they have come to some understanding about the nature of the subject". She characterised their memorising as a "stepping stone" or "scaffold" to support their growth in learning mathematics.

> In order to complete a piece of learning they might have memorised the theorem, and feel confident in the theorem, these are the students who get the picture of what mathematics is about. And in order to use it frequently and without judgement and without thought, they've memorised it. They can repeat the definition. But they use the definitions with insight whereas the other group merely have learned the words of the definition but they can't use it.

Thus, Mary perceived that memorising fulfilled the same function in learning for both groups of students but there were qualitatively different forms of memorising with differing longevity. Students who used memorising as a crutch were going to use that crutch forever, while students who used memorising as a scaffold for understanding ultimately removed that scaffold.

Case Study 3: Qian

To Qian memorising was a strategy that played a key role in learning mathematics and pervaded every aspect of his own learning mathematics.

> From my experience, definitely memorising is one of the very basic techniques for me, to go through my study. You just start to memorise from the first page to the last pages and exercises you produce. Memorising is very important.

He reported that memorising patterns of proofs and how to do things was essential for remembering in the long term, for problem solving, and for precision. He added that memorising, for him, was essential for speed in examinations as there was no time to derive the required formulae or prove the theorems. His strategy appears analogous to a master chess player deliberately memorising multiple chess games, forming a permanent memory base from which a play could be drawn and speedily applied to a game of chess in progress but also as a springboard for creating new plans and solving problems.

Doing problems was an important strategy in Qian's learning to see if he could apply what he had learned, to test his understanding of a mathematical concept and to identify gaps in understanding.

When asked how his students learned mathematics, Qian replied that students learned mathematics by pattern recognition and learning pattern types which they tried to imitate or copy. He considered that for the majority of students, learning was accomplished largely through doing examples and using those examples as templates for doing the same kind of examples. He estimated that only the top 20% of his students were attempting to learn how to reason mathematically.

Qian reported that memorising was a strategy used by all students. However, consistent with his view of memorising in his own learning, he emphasised memorising as being of prime importance for the top students in the senior years of their degree. He expected these advanced students to remember a lot of proofs and definitions and to apply these to solve complex problems. He emphasised that memorising a "pattern of proof" was an efficient and time saving examination technique for advanced students even though they had the skills to use derivation and deduction.

He considered that there was a qualitative difference in memorising as a strategy for the top students and for the majority of students.

> One is memorising with understanding and the second is memorising hopefully, I guess. Is this the formula, or is that the formula? They did not have understanding so they could not use it.

He stressed that those students who memorised a formula or technique with understanding would remember "forever", while students who memorised a formula "hopefully" and "for the sake of memorising, without even knowing anything", would quickly forget it.

DISCUSSION AND CONCLUSION

In reflecting about their own mathematical learning, the mathematicians expressed their engagement with mathematics at a high level. They were "mathematically curious" and wanted to see how everything fitted together with a focus on developing a personal understanding. For them learning mathematics was effortful. All agreed that it was vital to have a memory bank of precise definitions and mathematical concepts and techniques that could be recognised and retrieved as required. This enabled progress to be made in learning mathematics. Their learning strategies included rewriting lecture notes according to their understanding, solving problems, a line by line analysis of mathematical statements and, for some, memorising. Memorising was employed to learn patterns and proofs, for precision and to facilitate progress, for speed in examinations, for permanency and to enable application.

In discussing their students' learning, all our participants identified a large group of students whose learning was characterised by imitation. Students in this group were

perceived as learning mathematics by rote without an underlying basis of conceptual understanding. For these students the primary goal in learning mathematics was to pass assessments and the mathematicians considered these students to be the weaker students or the students who were struggling.

The mathematicians also identified a smaller group of students, described as the good, advanced or top students, who were motivated by understanding the mathematics and who worked at learning mathematics. Not all our participants perceived that memorising was a strategy used in learning mathematics by these students. The mathematicians who believed that memorising was used, and was an appropriate and valuable strategy for good students, perceived the purpose and quality of their memorising to be quite different to memorising as the empty repetition and imitation of the weaker majority.

The three case studies reveal diverse views on memorising in learning mathematics from memorising as rote learning or parroting to memorising as a form of permanent internalising with understanding. Further, while each individual conveyed strongly his or her own view about memorising in learning mathematics, apparently none considered or contemplated other possibilities for the role of memorising in learning mathematics.

The observations of the mathematicians about memorising in their students' learning appeared to be related to perceptions of memorising in their own learning as students. Moreover, there was little acknowledgement by our participants about the impact of their own learning experiences on their teaching. If personal experiences of learning mathematics are not recognised as being related to teaching mathematics at university we can expect to perpetuate the idea that mathematics is a pure body of knowledge that is learned through logic and rational thought alone, independent of social context, emotion, personal history and human interaction.

The case studies demonstrated little appreciation or nurturing of diversity in learning mathematics and the contributions of students' life experiences did not appear to be acknowledged in the mathematicians' reports. Burton (2001) distinguishes between mathematical 'knowledge' and mathematical 'knowing' and concludes that while the journey of coming to know is seen as important for research in mathematics it is notably absent from views of teaching mathematics at university. All three participants of our study talked about "good" students and "weak" students and their concepts of these appear to be in terms of acquiring knowledge or making an effort to acquire knowledge. A good student is one who works to gain the knowledge set out by the teacher — a poor student does not try or does not succeed in attaining this discipline knowledge.

Our participants reported constraints in the institutional context for teaching mathematics at university, which impacted on their teaching strategies, including pressures to have satisfactory pass rates, pressures to make things easy for students and colleagues, time constraints and large class numbers. Peter even proposed that

students' engagement with learning mathematics on entering university was systematically undermined during their undergraduate years. The impact of context is important to understanding memorising in learning mathematics and we are investigating this further (Gordon & Nicholas, In preparation).

Anecdotally mathematicians appear to expect students "to know" and remember the mathematics they are taught, to "have" a body of mathematical knowledge in their memories. Our data provides insights into the relationship between memorising and memory and suggests that it is not a simple connection. More research is needed to explicate the strategy and outcomes of memorising in learning mathematics.

Questions also arise as to the intentions of mathematics teachers at university — is the primary goal that students acquire a body of knowledge, whether by memorising or other learning strategies or do we focus on promoting mathematics enquiry (Burton, 2003) and diverse ways of learning mathematics? The rhetoric of mathematics education emphasises the importance of engaging students to construct mathematics and appreciate mathematical thinking but in our universities the rhetoric and the reality may not match.

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RECONSTRUCTING NORMS[1]

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Starting from the constructs 'cultural scripts' and 'social representations', and on the basis of the empirical research we have been developing until now, we revisit the construct norms from a sociocultural perspective. Norms, both sociomathematical norms and norms of the mathematical practice, as cultural scripts influenced by social representations, mediate the learning of mathematics in multicultural classrooms. When taking into account the particular circumstances in which mediation occurs, there is a need for a move from a cultural perspective to a broader sociocultural one.

THE SCENE

Because of the important recent waves of migration into Spain, our educational system must face the needs of a society where the plurality of cultures and languages is, and will be, a reality. However, the Spanish school system has a limited understanding of the sociocultural and linguistic aspects linked to teaching mathematics in multicultural situations. In general, mathematics teachers are not prepared to teach in multiethnic classrooms, the mathematics curriculum is intended for the 'native' groups and, at most, language, understood as everyday language, is the only 'problem to be solved' in multiethnic mathematics classrooms.

For more than six years now, we have been researching multiethnic mathematics classrooms in an effort to understand their complexity. The beginning of our project was the result of a request from the Catalan Ministry of Education. Its aims were a result of an initial negotiation with the educational administration, and they included, among others:

a) to know more about the knowledge that immigrant students bring with them to school and how this knowledge can be linked with the curriculum and its development in the mathematics classroom;

b) to uncover the values and expectations immigrant students associate with school, and out-of-school, mathematics and determine how these could help or interfere with the teaching and learning of mathematics, and

c) to develop both proposals and practical examples of how to adapt the school curriculum and the classroom organisation to the multiethnic classroom.

Since the very beginning, we have studied the interactions taking place in multiethnic mathematics classrooms while the students were working on problem solving. The

[1] The empirical study that sustains this theoretical research report has been documented in Planas & Gorgorió (2005)

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 65-72. Melbourne: PME.

mathematics classrooms under study were those of teachers that had volunteered to work collaboratively with us and had shown a sensitivity towards issues of equity and inclusiveness. Classroom observation and interviewing teachers and students had the purpose of uncovering and understanding the different social and cultural elements affecting the social and mathematical interactions taking place.

IN SEARCH OF THEORETICAL LENSES

When studying the interactions within the mathematics classroom, the idea of norm becomes an essential construct. When beginning our research (Planas & Gorgorió, 2001; Gorgorió et al., 2002), we used the construct norms as they had been established by Cobb and his colleagues. Cobb, Yackel and Wood (1992) introduced the idea of social norms as social constructs that involve a taken-as-shared idea of what constitutes an appropriate contribution to a discussion. The term sociomathematical norm was coined by Yackel and Cobb (1996) and has been widely used since. Sociomathematical norms have to do with the actual process by which students and teacher contribute to a discussion. They designate the classroom social constructs specific to mathematics that individuals negotiate in discussions to develop their personal understandings (Hershkowitz & Schwarz, 1999, p. 150), and are the result of legitimating explanations and justifications. Sociomathematical norms are also understood upon a taken-as-shared basis (Yackel & Cobb, op.cit.). In the different works by Cobb and his colleagues that we have been able to trace, social norms and sociomathematical norms are seen as different aspects of the classroom microculture. Social norms are used to interpret the classroom participation structure and are not specific to the mathematics classroom; sociomathematical norms deal with normative aspects of classroom action and interaction that are specific to mathematics (Cobb & Liao Hodge, 2002).

However, with the development of our study of immigrant students' transition processes (Gorgorió et al., 2002), it became apparent that the construct of norms did not fully allow us to interpret what we were observing: different understandings of the same norm within a mathematics classroom were difficult to reconcile, and could certainly not be taken as shared. In our study, as in the one reported in Cobb (1999), the norms and practices within the classroom were in conflict with those of the students' immediate contexts. The immigrant students in our classrooms had a different way –different from that of the local students and teachers– of understanding, valuing and using mathematics, differences that gave rise to cultural distances and cultural conflicts (Gorgorió &Planas, in press).

Norms refer to regularities of the practice and of the social interaction that are established by the individual and group interpretations of what is perceived as acceptable or desirable. In the multiethnic classrooms we studied, there were different perceptions of a particular contribution as 'being acceptable or desirable', a fact that was causing obstacles to communicative processes. The main issue was not to reach consensus on, for instance, what constitutes mathematical evidence, a good

hypothesis, or a good explanation. In the classrooms of our study, where immigrant children from different parts of the world were expected to work together with local children, there were other more basic, or prior, issues, also related to mathematics learning, on which agreement was needed. The meanings and values associated with mathematical knowledge and who is mathematically knowledgeable, the expected role of a mathematics teacher, the working organisation within the mathematics classroom, or the idea of learning mathematics in itself, were at the basis of the difficulties in the interaction processess and were not in the least 'taken-as-shared'.

The issue then, for us, was that the meaning of the word *social* in the social norms and sociomathematical norms needed to be revisited. From a sociocultural perspective the learning of mathematics is affected by what takes place within the classroom and in their nearest contexts. We could not understand anymore the word social as simply 'being conjointly constructed by the different participants in the classroom', without considering that all participants were, in turn, social individuals, with their own social and cultural experiences and expectations. Could we then still regard norms upon a 'taken-as-shared' basis?

RECONSTRUCTING NORMS

How do the different and multiple cultural and social histories of the individuals become apparent when they (are supposed to) work together in the mathematics classroom? It is widely recognized that different cultural artefacts, like different algorithms or number symbols, mediate the students' learning processes. It is our claim that the different ways of understanding the teaching and learning of mathematics itself and how it has to take place, or the value attributed to having or not having mathematical knowledge, are also cultural factors that shape how individuals act and interact within the mathematics classroom. The issues then are: 'How are norms established?', 'How can they be agreed on, negotiated or changed?', 'How is the 'desirable' or 'acceptable' established?'.

Our reconstruction of the concept norms is based in two well established constructs: those of 'cultural scripts' and 'social representations'. Cultural schemas (D'Andrade, 1990) make up the meaning system characteristic of any cultural group. Cultural schemas 'portray not only the world of physical objects and events, but also more abstract worlds of social interaction, discourse, and even word meaning' (D'Andrade, op. cit., p. 93). According to Cole (1996) 'a script is an event schema that specifies the people who appropriately participate in an event, the social rules they play, the objects they use, and the sequence of action and causal relations that applies' (p. 126). Scripts are to be treated as dual entities, one side of which is a mental representation, the other side of which is embodied in talk and action (Cole, op. cit., p. 129).

The idea of 'social representations' (Moscovici, 1983) also plays a significant role in our reconstruction of the concept norms. We understand social representations to be particular types of knowledge that allow people to organise their reality, both social

and physical, and to relate with other people and groups. They are reconstructions of reality, arising from communication between individuals; reconstructions which, in practice, regulate behaviour between and within groups. Social representations focus on, select and retain certain relevant facts of reality, according to the interests of the individual as inserted within a group. Selected aspects of the object of the social representation develop into an implicit theory that allows individuals to explain and assess their contexts. They constitute an operational guide to understanding complex or difficult situations, to facing problems and conflicts, to coping with unexpected realities, to justifying actions and to maintaining differences between groups when these differences seem to be fading. To our understanding, social representations are neither directly based on scientific knowledge, nor necessarily verified by means of empirical facts.

Our reconstruction of norms focusses on their social weight. The group's social valorisations shape the values, expectations, emotions and beliefs of the individuals who identify themselves with it. When the teacher calls on a certain norm, and the students tackle it, they all bring to the process their own interpretation of a social understanding about mathematical knowledge and mathematical knowledge ownership, and a social valorisation of mathematical practices. Broader social structures, like the educational system, impact on the classroom interactions through implicit messages about what are the legitimate norms within the classroom. In our reconstruction of norms we refer to sociomathematical norms and to norms of the mathematical practice as regulating actions and interactions within the mathematics classroom: the first when taking into account the individuals' and groups' social understanding and valuing of mathematical knowledge; the second when considering the individuals socially interacting with specific mathematical knowledge.

We refer to sociomathematical norms as the explicit or implicit regulations that influence participation within the mathematics classroom and the interactive structure of the development of the mathematical practice. They have to do with how the different participants value mathematical knowledge, and value and position themselves, the others and their group(s) with regard to mathematical practice(s) and knowledge. They arise from the individual's and group's interpretations of cultural scripts influenced by social representations of mathematical knowledge in relation to people having and using it. A sociomathematical norm explicitly stated by a teacher could be, for instance, 'In this class we work collaboratively and people must help each other'. When stating it, the teacher resorts to his/her understanding of an appropriate way of working in the mathematics classroom, which may come, for instance, from the collective image of a particular school culture. When putting into action this norm, the teacher has to take decisions about how to organise the students in small groups, and in doing so, s/he is borrowing meanings and values from the cultural scripts and social representations of a particular group. S/he may decide, a-priori, that a gypsy student does not need help when doing arithmetic. S/he may not know the student very well, but 'it is common knowledge' among mathematics

teachers that gypsy students are 'not so bad' at mental arithmetic. On the other hand, the gypsy student also has to tackle this norm. S/he is to agree or disagree, explicitly or silently, with the grouping suggested by the teacher. Her/his agreement will be built on the basis of how much help s/he feels that s/he needs. Her/his feeling about needing help is intimately linked to the identity that s/he is developing as a mathematics learner, which is also shaped through her/his understanding of how society values her/his cultural group as doers of mathematics.

Another sociomathematical norm could be 'In this classroom everybody may contribute with ideas'. Again, the sociomathematical norm is linked to the cultural script of the institution. The immigrant student, when deciding whether s/he contributes with a different solution to a problem, may feel conditioned because of previous experiences that tell her/him that the contribution will not be accepted, since s/he has not been recognised as a valid mathematics interlocutor on other occasions. Too often, the fact that a student identifies her/himself with the community of mathematical practice, or the fact that the group accepts her/him within it, has little to do with his real mathematical abilities, but with the others' interpretation of the social representations of his group in relation to mathematical practices.

Note that we consider as sociomathematical norms some of the norms that in Cobb's system would be regarded as social norms. We agree with Cobb and Liao Hodge (2002) that mathematics teachers, as well as history teachers and science teachers, may want students' participation. However, norms about participation in the mathematics class have other meanings and consequences than the same norms in history or science classes. The way teachers conceptualise the learning of mathematics constrains the prevalence of one norm over another. When establishing, for instance, 'who needs to work with whom' or 'who can benefit from a particular participation structure', mathematics teachers base their decisions on their conceptualisations of what teaching and learning mathematics is about. Their conceptualisations are unavoidably shaped by cultural scripts, social representations and valorisations of mathematical practices and of social groups in relation to mathematics.

We regard norms of the mathematical practice to be the norms that legitimate the mathematical activity, strategies, processes and certain ways of thinking within the classroom. They have to do with the rules and ways of doing of mathematics as a scientific discipline, and with how teachers and students interpret mathematics as a school subject. When teachers decide whether a content, procedure, task or strategy is appropriate as school mathematics, they borrow their meanings from the culture of the groups they are part of, be it an innovative association of teachers of mathematics, or a group of mathematicians educated in a certain way. They also borrow their meanings from the culture of the educational system and from their particular school cultures. The official intended curriculum, the syllabus and the textbook also convey to teachers cultural scripts of what constitutes school mathematics. Students interpret what mathematics is about through the lens of the culture(s) they have participated in

throughout their lives, be it the classroom(s) culture(s), the school(s) culture(s) or, at large, their home culture.

A norm of the mathematical practice explicitly stated by a teacher could be, for instance, 'In this classroom, a visual strategy is also a proper strategy to solve a problem' The teacher may have a particular understanding of mathematics, while her students may think that visual strategies are not 'proper mathematics' because they have never seen such a strategy used before. Another teacher may prefer an approximate solution while her students believe that 'exact' answers are more 'real mathematics', because this is part of their cultural scripts of what counts as mathematics, scripts brought from home or from their previous school history. Again, note that we would consider as norms of the mathematical practice those relating to what constitutes mathematical evidence, a good hypothesis, or a good explanation, norms which according to other authors would be regarded as sociomathematical norms. We would also like to make clear that, although in the examples that we have presented it is the teacher who explicitly states the norms, very often norms are established in implicit and less clear ways.

NORMS AS CULTURAL ARTEFACTS MEDIATING CLASSROOM INTERACTION

We understand norms as being secondary cultural artefacts as defined by Cole (1996). Sociomathematical norms are shaped by cultural schemas, representations and valorisations of mathematical knowledge and its ownership. They regulate and legitimise interactions and communication processes of mathematical practice. Norms of mathematical practice, as interpretations of cultural schemas about what mathematics in schools is/should be about, regulate the content of practice as legitimised within the classroom.

We consider a mediator to be an agent interposed in a process of change that can affect its path, either facilitating or hindering it. In Gorgorió and Planas (2005) we present our interpretation of how norms, shaped by social representations, act as mediators. Classroom practice and interactions, seen from the perspective of the individual, borrow their meanings from the social. Norms contribute to give shape to the way a person or a group makes sense of the mathematical practice, interactions and communication acts. Social valorisations of mathematical practices and groups shape the value individuals attribute to one another and to the knowledge they exhibit. Like Abreu and Elbers (2005), we view social mediation as an active process that occurs when individuals or groups are influenced (e.g., in their thinking, acting, feeling or identifying) by cultural tools, or individuals or groups resort to cultural tools to influence one another (e.g., when communicating or orchestrating interactions). The analysis of the mediational role of norms addresses issues around who (appropriately) participates, whose participation is (not) welcomed, and the different roles played by individuals within the mathematics conversation. The

mediation takes place in a multilayered way. It is not only the norms that shape the action, but the unique re-interpretation made of them by individuals.

In our reconstruction of the idea of norms, it is important to consider the interplay between cultural scripts and social representations. We agree with Cole (1996) that 'while culture is a source of tools for action, the individual must still engage in a good deal of interpretation in figuring out which schemas apply in what circumstances and how to implement them effectively' (p. 130), and that 'in order to give an account of culturally mediated thinking it is necessary to specify not only the artifacts through which behaviour is mediated but also the circumstances in which the thinking occurs' (p. 131). It is when taking into account these circumstances that we have to resort to the idea of social representations. The global scenario of our research is that of immigration in a country where, until recently, the only shared meaning for 'foreigner' was that of a tourist. When foreigners are no longer only tourists, and local people feel their rights and privileges at risk, having no scripts to orient action and interaction in the new multicultural situations, it is only through social representations that individuals can make meaning of those situations.

Representations coming from the educational institution and from the whole society that host the minority groups shape norms. Immigrant students, most of them socially at risk, tend to be stereotyped as less competent and their mathematical abilities have traditionally been considered from a deficit model approach. Therefore, in-transition students and their practices are more prone to be valued negatively due to a-priori assumptions socially constructed and this valuing interferes with the orchestration of the norms that should facilitate, or at least allow, their participation. The difficulties that immigrant students encounter when they are to understand and use 'new' norms may not lie only in their novelty, but also in the fact that norms are not neutral. To what extent do norms, as cultural artefacts of the dominant group(s), have as a possibly unintended effect the continuing of the culture and the social positioning?

FINAL REMARKS

Norms, being elements that regulate classroom action and interaction, are at the very basis of classroom discourse (see, for instance, Cobb & Liao Hodge, 2002, for a theoretical argument; and Planas & Gorgorió, 2004, for an empirical analysis). Understanding norms as constituent elements of discourse, they become valuable constructs in our empirical work to analyse the complexity of the multiethnic mathematics classroom from the complementarity of a cultural and social perspective. However, up to what point is it valid to change the interpretation of a theoretical construct, such as norms, without changing the word that represents it? Cobb (1999) already suggested the need to complement the 'classic view' about norms with a sociocultural perspective that places the classroom in its social context. More recently, Cobb and Liao Hodge (op. cit.) referred to their interpretive perspective 'as provisional and eminently revisable, particularly in response to empirical analyses' (p. 278).

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FORMING TEACHERS AS RESONANCE MEDIATORS

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Assumptions about knowledge construction, knowledge transmission and the nature of mathematics always underlie any teaching practice even if often unconsciously. In the paper we explain our theoretical assumptions about these cognitive and epistemological issues and derive from them a "model" of teacher. Finally we discuss why and how participation in a modelization process can constitute a suitable strategy for disciplinary and professional training of future teachers conforming to the model.

INTRODUCTION

In this paper we intend to bring into discussion a handful of theoretical remarks that we see as meaningful and relevant for the actual development of research on the crucial problem of teachers training in mathematics areas[1]. Our focus will be in particular on the search for a reasonably satisfactory model of cognitive dynamics, adjusted to non-specialistic knowledge levels: in fact, such a model plays a key role in both teachers formation and the teaching process. In planning, handling and evaluating teachers formation paths we recognize, very schematically, at least four basic "model-ingredients"[2]: i) a realistic, even rough, model of "natural" cognitive dynamics ii) a global, epistemologically founded view of mathematics as an internally structured scientific discipline; iii) a modulated view of the variety of interferences of mathematical thinking with other cultural fields (mainly scientific and technological ones), and with everyday culture(s); iv) a pragmatically successful, even rough model of cultural transmission in knowledge areas, in particular scientific ones. Such ingredients, obviously crucial to teaching profession, are obviously correlated to each other: in particular it is clear the basic framing role assumed by i) with respect to other aspects.

A distinctive feature of our cognitive modelling is the core relevance of basic resonance dynamics[3] assumed to work at the root of all the modulations (from

[1] The actuality of this debate is witnessed for example by the argument of the forthcoming ICMI study 15 (ICMI, 2004).

[2] But see also for example (Malara, 2003), where other influential variables like *emotion* and *belief* are put in evidence.

[3] We borrow the word *resonance* as a Physics metaphor: and in this sense resonance is actually much more than consonance. In his/her proposals, each teacher in fact will reasonably take into account the consonance of mathematical tools (equations, graphs, examples, an so on), employed in a particular context, in such a way that their notes are not played as *dissonant* with each other, and in reference to reality and thought structures. But when a complex interaction is driven by resonance dynamics, this implies that different

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 73-80. Melbourne: PME.

perception to abstract thinking) and interferences characterizing the knowledge of an individual. In our view, in fact, such dynamics always play the central role on the frontier of the progressive adjustment/fit between different, ever-present dimensions: the actual potentialities of individually developing cognitive structures, the framing patterns supplied by implicitly as well as explicitly codified cultures, the constraints of physical (at large) reality. The aim of the present work is therefore a twofold one: a) to discuss about the productivity of looking at the teacher's main role as one of resonance inducing mediation, on both planes of understanding and of motivation to understanding; b) to point out that critical awareness and a responsible assumption of such a role can actually be developed and supported by suitable teachers formation strategies, in turn resonance-exploiting vs individual learning experiences as well as resonance-emphasizing vs cooperative professional formation.

The next sections are organized according to the following path. After presenting some general theoretical references, we illustrate our "model" of the understanding process. From this we derive the necessity of characterizing the role of teachers as resonance mediators. Then we discuss some aspects of teachers formation strategy finalized to this purpose. Finally an example of teaching practices/events will be presented, in order to illustrate, as a conclusion, the potential impact and possible outcomes of our assumptions.

OUTLINE OF A GENERAL THEORETICAL FRAMEWORK

Before stressing in the next section some relevant "resonance features" characterizing our cognitive modelling, we feel it necessary to very briefly state its location/rooting within the complex landscape of the cognitive theories and interpretations available nowadays. In particular we would like to draw attention to the fact that many critical aspects of cognition have been variously noticed and variously entangled along times within different (often reciprocally contrasting) cognitive theories and/or epistemological positions. Actually, our basic research finding is that most of such aspects appear relevant in interpreting experimental teaching/learning evidences: and this directly implies their reciprocal complementarity. For example:

It is now quite common to refer to Vygotskij's views about the crucial role assumed by natural language in mediating natural culture, and viceversa, since the earliest ages. However such a mediation is only in a minor part an automatic, passive one: careful observation of cognitive transactions shows that strong resonance/ dissonance effects always take place on the fuzzy background of "implicit acculturation", and that an early, careful, active adult mediation plays a key role in fostering resonances and preventing dissonances ("misconceptions" appear at most as the result of missing/wrong/misleading mediations between developing cognition and culture).

elements (in cognition: an idea, a mental construct, an image, an action, etc.) are simultaneously activated whenever one is evoked, producing by reciprocal interference a mutual reinforcement effect. Resonance dynamics are obviously at work also along the paths of cultural evolution at large – but this is an even more complex subject.

Apart from the "stages" machinery, some insights by Piaget appear to be crucial to outline features of cognitive dynamics. Assimilation, accommodation, (temporary) equilibration… lively define the main modes of any resonance process, adjusting to each other partially mismatching external and internal terms, and recognizing as such the resulting reciprocal fit. Something similar, though in a "divergent" modality, correlates the dynamics of the "physical abstraction" to the ones of the "reflecting abstraction", ending up with separate though entangled scientific and formal models. Cognitive activity is, exactly, an activity – structured according to possible, effective, meaningful … internal actions, monitored by internal "convergence sense" and counterpointed by external action and discourse.

There is evidently no sense at all to counter Vygotskij's views to Piaget's: both their ways-to-look-at cognition account in fact for crucial aspects of what actually happens. The point is to correlate such views within a comprehensive dynamical model: and the resonance dynamics frame actually lends itself to account for many crucial correlation aspects. Something very similar can be said about most of the presently debated views about mathematically relevant cognitive structures: from the "embodied cognition" ones (Lakoff & Núñez, 2000), naturally referring to some neurocognitive studies (Changeux, 2000), (Dehaene, 1997), to the "language referred" (Sfard, 2000) or more generally "semiotic" ones (Radford, 2000), to the "information processing" ones … and so on. It is evident from our research experience, and it can be shown by careful analysis of learning experience, that all these ingredients appear to be crucial in some respects within resonantly converging dynamics of meaningful learning.

LEARNING THROUGH UNDERSTANDING

Our basic theoretical assumption about "learning through understanding" has been developed and refined, within the above wide theoretical framework, managing multi-year, multi-classroom action-research projects mainly devoted to a coherent reorganization of both school teaching and formation activity for pre-elementary and elementary in-service teachers in science-mathematics area[4]. We briefly synthesize as follows the main points of our model of cognitive dynamics and knowledge transmission[5]:

[4] We are mainly referring to the long-term Italian project *Capire si può* (*It is possible to understand*) partially documented in http://www5.indire.it:8080/set/capire_per_modelli/capire.htm

[5] Correspondent views of mathematics as discipline and as educational task are implicit in our model, too. The prevailing view of mathematics today, pervading most curricula, and in particular university curricula for future teachers, stressing its **apriori separateness** from other scientific areas, conflicts with natural cognitive processes, and appears to be at the origin of many students' difficulties. On the contrary, if mathematics is conceived as an **aposteriori abstraction** coming for example through a modelization process, its "cognitive" resonance stimulates students motivated interest toward its structural development and allows them to reach quite high levels of formalization.

a) Understanding, as different from learning, and motivation, as different from acceptance, are strictly correlated, for pupils as well as for teachers: based on feelings and feedback of competence in dealing with increasingly complex situations.

b) Learning through understanding is the result of a process of resonance between individual cognition, social culture and reality structures, along cognitive paths efficiently addressed and controlled in their meaning-driven dynamics. It requires, at any level, also resonance between various "dimensions" of *natural thinking* (Guidoni, 1985): perception, language, action, representation, planning, interpretation, etc.

c) Learning through understanding requires long-term, longitudinal along years, and wide-range, transversal across disciplines, processes, mediated and supported by a simultaneous development of language, with increasing awareness of usages and functions of all its components (syntax, semantics, pragmatics, semiotics).

Two kinds of activities appear as critical, self-developing keys for both teachers formation and pupils learning: modelization processes from everyday experience contexts and word problems.

As noticed in (Verschaffel, 2002), in mathematics education literature word problems have been used in many ways and with several different goals, dating back from the classical proposals by Pólya (Pólya, 1962). The same applies to modelization, where, moreover, the meanings themselves assigned to the word "modelization" considerably vary. Therefore, it is necessary to begin with explaining what "modelization" means for us (contrasting for example Verschaffel's definition), (although the example in the last section will better clarify our own point of view): we interpret it as a very complex, neither deterministic nor a one-way process where the formal structures are seen as one of the different correlated ways into which the cognitive reconstruction of external world structures take form. In other words what really counts is not a standard hierarchy of multi-representations (actions, words, graphs, and so on) whose top is identifiable by the algebraic formulation of a physical law: due to the subtended cognitive dynamics, what is most effective is a *continuous* (quasi subliminal, in expert situations) *shifting from one cognitive dimension to another in a mutual progressive enhancement.*

As for word problems, for reasons of space we will limit ourselves to the restrictions which make them appropriate for our goal (for further details see Guidoni *et al.*, 2003; Tortora, 2001): they ought to be problems which could be tackled in parallel through the use of symbolic tools and direct action, led in such a way as to illustrate that the solution process converges if sustained by a multiplicity of representations (in particular graphic: see Guidoni *et al.*, 2005) brought along by the individual cognitive dynamics themselves, and thus, reciprocally enlightening. The final task is extracting their structures which requires a double "variation of the subject": one is the varying of the numerical data in the same context; and one is facing "isomorphic" situations, recognizing them as such (for an analogous detailed analysis see Mason, 2001).

Therefore, it is possible to understand, and the attention devoted to the mechanics of understanding takes priority over the mechanics involved in non- understanding. However, on the condition that adult mediation be sufficiently flexible and incisive in order to generate the conditions which set off and fuel the comprehension process: assigning a crucial role to teaching mediation as *resonance inducing*.

TEACHER'S ROLE AND ITS IMPLEMENTATION

"*Pick them up where they are, then find a path which guides them to the place you want them to reach*". According to this famous Wittgenstein's mot, a teacher must manage, among other things, specific skills. This implies defining the space of cognitive configurations (multi-dimensional) and, based on available resources, the designing of possible learning trajectory paths. Overall to adopt teaching strategies that are progressive, coherent, not imposing but supportive of potentialities. The teacher should create, on a local level, the many possible links between individual cognition, social culture and reality structure through the use of dynamics of abstraction and de-abstraction (modelling and de-modelling) with coherence, flexibility and competence.

In working with in-service teachers (as in above mentioned projects – see footnote 4), since teachers and students are simultaneously involved, the critical awareness and the assumption of the teacher role are supported by the immediate and long term interaction with the learners cognitive processes. For the teachers in training this important support is missing, making the development of the above mentioned skills ever more complex. However, five years of research on our part in the formation of elementary school teachers in training, based on conceptual paths and didactic strategies gradually validated, have convinced us that in each case the guided collective participation in modelization or problem solving processes makes up a privileged entrance into the world of the combined acquisition of knowledge and professionalism. Here, we limit ourselves to illustrating through a single example, underlining some aspects which systematically emerge in the modelization process, the way in which the theoretical hypothesis of resonance mediation in cultural transmission is effective when put into practice in the classroom. It is important to highlight that, in this as in other work contexts experimented in different environments, the underlying cognitive dynamics put into play are very similar between both working in-service teachers and pre-service teachers in training, and in substance correspond to what takes place in class; likewise the crucial role played by a meta-cognitive attitude is analogous both on an individual and group scale. We have also noticed that it is important in all situations to alternate auto-directed work of manipulation and interpretation either individually or in small groups (including substantial homework) with collective guided work of comparison and analysis of partial results, yet leaving to the individual the final systemizing of results and interpretation of the processes being adopted.

The choice of the context to be explored is always addressed by some conditions. It is in fact important to deeply commit to the task both conscious perception/action, and the construction of elementary logical relationships to support in integrated way the complexity of the experience. In other words, the context must lend itself to approaches characterized, since the beginning of the cognitive path, by direct manipulations guided by reflection on what is being observed: a context at the same time complex enough to demand a careful, previous individuation of interacting systems as of pertinent variables, and simple enough to allow for an exploration not too rigidly guided.

AN EXAMPLE AND SOME CONCLUDING REMARKS

According to our experience, **springiness** can be a good example of a prototypical modelling context. As a matter of fact, the great pervasiveness of this family of phenomenologies within everyday experience is even marked by the corresponding structures of the natural language (in Italian, the adjective "elastic" is commonly used as a noun); while "what happens" to a stressed elastic object, together with the basic rules of such "happening", is first understood since childhood on the basis of direct bodily reference. As a consequence, a lot of pertinent thought-action-wording-representation aspects are actually available as crucial, interfering dimensions (not steps!) to support development and structuring of both phenomenological and formal competences, on the basis of an explicit restructuring of "what one already knows". In particular, it is important to remind some important aspects (not steps!) of the basic cognitive path: a rich, explicit, qualitative analysis of the behavioural patterns of different springing objects is always necessary: both to support more and more sophisticated modelling, and to correlate modelling itself to less and less evident springiness phenomena and features; once a general pattern of "forcing" is reasonably well controlled, and well represented by natural language, correlations between configuration variables, and between variables and systemic parameters, can be explored: first by "order relations and correlations" (the more… the less…; the more… the more…; the less… the less…; etc.); then by actual measuring what can be directly or indirectly measured, in properly arbitrary units. Always under the control of natural language, aspects of the explored situations can then be represented by qualitative line drawings, then by tables of number-pairs, gradually allowing also for (cautious) prediction of new facts.

As always, meaning really emerges from "*what is unchanged across change*" (Plato). To make cognitively explicit that here what does not change across systems and situations is a "form" of the relationship among pertinent variables and parameters, it is crucial that the relationship itself becomes explicitly and coherently multi-represented: by careful words; by symbols, standing for variables and parameters and their relationships; finally, by systematic use of the Cartesian plane. And the resonant interference of different representation features appears to allow for effectiveness and stability of the understanding.

Here is an excerpt from a "final" account of three pre-service teachers working with different rubber springs, differently configured in different stretching situations. From the initial "confusion" of random testing, a clear "rationalization" of observed patterns starts to emerge: not as a passive check within apriori imposed schemes and procedures, but as an active "reducing to stable, workable, schematic order" of the world observable variety.

"By comparing the stretching of single, serial and parallel rubber springs, we have observed, in spite of the inaccuracy of data, a piece of straight line in the central region of the broken lines. From this we have inferred that springs behave the same way in all three cases: a constant, direct proportionality shows up between the number of coins (or more in general the weight of the objects utilized) and the actual stretching. Let's represent what happens when y=x/10, where y corresponds to F (the force is the weight of the coins), x is $l-l_0$*, and 1/10 corresponds to k… In conclusion, we cannot rely upon experimental data alone: the proportionality between the two variables can be seen when the error margin is small enough, and anyhow in the central part of measurements (when weight is neither too small nor too large)".*

This way, one is giving sense at the same time to the linear function and to the physical law; to the intuition that the proportionality between force and stretch is but a part of a more complex behaviour, in both physical and formal terms; to the generalization of a "partially linear" behaviour as a powerful key of first interpretation; and so on. The (suggested) comparison/superposition of several graphs by overhead projection has then allowed to see, in the central part of the graphs, a real "linear cord" stemming from the merging of about twenty "broken lines": this simple contrivance incisively (resonantly) emphasizes both the abstraction process leading to the linear function, and all the "impediments" (in Galileo's words) which, from different origins, interpose themselves between the formal scheme, the reality of the spring and the practice of measurement.

"$F=k(l-l_0)$*. If I trust the formula, I will look for a straight line at any cost, trying even to transform a broken line in a straight one. On one side we have mathematics, on the other side we have phenomena: how putting them together? The mathematical straight line is an abstraction. The spring of physicists cannot be identified with our elastic band, it looks like a straight line more than our graphical "cord" does. To understand, I've used both things: mathematics and phenomena. If I don't know the law, all springs are different, and I cannot identify the small region of their common behaviour. What is a spring? We succeeded in defining by words a real spring, the physicists' spring has its origin in the real ones, but is represented by a mathematical formula:* $F=k(l-l_0)$*".*

From this example it is also emerging the meaning we attribute, within a reciprocally resonant convergency between phenomenology and formalism, to a cognitive process of "de-constructive de-modelling" seen as parallel and interfering to the one more commonly evoked of "constructive modelling". A continuous back-and-forth between what is "said" by facts and by their symbolization appears to be crucial. In particular we feel it important that contributions quite different in their physical

origin, whose superposition determines phenomenological observations and measurements (intrinsic not-linearity of the deformation function, within its power expansion; deformations at the boundaries of the observed interval resulting from physical behaviour and measurement processes; measuring inaccuracies and errors; etc.) are not acritically merged and confused below the cover of an all-purpose "formalization", at this point very poorly significant in its not transparent stiffness. And it is evident that along a critical, guided sharing and comparison of the research results new phenomenological doubts can emerge, together with linguistic and formal discrepancies and further cultural needs: altogether defining the action space for further systematization levels.

A question is still open: how many of our students shall assume in their teachers life the proposed role, and how many shall turn back to old, reassuring models?

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INVESTIGATING THE PROBLEM SOLVING COMPETENCY OF PRE SERVICE TEACHERS IN DYNAMIC GEOMETRY ENVIRONMENT

Shajahan Haja

This study investigated the problem-solving competency of four secondary pre service teachers (PSTs) of University of London as they explored geometry problems in dynamic geometry environment (DGE) in 2004. A constructivist experiment was designed in which dynamic software Cabri-Géomètre II (hereafter Cabri) was used as an interactive medium. Through participating in the study, the PSTs demonstrated their competency exemplified by using their subject knowledge to construct, to make conjectures and verify their conjectures on their way to solve the given geometric problems.

INTRODUCTION

Being a mathematics teacher educator I felt the gap between the technology infused learning experiences in geometry class and the current status of teacher preparation programs that in general pay little attention to this aspect. For a solid prescription, I needed an empirical data involving PSTs and DGE. The first issue that stroke me was: How competent are the PSTs in using subject knowledge to solve problems in DGE? I decided to pursue my investigation to find answer for this question.

Related Studies

There exists very few experiential data to inform us about the competency of PSTs in using DGE. Saads and Davis (1997) investigated the van Hiele levels in three-dimensional geometry and the spatial abilities of a group of secondary PSTs and found that PSTs' language depends on a combination of the PSTs' general geometric level, their spatial ability and their ability to express the properties of the shape using language. Pandiscio (2001) studied how secondary mathematics PSTs perceive the need for and the benefits of formal proof when given geometric tasks in the context of dynamic geometry software. Jiang (2002) observed the actual learning processes of two PSTs as they explored geometry problems with dynamic geometry software and the effects of using the software on developing their mathematical reasoning and proof abilities.

'Knowledge-in –Action' Design

The aim of the study is to find out the PSTs' problem solving competency in DGE. I developed a 'Knowledge–in–Action' Design (Fig 1) based on geometrical constructions in Cabri using dragging and other functions to illustrate the problem solving process of PSTs. The process of converting the conceptual understanding of the given geometric problems into action demands the PSTs to apply their content

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 81-87. Melbourne: PME.

knowledge to understand the given problem, construct the dynamic figures, make conjectures, verify the conjectures, and solve similar problems. Accordingly the competencies are defined as follows:

- Competency 1: PSTs construct figures in DGE.
- Competency 2: PSTs make conjectures in DGE.
- Competency 3: PSTs verify their conjectures in DGE.

It is been hypothesized that if the PSTs are able to solve the given problem using their subject knowledge by constructing, conjecturing and verifying the properties in DGE then their competency is testified. The study builds on the assumption that learning involves the active construction of knowledge through engagement and personal experience (Ernest, 1994; von Glasersfeld, 1995). The Van Hiele theory of learning geometry also provided a useful conceptual framework for this study.

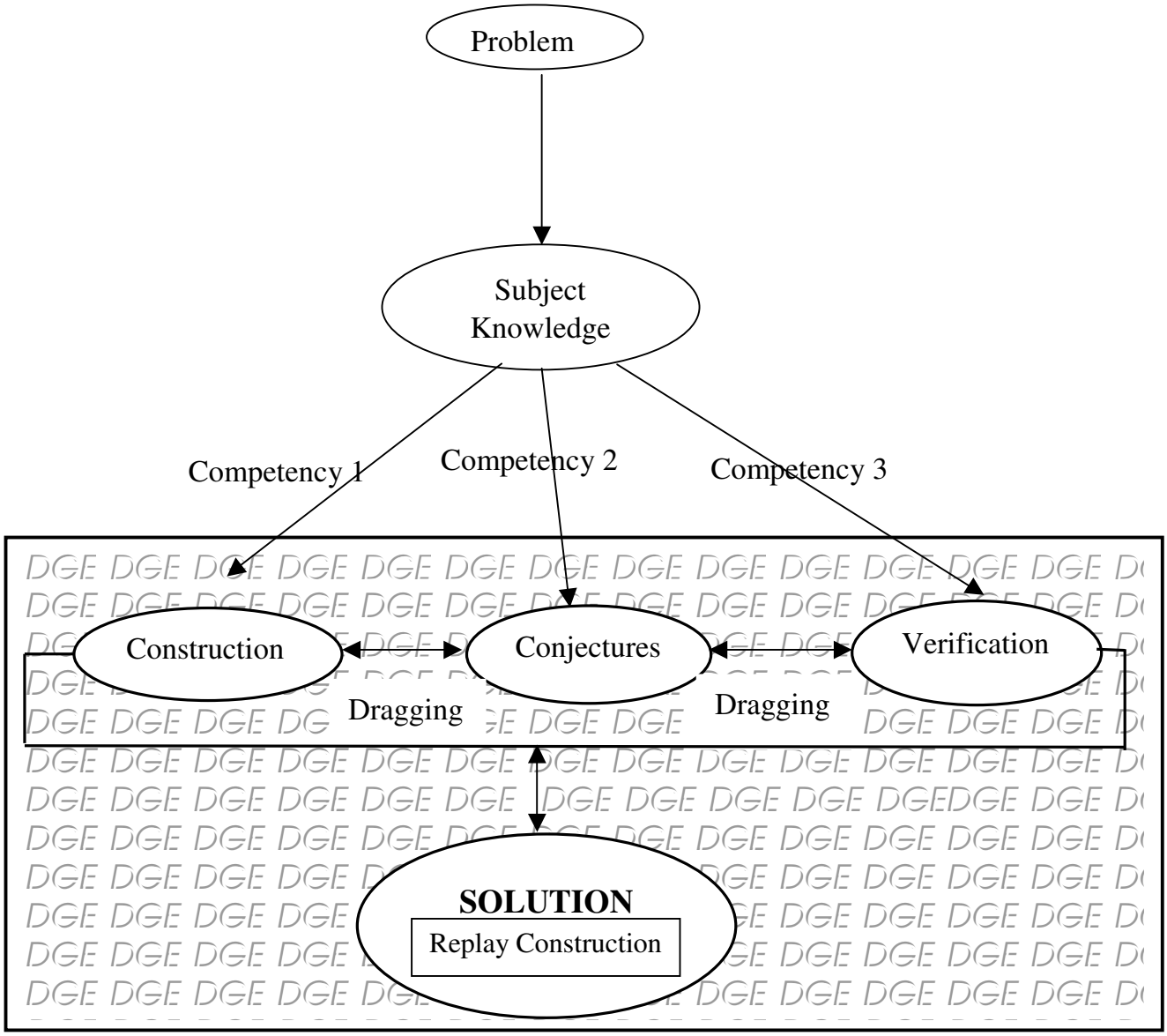


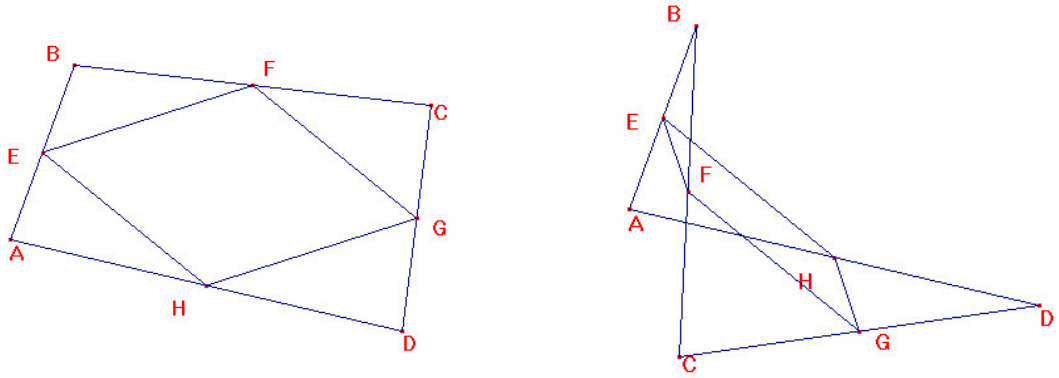
Figure 1: Knowledge-in-Action Design.

METHODOLOGY

The subjects of this experiment were four secondary PGCE mathematics students (3 males PST1, PST2, PST3 and 1 female PST4) of University of London. None of the PSTs had used Cabri before taking part in this experiment but they had studied geometry at the high school/O level. The experiment was conducted on one- to-one basis on different days in January 2004 at the Institute of Education, London. The experiment was recorded.

Constructive Experiment: Discussion

The constructive experiment designed for the study is an open-ended problem involving Varignon's quadrilateral and medial triangle. I analyze the Varignon's quadrilateral using non - conventional quadrilaterals (crossed/ bow tie) in DGE background, which is otherwise not possible with pencil and paper. Consider a quadrilateral ABCD (figure 2a) with the midpoints of its sides E, F, G and H. In the experiment, PSTs conjectured that EFGH is a parallelogram. By dragging the vertices of the quadrilateral ABCD they observed that EFGH remained a parallelogram no matter what type of quadrilateral ABCD was. And this result is valid for a crossed quadrilateral too (figure 2b). Also the area of the Varignon parallelogram EFGH is exactly half of the area of the original quadrilateral ABCD; even in the case of the crossed quadrilateral.



(2a) **(2b)**

Figure 2: Varignon's Quadrilateral.

Now let us consider three-sided polygon (n=3) in the above example; the quadrilateral is then replaced by a triangle. I see that the medial triangle formed by joining the adjacent midpoints is similar to the original triangle. PSTs explored this task with special cases: equilateral triangle, right triangle and isosceles triangle. Now, the midpoints of the sides of a triangle form a triangle DEF (see Fig 3) that is exactly one quarter of the area of the original triangle ABC.

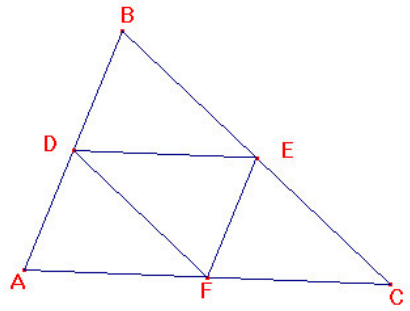


Figure 3: Medial Triangle.

Also the medial triangle is not only congruent to the other three small triangles (ADF, BDE, EFC) but also similar to ABC and the line segment connecting the midpoints of two sides of a triangle is parallel to the third side and is congruent to one half of the third side.

Now, how did the dynamic figures control the PSTs exploration? In paper-and-pencil geometry, all entities are static, but in dynamic geometry, they behave in a specific way to maintain their geometric properties. For instance, the vertices of quadrilateral ABCD are completely arbitrary and can be moved freely, while E, F, G and H are fixed as midpoints that cannot be altered. This is the same with triangles as well. For instance, in the right triangle the perpendicularity of two lines controlled the activities of PSTs.

Exploration by PSTs as Replay Construction – The Road to the Solution

In this experiment, the *Replay Construction* function in the edit menu of Cabri describes the construction process when a convincing solution was achieved by a set of physical manipulations by the PSTs. The following two illustrations describe step-by-step how each PST works.

Illustration 1: Medians of Isosceles Triangle

All the four PSTs were able to construct the isosceles triangle using different conjectures. PST 1 & PST3 used angle bisectors to construct the triangle while PST2 used the conjecture (*perpendicular bisector of a chord passes through the centre of the circle*) & PST4 used (*isosceles triangle has two equal sides*) circles to construct the triangles. PST2 & PST4 used the radii of circles as the two equal sides of the right triangle (Fig 4).

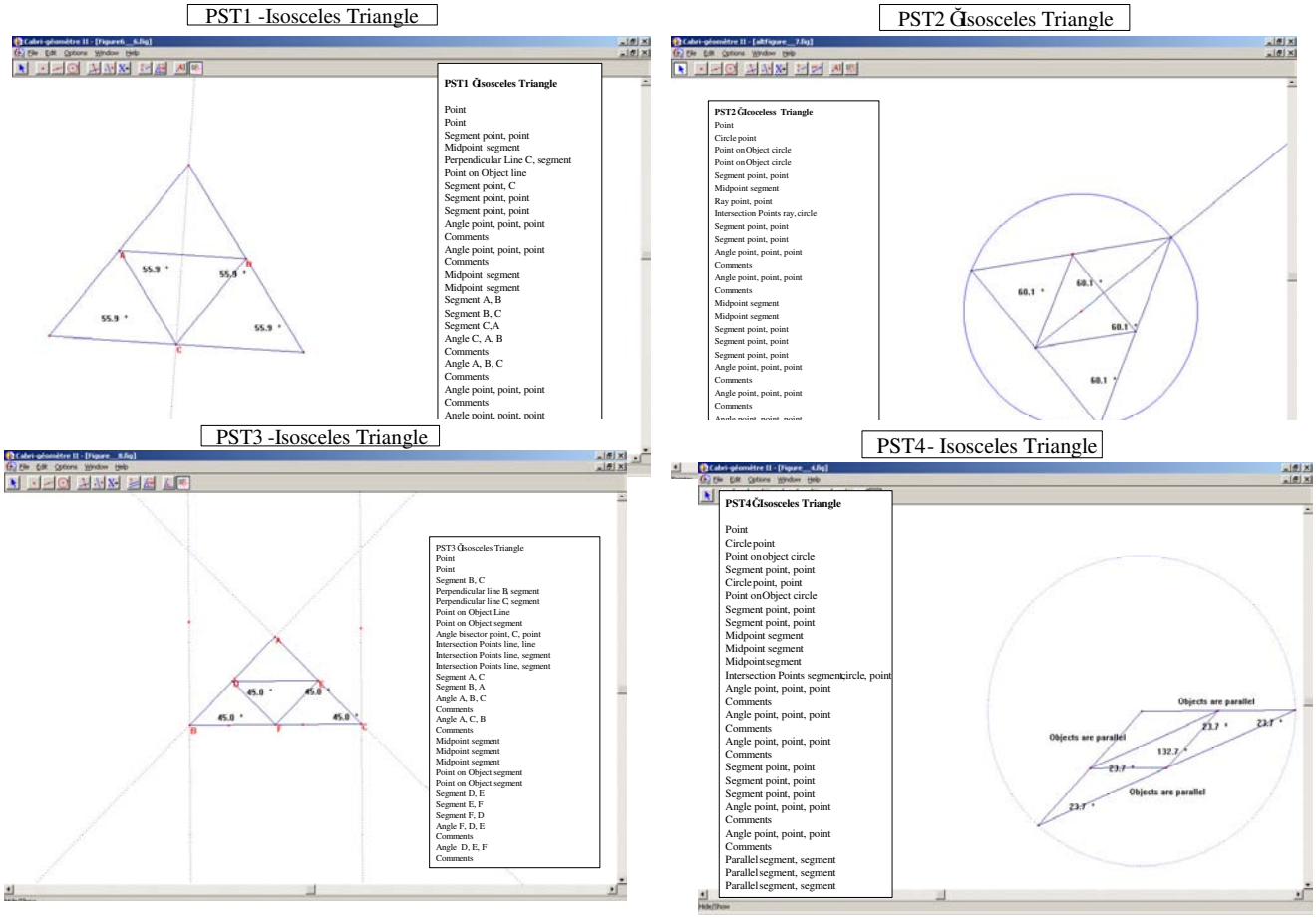


Figure 4: Medians of Isosceles Triangle.

Illustration 2: Medians of Quadrilateral

PST2 tried non-convex quadrilateral while others tried the convex. In either case PSTs found that the inner quadrilateral turned out to be a parallelogram (Fig 5). PST1 Used the conjecture *diagonals of a parallelogram bisect each other*. Others used the conjecture *the opposite sides of a parallelogram are parallel and opposite angles are congruent.* PST1 tried to compare the areas of inner and outer quadrilaterals by drawing the diagonals. He found that the sum of the areas of the four triangles was one-half of the area of the entire quadrilateral.

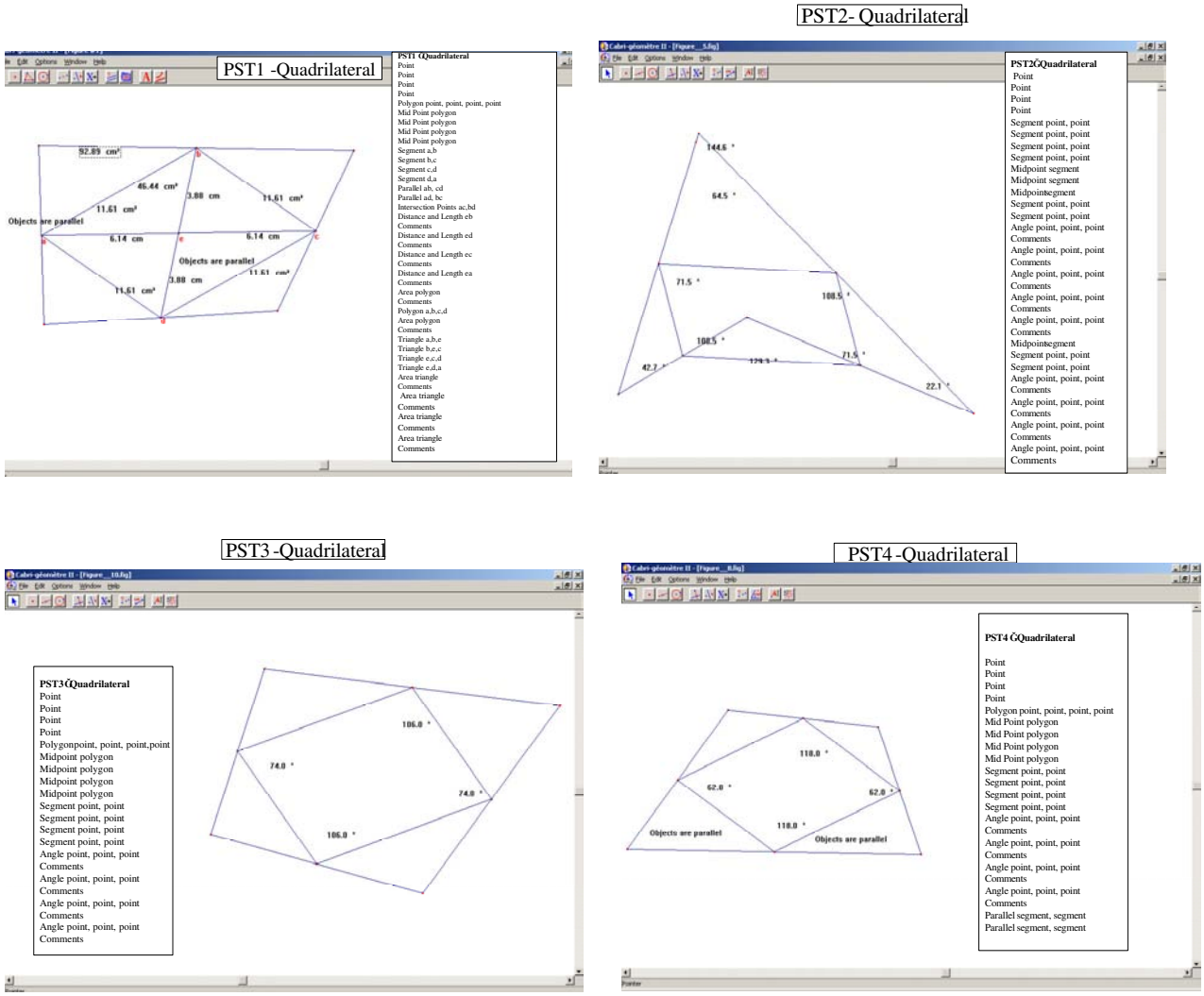


Figure 5: Medians of Quadrilateral.

CONCLUSION

In this experiment, geometrical facts about medial triangles and mid point quadrilaterals are examined by the PSTs. The emphasis was on tracing a solution to the given problem not a formal proof. As PSTs manipulated the dynamic figures, they could make conjectures about the properties of the shapes. Also, PSTs used on-screen measurements for side lengths and angles to transform their intuitive notions into more-precise formal ideas about geometric properties (Battista, 1998). For instance, PSTs used distance and angle measurement to check the geometric properties of medial triangles. I observed that PSTs are motivated to search for explanations driven by 'dragging' that guided them towards the solution. This is substantiated by PSTs' opinion about the experiential learning; to put it in their own words, '*This is a new experience and I thoroughly enjoyed it* (PST1)', '*It is wonderful that I could create my own way of finding a solution* (PST4)', '*I learned* (PST2)' and '*It is useful but time consuming* (PST3)'. Analysis of the PSTs solution process shows that learning took place in the form of construction, conjectures, and verification and thus validates the assumption of the study (Ernest, 1994; von

Glasersfeld, 1995). However the interaction with dynamic software can't be accepted as a formal proof as argued by Saads & Davis (1997). The problem solving behaviour exhibited by the four PSTs correlates with the Knowledge-in-Action design (Fig1) in terms of the components (construction, conjecture, and verification) described in the design.

The construction process showed that PSTs demonstrated adequate geometric reasoning in accordance with the hierarchical levels of Van Hiele viz., Visualization, Analysis, Abstraction, Deduction, and Rigor on the selected tasks. However, this can't be generalized to the PSTs understanding of other geometrical concepts (Mayberry, 1983). Observation of the problem solving processes of the four PSTs while they attempted to solve geometric problems with the help of Cabri suggests that:

a) PSTs' content knowledge are adequate to understand the given problems b) PSTs are competent enough to apply the content knowledge to construct the dynamic figures c) PSTs are competent enough to apply the content knowledge to make conjectures d) PSTs are competent enough to apply the content knowledge to verify the conjectures and e) PSTs are able to use DGE to justify their solution.

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STRUCTURE AND TYPICAL PROFILES OF ELEMENTARY TEACHER STUDENTS' VIEW OF MATHEMATICS

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The elementary school teachers' view of mathematics is important because it will influence the way they will teach mathematics. Based on a survey study in three Finnish universities we explored the structure of student teachers view of mathematics and also the different belief profiles that they had. The core of student teachers' view consisted of three correlated beliefs: belief of one's own talent, belief of the difficulty of mathematics, and one's liking of mathematics. Five other beliefs were also identified. Gender, grade, and mathematics course selection at high school each contributed to the variation in these factors. A cluster analysis produced three main types of belief profiles: positive, neutral, and negative view. Each of these was further divided into two subclasses.

INTRODUCTION

Elementary school teachers' competences in and view about mathematics have a profound significance, because they are the first ones to teach mathematics to children. A negative view can seriously interfere their becoming good mathematics teachers, unless they can either overcome their anxiety or find constructive coping strategies. Fortunately, efforts to promote positive view of mathematics in elementary school teacher students have often proven at least partially successful (e.g. Kaasila, 2000; Liljedahl, 2004; Pietilä, 2002; Uusimäki, 2004).

In our research project "Elementary teachers' mathematics", our aim is to explore the belief structure of teacher students and how it develops in three universities that use different approaches to promote positive affect among students. This report will focus on revealing the structure of teacher students' view of mathematics at the beginning of their studies and identifying their typical belief profiles.

The three basic elements of the human mind are emotion, cognition and motivation. Respectively, we need to pay attention to the students' feelings, beliefs, and wants as the elements of view of mathematics. The feeling aspect of the view of mathematics consists of the emotions one experiences while doing mathematics. However, in any survey study it is only possible to find about expectancies or memories of these emotions. For the term "belief", there is no single, exact definition. Here, we shall define beliefs as purely cognitive statements to which the holder attributes truth or applicability. According to this view, beliefs do not include an emotional component, although a belief can be associated with an emotion. The aspect of motivation, or wanting, relates to the goals and desires one has. (Hannula 2004).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 89-96. Melbourne: PME.

View of mathematics has a structure. We can distinguish between one's view of different objects, such as 1) mathematics education (mathematics as subject, mathematical learning and problem solving, mathematics teaching in general), 2) self (self-efficacy, control, task-value, goal-orientation), and 3) the social context (social and socio-mathematical norms in the class,) (Op't Eynde, De Corte & Verschaffel 2002). With regard to the social context, Op 't Eynde and DeCorte (2004) found later that the role and functioning of one's teacher are an important subcategory of it. The spectrum of an individual's view of mathematics is very wide, and they are usually grouped into clusters that influence each other. Some views depend on other ones, for the individual more important views. When discussing beliefs Green (1971) uses the term 'the quasi-logical structure of beliefs' which means that the individual himself defines the ordering rules. We assume that emotions, cognitions and motivations form a system that has a quasi-logical structure. The view of mathematics also has a hard core that contains the student's most fundamental views (cf. Green 1971: the psychological centrality of beliefs; Kaplan 1991: deep and surface beliefs). Only experiences that penetrate to the hard core can change the view of mathematics in an essential way (Pietilä, 2002).

Kaasila (2000) studied the school-time memories of elementary school teacher students (N = 60). He divided the students autobiographical narratives into five groups: 1) "*It was important to be the fastest to solve the exercises*" (15 %), 2) "*Mathematics provided AHA!-experiences*" (20 %), 3) "*I survived by learning by heart*" (9 %), 4) "*Mathematics was boring, I lost my interest*" (36 %), 5) "*I fell off the track*" (20 %). Respectively, Pietilä (2002) grouped elementary school teacher students (N = 80) based on their written responses into four groups: 1) "*Mathematics is challenging problem solving*" (13 %), 2) "*Mathematics is important and usually pleasant*" (36 %), 3) "*Mathematics is one subject among others*" (20 %), 4) "*Mathematics is difficult and unpleasant*" (31 %). These two qualitative studies suggest that 20 – 30 % of Finnish elementary school teacher students have negative view of mathematics at the beginning of their studies.

The first analysis based on the Fennema-Sherman self-confidence scale indicated that 22 % of teacher students in this study had a low self-confidence in mathematics (Kaasila, Hannula, Laine & Pehkonen, 2004). Similar problems have been identified in other countries as well (e.g., Uusimaki & Kidman, 2004; Liljedahl, 2004).

METHODS

The research draws on data collected on 269 trainee teachers at three Finnish universities (Helsinki, Turku, Rovaniemi). Two questionnaires were planned to measure students' mathematical beliefs and competences in the beginning of their studies.

The 'view of mathematics' indicator consisted mainly on items that were generated in a qualitative study on student teacher's mathematical beliefs (Pietilä, 2002) it also included a self-confidence scale containing 10 items from the Fennema-Sherman

mathematics attitude scale (Fennema & Sherman, 1976), four items from a 'success orientation' scale found in a study with pupils of comprehensive school (Nurmi, Hannula, Maijala & Pehkonen, 2003) and some background information about earlier success in mathematics and experience as a teacher.

The mathematical skills test contained altogether 12 mathematical tasks related to elementary level mathematics. Four tasks measured understanding of some key concepts and eight tasks measured calculation skills. The questionnaires were administered within the first lecture in mathematics education studies in all universities in autumn 2003. Students had altogether 60 minutes time for the tests and they were not allowed to use calculators.

For a principal component analysis we chose 63 items of the ‘view of mathematics’ indicator. The topic 'experiences as a teacher' was excluded, because almost half of the respondents had not had enough experience as teacher to answer these questions. We chose to use Maximum likelihood method with direct oblim rotation and examined a number of different solutions. A more detailed description of the method can be found in another article (Hannula, Kaasila, Laine & Pehkonen, 2005).

When a solution of principal components was found, the structure of the components was analysed. We calculated the correlations between components and also the effect of main background variables: gender, the selected course and grade at high school mathematics.

After extraction of principal components, a cluster analysis of the data was made in order to find different student teacher profiles. For the cluster analyses we chose 8 principal components of the student teachers' view of mathematics as the variables for clustering, squared Euclidian distance as measure, and Ward's method to determine clusters. Cluster solutions of three to seven clusters were examined.

RESULTS

Principal component analysis

In the principal component analysis we chose to use the ten-component solution, which yielded high alpha values for the components (Table 1). Two of the components had only two items loading on them and they were not included in further analysis. A full description of all components and their loadings can be found in Hannula *et al*. (2005).

Structure of student teachers' view of mathematics

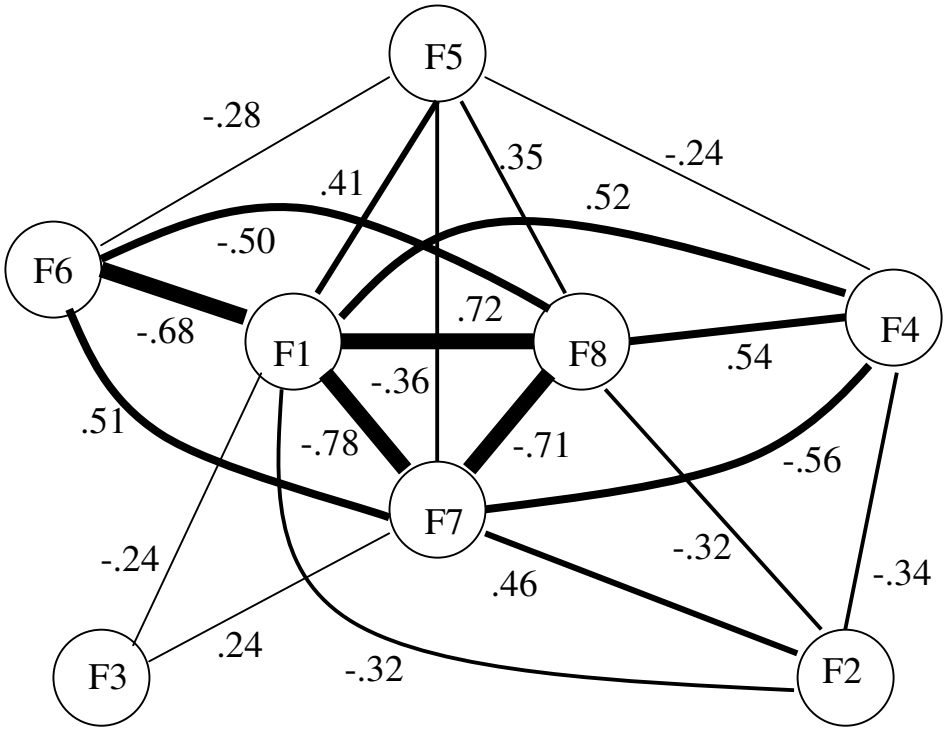
When we look at the correlations between the created components (Figure 1), we see that three of the components are closely related and form a core of the person's view of mathematics. This core consists of three aspects of person's general attitude towards mathematics. The first aspect (F1) focuses on beliefs about self, the second aspect (F8) on beliefs about mathematics, and the third aspect (F7) on the person's emotional relationship with mathematics. Around this core there are five factors, each

of which relate primarily to the core and some secondarily also to each other. The encouraging family (F 3) had only a minor effect on the core view, whereas experiences of poor teaching (F 4) related more closely to the core view and also to diligence (F 2) and insecurity as a teacher (F 5). The core had also a strong connection with future expectations (F 6), which probably differs from factor 1 in that the element of effort has a greater role in it.

Table 1. The 10 principal components of teacher students' view of mathematics.

Component number	Name of the component	Number of items	Cronbach's alpha
F1	I am not talented in mathematics	8	0.91
F2	I am hard-working and conscientious	5	0.81
F3	My family encouraged me	3	0.83
F4	I had a poor teacher in mathematics	8	0.84
F5	I am insecure as a mathematics teacher	4	0.74
F6	I can do well in mathematics	4	0.80
F7	I like mathematics	8	0.91
F8	Mathematics is difficult	6	0.78
F9	Mathematics is calculations	2	NA
F10	I am motivated	2	NA

Figure 1. Structure of student teacher's view of mathematics. The connection weights are Pearson correlations. The eight factors are described in Table 1.



We found significant gender differences in most of the variables examined. The largest difference was in that female students felt that they are more hard-working

and diligent (F2). Male students had higher self-confidence regarding their talent in mathematics (F1). However, there was no gender difference in students liking of mathematics (F7) or perceiving mathematics as difficult (F8). Female students had more critical image of their mathematics teachers (F4). According to a regression analyses, gender accounted for 20 % of the variation in the view of mathematics.

As assumed, the course selection in high school had affected students' view of mathematics. Those who had studied the more advanced mathematics course in high school had significantly higher self-confidence regarding their talent (F1). They also liked mathematics more (F7), but both groups perceived the subject equally difficult (F8). Those who had studied the more advanced track were less critical about their teachers (F4) and they had also received less encouragement from their families (F3). Surprisingly, the track had no effect on view of oneself as hard-working (F2), although the more advanced course is generally regarded to require a lot more work. According to a regression analyses, course selection accounts for 15 % of the variation in the view of mathematics. As there is usually a clear gender difference in course selection, we checked also for interaction effect between gender and course, but found this to be non-significant.

Those who had made it well in mathematics held more positive views about themselves and mathematics. Grade had a significant correlation with one's view of oneself as talented (F1) and hard-working (F2), as well as one's liking of mathematics (F7), and view of mathematics as a difficult subject (F8). Correlations were weaker, but still significant among female subjects with a positive view of one's teacher (F4) and one's chances to do well in mathematics (F6). All effects of grade were more pronounced among female subjects and those who have studied the more advanced mathematics course in high school. According to a regression analyses, course grade accounted for 12 % of the variation in the view of mathematics.

Cluster analysis of student views

In cluster analysis, the three-cluster solution separated students into groups based on the core of their view of mathematics: negative view, neutral view, and a positive view. In a six-cluster solution each of these was split into two subgroups, mainly based on the encouragement they got from their family and their view of themselves as hard-working (Figure 2).

A **positive view** included a view of oneself as talented in mathematics and hard-working, mathematics as easy and enjoyable, good memories of one's teachers, and confidence as a teacher and as a student of mathematics. The two subgroups of the positive view were the following:

Autonomous (21 %): These students had most positive view of all, they were rather hard-working but family had encouraged them only modestly.

Encouraged (22 %): These students had received the strongest encouragement from their family and they were the most hard-working group.

A **neutral view** fell on in the middle ground on most of the dimensions. These students had modest confidence on their own talent and they neither liked nor hated mathematics. The two subgroups of this category were the following:

Pushed (18 %): These students were encouraged by their family, but they did not work hard. They felt secure about their ability as a mathematics teacher.

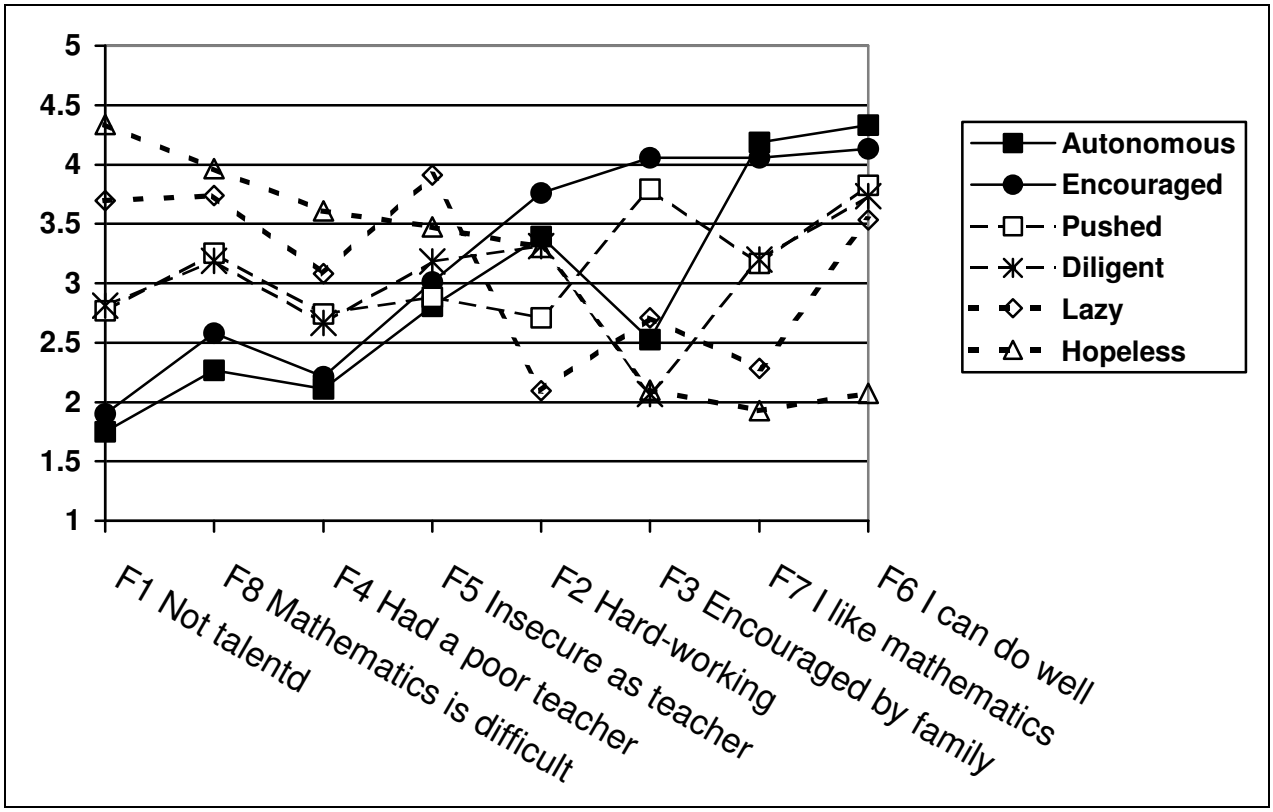
Diligent (18 %): These students were one of the two groups that received the least encouragement from their family, but they were rather hard-working.

A **negative view** included an image of oneself as not talented, dislike of mathematics, view of mathematics as difficult, negative view of one's teachers, and lack of confidence as a teacher of mathematics. The two subgroups of a negative view were the following:

Lazy (18 %): These students had a less extreme negative view compared to the other negative category. They were the least hard-working and the most insecure of all groups about teaching mathematics.

Hopeless (4 %): These students were the most extreme in their negative view. They were not encouraged by their family, yet they had worked hard. These students differed from all the other clusters in that they did not believe that they could learn mathematics.

Figure 2. The six clusters of student teachers according to their view of mathematics.



To provide a more detailed description of the clusters, we looked at how they differ in some of the other variables. We found out that students with different entrance

procedure fell unevenly into different clusters. Furthermore, those who had studied the more advanced mathematics in high school were more likely to fall into the cluster of 'encouraged' students while the students who had studied the less advanced mathematics were more likely to fall into the 'diligent' cluster. Gender was not a statistically significant factor in students' distribution in different clusters, but because the number of male students was low, we can not conclude gender to be a non-significant factor either.

Clusters also differ in test results and in mathematics grades, the successful having more positive views. The clusters differ also in motivation, those with positive view having higher motivation and the 'lazy' students having lower motivation than the remaining clusters. However, the clusters do not differ in teaching experience or age.

CONCLUSIONS

Regarding the structure of beliefs, we found a core of the view of mathematics, which consists of three closely related elements: belief of own talent, belief of difficulty of mathematics, and liking of mathematics. The result supports the view that although emotion is correlated with beliefs, it is a separable aspect of one's view of mathematics.

The three background variables: gender, course selection and grade are related to many of the variables, explaining a fair amount of the variation. Interestingly, while we found gender differences in self-confidence, we did not found those in liking mathematics or perceiving mathematics difficult. Hence, it seems worthwhile to separate the different aspects of the positive view of mathematics. Female students perceive themselves to be more hard-working and diligent than male students.

Students fall within three main categories. Some have positive (43 %), some neutral (36%) and some a negative (22 %) view towards mathematics. Each category was further divided into two subcategories mainly based on how much encouragement they got from their family and how hard-working they perceived themselves to have been. Some of the students with a negative view were seriously impaired as they felt that they have tried hard and failed. Consequently, they have adopted a belief that they can not learn mathematics.

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PRESERVICE TEACHERS' VIEW ON $y = x + 5$ AND $y = \pi x^2$ EXPRESSED THROUGH THE UTILIZATION OF CONCEPT MAPS: A STUDY OF THE CONCEPT OF FUNCTION

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This paper considers a group of preservice teachers' construction of concept maps derived from $y=x+5$ and $y=\pi x^2$ with emphasis on their conceptual understanding of function. The two statements are perceived to represent a number of different concepts with indications of compartmentalized knowledge structures that might prevent the preservice teachers from building rich conceptual structures. The preservice teachers' view on the concept of function contrasts with a view where the function concept is a unifying concept in mathematics with a large network of relations to other concepts. Different properties and categorizations of functions are less frequently recognized. In the preservice teachers' response of drawing concept maps there are signs of metacognitive activity.

INTRODUCTION

A limited number of studies have been conducted using concept maps in mathematics education, with only a minor part accessing students' conceptual understanding of function (e.g., Doerr & Bowers, 1999; Grevholm, 2000a, 2000b; Hansson, 2004; Hansson & Grevholm, 2003; McGowen & Tall, 1999; Williams, 1998). Williams (1998) concludes that concept maps assess conceptual knowledge in studying maps drawn by calculus students and professors with PhDs in mathematics. McGowen and Tall (1999) give further support to this conclusion in studying concept maps drawn by students on different levels of achievement in their studies. Other studies in mathematics education (e.g. Laturno, 1994) give further credibility to concept maps as an assessment technique.

Functions are present within all areas of mathematics, and it is important for preservice teachers in becoming successful dealing with the concept of function to have a well-developed conceptual understanding of function, including the concept's relations to other concepts and significance in mathematics (Cooney & Wilson, 1993; Eisenberg, 1992; Even & Tirosh, 2002; Vollrath, 1994). Mathematical statements like $y=x+5$ and $y=\pi x^2$ can be perceived to represent a number of concepts on a variety of levels, related to the preservice teachers' future teaching as well as more advanced concepts in mathematics. It is possible to let the statements represent concepts such as a straight line, parabola, equation, formula, proportionality, function (if we assume that domain and codomain also are considered), and others. The two statements $y=x+5$ and $y=\pi x^2$ can in particular be recognized as real functions of a real variable in using a representation not uncommon for the concept of function (Eisenberg, 1991, 1992; Tall, 1996). The function concept can in this context be related to different

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properties and classes of functions and recognized as a concept with a large network of relations to other concepts (Cooney & Wilson, 1993; Eisenberg, 1991, 1992).

The purpose of the current study is to examine preservice teachers' conceptual understanding of function in relation to $y=x+5$ and $y=\pi x^2$ through the utilization of concept maps.

THEORETICAL FRAMEWORK

In the chosen theoretical framework knowledge is considered as an individual construction built gradually. Knowledge is represented internally and described in terms of the way an individual's mental representation is structured (Goldin, 2002; Hiebert & Carpenter, 1992). Internal representations can be linked, metaphorically, forming dynamic networks of knowledge with different structure, especially in forms of webs and vertical hierarchies. Understanding grows as an individual's knowledge structures become larger and more organized, where existing knowledge influence relationships that is constructed. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. Hiebert and Carpenter (1992) describe the construction of larger and more organized networks of knowledge as learning with understanding. A similar notion is described by Ausubel (2000) as meaningful learning, as opposed to rote learning "only in rote learning does a simple arbitrary and nonsubstantive linkage occur with preexisting cognitive structure." (Ausubel, 2000, p. 3).

A central part of Ausubel's assimilation theory of meaningful learning is the idea that new meanings are acquired by the interaction of new, potentially meaningful ideas with what is previously learned. This interactional process results in a modification of both the potential meaning of new information and the meaning of the knowledge structure to which it becomes anchored. The process of assimilation results in progressive differentiation in the consequent refinement of meanings, and in enhanced potentially for providing anchorage for further learning.

The total cognitive structure that is associated with a concept in the mind of an individual is viewed in the way of Tall and Vinner (1981), as a concept image, including "all the mental pictures and associated properties and processes" (p. 152). When an individual meets an old concept in a new context it is the at that time evoked concept image – a portion of the concept image – with all the implicit assumptions abstracted from earlier contexts that respond to the task. In an individual's conceptual development Sfard (1991, 1992) suggests a process-object model. The formation of an operational conception, a process conception, precedes a more mature phase in the formation of a structural conception with focus on objects.

METHOD AND PROCEDURE

The current study is part of an ongoing study on preservice teachers' understanding of the function concept (Hansson, 2004). The preservice teachers that participate in the study are in the third year of a four and half year long teacher preparation program and specialize in mathematics and science, grades 4 to 9. During the sixth

term they enrolled in the final courses in mathematics of the program, and the study is conducted after calculus course where the concept of function is a central concept.

A group of twenty-five preservice teachers was given a presentation on concept maps during a lecture and showed examples of different types of concept maps such as non-hierarchical web-based maps and hierarchical maps, where the nodes represented concepts and the links were labeled in each case. (The hierarchical maps were constructed according to Novak & Gowin, 1984, and Novak, 1998). The maps that were displayed were often related to science education. Concept maps which were derived from mathematical concepts were largely avoided during the presentation, so as not to influence the contents of the maps which the students were to draw in the subsequent assignment.

After the introduction, the preservice students were each directed to draw concept maps based on the statements $y=x+5$ and $y=\pi x^2$, respectively. In the process of drawing maps with hierarchical structure they started to draw maps with a freely formatted structure. As a result, the preservice students constructed two maps for each mathematical statement; one freely formatted map and one hierarchical map. The preservice students were also asked to comment on their maps and their experiences of drawing the maps.

To draw concept maps based on $y=x+5$ and $y=\pi x^2$, respectively, differ from the use of concept maps derived from "function" (as in e.g. Doerr & Bowers, 1999; Grevholm, 2000b; McGowen & Tall, 1999; Williams, 1998) in that a concept is not explicitly stated. The statements can thus be perceived to represent a number of different concepts. The derived concept maps give the subject an opportunity to illustrate what concepts the statements are perceived to represent, and their different properties and relations.

All of the concept maps were analyzed. Each map was analyzed as an integrated unity in which the contents and structure of the map were noted. Furthermore, the manner in which the different sections of the map were related to each other was also studied. In particular, the manner in which the function concept was expressed on the maps was noted, its relationships to other concepts and properties that were assigned to functions were considered. The contents of the maps and the preservice students' comments on the maps were also compiled in tables.

The presentation took about 30 minutes, and the group received 60 to 80 minutes depending on when the students decided they had completed the assigned task. Twenty-four students in the group submitted all of their maps.

RESULTS

The statements on the freely formatted maps are usually placed at the center of the map; the map has a web-like structure, but very few cross-links i.e. links connecting different parts of a map (e.g. those drawn by F3, F5 and F18). The maps are less detailed than the hierarchical maps, and often served as a draft of the information that later would be included in the hierarchical maps. The hierarchical maps tend to branch off into substructures which often have few cross-links (as in those drawn by

F2, F8 and F16), implying that the concepts which are perceived to be represented by the statement often develop separately from each other. Although the maps have a hierarchical form, the conceptual structure is less hierarchical; thus, more general and inclusive concepts tend to be mixed with more specific concepts (a completely hierarchical conceptual structure is however not possible, since the maps are derived from specific examples). Furthermore, it is clear that when the preservice teachers construct maps for $y=x+5$ and $y=\pi x^2$ at the same occasion, it influenced the maps' contents and structure (as indicate by the maps and comments of e.g. F1, M8, F16).

All of the preservice teachers in the group have included the function concept in their maps. None of the preservice teachers discuss domain or codomain in relation to the function concept. They often express the function concept as a "relation" (e.g. F1, F12, F16) or a "dependence" (e.g. F3, F10, F11) between the variables x and y, reflecting a process conception (Eisenberg, 1991; Sfard, 1992). The node which contains the function concept is usually connected to the statement from which the map is derived and often gives rise to an underlying structure with very few links to other sections of the map (e.g. F2, F9, F10) – even if there are students who integrate the function concept in their maps to a great extent (e.g. F1, M4, F14).

Maps that contain fewer cross-links highlight to a lesser degree the manner in which the different concepts – which are represented by the statement – are related to each other. For example, F2 has the nodes "straight line" and "function" but does not state that the straight line is a graph of the function. Similarly, M7 links "curve" to "minimum value", but does not link "minimum value" and "function". Whereas preservice teachers who largely link the function concept to other nodes on the map, also link it to concepts which give the impression that they do not fully understand the function concept. The preservice teachers experience difficulties in distinguishing relations between the function concept and the equation concept (similar to e.g., Grevholm, 2000b; Leikin, Chazan & Yerushalmy, 2001; Williams, 1998), where e.g. M8 place "function" as a sub-concept[1] to "equation" in his hierarchical maps, M2 links "inverse" to "equation" and M17 links "equation" to "has derivative".

Concerning classes of functions, about half of the preservice teachers did not tie the function concept to any special function class (e.g. F1, M6, F18). While nearly a third gave "second degree function" as a polynomial function and "function" for $y=\pi x^2$ and $y=x+5$, respectively, and thus link $y=\pi x^2$ but not $y=x+5$ to a function class (e.g. M5, F14, F16). Other preservice teachers state that $y=\pi x^2$ is a "second degree function", and that $y=x+5$ is a "first degree function" (M4 and M8) or a "linear function" (M2, F15, F17). It was noted that there were preservice teachers who incorrectly associated $y=\pi x^2$ with other function classes, e.g. F13 who link it to "exponential function". Otherwise, the preservice teachers do not mention any properties of functions, implying that they view $y=x+5$ and $y=\pi x^2$ as belonging to

[1] M8 links "y=f(x)" to "function", the concept map implies that M8 interprets the notation "y=f(x)" as an equation and thus views "function" as a sub-concept to "equation".

categories of functions that they met during their calculus course like even, continuous, increasing, differentiable etc. In those cases where the derivative is discussed (only a few of the students mention derivatives in their maps, e.g. M1, M4, F18), it is not viewed as a property which gives rise to a class of functions, but instead as a procedural skill in the form of a calculation process. For example, just as M2 states that the derivative determines the slope or F5 states that the derivative determines the minimum point. Derivatives are however more common in the maps derived from $y=\pi x^2$ (6 maps) than for maps derived from y=x+5 (4 maps).

Only a few preservice teachers establish links between teaching and learning in their maps (similar to Doerr & Bowers, 1999). F5 for example, states that "$y=\pi x^2$" is a "second degree function" which is "difficult" and that "primary school students cannot manage it". This is in contrast to "y=x+5" which she labels a "simple function" which "is used in primary schools". The preservice teachers also expressed similar opinions on degree of difficulty without establishing any connection to learning: F3 states that y=x+5 is a "simple function" and M6 states that "y=x+5" is "easier" than "$y=\pi x^2$" (on his map for $y=\pi x^2$).

There are elements of an "algorithmic nature" in the maps (in common with McGowen & Tall, 1999; Williams, 1998) in which e.g. F3 states that $y=\pi x^2$ has two roots $x=\pm\sqrt{y/\pi}$, F8 establishes a relation to the area of a rectangle "A=ab" and a triangle "a=bh/2" (in relation to $y=\pi x^2$), and F5 states that "the derivative" of $y=\pi x^2$ gives a "minimum point". Moreover, the preservice teachers also raise matters which are trivial, to say the least, in their maps e.g. M1 states that y=x+5 is "easy to draw", F2 states that x^2 is x "multiplied by itself" or F8 who writes the Greek letters "ε, α, β" in relation to π.

The preservice students' written comments revealed clear perceptions that the maps promote metacognition and have a mediating role (Novak, 1998), e.g. M6 who states that "you see relations that you have not considered", F3 who writes that "one really has to consider the meaning of what the different things mean and where they lead", or F8 who says that one "learn from the map, facts are entered once again". But there are also signs that the maps evoke concept images with conflicting pieces of information, e.g., M5 with the opinion that "It only causes confusion...".

One preservice teacher's maps

The two concept maps below are examples of how the preservice teachers might draw their maps. They illustrate a tendency for the maps in the study to have few cross-links – or as in this case a lack of cross-links – where different parts of a map are developed separately. Moreover, the two maps illustrate a common feature of the preservice teachers' maps in containing graphical interpretations – although not always in relation to the concept of function in contrast to the maps below. They also show how preservice teachers less frequently recognize relations to different properties of functions. For instance, "k=1" (in figure 1) might be related to an increasing function. Moreover, "min" and "min max" (in figure 2) are related to the concept of "parabola" rather than "function" or "2nd degree function", respectively.

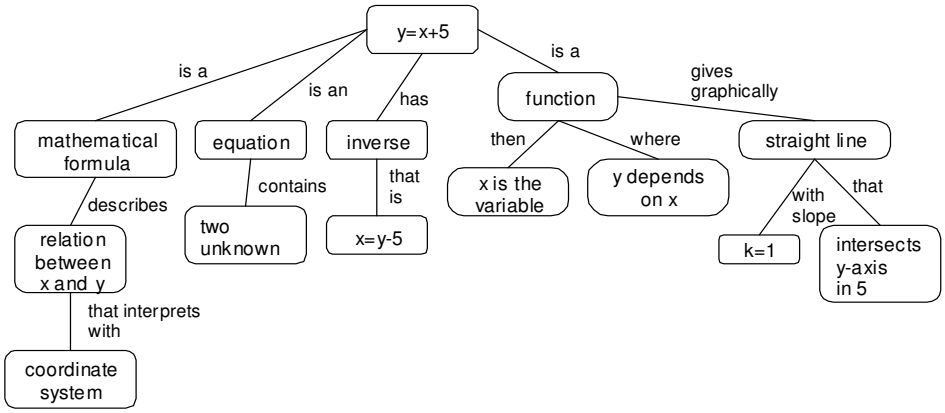

Figure 1. The hierarchical map of y=x+5 drawn by F16.

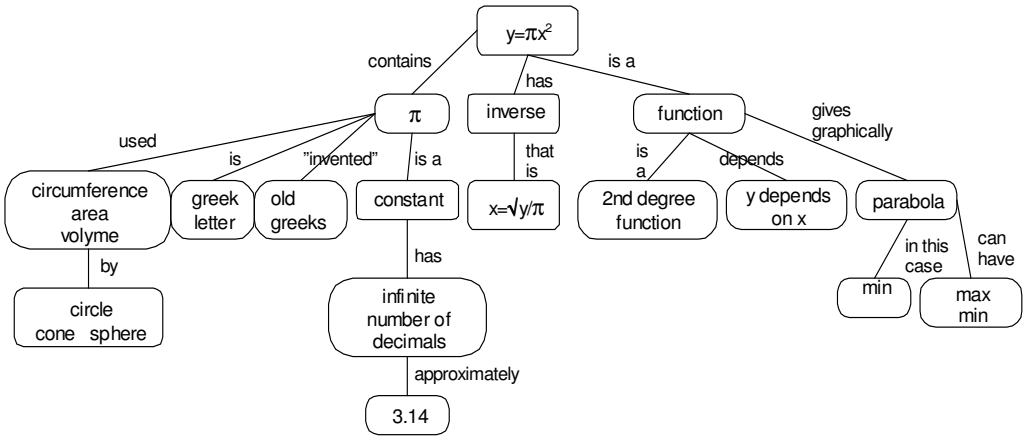

Figure 2. The hierarchical map of $y=\pi x^2$ drawn by F16.

The two maps also show individual characteristics as for example in the case of "inverse", where the function concept seems to be a less meaningful concept. This indicates rote learning (Ausubel, 2000) in particular as $y=\pi x^2$ does not have an inverse (when considered as a real function of a real variable).

DISCUSSION AND CONCLUSIONS

There are clear indications that the function concept often is developed independently with few relations to other parts of the maps. This may be an expression of compartmentalized knowledge structures that prevent the preservice teachers from building rich conceptual structures that form a basis for meaningful learning (Ausubel, 2000) and learning with understanding (Hiebert & Carpenter, 1992), with consequences for the preservice teachers' future teaching (Even & Tirosh, 2002; Vollrath, 1994). Moreover, the preservice teachers rarely relate to teaching and learning in their maps. This may be surprising, since their mathematics courses all contain parts related to mathematics education. Particularly as the characteristics of the two statements y=x+5 and $y=\pi x^2$ make them suitable for connection to different teaching scenarios which the preservice teachers will face as inservice teachers.

Preservice teachers often express the function concept as a dependency between the variables x and y – describing a process conception (Eisenberg, 1991; Sfard, 1991, 1992). In some cases, they give somewhat more elaborate explanations and state that an x gives one y. But none of the students discuss domain or codomain in association

with the function concept, and a more developed conceptual understanding in the form of an object with a set of properties is less frequent in the maps.

Elements that express procedural knowledge and skills of an algorithmic nature occur frequently (in agreement with results presented by e.g. Grevholm, 2000b; McGowen & Tall, 1999; Williams, 1998). The maps may also contain elements that are completely trivial at the expense of important concepts and relations between them, which indicates root learning (Ausubel, 2000). Moreover, several misconceptions related to the concept of function can be identified on the maps.

The function concept's large network of relations to other concepts is frequently not part of the concept maps. This is usually a consequence of that the preservice teachers do not observe and relate to different properties of $y=x+5$ and $y=\pi x^2$ when they are regarded as functions. In those cases the preservice students give a concept, or some characteristic of a concept, they do usually not observe the relations to the concept of function. The perception of the function concept that is illustrated in the concept maps contrasts highly with the idea of functions playing a central and unifying role in mathematics (Cooney & Wilson, 1993; Eisenberg, 1991, 1992).

The concept maps seem to reveal a need for the preservice students to reflect upon the function concept's relevance in mathematics, its different properties and its network of relations to other concepts. The preservice teachers' comments on drawing the maps indicate that the process of drawing concept maps supports metacognitive activities (Novak, 1998). Such activities might promote the preservice teachers' conceptual understanding of the function concept and its significance in mathematics and require further research.

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MISTAKE-HANDLING ACTIVITIES IN THE MATHEMATICS CLASSROOM

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The handling of mistakes in the mathematics classroom is an area in which only little research is done so far. In mathematics education there are, in fact, several investigations about students' mistakes from a diagnostic perspective, but hardly any studies on the question how teachers react in concrete mathematics lessons. In this article theoretical aspects and results of different video studies are summarized. Moreover, a student survey based on a questionnaire is presented in detail. The findings indicate that particularly low achieving students are not aware of the learning opportunities within mistake situations.

INTRODUCTION

In the German language there exists typical proverbs like "Aus Fehlern wird man klug." (You will become clever from mistakes) or "Aus Fehlern lernt man." (You will learn from mistakes). Though it seems that mistakes are generally accepted as a natural part of the learning process, it is unpleasant for individuals to make mistakes or to be caught when making mistakes. This fact is mainly based on two reasons: on the one hand, there is an affective component, i.e. to make a mistake means to be embarrassed. On the other hand, mistakes give in some sense evidence that there are some deficits in a certain individual competence, which in school is a disadvantage when being judged[1]. Thus, when approaching the topic of mistake in the classroom we must have in mind that there are different facets of mistake situations. In this contribution we will focus on cognitive and affective aspects of the mistake-handling in the classroom processes from the student and observer perspective.

THE ROLE OF MISTAKES IN THE LEARNING PROCESS

Within the last decades, in particular in the 1970s and 1980s, there was a lot of research in mathematics education concerning the question of students mistakes (e.g., Radatz, 1979). The main aim of this research was the identification of reasons for different typical students' mistakes and, consequently, the development of didactical ideas, material etc. to prevent these mistakes or to use these mistakes as a learning opportunity. Beyond this research approach which follows a diagnostic perspective, there is less research on the question what teachers know about typical student mistakes and what is a good way to deal with them in the lessons. Examples are the intervention program of the Italian group for detecting and overcoming conceptual mistakes (e.g., Garuti, Boero, Chiappini, 1999) or the study of the Israeli group on

[1] This led to the fact that the word "mistake" has a negative connotation and is often avoided. We use the word mistake anyway and stress the positive aspects of mistakes in the learning process.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 105-112. Melbourne: PME.

teacher awareness of students' difficulties when manipulating with rational numbers (e.g., Klein, Barkai, Tirosh & Tsamir, 1998). However, if we address the issue of mistake-handling activities of teachers in mathematics lesson, particularly in the students' perception, only very few studies are known. Thus, we do not really know how mathematics teachers and their students handle mistake situations.

Learning from mistakes – the theory of negative expertise

As already mentioned in the introduction it is generally accepted (in the sense of a public opinion) that one can learn by mistakes. However, we rarely find a detailed description or idea how this learning process takes place. Mainly, we find the diffuse idea which can be described as "a mistake is only made once", which is obviously not true. A deeper theoretical approach to the topic "learning by mistakes" is given by the research group of the educational psychologist Fritz Oser which worked on this question in a three year research project. The basic idea of Osers group is the theory of negative expertise.

For the theory of negative expertise the notion "mistake" was defined as follows: *A mistake is a process or a fact that does not comply with the norm. It is necessary for the identification of the line of demarcation to the correct process or fact that complies with the norm.* In other words: Mistakes are necessary to sharpen the individual idea about what is false and what is correct (according to a given norm). If we compare this with the role of examples and counterexamples for the learning of mathematical concepts, then mistakes play the role of counterexamples. Thus, it is not enough that an individual knows what is correct. She/he also has to know what is incorrect, because otherwise it is not possible to identify at which point the correct ends and the incorrect begins. From this point of view the knowledge of incorrect facts and processes is necessary and this negative expertise completes the knowledge of correct facts and processes, i.e. it completes the positive expertise (cf. Oser, Hascher & Spychiger, 1999).

Mistakes are essential for the acquisition of negative knowledge and, consequently, mistakes are necessary components of the learning process. However, to make a mistake does not mean automatically to acquire negative expertise, which can be used to prevent further mistakes. For using a mistake in a productive way by Oser et al. (1999) it is necessary that an individual is able to realize, to analyze and to correct the mistake and, moreover, that she/he uses the mistake to develop a strategy for the prevention of further mistakes. At this point we are facing a lot of open questions. For example, the question if an individual has to make a mistake by itself or if it is sufficient to participate in a mistake situation of another person. Moreover, the question is how to foster the individual productive use of mistakes in the mathematics classroom. And finally, what about different types of mistakes? If a mistake is part of the procedural knowledge of a student, then often she/he will make this mistake again and again and it might be more difficult to learn from this mistake than from mistakes which are part of non-procedural knowledge.

From the previous description it becomes clear that the theory of negative expertise with its positive role of mistakes corresponds to a constructivist view of learning. While the behaviouristic approach avoids mistakes and tries to stress only successful students activities (i.e. only positive knowledge is important), this approach realizes mistakes as learning opportunities which are necessary and unavoidable.

Mistake-handling in mathematics classroom – empirical results

There are only a few studies which focus on the question of mistake-handling activities in mathematics lessons from an empirical perspective. Until now we do not know much about teacher and student behaviour within mistake situations in the real mathematics classroom. It is obvious that there is a difference between mistake-handling activities in so called private and public situations. For example, if an individual mistake is discussed with the whole class, then the student may feel to be exposed. In a private situation, i.e. the teacher discusses a mistake with one student, there may be a better chance for an individual learning progress. Thus, before presenting findings of previous investigations on this topic from Germany, Switzerland, USA and Italy, I will give a short overview about basic data on the organisation of the mathematics lessons in these countries (cf. Stigler et al, 1999; Hiebert et al, 2003; Santagata, in press).

As one can see in Table 1 in all four countries most of the lesson time is devoted to public work, i.e. phases in the lessons in which the whole class is working together. Generally, in this phase the teacher is talking to the students or tries to progress with the content of the lesson by asking questions to the students. Private work means that students are working on their own (individual) or in small groups/pairs. However, these phases are mainly used for practicing routine procedures, making drawings etc. and rarely for explorative work or the introduction of new content.

	private work		public work
	individual	group/pairs	
Germany	23%	3%	73%
Switzerland	33%	11%	54%
Italy	10%		82%
USA	29%	9%	61%

Table 1: Percentages of lesson time for private and public work

The following results are from three video-based investigations of mistake-handling in mathematics classroom. The German study by Heinze (2004) comprises 22 lessons from grade 8, the Swiss study by Oser et al. (1998) ten lessons (different grades on the lower secondary level) and the study from USA and Italy by Santagata (in press) comprises 30 lessons from grade 8 in each country. All studies analyzed the lessons on the basis of mistake situations which were identified by the teacher reaction to a student contribution. A further characterization of the mistake situations was then made in different ways.

Table 2 shows the average number of mistakes in the mathematics lessons of the different countries in public and private situations (duration of the lessons: 45-50 min-

utes). For the German lessons we do not have data for the private situations, because the sound recording of the students' voices during private work is of bad quality.

From the table we can see that the number of mistakes in the public teacher-students interaction in Germany, Switzerland and the USA is comparatively low. Even if we take into account that in Italy 82% of the lesson time is used for public class work and in Switzerland only 54% of the lesson time (cf. Table 1), we can state that in Italian mathematics lesson proportionately more public mistakes appear than in Swiss lessons. The reason for the comparatively high number of public mistakes in Italy goes back to the so called blackboard activity: nearly half of these mistakes occur when an individual student is asked to solve a problem at the black board (Santagata, in press). Such activities do not play an important role in the other three countries. If we take a closer look to the relation between students mistakes and students contributions in a lesson, then we have for the German sample an average number of 47.3 contributions of students in a lesson, i.e. in 10% of the students' contributions the teacher identified a mistake.

Mistakes per lesson	Public	Private
Germany (N=22)	4.7	no data
Switzerland (N=10)	3.5	1.6
Italy (N=30)	10.7	0.3
USA (N=30)	4.6	3.2

Table 2: Average number of mistakes per lesson in private or public situations.

The public mistake-handling activities in Italian, US and German mathematics lessons are clearly directed by the teacher (93% USA and Italy, 88% Germany, no data for Switzerland). As shown in Table 3 a quarter to a third of the mistake situations are directly solved by the teacher; between 48% and 62% are returned by the teacher to the students as a challenge. In the German lessons 15.4% of the mistakes were directly corrected by students without a teacher activity and about 10% were ignored, i.e., no corrections could be observed though the mistakes were identified by the teacher or students.

	Teacher corrects	Teacher asks students
Germany	26.9%	48.5%
Italy	31.6%	53.6%
USA	25.1%	62.2%
Remaining cases: Ignoring, student corrects directly etc.		

Table 3: Mistake-handling by Teachers.

A last topic that should be addressed here is the question how the mistake situations in the German mathematics lessons are finally solved. In 12.5% of the cases there was no clear correction, because the mistake was ignored or the correction of the teacher/students was obviously not sufficient. For 46% of the mistakes there was a simple correction and for 41.3% a detailed explanation was given. A second rating of the mistake situations showed that 44% of the mistake-handling activities mainly focus on the individual learning progress and also 44% were mainly oriented to the continuation of the intended course of the lesson (12% other or undecidable).

Mistake-handling activities from the perspective of students were investigated in the Swiss study (c.f. Spychiger, Mahler, Hascher, Oser, 1998). A questionnaire with 27 items (four-point Likert scale) was developed and administered in a sample of 295 students from grade 4 to 9. A factor analysis yielded three main components which describes the teacher behaviour in mistake situations, the individual use of mistakes in the learning process and the individual emotions in mistake situations. The results can be summarized as follows: The teacher behaviour and the individual emotions in mistake situations were rated comparatively positive by the students (cf. Table 4). In contrast to this the finding for the second component indicates that the students hardly use mistakes as an individual learning opportunity. Spychiger et al. (1998) consider this result as a possible starting point for an intervention to improve students' individual mistake handling.

RESEARCH QUESTIONS AND DESIGN

Based on the theoretical approach and the empirical results described in the previous sections we addressed in our research the student perception of the mistake-handling activities in the German mathematics classroom. Our study was guided by the following research questions:

1. What is the perception of the students regarding mistake-handling activities in the mathematics lessons? What do the students think about the teacher behaviour and are they afraid of making mistakes in public situations?
2. How do students individually deal with (their own) mistakes? Do they use mistakes as a learning opportunity?
3. Which kind of mistakes are "permitted" or "forbidden" in the perception of students (in the sense of negative teacher reactions)?

The sample of the study comprises 85 students from grade 8 and grade 9 from three different classes of one school. For the data collection the questionnaire from the Swiss study described above (Spychiger et al, 1998) was adapted. At the end of the questionnaire two open questions were added: (1) *In which situations (except exams, tests etc.) are mistakes forbidden?* and (2) *In which situations (except exams, tests etc.) are mistakes permitted?*. Moreover, the grade for mathematics of the last school report was asked.

RESULTS

Like in the Swiss study a factor analysis yielded three main components which basically coincide with the components described above. However, in our study some of the items of the Swiss component *Teacher behaviour* also loaded on the component *Individual emotions in mistake situations*. These were particularly items like "I have the feeling that it is not allowed to make mistakes, because our mathematics teacher doesn't like this." Thus, we denoted the emotions component in our study *Fear of making mistakes* (10 items). The other two components were denoted like in the Swiss study *Teacher behaviour* (7 items) and *Individual use of mistakes*

(8 items). The reliability turned out to be satisfactory (Cronbach's α between 0.73 and 0.83).

Table 4 gives the mean values for the scales for the German and the Swiss study. Regarding the component "Individual use of mistakes for the learning process" and "Fear of making mistakes" the values are similar: On the one hand, the students are rarely afraid of making mistakes and on the other hand, they do not use their mistakes very productively.

	Teacher behaviour	Individual use of mistakes	Fear of making mistakes
German study	2.93	2.86	1.74
Swiss study (Spychiger et al. 1998)	3.38	2.75	1.82*
Likert scale: 1=strongly disagree, 2=disagree, 3=agree, 4=strongly agree *Calculated from the reverse component "Individual emotions in mistake situations" with mean value 3.18.			

Table 4: Mistake-handling from students' perspective (mean values).

The rating of the teacher behaviour is in the German sample worse than in the Swiss sample; however, from the students' perspective the teacher behaviour is acceptable.

If we consider the histograms of the three components in Figure 1, we can see that the positive results for the teacher behaviour and the fear of making mistakes is shared by most of the students in the sample. Moreover, we can learn that there is a slightly broader range regarding the individual use of mistakes for the learning process (e.g. nearly 20 students show values about 2.5, the mean of the Likert scale).

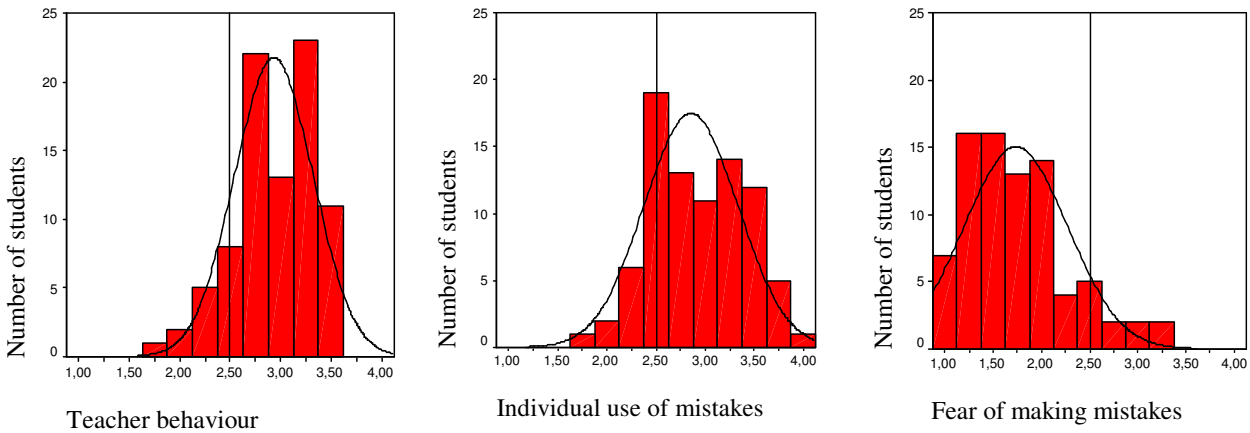

Figure 1: Distributions of students for the three components regarding mistake-handling.

If we consider the relation between the students' achievement (measured by the last grade[2] in mathematics on the school report) and the students' rating for the three mistake-handling components, then we find only one correlation. The grade in mathematics correlates significantly with the rating for the scale *Individual use of mistakes*

[2] School grades in Germany: 1 = very good, 2 = good, …, 6 = insufficient.

(r = -.235, p=0.006, Kendall-τ). This correlation indicates that high achieving students are better in using mistakes for their learning processes than low achieving students (cf. Figure 2).

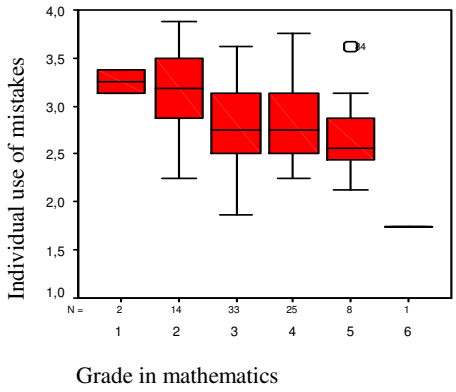

Figure 2: Grade and individual use of mistakes for the learning process.

The last part of the student questionnaire was dedicated to typical situations in mathematics lesson, in which mistakes are (1) forbidden or (2) permitted. The students' responds were categorized by a bottom-up analysis and we got the results of the following Table 5.

From the students point of view basic knowledge and repetitions are areas in which an individual cannot afford to make mistakes. In contrast, new content, challenging tasks and the official opportunities to practise the acquired knowledge are areas in which mistakes are permitted. However, the students have different opinions on that question, because 14 students think that mistakes are (nearly) always permitted.

forbidden	students	permitted	students
basic knowledge	28	new content	39
repetition	19	challenging tasks	18
hardly any / none	9	homework	17
at the blackboard	7	exercises	15
other	6	(nearly) always	14

Table 5: Forbidden / permitted situations for mistakes

DISCUSSION

The findings of the different studies described in the previous sections discover positive and problematic aspects of the mistake-handling activities in the mathematics classroom. Regarding the German situation the video analysis shows that there are comparatively few mistake situations in the lessons (the same problem is observed in Switzerland and the USA, cf. Table 2). If mistakes are considered as an essential part of the learning process, like the theory of negative expertise states, then this small amount of mistakes is problematic. As Santagata (in press) supposes the mistake-handling activities by the US teachers is still influenced by the behaviouristic approach concerning learning, i.e. avoiding mistakes and reinforce correct answers. A similar explanation may be valid for the German situation since 44% of the mistakes-handling activities aimed at continuing the indented course of the lesson. Nevertheless, we could observe that in the German lessons for 41.3% of the mistakes a detailed explanation was given which led to a productive solution of the mistake situation. Thus, there are only few mistakes in the lessons, but if a public mistake appears, then there is a good chance that it will be explained and corrected. Until now we do not know which types of mistakes are treated more deeply by the teachers.

From the students perspective we got a comparatively positive image regarding the mistake-handling activities in the German mathematics classroom. Generally, the students do not fear to make mistakes and in their opinion the teacher behaviour is acceptable. However, there is some potential to improve the students' individual

mistake-handling activities. Thus, the findings of the German study replicate the results of the Swiss study (cf. Table 4). Moreover, the correlation between the achievement and the individual mistake-handling activities indicates that this is a particular problem of the medium and low achievement students (Figure 2). Here we can think about the development of special teaching elements.

The students' views on forbidden and permitted mistake situations must be seen critical. In particular, the fact that mistakes are considered as forbidden when basic knowledge or repetitions are treated in mathematics lessons, contradicts the idea that teachers should assist students in their individual knowledge construction. Since knowledge construction substantially involves the pre-knowledge, these "forbidden mistakes" give the opportunity to detect critical gaps. Thus, a starting point for productive mistake-handling activities by teachers is to avoid the students' impression that certain mistakes are forbidden.

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ONE TEACHER'S ROLE IN PROMOTING UNDERSTANDING IN MENTAL COMPUTATION

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This paper reports the teacher actions that promoted the development of students' mental computation. A Year 3 teacher engaged her class in developing mental computation strategies over a ten-week period. Two overarching issues that appeared to support learning were establishing connections and encouraging strategic thinking.

While a growing interest in mental computation as a vehicle for developing number sense has become a focus in many international mathematics curricular (e.g., Maclellan, 2001; McIntosh, 1998; Reys, Reys, Nohda, & Emori, 1995), mental computation is new to the Queensland (Australia) scene. In fact, many schools in Queensland have not introduced mental computation into their mathematics programs to date, as the new syllabus (Queensland Studies Authority (QSA), 2004) will not be mandated until the year 2007. However, some schools have been keen to embark on the development of mental computation. Certainly, text book writers have been quick to publish new mathematics texts that include mental computation exercises. The student books provide practice for students to apply particular strategies they have been taught. Often, the focus is on one or more specific strategies; therefore, the students practise the strategies, rather than engage in the thinking involved. This often results in a routine approach to teaching mental computation. In reality, it is easy to see why text books could become popular in the teaching of mental computation, as teachers often do not have the knowledge to sequence and present worthwhile mental computation activities.

In the context of this study, mental computation refers to efficient mental calculation of two- and three-digit addition and subtraction examples. Mental computation does not refer to the calculation of number facts. This is in contrast to the discussion of mental computation in the new syllabus (QSA, 2004), where mental computation strategies for Levels 1 and 2 (relevant to the children in this study) refer to basic facts strategies (e.g., count on, count back, doubles, near doubles, make to 10). Even Level 3 'mental strategies' do not include strategies that have been identified elsewhere as appropriate for young children to develop, for example, *compensation (N10C)* (e.g., Beishuizen, 1999; Thompson, 1999).

At present in Queensland (Department of Education, Queensland, 1987), children in Year 3 (approximately 8 years of age) are expected to be able to complete addition and subtraction two-digit with and without regrouping and three-digit without regrouping written algorithms. The final product is generally procedural with little understanding.

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One school that has embarked on the development of mental computation (in the early years – Years 1–3) is the one described in this paper. For the purposes of this paper, only the work conducted in the Year 3 class will be discussed. In 2004, the researcher worked with the Year 3 teacher to develop a program to enhance mental computation. This teacher had also been involved in a similar study in the previous year (reported in Heirdsfield, 2004a, b). The previous year's work impacted on the present study, as the teacher had already been introduced to the literature on mental computation; conducted some pre-interviews with her students to establish their base knowledge; plan a mental computation instructional program in conjunction with the researcher; and, then, implement the program. The researcher acted as a critical friend. Finally, the teacher conducted some post-interviews to identify growth in students' mental computation, measured by strategy choice and accuracy; and reflected on the project; for instance, identification of effective models (e.g., empty number line, hundred chart), sequencing, and questioning; and level of student participation and interaction. Therefore, the teacher already had some knowledge about what constituted an effective mental computation program.

Several research studies investigating successful instructional programs (e.g., Blöte, Klein, & Beishuizen, 2000; Buzeika, 1999; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Hedrén, 1999; Kamii & Dominick, 1998) have indicated that the emphasis of instruction should be strategic flexibility and students' exploring, discussing, and justifying their strategies and solutions. In addition to student behaviour, teacher competence is also an important factor in successful instruction (e.g., Askew, 1999; Brown, Askew, Baker, Denvir, & Millett, 1998; Brown, Askew, Rhodes, Denvir, Ranson, & William, 2001; Brown & Campione, 1994; Diezmann, Watters, & English, 2004). Summarising these studies, important factors in effective teaching include teacher expectations, instruction as systemised and connected, and the four teaching characteristics of Brown et al. (2001) – tasks, talk, tools, and relationships and norms. Therefore, teacher competence is a key factor in students' quest for understanding.

The purpose of the project was to enhance Year 3 students' mental computation performance. The specific aims were to collaboratively design an instructional program to build on students' existing strategies, and to identify and monitor students' mental computation performance. The instructional program was based on students' prior knowledge (identified from individual interviews). This paper focuses on the identification of teacher actions that promoted the development of mental computation.

THE STUDY

The research adopted a case study design (Yin, 1994) in which a teaching experiment (Steffe & Thompson, 2000) was conducted with the aim of developing Year 3 children's mental computation performance. The study was conducted in a Year 3 class (7-8 year olds) consisting of 30 students, in a school serving a predominantly

middle class community in an outer suburb of Brisbane. Students engaged in 30 to 45 minute lessons once a week for 10 weeks. These lessons focussed on the development of mental computation strategies for 2- and 3-digit addition and subtraction. The teacher and researcher had worked together in the previous year on a similar project, when the teacher was

A similar approach was taken in 2004. In addition, it was decided that teaching of the traditional pen and paper algorithm (which is still used in Queensland schools) would be avoided for the duration of the project. Pre- and post-interviews were conducted by the researcher, teacher and a research assistant. The teacher incorporated learning from the previous year into the instructional program. Each lesson was videotaped; and observations (including comments) of the lessons were documented by the researcher. The focus was on identifying the connections and sequencing of the lesson, student participation and communication, the sense that students were making during the lesson, questioning, and quality of interaction, in general. The researcher was a participant observer, who interacted with individual students and small groups during the lesson. Each lesson was followed by a brief discussion between the teacher and the researcher, where clarification of the aims and perceived outcomes was sought, and inhibiting factors and avenues to pursue were identified. The teacher was also provided with a copy of the researcher's notes for further consideration, and as a record of the lesson from an observer's view. The videotapes were later analysed for further insight. Data comprised videotaped lessons, the researcher's field notes, student work samples, the teacher's lesson plans and reflections, and the pre- and post-interviews. Data were analysed to identify emerging issues related to the students' mental computation reasoning.

RESULTS

Analysis of the teacher's actions revealed two issues that influenced student mental computation performance. Well planned questioning; provision of appropriate tasks and models; a great deal of exploration, discussion, and critiquing of strategies; and careful sequencing were used to establish <u>connections</u> and encourage <u>strategic thinking</u>.

Connections

From the previous year's project, the teacher became aware of the importance of sequencing both within a lesson and between lessons. The researcher formulated a suggested sequence for introducing number combinations in conjunction with appropriate models (empty number line, hundred chart, & 99 chart):

1. jumping in tens forwards and backwards from multiples of ten (e.g., start with 40 – jump forwards or backwards in tens);
2. jumping in tens forwards and backwards (e.g., start with 43 – jump forwards or backwards in tens);

3. relate the previous step to addition and subtraction (e.g., start with 43 – add 10, add 20, add 30; take away 10, take away 20, etc);
4. further addition and subtraction, without bridging tens (e.g., 43±22);
5. further addition and subtraction, bridging tens (e.g.,47±28; 47±19 as a special case). For an example of the type 47-28, only the ENL might be used, as it supports a *going-through tens* strategy (Thompson, 1997). A hundred chart cannot easily be used for this strategy (for subtraction); although, a 99 chart can be used.

Progress through steps one and two were easily completed in one lesson, but progress to step three, for some students, required making the connections explicit. The teacher successfully scaffolded these students learning with careful questioning.

> Start at 33 (on the hundred chart) and jump to 53. How far is that?

Some students responded with "twenty" and others responded with "two tens". Both responses were accepted. For others who were hesitant, a further line of questioning was pursued.

> Start at 33 (on the hundred chart). Add 10 more. Where are you now? Where did you start? What did you add on? Now add another 10. Where are you now? Where did you start? What did you add on altogether?

As well as the teacher scaffolding the slower students, class discussion was encouraged. Students who originally were hesitant started to make connections by participating in this discussion.

> To do 66 and 20 more, I said that's the same as ten and ten more.
>
> I said that's the same as two tens.

By the time, students were presented with examples of the type at steps 4 and 5, the teacher documented students' strategies using equations, as they explained their strategies; for instance,

	86-45	
86-5=81	or	86-40=46
81-40=41		46-5=41

Although the students had viewed this documentation several times, there was no smooth progression to the students' documenting their own strategies in the same way. So, the students were placed in small groups, made up of a recorder, demonstrator and speaker. The recorder (who documented the equations) and speaker (who was to present the strategies to the class) had to listen very carefully to the demonstrator while the strategy was being described and check that all steps had been documented. The researcher and teacher scaffolded many groups through this process. However, success was achieved (see Figure 1).

A final example of making connections concerns the use of the empty number line. In Queensland, students have had no experience with the empty number line: although, now, some teachers are using this model. The teacher introduced the empty number line by firstly providing the students with number lines where tens were labelled and divisions between tens marked (see Figure 2).

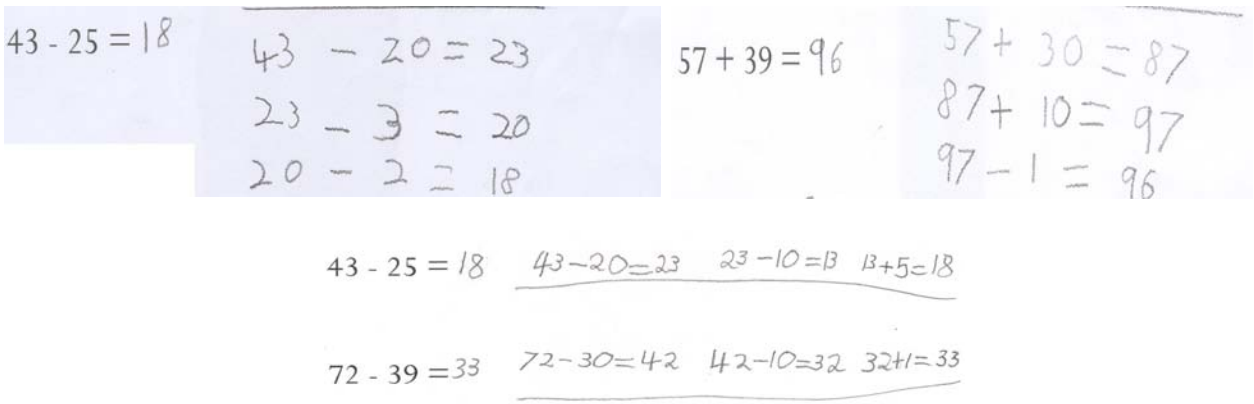

Figure 1. Examples of students' written documentation of strategies

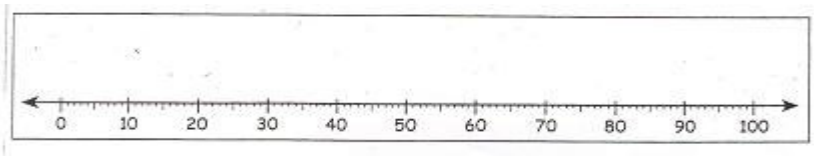

Figure 2. Number line used to introduce the empty number line

In addition, the teacher used a white board drawn up with number lines where tens were marked. A large clear plastic sheet sat over the whiteboard, so that jottings on the plastic sheet could be removed without affecting the number lines drawn on the whiteboard. While the students worked on their number lines, the teacher and individual students worked on the number line on the whiteboard. The students were directed to find/mark numbers on their number lines, and explain how they knew how to find the numbers. They then jumped on from or backwards from these numbers in tens. Finally, the connections were made between jumping in tens and adding and subtracting multiples of tens (e.g., 73-40). Again, scaffolding questions were required for some students.

> Start at 33 (on the number line). Add 10 more. Where are you now? Where did you start? What did you add on? Now add another 10. Where are you now? Where did you start? What did you add on altogether?

The empty number line was introduced by the need to use a more flexible number line. The teacher drew a straight line (with no markings) on the blackboard, and the example "95+30" was written above the line. Discussion was opened up to the class to decide how best to use the number line to solve the problem. One student suggested placing 95 towards the right of the line "because that's where 95 is".

However, others suggested that the line would then need to be extended to permit the calculation to be recorded. One student suggested placing 95 to the left end, to permit the jumps to the right to be completed. The remainder of the class agreed with this solution. That student was then invited to draw the solution on the empty number line. There was also discussion about possible solutions – some suggested jumping in tens; while some suggested they could jump 30 in one go. From there, steps 4 and 5 (see above) were followed for the empty number line.

Strategic thinking

While students were introduced to models (hundred chart, ninety-nine chart, empty number line) to aid the development of mental computation strategies, the focus was not the models, but the strategic thinking. Therefore, students were free to choose any model (or no model) to solve the examples. Further, they were constantly encouraged to explain their reasoning, compare their own strategies with others' strategies, and critique the strategies. Apart from a means of solution, the models were also used as a means of communication. Sometimes, when the students discussed their strategies, the teacher documented the strategies using the models, and sometimes she documented the strategies in equations. Further, students were permitted to use any model or no model. In fact, during two lessons, students were provided with a page of empty number lines (it was found that students wasted precious time if they drew their own number line – they were obsessed with using rulers), and a sheet with a hundred chart and a ninety-nine chart. The teacher presented the students with examples to solve, and individual students were invited to present their own examples for the class to solve. They were permitted to use any model (but were asked to identify the model that they used) or no model if they chose to work solely in their head, and they were asked to explain their strategy.

Students decided that the number combinations often determined (for them, individually) what model they might use. For instance, to solve 47+26, some students preferred the hundred chart, as a *going-through tens* strategy could easily be employed. However, others preferred the empty number line for the same reason. In contrast, to solve the subtraction example 64-28, the ninety-nine chart was preferred by some, again because the *going-through tens* strategy could be employed; while, others preferred the empty number for the same reason. When three-digit examples were presented, for instance, 192-28, some students suggested constructing hundred and ninety-nine charts that covered these numbers; while others suggested the empty number line was more appropriate, because of its flexibility. By this stage of the project, however, some students were beginning to solve examples without models, and using strategies that did not reflect the support of models.

> I did 99+47 by saying, that's the same as 100+47, but then took one away.

CONCLUDING COMMENTS

The focus in this teaching experiment was not merely on developing mental computation strategies, but on higher order thinking – reasoning, critiquing, engaging in sense making, both in what they did and in what they said. The teacher suggested that there were higher participation rates and enthusiasm on the part of the students compared with previous mathematics lessons. Strategic thinking was encouraged, rather than merely "getting the right answer". The teacher reported that in other number work students were exhibiting a sense of number – they were talking about numbers in more flexible ways and making more sense of computations. When students were reintroduced to formal written algorithms (after the completion of the teaching experiment), they made sense of the algorithms – rather than merely following procedures. The students were making connections.

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IN THE MIDDLE OF NOWHERE: HOW A TEXTBOOK CAN POSITION THE MATHEMATICS LEARNER

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We outline a framework for investigating how a mathematics textbook positions the mathematics learner. We use tools and concepts from discourse analysis, a field of linguistic scholarship, to illustrate the ways in which a can textbook position people in relation to mathematics and how the text can position the mathematics learner in relation to classmates and to the world outside of the classroom. We conclude with a general consideration of alternative language forms, which, we suggest, ought to include linguistic recognition of the moves associated with mathematisation.

INTRODUCTION

> The equations below express the relationship between a person's height, H, and femur length, F. Males: H = 27.5 + 2.24F. Females: H = 24 + 2.32F. […] If a man's femur is 20 inches long, about how tall is he? (Lappan *et al*, 1998, p. 21)

A student could respond in many ways to this prompt from a mathematics textbook. She might just follow the instructions, or she might pose further questions. Her questions could be mathematical or *about* mathematics, such as the following: Who found these equations? How? Why would someone want to use these equations? For whom and why am I finding the man's height? Should my answer depend on the situation?

Questions such as these will be addressed in this report's investigation of mathematics textbooks and how they position students. In this paper, we use tools and concepts from discourse analysis, a field of linguistic scholarship, to demonstrate a framework for examining the way a textbook can influence a mathematics learner's sense of mathematics. We are interested in the way textbooks position people in relation to mathematics and how the text positions mathematics learners in relation to their classmates and teachers and to their world outside of the classroom.

CONTEXT AND METHOD OF THIS INVESTIGATION

To illustrate our textbook analysis framework, we draw examples from the *Connected Mathematics Project* (CMP) (Lappan *et al*, 1998). This middle school mathematics curriculum was developed and piloted in the 1990s in the United States with funding from the National Science Foundation and was recently classified as 'exemplary' by the U.S. Department of Education's Mathematics and Science Expert Panel (1999). CMP textbooks were written to embody the mathematics *Standards* (1989, 1991) published by the National Council of Teachers of Mathematics, an organization comprising mathematics educators from across Canada and the USA.

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Following the suggestions in the *Standards* documents, this CMP curriculum uses problem solving contexts to introduce students to big ideas in mathematics and the curriculum encourages exploration and discussion. Because it makes extensive use of problem solving contexts, this text material is significantly more verbose than more conventional mathematics textbooks.

There are many aspects of this text material that we appreciate. The materials are engaging, mathematically rich and based on constructivist principles within which learners are encouraged to develop and discover big mathematical ideas. The text series' wordiness makes linguistic analysis much easier in many ways – because there are many sentences to analyse – but also much more complex. As researchers who are interested in discourse, ideology, and power, of which, student positioning is a part, we assert the importance of analysing the way students are positioned by this textbook series and other textbooks.

We chose to use CMP to demonstrate our framework because we both, independent from each other, have been involved in detailed critical linguistic analyses of its curriculum materials. (See, for example, Herbel-Eisenmann, 2004.) Indeed, the series' author team has demonstrated its interest in healthy classroom discourse by asking for this linguistic analysis. In our analyses, we examined the materials as a whole made up of parts. We examined the text word-by-word, sentence-by-sentence, and section-by-section to identify and classify particular linguistic forms. We also considered linguistic forms as parts of a whole (i.e., each word is part of a sentence as well as part of a section within a unit) and interpreted the function of language forms in the context of their location within the text. We particularly attended to words and phrases that constructed roles for the reader, relationships with the author and mathematics from a particular epistemological stance.

Due to space limitations, for this report, we draw on our analyses of one unit in the curriculum: *Thinking with Mathematical Models (TMM)* – a 64-page soft-bound booklet. This particular unit focuses on mathematical modelling through the use of experiences that require data collection (e.g., measuring the breaking weight of a set of paper bridges based on the number of pennies each could hold) as well as predefined modelling situations (e.g., different equations that model the economic impact of raising the price of cookies in a bakery). We draw on our textual analyses to show how the unit positions the mathematics learner.

PERSONAL POSITIONING IN RELATION TO MATHEMATICS

To develop an understanding of how the people are positioned with respect to mathematics, we focus on two language forms: personal pronouns and modality. With these forms we see who the text recognises as the people associated with mathematics and how they stand in relation to the mathematics.

First person pronouns (*I* and *we*) indicate an author's personal involvement with the mathematics. Textbook authors can use the pronoun *I* to model an actual person

doing mathematics, and can draw readers into the picture by using the pronoun *we* (though there is some vagueness with regard to whom *we* refers). The use of the second person pronoun *you* also connects the reader to the mathematics because it speaks to the reader directly, though it can be used in a general sense, not referring to any person in particular (see Rowland, 2000)

In *TMM*, first person pronouns were entirely absent. Morgan (1996) notes that such an absence obscures the presence of human beings in a text and affects "not only […] the picture of the nature of mathematical activity but also distances the author from the reader, setting up a formal relationship between them" (p. 6).

The second person pronoun *you* appears 263 times in *TMM*. Two of these forms, in particular, are relevant here: 1) ***you* + a verb** (165 times); and 2) **an inanimate object + an animate verb + *you* (as direct object)** (37 times). The most pervasive form, ***you* + a verb**, includes such phrases as 'you find', 'you know' and 'you think'. In these statements, the authors tell the readers about themselves, defining and controlling what linguists Edwards and Mercer (1987) call the 'common knowledge'. The textbook authors use such control to point out the mathematics they hope (or assume) the students are constructing.

Linguists use the term *nominalisation* to describe language structure that obscures human agency. In *TMM*, the other common *you*-construction (**an inanimate object + an animate verb + *you* (as direct object)**) provides a striking example of nominalisation in the text. In this text, inanimate objects perform activities that are typically associated with people, masking human agency – for example, 'The graph shows you…' and 'The equation tells you...'. In reality graphs cannot show you anything; it is the person who is reading the graph who determines what the graph 'shows'. This type of nominalisation depicts an absolutist image of mathematics, portraying mathematical activity as something that can occur on its own, without humans. Scattered throughout these instances of nominalisation, however, the text refers to human actors in the mathematical problems. These actors are named by occupation, by group (e.g. 'riders on a bike tour'), or by fictional name (e.g. 'Chantal'). The combination of nominalisation alongside these human actors may send mixed messages to the reader about the role of human beings in mathematics.

The modality of a text also points to the text's construction of the role of humans in relation to mathematics. The modality of the text includes "indications of the degree of likelihood, probability, weight or authority the speaker attaches to an utterance" (Hodge and Kress, 1993, p. 9). Modality can be found in the "use of modal auxiliary verbs (*must, will, could*, etc.), adverbs (*certainly, possibly*) or adjectives (e.g., *I am sure that*…)" (Morgan, 1996, p. 6). With such forms, the text suggests that mathematics is something that people can be sure about (for example, with the adverb *certainly*) or that people can imagine possibilities within mathematics (for example, with the adverb *possibly*).

Linguists use the term *hedges* to describe words that point at uncertainty. For instance, because of the hedge *might* in 'That function *might* be linear', there is less certainty than in the unhedged 'That function is linear'. The most common hedge in the *TMM* text is *about* (12 instances), followed by *might* (7 instances) and *may* (5 instances). This kind of hedging could raise question about the certainty of what is being expressed and could also, as Rowland (1997) asserts, open up for readers an awareness of the value of conjecture.

Modality also appears in verb choice. The modal verbs in this text include *would* (55 times), *can* and *will* (40 times each), *could* (13 times), and *should* (11 times). The frequency of these different modal verbs indicate an amplified voice of certainty because the verbs that express stronger conviction (*would*, *can*, and *will*) are much more common than those that communicate weaker conviction (*could* and *should*). The strong modal verbs, coupled with the lack of hedging, suggest that mathematical knowledge can be and ought to be expressed with certainty.

HOW TEXT CONSTRUCTS AND POSITIONS A MODEL READER

In the second half of this report, we draw attention to how the *TMM* booklet speaks to and constructs what Eco (1994) calls the *model reader*. Any text gives linguistic and other clues about the audience-in-mind, which can be described both as the intended reader or the reader the text is trying to create. While textbooks are written for both teachers and students to read, we concentrate only on the positioning of the student–reader, whom we call the text's *model student*. To do this, we focus on the language 'choices' made by the authors (though we recognise that most choices are subconscious). Morgan (1996, 1998), in her extensive study of mostly student-authored mathematical writing, also focuses on author choices in her analyses:

> Whenever an utterance is made, the speaker or writer makes choices (not necessarily consciously) between alternative structures and contents. Each choice affects the ways the functions are fulfilled and the meanings that listeners or readers may construct from the utterance. […] The writer has a set of resources which constrain the possibilities available, arising from her current positioning within the discourse in which the text is produced. (Morgan, 1996, p. 3)

Language indirectly indexes particular dispositions, understandings, values, and beliefs (Ochs, 1990). By examining the language choices authors make, we can see how they construct the model reader and position mathematics students. From this, we can infer the student's experience of mathematics.

Student positioning in relation to particular people

Though the *TMM* booklet presents mathematics as impersonal and pre-existent (or, at best, coming from an obscured body of impersonal mathematicians), most students actually learn mathematics within a social environment. Given their experience of this community, we ask how the text positions students in relation to the people around them – their teachers and their peers. Various aspects of the text, including the

graphical work and the ways the authors directly address the reader, position the model student in relation to the people around her.

Pictures alongside verbal text strongly impact the reader's experience of the text. In the analysed text, for example, there are 24 graphical images (in addition to graphical representations of particular mathematical expressions). Of these, 17 are 'generic' drawings and seven are 'particular' photographs. To illustrate, a *drawing* of a boy could represent any boy, whereas a *photograph* of a boy is a representation of only the one boy in the photograph. The textbook's preference for the more generic drawings mirrors its linguistic obfuscation of particular people, as we detailed above.

This obfuscation is heightened by the absence of people in the graphic images. Only one quarter of the images have people in them. In these, we find people playing on teeter-totters and operating cranes (among other things), but only one image of a person doing mathematics. Significantly, this image is a drawing (which makes the person generic) and the image only shows a person's hand conducting a mathematical investigation. This disembodied, generic hand parallels the lost face of the mathematician in agency-masking sentences such as the ones discussed above.

Mathematics addresses the student literally too, with sentences structured in the imperative mood. Morgan (1996) asserts such imperatives tacitly mark the reader as a capable member of the mathematics community. However, we suggest that such positioning is not clear from the mere presence of imperatives. Rotman (1988) distinguishes between what he calls *inclusive* imperatives (e.g. *describe*, *explain*, *prove*), which ask the reader to be a *thinker*, and what he calls *exclusive* imperatives (e.g. *write*, *calculate*, *copy*), which ask the reader to be a *scribbler*. Mathematicians think and scribble.

We find significance in Rotman's terms: *inclusive* and *exclusive*. The thinker imperatives construct a reader whose actions are *included* in a community of people doing mathematics, whereas the scribbler imperatives construct one whose actions can be *excluded* from such a community. The student who 'scribbles' can work independent from other people (including her teacher and peers). A drawing in this textbook seems to capture the text's view of the model student's relation to other people while doing mathematics. A generic boy sits alone in an empty theatre, with his hand up, as if to hold other people away (p. 15).

Though the textbook we are examining has many more exclusive imperatives than inclusive (221 exclusive and 94 inclusive), we caution against premature characterization of the text's preference for scribbling. When we follow the flow of imperatives in a text, we can form a better picture of the model student constructed by it. For example, in this textbook's initial 'investigation', we find the following stream of imperatives (pp. 5, 6): *make* a bridge, *fold* the paper, *suspend* the bridge, *place* the cup, *put* pennies in, *record* the number, *put* strips together, *find* the weight, *repeat*, *do* the experiment, *make* a table, *graph* your data, *describe* the pattern, *suppose* you use. Of these 14 imperatives, the first 12 are exclusive and the last two

inclusive. Does this mean that scribbling is privileged over thinking because of the ratio 12:2? Not necessarily. It is appropriate to scribble before thinking. Furthermore, if a text makes too many thinker demands, it could be tacitly encouraging students to jump from one thought to another without dwelling on any of the thinker demands.

But this raises a further question about the model student's relation to the people around her. The text makes no mention of the other people involved in the thinking imperatives. Looking again at the closing imperatives in the above sequence, we note that a student could ask, 'Describe? Describe to whom?' The same goes for *TMM*'s most prevalent thinker imperative: 42 of the 94 inclusive imperatives are *explain*. If the students are expected to explain their reasoning, they might reasonably expect some direction from the text with regard to their audience. Morgan (1998), in her analysis of students' mathematical writing, notes the significance of the students' sense of who their audience is.

There is an exception to this absence of reference to the people around the model student. At the end of each section, the text includes a page called 'Mathematical Reflections'. In all of these sections, the same sentence follows the prompts for reflection: "Think about your answers to these questions, discuss your ideas with other students and your teacher, and then write a summary of your findings in your journal" (e.g. p. 25). Here the text does recognise the presence of other people around the student. From this, the model student might see herself doing mathematics independent from other people, but thinking *about* her mathematics in community. Only metacognition is interpersonal, it seems.

Student positioning in relation to the world

While we suggest that explicit linguistic reference to the student's audience would be appropriate in this mathematics textbook, we recognise that such reference may raise further questions. As soon as a student is led to explain something to other people, she is likely to require a reason and context for the explanation. She would want to tailor her explanation to the situation and the people's needs within the situation.

Most of the questions and imperatives in the analysed textbook are referred to as 'real life', 'applications' (applications of mathematics to real life, we presume) and 'connections' (connections between mathematics and real life, we presume). Though the textbook consistently places its mathematics in 'real' contexts (with few exceptions), linguistic and other clues still point to an insignificant relationship between the student and her world. When we compare the instances of low modality (expressing low levels of certainty) with those of high modality, we begin to see what experiences the text foregrounds. First, we look at references to the student's past experiences. The text refers with uncertainty to the student's experiences outside the classroom using hedging words like *probably* or *might* – for example, when referring students to their experiences with a teeter-totter the text tells the student "You *might* have found the balance point by trial and error" (p. 28, emphasis ours). However, the text expresses certainty about the student's abstract mathematical experiences, as in

"In your earlier work, you saw that linear relationships can be described by equations of the form $y = mx + b$" (p. 9). This certainty may seem odd, considering that there are no indicators about sequencing in the textbook series, but there are pragmatic reasons for such assertions. Because the authors know what the curriculum offers, they can work under the assumption that the student has learned particular mathematical ideas. Yet, the authors cannot really know who their readers are. *TMM*'s authors hedge statements about students' experiences to acknowledge this. We are left wondering what message the model reader could take away from this. Would the reader think that her *everyday* experiences matter less than her *mathematical* experiences?

This possible suggestion, that everyday experience is less important, is substantiated by the word problem settings in this textbook. Though most problems are situated in everyday contexts, the sequencing of problems jumps from one setting to another. For example, in the first set of 'applications, connections and extensions' (pp. 15f), the student is told to exercise her mathematics on bridge design, bus trip planning, economic forecasting, fuel monitoring, and the list goes on. Similar to our concern for well-placed, inclusive imperatives to promote students' careful thoughtfulness, we suggest that longer series of problems relating to any particular context would guide students to see real life in its complexity, as opposed to operating on the basis of shallow glimpses of small-scoped data sets. Unlike the problem sets, this textbook's 'investigations' do dwell longer on given problem contexts.

REVISIONING MATHEMATICS TEXT

In summary, the analysed textbooks seem to construct a model reader that is independent of personal voices in mathematics. There are no mathematicians but there are people who use mathematics: pure mathematics is impersonal and applied mathematics is more personal. There is an impersonal body of mathematics and there are some children doing some of this mathematics. The model reader is expected to do mathematics independent of the people surrounding her and independent of her classmates and teacher, but to reflect on it within the classroom community. The reader's mathematics is also independent from her environment.

This sense of detachment should come as no surprise because mathematics is characterised by abstraction: Balacheff (1988) has noted the necessity of decontextualisation, depersonalisation and detemporalisation in logical reasoning. We draw attention to his prefix *de* (as opposed to the alternative prefix *a*). Balacheff does not call for apersonal, atemporal, acontextual language. Rather, mathematics requires the *move* from personal to impersonal, and perhaps back. It is the *move* from being situated in a physical and temporal context to finding truth independent of context. Just as mathematics is about the moves from the particular to general and back again, we see mathematisation as the moves between the personal and impersonal, between context and abstraction.

Though our textbooks do not typically recognise the moves we associate with mathematisation, we see room in textbook use for the recognition of mathematisation. In typical classrooms, the textbook is mediated through a person (the teacher) in a conversation amongst many persons (the students). In such a community, even if the textual material at hand does not recognise the role of persons in mathematics, there is room to attend to persons and contexts. There is room to draw awareness to the dance of agency between particular persons (whether they be historical or modern, professional or novice mathematicians) and the apparently abstract, static discipline of mathematics (c.f. Wagner, 2004). For textbook writers, there is much room for change. Considering this possibility, we wonder how students' experiences of mathematics would differ if their textbooks recognised persons, their contexts and mathematisation more.

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CHINESE WHISPERS – ALGEBRA STYLE: GRAMMATICAL, NOTATIONAL, MATHEMATICAL AND ACTIVITY TENSIONS

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This paper analyses students' written work from an activity based on two well known games in the UK: Chinese Whispers and Consequences. Within this activity students were asked to translate formal algebraic equations into word statements and vice versa. Using the framework of affordances and constraints to offer an account for what the students' wrote, I have identified some tensions between four different aspects of the activity: the grammar of the word statements; notational conventions; mathematical sense-making; and the rules of the activity itself. Through an increased awareness of these tensions I surmise that such tensions are not special to this activity and may be taking place during daily mathematical activity in classrooms.

INTRODUCTION

I have argued elsewhere (Hewitt, 2001) for a distinction between students' ability to work algebraically and students' competence and confidence with formal algebraic notation. It is the latter what often presents a barrier for students, both when being asked to write formal notation and when they have to interpret it. Radford (2000), for example, said that students tended to state generalities in words rather than in formal notation and seemed to lose a sense of the original figures from which the generalisations were made when trying to write in formal notation. Zazkis and Liljedahl (2002) and Neria and Amit (2004) also report students finding it difficult to express themselves in algebraic notation and that it is still not natural for high achievers to use formal notation even after years of instruction. The reverse process of interpreting formal algebraic notation is also problematic for most learners (Sáenz-Ludlow and Walgamuth, 1998). Formal notation has a particular structure determined not by students themselves but by mathematicians long in the past. The reasons behind decisions made are not transparent and so the conventions of formal notation can appear quite arbitrary for students who are then left only with the surface features of the notation. These surface features rarely point to the underlying systemic structure (Kieran, 1989) such as order of operations or commutativity. Zazkis and Sirotic (2004) have used the terms *transparent* and *opaque* to indicate whether a certain property can be seen or derived from a given representation or not. Of course, this is dependent upon the awareness of the individual looking at a representation. For example, certain properties are transparent in the expression $2(3+4)$ for those who know about brackets and implied multiplication and opaque for others. Linchevski and Livneh (1999) have looked at whether students have a structure sense when working with arithmetic statements and Hoch and Dreyfus (2004) have looked

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 129-136. Melbourne: PME.

at this with respect to algebraic statements and manipulations. The paper I present here looks at ways in which students structure their reading and writing of algebraic equations and written word statements of such equations. In particular the tensions which can exist when asked to translate from one form to the another.

METHODOLOGY

The original interest behind this study was to look at the ways in which students translate word statements into formal algebraic notation and vice versa. The vehicle I used for this was based on two well-known games in the UK. The first is called Chinese Whispers, where someone whispers a message to another person, who in turn whispers what they heard to a third person, etc., until the message goes round a loop and returns to the original person. This person then states the original message and the message at the end of the loop (these usually being different). The second game is called Consequences. This starts off with a group of people (again in a form of a loop) who all write down on their individual piece of paper a boy's name. Then the paper is folded over to hide what has just been written and passed onto the next person, who writes down a girl's name. Again the paper is folded and passed on. This continues with the following being written: (i) a boy's name; (ii) a girl's name; (iii) where they met; (iv) what he said to her; (v) what she said to him; and (vi) what the consequences were. The paper is then unfolded and each person reads out the whole sequence on their sheet.

For my activity each student received a unique sheet where there was an original algebraic or arithmetic equation (different for each student) given at the top of the sheet. Students were then asked to translate this equation into a word statement (not using numerals or mathematical signs, except for letters such as x), fold over the sheet so that only their word statement could be seen and pass it on to the next person in the loop. This continued with a sheet going round the loop with a different person each time translating from equation to word statement, then back to equation, then word statement, etc., for up to ten translations. Each time only the last person's writing could be seen.

The equations at the top of each of the sheets were chosen to address particular issues which are not relevant to this paper and so I will not go into detail about these here. Before having one of these issues in mind, I looked through the students' sheets and noted anything which struck me to be of interest. This was working within a Discipline of Noticing (Mason, 2002) where what I noticed related to the awareness I had at the time of reading and that the reading informed that awareness. In Mason's language, I not only noticed but also marked and recorded and began to form links between the separate noticings with strong links being given labels. One of those links was the issue of students appearing to stress the visual symbols or words they saw on paper, rather than the mathematical operations which they represented or which were implied through those symbols/words. For example, whether a bracket in $2(3+4)=14$ was seen as merely a bracket, or as an indication of order of operations

and that a multiplication was taking place. Since this concerns two different aspects, notation and mathematical structure, and since the some of the students' written equations became notationally incorrect, I did not find either the framework of surface and systemic structure (Kieran, 1989), nor structure sense (Linchevski and Livneh, 1999) particularly helpful for this situation. Instead I began to analyse the students' sheets making use of Greeno's (1994) identification of affordances, constraints and attunements which has been developed recently in a mathematical context by Watson (2004). The grammar of word statements has certain affordances which allow possible readings and parsing of a given statement. There are also constraints due to the rules of grammar, and any particular individual may have regular patterns in the way in which they respond to certain grammatical forms, which are examples of attunements. Of course, mathematical language will have its own grammar, for example it might not have a verb. So the grammar of these word statements will be different to the grammar of a novel.

There will also be affordances, constraints and attunements relating to the syntactic conventions of formal algebraic notation, and again for any mathematical sense-making developed when looking at an equation or a word statement. The notion of affordances, constraints and attunements provided a framework within which I have tried to account for some of the students' writings. In this paper I am not reporting the tracking of individual students through their writing across several sheets, which is where the attunements of individual students to aspects of this activity were to be found. So for this paper I have used a framework for my analysis of affordances and constraints only. Through working with this analysis I re-articulated my area of interest as concerning the apparent tension between four aspects (each of which has its own related affordances and constraints): the grammar of the word statements; the conventions of formal algebraic notation; mathematical sense-making for these statements/equations; and the rules of the Chinese Whispers-type activity. Of course, the extent to which someone will explore the affordances which something offers is down to the awareness they have, likewise the extent to which someone feels constrained will be down to the awareness they have of possibilities going against those constraints. So, although I will talk about affordances and constraints of certain aspects of the activity, the extent to which they were explored or experienced by students will have varied.

Teachers from 17 schools gave one of their classes the activity outlined above. The teachers were following a taught Masters course on the teaching and learning of algebra and were invited to carry out the activity on a class of students with whom some algebra had been taught including the conventions of mathematical notation. The schools were mainly from the West Midlands area and included two selective schools. The classes varied between year 7 and year 10 with just one class being year 11. The National Curriculum levels of the students ranged from four to eight.

DISCUSSION

In reporting some of the detail of what students from a particular school wrote, I will use the convention of putting in brackets after the name of the first person from that school with a letter to stand for the school they were from. I will also include the year of the class and a number which represents a guide to the national curriculum (NC) level to which their teacher felt they were performing (e.g. Dave (C, Yr7, NC6)). Future references to students will assume the same school, year and NC level until I indicate otherwise by including a bracket with the new school letter, year and NC level. Then further students will be assumed to be from this new school, etc.

I will offer a few examples of situations where there appears to be tensions between the grammar, the notational conventions, and mathematical sense-making along with the rules of the activity itself. With nearly half of all the sheets ending with an equation different to the original equations on the sheet, these are not isolated examples of the points being made. However, my interest here is in analysing certain examples rather than making more general quantitative statements and in doing so educate my own awareness as to the tensions which can exist within different aspects of any activity.

Stressing the activity

As a generality, the rules of the activity (to translate from words to equation and vice versa) seemed to dominate over whether the equation students were writing was mathematically correct or not. Eight of the 28 sheets had starting equations which were purely numeric and of those, five were mathematically incorrect equations. After students had completed the activity there was nearly the same percentage of sheets which had their last equation the same as the original one on the sheet, whether those equations were initially mathematically correct or not (52% for correct equations and 54% for incorrect equations). So generally the students appeared to engage with the task of faithfully translating an equation irrespective of its mathematical correctness.

I offer here some examples of when the rules of the activity may have been stressed. Tim (D, Yr9, NC4) read *eight minus brackets two + three equals three* and translated this into $8-(2+3=3$. Here he appears to have carried out a one-to-one translation with each word being translated into a symbol:

eight	*minus*	*brackets*	*two*	+	*three*	*equals*	*three*
↓	↓	↓	↓	↓	↓	↓	↓
8	–	(	2	+	3	=	3

Affordances of the activity allowed students to choose their own way of expressing whilst a stated constraint is that students were to write what they saw (in a different notational form) and not just anything. Tim was successful in that he wrote what he saw and this appears to have overridden a global view of what role a bracket plays

within a mathematical expression and the mathematical meaning of the equation as a whole. This is not a perfect one-to-one translation however, as the word *brackets* is plural but a single bracket was written. There is a potential tension here, irrespective of whether Tim felt the tension or not, that plurality is stated yet a pair of brackets '()' are not normally written next to one another in a mathematical expression. Also the words *bracket* or *brackets* do not appear again in the word statement and so multiple left-brackets cannot be closed later on by associated right-brackets. So to follow the grammar of the word statement (more than one bracket) conflicts with the conventional constraints of formal notation and so Tim is in a 'no-win' situation.

The next person in the loop, Emily, also appears to have performed a one-to-one translation by writing *open brackets* for the '(' symbol in her word statement of *eight take away open brackets two add three equals three* and did not include a written *closed brackets*. Again, Emily obeyed the constraints of the activity and in doing so broke the constraints of notational convention.

Stressing grammar and notation

Sue (A, Yr8, NC4/5) inherited $3(4+2=80$, after a sequence which had gone: $3(4+2)=18$ → *three brackets fore add two equs eightty* [sic] → $3(4+2=80$. She wrote *three bracket four plus two bracket equals eighty* demonstrating an awareness of the notational need for a second bracket and where it would be placed. In contrast to Tim and Emily, she broke the constraints of the activity – that she was to write in words what she saw in symbolic notation – rather than breaking the constraints of the notation.

The fact that this was a paper activity with no-one checking what was written meant that one of the affordances of the activity was that the rules could be broken! This would be different if, for example, this activity was carried out on a computer with a program that would not accept a statement which included anything extra to what was already there on the line above. So although the activity had stated constraints, as laid out by the rules, the medium through which it was played meant that these theoretical constraints could be ignored. In some isolated examples this was taken to an extreme with one student writing down his own made-up equation which was unconnected to what had come before. In Sue's case, the mathematical sense was also ignored with no attempt to make the equation mathematically correct, only notationally correct.

Continuing this sequence, Joe inherited Sue's statement of *three bracket four plus two bracket equals eighty* and wrote $(3)4+2(=80)$ showing little awareness of the notational conventions or the mathematical role that brackets play in expressions. Once again, the affordances of the activity, by being played on paper without some form of checking, meant that the constraints of formal notation could be broken. However, the sequence was then finished in the following way:

Joe: $(3)4+2(=80)$ → Alan: *three in brackets four add two brackets equals eighty* → Nigel: $3(4+2)=80$.

There are many ways in which Alan's statement could be read, three of which are:

three in brackets four add two brackets equals eighty; or

three in brackets four add two brackets equals eighty; or

three in brackets four add two brackets equals eighty

The first was the way in which Alan had interpreted the statement and the second was the way in which Nigel interpreted the statement. I assume that Nigel didn't choose to interpret the statement the third way because this did not fit with notational conventions. The constraints of notation provide a context which can influence the way a word statement is read. The affordances of grammar are quite different to that of formal notation. The latter offers a strength in clarifying a particular way of reading and interpreting. The former offers a strength in allowing exploration of poetry and humour through the fact that there can be different ways of reading and interpreting text. For example, a Christmas cracker joke is as follows:

Why couldn't the skeleton go to the Christmas Party?
He had no body to go with!

The joke would not be possible without two ways of reading *no body*, one where they are associated together to form a single word and one where they are separate.

Pimm (1995, p.xii) noted that with plane spotting and car number plates the actual symbols are often the focus of interest rather than the referents and here notation was usually the focus rather than a mathematical operation or property implied by those symbols. For example, consider the following sequence from School C (Yr8, NC6). Note this is after the initial equation of $6x-1=y-2+3$ had already been changed into just an expression:

$6x-y-(2+3)$ → *six x minus y minus 2 add three in brackets* → $6x-y(-2+3)$ → *six x take away y bracket minus two plus three bracket*

Six x is stated rather than *six times x*. The notation is described rather than the underlying mathematics. The use of the words *bracket/brackets* describe a notational sign rather than what it means in terms of mathematical operations. The ambiguity in the word statement of exactly where the brackets are placed has led to an interpretation which changes a mathematical operation and the sign of one of the numbers. An alternative for the last word statement could have stressed the mathematics by saying something like: *six times x minus y multiplied by minus two and three added together*. This is also ambiguous and I am not suggesting it as a preferred option, only as an example of a case where the mathematics is stressed rather than the notation. This was rarely done by any student.

Stressing mathematics

20 of the 28 sheets had starting equations which involved letters. This meant that there was not such an issue about whether these were mathematically correct or not. It was more a matter of whether they were notationally correct. Notation can be considered as 'looking right' even without someone carrying out mathematical calculations in making that judgement. However, even within the equations containing letters there were still examples of students stressing the mathematics. In one sequence the start equation was $3(x+2)=5$ and Bridget (A, Yr8, NC4/5) translated this as follows:

three (x plus two) = five
x = 0

She had broken some of the rules of the activity by using symbols for brackets and the equals sign. Then, having written out the word statement, she appears to have then tried to solve it, which was certainly not within the stated constraints of the activity. Looking at her word statement, if the strange curly symbols of x and brackets were ignored then the statement is mathematically correct: three plus two does equal five. So, I interpret that Bridget wanted to ignore the x and so she implied it must equal zero. Irrespective of this incorrectness, Bridget considered and tried to address the mathematics of the statement and shown she has made mathematical sense of the equation. An affordance of working mathematically is that new statements can be deduced from other statements and this is what Bridget has done.

Later in the same sequence, Martin translated *three add x add two equals five. x equals zero* to the following:

3+0+2=5 X=0

Here, I suggest that Martin has not solved the equation but substituted in the given value of $x=0$. Again, he has attended to the mathematics of the situation and stressed this above the stated constraints for the activity.

SUMMARY AND REFLECTIONS

Retrospectively I would like to have spoken with the students about their writing and to explore tensions they may have experienced. Partly due to the organic process through which this particular focus appeared, it was too late to go back and arrange for those conversations to take place. However, the existing analysis of the written sheets has raised for me an increased awareness of the tensions between the grammatical, notational and mathematical aspects of this activity. The affordances of one of these gives opportunities which can conflict with the constraints of others. Although this has been an artificially created activity, it has brought to my attention the potential tensions between different aspects of any classroom activity. How often have I heard a teacher (and myself) say to a student "You were not supposed to do

that". Yet a student may just have been stressing one aspect of the given activity and exploring the affordances it offers and this has ended up going against the constraints of a different aspect of the activity. It is sometimes difficult for students to find a way of meeting the constraints of each aspect of the activity at the same time. Even if it is possible it may be at the expense of exploring affordances. So a tension arises for us as educators: do we want students to explore affordances? If so, we have to accept that such exploration may lead to conflicts with other aspects of the activity.

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PEDAGOGY OF FACILITATION: HOW DO WE BEST HELP TEACHERS OF MATHEMATICS WITH NEW PRACTICES?

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Using comparative analysis two orientations to helping teachers implement new practices in mathematics are examined through four characteristics of practice. A design adherence orientation to facilitation emphasizes classroom activity following the guidelines of the teacher manual. By contrast, a contextually responsive orientation using structural elements of a programme emphasizes student understanding. The paper raises questions about attributes of effective facilitation.

PREAMBLE

Since the release of the results of the Third International Mathematics and Science Study (TIMSS) there have been concerns about student achievement in mathematics in many western English-speaking countries, including New Zealand. The TIMSS results for New Zealand (Garden, 1997) prompted a professional development initiative that aimed to improve teachers' classroom practice in mathematics.

One major kind of response in western education systems has been the declaration of "standards" and the increased attention to teachers' content knowledge. The concerns and their implications have continued to intensify with increased attention to new theories of learning. Specifically, the extensive interest in "mediated learning experiences" presented by Vygotsky (1978) and Newman et al. (1989) have created new kinds of dilemmas about the management of the learning environment.

Yet in the mass of reform efforts there has been little attention paid to helping teachers implement new practices in everyday teaching aimed at improved student outcomes. This paper explores what appears to be a vacuum in the support of teachers and examines the "pedagogy of facilitation". In other words, in what ways can teachers be helped to effectively implement new practices in the classroom? Specifically examined is how the help or facilitation is influenced by the orientation of the facilitator's pedagogy to various characteristics of new practice.

This paper posits that ways of helping teachers implement a reform can be usefully shaped by the concepts and strategies that underlie the materials or activities. Following Sewell's (1992) work on structure and agency, the structural elements provide schema and resources for a contextually responsive approach to improving practice. The degree of knowledge of the discipline, progressions in learning and pedagogy are also critical influences on the impact of facilitation on teachers' practice and children's learning.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 137-144. Melbourne: PME.

A COMPARATIVE ANALYSIS: TWO ORIENTATIONS TO FACILITATION

It is generally assumed that for a new practice to be successfully established across a system, such as primary education, teachers need support. This paper compares two approaches to the "pedagogy of facilitation". It has been common to emphasize material-based experiences for children when helping teachers. This highlights getting students engaged with the mathematical activity defined by the materials' designers. In this approach the assumption is that the help or guidance is built into the teachers' handbook and their literal knowledge of the materials-based activities. Facilitators often, in respect to materials' designers, emphasize the materials' attraction for children and the need to follow the handbook sequencing.

By contrast, a second approach to facilitation is to emphasize guidance for teachers centering on structural components so that they gain skills needed for flexibility in classroom use.

Four characteristics common to new programmes and practices are examined in terms of the emphasis they are given for helping teachers' classroom implementation. Each characteristic is examined in terms of two views a facilitator may follow in working with teachers in presenting the new initiative.

The comparison of the two approaches seeks to highlight key differences through examining kinds of emphases for each orientation (see Table 1). The four characteristics of new practice include teachers' manual or handbook; materials (activities); teaching method; and modelling new practice.

Characteristic of new practice	**Orientation of facilitator's actions** A: Facilitation disposed towards design adherence	**Orientation of facilitator's actions** B: Facilitation disposed towards contextual responsiveness
Teachers' manual or handbook	Emphasis is given to adhering to the programme design and the handbook.	Emphasis is given to using structural elements to interpret the handbook.
Materials (activities)	Emphasis is given to engaging students actively with the materials.	Emphasis is given to teachers' understanding of the mathematical purposes and concepts underlying the materials.
Teaching method	Emphasis is given to the experiential effect of activities.	Emphasis is given to students' representations of their mathematical understandings.
Modelling new practice	Emphasis is given to students' "proper" use of the materials.	Emphasis is given to extending concepts in response to students' actions and explanations.

Table 1: Comparison of orientations of facilitator's actions towards changing mathematics teaching practice.

Each characteristic is discussed in terms of the kinds of emphasis it is given for classroom teaching practices. The order of the discussion is parallel for each characteristic: that is, first, the emphasis given to "design adherence" facilitation mode; second, emphasis given to a "contextual responsiveness" mode[1].

COMPARISON OF ORIENTATIONS TOWARDS CHANGING MATHEMATICS TEACHING

Teachers' Manual or Handbook: Orientation A

Historically it has been presumed that the provision of curriculum materials (such as a handbook)**,** including teaching activities, was sufficient to shift teachers' practice. By following the manual it was assumed that teachers would adopt new practices. For instance, in an initiative of the 1980s teachers were directed to start all students at the same point on a programme of work rather than decide a starting point based on a student's prior knowledge (Young-Loveridge, 1987). The quotes that follow contrast a step-by-step approach in following a manual with that of working from the manual's inherent structural elements.

> This is not something *you start from A and go to Z and work your way through.* This is not telling you to throw out everything you know about teaching. You already did good things in your teaching. This is adding to your tool-box of information and adding to your knowledge as a professional. (Kay, Facilitator, ANP 2001)

> That's what we're doing, we're giving teachers a structure without giving them *"You will do page this or that"*. … I'm in favour of resources … whatever you might be using, [but] it's which bit are you going to choose. (Emma, Facilitator, ANP 2001)

Underlying adherence to following the programme from beginning to end is reassurance for teachers that important aspects necessary for building student understanding are not omitted.

Teachers' Manual or Handbook: Orientation B

Mathematical principles and practices such as stages of students' mathematical thinking and ways of advancing these can be displayed for teachers in a manual through frameworks and/or models (Ministry of Education, 2004). The important thing is how explicitly a framework is presented to teachers; that is, the actions taken by facilitators in presenting the structural elements of a programme.

> I think it's really important that they get to know the framework … to me it's one of the key things. … I am constantly bringing them back to the framework. (Nancy, Facilitator, ANP 2003)

> It's about giving simple, understandable, credible, reasonable structures for teachers to use. (Roger, Facilitator, ANP 2003)

[1] For the comparison data will be drawn from Advanced Numeracy Project (ANP) research reports (Higgins 2002, 2003, 2004). Details of the methodology used are described in these reports. This paper raises theoretical issues about the pedagogy of facilitation.

These facilitators' comments highlight their orientation to presenting the framework as a useful structure for guiding teacher decisions in implementing the programme in their classrooms. It is clear from their comments that they are explicit in their reference to the structures during the process of the teachers becoming familiar with the detail of the programme and internalising the new practice.

Materials (Activities): Orientation A

Engaging students in activity has been an idea with a long history in education in New Zealand (Nuthall, 2000). Many initiatives in mathematics education have provided teachers with resources to use in their classrooms. This orientation assumes that once teachers become familiar with new resources and incorporate them into their classroom programmes, practice will improve and student learning will be enhanced. Helping teachers from a "design adherence" orientation is expressed in the following views of teachers, especially those who might be expecting this orientation from their facilitator.

> [We need] more resources – resources/games with better instructions. (Teacher, ANP 2001)

> [We need] more activities/resources that could be just picked up and used. (Teacher, ANP 2001)

The comments reflect some teachers' focus on the surface features of activities. These teachers appear to be concerned with keeping their students active without attention to the underlying mathematics for which the activity is designed. This often leads to a desire for a never-ending supply of easily implemented activities for students to use.

Materials (Activities): Orientation B

The orientation towards materials where facilitation is contextually responsive suggests that the effectiveness of an activity or materials is judged against a framework of stages of mathematical thinking. The choice of activity now incorporates an underlying rationale with the usefulness of an activity or manipulative to teach a particular concept as paramount. It also enables a teacher to ensure an activity's mathematical integrity is retained when adapting it for the context within which they are working.

> If you keep your sights on the framework [you have] a clear sense of those progressions embedded in your head. It doesn't matter what resources you particularly have in front of you or don't [have]. (Kay, Facilitator, ANP 2001)

> You are able to talk with the teacher … not just show what the activity is, but talk about underlying concepts because we still have a lot of teachers teaching an activity without any concept. (Emma, Facilitator, ANP 2001)

A contextually responsive orientation assumes teachers use materials to guide, not prescribe, student activities in terms of the mathematical needs of the students being taught. This orientation suggests that teachers are able to internalize the new practice

through observing and interpreting children's language and representations examining the efficacy of various activities in teaching mathematical concepts.

Teaching Method: Orientation A

It has been generally accepted in a child-centred approach that it is important to present learners with a range of experiences. The rationale underlying this assumption is that this is a way to address individual learner needs.

> For the purposes of the kids going on to intermediate [school] now we have another issue because if they are going on to a school where they will be presented with [algorithms] in textbooks and they are going to be streamed on the basis of a test on vertical algorithms for number, then I think we have got a responsibility to those kids to teach them how to do it that way as well. (Emma, Facilitator, ANP 2001)

> I have changed my style of teaching maths. … Just trying to bring out more language from them whereas before it was a lot of bookwork but now I find that we hardly do a lot of things in the books. (Teacher, ANP 2002)

The perceived needs of the students are often age-bound. For younger students experiences with practical activities are promoted. For older students of mathematics more abstract exercises better suited to teacher explanation followed by pencil and paper exercises are typically followed.

Teaching Method: Orientation B

A contextually responsive approach to helping teachers with a method of teaching uses a model of representations based on the work of Pirie and Kieren's (1989) model of development of students' understanding of mathematical ideas.

> I had never thought about the middle section [imaging] before … I think I always understood that kids had to manipulate materials first … to get the concept and then when you have done that part and you can do that problem. (Teacher, ANP, 2002)

> I now encourage children to share their strategies with the class - the focus in teaching maths has changed to "how did you find out the answer?" not, "what was the answer?" (Teacher, ANP, 2001)

The structure of the model provides an important anchor point for teachers when learning new practices in teaching mathematics. The frames provide teachers with a range of professional choices based on the mathematical understanding of the students.

Modelling New Practice: Orientation A

According to this orientation the facilitator's role tends to emphasize adherence to the procedures of activities usually explained in detail in the teachers' handbook. Teachers often expect and request clear demonstration of the procedures as they then feel reassured about what they are expected to do and they feel they can be accountable to the procedures.

> Practical demonstrations are much easier to follow than the manual. (Teacher, ANP 2001)
>
> Observing the facilitator - it is much easier to watch someone than to try to interpret the instructions. (Teacher, ANP 2001)

The notion of delivery and demonstration suggest a programme that deliberately limits opportunities for variations.

Modelling New Practice: Orientation B

From a contextually responsive orientation a facilitator's modelling of practice includes highlighting the subtleties of questioning techniques to advance student understanding. The choice of pedagogy arises from the structural elements of the number framework as well as the teaching model (Ministry of Education, 2004).

> I've been watching Emma take lessons … Like they give her an answer but she'll always come back and ask them that extra step … you then start realising what your own kids are capable of. … We were stopping children because I think we were afraid that our own knowledge wouldn't go that far. (Teacher, ANP, 2001)
>
> … to model the pedagogy … a teacher gives me a group … I will raise [the activity] up or down depending on where the children are at … (Emma, Facilitator, ANP 2001)

In essence modelling from a contextually responsive orientation places facilitators in the role of mediating viewpoints, encouraging discussion, and inviting multiple perspectives.

DISCUSSION

The emphasis of the design adherence orientation is focused on procedural classroom practices. The expected procedures are usually unambiguously stated in the teachers' handbook. In essence, the activity is viewed as paramount.

In contrast, the emphasis of the contextually responsive orientation is focused on students' strategies, meaningful activities and multiple representations. In essence, the students' understanding and thoughtful investigation is paramount.

This analysis has given a focus on the pedagogy of facilitation. Having created this perspective, key points need to be investigated. In general, we need to know more about the attributes of contextually responsive facilitation through asking the question, What knowledge do we have of the pedagogy of facilitation and how can we best employ this knowledge to improve teachers' classroom practice in numeracy? Further research needs to be conducted to investigate the reasons for the confusion around the attributes of effective facilitation.

Specific questions are:

1. What dimensions of structure support a contextually responsive orientation to facilitation and enable teachers to internalize new practice for their own working context?

2. What actions of facilitators best enable an evolutionary dynamic to develop in a school that leads to new practices being sustained?
3. How are facilitators best prepared for a contextually responsive orientation?

This paper examined the notion of pedagogy of facilitation using a comparative structure to contrast two views of facilitation. Typically educators have viewed effective facilitation as that which seeks adherence to the design features of a programme of work. A new view of facilitation, that has evolved through the implementation of the Numeracy Project (Ministry of Education, 2004), suggests that through introducing a framework of ideas teachers are able to internalize the changes to their practice and sustain the programme in terms of the context within which they work.

> It's about confidence and a change in their articulating from "What do I do next with this bit?" to the kinds of things that they say about why they had done the things … about why they have changed, why they have chosen particular activities, about why they abandoned particular activities. (Emma, Facilitator, ANP 2001)

> I think we were stopping before at the talking the talk and the teachers' intellectual knowledge and we were hoping it translated into practice. (Neil, Principal, ANP 2003)

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STUDENTS' DIFFICULTIES WITH APPLYING A FAMILIAR FORMULA IN AN UNFAMILIAR CONTEXT

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This paper discusses problems many tenth grade students have when asked to apply a familiar formula in an unfamiliar context: specifically factoring a compound expression. Some of their attempts at solution are presented and discussed in terms of structure sense.

Students who have previously displayed proficiency at using algebraic techniques often have difficulty in applying these techniques in unfamiliar contexts. Specifically we have observed that many students are unable to use the formula $a^2 - b^2 = (a-b)(a+b)$, which is familiar to them, to factor the expression $(x-3)^4 - (x+3)^4$, which they have probably never met before.

Rauff (1994) has classified students' beliefs about factoring and shows how these beliefs affect how students approach a factoring exercise. Rauff's categories of students' beliefs about factoring are: reversal – the process of undoing a multiplication; deconstructive - the process of breaking down or simplifying; evaluative – a process that can be checked by multiplying out; formal – a factored expression is in the form of a product; numeric – decomposing numbers into products of primes. We will return to this classification when presenting and analysing our data.

Most of the literature on algebraic techniques is about learning to use them, not about how students use these techniques later. Most of the current research is on the use of computer algebra systems. The emphasis in the research concerning CAS is on conceptual understanding rather than on skill development. However, Pierce and Stacey (2001, 2002) have developed a framework for algebraic insight which is described as two groups of skills needed to use computer algebra systems: algebraic expectation which consists of recognition of conventions and basic properties, and identification of structure and of key features; and ability to link representations. Pierce and Stacey might explain the above difficulty in factoring as a failure of algebraic expectation.

Skemp (1976) might have called this difficulty in factoring a use of instrumental rather than relational understanding. Zazkis and Gadowsky (2001) might attribute it to opaque rather than transparent representations. Tall and Thomas (1991) might view it as a lack of versatile thinking. Another possible way of explaining the difficulty is through concept image (see for example Tall and Vinner, 1981). If a student has never seen an example similar to $(x-3)^4 - (x+3)^4$ then this kind of expression does not fit into his concept image of "the difference of squares".

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We attribute the difficulty to a lack of structure sense. In this paper structure sense is defined, and students' attempts at factoring the above expression are discussed in terms of structure sense.

Linchevski and Livneh (1999) first used the term structure sense when describing students' difficulties with using knowledge of arithmetic structures at the early stages of learning algebra. Hoch (2003) suggested that structure sense is a collection of abilities, separate from manipulative ability, which enables students to make better use of previously learned algebraic techniques. Hoch & Dreyfus (2004) described algebraic structure as it applies to high school algebra and gave a tentative definition of structure sense. Here we present a refined definition of structure sense.

STRUCTURE SENSE FOR HIGH SCHOOL ALGEBRA

A student is said to display structure sense (SS) if s/he can:

- Deal with a compound literal term as a single entity. (SS1)
- Recognise equivalence to familiar structures. (SS2)
- Choose appropriate manipulations to make best use of the structure. (SS3)

Examples to illustrate the definition

SS1 Deal with a compound literal term as a single entity.

Example: Solve: $5(3x-14)^2 = 20$. Here the student is said to display SS1 if s/he can relate to the term 3x-14 as a single entity or object.

Example: Factor: $6(5-x)+2x(5-x)$. Here the student is said to display SS1 if s/he can extract the term 5-x as a common factor (which implies relating to it as a single entity).

SS2 Recognise equivalence to familiar structures.

Example: Factor: $64x^6 - 36y^4$. Here the student is said to display SS2 if s/he can recognise the expression as possessing the structure "the difference of two squares".

Example: Solve: $(x+17)(x-12)=0$. Here the student is said to display SS2 if s/he can recognise the equation as possessing the structure ab=0.

Example: Solve: $(x^2+2x)^2+(x^2+2x)-6=0$. Here the student is said to display SS2 if s/he can first relate to the term x^2+2x as a single entity (this is in fact ability SS1), and then recognise the equation as being quadratic in that entity.

SS3 Recognise which manipulations it is useful to perform in order to make use of structure.

Example: Simplify: $(4x^2-25x+13x^2)(7x-4-7x)$. Here the student is said to display SS3 if s/he can recognise the advantage of first simplifying each term in the product before opening brackets.

Example: Find the derivative of: $f(x)=\frac{(x^3+x)^2}{x^2+1}$. A student with the ability SS3 will see the advantage of simplifying the function before differentiating. Having recognised that the function possesses the structure of a ratio of two polynomials (SS2), s/he will be able to cancel the term x^2+1 from the numerator where it appears as a factor and from the denominator (SS1).

In some of these examples the partially hierarchical structure of structure sense can be seen.

SS1 = see entity

SS2 = SS1 + recognise structure

SS3 = SS2 + useful manipulations

METHODOLOGY

A group of 190 students (from seven tenth grade classes, intermediate/advanced stream), were asked to factor the expression x^4-y^4. A similar group of 160 students (from six tenth grade classes, intermediate/advanced stream), were asked to factor the expression $(x-3)^4-(x+3)^4$. Both groups were encouraged to use the formula $a^2-b^2=(a-b)(a+b)$. In both groups the question was posed within the framework of a larger questionnaire containing several factoring exercises.

Subsequently ten students from the second group were interviewed. Students were chosen on the basis of willingness to be interviewed, and on recommendations from their teachers as to their ability to express themselves verbally.

The students' proficiency in using algebraic techniques is assumed on the basis of the fact that the target population consists of students whose grades in the previous school year were well above average. Their familiarity with use of the formula $a^2-b^2=(a-b)(a+b)$ is based on the fact that the majority of the first, similar group succeeded in using it. The assumption that the context is unfamiliar is based on talks with teachers and a survey of the textbooks used.

RESULTS

Whereas 146 out of 190 (77%) students factored x^4-y^4 correctly, only 12 out of 160 (7.5%) students factored $(x-3)^4-(x+3)^4$ correctly. We decided to study in greater detail the responses of the 148 students who didn't manage to factor $(x-3)^4-(x+3)^4$ correctly.

Forty-seven students made no (written) attempt whatsoever, leaving a blank. The responses of the remaining 101 students are summarized in Table I.

Table I Unsuccessful attempts to factor $(x-3)^4-(x+3)^4$

Belief about factoring	**Type of response**	**Number of students**	Examples
Deconstructive (the process of breaking down or simplifying)	Opens brackets incorrectly	**27**	$x^4-12x+81-x^4-12x-81=-24x$
	Obtains difference of squares, does not use formula	**6**	$(x^2-6x+9)^2-(x^2+6x+9)^2$ $=x^4-36x^2+81-x^4-36x^2-81$
Formal (a factored expression is in the form of a product) Or Reversal (the process of undoing a multiplication)	Manipulates powers incorrectly	**29**	$(x^2-9)^4$
			$(x-3-x-3)(x-3+x+3)$
			$(x-3)^2(x+3)^2$
			$(x-3)^4(x+3)^4$
	Applies formula incorrectly	**16**	$(x-3-x-3)^2(x-3+x+3)^2$
			$[(x-3)^2-(x+3)^2][(x+3)^2-(x-3)^2]$
			$a=x-3$ *and* $b=x+3$ but then $[(x-3)(x+3)]^8$
			$a=(x-3)^4$ *and* $b=(x+3)^4$ and then $[(x-3)^4-(x+3)^4][(x-3)^4+(x+3)^4]$
Numeric (decomposing numbers into products of primes)	Does not use $[(x-3)^2]^2$	**17**	$(x-3)^2(x-3)^2-(x+3)^2(x+3)^2$
	Writes 4th power as product	**6**	$(x-3)(x-3)(x-3)(x-3)-(x+3)(x+3)(x+3)(x+3)$

The responses are arranged according to Rauff's classification of students' beliefs about factoring, as far as it is possible to determine from the written answers.

We were unable to differentiate between a written answer belonging to the formal and to the reversal categories, as both result in a product. Opening brackets clearly belongs to the deconstructive category, and writing the fouth power as a product to the numeric category. We found no example of the evaluative category.

Some of these responses will be examined in greater detail, and analysed in terms of structure sense.

INTERPRETATION IN TERMS OF STRUCTURE SENSE

The data were arranged in the above table according to what they might reveal about students' beliefs about factoring. In this section we will discuss what these students' responses might tell us about their structure sense, and how they use formulas.

The huge difference between results on the two tasks is surprising: 77% success as opposed to 7.5%. The formula was given to the students and they applied it easily in the case of $x^4 - y^4$. What makes $(x-3)^4 - (x+3)^4$ so much more difficult? Students were given the formula (the structure) but were unable to apply it in this instance. It seems that they did not see the connection between the expression and the structure. They were unable to relate to $(x-3)^4 - (x+3)^4$ as being an instance of $a^2 - b^2$.

In the interview some of the students attributed their difficulty to "*We haven't learned this*" or "*I can't remember*". Despite the fact that they succeeded in factoring simple expressions, they were confused when asked what it means to factor. Their answers ranged from "*to get to some formulas*" to "*to break up a number into smaller parts*" to "*to cancel or to simplify as much as possible*". None of the ten students described factoring as writing an expression as a product.

Display of structure sense

The 12 successful students (not shown in the table) displayed structure sense. They were able to relate to x-3 and x+3 as entities (SS1) and recognised the structure $a^2 - b^2$ in the expression $(x-3)^4 - (x+3)^4$ (SS2).

Lack of structure sense

We can say nothing about the 47 students who left a blank, although we may suspect they lack structure sense. Another group who seem to lack structure sense are those 29 students who manipulated the powers incorrectly. They seem to be aiming to simplify the expression, using an "anything goes" strategy.

The 6 students who expressed the fourth power as a product of four factors appeared to be feeling their way, unsure of what to do but attempting to find a meaning for the expression. In the interview Gal was asked why students found this exercise difficult. She said "*when they see the 4th power they think they have to do multiplication between the terms*". 17 students were able to identify $(x-3)^4$ as $(x-3)^2 \cdot (x-3)^2$ but not as $\left[(x-3)^2\right]^2$. These students too are lacking structure sense.

Partial structure sense

We could claim that the 27 students who opened brackets lack structure sense. It is possible that they expect to obtain a simpler expression after opening brackets and collecting like terms. When interviewed Dana said "*I opened brackets to collect like terms, in order to find the common factor*". However, in the process of opening the brackets they are displaying a kind of, if not very useful, structure sense. Some of them "recognised" the expression $(x-3)^4$ as having a structure similar to $(a-b)^2$ and then used a false generalisation of the formula $(a-b)^2 = a^2 - 2ab + b^2$ to get $x^4 - 12x + 81$. As Natalie explained in the interview "*When it's squared you take that squared then that times that times the power, then that to the power. …(In this case) We do x to the fourth, then 3 times 4, that's 12x, then 3 to the fourth. That's how I was taught*".

A very interesting group is the one consisting of 6 students who apparently were able to identify $(x-3)^4$ as $\left[(x-3)^2\right]^2$ since they opened the interior brackets and obtained $\left(x^2-6x+9\right)^2 - \left(x^2+6x+9\right)^2$. However they quite clearly lacked some structure sense as they did not recognise this as having the structure $a^2 - b^2$. Why is it so difficult for them to see what seems to us so obvious? Remember, the formula (structure) was given to them. Is it the compound expression that hides the connection between the expression and the given formula? Or is it the brackets? Is it a failure of SS1 – seeing the compound expression as an entity – which causes them to fail in SS2 – recognising the structure?

Finally we have the 16 students who valiantly tried to apply the given formula. They are told that the structures are equivalent, but they cannot find the connection. Their structure sense, at least the SS2, is clearly deficient. Let us look at some of their attempts.

The students who wrote $(x-3-x-3)(x-3+x+3)$ seem to have read the fourth power as a second power – perhaps they thought there was a misprint in the question. However they definitely displayed SS1, relating to x-3 and x+3 as entities.

The students who wrote $(x-3-x-3)^2(x-3+x+3)^2$ applied the formula wrongly. They apparently assumed that if $a^2 - b^2 = (a-b)(a+b)$ then $a^4 - b^4$ must be equal to $(a-b)^2(a+b)^2$. These students are also displaying SS1.

It is difficult to interpret the student who wrote $[(x-3)^2 - (x+3)^2][(x+3)^2 - (x-3)^2]$. Perhaps s/he is using a rule wrongly, like Roy who said that $(x-3)^4$ is equal to $[(x-3)(x+3)]^2$ because "*minus times plus gives minus*" and $(x+3)^4$ is equal to $[(x-3)(x-3)]^2$ because "*minus times minus gives plus*".

The student who wrote $a = x-3$ and $b = x+3$ but then $[(x-3)(x+3)]^8$ displays SS1 but an inability to use the formula.

The student who wrote $a=(x-3)^4$ and $b=(x+3)^4$ and then

$[(x-3)^4-(x+3)^4][(x-3)^4+(x+3)^4]$ could be seen to be displaying SS1 and SS2, and was probably just being careless in writing the exponent as four instead of two. In the interview Michal gave this identical answer, and then later corrected herself: "*oh, it should be squared*".

The difficulties of those students who seemed to be able to relate to x-3 and x+3 as entities (SS1) but did not see the a^2-b^2 structure may be attributed to their lack of SS2.

CONCLUSION

What do the above data tell us? Many students are unsure what factoring is all about. Some students have difficulties with using a given formula. Very few of the 160 students in this study used structure sense. It should not be inferred that structure sense is solely about using formulas: an ease with using formulas is just one result of a healthy structure sense.

A recurring theme was the inability to relate to *x*-3 and *x*+3 as entities – a lack of SS1. When students were asked to give an example of something "complicated" that *a* could stand for in the formula $a^2-b^2=(a-b)(a+b)$ they only gave products (e.g., 2*x* or *xy*), not sums (e.g., *x*+2). As Ben put it "*you have to see that this term represents a variable … you simply have to look at it differently*".

Another common response was "*we didn't learn this*". Since these same students were able to answer standard factoring exercises correctly, this could be interpreted as a lack of SS3 (recognise which manipulations it is useful to perform).

Is structure sense something that can be taught? Should it be taught? We feel that the last two questions should be answered in the affirmative but are not yet ready with an answer for the obvious next question: How should structure sense be taught?

We are convinced of the importance of drawing students' attention to structure. The above definitions should be useful as guidelines for further research and didactic design.

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LEARNING MATHEMATICS: PERSPECTIVES OF AUSTRALIAN ABORIGINAL CHILDREN AND THEIR TEACHERS

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Two key stakeholders in enhancing and building Aboriginal children's capacity to learn mathematics are teachers and the Aboriginal children themselves. In Australian schools it is often the case that the two groups come from different cultural backgrounds with very differing life experiences. This paper reports on an ethnographic study and focuses on the beliefs of Aboriginal children and their teachers about learning mathematics. It suggests significant issues that teachers need to acknowledge in providing appropriate learning environments for Aboriginal children.

BACKGROUND

Australian Aboriginal societies are both person-oriented and information-oriented, emphasising the personal and relational connections between "teacher" and "learner". This is the society that Aboriginal children experience for their first 5 years. For many of these Aboriginal children school is very different to their life experiences before entering school. They are often confronted with new issues that impact upon their learning. Within the classroom there are often differences in the language of the Aboriginal children and the teacher, the learning styles of Aboriginal children, the teaching styles and in the expectations, understandings and appreciations of the Aboriginal child and the teacher (Howard, 2001; NSW Department of Education and Training, 2004). Aboriginal children are learners in two worlds. Their mathematical programs have to both appreciate and utilise the Aboriginal child's context, community, skills and knowledge (Howard, Perry, Lowe, Ziems, & McKnight, 2003). But do they?

Learning is "fundamentally a social process and the recognition of the social purposes of learning as well as the social learning processes through which such learning is achieved is important in the teaching of Aboriginal children" (Gray, 1990, p.135). Without reform in the mathematics classrooms of Aboriginal children, pedagogy "will tend to reproduce social inequalities of achievement and subordinate individual development to social domination" (Teese, 2000, p. 8). Banks (1993) suggests that "the ethnic and cultural experiences of the knower are epistemologically significant because these factors influence knowledge construction, use and interpretation" (p. 6). Many Aboriginal children struggle with school mathematics and this brings about high anxiety, worries of failure and feelings of inadequacy (Frigo & Simpson, 2000). Educational policy and decisions that affect Aboriginal children's mathematical learning are best made through consultation and data

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 153-160. Melbourne: PME.

collection involving Aboriginal people (Mellor & Corrigan, 2004). This paper reports on the voices of Aboriginal children and their teachers as they talk about the learning of mathematics in primary school classrooms.

DATA COLLECTION

Data for this paper were collected as part of a 9-month ethnographic study involving Aboriginal students, their parents, Aboriginal educators and non-Aboriginal teachers living in a remote rural community in New South Wales, Australia. The study investigated the espoused beliefs about the nature and learning of mathematics in Years 5 and 6 of a primary school in Tremayne, a farming town dependent upon wool, wheat and cotton. The town's population comprised thirteen ethnic groups with the two significant groups of people being Aboriginal and non-Aboriginal. The town has had a history of recurring conflicts between the two groups. Some Aboriginal people would suggest that the overt racism within the town had been replaced by a subtler, covert form of racism that affected how Aboriginal people were accepted in the town. The last twenty years has seen improvement in medical, legal and educational facilities provided for and administered by Aboriginal people.

Ellen Road Public School was a two stream primary school, with a staff of nineteen teachers [18 non-Aboriginal; 1 Aboriginal] and an enrolment of 412 students, 32% of whom were Aboriginal. The school had three Aboriginal Education Assistants with specific roles and responsibilities in Aboriginal education. Over a period of 6 years, the lead author had come to know the community and be known by the teachers and Aboriginal educators working in the school. Mathematics classes within the school were graded with the higher ability class of Year 5/6 comprising 30 students, one of whom was Aboriginal. The lower ability class had 25 students, 12 of whom were Aboriginal.

Conversational interviews were held with all participants, then transcribed and analysed using a grounded theory approach. Teacher interviews involved the five non-Aboriginal teachers on Years 5 and 6. Student interviews involved all Aboriginal students [13 girls; 8 boys] in Years 5 and 6 who were given parental permission to be part of the study. Sixteen categories of responses were determined. In this paper, one of the categories-the learning of mathematics-is considered from the perspective of Aboriginal children and their teachers. Pseudonyms were used for all participants and locations.

RESULTS - TEACHERS

While this category reported the espoused teacher beliefs about mathematics learning, there was not a general focus on Aboriginal children. This seems paradoxical given that there were Aboriginal children in the teachers' classes and that they acknowledged that these children were not learning to their potential.

Motivation

Teachers raised factors such as motivation, attentiveness and the apparent unwillingness on the part of some of the Aboriginal students to learn and their own resultant frustration when the children did not respond to a prepared lesson. One teacher raised the concern that both Aboriginal and non-Aboriginal children did not want to think or work out mathematics problems. They preferred doing stencils where it was all written for them.

> I don't know why but a lot of these children are unmotivated, both Aboriginal and non-Aboriginal. They don't want to think either. They like to do the easy things but they don't want to do anything that requires thinking and working things out. They like doing stencils because it's all there for them but they don't like writing things. (Mrs Allan)

> I have a few in my group who might not be attentive. There are some kids who tune out, don't listen, and then don't know how to do something. I'd get cross because they hadn't been listening. You try to make things motivating but some kids get excited about mundane and boring things. You think of something you think is exciting and they look at you and say, "Do we have to do that? (Mrs Allan)

Maturity

One teacher talked about the timing of learning and that it might be related to maturity and learning readiness.

> You can see that they're really different in their eyes. It's just not evident. I think it's a maturity thing with a lot of kids, maths. It's like reading that you can introduce things at certain times and it's the wrong time and if you just leave it and don't worry about it 6 months down the track it might fit in and they just weren't ready for it. (Mrs Cotter)

Thinking

One teacher, initially, doubted if all children were capable of thinking mathematically. However, she espoused the belief that they all have the ability to some degree and acknowledged that Aboriginal children try hard in their mathematics to please the teacher. Another believed that many children were unwilling to attempt mathematical problems and that society was not teaching children perseverance.

> Some have a lot and some have none. No they all have some. Some kids are too keen to please you rather than do their work. Some kids apologise when they get things wrong. They say they're sorry because they are trying so hard to please. (Mrs Martin)

> I just thought they had more thinking ability than what came across. I know they had the basics. They have all the skills but I obviously haven't extended them enough into the thinking. That's why we have to finish the curriculum earlier so that we can go into that. By the time you deal with the basics and do some extension you don't always have a lot of time to do the thinking.(Mrs Cotter)

> The problem solving…and seeing the different ways that they can do it. They don't often…what's the word, tunnel vision. I think society, I blame them, doesn't teach them perseverance. I don't think that's something that they can get from school. A lot of

children say I can't do it and straight away don't want to do it. I think that's the old style way of society. (Mrs Cotter)

Prior learning

A new teacher to the school felt she had been given little information on the mathematics the children had learnt previously. She did not know if the children had been allowed to use materials in their learning of mathematics, did not know the past mathematical school experiences of the children and thought that past teachers in mathematics lessons may have been quite strict on the children.

> I think I try to do this with them in mathematics and I don't know if they've ever been allowed to use materials. They must have been like that all the way through so the teachers have kept them under the thumb. I don't really know what goes on in the infants. This is my first year. (Mrs Jones)

Cross cultural learning

One teacher talked of issues related to cross-cultural learning.

> They talk about different learning styles for Aboriginal children. I don't know sometimes if I'm catering for them because a lot of Aboriginal children don't appear to go well with maths. Sometimes you wonder why? Then you get a good kid and you think this person is special and you shouldn't have to think of that. Then you have other kids who perform well in class and you put a test in front of them and they just do poorly. I don't know why. If I sat down with them they would be able to do it. I don't know if it's the concentration, they have to sit still or what it is. (Mrs Allan)

RESULTS – ABORIGINAL CHILDREN

The Aboriginal children depended on listening to learn mathematics, "I just listen and learn them and that and go back to my tables and do them sometimes." Colin commented that "some people think I'm dumb and I try to prove that I'm not dumb by listening and listening." When Dennis learnt mathematics, he depended on watching, believing that "the teacher shows you how to do it and you just watch her the right way to do it." Natalie tried her hardest in mathematics except on those days when she was not in the mood. She expressed her beliefs on how she learnt mathematics.

> Yeah. Well I just mainly listen and keep my ears open so I can learn more than what I've already learnt cause I may think that I haven't learnt enough so I need to learn more so just listening to the teacher and stuff.

Richard expressed a definite view of how he learned mathematics.

*Richard: Teacher just puts stuff on the board and if we don't do it you get put on detention…get in trouble. Teacher says if we don't do it he'll just make us.

*Interviewer: How does it get into your head?

*Richard: The teacher learns me. The teacher learns us.

Vivian believed that "the teacher teaches us" and that what the Aboriginal children have to do is "listen and learn". Lynda believed that her learning "just popped into her head as the teacher was talking." Tom believed that he learned mathematics through "getting teached." Lucy was very good at her times tables and when asked how she had learnt them she responded, "I just practised them. Sir says them with us some times and then I practise." Susan was sure of how she learned mathematics.

*Susan: I learn by just getting my paper and I tell mum to write out about two pages of sums. I do them and then I give them to her and tell her which ones are right and which ones wrong. The ones that are wrong I tell her and then she tells me to go back and do them again and when they're right that's how I learn. I learn at school from things that mum doesn't show me and that.

*I: So how do you learn at school then?

*Susan: Learn by the teacher. By the way they explain it and that and the way they talk about it and you got to listen and that.

Meryl believed that she learnt mathematics through listening to the teachers, her dad and to other Aboriginal children. She became annoyed when other Aboriginal children misbehaved "cause when someone's trying to learn something other kids don't want to learn and they don't know that you want to learn and they muck up anyway." Natalie believed that her teacher taught her something new most days "because she is understanding and she knows what you're talking about and she helps you out a lot." Judy liked learning new mathematics but thought she knew a lot of what was being taught. Susan also liked learning new mathematics because "you don't get to learn things every day of the week." Natalie believed that there was more mathematics to learn than any other subject, especially in high school, and that it was going to get quite hard. Lynda talked about mathematics getting harder in high school.

*I: Do you think maths is going to get easier or harder?

*Lynda: Harder cause when you get older and you get a bit good at maths they put you in a higher maths group and you might not be able to do it. I can do some of the top maths class' work.

DISCUSSION

Teachers believed Aboriginal children's levels of attention, motivation and willingness to learn were important factors in the learning of mathematics. There was an espoused belief about the influence of children's maturity and that the timing of learning mathematics would impact upon how quickly children would learn. Only one teacher, Mrs Allan, talked specifically about varying learning styles and wondered if she was catering for the Aboriginal children because "many just did not appear to do well in mathematics". She did not know if it was concentration or confidence or whether the Aboriginal children just did not think they would perform well so did not try. Not knowing the mathematical learning backgrounds and

experiences of the students had implications for both the children's learning and the teacher's teaching. There were teacher feelings of pressure, frustration and personal annoyance in having to cover the mathematics curriculum content, reducing the amount of time to concentrate on helping the children develop their mathematical thinking. Teachers raised the importance of and yet the apparent lack of children's motivation and perseverance in thinking mathematically, particularly in problem solving. Perhaps if lessons were more interesting, relevant and tangible, and less reliant on the use of mathematics textbooks and stencils in Years 5 and 6, Aboriginal children may well be encouraged in their mathematics thinking (Matthews, Howard & Perry, 2003).

The Aboriginal children were most able to express their views and beliefs about learning mathematics. When learning mathematics, the Aboriginal children believed in the importance of listening, watching and working out the mathematics on paper. The teacher was the main source of Aboriginal children's learning of mathematics. The Aboriginal children also believed that the teacher puts mathematics into their heads through writing it down. Mathematics could also be learnt through textbooks and by redoing incorrect work. The Aboriginal children were not aware of their own mathematical competencies, strategies and problem solving abilities they used when learning mathematics. It would be purposeful for teachers to take time to discuss and enable Aboriginal children to identify the learning tools they utilised in their learning of mathematics.

Aboriginal children find learning mathematics hard, that there was more to learn in mathematics than any other subject and that they could not do some parts of mathematics even though they knew a lot. If Aboriginal children have negative beliefs about mathematics and themselves as learners of mathematics, then appropriate mathematics programs and teaching strategies need to be developed to help overcome such views (Howard, Perry, Lowe, Ziems, & McKnight, 2003). Such mathematics programs, that accentuated the children's life experiences and contexts, would bring a relevance to their learning resulting in increased motivation and engagement, because Aboriginal children really liked learning new things in mathematics.

These data suggest that even though there were significant numbers of Aboriginal children in their classes, teachers were not considering them and appreciating their cultural needs in the mathematics classroom. Teaching and learning suffer when the cultural needs of a specific group of students are not addressed. Teachers have to become aware of and appreciate the cultural diversity and hence the cultural conflict that can occur between the teacher, Aboriginal children and the mathematics curriculum content. Such cultural conflicts are critical elements in the reasons why Aboriginal children are not achieving to their potential in learning mathematics (Bishop, 1994; Kemp, 1999). If there is a mismatch in the set of beliefs between children and teachers towards the learning of mathematics and the interactions within the mathematics classroom, mathematics anxiety and frustration for both will occur.

If teachers view mathematics only as rule learning they will not consider how children learn mathematics (Battista, 1994). As Aboriginal people and teachers become more aware of the effect on mathematics learning of cultural diversity, structured professional development, policy and planning will need to occur for appropriate mathematics curriculum and teaching practices to develop (Howard, 2001; Mellor & Corrigan, 2004).

CONCLUSION

The reality is that Aboriginal children are often poor and from families where the economic support for children's learning of mathematics may not be available and where mismatches between school and home expectations may exist. Teachers cannot devise appropriate educational programs for marginalised children on their own. They have to be "devised in consultation with adults who share their culture" (Delpit, 1988, p. 296). Aboriginal children bring a cultural heritage and context into schools and mathematics classrooms that can be implemented into effective mathematics curriculum activities for all children. The ideas of disadvantage and deficit linked to Aboriginal children's learning need to be replaced with beliefs about their competencies.

In co-operation with Aboriginal people, teachers need to become more aware of individual's learning styles to foster appropriate learning environments in mathematics classrooms. Such co-operation between the community, children and teachers can help bridge the difficult social and learning experiences that many Aboriginal children face in the mathematics classroom. Rather than making "the learner [Aboriginal student] fit the system, a preferred focus is on how the system can better meet the learner's needs" (Frigo & Simpson, 2000, p. 6). If some Aboriginal children have negative beliefs about mathematics and themselves as learners of mathematics, then appropriate mathematics programs and teaching strategies need to be designed to help overcome such views.

The lack of a mathematics curriculum that emphasises Aboriginal children's experiences, culture and traditions is viewed as a barrier to achieving mathematics equity amongst varying cultural groups (Tate, 1995). An inclusive mathematics curriculum is needed that supports and empowers teachers to consider how they value the experiences, cultural background and language that Aboriginal students bring into the mathematics classroom (Frigo & Simpson, 2000). Aboriginal educators, Aboriginal children, parents of Aboriginal children and teachers should collaborate to:

- talk about, reflect upon and make decisions about appropriate actions that need to take place within schools and homes to enhance Aboriginal children's mathematics learning;
- assist teachers in discussing and identifying ways in which Aboriginal children learn mathematics.

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VERIFICATION OR PROOF: JUSTIFICATION OF PYTHAGORAS' THEOREM IN CHINESE MATHEMATICS CLASSROOMS

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This paper presents key findings of my research on the approaches to justification by investigating how a sample of teachers in Hong Kong and Shanghai taught the topic Pythagoras theorem. In this study, 8 Hong Kong videos taken from TIMSS 1999 Video Study and 11 Shanghai videos videotaped by the researcher comprised the database. It was found that the teachers in Hong Kong and Shanghai emphasized justification of the theorem; one striking difference is the former stressed visual verification, while the latter paid close attention to mathematical proof.

INTRODUCTION

The superior performance of students from Confucian Heritage Culture (i.e., CHC) [1] communities in international comparative studies of mathematics achievement (Mullis *et al.*, 2000, 2004) has led researchers to explore what factors possibly account for the superiority of students in CHC (Fan, *et al.*, 2004; Ho, 1986; Leung, 2001). Recently, these studies focused on classroom instruction (Stigler & Hiebert, 1999; Hiebert *et al.*, 2003). Since mathematical reasoning is a key element in school mathematics(National council of Teachers of Mathematics, 2000), and it was found that Japan and Hong Kong students worked on problems involved more proof than the students in other countries (Hiebert, J. *et al.,* 2003), a contribution to understanding mathematics teaching in CHC, is to describe how proof is dealt with in classrooms.

Although the notions on proof and the ways of introducing proof are diverse, a great body of literatures on the natures of proof and the functions of proof suggests the following functions of proof and proving (Hanna, 1990, 2001; Hersh, 1993): (1) to verify that a statement is true; (2) to explain why a statement is true; (3) to communicate mathematical knowledge; (4) to discover or create new mathematics, or (5) to systematize statements into an axiomatic system.

The above statements seem to suggest that the justification should include convincing, explaining and understanding. Justification refers to providing reasons why the theorem is true. The validity of a proof does not depend on a formal presentation within a more or less axiomatic-deductive setting, not on the written form, but on the logical coherence of conceptual relationships, that serve not only to

[1] Confucian Heritage Culture communities refer to Chinese Taiwan, Korea, Japan, and Singapore (Biggs & Watkins, 1996).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 161-168. Melbourne: PME.

convince others that the theorem is true, but also to explain why it is true (Cooney *et al*, 1996; Hanna, 1990, 2001). "Proof" in this study is grouped into two categories. One is called *mathematical proof*, in which the theorem is proved deductively and logically by using geometrical properties and theories or the operations of algebraic expressions. The other is called "*verification*", in which the theorem is shown to be true by using certain evidences such as solving a puzzle or demonstrating some cases.

In the research, the focus will be on the practice of teaching of proof in Chinese classrooms (Hong Kong and Shanghai) when Pythagoras theorem was taught. It attempts to address the following two questions: (1) How the theorem was justified in the lessons? (2) How were those justifications carried out?

METHOD

In this study, the Chinese cities of Hong Kong and Shanghai were selected as two cases to be investigated. Within each setting, several lessons were selected, as discussed below. The Hong Kong videos from the TIMSS 1999 Video Study on the teaching of Pythagoras' theorem were chosen. Thus, the Hong Kong data for the present study consists of eight CDs of the sampled lessons and relevant documents such as questionnaires for understanding teachers' background, sample of students' work in the lessons etc. Correspondingly, adopting the procedure of videotaping designed by the TIMSS 1999 Video Study, eleven Shanghai lessons which come from compatible Shanghai schools were videotaped. As in Hong Kong, supplementary documents were collected.

The CDs of the Hong Kong lessons and relevant multimedia database with English transcripts from LessonLab, Inc. of Los Angeles were already available to the researchers. For Shanghai, all videos of the lessons were digitized into CDs. The teachers who delivered the lessons respectively transcribed the CDs verbatim in Chinese and the data analysis mainly depended on the Chinese transcripts and the original CDs. During the process of data analysis, the CDs were referred to from time to time, to ensure that the description represented the reality as closely as possible.

RESULT

Approaches to justification

The approaches to justification can be summarized in Table 1. The most prominent difference between these two cities regarding justification is that the Shanghai teachers paid considerable attention to the introduction of **mathematical proofs.** On the contrary, the Hong Kong teachers seemed to hold different attitudes toward justification. Six out of the eight teachers tended to **verify the theorem** either through exploring activities for discovering the theorem or certain other activities for verifying the theorem once it was found.

Types of Justification	Hong Kong * (8)	Shanghai (11)
Visual verification	6	0
Single mathematical proof	2	3
Multiple mathematical proofs	1	8

*The sum of the ways of justification is not equal to the total of the teachers since there are some teachers who gave different kinds of justifications.

Table 1: Distribution of the ways of justification

Verification

It was found that six out of the eight Hong Kong teachers verified the theorem visually. The typical way of verification is which serves as both discovering and verifying.

Given a bag containing five pieces of puzzles and a diagram as shown in Figure 1(1), students are asked to(1)fit the 5 pieces into square C in the diagram in the shortest time;(2)rearrange the above 5 pieces into two smaller squares A and B in the diagram

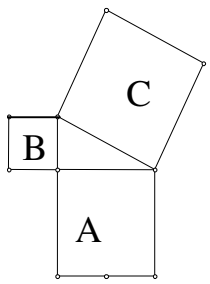

Figure 1(1)

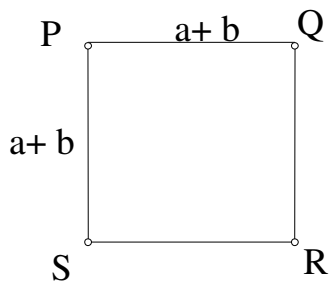

Figure 1(2)

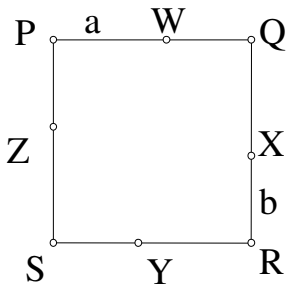

Figure 1(3)

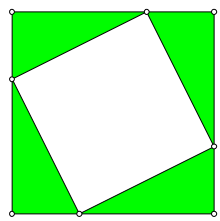

Figure 1(4)

It is expected that the students by solving the puzzle in different ways would discern the invariant area relationship – the area of the square on the hypotenuse is equal to the sum of the areas of the two smaller squares on the two adjacent sides**.** Moreover, by associating with the area formula of the square, the relationship among the areas of the three squares was transformed into the relationship among the lengths of the three sides of the right-angled triangle, which is Pythagoras' Theorem.

Mathematical proof

All the Shanghai teachers introduced mathematical proof. In particular, three quarter of teachers introduced more than two mathematical proofs. However, all these different proofs are based on a strategy, i.e., calculating the area of the same figure by using different methods. Although only a half of the Hong Kong teachers introduced mathematical proof, there are different strategies to prove. The following is one example.

Consider a square *PQRS* (Figure 1(2) with side $a + b$ and prove Pythagoras' Theorem by finding the area of *PQRS* with two different methods.

Method 1: Find the area of *PQRS* directly.

Method 2: Hint: Join *WX*, *XY*, *YZ,* and *ZW* (Figure 1(3)). *PQRS* is divided into four _____ and one _____ *WXYZ*.

Let the hypotenuse of the right-angled triangle *WPZ* be *c*, then Area of *PQRS* = _____.

In the Shanghai lessons, the students were asked to fit the squares together by using four congruent right-angled triangles. Then, they were asked to calculate the areas of the squares in different ways. Based on these kinds of activity, several proof were found. The following is one typical method (Figure 1(4)):

$S_{\text{large square}} = S_{\text{small squares}} + 4S_{\text{triangles}}$

$(a+b)^2 = c^2 + 4\times\frac{1}{2}ab$

Simplifying: $a^2 + b^2 = c^2$

From a mathematical point of view, the above two proofs are essentially the same. However, it may make differences in students' learning when the proof is introduced differently. The following excerpts demonstrate the kind of classroom interactions when one method of calculating the area of the square was introduced.

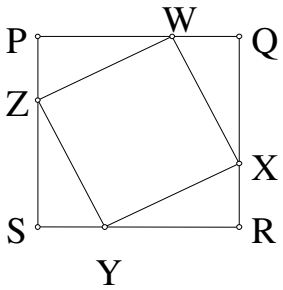

Figure 2(1)

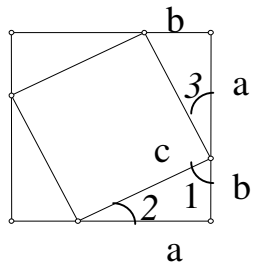

Figure 2(2)

In Hong Kong, the first method (namely, calculating the area of *PQRS* directly) was discussed. After the students have found that the square *PQRS* was divided into four congruent right-angled triangles (see figure 2(1)), the discussion then moved on to the following episode:

1 T: Congruent. Right. So you may say, *PQRS*, now this time, after you joined the four sides, you will have to divide it into four congruent right-angled triangles.

2 T: So *PQRS* is divided into four congruent right-angled triangles and also, another figure, *WXYZ*, that's a?

3 Ss: Square.

4 T: Square. So now, this is the second method. The second method in finding about the area of this square with the side *a* plus *b*, so now, this time, how to find about its area?

5 T: There are four congruent triangles. [They are] Right-angled triangles with the side, base is *a*, height is?

6 Ss: *b*.

7 T: *b*. You know the area *ab* over two for each triangle. And then four of them. Plus?

8 Ss: *c* square.

9 T: *c* square, right. The smaller square that is inside. So you may find, they're two *ab* plus *c* square. Two *ab* plus *c* square. How about the last step? By considering the two different methods in finding the area of the same square, what do you find?

10 T: Method one, you find *a* square plus two *ab* plus *b* square. How about the second one? Method two you find two *ab* plus *c* square. So what do you find?

11 T: Oh, both sides they're two *ab*, so you cancel this, and finally you find *a* squared plus *b* squared is *c* squared. Is it?

The Hong Kong episode shows that the teacher led students to achieve the proof expected by her through asking a sequence of simple and close questionings (2, 5, and 7). Even though, sometimes, the teacher asked an open-ended question (4), finally, the question was rephrased into two closed questions (5, 7). Meanwhile, the teacher often answered her own questions (9~11). Thus, the teachers tended to ***tell*** her students the proof.

In Shanghai lesson, after students created a figure by using four given congruent right-angled triangles, the teacher presented the diagram on a small blackboard (see figure 2(2). The students were then asked to calculate the area of the diagrams using different methods, as shown below:

1 T: This is what he put. He cut a big square outside and a small square inside. Then put the big one on the small one. Ensure that the vertexes of the small square are on the four sides of the big one. Would you please prove it? (Nominating a student)

2 S: Sorry, I can't

3 T: You have put it (together), but you can't prove it? Who can? *You, please.*

4 S: (Coming to the front and proving on the blackboard)

$S_{\text{large square}} = (a+b)^2$.

5 T: How do you know that this side is a?

6 S: Because the two triangles are congruent.

7 T: How to prove they are congruent? Don't be nervous.

8 S: I've forgotten.

9 T: He's forgotten. Please give him a hand. Go back to your seat, please. Tao Li.

10 S: I've forgotten.

11 T: You've all forgotten. How to prove? Liu Wenju, would you please prove it? Be brave.

12 S: There's one that is equal. $\angle 1+\angle 2=90°$, $\angle 1+\angle 3=90°$, $\angle 2=\angle 3$.

13 T: What's the reason?

14 S: (Together) the complementary angles are equal.

15 S: We can get congruence according to ASA.

16 T: With the same reason, we can get the four triangles equal. Now continue, please.

17 S: $S_{\text{large square}} = (a+b)^2$.

In the Shanghai lessons, the teacher encouraged her students to focus on critical aspects, express their understanding, and finally find their solutions through asking a sequence of open-ended questions (1, 11, and 13). Moreover, when students faced difficulties in answering teacher's questions, the teacher asked other students until eliciting desirable answers (3, 7, and 11). Thus, in the Shanghai lesson, the teachers tried to encourage and foster students to *construct their proof* instead of telling them the proof as the Hong Kong teacher did.

There are two critical aspects in this proof. One aspect is to dissect the square into one smaller square and four congruent right-angled triangles. Another aspect is to calculate the area of the square by two different methods and simplify the expression of an area relationship into a side relationship. Regarding the first aspect, in the Hong Kong lesson, the students got used to following the teacher's instructions and the worksheet seemed to limit students' thinking. Furthermore, the teacher did not provide justification as to why the central figure was a square. However, in the Shanghai lesson, the students were not only asked to present the diagram, but also to justify the statement on the diagram. In this sense, there is an obvious difference in experiencing the diagram: **taking for granted and justifying the diagram logically.** Regarding the second aspect, basically the Hong Kong teacher stated the relevant formulae and simplified the expression on his own. Thus, the students were seldom given a chance to express their own understanding and thinking. On the contrary, the Shanghai teacher always let the students express the formula and the relevant transformation verbally. In this regard, the Shanghai lesson seemed to have created more **space for students to verbalize the process of deductive reasoning.**

DISCUSSION AND CONCLUSION

The findings of this study show both similarities and differences in terms of the approach to the justification of Pythagoras theorem between Hong Kong and Shanghai. Although the teachers in both places emphasize on justification of the theorem by various activities, the following differences are noticeable: the Hong Kong teachers are visual verification-orientated, while the Shanghai teachers are mathematical proof-orientated. Moreover, compared with the Hong Kong teachers,

the Shanghai teachers made more efforts to encourage students to speak and construct the proof.

What are the possible explanations of the finding that the Hong Kong teachers emphasized visual verification, while the Shanghai teachers emphasized mathematical proof? Firstly, it was repeatedly found that Chinese and Japanese students tended to use more abstract and symbolic representation while the U.S. students tended to use more concrete and visual representation (Cai, 2001). However, within Chinese culture, the Hong Kong teachers valued visual presentation more than the Shanghai teachers did which might be partly due to the British cultural impact on Hong Kong for more than one century. Secondly, even in the current mathematics curriculum in Shanghai, the emphasis is put on abstract presentation and logical deductive thinking. It is an expected phenomenon that when teaching geometry, particularly when teaching Pythagoras' Theorem, geometric proof is stressed since it is believed that geometric proof is a good way "to master the mathematical method and to develop the ability to think" (Mannana & Villani, 1998, p.256). In the official textbook, two different proofs of Pythagoras' Theorem are introduced. The textbooks used Hong Kong, some visual exploring activities are presented and one proof is introduced as well. However, the teachers in Hong Kong seemed to pay more attention to verifying the theorem through exploring activities, rather than proving the theorem.

It seems that there is a consensus that deductive reasoning (or in the classroom jargon 'proving') still has a central role in geometry learning, However, the classical approach is now enriched by new facets and roles such as verification, convincing and explanation (Mannana & Villani, 1998, p.31). The teacher's classroom challenge is to exploit the excitement and enjoyment of exploration to motivate students to supply a proof, or at least to make an effort to follow a proof supplied (Hanna, 2000, p.14).

As demonstrated in this study, there are multiple approaches to justification: visual verification, one mathematical proof, multiple mathematical proofs etc.. If regarding the ways of justification as continuity spectrum in terms of the rigor: one pole is the visual reasoning and the opposite is the deductive reasoning, then this study shows that the approaches in Hong Kong seem to be quite near to the visual reasoning, while those in Shanghai seems to be near to the deductive reasoning side. However, it seems to suggest that the teachers in Hong Kong and Shanghai are making effort to strike a balance between visual reasoning and deductive reasoning which seems to be a direction to pursue. From the classroom practice in those Chinese mathematics lessons at eight-grade, the students are able to explore the theorem and prove it in certain ways if the teachers organize the teaching appropriately. Although we have no intention to apply these findings to any other cultures, we try to argue that effective teaching of justifications in Chinese classrooms is possible which might contribute to high mathematical performance of Chinese students.

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MOZAMBICAN TEACHERS' PROFESSIONAL KNOWLEDGE ABOUT LIMITS OF FUNCTIONS

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This paper analyses five Mozambican secondary school teachers' professional knowledge about limits of functions. Building up on Even's analysis of SMK, a new framework has been put together for this analysis. Interviews have been held with the teachers to analyse their knowledge according to this framework. Data suggest that the teachers have a very weak knowledge of the limit concept.

INTRODUCTION

This work is part of a research project that aims to investigate how high school mathematics teachers' knowledge of limits of functions evolves through their participation in a research group.

The limit concept has been chosen because it is the first higher Mathematics concept met by students in secondary schools. It is a very abstract concept and difficulties experienced by students have been explained as caused by a gap between the concept-definition and the concept-image (Tall & Vinner, 1981). Several studies in Mathematics Education about the concept of limits have already been done. In a European context, these studies relate to students conceptions (Tall & Vinner, 1981; Cornu, 1984; Sierpinska, 1987), epistemological issues (Cornu, 1991; Sierpinska, 1985; Schneider, 2001) or results of didactical engineering (Robinet, 1983; Trouche, 1996). In the Mozambican context, they relate to the institutional relation (Chevallard, 1992) of secondary education (Mutemba & Huillet, 1999), the personal relation of some teachers (Huillet & Mutemba, 2000), and students' understanding of limits (Mutemba, 2002). In my research, I focus on the professional knowledge of mathematics teachers on this concept.

What kind of knowledge does a teacher need to teach a mathematical topic? Several authors tried to answer this question. In particular, Even (1993) considers teachers' knowledge about a mathematical topic as having two main components: teachers' subject-matter knowledge (SMK) and pedagogical content knowledge (PCK). She states that few years ago, teachers' subject-matter knowledge was defined in quantitative terms but that,

> in recent years, teachers' subject-matter knowledge has been analysed and approached more qualitatively, emphasising knowledge and understanding of facts, concepts and principles and the ways in which they are organised, as well as knowledge about the discipline (p. 94).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 169-176. Melbourne: PME.

Pedagogical-matter knowledge

> is described as knowing the ways of representing and formulating the subject matter that make comprehensible to others as well as understanding what makes the learning of specific topics easy or difficult (pp. 94-95).

She (Even, 1990) built an analytic framework of SMK for teaching a specific topic in Mathematics, applied to the study of the function concept, that considers seven main facets of this knowledge: Essential Features, Different Representations, Alternative Ways of Approaching, The Strength of the Concept, Basic Repertoire, Knowledge and Understanding of the Concept, and Knowledge about Mathematics.

Looking at teachers' pedagogical content knowledge of Geometry, Rossouw & Smith (1998) describe PCK as "a means to identify teaching expertise which is local, part of the teachers' personal knowledge and experience" including

> (a) the different ways of representing and formulating the subject matter to make it comprehensible to others, (b) understanding what makes the teaching of specific topics easy or difficult and (c) knowing the conceptions and pre-conceptions that learners bring to the learning situation (p. 57-58).

They also referred to Mark (1990), who

> has painted a portrait of PCK as composed of four major areas: (a) knowledge of subject matter, (b) knowledge of student understanding, (c) knowledge of the instructional process and (d) knowledge of the media for instruction (p. 58).

TEACHERS' PROFESSIONAL KNOWLEDGE ABOUT LIMITS OF FUNCTIONS

Analysing these frameworks and applying them to the limit concept, I found that the boundary between SMK and PCK was not very strict, and that this distinction was not so relevant to my study. In order to teach limits, a teacher needs to have a deep knowledge and understanding of this topic, and this knowledge must be oriented towards teaching it to specific students in a specific context. Consequently, in my study of teachers' knowledge about limits of functions, I do not distinguish between SMK and PCK, but consider what kind of knowledge a teacher needs for teaching limits of functions in Mozambican secondary school. My framework includes some of the topics of Even's SMK, and the part related to students understanding, conceptions and difficulties, taken from Rossouw and Mark. In this way, the components of teachers' knowledge about limits of functions that I consider are: (a) Essential Features; (b) Different Settings and Models; (c) Different Ways of Introducing the Concept; (d) The Strength of the Concept; (e) Basic Repertoire; (f) Knowledge about Mathematics; (g) Students Conceptions and Difficulties. Obviously these seven aspects are not strictly separated, and some of them are strongly interrelated.

I used this framework for interviewing the teachers and for defining the topics for their personal research.

METHODOLOGY

At the beginning of the study, six teachers were selected (*A* to *F*), each one researching a specific aspect of the limit concept. One of them (*B*) dropped out at the very beginning of the work. The methodology for the whole research included three interviews with each teacher, individual supervision sessions, and periodic seminars where they discussed their personal research or a specific aspect of limits. The results presented here come from the first interview, held at the beginning of the whole process. It was my first individual contact with the teachers after they decided to join the group. In order to create a good relationship with them at this early phase of our work together, I did not want them to consider this interview as trying to test their knowledge about limits of functions, but as a conversation about this concept. For this reason I conducted semi-structured interviews focusing on the story of their contact with "limits of functions" through the several institutions where they had met this concept (Chevallard, 1992), as well as on their personal ideas about the teaching and learning of limits of functions at school. During the interview, the teachers were shown several definitions and several tasks about limits in different settings (numerical, graphical, and algebraic) and were asked which of these definitions or tasks they would use to teach in secondary school. They were not asked to solve the tasks, but a few of them voluntarily engaged in solving them.

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In this section, I will present some results of the analysis of the first interview, according to the seven aspects of my framework.

Essential Features

The essential features pay attention to the essence of the concept and deal with the concept image. I consider three main aspects of the limit concept, which emerge from the history of the limit concept and have already been underscored by Trouche (1996). These are:

- A dynamic or "cinematic" point of view, related to the idea of movement;
- A static or "approximation" point of view: the approximation of the variable depends on the degree of approximation needed;
- An operational point of view: the limit works in accordance with specific rules.

Through their answers to the question "*How would you explain the limit concept to a person who doesn't know Mathematics, for example a Portuguese teacher?*" and other statements during the interview, the teachers showed that they mainly consider limits as algebraic calculations. They also evidenced a static view of limits, as they described it as an unreachable approximation (*A*), a boundary (*C*), a repetition (*D*), a maximum or a minimum (*E*), and a limitation or restriction (*F*). Two of them also spoke about a dynamic point of view (*A* and *C*).

Different Settings and Models

Several authors pointed out that a mathematical object can be represented using different models (Chevallard, 1989) within different settings (Douady, 1986) or representations (Janvier, 1987). Changes of settings, or shifts from one model to another, allow the learner to access new information about the concept and, consequently, to construct a deeper understanding of this concept.

The limit concept can be studied in many different settings: geometrical (areas and volumes), numerical (sequences, decimals and real numbers, series), cinematic (instantaneous velocity and acceleration), functional (maximum and minimum problems), graphical (tangent line, asymptotes, sketching the graph of a function), formal (ε-δ definition), topological (topological definition, concept of neighbourhood), linguistic (link between natural and symbolic languages of limits), algebraic (limits calculations). Each of these settings underscores a specific feature of the limit concept.

In Mozambican secondary schools, the algebraic setting is dominant (Mutemba & Huillet, 1999). To analyse the teachers' knowledge about settings, I used different tasks in different settings, in particular numerical and graphical setting, which are unusual in Mozambique, and asked the teachers their opinion on the use of these tasks at secondary school. Some of them tried to solve the tasks.

The teachers showed that they were able to solve tasks in an algebraic setting.

C and *D* did not understand the tasks in a numerical setting. *A*, *E* and *F* recognised some numerical tasks but were not very used to solve them.

The graphical setting is where the teachers faced more difficulties. I showed them several tasks, some of them to read limits from graphs, and other to sketch a graph using limits, without analytical expressions. Three teachers (*a*, *e* and *f*) did not try to solve the tasks to read limits. *A* tried to relate all graphs to some analytical expression. *C* was able to solve most of these tasks, sometimes after some hesitation. *D* tried to solve some of the tasks but faced many difficulties and said that he had never done it before. *E* and *F* did not even try to solve any of the tasks.

Regarding the tasks on sketching graphs using limits, *A* only tried to solve one of them and faced many difficulties. *C* solved correctly one of them, showed interest for this kind of task, but did not try to solve the other ones. *D* tried to solve one of the tasks but sketched a graph that did not represent a function. *E* and *F* did not try to solve any of these tasks.

Their knowledge of the formal setting is also very weak (Huillet, 2004).

Alternative Ways of Introducing

There are several ways of introducing a concept at school. For instance, the limit concept can be introduced through sequences, through the tangent line problem, through problems of maximum or minimum, through instantaneous velocity or

through the formal definition. These different ways of introduction correspond to different settings and underscore a specific feature of the limit concept.

During the interview, I asked the teachers the following questions:

> At secondary school in Mozambique, limits are usually introduced through sequences. What do you think about this way of approaching limits? What other ways of approaching limits do you think could be used at school? Which one do you think more appropriate to secondary school level?

All teachers know the way limits of functions are usually introduced in Mozambique, according to the syllabus. None of them presented any alternative to this method. Two of them spoke about using more graphs (*C* and *E*) but without explaining how to do it.

The Strength of the Concept

The strength of a concept deals with the importance and power of this concept, with what make this concept unique. In that sense the concept of limits of functions is a very strong concept. It has strong links with other mathematical concepts, such as the function and the infinity concepts. It is also a basic concept for differential and integral calculus. Furthermore, it has many applications in other disciplines, such as Physics, Biology and Economics.

To analyse this aspect, I asked the teachers the following questions:

> In your opinion, what kind of applications of the limit concept should be taught at school? Do you think that it is useful to teach limits of functions at secondary school level? Why? How do you think the students will use this concept later, during their studies at university for example? In which disciplines? In which areas?

The teachers pointed the following applications of limits: in physics (*A*, *E* and *F*), in geometry (*A*), in derivatives (*C* and *F*), to locate intervals of increase and decrease of a function (*D*) and in the convergence of a series (*F*). *C* said that he does not understand the importance of the limit concept and *D* that he does not see it as a special concept.

Basic Repertoire

According to Even (1990), the basic repertoire of a mathematical topic or concept

> includes powerful examples that illustrate important principles, properties, theorems, etc. Acquiring the basic repertoire gives insights into and a deeper understanding of general and more complicated knowledge" (p. 525).

In Mozambique two kinds of tasks are usually solved in secondary schools: those to calculate limits and those to study the continuity of a function (Mutemba & Huillet, 1999). During the interview, it became clear that, either through their own experience with the concept or through their opinion on the tasks presented to them, the basic repertoire of the teachers was limited to the kind of tasks usually solved in

Mozambican schools: algebraic tasks to calculate indeterminate forms, and some tasks to apply the ε-δ definition.

Students Conceptions and Difficulties

When teaching a mathematical topic, it is important that the teacher is aware of the different conceptions, and even "misconceptions" or "alternative conceptions", held by the students, as well as the difficulties they face. Some of the conceptions and difficulties of the students when learning the limit concept have already been highlighted by several researchers (Cornu, 1983; 1991; Sierpinska, 1985; 1987; Monaghan, 1991).

In the interviews, I asked the questions:

> Which difficulties do you think that students meet when they study the limit concept? How do you explain these difficulties?

In answering these questions, the two teachers (A and E) who already taught limits at school used their experience as teachers and the others (C, D and F) their own experience as students. They pointed out students' difficulties to understand the ε-δ definition (A, C, E), specifically because of the use of Greek letters (E), to use certain techniques to calculate some indeterminate limits (E) and to read graphs (F).

Knowledge about Mathematics

In this section, I analysed what kind of knowledge about Mathematics can be helpful to learn limits of functions and, at the same time, how teachers' knowledge about mathematics can be developed through the study of this topic. One important aspect, through the study of the ε-δ definition, would be reflecting on the role of definitions in mathematics. It would also help them to reflect on the role of proofs: why is it necessary to prove that the limit is *b*, using the definition, if we already calculated the limit and found *b*? Another important aspect is the connectedness, as stated by Ball et al. (2004):

> Another important aspect of knowledge for teaching is its **connectedness**, both across mathematical domains at a given level, and across time as mathematical ideas develop and extend. Teaching requires teachers to help students connect ideas they are learning. […] Teaching involves making connections across mathematical domains, helping students build links and coherence in their knowledge (p. 59-60).

The limit concept has different features, can be studied in several settings and has strong links with other mathematical concepts. This should help the students to build links and coherence in their knowledge.

I did not ask specific questions about this aspect, but from the teachers' discourse it was clear to me that their knowledge about mathematics is very weak. They are used to learn rules without demonstrations and they are not able to make the connection between different concepts or between different settings.

CONCLUSION

Summarising the main results of the interviews according to my framework on teachers' professional knowledge about limits of functions, I would say that:

- They mainly consider limits according to operational and static points of view;
- They are used to work with limits in an algebraic setting, and face many difficulties in linking it with a graphical setting; they also face some difficulties when working in numerical and formal settings;
- They only know the way of introducing limits that is stated by the Mozambican syllabus;
- They do not understand the strength of this concept, as they know very few applications in mathematics and in other sciences;
- Their basic repertoire is limited to algebraic tasks and some tasks with the ε-δ definition;
- Their knowledge of students conceptions and difficulties is limited by their own knowledge about limits;
- Their knowledge about mathematics is also very weak.

As a conclusion, I would say that these teachers showed a weak knowledge of the limit concept, mainly shaped by the institutional relation of Mozambican secondary school to this concept.

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HEURISTIC BIASES IN MATHEMATICAL REASONING

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In this paper we briefly describe the dual process account of reasoning, and explain the role of heuristic biases in human thought. Concentrating on the so-called matching bias effect, we describe a piece of research that indicates a correlation between success at advanced level mathematics and an ability to override innate and misleading heuristic biases. Implications for the importance of teaching and learning mathematics are discussed.

Over the last thirty years psychologists have been very interested in the question of rationality. Do human beings reason according to normative logical rules? Most research answers this question in the negative, and the goal has shifted to explaining why. Recently, dual process theory has attempted to answer this question by positing the existence of heuristic biases: fundamental features of the brain that steer human cognition towards certain computational strategies (Stanovich, 2003).

The goal of this paper is to compare how matching bias, one particular heuristic bias, affects successful mathematicians compared to its effect on the general well-educated population.

DUAL PROCESS ACCOUNTS OF REASONING

In recent years, cognitive psychologists have brought together several theories that attempt to explain reasoning under one heading: dual process theory. In essence, the theory puts forward the idea that there are two quite separate parts to the human brain that perform different tasks in different ways (e.g., Evans, 2003; Stanovich, 2004; Stanovich & West, 2000).

One part, System 1, operates in a quick and automatic manner, and, very roughly speaking, corresponds with intuitive thought. System 1 is thought to be a large collection of autonomous subsystems, most of which are innate to all humans, but some of which may have been acquired through experience. The subsystems' processes are subconscious in nature and only the results can be actively reflected upon.

System 2, in contrast, is slow, sequential and conscious. It is unique to humans and is believed to have evolved relatively recently. It is this part of the brain that allows complex hypothetical thought, including abstract logic. System 2 is also involved in expressing the output of System 1, and it can monitor and possibly override these intuitive responses, although, as we shall see, this does not always happen.

The evidence for the dual process account comes from two different research programs. Firstly, some persuasive evidence has come from cognitive neuroscience.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 177-184. Melbourne: PME.

Both Goel & Dolan (2003) and Houdé, Zago, Mellet, Moutier, Pineau, Mazoyer & Tzourio-Mazoyer (2000) have used fMRI scans to reveal that people do actually use different parts of their brain when exhibiting responses that are traditionally attributed to System 1 or System 2. In particular, they found that extra activity in the right prefrontal cortex appears to be critical to detecting and overriding mistakes from System 1 reasoning.

A second source of evidence has come from mainstream psychological research into human reasoning. It has been found that many people respond in an apparently irrational, non-normative fashion when given straightforward logical reasoning tasks. For example, experimenters have found that people are much more likely to endorse logical arguments as valid if the conclusions are believable. Conversely, it is much harder to correctly evaluate logically valid arguments when the conclusion is unbelievable (Evans, Barston & Pollard, 1983).

This result makes sense when analysed from a dual process perspective. System 1 has a built in 'belief bias' heuristic, that causes the reasoner to be more uncritically accepting of arguments with conclusions that they believe to be true. Evolutionarily, belief bias makes sense: why spend time and resources evaluating the validity of an argument that leads to things that you already know are true? It is only if System 2 is effectively monitoring and overriding the automatic System 1 heuristics that invalid arguments might be detected.

Stanovich (2003) reviews a number of different heuristic biases which have been identified by psychologists. These include:

- the tendency to socialise abstract problems (Tversky & Kahnemann, 1983).
- the tendency to infer intention in random events (Levinson, 1995).
- the tendency to focus attention on items that *appear* relevant, but may not actually *be* relevant (Wilson & Sperber, 2004).[1]

It should be noted here that the use of the word 'bias' here is not to be equated with 'error'. Heuristic biases need not result in processing errors, and in fact in the vast majority of cases they don't – their relative success for small resource cost is what ensures their evolutionary survival. The word 'bias', in this paper at least, should be understood in the sense of a default cognitive tendency as opposed to a processing error.

MATCHING BIAS ON THE WASON SELECTION TASK

One of the most puzzling heuristic biases found in System 1 is the so-called matching bias effect (Evans & Lynch, 1973; Evans, 1998) – roughly speaking, the effect of disproportionately concentrating on the items directly referred to in a situation. This effect was first found using the famous Wason Selection Task (Wason, 1968), but has since been extended into other arenas.

Participants in the selection task are shown a set of cards, each of which has a letter on one side and a number on the other. Four cards are then placed on a table:

The participants are given the following instructions:

> Here is a rule: "if a card has a D on one side, then it has a 3 on the other". Your task is to select all those cards, but only those cards, which you would have to turn over in order to find out whether the rule is true or false.

When the rule is "if *P* then *Q*", the logically correct answer is to pick the *P* and ¬*Q* (not *Q*) cards – D and 7 in this example – but across a wide range of published literature only around 10% of the general population make this selection. Instead most make the 'standard mistake' of picking the *P* and *Q* cards. Indeed, Wason (1968) suggested that about 65% incorrectly select the *Q* card.

A large and controversial literature (reviewed by Manktelow, 1999) has built up over the last thirty years that attempts to explain this range of answers. One of the most robust, and counterintuitive findings in this research program is associated with the 'matching bias' effect.

Evans & Lynch (1973) discovered that when the rule was changed to "if a card has a D on one side, then it *does not have* a 3 on the other" (our emphasis), performance is dramatically improved. They found that 61% selected the correct answer with this version. With the added negative, the *P* and ¬*Q* cards are now D and 3. Intriguingly, the 'standard' mistake (that of selecting the *P* and *Q* cards) was made by virtually no one.

After testing various different versions of the rule by rotating the negatives (i.e. as well as D⇒3 and D⇒¬3, they used ¬D⇒3 and ¬D⇒¬3), Evans & Lynch concluded that participants were systematically biased in favour of selecting cards mentioned in the rule, regardless of the normatively correct answer. So, regardless of the where the negatives were in the rule, the D3 selection was always prominent. This effect has become known as matching bias, and has been found to be highly robust in abstract non-deontic contexts.

The dual process account suggests that matching bias is the result of a System 1 heuristic that appears to select salient features of the environment to spend processing time on. System 1 cues certain cards as relevant: namely the ones that match the lexical content of the rule, regardless of the presence of negations (Evans, 2003). System 2 then analyses the relevant cards and outputs the result.

In the D⇒3 version of the task, System 2 needs to successfully detect the logical mistake, something it seems to do rarely. Evans & Over (2004) note that, rather than finding this System 1 mistake, many respondents to the task simply rationalise and

then output their intuitive first response: D and 3. In the D⇒¬3 version, however, the same rationalisation and output process leads to the correct answer: D and 3.

The current study examined whether those who are highly successful at mathematics exhibit the same heuristic biases as have been found in the general population.

One of the major historical reasons why mathematics is a compulsory subject at school level is that, in the UK at least, it has been assumed to help to develop 'thinking skills': the ability to think rationally and logically. This belief is all-pervasive in mathematical world. For example, in their subject benchmark standards for mathematics degrees, the QAA (the UK higher education regulator) write that:

[Mathematics] graduates are rightly seen as possessing considerable skill in abstract reasoning, logical deduction and problem solving, and for this reason they find employment in a great variety of careers and professions. (QAA, 2002).

But is this really true? Adopting a dual process framework, it is clear that one major sources of failure in abstract reasoning can be attributed to System 1 heuristics such as matching bias. So, if the QAA's claim were true, one might expect those successful in studying mathematics to be less affected by System 1 heuristic biases than the general population. This research design reported in the next section of this paper attempted to test this hypothesis.

METHODOLOGY

As our mathematical sample we used first year undergraduate students from a high ranking UK university mathematics department. All these students had been highly successful in their school mathematics career, having received top grades in their pre-university examinations. In order to have a 'general population' sample, we used trainee primary school teachers, again from a high ranking UK university. Whilst these trainee teachers came from a wide variety of subject backgrounds, and were not specialising in mathematics, it should be noted that they all had been educated to degree level, and thus cannot be said to be truly representative of the population at large. It is reasonable, however, to claim that they are representative of a general, well-educated population.

We asked half of each of our samples to complete a standard D⇒3 selection task, and the other half to complete a rotated negative version (D⇒¬3). The exact wording used was the same as that given above, and is identical to Wason's (1968). The experimental sample size was large, with a total of 293 people taking part.

Each group were given approximately five minutes at the start of a lecture to tackle a questionnaire with two problems, one of which was the selection task.[2]

RESULTS

The results are shown in table 1. The first result to comment upon is that the mathematics students came out with a significantly different set of answers to the

general population on the D⇒3 task (χ^2=14.1, df=4, p<0.01). More mathematicians selected the correct answer (13% v 4%) and fewer selected the 'standard mistake' (24% v 45%) than the general population. These findings are in line with previous work that has looked at mathematicians' performance on the selection task (Inglis & Simpson, 2004).

	Maths Students				General Population			
Selection	D⇒3		D⇒¬3		D⇒3		D⇒¬3	
P	25	36%	19	17%	12	23%	5	9%
P,Q	17	24%	0	0%	24	45%	0	0%
P,¬Q *	9	13%	80	71%	2	4%	45	78%
P,Q,¬P,¬Q	5	7%	5	4%	10	19%	3	2%
other	14	20%	8	7%	5	9%	5	9%
n	70		112		53		58	

Table 1: The results (*correct answer).

However, whereas Inglis & Simpson found that 28% of mathematics undergraduates selected the correct answer, only 13% of the current sample made this choice. This can probably be attributed to the differing experimental settings, possibly to the amount of time available to participants. Inglis & Simpson used an internet based methodology where participants were given unlimited time to make their selection, whereas in the current study thinking time was limited to at most a few minutes. It may be that a tacit understanding of the problem as a 'two-minute' or a 'ten-minute' one affects the likelihood that System 2 may find and correct a System 1 error – an interesting question still to be answered. Certainly care needs to be taken in comparing results *across* studies which use even slightly different methods.

However, since all the results in the current study were conducted with exactly the same methods, comparisons between our two groups are valid in this case.

On the D⇒3 version the mathematicians performed significantly better than the general population. However, on the D⇒¬3 version, the performances for both groups were dramatically higher, and there was no significant difference between the groups. On this version, 71% of mathematicians and 78% of the general population made the correct selection. Remarkably, nobody in either group made the 'standard mistake' of selecting the *P* and *Q* cards on the D⇒¬3 version.

To determine what effect the group that each individual came from had upon their selections, the data were analysed using a general linear models analysis, employing a saturated logistic model with a binomial error distribution. The χ^2 and p values reported are derived from this analysis.

The test version that the individual tackled was, as expected, highly significant in determining whether they selected the correct answer (χ^2=139.9, df=1, p<0.001). The group main effect was not significant (χ^2=0.47, df=1). However, the group × test interaction effect *was* significant (χ^2=5.94, df=1, p<0.02). This indicates that the effect of changing the test had a significantly *different* effect for the two groups.

Looking at the raw percentages reveals what this significant difference means. Whilst changing the test from the D⇒3 version to the D⇒¬3 version increased the success rate for the general population from 4% to 78%, it had a significantly smaller effect on the success rate for the mathematicians, which increased from 13% to 71%. That is, the mathematics sample was significantly less affected by matching bias than the general population.

DISCUSSION

Thus those who have studied mathematics to an advanced level appear to be less affected by the inbuilt System 1 heuristic known as matching bias than the general population. Note that although some researchers suggest that there is a correlation between measures of 'general intelligence' and the ability to inhibit System 1 responses (Stanovich & West, 1998), this effect cannot explain our results. There is no reason to suppose that the mathematics undergraduates are substantially higher in *general* intelligence than successful graduates from other academic disciplines. The only difference that we can be certain of is that the mathematics students have been educated in mathematics to a higher level than the trainee teachers.

What can explain this result? Does an advanced education in mathematics *cause* a reduction in the effect of matching bias, or are those who are naturally less prone to be affected by it more likely to be filtered into a mathematics degree? Additional research, in the form of a longitudinal study, would be needed to provide a satisfactorily answer to this question.

The results presented here raise a number of important questions, including:

- If an advanced education in mathematics did cause this effect, what parts of the syllabus are most important?
- To what level of education in mathematics must a student reach before the effect becomes significant?

Overriding misleading System 1 heuristics is a vital life skill. It is not sufficient to reject problems such as the selection task as being irrelevant laboratory experiments that have nothing to do with 'real life'. Stanovich (2003, p.53) writes:

> The issue is that, ironically, the argument that the laboratory tasks and tests are not like "real life" is becoming less and less true. "Life," in fact, is becoming more like the tests!

Those who struggle with abstract reasoning are vulnerable. They can find complex electricity bills problematic; they can struggle to follow convoluted instructions associated with tax returns or funding applications; and they can be misled by

advertising. In short, they can easily fall foul of the many highly abstract rules and regulations that govern modern society. Whilst System 1 heuristics may be well adapted to the environment in which early man lived, they do not always result in maximisation of individual utility in the modern era.

A well developed ability at using System 2 to monitor and override misleading System 1 output is important. This is exactly the sort of skill that, anecdotally, mathematics is supposed to develop (QAA, 2002). Perhaps strangely, however, whether or not it actually does, appears not to have been the focus of much empirical research.

Acknowledgements

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[1] It has been argued that the matching bias effect is a special case of a more general relevance heuristic (Evans, 1995). Indeed, some argue that the notion of relevance is fundamental to all cognition – meaning System 1 *and* System 2, cognition (Sperber, Cara & Girotto, 1995; Wilson & Sperber, 2004).

[2] The questionnaire consisted of two parts, each of roughly equal length. The other part is not related to this research, and is not reported here.

THREE UTILITIES FOR THE EQUAL SIGN

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We compare the activity of young children using a microworld and a JavaScript relational calculator with the literature on children using traditional calculators. We describe how the children constructed different meanings for the equal sign in each setting. It appears that the nature of the meaning constructed is highly dependent on specificities of the task design and the tools available. In particular, the microworld offers the potential for children to adopt a meaning of equivalence for the equal sign.

CHILDREN'S PERCEPTIONS OF THE EQUAL SIGN

There exist "two intuitive meanings of equality among preschoolers" that correspond to equivalence and operation (Kieran, 1981). When conceived of as an operator the equal sign would be expected to generate an answer from a sum. In contrast, when conceived of as equivalence the equal sign would be regarded as a static relation between two expressions that are the same in value.

However school children attend to the operator meaning more readily than equivalence (Behr et al, 1980). The reasons for this have been identified by Ginsburg (1977) as based in the way the equal sign is encountered in the classroom. Invariably it is the operator meaning of the equal sign that is appealed to through the way sums are framed in textbooks, as well as the use of the = button on calculators.

Hughes (1986) points out that in the case of using a calculator it is actually valid to consider the equal sign as an operator. The = button can be viewed as a function that takes a 'number-operator sequence' and outputs a 'number'. In order to discuss this functionality in contrast to others desribed later we have found it helpful to consider the notion of 'utility' (Ainley et al, in press), specifically the construction of meaning for the ways in which the equal sign is useful. The utility of the equal sign in calculators is to obtain the answer to programmed sequences. Calculators arguably contribute significantly to children's adoption of the operator meaning. Behr et al. (1980) in interviews of six to twelve year olds report

> children consider the symbol = as a 'do something signal' that 'gives the answer' on the right hand side. There is a strong tendency among all the children to view the = symbol as being acceptable when one (or more) operation signs precede it.

The correlation between the function of calculator = buttons and children's conceptions of the equal sign is clear. In addition Behr, Erlwanger and Nichols note that children consider the equal sign as unacceptable without an operator and respond by "completing the sum". For example, when a 12 year old child was asked about the meaning of 3 = 3 she said that it could mean "0 minus 3 equals 3". This is consistent with the = button on a calculator lacking utility if the input is a single number.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 185-192. Melbourne: PME.

Calculators may also contribute to children's "conception of an equation as a temporal event, corresponding to a verbal left-right reading, rather than a static state" (Pirie & Martin, 1997). The programming of a sequence of operations followed by pressing = and the appearance of a 'final' answer can be expected to reinforce dynamic, directional conceptions of equations.

In this paper we explore the implications of exposing children to technologies which ascribe alternative functionalities to the equal sign. One of those technologies is a microworld in which an = object takes two number inputs and outputs a Boolean state; the other is a 'relational calculator' in which the = button takes two 'number-operator sequence' inputs and outputs a graphic, which may be blank or an =. In functional notation the role of = can be considered as follows:

Traditional Calculator	$e^{C}(num - op\ sequence) \rightarrow number$
Microworld	$e^{M}(num, num) \rightarrow Boolean$
Relational Calculator	$e^{RC}\ num - op\ sequence, num - op\ sequence) \rightarrow [graphic]$

Our discussion of children's conceptions in relation to different functionalities of the equal sign will adopt a perspective in which various meanings can be held simultaneously though with varying priorities. Accordingly, we are using a theoretical framework based on diSessa (1993), in which knowledge is seen as made up of a multitude of small pieces, which gradually become connected through learning. We believe this approach will enable fruitful analysis of the subtleties and apparent inconsistencies to be expected when children interact with technologically supported representations of mathematical equality.

THE MICROWORLD

The *Visual Fractions* microworld allows interaction with a variety of arithmetical objects including fractions, operators, relations, regions and Boolean flags (Figure 1). The user may create, manipulate, destroy and connect objects on screen. All objects have functionality: they can take one or more inputs and produce a numerical or Boolean output.

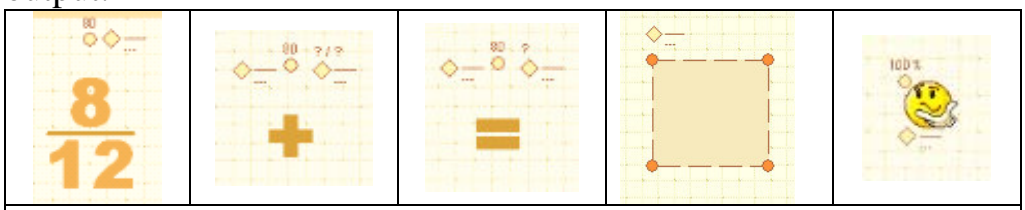


Figure 1 Arithmetical objects: Fraction; Operator; Relation; Region; Flag

The user creates an input by dragging the small diamond above an object to another object. For example, in Figure 2 the + object takes the inputs $1\frac{3}{6}$ and $\frac{8}{12}$ and produces the output $\frac{26}{12}$ which is displayed just above the right diamond. (Note the curved arrows have been added to screen excerpts for the purposes of clarity.)

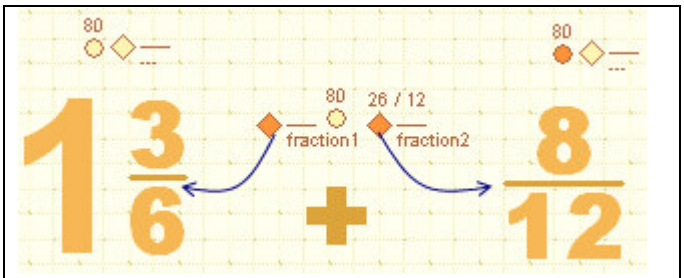


Figure 2 An operator takes two numbers as inputs and produces a number as output

A fraction object connected to the + object takes as input the output from + and produces $\frac{26}{12}$ which is displayed as the fraction itself (Figure 3). This fraction is dependent on the output of the + object and changes with it.

An = object connected to the + object and a fraction object takes the inputs $\frac{26}{12}$ and $\frac{26}{12}$ and produces the Boolean output 'true' which is displayed just above the right diamond (Figure 4). Regions (Figure 1) have a similar functionality to relations, taking two or more numbers as inputs and producing a Boolean output. Flags (Figure 1) take a Boolean input and simply display that input graphically as a 'thumbs-up' for true and 'thumbs-down' for false.

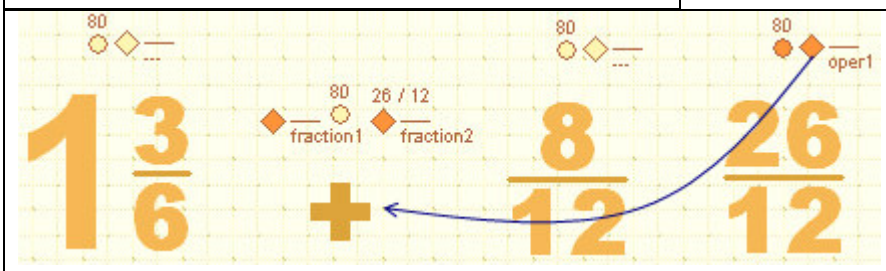


Figure 3 A fractions object takes a number as input and produces a number as output

The function of the equal sign is different to that of a calculator's = button. Thus, changing its position allows construction of an expression that lacks left-right spatial sequencing.

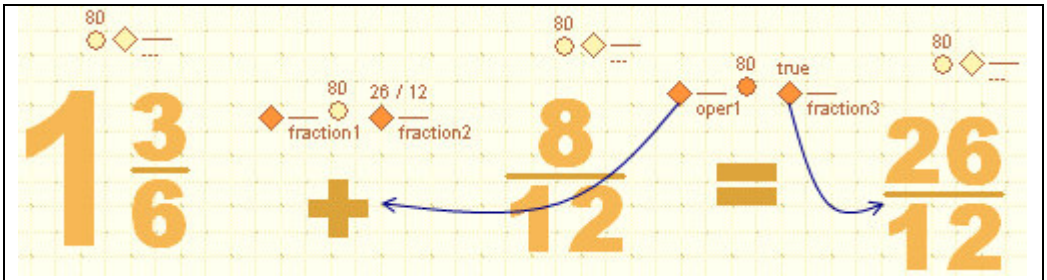


Figure 4 A relation takes two numbers as inputs and produces a Boolean output

The Microworld - Trial

A trial of the microworld took place with two girls, C & L, aged 13 years. The activity lasted 80 minutes and was recorded using a microphone and screen capture software. A researcher introduced the addition operator and asked C & L to investigate its connector ports. After practising a little they were challenged to design a task for peers of the same age. They went on to construct an arithmetic task in which users are required to correct answers to expressions. The activity was transcribed and analysed by identifying key incidents associated with the equal sign. A trace of the evolution of conceptions for the equal sign was subsequently constructed.

The Microworld - Findings

Initially the students placed two fractions and an operator on screen. The operator's constraint of possessing two connector ports guided the students to connecting the three objects correctly. An equal sign was placed on screen and the students discussed possible connections. Figure 5 captures L attempting to drag the second equal sign connector to empty space [see mouse cursor]:

1 L: I don't know where you connect that.

2 C: No you leave that one…put a box or something.

3 L: We really need to put a region

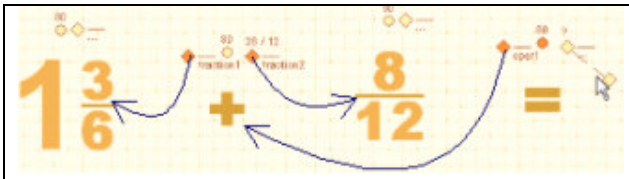


Figure 5 C & L tried to connect the equal sign

Lines 1-3 show that the existence of two connector ports above = affords making two connections. It seems the girls wanted an object to which = could be connected and decided upon using a region. A fraction was positioned inside the region to act as the answer. The equal sign was then connected to it in a way that parallels the operator's connections. Next a flag was placed on screen and connected to the region (Figure 6).

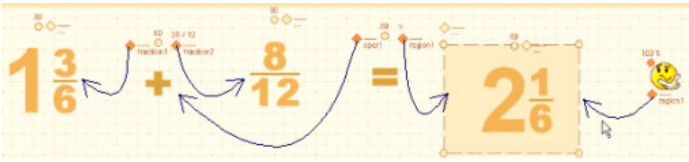


Figure 6 C & L connect the equal object and a flag to a region containing a fraction

At this point the students had made progress largely due to experimenting with objects that afford connection. However the region was technically superfluous to realising their design and consequently the flag did not display 'true' as the students expected. As it emerged that experimenting with possible connections was a flawed strategy, a switch to thinking mathematically was evoked. C considered how else the flag might be connected, at which point L attempted to connect the answer fraction to =.

Figure 7 The students connect the flag to the equal sign

The connection failed and an error sound played. The students had now hit an impasse, and went on to try connecting the answer to = another three times throughout the trial - despite the clear error sound that played on each attempt. A researcher suggested that they try without using the region and the students deleted it. Further attempts to attach the answer to the equal sign were disallowed and the students reconnected the objects from scratch, this time with both the equal sign's connector ports linked to the operator and the flag connected to the equal sign (Figure 7). The flag became true and the students believed that they had

succeeded, but when they tested it by changing the answer the flag remained true. The students tried to fix the expression, starting with a disallowed attempt to connect the answer to the flag. The experimental nature of their trials is evidenced by dialogue:

4 L: Maybe that needs to be on the other side.

5 C: This [answer] isn't connected to anything.

6 L: Perhaps if you do that to the smiley face. Or try it to the equals

After another failed attempt to connect the answer to = a researcher asked what was written above the equal sign.

7 C: It says true. But we don't know what...

8 L: Click on the true thing [text above =]

C clicked on the word true and the equal sign's right-hand connector was disconnected and the word true changed to ?. C connected = to the answer and the word true reappeared. This left the flag unconnected and an attempt to connect it to the answer was made. The students did not relate the required Boolean state of the flag to the word 'true' written above =. It would seem the girls expected the flag to simply 'know' if the expression was correct by virtue of being tagged on to the end. When the attempted connection failed, C momentarily dragged the flag's connector round the screen and placed it on the equals. This circuitous route suggested her choice of where to place the connection was somewhat experimental and playful. The flag became true.

9 L: He still always has his thumbs up

10 C: No he won't. It's worked now because look it says now. There, that's done it now.

The students tested the activity by changing the value of the answer. The flag (and =) displayed false.

THE RELATIONAL CALCULATOR

The first author is developing a JavaScript relational calculator, which allows number-operator sequences to be entered into left and right screens using two keypads (Figure 8). The user may click on the operators that appear in the screens in order to collapse 'number-operator-number' trios into equivalent single numbers (Figure 9).The

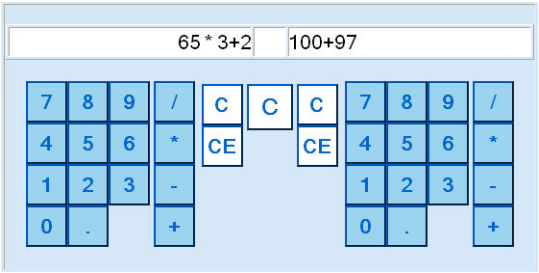


Figure 8: Number-operator sequences are entered into two screens

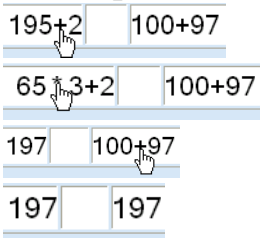


Figure 9: Effect of clicking on operators

user may click on the smaller, central screen to establish the relation between the left and right screens. Doing so causes a graphic to appear (Figure 10). Further adjustments to the left or right screens clear the central screen.

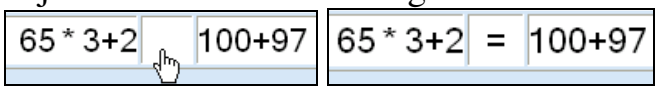


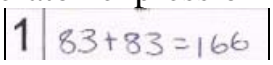
Figure 10: Clicking on the central screen displays =

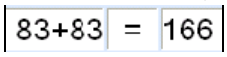
The Relational Calculator – Trial

Three trials of the microworld took place with pairs of students, aged 8 and 9 years. Each trial lasted about 60 minutes and was recorded using a microphone and screen capture software. The students were allowed free exploration of the calculator and the researcher pointed out features as appropriate. They were then challenged to create a worksheet of up to 20 'really hard' number sentences for other children to check using the relational calculator. Each number sentence created on the relational calculator had to be hand-written on a worksheet.

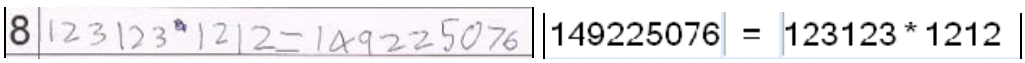
The Relational Calculator - Findings

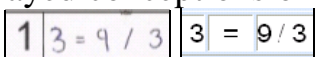
The relational calculator data offer no window on student interactions (Noss & Hoyles, 1996) but do provide insights into students' interpretations and responses to = as an outputted graphic.

Temporal Sequencing Conceptions. One pair of students, Ch & H, began by creating a single operator expression and writing it down exactly as it appeared on screen: *Worksheet:* 

Screenshot: 

When the researcher later suggested they write down the answer to the left of = on their worksheet they were reluctant to do so despite having seen the relational calculator allow this on occasion:



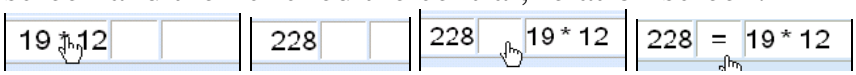
Another pair, U & Lu, displayed conceptions of right-left directionality when writing down their first expression: 

11 U: But we write it that way [9/3=3], not three equals nine.

12 Lu: I'd do it that way [3=9/3] so it would be harder for them.

However, left-right conceptions became prioritised in U & Lu's later expressions, perhaps because they attended to numeracy as they tried to make the worksheet progressively more difficult by using longer numbers. Left-right temporal sequencing is also evident when the researcher requested they create an expression with operators on each side of =, resulting in '6781+2976=9757+39=9796'.

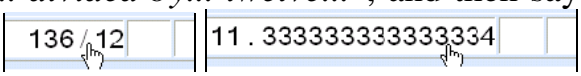
The final pair, K & M, began entering '19*12' into the left screen and clicked the operator. They decided to test it and without comment entered '19*12' into the right screen and then clicked the central, relation screen.

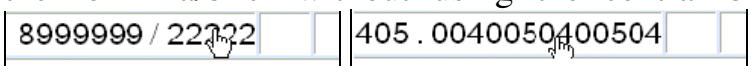


The appearance of '=' satisfied K & M that the expression was correct; but they wrote it on the worksheet as '19*12=228' (i.e. with left-right temporal sequencing). It would seem that K & M's conception of left-right directionality prioritises itself over and above the on-screen right-left directionality. Left-right directionality is also evident when K & M were requested to use an operator on each side of an expression, resulting in '1200-34=1162+4=1166'. A principle factor of prioritisation of left-right conceptions would appear to be the students' requirement for the equal sign to be followed by an answer. When the researcher requested K & M write down '1200-34=1162+4' as it appeared on screen they considered it to be impossible for other children to understand:

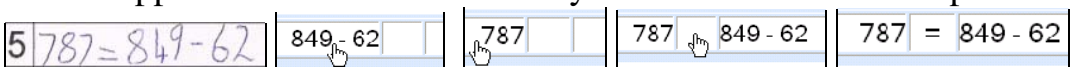
13 M: Because otherwise the answer's going to be in the middle of the... and then the sum's going to be on the side and they won't know which...

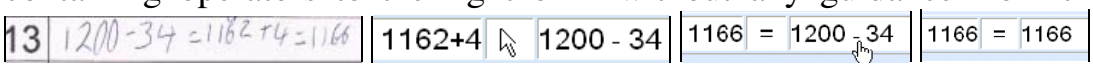
14 K: People aren't going to know which answer's which.

Operational Conceptions. The relational calculator equal sign affords no interactivity. Nonetheless, the data contain some evidence of students attending to operator notions. For example at one point Lu reads an expression while inputting it, "*One three six... divided by... twelve...*", and then says "*equals*" while clicking '/' to get the answer: 

The data contain examples of all pairs of students saying "*equals*" while clicking on an operator. However, perhaps more interestingly, the data abound with evidence of students not attending to = at all. Ch & H created and wrote down an expression of the form 'a/b=c' without using the central button or even mentioning equals: 

Ch & H went on to create a further four expressions without any attention to =. The data contain examples of all students doing this to a greater or lesser extent. It would seem that when = lacks affordance there are times children do not attend to = at all.

Relational Conceptions. K & M were the only pair to make substantial use of the = button of their own volition throughout most of the trial. Typically they got an answer by clicking on an operator then re-entered the expression to check that = would appear. When it did so they were satisfied the expression was correct: 

In addition they were the only pair who managed to create a valid expression containing operators to the right of = without any guidance from the researcher: 

Although it cannot be inferred whether or not K & M actually prioritised or even attended to relational meanings during the trial it is notable that they are the only students who did not repeatedly violate equivalence.

DISCUSSION

In the case of traditional calculators the equal sign has the function $e^{C}(num - op\ sequence) \rightarrow number$. The outputted number is inevitably an answer to a

sum and it is this aspect that has been shown to generate the operational utility (Hughes, 1986; Behr et al, 1980). Traditional calculators also reinforce left-right temporal sequencing as experienced in writing.

In the case of the microworld the equal sign has the function $e^{M}(num, num) \rightarrow Boolean$. The Boolean output has mathematical potential to be re-employed in developmental sequences. This potential seems to allow, through experimentation, a utility for the equal sign of equivalence and can challenge students' prioritisation of operational utility of the equal sign. The mathematical potential of interactions with the equal sign can invoke thinking about relational meanings.

In the case of the relational calculator the equal sign has the function $e^{RC} num - op\ sequence, num - op\ sequence) \rightarrow [graphic]$. The output has no potential for further interactivity and sometimes leads to children ignoring = altogether. There is a potential utility of allowing expressions to be checked for equivalence, though in practice only one pair seemed to appreciate this possibility. The relational calculator can however challenge left-right temporal sequencing conceptions**.** We now wonder whether the next design of the relational calculator should automatically display <, = or > according to the relative value of the two expressions. Perhaps students would then be less inclined to ignore the equal sign though there would still be a lack of potential for the graphic to be used in contrast either to the outputted number on a traditional calculator or the outputted Boolean in the microworld.

Our experience of using the relational calculator and the Visual Fractions microworld has highlighted the close relationship between the particular utility that is constructed and the nature of the task and the structuring resources in the setting (Lave, 1988). We see the pedagogic challenge not as one of eliminating the operational utility but as providing new experiences, carefully designed, to optimise the possibility that the child may construct new utilities for the equal sign such as that of equivalence.

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COMPUTER ALGEBRA SYSTEMS (CAS) AS A TOOL FOR COAXING THE EMERGENCE OF REASONING ABOUT EQUIVALENCE OF ALGEBRAIC EXPRESSIONS

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Equivalence of algebraic expressions is at the heart of transformational work in algebra. However, we know very little about students' understanding of equivalence. This study is part of a larger project that explores the use of CAS as a didactical tool for promoting both technical and conceptual growth in high school algebra with tasks specially designed by the research team. We report on a class of 10th graders coming to grips with the underlying theoretical ideas related to algebraic equivalence and on the role that the CAS played in the shaping of these newly emerging notions. Two different notions of equivalence of expressions were found to emerge: one was purely numerical and entailed reasoning about expressions for which some but not necessarily all numerical substitutions would yield equal values; the other entailed both numerical and common-form reasoning. Interpretation of CAS outputs (the equality test, in particular) played a role in occasioning discussions that don't normally occur in algebra classrooms and led to clarifying distinctions about equivalence for many.

PAST RESEARCH IN THIS AREA

While computers and calculators enabled with symbol-manipulating capabilities have been considered quite appropriate for student use in tertiary-level mathematics courses (Heid, 1988), such has generally not been the case for the secondary-level. Many teachers have tended to stay away from such technology in their classrooms, preferring that their students first develop paper-and-pencil skills in algebra (NCTM, 1999). In contrast, graphing calculators have been adopted on a wide scale -- their use supported by the large number of studies providing evidence of the role that graphical representations can play in enhancing student understanding in algebra (Kieran & Yerushalmy, 2004). This has, of late, encouraged some researchers to begin to investigate whether and how CAS technology can contribute to student learning of secondary school algebra.

Recently, researchers (e.g., Artigue, 2002; Guin & Trouche, 1999; J.-B. Lagrange, 2000) have argued that these new technological tools promote both conceptual and technical growth in mathematics, as long as the technical aspects are not ignored.

[1] The data that are reported in this paper were collected and analyzed while Luis Saldanha was a post-doctoral fellow at the Université du Québec à Montréal. The authors express their appreciation to the Social Sciences and Humanities Research Council of Canada, which funded this research, and to the other members of the research team who collaborated in the design of the tasks: André Boileau, José Guzmán, Fernando Hitt, Denis Tanguay, and also Michèle Artigue who served as consultant on this project, and the participating classroom teachers who provided feedback on a regular basis throughout the successive cycles of task development.

More specifically, Lagrange (2000) has elaborated the notion that technique is the bridge between task and theory. In other words, as students develop techniques in response to certain tasks, they engage in a process of theory building. The technological tools that students use in this theory building become instruments of mathematical thought (Verillon & Rabardel, 1995), permitting both conceptual amplification and reorganization (Pea, 1987). Instrumented techniques thus have an epistemic value, contributing to the understanding of the objects they involve. Instrumentation theory, which articulates the relation between tool use and conceptual development has been applied to the learning of not only calculus but also high school algebra. For example, Drijvers (2003), who focused on the relation between instrumented techniques in a CAS environment and the learning of parameters, noted that the obstacles raised by student difficulty in reconciling CAS output with expected results presented opportunities for learning, when addressed within whole-class demonstrations and discussions.

In another CAS study (Ball, Pierce, & Stacey, 2003) dealing explicitly with equivalence of expressions, researchers found that students could not recognize equivalent expressions, even simple cases, and noted that, "the ability to recognize equivalent forms of algebraic expressions is a central part of working with CAS and one that is likely to take on new importance in future curricula" (p. 4-16). Similarly, Artigue (2002) drew upon students' work involving the passage from one given form of an expression to another to illustrate how the research team paid specific attention to the fact that "equivalence problems arise which go far beyond what is usual for the classroom." She used the CAS as a "lever to promote work on the syntax of algebraic expressions, which is something very difficult to motivate in standard environments" (p. 265), adding that its use obliges students to face equivalence and simplification issues. According to Nicaud et al. (2004), "reasoning by equivalence is a major reasoning mode in algebra; it consists of searching for the solution of a problem by replacing the algebraic expression of the problem by equivalent expressions … identities allowing for the transformation of expressions while maintaining equivalence" (pp. 171-2). The importance accorded to the notion of equivalence of expressions, as well as students' reported difficulties in this area, suggests that we need to know a great deal more about the ways in which algebra learners think about equivalence of expressions. Wishing to build upon the recent work that has been initiated by researchers in this area, we designed a study that uses CAS as a lever for promoting student work, *and reflection,* on equivalence of expressions.

THE STUDY

The study reported in this paper is part of an ongoing three-year project involving five intact classes of 10th graders (15-year-olds). These students have been following an integrated program of mathematics since the 7th grade, which means that algebra is part of the course of study each year. One of these five classes is featured in this report. Students in this class had learned basic techniques of factoring and the solving of linear and factorable quadratic equations during the previous year and had used

graphing calculators on a regular basis; however, they had not had any experience with the notion of equivalence or with symbol-manipulating calculators. They were quite skilled in algebraic manipulation, as was borne out by the results of a pre-test administered at the outset of the study. It was during the algebra part of their 10th grade mathematics course, when the activities designed by the research team, accompanied by CAS technology (TI-92 Plus calculators), were integrated into their regular program of mathematics and taught by the classroom mathematics teacher.

The Design of the Activities

Of the eight activities created by the research team, each one designed to take up about two 65-minute-long class periods, three dealt with equivalence of expressions. Each activity was punctuated by parts, each part including presentation of student work and discussion of the main issues raised by the tasks in the given part. Tasks were of three types that involved either work with CAS, or with paper/pencil, or were of a reflective nature. For every activity, there was an accompanying teacher version that included suggestions for classroom discussion. In designing these tasks, we took seriously both the students' background knowledge and the fact that these tasks were to fit into an existing curriculum; but we also moved to ensure that they would unfold in a particular classroom culture that reflected a certain priority given to discussion of serious mathematical issues.

Activity 1: Equivalence of Expressions
- Part I (with CAS): Comparing expressions by numerical evaluation
- Part II (with paper/pencil): Comparing expressions by algebraic manipulation
- Part III (with CAS): Testing for equivalence by re-expressing the form of an expression – using the EXPAND command
- Part IV (with CAS): Testing for equivalence without re-expressing the form of an expression – using a test of equality
- Part V (with CAS): Testing for equivalence – using either CAS method

Activity 2: Continuation of Equivalence of Expressions
- Part I: Exploring and interpreting the effects of the ENTER button, and the EXPAND and FACTOR commands
- Part II: Showing equivalence of expressions by using various CAS approaches

Activity 3: Transition from Expressions to Equations
- Part I (with CAS): Introduction to the use of the SOLVE command
- Part II (with CAS): Expressions revisited, and their subsequent integration into equations
- Part III (paper/pencil): Constructing equations and identities
- Part IV (with CAS): Synthesis of various equation types

Figure 1: Outline of content of the three activities related to equivalent expressions

In these activities, *equivalent expressions* were defined as follows: "We specify a set of admissible numbers for x (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces x, each of the expressions gives the same value, we say that these expressions are equivalent on the

set of admissible values." As seen in Figure 1, which displays in outline form the content of the three activities related to equivalence, numerical evaluation of expressions served as the entry point for discussions of equivalence. The impossibility of testing all possible numerical replacements in order to determine equivalence motivated the use of algebraic manipulation and the search for common forms. Discussion included attention to restrictions on equivalence. The relation between equivalent/non-equivalent expressions and equation solutions was then explored in both CAS and paper/pencil tasks.

Classroom Set-Up and Data Sources

Two video cameras were set up in the classroom, one in the front and fixed in position, and a second one in the rear that pivoted and zoomed. Two ceiling-mounted microphones provided sound that was mixed and directed into each camera. One or two researchers took field notes during each class period. Two students were interviewed (and videotaped) after each class period in order to further probe their thinking. A post-test involving CAS use was administered after the three sets of activities had been completed. Thus, data sources for the segment of the study analyzed for this report include the DVD-captured videotape of all the classroom lessons involving the three activities, individual interviews with nine students, transcripts of selected video segments, the completed activity sheets of all students, written pre- and post-test responses, and researcher field notes.

RESULTS AND DISCUSSION

Responses to one of the post-test questions in particular, Question 5(iii) (see Figure 2), suggested that students' emerging ways of thinking about equivalence were being confounded with notions of equality. While for part (i) 93% of the students were coded with a correct score, and 100% of them used a valid CAS method to show that there are no other solutions in part (ii), only 60% of them correctly answered part (iii). Examples of students' correct answers to Q.5(iii) were:

- "They are not equivalent, because when x stays as x and both sides are factored out they are not identical. These expressions can be equal when x is replaced by 2 or 2/3, because the two sides would be identical."
- "They are not equivalent, as only when 2 and 2/3 are plugged in as values of x are the expressions equal. They cannot be put into common form."

Incorrect answers included the following:

- "They are equivalent when the numbers you are substituting are $x = 2$ or $x = 2/3$. Not any other numbers."
- "The expressions are equivalent since they both have the same solutions."

While it could be argued that students' incorrect responses were merely an indication of linguistic error (i.e., using "equivalent" instead of "equal"), we wondered whether the problem might be deeper than that and might reflect an interpretation of equivalence purely in numerical terms and an absence of the relevance of the idea of algebraic form as a tool of algebraic thinking. Furthermore, we asked ourselves

whether the ways in which they were using the CAS tool or whether the ways in which the CAS displayed its outputs could be serving to reinforce numeric interpretations to the exclusion of those related to form.

Q.5 The following equation has $x = 2$ and $x = 2/3$ as solutions:

$$x(2x-4)+(-x+2)^2 = -3x^2+8x-4$$

(i) Precisely what does it mean to say that, "the values 2 and 2/3 are solutions of this equation"?
(ii) Use the CAS to show that: (a) the two values above are indeed solutions, <u>and</u>
(b) there are no other solutions.

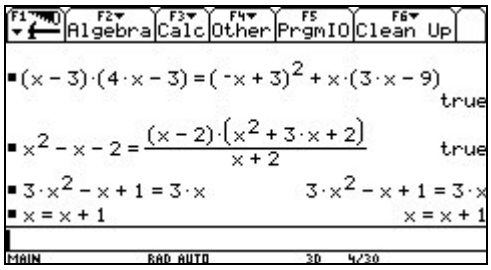
What I entered into the CAS:

What the CAS displays <u>and</u> my interpretation of it:
(iii) Are the expressions on the left- and right-hand sides of this equation equivalent? Please explain.

Figure 2: Post-test question (see part iii) that revealed interpretations of equivalence

Correct answers to Q.5(iii) had generally incorporated two components, that of being able to represent equivalent expressions in some common form and that of producing the same numerical value for equivalent expressions on substitution by each of an infinite set of values. Incorrect answers did not refer to common form. Of the 60% of those who were successful in answering Q.5(iii) on the non-equivalence of the two given expressions, 88% of them used some type of common-form argument in their reasoning. (See Saldanha & Kieran, 2004, for further discussion of this result.)

We note also that only two students used the equality test to determine whether the two given expressions are re-expressible in a common form. Indeed, our analysis of students' work sheets for the activities preceding the post-test, as well as the interview data, revealed that many students were puzzled by the equality test.



(i)

(ii)

(iii)

Figure 3: Output from the TI-92 Plus for four cases of equivalent and non-equivalent expressions using the equality test

Figure 3 illustrates the TI-92 Plus output when the equality test is used for: (i) two equivalent expressions with no restrictions, (ii) two equivalent expressions with one restriction, (iii) two non-equivalent expressions that are equal only when $x = 1$ or $x = 1/3$, and (iv) two non-equivalent expressions that are never equal. As can be seen, this test produces *true* when an equation is entered that is formed from two equivalent expressions (note, however, that the restriction to the equivalence is not displayed); on the other hand, the CAS simply displays the input-equation when the left- and right-hand-sides of the equation are not equivalent expressions. Thus, with its two

distinct outputs – either *true* or the input-equation -- the equality test can serve as a test for equivalence of expressions.

The latter output was, however, difficult for students to interpret. This is illustrated by the following excerpt, drawn from an interview with a student (S) immediately after classroom work on Activity 1:

S: When I see "true", I figure that that there aren't any exceptions [having entered: $(3x-1)(x^2-x-2)(x+5) = (x^2+3x-10)(3x-1)(x^2+3x+2) / (x+2)$]. Like, I figure if it says "true" all the time, it would always, no matter what you put x as, it would be equivalent.
I: Ok. Let's go on to the next part here. [turning page] Alright, do you remember that one?
S: Uhm, I entered the problem [$(x^2+x-20)(3x^2+2x-1)=(3x-1)(x^2-x-2)(x+5)$] and it gave me pretty much the same problem back, but rearranged, it's the same answer. When you think that the other one said "true," it is kind of puzzling. ... The answer that it gave me. I figure that that's this statement, like the first expression equals the second expression is true. … When I see an equal sign, I figure they are equivalent, the same.
I: Is there anything you might do to check whether your hunch about what that means is right indeed?
S: I'd expand it. Then you see all the parts.
I: Did you actually do that as a follow up?
S: I did, myself, but I don't think we did, as a class.
I: And did it corroborate what you
S: No, because when you expand them, they're different.
I: Would you revise your response to this in light of that?
S: Yes, but I still don't understand why it would tell me that it's equivalent.
I: So, if I understand you correctly, to you it's sort of mysterious why
S: It's misleading.

While this student evidently thought that, when a statement with an equal sign is returned, such a display should still signal equivalence of the two given expressions, others did not know what to make of such output. They thought that the CAS should display *false* if the two expressions that formed the equation were not equivalent, especially for cases such as $x = x+1$, which they argued "could never be true." They had seen *false* displayed when they had substituted certain values into equations containing restrictions, and thus expected *false* for certain cases when using the equality test. Naturally, there was considerable classroom discussion around this issue when it was first encountered during Activity 1. Teacher and students alike both referred to some of the prior numerical substitutions they had made with these expressions, pointing out that equations made from such expressions were sometimes true and sometimes false, depending on the numerical value being substituted. If an equation formed of two expressions was not always true (subject, of course, to a few restrictions), they argued, then the expressions forming that equation were not equivalent and never could be. But because there were sometimes exceptions to the equivalence (in the case of inadmissible values, which were not always obvious to the students), some believed there might be exceptions to the "non-equivalence of expressions," which they were similarly not seeing. Thus, the fragility of students'

emerging knowledge of equivalence was being exposed by their difficulty with interpreting some of the CAS displays.

Activity 3 introduced students to the SOLVE command and its use in interpreting the equivalence of the given expressions of an equation. Students seemed more at ease with the CAS outputs of SOLVE: *true*, some solutions, or *false*. They were on more familiar ground; but it was a numerical ground (solutions that included all real numbers, only some real numbers, or no real numbers). Thus, those equations that had been returned "as is" by the equality test, and which had been uninterpretable by many, were now seen in a clearer light. However, as illustrated by some of the post-test responses, the expressions on the two sides of an equation for which there were some solutions were being considered as equivalent expressions, that is, "equivalent for certain values of x." With their purely numerical interpretations of equivalence, they tended to lump into the category of equivalence both pairs of expressions whose equations were true for all, as well as those with only a few solutions. In contrast, those students whose numerical interpretations were accompanied by "common-forms" reasoning (i.e., only those pairs of expressions forming equations that were true for all could be expressed in common form) correctly distinguished equivalent from non-equivalent expressions. The ramifications of the former kind of reasoning on students' understanding of the process of equation solving where sub-expressions of the equation are to be replaced by equivalent expressions must surely be obvious.

While the CAS equality test, with its "mysterious" output, alerted students to something that they needed to think deeply about and brought their conceptual difficulties to the surface, clearly not all of them were resolved. As students continue to work with equation-solving procedures during the year and more attention is paid to the role of common-form arguments in talking about equivalent expressions, equivalence of expressions may slowly become disentangled from equality of expressions. As students gain more experience with the CAS, along with and in interaction with their emerging knowledge on equivalence of expressions, they may come to use the equality test, among others, with confidence. Ball, Pierce, and Stacey (2003) have pointed out that the students in their CAS study had a great deal of difficulty in recognizing equivalent expressions. To compensate for this difficulty, we suggest that students need to be adept at using various CAS tests in order to determine equivalence of expressions, but they also must have a clear idea of what is meant by equivalent expressions. The report of this part of our study has described our initial efforts toward understanding students' grappling with both these issues.

CONCLUDING REMARKS

In this study, CAS was not intended to replace paper/pencil work as a technical tool. Nor was it to be simply a means for checking paper/pencil work. It was, however, intended for use as a didactical tool for coming to grips with underlying theoretical ideas in algebra. Some of the components of our activities that made CAS a didactical tool were questions that used the machine to occasion discussions that don't normally happen in mathematics classes. Tasks that asked students to write

about how they were interpreting their work and the related CAS displays bring mathematical notions to the surface, making ideas and distinctions much clearer, in ways that simply "doing mathematics" may not require. If CAS is to be effective at the high school level, it is precisely this kind of usage that needs to be considered.

Acknowledgments

Our deepest appreciation to the students and teacher of the class reported herein. They gracefully put up with all of our videotaping, observing, and interviewing over a period of several months. We are grateful to Texas Instruments for providing the calculators used in this study. We thank Andrew Izsák and John Olive for their advice with respect to technical equipment and its set-up at the various research sites.

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STUDENTS' COLLOQUIAL AND MATHEMATICAL DISCOURSES ON INFINITY AND LIMIT

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The purpose of the study presented in this paper is to investigate how students deal with the concepts of infinity and limit. Based on the communicational approach to cognition, according to which mathematics is a kind of discourse, we try to identify the characteristics of students' discourse on the topic. Four American and four Korean students were interviewed in English on limits and infinity and their discourse is scrutinized with an eye to the common characteristics as well as culture, age, and education-related differences.

INTRODUCTION

Infinity is the conceptual basis for many mathematical topics such as the number line and infinite decimals. Since the 19th century, the concept of limit has been foundational to mathematical analysis. As is known to teachers and as confirmed by researchers, most students have considerable difficulty with both of these notions.

In this study, students' thinking about infinity and limit is investigated based on the *communicational approach to cognition*, according to which mathematics is a kind of discourse. There are several reasons why this kind of study may be important. First, there has been little research on the mathematical concepts of infinity and limit using discourse analysis as a methodology. Discourse analysis holds promise of answering some previously unanswered questions. Second, such investigations may lead to methods for helping students overcome their difficulties, something that will also have implications for teacher education and K-16 curriculum. Third, this approach may have implications for investigating advanced mathematical thinking in other areas. Finally, applying the communicational method to culturally different groups of students may shed light on the impact of culture on how students learn the most advanced of mathematical notions.

THEORETICAL BACKGROUND

Epistemology and History of the Notions of Infinity and Limit

The histories of the mathematical concepts of *infinity* and *limit* have been interwoven ever since their beginning. The story of infinity begins with the ancient Greeks. The Greek word '*apeiron*' meant *unbounded*, *indefinite*, or *undefined* (Boyer, 1949). For the Greeks, infinity did not exist in actuality, but rather as a potential construct. Although there was the notion of bounded processes, there was no concept of limit as a concrete bounding entity.

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In the Middle Ages, Christianity came to value infinity as a divine property. With the developments of astronomy and dynamics in the 16th century, there was an urgent need to find methods for calculating the area, volume, and length of a curved figure. In the 17th century, to find the areas of fan-shaped figures and the volumes of solids such as apples, Kepler used infinitesimal methods (Boyer, 1949). Throughout the 18th century, calculus lacked firm conceptual foundations. At the end of the 18th century, mathematicians became acutely aware of inconsistencies with which the theory of infinitesimal magnitudes seemed to be fraught (Rotman, 1993).

Today's notion of limit emerged gradually in the 19th century as a result of attempts to remedy the uncertainties with which the mathematical analysis was ridden at that time. Cauchy and Weierstrass were pioneers of the movement toward a rigorous calculus (Moore, 1990). By mid-19th century the concept of limit became the basic concept of the calculus (Kleiner, 2001). At this time, limit turned into an arithmetical rather than geometrical concept, as it was before, in the context of infinitesimals. Infinity was now actual rather than potential. In order to complete Weierstrass' foundations of arithmetic, Dedekind and Cantor developed the theory of the infinite set (Boyer, 1949).

In spite of the mutual interdependence of the concepts of limit and infinity, there has been little research to examine students' understandings and difficulties of both of them simultaneously.

Leaning the Mathematical Notions of Infinity and Limit

Various aspects of the learning about infinity and limit have been investigated over the last few decades.

Anchoring their research in the analysis *mathematical structure* of the notions, Cottrill et al. (1996) report that there are two reasons for student difficulties with limits. One reason is the need to mentally coordinate two processes ($x \to a,\ f(x) \to L$). The other is the need for a good understanding of quantification related to ε and δ. Borasi (1985) suggests several alternative rules about how to compare infinities based on students' intuitive notions (within this tradition, see also Cornu, 1992; Tall, 1992).

Other research focused on *misconceptions* and *cognitive obstacles* related to infinity and limit. Fischbein, Tirosh, & Hass (1979) and Tall (1992) emphasized the role of intuition. One source of difficulty is the belief that a part must be smaller than the whole. Other researchers (Cornu, 1992; Davis & Vinner, 1986) stress the influence of language. Students might have had many life experiences with boundaries, speed limits, minimum wages, etc. that involved the word "limit". These everyday linguistic uses interfere with students' mathematical understandings (Davis & Vinner, 1986). Przenioslo focuses on the key elements of students' *concept images* of the limits of functions. Still others have focused on *informal models* that act as cognitive obstacles (Fischbein, 2001; Williams; 2001). According to Williams, informal

models based on the notion of actual infinity are a primary cognitive obstacle to students' learning.

Finally, some researchers address students' difficulties through the lens of the theory of actions, processes, objects, and schemas (APOS; see Weller, Brown, Dubinsky, McDonald, & Stenger, 2004). Weller et al. speak about the cognitive mechanisms *interiorization*, *encapsulation*, and *thematization* that are used to build and connect actions, processes, objects, and schemas.

Conceptual Framework for This Research

The point of departure for the present study is the realization of the fact that when students come to the classroom to learn the notions of infinity and limit, they already have a certain amount of knowledge that comes from daily experience. The use of a given concept in everyday language can be crucial for students' future learning. For those who are supposed to teach the subject it is therefore important to find out how students use the notions of infinity and limit in colloquial discourse.

Most of the past research on learning limits and infinity was grounded in a neo-Piagetian, cognitivist framework which does not seem quite appropriate for this type of study as it underestimates not only the inherently social nature of student thinking, but also the role of discourse and communication in learning and other intellectual activities. Our project is guided by the conceptual framework within which school learning is equated to a change in ways of communicating. In particular, learning mathematics is seen as tantamount to becoming more skilful in the discourse regarded as mathematical. The word *discourse* signifies any type of communicative activities, whether with others or with oneself, whether verbal or not. Four distinctive features of mathematical discourses are often considered whenever discourses are being analysed, compared, and watched for changes over time: *words and their use*, *discursive routines, endorsed narratives*, and *mediators and their use* (Ben-Yehuda et al., 2004). In our ongoing study, particular attention has been paid to the participants' uses of the keywords *limit* and *infinity* in colloquial and mathematical discourses; to *discursive routines*, that is, repetitive patterns of both these discourses; and to *endorsed narratives* about limits and identity, that is, to propositions that the participants accepted as true.

DESIGN OF STUDY

Research Questions

Our interest in characterizing the mechanisms of students' thinking about infinity and limit led to the following research questions:

• What are the leading characteristics (in terms of word use, endorsed narratives, and routines) of students' colloquial and literate (school) discourse about infinity and limit?

• Do the students' colloquial and literate discourse on infinity and limit change with age and education?

• Are there any salient differences between the discourse of native English and Korean speakers on infinity and limit? Can these differences be accounted for in terms of the differences in the colloquial uses of these words in English and in Korean?

Methodology

Each ethnically distinct group included one elementary student, one middle school student, one high school student, and one university undergraduate (to refer to groups' members, we use symbols such as A_5 for the American 5th grader, K_{10} for the Korean 10th grader, and A_U for the American undergraduate.) The four American students were English speakers from the United States while the four Korean students were non-native English speakers from South Korea whose first language is Korean. Since the interviews were conducted in English, the four Korean students who were selected had been living in the United States and attending US schools more than 3 years.

The interview questionnaire consisted of 29 questions, organized into eight categories. The first two categories aimed at scrutinizing students' *colloquial* discourses on infinity and limit, whereas the rest were targeted at investigating students' *mathematical* discourses on the topic. Examples of the interview questions are shown in Figure 1.

I. Create a sentence with the following word: (a) *Infinite*, **(b)** *Infinity*.

II. Say the same thing without using the underlined word.

(b) Eyeglasses are for people with **limited** eyesight.

III. Which is a greater amount and how do you know?

(d) A: odd numbers, B: Integers

IV. $\frac{1}{4} = 0.25,\ \frac{2}{8} = 0.25,\ \frac{3}{12} = 0.25, \ldots$

How many such equalities can you write?

V. What do you think will happen later in this table? How do you know?

VI. (a) What is the limit of the following $\frac{1}{x}$ **when x goes to infinity?**

VII. Read aloud: $\lim_{x\to\infty} \frac{x^2+3x}{2x^2} = \frac{1}{2}$. **Explain what it says.**

VIII. (a) What is infinity?

Figure 1: Representative samples of questions from each category.

The interviews, which were conducted in English, lasted 30 to 40 minutes. The conversations were audio- and video-taped and then transcribed in their entirety.

Data were analyzed so as to identify and describe the three distinctive features of the respondents' discourses: word use, routines, and endorsed narratives. At the next stage, several comparisons were made: (a) for the two ethnic groups, we looked for similarities and differences between the group's colloquial and mathematical uses of the keywords *infinity* and *limit*; (b) we searched for differences in the mathematical discourses of different age groups within and across the ethnic groups (c) we compared the colloquial and mathematical uses of the words limit and infinity of the two ethnic groups. In this paper, we present the respondents' uses of the words *infinity* and *infinite*, and compare the results obtained in the two ethnic groups (comparison (c)).

SELECTED FINDINGS

The Use of the Words Infinity and Infinite

The Korean words *infinity* and *infinite* are mathematical and less often used in colloquial Korean than colloquial American. All students, with the exception of two undergraduates who took a calculus course, did not have the formal education about mathematical infinity.

In the first category, students were asked to create sentences with the words *infinite* and *infinity* (a separate sentence for each of these words). Their responses are summarized in Table. 1.

Students		I. Create a sentence with the following word: (a) Infinite		(b) Infinity	
American	A_5	[1]	They have an infinite amount of movies	[5]	The years go to infinity
	A_7	[2]	Outer-space is infinite and forever	[6]	Infinity is a concept…not a number
	A_{10}	[3]	There is an infinite amount of numbers	[7]	Infinity is the largest number
	A_U	[4]	There are infinite ways to spell certain things	[8]	I love you more than infinity
Korean	K_4	[9]	Numbers are infinite	[13]	One times infinity is infinity
	K_7	[10]	Numbers are infinite	[14]	A line goes to infinity
	K_{10}	[11]	We don't have infinite amount of natural resource in the planet	[15]	I don't think there is not just a thing in infinity.
	K_U	[12]	Some people like infinite space	[16]	We have to study infinity

Table 1: The summary of answer to the question that requires a sentence.

The first thing to note is that all the students, even those who are too young to have met the notion of infinity in the context of school mathematics, are capable of creating sentences with the words *infinite* and *infinity* – a fact that testifies to these words being a part of everybody's colloquial English discourse.

This said, there is a considerable difference between the American and Korean groups in the context in which the words are mentioned. In the American group, *infinite* is used in conjunction with *amount* in [1], [3], [11] and both words are applied mainly in the context of real-life phenomena involving large magnitudes: outer-space [2], ways to spell words, [4], number of years [5], love [8], etc. In the Korean group, the context of the sentences is predominantly abstract and mathematical (the sentences mention numbers [9], [10], lines [14], operations on infinity [13], and the infinity as an object of study [16]), whereas the relation to magnitudes and to large amounts is less pronounced.

There is also a delicate ontological difference between the groups in their application of the word *infinite* to numbers: While all three students who apply the word *infinite* to numbers (A_{10}, K_4 and K_7) seem to be saying the same thing – that numbers can be "infinite", only A_{10} makes it clear that he means the size *of the set* of all numbers. There is no reference to *the set* of all numbers in the utterances of the Korean students, and these utterances may be interpreted as saying that these are the numbers themselves (as opposed to the *set of numbers*, which is a second order construct) that are unlimited in their size.

The two differences noted above may be explained on the basis of the fact that the Korean mathematical words for *infinity* and *infinite* and *set*, with their origins in Chinese characters, do not appear in the Korean colloquial language and the Korean students do not associate them, or even their English counterparts, with anything in particular in the colloquial discourse. One can conjecture that for American students, the colloquial use of the English word *infinite* precedes the mathematical, whereas for the Korean students it may go the other way round.

The Use of the Word Infinity through Definitions

Students		VIII. (a) What is infinity?
American	A_5	Infinity will go on forever.
	A_7	Infinity is a concept that goes on forever
	A_{10}	Infinity keeps going on and increasing. Infinity has no limit.
	A_U	Infinity is never-ending…has no beginning and no end. There is not one thing that is infinite in the world. It's just a concept.
Korean	K_4	Infinity is like a number that never ends or something that never ends; infinity is not the number
	K_7	Infinity is like the furthest number keep going…like never-ending. Infinity is like it goes forever like there is no end.
	K_{10}	It's not a number…it's same like it's not limited…same that never end. The number system that never ends and keeps coming.
	K_U	It's *not a number* because it is a very large amount and cannot be explained to the number.

Table 2: The summary of the definitions about *infinity*.

In this part, the focus is on the *mathematical* discourse on infinity. The students were asked to define the notion and their responses are presented in Table 2.

The prevalent feature of the definitions given by the American students is that they take the object-like character of infinity for granted and characterize this object by saying what it is doing: "go on forever" (A_5, A_7), "keeps going and increasing" (A_{10}), "is never-ending" (A_U).

The Korean students begin with an attempt to specify the category to which infinity belongs, and they usually do it with the help of comparative or negative sentences, such as "It is like a number" (K_4, K_7) or "It's not a number" (K_{10}, K_U). Thus, a common property of all the answers in this group is that while stating some number-like properties of infinity, they also deny its being a number. Such explicit comparison to number (or to any other entity, for that matter) is absent from the American answers.

We have reasons, once again, to speculate that the Korean students' acquaintance with the English word *infinity*, unlike that of the American students, came primarily from the formal mathematical discourse. The more rigorous structure of their descriptions, which, unlike those of the American students, begin with the attempt to specify the general category and continue with the presentation of specific features, may be yet additional evidence that these students' were introduced to the discourse on infinity through mathematical, or as-if mathematical, definitions rather than through casual use.

CONCLUSION

Although our sample is too small to allow for generalizations, what we find in this study may serve as a basis for hypotheses to be tested in a future, more extensive project. On the grounds of our findings so far (and these were only sampled in the present report), we can conclude that colloquial discourse does seem to have an impact on mathematical discourse. This fact was evidenced by certain clear differences between the mathematical discourses on infinity of the American and Korean students, differences that we ascribed to the fact that only in English do the mathematical words *infinity* and *infinite* (as well as *set*) appear also in the colloquial discourse. With the colloquial discourse being the primary source of the American student's acquaintance with the notion, this discourse may have an impact not only on the students' later use of the mathematical keywords, but also on other aspects of their mathematical discourse, such as routines, use of mediators, and endorsed narratives. Our preliminary findings presented in this report justify additional attempts to test this conjecture.

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THE EFFECT OF MATHMAGIC ON THE ALGEBRAIC KNOWLEDGE AND SKILLS OF LOW-PERFORMING HIGH SCHOOL STUDENTS

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Algebra is considered one of the most important areas of school mathematics. Despite its importance, students find it difficult to understand simple algebraic concepts such as variables, expressions, and equivalence. Although basic algebraic concepts are introduced at the elementary and middle school levels, some high school students cannot understand algebra because they find it abstract and difficult. Based on the study of an algebra class of 23 high school freshman students, this paper claims that student understanding of basic algebraic knowledge and skills is enhanced when it is taught using mathmagic. It is suggested that the teaching of algebra should provide an opportunity for students to engage in mathmagic activities and connect them to the learning of variables, expressions, and equations.

INTRODUCTION AND THEORETICAL FRAMEWORK

Teachers, mathematics educators, and mathematicians consider algebra to be one of the most important areas of school mathematics. Despite the importance placed on algebra in school mathematics curricula, many students find it abstract and difficult to comprehend (Witzel, Mercer, & Miller, 2003). They cannot understand simple algebraic concepts such as variables, expressions and equivalence. A substantial amount of research has been devoted to the learning and teaching of algebra at the elementary and secondary levels (Carraher, 2001; Kieran, 1992; Malara, 2003). A quick survey of the PME proceedings in the last 10 years provides evidence that algebra has been one of the most widely discussed topics in its annual conferences. A considerable amount of research in PME and other literature is focused on how students can transit from arithmetic to algebra. Some researchers argue that many students at the elementary level face "difficulties in moving from an arithmetic world to an algebraic world" mainly because they cannot understand the structure and patterns of arithmetic (Warren & Cooper, 2003, p. 10). In order to connect number patterns and structures to more abstract concepts in algebra, the National Council of Teachers of Mathematics (NCTM, 2000) recommends that children in elementary and middle grades should be provided an opportunity to describe, extend, and generalize numeric patterns.

A substantial amount of research in learning and teaching of algebra is focused on the effectiveness of manipulatives (Raymond & Leinebach, 2000; Witzel et al., 2003) and computer applications (Glickman & Dixon, 2002; O'Callaghan, 1998) on student

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ability to generalize patterns and solve algebraic relationships. Although these studies have produced mixed results, the majority of them favor the use of manipulatives or computer applications in developing student abilities and interest in solving algebraic problems. Although the research on the effect of manipulatives and computer applications is not hard to find, the research on the effectiveness of number games such as mathmagic in the learning and teaching of algebra is scarce.

Mathmagic is a game in which students are invited to play with numbers in which the students "think of a number", "add 10", "multiply it by 3", and so on (Koirala & Goodwin, 2000). Utilizing basic algebraic knowledge, the mathmagician then figures out the final number that a student is thinking of. Provided below is an example of a mathmagic.

Think of a number.

Add 10.

Multiply by 3.

Subtract 3.

Divide by 3.

Subtract 5.

Subtract your original number.

Map the digit to a letter in the alphabet 1=A, 2=B, 3=C, etc…

Pick a name of a country in Europe that begins with that letter.

Take the second letter in the country name and think of an animal that begins with that letter.

Think of the color of that animal.

The mathmagician then predicts that the students would be thinking of a "Grey Elephant from Denmark." When this magic is completed the students attempt to make a link between working principles of the magic and algebra. Students need to understand the concept of variables in algebra to be able to complete the magic successfully. They need to translate the sentence "think of a number" to a variable, for example n, and then extend it to expressions such as $n+10$ and $3(n+10)$. The directions for mathmagic and the corresponding algebraic expressions are shown in Table 1.

A few studies have reported that this simple magic generates tremendous amount of excitement and interest in students (Lovitt & Clarke, 1988; Koirala & Goodwin, 2000). Despite a strong potential to motivate students towards the learning of algebra, the research on the effect of mathmagic on student achievement and attitude is inadequate. More systematic studies are needed to determine the effectiveness of mathmagic on student learning of algebra. This paper contributes to the existing literature by adding research on the effectiveness of mathmagic on the learning and

teaching of algebra. Based on quantitative and qualitative data, it provides insights into students' attitude and achievement towards algebra.

	How a student's number changes	Algebraic expressions
a. Think of a number.	7	n
b. Add 10.	17	n+10
c. Multiply by 3.	51	3n+30
d. Subtract 3.	48	3n+27
e. Divide by 3.	16	n+9
f. Subtract 5.	11	n+4
g. Subtract your original number.	4	4

Table 1: Mathmagic Directions and Corresponding Algebraic Expressions

Purpose and Research Questions

The purpose of this project was to teach algebra using mathmagic and determine its effectiveness in helping students to understand basic algebraic concepts. More specifically, the project sought the answers to the following research questions: Does mathmagic improve algebraic knowledge and skills of low-performing high school students? In what ways does mathmagic help students to develop their confidence and interest in learning algebraic concepts?

To answer the first question, this study hypothesized that the use of mathmagic would improve algebraic knowledge and skills of low-performing high school students. The second question is answered mostly through the analysis of student work samples, researcher's field notes, and the interviews with the teacher and students.

RESEARCH METHODOLOGY

This is a case study (Yin, 1989) of a high school freshman algebra class. It follows the mixed methods research paradigm as proposed by Johnson and Onwuegbuzie (2004). This paradigm calls for the combination of traditional quantitative and qualitative research.

Participants

A total of 23 ninth grade (14 years old) students with varied mathematical and algebraic experiences participated in this project. According to the classroom teacher, seven of these students were identified with various special needs and the majority of remaining students were considered "at-risk." Eight out of these 23 students had an exposure to pre-algebra from a previous class. The remaining 15 students had no exposure to pre-algebra. According to the classroom teacher, the students were not

motivated to learn algebra. Most of these students did not see how algebra would make a difference in their everyday lives.

Data Collection Procedure

Consistent with the mixed method paradigm, this study collected data by using both quantitative and qualitative techniques. Quantitative data were collected through pretest and posttest. Qualitative data were collected through student surveys, work samples, researcher's field notes and interviews with the teacher and selected students. This study took place in late September/early October and students in this class had not begun their work in algebra. They were simply reviewing other concepts such as fractions, decimals, and measurements.

Surveys and Pretest/Posttest

The data collection process began with a student survey. In the survey, students were asked if they had learned any algebra prior to the class and if they liked or disliked the subject. The survey provided background information about students' prior knowledge and attitude towards algebra. They were also given a postsurvey at the end of the study to find out their thoughts about the mathmagic activities that were taught during the study.

Immediately after the initial survey, all the participating students were given a pretest to determine their level of algebraic knowledge and skills. The students were also given a posttest at the end of the study to determine whether or not they gained algebraic knowledge and skills after the use of mathmagic in their class. The questions in the pretest and posttest were similar in terms of their structure and difficulty. Some sample pretest/posttest items are provided below.

Simplify: $6-3(2-5)$

What is a variable? Explain with an example.

Write an algebraic expression for each of the following phrases and statements:

17 MORE THAN 3 TIMES A NUMBER.

Subtract 3 from x and divide the difference by 2.

Simplify the following expressions:

$3(x-5)$

$9+2(5y+11)$

Factor each expression:

$5m+15$

$100c^2+10c$

Interviews

At the end of the project 8 students were selected for interviews. The students were selected purposively to represent the ability level of the whole class. The students were asked to explain what they thought about mathmagic activities in the class and if they could perform a mathmagic with the interviewer. The interviews allowed for the exploration of students' ability to connect mathmagic activities with their algebra learning. Each interview lasted approximately 15-30 minutes. All of these interviews were audiotaped and transcribed. In addition to the student interviews, the teacher was also interviewed two times during the study.

Classroom Planning and Teaching

The teacher and researcher collaborated in planning lessons and teaching. The researcher observed each class when mathmagic was used and made field notes. All the notes and other aspects of the course were shared, discussed, and reflected with the teacher for further planning.

Data Analysis Procedure

This study used a paired *t*-test of scores to determine the statistical significance of the research hypothesis which assumed that the use of mathmagic would increase student achievement in algebra. The data from field notes and interview transcripts were analyzed qualitatively using the constant comparative method (Guba & Lincoln, 1989).

RESULTS

Table 2 displays means and standard deviations of both pretest and posttest scores. The scores indicate that the students performed better in the posttest than in the pretest. The difference between the pretest and the posttest was statistically significant ($t(22)=5.63$, $p<.01$). This result indicates that mathmagic was effective in helping students solve basic algebraic problems.

	n	M	SD
Pre-test	23	4.78	3.21
Pre-test	23	8.33	3.65

Table 2: Means and Standard Deviations of Pretest and Posttest Scores

The researcher's fieldnotes from the classroom indicated that the students were very excited about mathmagic. When the students completed their computations in mathmagic activities they were surprised that they all ended up with the same number. The students then mapped their final numbers with letters in the English alphabet and created different words and phrases for the amazement of the class. The mathmagic activities were highly motivating to students and helpful in making sense of algebraic concepts such as variables, expressions, and the distributive property of multiplication over addition.

The interviews of the teacher and the selected students added further insights into student motivation and learning of algebra. The teacher agreed that mathmagic motivated the students. For example, in one of the interviews the teacher stated:

> I've been pleasantly surprised on how sophisticatedly they were able to think of the algebra skills. To be honest with you, I thought that they would all get very lost and very bored and start to act out more than they were. But many of the students understood how you were getting rid of the initial number they were thinking of. To get rid of that is a fairly complex thinking. Okay, if you multiply your original number by 2 then either you have to subtract off your original number, or somehow you have to make the original number disappear from the equation. I think that is a skill that I would have never guessed that any of those kids was capable of doing.

All of the students who were interviewed agreed that they had fun with mathmagic and six of the eight claimed that mathmagic helped them to learn algebra better. Provided below is a sample interview with one of the students:

Interviewer:	In this class you had some opportunity to learn about mathmagic. Can you explain what you thought about mathmagic?
Student:	I think it was fun and helpful when learning algebra. It helped me a lot.
Interviewer:	Why was it fun?
Student:	It is something we are not used to doing everyday. Last year when I was in math class, we didn't do anything like this; so it was new. Plus it was easy for me after seeing it a couple of times on the board. We all got the same number at the end and were wondering how.
Interviewer:	Any other reasons?
Student:	I like working with other kids a lot. We could play mathmagic with each other.
Interviewer:	How did you learn algebra with mathmagic?
Student:	Because we had to use a variable and add, subtract, and multiply with numbers and variables.

This student was very excited about mathmagic. Not only was he excited but was also able to successfully perform mathmagic with the researcher during the interview. Other students consistently stated that mathmagic was fun and it should be a part of their algebra class. Only two of the eight students did not believe that mathmagic helped them to learn algebra. It was interesting to note that both of these students had high pretest scores and the their posttest scores did not increase after the study period. Nevertheless, they stated that they liked the activities. Most of the students, including these two, were excited to try mathmagic with their parents, siblings, and friends.

CONCLUSIONS AND IMPLICATIONS

Both the quantitative and qualitative data in this study indicate that mathmagic motivates students to learn basic algebraic concepts. The students who usually

struggle with mathematics were enthusiastic to learn mathmagic so that they could try it with their friends. They understood that mathmagic does not work if the arithmetic computations are not correct. They were interested to learn algebraic skills because learning the skills would help them to become a successful mathmagician. Their engagement in mathmagic activities enhanced their understanding of variables and expressions. They were able to add and subtract like terms and use the distributive property. They understood that the use of a variable is central to successful mathmagic. The posttest indicated that they were more successful in manipulating and evaluating algebraic expressions than in the beginning of the study. This study indicates that algebra can be more accessible to students through mathmagic, especially to those who do not perform well in mathematics.

Although mathmagic was successful in bringing positive results in student motivation and achievement there are some limitations with this study. There was no control group so the student achievement could not be compared with another group of students who were taught in a traditional manner. Also, this study was effective as a short-term intervention. The mathmagic activities lasted only 2 weeks, and covered only 5 sessions of approximately 50 minute class period. Although the posttest indicated higher student achievement, the scores were not very high. The mean posttest score was only 8.33 out of possible 15. Many students could not factor the polynomials even after the completion of mathmagic activities.

In order to improve student interest and confidence in algebraic knowledge and skills, mathmagic needs to be directly connected to important topics such as solving equations. Although mathmagic can be used to create and solve equations such as $3(n+10)=17$, the duration of this study was not sufficient to make these connections. Despite these limitations the data in this study implies that mathmagic activities not only motivate but also improve algebraic knowledge and skills of low-performing high school students. As indicated in the literature extensive effort has been made to help students to improve their algebraic knowledge and skills. Yet, many of these efforts have failed to produce desired results. If mathmagic can improve students' attitude and achievement in algebra it needs to be utilized more consistently in their classrooms.

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LEARNING AND TEACHING EARLY NUMBER: TEACHERS' PERCEPTIONS

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The context of this study is the current curriculum reform in South Africa. Teachers' didactical knowledge is regarded as playing a crucial role in students' learning and this study investigated teachers' perceptions about ways in which children learn number skills and concepts. Six Foundation Phase teachers from schools in the Cape West Coast Winelands region of South Africa were involved in the study. The data was collected by means of stimulated-recall interviews based on constructed classroom vignettes and teachers' comments were analyzed in terms of accepted theories on learning and acquisition of number. The study revealed that teachers have a limited understanding of how children learn number and gives support to the idea that a learning pathway description may assist with broadening understanding of learning and teaching early number.

INTRODUCTION

The background to this study is the reform movement in mathematics education in South Africa over the past 10 years, the launch of Curriculum 2005 (C2005) in 1997 and the Revised National Curriculum Statement (RNCS) in January 2004. The introduction of the new curriculum meant that some 1400 registered syllabi (Chisholm, 2001) had to make way for the creation of a core national curriculum that aims at an integrated system of education and training and a single national qualification framework while attempting to address issues of inequality, principles of multiculturalism and notions of citizenship in South Africa (Young, 2001).

A key feature of C2005 was the shift in emphasis from traditional subject-based knowledge and skills, to the statement of broader overarching outcomes. The formal presentation of content was removed and teachers were expected to plan their own instructional sequences. Teachers' didactical knowledge necessarily played a crucial role in the implementation of the new curriculum and approach to teaching mathematics. This required a good understanding of conceptual development and progression of content. Ultimately this feature of C2005 'disabled' many teachers who already had difficulty selecting and teaching appropriate content. The RNCS specifies content to be covered for each grade in the form of minimum standards. Understanding teachers' knowledge is necessary for effective professional development and in-service support for the implementation the new curriculum and improved teaching and learning. The purpose of the study reported here was to gain information about teachers' knowledge and their perceptions of learning and teaching

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early number. Number-related strands in the RNCS Mathematics Learning Area form the basis of all learning in mathematics at the Foundation Phase and receive the greatest learning and teaching attention and time allocation.

CHARACTERIZATION OF THE STUDY

An exploratory quantitative study was undertaken with six Foundation Phase (Grade R – 3) teachers from thirteen rural and peri-urban primary schools who were interviewed in order to elicit their understanding of the ways in which young children learn number concepts. The discourse that teachers use when describing their knowledge and understanding of number acquisition in the early grades was of particular interest. For the data collection, stimulated-recall interviews were used, based on four constructed classroom vignettes and related open-ended questions. The vignettes and questions were based on current theories on learning and number acquisition and were constructed in such a way that they could act as prompts to encourage teachers to reflect on, and express their opinions about, certain aspects of learning and teaching number in different contexts.

THEORETICAL FRAMEWORK

A review of research and programmes on early number (Kühne 2004) engaged with Cognitively Guided Instruction (CGI) (e.g. Carpenter, Fennema, & Franke, 1996), the Problem Centered Primary Mathematics Program (PCM) (Murray, Olivier, & Human, 1998), Realistic Mathematics Education (RME) (e.g. Van den Heuvel-Panhuizen, 2001a), Mathematics Recovery (e.g. Wright, 1994) and Count Me In Too (e.g. Stewart, Wright, & Gould, 1998). These programmes have in common a focus on helping teachers understand the mathematics of specific content domains and children's mathematical thinking in those domains. Each programme operates from a perspective that teachers' knowledge and understanding of children's mathematical thinking is a critical factor in supporting children's mathematical learning. These programmes and other literature associated with learning and teaching early number were significant for this study as very little systematic research on early number acquisition exists in South Africa. The theoretical framework used for the analysis in this study drew on Steffe's (1992, 2000) model which outlines four basic counting schemes or stages, and on the notion of emergent counting as used by Wright (1998). The framework was founded on the idea that the advancement through these stages begins from the ability to invent informal context-related solutions, to the creation of various levels of efficient solution strategies (short cuts) and schematisations, and the acquisition of insight into the underlying principles and the discernment of even broader relationships (Van den Heuvel-Panhuizen, 2001a).

The framework identifies the following stages for learning early number:

1. Emergent Counting
2. Perceptual Stage
3. Figurative Stage
4. Initial number Sequence Stage
5. Tacitly-nested Number Sequence
6. Explicitly-nested Number Sequence Stage.

In addition to the hierarchy of stages, the framework contains an explicit description of behaviour which can be expected of most children at each stage and an overview of children's knowledge, strategies and solutions to number problems.

METHOD

Instruments

A stimulated-recall interview method (Dunkin et al., 1998) was used to elicit data. This required research participants to respond to hypothetical learner and classroom episodes. The interview instrument developed for the present study contained four parts. Below is an example of one vignette used in the interview.

Interview 1: Part 1 (I-1)

> A class is investigating addition by using counting rods. The learners are working in pairs. The teacher asks one learner in each pair to represent and write the number 34, and the other learner to represent and write the number 27. The learners are then asked to combine the rods and tell her how many there are altogether. She asks several pairs to explain how they found their answer. She then demonstrates a method for recording what the learners have done with the rods.
>
> Please consider the following questions:
> 1. What do you think the teacher is hoping to achieve from this activity?
> 2. What number concepts do you think are being developed?
> 3. How do you think learners will represent the numbers?
> 4. What method do you think the teacher would have demonstrated?

The vignettes and related questions were intended to give participants an opportunity to discuss particular features of children's number development, elicit information about their knowledge and understanding of the developmental processes underpinning the learning of number and the didactics they use when mediating this learning in the classroom. The four interview parts (I-1, I-2, I-3 and I-4) include several stages of number development (see Table 1).

Stages	I-1	I-2	I-3	I-4
Emergent Counting		*	*	*
Perceptual Stage		*	*	*
Figurative Stage	*	*		*
Initial Number Sequence stage	*	*		*
Tacitly-nested Number Sequence	*	*		*
Explicitly-nested Number Sequence Stage	*			*

Table 1: Stages across interviews

Vignette 1 was designed to encourage discussion about knowledge of number structure, basic calculation skills and strategies necessary to understand and successfully solve problems involving numbers up to 100. Important aspects of the discussion were expected to include: the sensory activity of counting-all that is still

needed for some children (Figurative stage); the notion that children have constructed numerical counting concepts and schemes and use counting-on strategies (Initial number sequence stage); the idea that they have developed ways of keeping track of their counting acts (Tacitly-nested number sequence stage); an understanding that children apply part-to-whole reasoning and have an awareness of addition and subtraction as inverse operations (Explicitly-nested number sequence stage). In order to check the validity of the interview instrument the vignettes were validated and piloted prior to use.

Sample

Six teachers from schools in the Cape West Coast Winelands region of South Africa participated in this study. Four participants in this study taught at Afrikaans medium primary schools located in a small peri-urban area. One teacher taught at a small Afrikaans medium primary school situated on a wine farm and another at a large Xhosa medium school situated in an informal settlement outside a small rural town. The number of children in the Foundation Phase classes at these schools ranges from 40 to over 70. All the teachers involved in this study were women, each of them had taught at the primary level for more than 10 years, and each had experience teaching Grade 1.

The highest formal qualification held by three teachers was a senior school certificate (matriculation) plus three additional years of tertiary training (M+3). Two teachers had completed a matriculation certificate and had studied for a further four years (M+4). One teacher received her teacher training qualification prior to completing her matriculation certificate.

Data collection and analysis

The data was collected in the second term of 2003. Participants were first asked to complete the teacher questionnaire in order to establish accurate teacher profiles. Each participant was interviewed once; each interview lasting between one and two hours. The analysis of the interview data was conducted by one of the authors of this paper working both deductively and inductively in developing a theoretical framework. Initially she worked with the transcripts deductively, using the elements of the theoretical framework to describe and classify teachers' comments according to *elements of the stages in number development*. She also worked more inductively with the data in order to identify participants' comments on *other didactic themes* related to learning and teaching number.

RESULTS

The first part of the analysis disclosed how often the participants (P1-6) reflected knowledge of stages in number development (Table 2) and made references to other didactic themes (Table 3) in their comments to the vignettes and questions when they were interviewed.

	Number of times participant reflected knowledge of stages in number development						
Stages	P1	P2	P3	P4	P5	P6	Total
Emergent Counting			1			1	2
Perceptual Stage	2		1				3
Figurative Stage	4					1	5
Initial Number Sequence Stage	3		2	1	1	1	8
Tacitly-nested Number Sequence	4	5			1	7	17
Explicitly-nested Number Sequence	1		3				4
Total	14	5	7	1	2	10	

Table 2: Results related to elements of the stages in number development

The analysis of stages in number development revealed that the participants in general reflected an understanding of number development. However, they did not mention specific stages, levels or categories as included in the theoretical framework.

References to the early stages of number development – emergent numeracy, perceptual stage and figurative stage – were mostly contained in participant remarks in relation to weak number concepts or poor learner performance. The following quotes are examples of comments which illustrate this point:

> ..can't write any number or show to a number or even count out. He can count without understanding, he's counting like the parrot but he doesn't understand what he is really doing. (P3:63– 65)

> The one with the poor concept of number must start from the beginning, he must every time have the picture from the beginning again and you have to tell him start in 3's from 3 to 18, then he'll say 3, then he'll count on his fingers to get to the 6 and maybe count again to get to the other one, but he won't off-hand be able to count (P1: 205-209)

Comments which related to the initial number sequence stage, tacitly-nested number sequence and explicitly-nested number sequence stage were made in relation to performance and skills which participants considered most desirable. The majority of comments pertained to the tacitly-nested number sequence. In particular, as the following examples illustrates, that automated number facts are more desirable than counting based strategies.

> And then Freda is also using a long method because she is counting with her hands. Zondi is also using the long method. Lisa is using a little bit a short method because she didn't use a lot of coins to come up with that 12 (P6: 46-48)

> Ayanda was very fast! Ayanda didn't use any concrete objects to come to that 12. Fran was also a little bit ...very…he was also fast because he just count 5 and 4 is 9 plus 1 is 10 and then 2 is 12. Ayanda was marvellous! ..the one that is going to take a loooong method and the one who is just counting and tell you, even when you doing some demonstration on the board just NO, they are calling me dadobawo (aunt), dadobawo that is 8 or that is 7 (P6: 49-54)

A large number of references were also made to the initial number sequence stage. The following quote highlights a discussion regarding addition strategies:

> But if they going to put it together they'll also just, if they clever enough, the one will take the 34 and they'll just count on, just add on the 27 to get to the new number. (P1: 24-26)

The second part of the analysis showed that the participant comments include four other didactic themes; included the teachers' role, sequencing material, perceptions of good and weak learners and the effect of environment on learning and teaching (Table 3).

		Number of times participant referred to other didactic themes in number development						
Didactic Themes		P1	P2	P3	P4	P5	P6	Total
Teachers' Role	Direct instruction	1	2			3	3	9
	Indirect instruction	3	3		2	1	1	10
Sequencing Material	Concrete to abstract	1	1	1	4			7
	Representation	1			1		4	6
	Range	6	1	1	2	3	3	16
	Prior-knowledge	3	2	2	3	3	3	16
	Planning by age/time	1	1			3	3	6
Perceptions of good and weak learners		12	3	2	3	1		21
Effect environment on learning/teaching				1	2	3		6
Total		28	13	7	17	21	17	

Table 3: Results related to other didactics themes in number development

The comments indicated that participants considered various ways of supporting number development, such as organizing and sequencing material and learning in particular ways. Participants' comments also revealed an awareness that understanding and acquisition of skills are linked to various forms of representations and that this is a structured process that develops gradually from the use of concrete manipulatives to more abstract symbolic forms of (mental) representation. An example is this comment which indicates a strong position on the use of concrete manipulatives as a pre-requisite to understanding number in its related abstract forms:

> I suggest is that you should use concrete things to learn a child. He must use the real thing to count, we can't just count and we can't just do sums. Every time we do a sum when it comes to number you must give the child a problem, because if we just work from, if the child just see that number on the board or I just give them the number and he put out the sum, and there isn't a problem attached to it, he will never understand, he will never be able to make a, be a critical thinker when it comes to maths. Because if they understand number, they will be able to understand everything else in maths, because it starts with number. (P1: 245-253)

All participants in this study suggested that learners should be provided with a range of learning experiences that facilitate the move from concrete to abstract forms of representation, and that these should include pictorial, concrete and symbolic forms. Most participants' comments indicated an awareness of the idea of progression. This

progression was expressed in terms of the use of manipulatives, building on prior-knowledge and experience and ways of organizing learning according to time-related aspects (including age or grade). The analysis of comments further revealed that participants recognize certain factors that impact on learners' performance. These factors include teaching styles and the role of the teacher in facilitating or supporting learning, and the effect of environments on children's learning.

CONCLUDING REMARKS

This study has a number of limitations, such as not including classroom observations and not interviewing teachers in their mother tongue, therefore the results should be treated with prudence. Furthermore, the limited number of teachers involved does not allow wider generalizations. The study showed that the participating teachers have a limited understanding of how children learn number and the complexity and diversity of this process. It also revealed that they understand children's number development in particular ways, which is expressed within a framework of their own classroom experience. Teachers do not share a common discourse about number development. It is also evident from the analysis of teachers' comments that they do not describe an overview of how number develops or what can be expected in the learning process from a particular understanding.

Although teachers in this study recognize that learning should be organised and sequenced in particular ways, this was mainly described in terms of notions of concrete to abstract representations. Discussions about sequencing number concepts progressively were not based on cognitive development but rather on generic teaching approaches. Teachers' did not discuss the process of modeling concepts and operations which enable children to bridge the gap between informal, context-bound strategies and formal standardized operations. From this study it does not appear that teachers are able to describe a long-term overview of the process of learning and teaching number that connects the different development stages and offers a framework for didactical decision making. These findings are perhaps surprising given that all teachers had attended in-service mathematics courses. This suggests the need for a more elaborated form of in-service education with a framework or trajectory for learning and teaching number, such a trajectory would present mathematical understanding and content in a progressive and structured way and encompasses the cognitive and didactic continuum. A trajectory that emphasises relationships between various forms of representation, highlights different learner strategies and 'bridges' the RNCS and classroom practices would be a useful tool to assist in pre and in-service teacher education. This instrument would also support the arduous task of selecting and sequencing content and classroom activities.

Recently a research and development project, inspired by the Dutch TAL Learning-Teaching Trajectory for calculation with whole numbers (Van den Heuvel-Panhuizen, 2001b), was initiated which aims to develop and measure the impact of a Learning Pathway for Number (LPN) at the Foundation Phase.

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SITUATION-SPECIFIC AND GENERALIZED COMPONENTS OF PROFESSIONAL KNOWLEDGE OF MATHEMATICS TEACHERS: RESEARCH ON A VIDEO-BASED IN-SERVICE TEACHER LEARNING PROGRAM [1]

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We report on first results of a study with mathematics teachers from Germany and Switzerland who took part in a video-based in-service teacher education program. We wanted to find out whether the interpretation of videotaped classroom situations was influenced by the teachers' professional knowledge and instruction-related beliefs. The findings indicate that cognitive constructivist or direct-transmission views of teaching and learning might have an impact on situated beliefs of teachers and their interpretation of videotaped classroom situations. For video-based reflections of teaching in teacher learning projects, the results emphasise the importance of finding and defining a way how to look at classroom situations in cooperation with the participating teachers.

THEORETICAL BACKGROUND

The video-based analysis of teaching might encourage mathematics teachers to reflect on the quality of their instruction and to improve it. Confronted with videotaped instructional situations, teachers will activate their professional knowledge and their instruction-related beliefs in the process of interpreting the classroom situations presented. Accordingly, this study refers to a theoretical background on domains of professional knowledge and their instruction-related beliefs. According to Shulman (1986, 1987) and Bromme (1992, 1997), one can distinguish between different domains of professional teaching knowledge: subject matter knowledge, pedagogical knowledge and curricular knowledge. These domains include both declarative knowledge and individual beliefs. For example, constructs like the cognitive constructivist or the direct-transmission views of teaching and learning (Staub & Stern, 2002; Stern & Staub, 2000) can be described as pedagogical content beliefs on a rather generalized, non-situation-specific level. Similarly, epistemological beliefs (Grigutsch, Raatz & Törner, 1995; Klieme & Ramseier, 2001) concern beliefs on mathematics as a whole. Diedrich, Thußbas & Klieme (2002) as well as Lipowsky, Thußbas, Klieme, Reusser & Pauli (2003) found that there was an interdependence of such constructs in the form of "syndromes".

Beyond those generalized constructs, professional knowledge also encompasses episodically organized, situation-specific and content-specific cognitions and beliefs

[1] This study was funded by the Robert-Bosch-Stiftung

(Bromme, 1997; Leinhardt & Greeno, 1986; Escudero & Sanchez, 1999). For instance, knowledge on specific contents can be linked to mental representations of instructional situations concerning these contents. Because of its relevance for instructional practice, professional knowledge also plays an important role for in-service teacher learning: The making of decisions by the teacher involves general, situation-specific, and content-specific cognitions and beliefs (Malara, 2003). In particular, this seems to be the case for decisions teachers make in instructional classroom situations.

For the interpretation of instructional situations, teachers' individual theories on instructional quality might also play an important role: For example, individual criteria for observable characteristics of "good mathematics lessons" could influence judgements on classroom situations. As a reference for criteria of instructional quality, we took the study of Clausen, Reusser & Klieme (2003), in which four basic dimensions of instructional quality were established in a Swiss-German video study using high-inference rating methods.

On this theoretical background, we assume the following model for describing possible interactions between the judgements of classroom situations and components of professional knowledge:

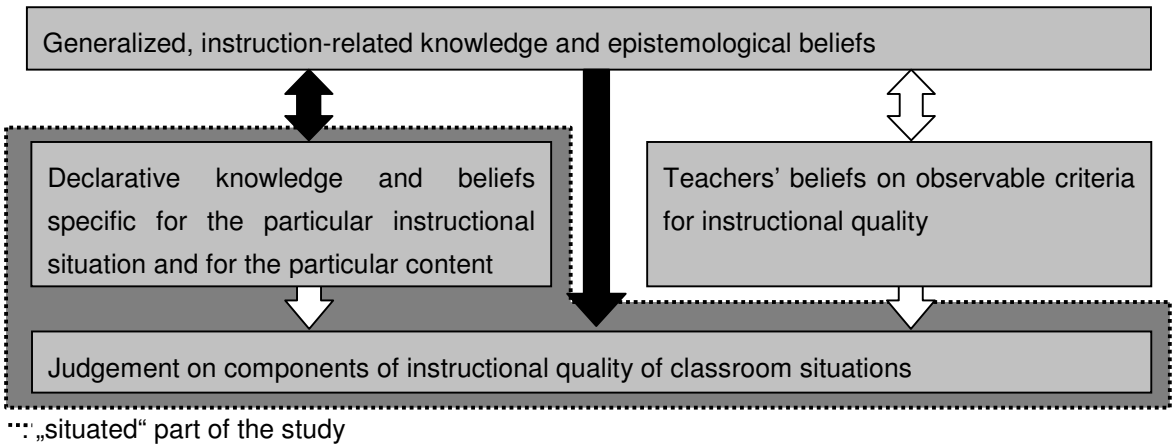

Figure 1: Model for possible interactions between the judgements of classroom situations and components of professional knowledge

In the following, we focus on observations linked to the black arrows in Figure 1.

The part of the video-based teacher learning project in which this study took place concentrated on the situation-specific context of geometrical proof (Reiss, Klieme, & Heinze, 2001; Kuntze & Reiss, 2004). As a consequence of this orientation and based on the results of Clausen, Reusser, & Klieme (2003), we focused on "cognitive activation", "intensity of argumentation in the classroom", and "learning from mistakes" for the teacher learning project and for the judgements teachers were to make on the classroom situations in this study.

RESEARCH QUESTIONS

The study aims at providing evidence for the following research questions:

(i) Is there a correspondence between situation-specific instruction-related beliefs on geometrical proof and more general components of professional knowledge?

(ii) How do teachers judge the quality of instruction in videotaped mathematics lessons? Do these judgements depend on professional knowledge and instruction-related beliefs?

DESIGN OF THE STUDY AND METHODS USED FOR THE ANALYSIS OF RESULTS

In our study, 53 Swiss and German teachers were asked to complete several paper-and-pencil questionnaires.

Before the start of the project, a first questionnaire focused on professional knowledge and instruction-related beliefs. The part of this questionnaire dealing with cognitive constructivist or direct-transmission views of teaching and learning was an adaptation of the instrument used by Staub and Stern (2002) which is based on a questionnaire by Fennema et al. (1990) and scales by Peterson et al. (1989).

Before the presentation of the videotaped classroom situations, the teachers had to answer a second questionnaire and to activate situated components of their professional knowledge about introductory lessons on proof in geometry. For instance, the teachers were asked about preferred characteristics of instruction when introducing geometrical proof, like e.g. the importance they would attribute to exactness. Sample items of three of the scales are shown in table 1.

Scale	Sample item	Number of items	Cronbach's α
argumen-tational discourse	"It is important to me that many students report on their prior knowledge about argumentation and proof and integrate it in the discussion on proof problems, even if it is probable, that misconceptions appear in the classroom without being corrected instantly."	3	.66
advancing by small steps	"In order to encourage students' contributions to the development of proofs, I would divide the argumentation into small steps, so that the students can contribute that steps in the classroom."	2	.44
initial tolerance with respect to exactness	"It is important to me that the students find ways of argumentation that are relevant to them, in order to encourage them to share the ideas of the argumentational problem. Exact proofs can be approached later."	3	.64

Table 1: Three scales on situated instruction-related beliefs (geometrical proof).

After having activated their content-specific and situation-specific pedagogical knowledge, the teachers were shown two videotaped classroom situations, both dealing with introductions to geometrical proof. According to our approach (Kuntze & Reiss, 2004), video A showed patterns of interaction marked by discourse and argumentational exchange between the students and the teacher, whereas video B could be characterized as a teacher-centered interaction comparable to the dominant teacher script in Germany described in the TIMS Study (Baumert, Lehmann, et al. 1995).

Immediately after having seen the videos, the teachers had to give judgements about these two classroom situations in a third questionnaire. In multiple-choice and open items, the teachers were asked about particular components of instructional quality, about how similar their own instructional practice was to the classroom situations and about further observations.

In the following, we concentrate on the quantitative results of the multiple-choice items. We focus on the group of German teachers (N = 42) in order to be culture-fair.

RESULTS

The results concerning the situation-specific pedagogical knowledge on teaching geometrical proof asked in the second questionnaire indicate that these situation-specific domains correlate with the general cognitive constructivist or direct-transmission views of teaching and learning. According to these views of teaching and learning, we divided the teachers in thirds of a lower, mediocre, and higher cognitive constructivist or direct-transmission view, respectively. Boxplots of the different scales of situation-specific beliefs are shown in figures 2, 3, and 4. Some of the tendencies in these figures are reflected in significant correlations between the variables (cf. table 2).

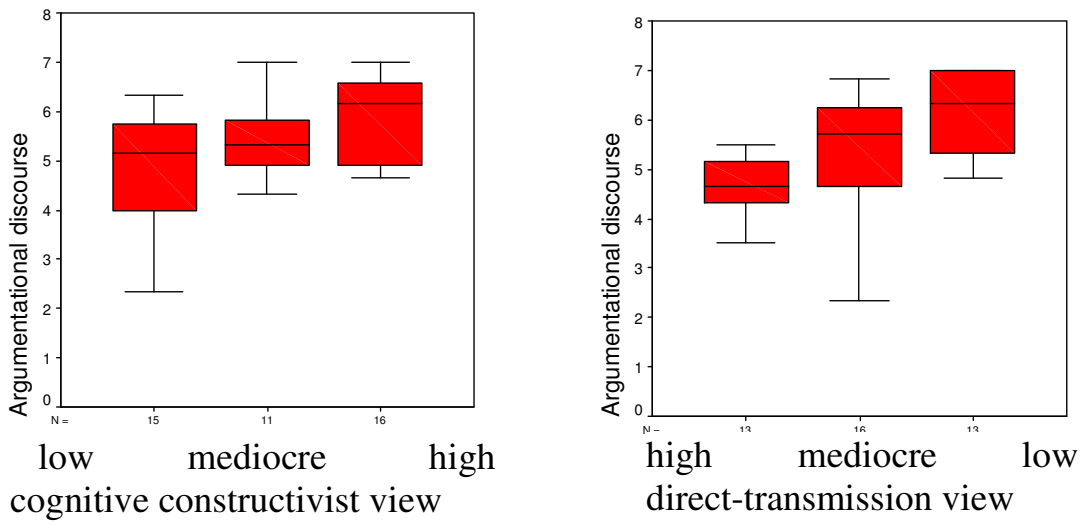

Figure 2: Argumentational discourse

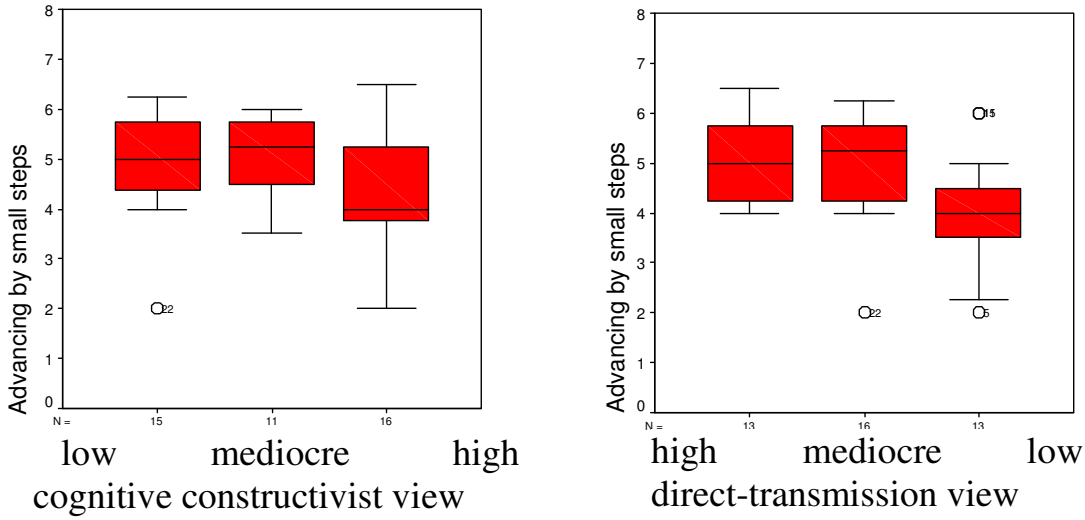

Figure 3: Advancing by small steps

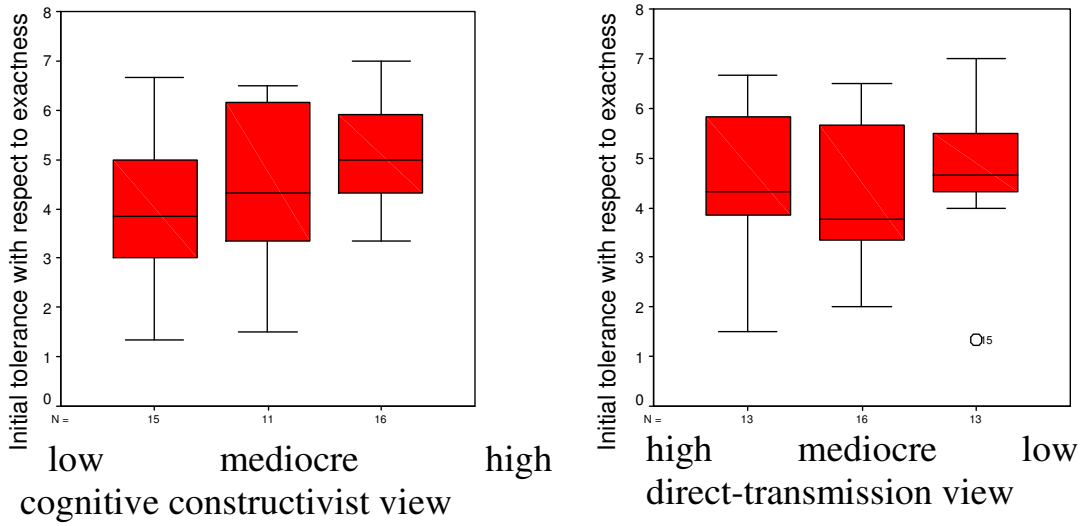

Figure 4: Initial tolerance with respect to exactness

	cognitive constructivist view	direct-transmission view
direct-transmission view	-.400**	
argumentational discourse	.482**	-.503**
advancing by small steps		.449**
initial tolerance with respect to exactness	.362*	

Table 2: Correlations between generalized and situation-specific components of professional knowledge.

In the third questionnaire, the teachers were asked to judge on the instructional quality of the two videotaped classroom situations. In Figure 5, we give two examples for the obtained results. These examples concern video B, in which was shown a situation very close to the dominant teacher script in Germany. The two subgroups of the participants seem to have judged the videotaped classroom situations differently according to their cognitive constructivist or direct-transmission views of teaching and learning: Teachers with a low direct-transmission orientation in their view of teaching and learning rated video B to contain less cognitively activating situations, less argumentational exchange, and less possibilities for the students to learn from their mistakes than teachers with a high direct-transmission orientation. The differences between the groups distinguished in Figure 5 are highly significant for the judgements on cognitive activation.

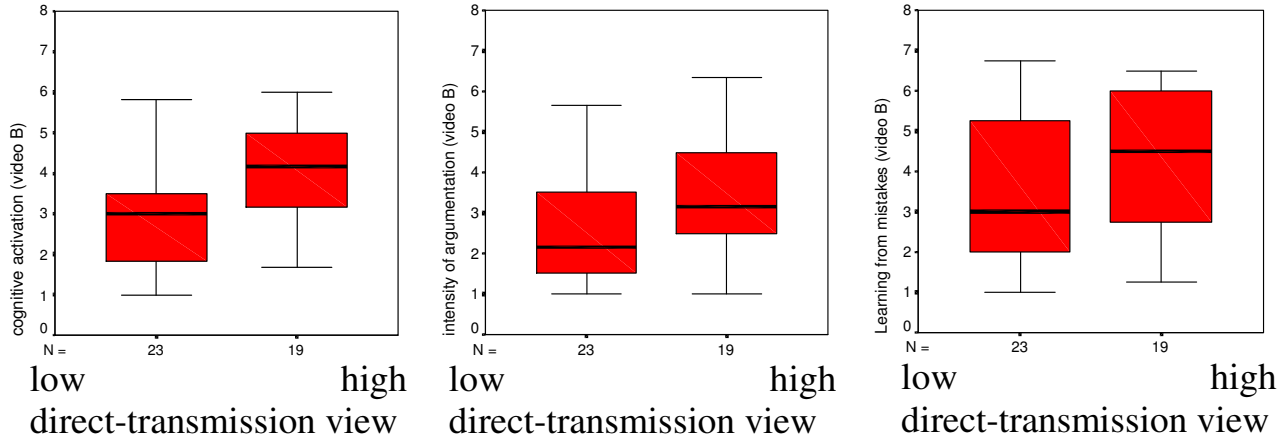

Figure 5: Examples for judgements on instructional quality

When asked to compare the videotaped classroom situations to their own teaching, all subgroups of teachers agreed on average that video B was closer to their own instructional practice than video A (cf. figure 6). Moreover, there is a correlation of .410** between the direct-transmission view of teaching and learning and the reported similarity to video B.

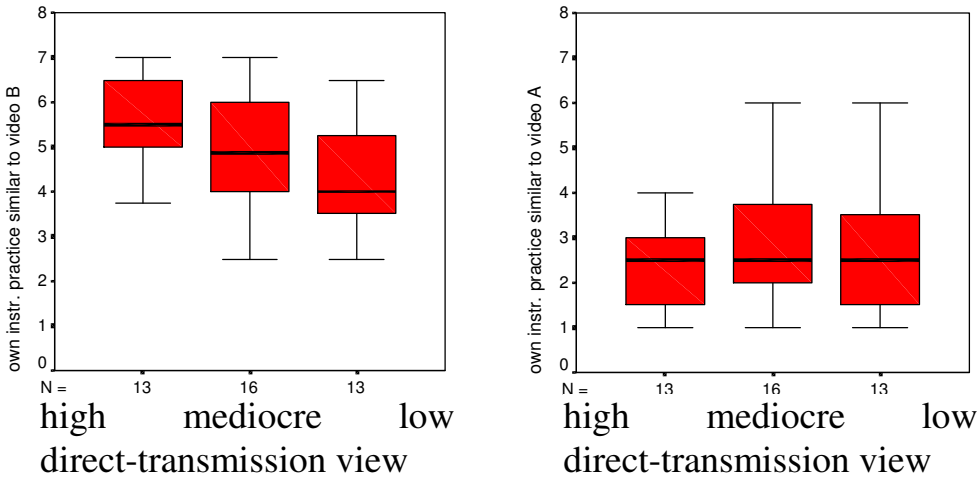

Figure 6: Comparison to the teachers' own instructional practice

INTERPRETATION OF THE RESULTS AND IMPLICATIONS FOR THE THEORETICAL AND PRACTICAL CONTEXT

The results indicate that situation-specific components of pedagogical knowledge on geometrical proof seem to reflect cognitive constructivist and direct-transmission views of teaching and learning, which belong to more generalized components of professional knowledge and their instruction-related beliefs.

Similarly, the ratings of instructional quality in videotaped classroom situations seem to be interdependent with the individual professional knowledge of the teachers. For this reason, video-based reflection of teaching in teacher learning projects should take into account that the participants might see videotaped classroom situations differently according to components of their professional knowledge. The cognitive constructivist respectively the direct-transmission views of teaching and learning could have explanatory power for differences in judgements about the instructional quality of classroom situations.

Comparisons teachers make to their own instructional practice might play a role, too: Own instructional experiences probably serve as a reference for instruction-related beliefs and situation-specific components of professional knowledge. The findings indicate for instance, that the teachers' reported own instructional practice in comparison to video B correlates with the direct-transmission view of teaching and learning.

On the theoretical level, the results on teachers' rating of instructional quality in the videotaped classroom situations could also contribute to explain why cognitive constructivist or direct-transmission views of teaching and learning might have an impact on instruction and on achievement gains like those reported by Staub & Stern (2002).

Practical implications of the results concern video-based teacher learning projects: For video-based reflection of teaching, it might be of primary importance to develop a common basis of reference shared by the participants, in order to get a guideline for observation, interpretation, and reflection of classroom situations.

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ASSESSING PROFESSIONAL GROWTH OF PRE-SERVICE TEACHERS USING COMPARISON BETWEEN THEORETICAL AND PRACTICAL IMAGE OF THE 'GOOD TEACHER'

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Atara Shriki — Oranim Academic College of Education

The aim of the study was to assess the professional growth of third year pre-service mathematics teachers (PST) during and after the implementation of a computerized project-based-learning (CPBL) approach into a didactical course. For this purpose we implemented an integrated tool - class discussion and portfolio. From the portfolios we could learn about the theoretical image of the 'good teacher' as perceived by the PST and from the students' presentations which were followed by class discussion, we learned about the practical image of the 'good teacher'. In order to learn about the impact of the CPBL on the PST's professional growth, we compared the resulting two images and received unexpected results.

INTRODUCTION

Following the NCTM's (2000) recommendations for integrating inquiry activities into the mathematics curriculum, we exposed our PST to innovative teaching methods and educational theories, hoping to support their professional growth. However, we had no indication whether this exposure did indeed contribute to their professional growth. Therefore, we looked for valid tools that would provide us with maximal information regarding each student's gradual development and growth. Based on our diverse experience in assessing PST, we decided to use an integrated approach for assessing our PST professional growth: portfolio and classroom discussion.

In the current paper we focus on the information gained from the portfolios and the class presentations, which were followed by class discussions. The comparison between the data received from the portfolios during the period of their experience in the CPBL approach and the data received from the class presentations enabled us to assess the professional growth of our PST. Professional growth can be expressed in various manners, such as developing perceptions regarding: content knowledge, didactical knowledge, and knowledge concerning the learners. In the scope of this paper we chose to focus on the changes in the perception of the image of the 'good teacher' as an indicator for the PST's professional growth. We focus especially on changes in the PST's perceptions regarding didactical knowledge a 'good teacher' should hold and exhibit.

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Contextual framework and background

One of the courses we teach deals with theories and didactical methods implemented in teaching and studying geometry and algebra in junior high school. The vision that was generated by the NCTM's (2000) standards regarding the image of the new generation of mathematics teachers had made innovative approaches to teacher education compulsory. In order to become aware of the various processes associated with the implementation of novel methods, we believe that the PST should experience them over a long period of time. One of those recommended methods is learning through self-exploration. To engage our PST in a genuine process of exploring important and meaningful questions we implemented the method of studying through Project-Based-Learning (PBL). PBL is a teaching and learning strategy, which enables students to work relatively autonomously over an extended period of time, during which they pose questions, make predictions and decisions, design investigations, collect and analyze data, use technology, share ideas, build their own knowledge by active learning, and so on (Krajcik, Czerniak and Berger, 1999). We termed our approach as CPBL (Computerized-Project-Based-Learning) since it rested heavily on the use of computer software.

Assessment of professional growth

Considering this context we wished to assess each student's: a. Gradual improvement of mathematical knowledge; b. Development of inquiry skills; c. Conceptual formation of didactical knowledge. All three dimensions are rather difficult to define and assess. Since each aspect requires a lengthy discussion we shall limit ourselves to the assessment of the third one, which is recognized as a significant and important component of teachers' knowledge as a whole (Even & Tirosh, 1995).

Consequently, two main questions are raised: what is 'didactical knowledge', and how to assess its formation? As for the first question, Shulman (1987) refers to didactical knowledge as an essential component of teachers' knowledge. Shulman (ibid) suggested seven categories related to teachers' knowledge, such as: content knowledge, general pedagogical knowledge, knowledge of learners and their characteristics etc. As to the second question, there is no agreed way for assessing the formation of didactical knowledge. Thus, we chose to formulate our own belief regarding that issue. For that matter, we focused on the way the PST perceived the image of the 'good teacher' before and after experiencing the CPBL. We believe that in referring to that image one must consider various aspects regarding didactical knowledge.

Assessment tools – portfolio and classroom discussion

Choosing the appropriate tool was important for us to enable the PST to reflect on the processes they were experiencing, since personal development occurs by reflecting on activities (Cooney & Krainer, 1996). Additionally, through reflection students may become aware of the viewpoints that lie behind their mathematical performances in terms of what it means to solve problems and to reason (NCTM, 2000), as well as

to develop new ways of making sense of what it means to do and teach mathematics (Simon, 1994).

Holding in mind the important role of reflection we decided to employ an integrative assessment tool: classroom discussion and written portfolios, since we believe these tools provide the opportunity to reflect on processes. As follows we shall briefly review some of the characteristics of classroom discussion and written portfolios.

Classroom discussions provide insight into students' thinking (NCTM, 2000). Discussions in which students communicate mathematically, present and evaluate different approaches to solving complex problems can develop their sense of criticism towards the quality of solutions. Consequently the capacity to reflect on solutions and to engage in self-assessment is increased. Teachers' questions during class discussions promote the abstraction of ideas, so that extension of the new ideas may derive spontaneously from the class discussion (Simon, 1994).

Portfolio is a record of one's process of learning. It is a purposeful collection of examples of work collected over a period of time. Portfolio includes what one has learned, how one thinks, how one creates and analyzes things. As such, it enables the evaluation of the learner's progress and performance (Arter & Spandel, 1991). In using portfolio as an assessment tool the focus is on the learner's successes rather than his failures. As a consequence portfolio has the potential to motivate students and to advance their ability to reflect on the processes they are going through and to carry out self-evaluation. Various studies on the use of portfolios have indicated that they could usefully serve purposes of assessment of professional competence and development (Campbell et al, 1997).

METHODOLOGY

The research data came from two main sources: *a*. The PST portfolios which included continuing reflections regarding the process they were experiencing and their responses to a written questionnaire that was given before and after the implementation of CPBL. One of the issues they had to relate to in those questionnaires was the image of the 'good teacher'; *b*. our observations of the students' presentations in which they reported their progress in the project and the class discussions that followed them.

Each resource had its unique contribution to the assessment of the PST professional growth. However, since they mutually stimulated each other we could gain a view of each student's current state as well as the processes that led her/him to that state. Looking for phenomenological categories, we applied inductive analysis (Goetz & Lecompte, 1984) on the data collected from the written portfolios referring to the students' experience with CPBL and the class presentations and discussions. We considered all the students' utterances through the lenses that concerned their perception of the image of a 'good teacher', before and after working on the project,

as well as the reflection on their experience with the CPBL approach, and their presentations in class.

From the portfolios we could gain a sense of the PST's 'theoretical' image of the 'good teacher', before and after experiencing CPBL, while during the class sessions we could observe the 'practical image', as was exhibited by the PST. In order to assess the entire professional growth as was reflected from the resulting three images, we made a comparison between them.

THE STUDY

25 PST in their third year of studying towards a B.A. degree in mathematics teaching participated in the course. CPBL is one of the main methods of learning in that course. The students could choose to work individually or in pairs. The project was geometrical by nature, and the students used dynamic geometrical software in the various stages of the project. The project included the following phases (Lavy & Shriki, 2003): 1. Solving a given geometrical problem, which served as a starting point for the project; 2. Using the 'what if not?' strategy (Brown & Walters, 1990), to create various new problem situations on the basis of the given problem; 3. Choosing one of the new problem situations and posing as many relevant questions as possible; 4. Concentrating on one of the posed questions and looking for suitable strategies for solving it; 5. Raising assumptions and verifying/refuting them; 6. Generalizing findings and drawing conclusions; 7. Repeating stages 3-6, up to the point in which the student decided that the project had been exhausted. In each class meeting part of the time was allocated to class discussion, during which the students raised their questions and doubts, asked for their classmates' advice, and presented parts of their work. The students were required to write in their portfolio at least once a week and to send us a copy by e-mail. We then returned our feedback. The problem we were facing was which items should be included in the portfolio. We decided to give the PST the autonomy to include anything they wished in their portfolios in order to learn which aspects they believed they had to reflect on. The only instruction that was given was a general one, namely to describe all the aspects that concerned their way of thinking and working, including their progress, difficulties, experiments, conjectures, doubts and so on, and to reflect on them. Our feedback was not a judgmental one. We asked mainly for further clarifications and elaborations and suggested new viewpoints.

RESULTS AND DISCUSSION

We first refer to the professional growth as was viewed from the portfolios, then as was observed during the presentations, and finally we compare between them. In the scope of this paper we refer only to aspects relating to the didactical knowledge the 'good teacher' should poses and exhibit according to our PST.

Professional development as was viewed from the portfolios

In this section we refer to the PST's utterances regarding the didactical knowledge the 'good teacher' should hold and exhibit. These utterances helped us to draw the 'theoretical image' of the 'good teacher' as was perceived by the PST.

A. References to the didactical knowledge of the 'good teacher' before experiencing the CPBL

Before experiencing the CPBL the most common utterances of our PST regarding the didactical knowledge of the 'good teacher' were (indicated by b_i): b_1. *Knows to explain well*; b_2. *Teaches in an interesting manner*; b_3. *Adjusts the learning materials to the students' level*; b_4. *Tries various teaching methods in order to find the optimal one*; b_5. *Uses many examples and games*; b_6. *Learns from his mistakes and avoids repeating them*; b_7. *Can solve every mathematical problem.*

We distinguished between declarative and operatively oriented utterances. By declarative utterance we mean a general statement without recommendations for implementation, and by operatively oriented utterance we mean utterances that imply or include practical operations. Part of the above utterances is declarative (b_1, b_2, b_3, b_4) and part of them is operatively oriented (b_5, b_6, b_7). Declarative utterances were more frequent than the operatively oriented ones. Among them, the most frequent utterances related to knowledge about teaching methods and strategies (b_2, b_4, b_5). A minor part of the PST's utterances implied to knowledge about the students (b_3).

B. References to the didactical knowledge of the 'good teacher' after the experiencing of the CPBL

After experiencing the CPBL the most common utterances regarding the didactical knowledge of the 'good teacher' were (indicated by a_i): a_1.*Creates a good atmosphere in the mathematics lessons;* a_2. *Inspires the students to look for a deep understanding and not just success;* a_3. *Teaches attractively;* a_4. *Develops creative thinking;* a_5. *Encourages his students to ask questions;* a_6. *Motivates his students to experience the process that a mathematician goes through while looking for a mathematical regularity;* a_7. *Spends a great amount of thinking in the design of a lesson*; a_8. *Provides his students the opportunity to explore mathematics, and does not give them the solutions right away;* a_9. *Lets the students think first and then establishes the subject matter*. a_{10}. *Explains each topic clearly and simply*; a_{11}. *Uses many examples in class for clarification;* a_{12}. *Integrates interesting activities in his teaching;* a_{13}. *Enables all the students to take an active part in the mathematics lessons;* a_{14}. *Uses many class discussion*

From the above utterances we can see again that they can be divided into two kinds: declarative utterances (a_1, a_2, a_3, a_4, a_5 and a_6), and operatively oriented utterances (a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13} and a_{14}).

After experiencing the CPBL approach, operatively oriented utterances were more frequent than the declarative ones. The utterances pointed to a much more

comprehensive didactical knowledge, and related more broadly and specifically to knowledge about teaching methods and strategies (the most frequent were utterances a_8, a_9, a_{11}, a_{12}) as well as knowledge about the students and knowledge about class management (a_{13},a_{14}). The ability to give a detailed list of characteristics that relates to knowledge about the students, knowledge about teaching methods and strategies and knowledge about class management the 'good teacher' should hold, point to the PST's professional growth.

Professional development as was viewed from the presentations

The class presentations and the classroom discussions that followed them enabled the tracing of the practical implementation of the changes in the PST's perception regarding the image of the 'good teacher'. These presentations helped us draw the PST's 'practical image' of the 'good teacher'.

The class presentations began approximately at the middle of the semester. Each pair of PST had to present his work in class, to describe the problem situation that was chosen, results that were discovered, and difficulties that accompanied the process. For the presentation, most PST used Power Point and the dynamic geometrical software, which helped them to demonstrate clearly their geometrical problems and ideas.

The most visible change during the class presentations was the shift from 'lecture based' presentations into interactive ones. That is, during the first few presentations, the presenters focused primarily on talking about the difficulties they encountered while working on the project. Moreover, the presentations were rather vague and unfocused, and we had to function as mediators between the presenters and the class.

As time passed, the presentations became more structured. The presenters succeeded in handling a classroom discussion, and could explain clearly their ideas and doubts. They become aware of their classmates' comments, listened carefully and asked for further clarifications when needed. Nevertheless, many of them asked us to postpone their presentation claiming that they had not reached complete results, and said that they were afraid of being asked questions they would not be able to answer.

Theoretical vs. practical image

Tracing the revealed PST's 'theoretical image' before and after experiencing the CPBL, implies professional development. There was an obvious observed shift from declarative to operatively oriented utterances. Moreover, the utterances were more specific regarding the didactical knowledge the 'good teacher' should hold and exhibit in his class.

However, observations of the presentations revealed a different picture. Although there was a shift from perceiving the 'good teacher' as one who has to lecture his ideas to one who has to be in a constant interaction with his students, still most of the PST exhibited an approach which was consistent with the theoretical image that was drawn before the experiencing of the CPBL. Namely, the interactions were limited

only to situations in which the presenters felt safe with the problems they brought into the discussion. Moreover, the PST did not bring to the discussion any problems that they were not sure of their solutions. Relating to the utterances mentioned above, it is clear that the PST still believe in utterance b_7. In addition, they did not exhibit any behavior regarding the contents of a_2, a_4, a_5, a_6, a_8, and a_9.

To summarize, it might be said that the analysis of the portfolios revealed that the 'theoretical image' of the 'good teacher' had undergone a meaningful change as a consequence of experiencing the CPBL. Yet, observing the 'practical image' of the 'good teacher' as was drawn from the presentations and the class discussions that followed them leads to a different conclusion. That is, there was not a real change in the PST's general perception regarding what good teaching really means.

CONCLUDING REMARKS

The use of the integrated tool for assessing our PST's professional growth revealed unexpected results. When we examined only the outcomes that were received from the portfolio regarding the PST's perceived image of the 'good teacher' before and after experiencing the CPBL, we could observe professional growth. This professional growth was concluded from both the shift from declarative to operatively oriented utterances and from the nature of the utterances after experiencing the CPBL. Namely, with comparison to the utterances that were uttered at the beginning of the semester, the PST's utterances after experiencing the CPBL were more comprehensive on the one hand and more detailed on the other. The utterances referred to a larger scope of attributes regarding the image of the 'good teacher' than the ones that were uttered at the beginning of the semester. However, when we examined the image of the 'good teacher' as was revealed from the presentations and the class discussions that followed them, we realized that the PST demonstrated a behavior that matched the image of the 'good teacher' as was perceived before experiencing the CPBL and not after it. Consequently, we could observe a wide gap between the theoretical and the practical image of the 'good teacher' after the implementation of the CPBL approach.

Thus we suggest that any evaluation program aimed at assessing professional growth of PST should include both components: one that enables the assessment of the development of theoretical ideas and one that enables the assessment of the development of practical experiences. The comparison of the results received from both components will enable the evaluator to draw a much more reliable picture of the actual professional growth of the PST.

Further research is needed in order to understand the reasons that underlie the gap that was found in our study regarding the PST's perceived theoretical and the practical image of the 'good teacher'.

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MATHEMATICALLY GIFTED STUDENTS' GEOMETRICAL REASONING AND INFORMAL PROOF

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This paper provides an analysis of a teaching experiment designed to foster students' geometrical reasoning and verification in small group. The purposes of the teaching experiment in this paper were to characterize gifted students' proof constructions and to contribute to the theoretical body of knowledge about gifted students' mathematical thinking. The experiment was conducted as part of education for gifted sixth-graders (12 years of age). The analysis of the students' responses in this paper documents the evolution of the students' proving ability as they participated in activities from an instructional sequence designed to support geometrical reasoning. Three types of reasoning (pragmatic, semantic, intellectual) and creative informal proofs were identified in the analysis. In order for mathematically gifted students to develop their proving ability, teachers need to draw explicit attention to the value of informal proofs. Likewise, in order for students to develop their sense of geometrical reasoning, they need a lot of experience in conjecturing, testing, and then verifying in a mathematical way.

INTRODUCTION

There is a widespread agreement that students have difficulties with constructing proofs (Senk, S. L., 1982; Chazan, D. & Lehrer, R. (Eds.), 1998; English, L. D. (Ed.), 1997;Weber, K. 2001). A great deal of educational research investigating students' proving abilities were conducted. Much of the research on proof has examined both the valid and invalid proofs. More recently, some researchers have paid less attention to the proofs that students produce and have focused instead on the processes that students use to create those proofs (Graves, B., & Zack, V., 1997;Artzt, A., & Yoloz-Femia, S., 1999;Weber, K., 2003). The aim of this study is also to describe and investigate the proving processes that students produce.

The traditional view of proof has been and still is, largely determined by a kind of philosophical rationalism, namely, that the formalist view that mathematics in general (and proof in particular) is absolutely precise, rigorous and certain. Although this rationalistic view has been strongly challenged in recent years by the fallibilist views of, for example, Lakatos (1976), Davis and Hersh (1986), and Ernest (1991), it is probably still held by the vast majority of mathematics teachers and mathematicians. In an extreme version of this view, the function (or purpose) of proof is seen as only that of the verification (conviction or justification) of the correctness of the mathematical statements (Chazan, D. & Lehrer, R. (eds), 1998).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 241-248. Melbourne: PME.

This paper is based on the fallibilist perspective and Brousseau's (1997) theory of didactic situation. According to Lakatos (1976), mathematics is not a purely deductive science proceeding from accepted axioms to established truths. And he believes that the uncertainty in proofs is the basis for mathematical activity, not an impediment to it, because uncertainty makes possible the process of proof analysis. On the other hand, Brousseau (1997) documented that an attitude of proof exists and is developed by particular didactic situations. According to Brousseau, proof must be formulated and presented while being considered, and therefore most often written, and must be able to be compared with other written proofs also dealing with the same situation. The didactic situation in this study was designed to motivate students to discuss and favor the formulation of their implicit validations and informal proofs, even if the students' reasoning is incorrect or imperfect.

METHODOLOGY

Participants in the study were two groups of mathematically gifted sixth-graders of 16 students per group, studied geometry for three hours each day emphasizing mathematical argumentation and validation. The students' individual written proof constructions were collected. Classroom instruction and task-based, semi-structured individual interviews with 3 students were videotaped. Students were asked to explain their reasoning and challenge other students' explanations and validations in the instruction. The instructor asked to find the answers or representations, even though it consists of one word or visual images only. The necessity of proof was proposed by students who met uncertainty about the truth of mathematical propositions. In this process, students' pragmatic, semantic, intellectual reasoning, which Brousseau (1997) has distinguished as such, and informal proofs were identified and discussed.

Analysis was conducted by the theory of didactic situations as described by Brousseau (1997) and the interpretive framework developed by Strauss, A., & Corbin, J. (1990). Transcriptions of the classes and interviews, written proof constructions were summarized per student per task. Analyses of these summaries were discussed with the instructor in order to minimize inappropriate interpretations. The analyses led to the identification of many creative mathematical ideas, reasoning and informal proofs.

The task below provides an example of the question sets presented to the students. Observing, conjecturing, testing, generalizing and validating occurred while confronting the problematic issues in these questions.

1. Let's observe a football. You need to imagine a football as a polyhedron made of regular pentagons and regular hexagons. Then how many regular pentagons and regular hexagons are used?
2. How many vertices are there? Explain how you found it.
3. How many edges are there? Explain how you found it.

4. Looking at one vertex, what kinds of polygons are there? Is it all the same for every vertex? What is the sum of the interior angles collected at one vertex?
5. How many spherical solids can be made if we use regular triangles?
6. How many spherical solids can be made if we use squares?
7. How many spherical solids can be made if we use regular pentagons?
8. How many spherical solids can be made if we use regular hexagons?
9. How many spherical solids can be made if we use two kinds of regular polygons such as a football?

There are many possibilities for making spherical solids, so it is necessary to establish criteria for constructional method and their justification or verification. They may discuss what kind of regular polygons can and cannot be used simultaneously for making a spherical polyhedron. Of course they can investigate and announce the reasons, and predict another possible form for a football, discuss their strengths and weaknesses. The intent of the instructional sequence is to support students' development of sophisticated ways to reason geometrically about polyhedron in space and represent the consequences mathematically. The instructor encouraged students to observe regularity, pattern, or law and yield worthwhile results by insight. Any particular case or consequence was actively examined and verified, so that students acquired the credit of the conjecture they produced.

RESULTS AND DISCUSSION

The responses were marked and coded in terms of the number of reasoning types (intellectual, semantic, pragmatic) and in terms of the process of reasoning, whether it was attempted, incomplete and invalid. The transcriptions were categorized in terms of the central issue being considered.

Questions	Pragmatic	Semantic	Intellectual
How many regular pentagons and regular hexagons are used?	20/32 (62.5%) counted directly	3/32 (9.4%) examined other students' answers	9/32 (28.1%) logic and calculation
How many vertices are there?	19/32 (59.4%) counted	4/32 (12.5%) systematic counting	9/32(28.1%) 12 times 5
How many edges are there?	9/32 (28.1%) Counted	12/32 (37.5%) systematic counting	11/32 (34.4%) sum of 60, 30
Is it all the same for every vertex?	21/32 (65.6%) constructed	0/32 (0%)	11/32 (34.4%) what if not

How many spherical solids if we use regular triangles?	4/32 (12.5%) constructed	18/32 (56.3%) systematic	10/32 (31.3%) sum of angles
How many spherical solids if we use squares?	3/32 (9.4%) constructed	6/32 (2.2%) systematic	23/32 (71.9%) sum of angles
How many spherical solids if we use regular pentagons?	1/32 (3.1%) constructed	4/32 (12.5%) systematic	27/32 (84.4%) sum of angles
How many spherical solids if we use regular hexagons?	0/32 (0%)	2/32 (6.3%) systematic	30/32 (93.8%) sum of angles
How many spherical solids if we use two kinds of regular polygons such as a football?	28/32 (87.5%) constructed	2/32 (6.3%) systematic	2/32 (6.3%) sum of angles analogy induction

Table 1: Percentages of reasoning type students used (N=32)

The above table shows percentages of reasoning types followed by students per question. None of the students used pragmatic reasoning after similar questions were presented repeatedly (see shaded parts in Table 1). This indicates that students could judge the value or the level of reasoning and reflected their reasoning processes when they proceeded (see the last row of Table 1). However, if the problem situation would be changed completely, most students went back to pragmatic reasoning. Pragmatic reasoning is not perfect and often clumsy but foster students to revise their idea or think alternatively. The manipulative material named "Polydron" was used in this study, which helped students construct solids, guess, reflect on their construction, and test their guesses in various ways. It is argued that appropriate manipulative material can foster pragmatic reasoning that facilitates semantic and intellectual reasoning.

It was extremely difficult to distinguish between semantic and intellectual reasoning because students often reasoned in a mixed way. Both semantic and intellectual reasoning given by one or a group of students usually were discussed for a long time and were connected to creative informal proofs. The episodes described here provide explanations on that.

Episode: pragmatic reasoning and semantic reasoning

The episode below, which lasts about 3 minutes, comes from discussions in a group of five students. These students have studied independently on the first question, and now are comparing their answers.

[Codes]

1 S1: So what is your answer? Mine is 12.
2 S2: Regular pentagon? 12. I counted the regular hexagon first. It's 19.
3 S3: So did I, but in my case, it's different, it's 20.
4 S1: How did you count?
5 S2: Well, I started at this face, let's count again, one, two, …, twenty.
6 Oh, it's strange, what's happening here! **[Pragmatic]**
7 S4: I think 20 is correct because there were no mistakes before. Maybe you've
8. missed one.
9 S2: I need to count once again. By the way, all of you got 20?
10 S1, S5: Yeah.
11 S1: Why don't you count by following different directions? It might be
12. helpful. **[Semantic]**
13 S5: [speaking to S1] Directions? Why do we consider directions?
14 S2: If we collect lots of evidence, then we can believe a lot. Is it correct?
15 S1: In addition to that, there would not be mistakes if we insist on a direction
16 while counting.
17 S5: Oh! That's a good idea. Then we had better investigate how many directions are there.
18 S1: [speaking to teacher] We have discovered interesting aspects.

S2 explicitly states that he reasoned pragmatically not by mathematical calculation but by direct counting only ("I started at this face." Line 5) and he tried to prove by counting in front of the peers again. It reveals how he reasoned and how he felt certainty in his thinking at the same time. When S1 says that 20 is correct because there were no mistakes in S2's counting (Line 7), it indicates that S1 observed S2's way of counting and sought mathematical or systematic approach. His remark on directions (Line 11-12) is sufficient to show that he proceeded to semantic reasoning and promote other students' understanding of the problem.

In the above episode, S1 not only proposed a good solution to the problem but also presented a systematic way of counting which can diminish uncertainty in their processes. His description of the counting pattern does not need to be tested and is sufficient to suggest that they found out interesting aspects of semi-regular solids. In the next episode, on the spherical solids made of squares, another example of pragmatic and semantic reasoning is provided

[Codes]

1 S6: Only cube, cube can be made if we use squares.
2 S7: How do you know?
3 S6: Just because, …, at any rate, I remember what I did. **[Pragmatic]**
4 S8: We need to explain mathematically. I am thinking on that.

5 S9: It is a sort of order. I mean we have to consider each one from the case
6 with three squares at each vertex by increasing squares. **[Semantic]**
7 S10: Four. Then, four.
8 S9: Okay. Isn't it obvious? There are no solids with four squares at each
9 vertex.
10 [S1 is trying to make a plane with four squares]
11 S6: Okay, okay. It always becomes a plane if we use four squares. So we are
12 done, it was proved finally.

Although the above episode occurred after students spent considerable time to guess, test, and verify, S6 still seems to be in the pragmatic thinking level (Line 3). However, S8 and S9 helped him quickly grasp key ideas of justification (Line 4-6), so that he describes or formulates the problem situation meaningfully (Line 11). The pragmatic reasoning itself is insufficient but can be seen as an important part of mathematical reasoning from the above episode. When S9 states an order of consideration ("a sort of order" Line 5), S6 not only understood what she meant but was also convinced that it is sufficient for verification.

Episode: intellectual reasoning

According to Brousseau (1997), students adopt false theories, accept insufficient or false proofs and the didactic situation must lead them to evolve, revise their opinion and replace their false theory with a true one. He emphasized that a mathematical theory is progressively constructed. This point of view was identified in the previous section on pragmatic reasoning. Students who participated in this study often presented false conjectures and tested them insufficiently. It was impossible to find students who can reason intellectually from the beginning. The next episode occurred after students spent much time reasoning pragmatically or semantically.

[Codes]

1 S1: It is easy to find the number of regular hexagons by using the number
2 of regular pentagons in a football.
3 S2: How?
4 S1: 12 times 5 divided by 3, and then we have 20. Because, uh, uh, every regular
5 pentagon is surrounded by regular hexagons and every regular hexagon
6 has three regular pentagons. **[Intellectual]**
7 S3: Fantastic! Fantastic! It makes sense.
8 S4: Say that again please. Why do you divide 60 by 3?
9 S1: Because we counted three times.

S1 arrives at a mathematical expression which is an essential idea for developing geometrical reasoning in the given problem situation ("12*5÷3=20" in his worksheet & Line 3). Led by him, students check their solution and expressions in terms of

efficiency or sufficiency. This intellectual work of S1 must similar to mathematicians' activity, which is a faithful reproduction of a scientific activity, as Brousseau (1997) emphasized. S9 too, reasons intellectually in the next episode when she solves the last question on semi-regular polyhedron.

[Codes]

1	S9:	I am thinking of a good expression for them. Always the same kind of polygons
2		are at each vertex, …, in this case, three squares, in case of …
3	S6:	Right. In case of a football, there are two regular hexagons and one
4		regular pentagon.
5	S10:	That kind of information varies in all situations. I don't know.
6	S6:	That kind of information varies… in all situations?
7	S9:	It's not important … I mean… If I say … five, six, six, then all of you
8		are able to know what solid I am thinking. Okay, let's express it like this
9		from now on. [writes as "(5, 6, 6)" and shows it to other members] Then
10		we are ready to find another solid like a football. **[Intellectual]**

S9 drew a table and check several cases which can be semi-regular solids using her own expressions (Line 9), that made her discover meaningful ideas. Intellectual reasoning is usually accompanied by useful expressions. S9 proposed excellent expressions or informal proofs while solving other problems, too. (see Figure 1) She demonstrated that not all squares have circumscribed circles by Figure 1. She said in the interview after the class, "Triangles are very special polygons because we can find circumscribed circles in any of them." Her intellectual reasoning was informal and insufficient as in the above episode or Figure 1, but it contributed to successive verification and justification, which is essential for developing geometrical reasoning.

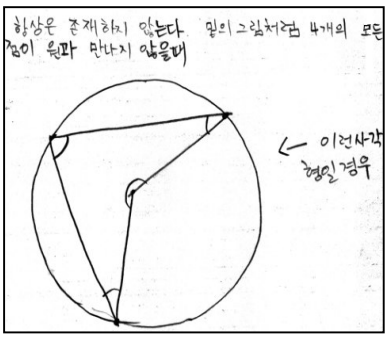

Figure 1

CONCLUSION

The mathematically gifted students demonstrated pragmatic, semantic, and intellectual reasoning in this study. Nine (including S1, S9 in the previous sections) of 32 students consistently reasoned intellectually in most of all the settings via pragmatic and semantic reasoning. Furthermore, they presented crucial ideas for formal proving or validation. These ideas gave chances to discuss what proof means and why proof is needed in mathematical situations. The students who had reasoned pragmatically at first have learned from discussions with other students on verifications, and consequently changed their ways of reasoning. Thus, it is argued that the didactic situation in this study permitted the evolution and the organization of informal but close to formal proofs by means of various kinds of reasoning. The example episodes also illustrate that peer interaction in small group is critical to

leading students in which they reflect and simplify their idea while explaining to peers. The theoretical purposes of the teaching experiment were to characterize gifted students' proof constructions and to contribute to the theoretical body of knowledge about gifted students' mathematical thinking. The connection of informal and formal proofs by students is an area in need of additional research.

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INVESTIGATION ON AN ELEMENTARY TEACHER'S MATHEMATICS PEDAGOGICAL VALUES THROUGH HER APPROACH TO STUDENTS' ERRORS

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Based on a three-year case study on the subject teacher (Ms. Lin), this article investigates how an elementary teacher's mathematics pedagogical values are manifested in her approach to students' mathematics errors. Ms. Lin's initial mathematics pedagogical value was "learning the knowledge in the textbook." She evaluated students' performance by whether they can solve the problems correctly and students' errors were taken as mere indications of failure and should be ignored and not discussed in class. After some value-cultivating programs, Ms. Lin's teaching belief and behaviors began to change to "learning a method of thinking and debating." She realized that discussing students' errors was important for clarification of concepts and improvement on the ability to reflect. In the end, the article points out that it would be a feasible way for teacher educators to probe into teachers' mathematics pedagogical values by studying how teachers' approach to students' mathematics errors.

INTRODUCTION

The teaching of mathematics involves three core elements: mathematics, teachers and students. These three elements are not value-free; in fact, they are value-carriers. In the aspect of mathematics, it changes with time and it carries certain contemporary values. One example is the elementary mathematics curriculum reform in Taiwan in 1996. The Old Elementary Mathematics Curriculum (OEMC) emphasized on acquisition of mathematics knowledge and mastery of calculation skills. The New Elementary Mathematics Curriculum (NEMC), however, not only focuses on " … to guide children to obtain mathematics knowledge from their daily life experiences" but also emphasizes on "… to develop the consciousness to communicate, discuss, rationalize and criticize in mathematics language" and the problem solving skills (ME, 1993, p.91).

The OEMC and NEMC in Taiwan's elementary mathematics curriculum reflects the two education views proposed by Borasi(1996): transmission model and inquiry approach. The transmission model regards mathematical knowledge as "a body of established facts and techniques that are hierarchically organized, context-free, value-free, and thus able to be broken down and passed along by experts to novices." The teaching approach of transmission model is " (a) direct transmission of knowledge

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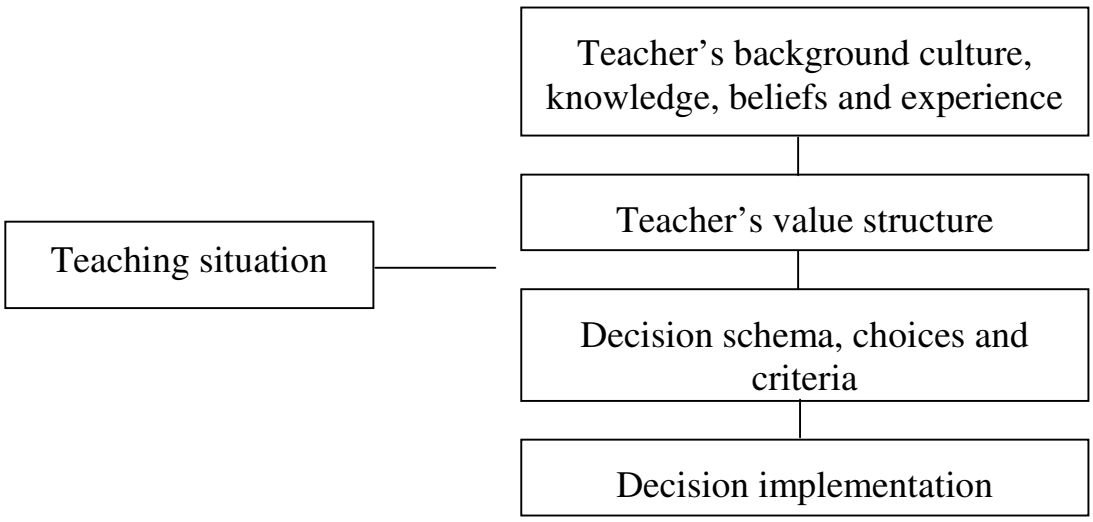
that can be achieved effectively as long as the teacher provides clear explanations and the students pay attention to them and follow them with memorization and practice. ” On the contrary, the inquiry approach views mathematics knowledge as “a humanistic discipline” and the construct of knowledge is through “a process of inquiry where uncertainty, conflict and doubt provide the motivation for the continuous search for a more and more refined understanding of the world.” The teaching approach of the inquiry approach is “(to) stimulate and support the students’ own inquiry and establishing a learning environment conducive to such inquiry.”

The present research is intended to assist elementary school teachers on professional development so they can implement the new curriculum successfully into their classrooms under the elementary mathematics curriculum reform in Taiwan. The specific goal of this research is to understand whether the mathematics pedagogical values of elementary teachers have changed under the curriculum reform. Since students usually make various kinds of errors when they are learning mathematics, one manifestation of a teacher’s mathematics pedagogical values is his/her approach to students’ errors. Under Borasi’s transmission model, errors can only provokes teacher and students’ negative feelings but errors are seen as “a prototypical example of an anomaly” under the inquiry approach (Borasi, 1996). The present research is to access and to investigate an elementary teacher’s mathematic pedagogical values through her approach on students’ errors.

THEORETICAL BACKGROUND

Mathematics curriculum is realized by mathematics instruction, and mathematics teaching carries implicit and explicit values (Bishop, 1988; Bishop, FitzSimons, Seah, & Clarkson, 2001; Chin, & Lin, 2001). Bishop (2001, p.241) proposed a diagram to illustrate how teacher’s value structure affects mathematics instruction (i.e. decision implementation).



According to the diagram, Bishop(2001) stated “the teacher’s value structure monitors and mediates the on-going teaching situation, constructing options and choices together with criteria for evaluating them. The teacher thus is able to

implement the decisions in a consistent manner." Consequently, mathematics teaching is not value-free and researchers can retrace a teacher's value structure through his/her mathematics instructions (i.e. decision implementation).

The statement made by Bishop (2001) has been supported by research data. For an instance, Ms. Chen, the sample teacher under Leu & Wu's experiment (Leu & Wu, 2002), would hint her students that they made some error(s) when they solved a mathematics problem wrong but she would not point out explicitly their error(s). It's her intention to develop students' practice of self-reflect and self-correct. Ms. Chen's particular teaching behavior can be explained by one of her main mathematics pedagogical values, that is, "the purpose of education is to reinstate students' enlightment."

In this study, the valuing theory (Raths, Harmin, & Simon, 1987) served as the foundation for exploring teachers' value-driven mathematical teaching. Raths, et al. defined values as "any beliefs, attitudes, activities or feelings that satisfy the following three criteria: choosing, prizing and acting." The criterion of choosing includes choosing freely, choosing from alternatives and choosing after thoughtful consideration of the consequences of each alternative. The criterion of prizing includes prizing, cherishing and affirming. The criterion of acting includes acting upon choices and repeating.

METHODOLOGY

In this case study, data were collected by once-a-week, whole-unit and un-scheduled observations and interviews. Various methods over a variety of schedules and topics were observed and interviewed to prevent unjustified influence of any single method, mathematics topic or instructional event and to make claims across multiple sources and to allow the triangulation of data.

The research subject of this case study is Ms. Lin who has had nine years of teaching experiences in elementary school at the time she joined the research in April 1999. During the three years of research, she taught 5th and 6th graders. For the first year, she taught the OEMC and the following two years she taught NEMC.

Based on Raths et al's theory, researchers used classroom observations to notice repeated behavioral patterns during her mathematics lessons. The purpose of the interviews is to recognize the reasons why Ms. Lin developed these behavioral patterns and to formulate some value indicators, as well as to examine if the value indicators met the criteria of "choosing" and "prizing". Over the course of three years, the researchers did 37 classroom observations and 36 interviews with the teacher.

During the first year of the research, the goal is to find out Ms. Lin's teaching method and mathematics pedagogical values under OEMC and to start some program to assist Ms. Lin's professional development in mathematics teaching. The design of the value-cultivating program was to select teaching videos where the demonstration

teachers have distinctively different pedagogical values and teaching strategies from Ms. Lin and where group discussion is used as a main classroom activity. After viewing the videos, Ms. Lin was asked to evaluate and comment on the advantages and disadvantages of the presented pedagogical values and strategies.

The research took a teacher's approach to students' errors in real classroom instruction as a probe to the underlying mathematics pedagogical values. It is to investigate whether some behavioral and belief changes has occurred on her approach to students' errors as researchers assisted the subject teacher on her professional development.

RESEARCH RESULTS

The initial Meaning of Error

We recorded a whole mathematics unit on circumference during June 1999 and during a total of 4 periods (160 minutes in total), students made 8 errors as they explained their problem solving strategies on the blackboard. However, Ms. Lin regarded these errors as the result of carelessness or forgetfulness. She didn't make use of students' errors as discussion prompts to further explain/clarify the possible underlying concepts behind these errors. In below dialogs, "T" is for Ms. Lin (the teacher), "S" represents individual student.

Example 1:

(The question is a word problem that reads "What is the area of a circle with a diameter of 20 cm?" Ms. Lin assigned S1 to solve the problem on the blackboard but S1 did it wrong. S1's answer was 20 X 2 X 3.14 = 125.6).

T: Let's see. The question states that there is a circle with a diameter is 20 cm and S1 just calculated out the area of that circle. S4 (one random student), what is the formula of the area of a circle?

S4: Radius X Radius X 3.14.

T: Very Good. Please sit down. S1, look at your answer, what were you thinking? Did you use the formula? NO! You didn't. This is the diameter and you multiplied the diameter by 2! What did you think it was? What were you calculating? You thought 20 cm was the radius and you calculated the circumference. Am I right? (Ms. Lin marks a cross on S1's calculation.) The least thing we want (when you are doing mathematics problems) is calculation mistakes. BE CAREFUL.

From the above incidence, Ms. Lin didn't analyze why students got confused between the area formula and circumference formula but she regarded this kind of error as calculation mistake and asked students to correct immediately. When the researchers asked Ms. Lin why she didn't make use of students' error as a discussion prompt, she answered

> … it's because when I presented the wrong structure of a Chinese character on the blackboard (to ask students not to make the same mistake) during my Mandarin lessons, students would just pick up the wrong version instead of preventing the error. That is

> why I think that it's better to let students find out their mistakes and correct their errors privately. Otherwise other students will focus on the error(s) a*nd cause some negative effects.*

As a result, various errors were merely mistakes to Ms. Lin. To her, errors didn't reflect different levels of students' conceptual understanding nor did errors function as a source of concept clarification. Under Ms. Lin's value of "learning the knowledge in the textbook"(Leu & Wu, 2002), errors are only indications of failures with no any other functions.

Loose up the Old Beliefs

Regarding to the issue of the meaning of errors in problem solving, Ms. Lin's beliefs loosened prior to her behavioral changes. At the beginning of the research, Ms. Lin had some doubts about using paper-pencil tests as the sole indication of "learning the knowledge." She stated that

> I was impressed by one of the questions you asked, 'Do you think the result of paper-pencil tests can really represent students' understanding?' In the past, I would say if students could do it right on a test, everything would be OK. …But now I wonder whether student really understand or they are just a copy machine that reproduced what the teacher has taught or corrected from repeated practice and rote memorization.

Based on this doubt, Ms. Lin started to discuss some wrong problem solving strategies in class. The following is an example excerpted from a workbook problem after Ms. Lin had finished teaching the unit on fraction.

> Example 2
>
> (The question is a picture with a basket of 8 eggs in total and 5 of them are hatched into chicks. When students were doing this problem, some got confused with the unit.)
>
> T: The question asks, "A hen has a basket of eggs. How many chicks are hatched?" Some students' answer was 5/8 chicks. Let me ask you a question. When you were born, was there a fraction of you that were born and the rest of you was left in you mother's tummy? When you were writing this answer, have you not noticed that your answer is wrong?
>
> Ss: No!
>
> T: What is the meaning of 5/8 chicks? A chick is divided into 8 pieces. … You can tell it's wrong by the first sight. It must be wrong. The topic of this unit is fraction so that your answers are all in fraction. S8, please answer me, how many eggs are hatched into chicks in this basket?
>
> S8: 5 chicks.
>
> T: … Good. This is a basket of egg, right? So what can be the unit of our answer? A basket. The unit is in basket. All right. Now there are 8 eggs in the basket and 5 of them are hatched, right? How many baskets of chicks are hatched?
>
> Ss: 5/8.
>
> *(2000.11.17)*

In Example 2, Ms. Lin would indicate which problem solving strategy was wrong and lead students to identify the errors and come up with the correct problem solving strategy step by step. However, from the excerpted dialogue, Ms. Lin and the students were speaking by turns and she was actually saying much more words than the students. At this stage, Ms. Lin started to discuss students' errors but she did not really know how to lead a classroom discussion. Moreover, Ms. Lin was not confident and assertive about the meaning of discussion on errors. She stated that

> Frankly speaking, I am still in the experiment stage as I dare to present errors to my students during this semester. I am not sure whether the result will come out definitely positive, but it is OK so far. Kids have their ways through.

Even Ms. Lin had been still experimenting and waiting for some positive results from conducting discussions on students' errors, from her words, it's clear that Ms. Lin discovered that discussion on students' errors would not cause more students make the same error and the stereotype and negative image of errors she had established from correcting students' Chinese characters has dissolved.

FIRST CHANGES IN BELIEF THEN CHANGES IN BEHAVIORS

From the interviews about the meaning of errors with Ms. Lin in May 2001, the researchers found that Ms. Lin had more concrete ideas and positive attitudes about discussing students' errors. She stated that,

> I would point out errors on purpose and help students better understand the concept through group interaction. This way of teaching is very different what I used to. I want to train my students to have the ability to digest different opinions. They should be able to compare the differences and transfer the differences into a more justified and refined thinking process. This ability not only can be applied to mathematics, but also covers a wide range, such as correcting his/her own behaviors and communication skills.

Ms. Lin not only upheld the positive meaning of conducting discussions on students' errors, she also started to provide discussion opportunities for students in class, as demonstrated in the following example.

Example 3

(It is a question from workbook and it is to draw out the right half of a symmetric figure in Figure 1. The solution provided by S6 is to draw out a right triangle △AMB and measure the lengths of line $\overline{AM}$ and $\overline{MB}$. Then through the reduplication of △AMD to form △AND (△AND ≅ △AMB), point D is found and lines $\overline{AD}$ and $\overline{DC}$ are connected (Figure 2)).

T: Any questions on this figure?

S8: I have a question to ask S6. (S8 stepping forward to point at the blackboard). He said that he drew the line here (Line $\overline{AD}$) and made a symmetric figure. I did the same thing. I drew a rectangular outside but when I drew out the figure it became like this (**i.e.** $\overline{AE}$). It is not symmetrical. What should I do?

S6: This is simple. You just need to find out how long line $\overline{MB}$ is and go here (line $\overline{ND}$) and make a mark (Point D). Then you just connect here to here (Point A to Point D to make Line $\overline{AD}$).

(2001.10.22)

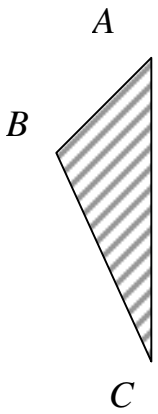


Figure 1

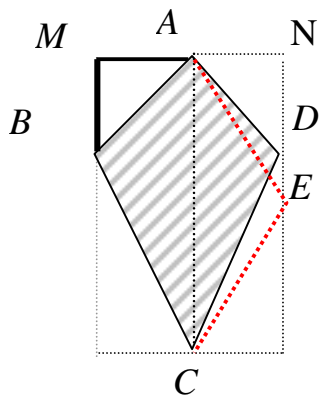


Figure 2

Ms. Lin's thoughts and teaching behaviors reflect her mathematics pedagogical values. It shifts from "learning the knowledge in the textbook" toward "learning a method of thinking and debating". Nevertheless, when facing the pressure such as limited class time constraint and/or students' high score attainment on tests, the practice of conducting a discussion on students' errors would likely to be forfeited. For an instance, when Ms. Lin was interviewed on November 7th, 2001 and was shown the videotape of her class on October 22, she said that she found students could not tell the key points she expected in the free discussion on first problem. That's why she had to lead her students step-by-step in the following discussion on other questions. Ms. Lin admitted that waiting for the expected response was a great challenge for her. She had to keep reminding herself not to interfere and to withhold the urge to tell the answer directly. The confession from Ms. Lin indicated that even though she realized the importance of presenting and discussing students' errors in class, Ms. Lin's initial value of "learning the knowledge in the text book" was still influencing her teaching behaviors, seesawing with the value of "learning a method of critical thinking and debating."

IMPLICATION

Through Ms. Lin's case study, this paper presents an alternative source to collect a teacher's teaching beliefs and to understand his/her teaching behaviors through his/her approach to students' errors. It can also be used as a probe to investigate a teacher's mathematics pedagogical values. A teacher's approach to students' errors is related to how he/she defines mathematics, knowledge, children's learning and proper teaching strategies. A teacher can find students' errors in doing mathematics problems everywhere, from his/her teaching processes to students' homework. When students' errors are the discussion topic, the interviewed teachers are usually more willing to express openly about their opinions and this allows researchers to perceive teacher's deep values more easily. Besides, during the course of teacher education

programs, the relationship between the meaning of errors and mathematics pedagogical values can be discussed among pre- and/or in-service elementary school teachers and it provides a topic of professional development.

Furthermore, this research presents a fact: it is not easy to change a teacher's mathematics pedagogical values. With the researchers' assistance and value-cultivating programs, Ms. Lin changed her beliefs in two years, under her free will and active choice of teaching beliefs and strategies. Even though Ms. Lin's teaching behaviors did change but they are feeble; they are still influenced by extrinsic factors such as time constraint and/or pressure of students' high score attainment. The question of how to shorten the adjustment time required for changes in mathematics pedagogical values can be the next research topic. In addition, it is found that teaching belief and behaviors do not change concurrently during a teacher's professional development. Changes in teaching behaviors need to be supported by relevant teaching knowledge and this present research finds that belief changes precede knowledge changes. Further researcher is needed on the area of a teacher's professional developmental process and patterns.

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HOW THE CALCULATOR-ASSISTED INSTRUCTION ENHANCES TWO FIFTH GRADE STUDENTS' LEARNING NUMBER SENSE IN TAIWAN

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This paper reviews a calculator-assisted instruction programme devised for two fifth grade pupils, who were low achievers in elementary arithmetic, for learning number sense. The programme was designed in three phases: (1)pre-test for examining what kinds of number sense the subjects were lack of; (2)instruction activities to guide them to develop the relative number sense; (3)post-test for confirming if they have developed the relative number sense. The results reveal that appropriate calculator-assisted instruction could enhance low achievers in arithmetic learning number sense. By providing a non-counting dependent procedure, pupils can concentrate on thinking the problem without too much cognitive load under the teacher's guidance.

INTRODUCTION

Numbers and operations are one of the main topics in school mathematics. When teaching this topic, teachers should not only introduce numerical conceptions and computational algorithms, but also teach relationships, estimations, awareness of what numbers are, ……, etc. These relative conceptions are all about what should be called "number sense". Many research reports emphasise the importance of teaching and learning number sense in school (e.g. Markovits & Sowder, 1994; McIntosh, Reys, Reys, Bana & Farrel, 1997). National Council of Teachers of Mathematics [NCTM] even indicates directly that "central to the Number and Operations Standard is the development of number sense" (NCTM, 2000, p.32). Number sense refers to an intuitive reasoning for numbers and their various uses and interpretations, as well as an appreciation for various level of accuracy when one computes (Reys, 1994). In general, number sense reflects a person's general understanding of numbers and operations including the ability and inclination to use the understanding in a flexible way to make mathematical judgments and develop appropriate strategies (e.g. mental computation and estimation) for manipulating numbers and operations (Howden, 1989; McIntosh, Reys, & Reys, 1992; Reys, 1994; Reys & Yang, 1998; Sowder, 1992; Treffers, 1991). Students with number sense could develop a holistic perspective of numbers, who "naturally decompose numbers, use particular numbers as referents, solve problems using the relationships among operations and knowledge about the base-ten system, estimate a reasonable result for a problem, and have a disposition to make sense of numbers, problems, and results" (NCTM, 2000, p.32).

The TIMSS relevant technical reports reveal that Taiwanese student mathematics achievements are highly ranked (Mullis, Martin, Gonzalez, & Chrostowski, 2003). However, having good computation abilities do not ensure having good number sense (Reys and Yang, 1998), that seems to be an echo of Mack's study (1990), who

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 257-264. Melbourne: PME.

suggests that relying on computation rules too much affects pupil's thinking. For instance, Chih (1996) finds that when sixteen fourth grade Taiwanese students were asked to consider which one is wrong of the following questions: 47×9 = 423, 98×16 = 948, 38×12 = 456, only four out of the sixteen could get the correct answer directly while the other twelve were busy in computing with paper and pencil.

Gray and Pitta (1997) conduct a study about helping a low achiever in elementary arithmetic change the quality of imagery associated with numerical symbols by using a calculator. Gray and Pitta's study adds a further dimension to notions that the use of calculators not only does not affect student computational ability but supports their concept development (e.g., Goldenberg, 1991; Huinker, 1992, 2002; Shuard, Walsh, Goodwin, and Worcester, 1991; Shumway, 1990; …, etc.). Further studies of the literature about calculator-assisted instruction also show that appropriate calculator-based programmes can help students develop problem solving ability (e.g., Dunham & Dick, 1994; Frick, 1989; Keller & Russell, 1997; …, etc.) and enhance their learning motivation (e.g., Hembree & Dessart, 1992; …, etc.). However, using calculator in the classroom is not popular in Taiwan. Two decades ago, students were prohibited from using calculator when learning mathematics (except statistics). Even in the newly published Grade 1-9 Curriculum, at the elementary level, the use of calculator is only for checking the correctness of the student's answer and dealing with the basic computation with large numbers.

Based on the above literature review, the study presented in this paper was designed to investigate how the calculator-assisted instruction helps two fifth grade students, who were low achievers in arithmetic, develop number sense.

RESEARCH DESIGN AND METHODOLOGY

In general, the whole research design includes picking two low achievers to be our subjects; developing a questionnaire with four questions accompanied with corresponding activity design; using the method of qualitative semi-structural interview to collect data while the two subjects meet pre-test, instruction activities, and post-test; as well as analysing and interpreting the collected data.

During the procedure of finding the two subjects, we firstly tried to find a case class of grade five whose teacher has strong will to collaborate in the research. Then we defined the nine low achievers by classifying those whose last four years mathematics marks were the last 20% from the whole class. We consulted the teacher about these nine pupils' computational abilities and interviewed them all. Finally we picked Howard and Jane who could not do arithmetic well but had good oral expression abilities which should be beneficial to our data collection (through qualitative semi-structural interviews).

Central to the whole research is the design of the questionnaire with corresponding calculator-assisted instruction activities. There were four main ways we would like to help Howard and Jane to learn in the study: (1)using a specific number as a referent to make estimation; (2)discovering the relation between division and numbers; (3)recognising number patterns; (4) discovering the relation between multiplication

and numbers. Based on the above four points, four groups of questions were designed in the questionnaire for examining if Howard and Jane have the abilities. Some examples, corresponding to the four points respectively, are as follows:

(1) Before having the mid-term examination, Andrew's parents set a standard of award for encouraging him to study harder. The standard is: If the total marks of the six subjects are above 450, Andrew can win a present from Mum; if above 550, an extra present will be given from Dad. What is the lowest average mark Andrew has to get to win the extra present? What is the highest average mark Andrew gets when he wins nothing? What are the average marks Andrew can only win one present?

(2) What are the relations between the numerator and the denominator when the quotient is greater that, equal to, and less than 1?

(3) Using calculator to find (a)15x2, 15x4, 15x8; (b)24x5, 24x15, 24x25; (c)555x12, 555x36, 555x72.

(4) Which of the following is the biggest: 15x0.699, 2x0.699, 18x0.699, 16x0.699?

Consulting Wheatley and Clements (1990) and Waits and Demana (1998), four corresponding activities were also designed (Table 1). During the development of the research implements (questionnaire and instruction activity design), another mathematics educator and two senior mathematics teachers (with more than ten years teaching experiences) were consulted for the affirmation of the content validity.

Table 1. Instruction activity design

Activity	Content abstract	Number sense to develop
Range game	Finding the multiples of a given number that will fall in a given range.	Using a specific number as a referent to make estimation
Number guessing	Interviewer A picks a number x in his mind, the interviewee B guesses the number as y and keys it in in the calculator, then A shows B the answer of $y \div x$. Duplicating the procedures by modifying the guessing number y, until B finds the number x.	Discovering the relation between division and numbers
Observing the pattern	Discovering the regularity of the products of a series of two numbers.	Decomposing and integrating numbers, developing multiple and flexible strategies
Multiplying your expectation	Estimating the products of two numbers which are greater or less than 1.	Discovering the relation between multiplication and numbers

The whole study was conducted through one-to-one interviews during the lunch time (about an hour) of four days every week for ten weeks, while one week for pre-test, eight weeks for instruction, and one week for post-tests. In the pre-test, Howard and Jane were asked to answer the questionnaire under prohibition of using calculator without the time limit. The purpose of the pre-test is to find out what kinds of number sense they were lack of, thus the interviewer might interrupt the subject's answering the questionnaire in order to understand what he/she was thinking about. After analysing their answers to the questionnaire, the first author started to conduct the

calculator-assisted instruction with Howard and Jane one by one personally for helping them develop the relevent number sense. In the instruction phase, the main teaching strategy is to guide the two subjects to probe. That is not only to find the answers, but also to think of the questions holistically. Finally, we used the same questionnaire given to the two subjects in the post-test to examine if the instruction activities we designed were effective. Of course, they were prohibited to use a calculator in the post-test. Probably there will be doubt that the pupils might be able to remember some of the questions when taking the post-test. However, we do not consider this is a possibility as the way the study was conducted should not motivate them to do so. In addition, they are only in year five and very naïve. Nevertheless, we would pay attention to whether it will happen in the post-test especially.

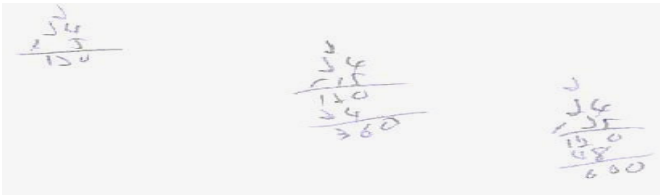
RESULTS

In the pre-test, Howard and Jane both passed the second question since they could directly give the answer that when the numerator and the denominator are the same, the quotient will be 1; when the numerator is greater than the denominator, the quotient will be greater than 1; when the numerator is less than the denominator, the quotient will be less than 1. However, they both failed the other three questions. It should be noticed that both Howard and Jane just immediately started to use paper-and-pencil computation to deal with the questions one by one without showing using other strategies when facing the three questions. The following is an episode of the interview with Howard when he was answering Question 3:

Interviewer: How do you solve the three sub-questions?

$$(1)\ 24 \times 5 = \quad (2)\ 24 \times 15 = \quad (3)\ 24 \times 25 =$$

Howard wrote the answers by using paper-and-pencil computation immediately.



Interviewer: Jolly good, you make a correct answer. But do you have any other strategies?
Howard: (Thinking for about 30 seconds.) No, I don't.
Interviewer: Do you notice there are three sub-questions of this question?
Howard: Yes.
Interviewer: And how do you think about it?
Howard: What should I think?
Interviewer: You could see the three sub-questions as a whole.
Howard: (Keeping pondering for a while, and saying nothing.)
Interviewer: So can you think of any other strategies now?
Howard: No! (Keeping shaking his head.)

In the following instruction phase, we only focused on three activities which were corresponding to Questions 1, 3, and 4 in the questionnaire. Another episode of

Jane's interview, as an example, in the activity "Observing the patterns" is presented as follows. It can be noticed in the conversation that following the interviewer's guidance, Jane could observe the pattern of the series of questions (relation of multiples) and progressively applied it to answering the following questions. She even tried to use mental calculation although the answer she got was incorrect.

(Jane was asked to answer the following series of questions: 15×4, 30×4, 45×16, 375×8, 375×48, 15×308, 1395×16, 150 24, 120×16, 495×36.)

Interviewer: Well, you just use calculator to get all these answers. Perhaps mental calculation is more useful for some of the questions.

Jane: I'm not good at mental calculation.

Interviewer: O.K. Have a look at 15×4=60 and 30×4=120, can you notice any relation between the questions and their answers?

Jane: They are multiples!

Interviewer: What multiples? Can you explain it more clearly?

Jane: Just like "30 is double of 15"!

Interviewer: Right! How about the answer?

Jane: It's double as well! (Note: She means 120 is double of 60.)

Interviewer: If 15×4=60 is given, can you answer 30×4 without using paper-and-pencil calculation?

Jane: Let me think. (Pondering for about two minutes.)

Interviewer: You can try to think of the relation of multiples.

Jane: I got it! 120.

Interviewer: How does the answer come from?

Jane: Since 30 is double of 15, so is the answer. Therefore, two 60s make 120.

Interviewer: Great! Let's try one more question, O.K. If given 15×4=60, can you answer 15×308?

Jane: Do I need to apply the same strategy?

Interviewer: Yes, if you think it's useful.

Jane: O.K. Let me see. 308 is 77 times of 4 (by using calculator), so the answer of 15×308 must multiply 77.

Interviewer: Can you find out how much it is?

Jane: No problem! (Using the calculator to count 60×77) 4620.

Interviewer: O.K. Can you count 375×48 if given 375×8=3000?

Jane: (keying in the calculator 48÷8) 6 times. So the answer of 375×48 must be 6 times of 3000, it's 1800 (Using mental calculation.).

Interviewer: How come the answer of 375×48 is less than 375×8 (=3000).

Jane: (Pondering for a while and using the calculator to check the answer.) Aha! I see. There's a zero missing.

Howard's performance in the series of "range game" activity is also worth of mentioning. The activity was about finding the multiples of a given number (e.g. 20) which will fall in a given range (e.g., 250-430). At the beginning of the instruction activity, Howard was just trying to compute 20×1, 20×2, 20×3, …, etc. The interviewer posed some probing questions to guide him to think of a more efficient way. Thus Howard started to change his strategy by trying 20×10 as a referent number to make estimation. The following conversation was happened at the end of

the activity. It is obviously that Howard became quite skilful in finding a specific number as referent to make estimation.

(The question is "What are the multiples of 25 which will fall in the range 230-590?".)

Howard: (Keying in 25×10 and getting 250, then murmuring ...) 10, 11, 12, let's try 20 (Keying in 25×20). Yah! It works. So 21, 22, (Pondering for about ten seconds.) 22 is 550, so we could only have one more, 23. That's all.

Interviewer: How do you know 23 is one of the numbers?

Howard: Because 25×20 is 500. Then add 50 is 550. So 21, 22 are the answer. Then I add 25 to 550 is 575. It also works. But add another 25 will be over 590. Therefore, I know 23 is the biggest number.

Interviewer: Brilliant! But how do you know 10 is the smallest number in the answer?

Howard: 25×10 is just adding a zero to 25. But 250–25 is ... (Pondering a while then using mental computation.) 225. It's less than 230. So 10 is the smallest one.

After the eight weeks instruction, Howard and Jane were given the same questionnaire without Question 2 the second time for the post-test. They both passed Questions 3 and 4, but unfortunately failed Question 1 which was beyond our calculation (p.s. further analysis will appear in the following section). That means both of them appeared to have developed the ability for decomposing and integrating numbers to use multiple and flexible strategies, as well as discovering the relation between multiplication and numbers after having the calculator assisted instruction. The following conversation presents an episode of Howard who was answering Question 3 in the post-test. Compared with his former performance in the pre-test (see the earlier quoted conversation), Howard could observe the pattern and use the relation to consider the whole series of questions this time.

Interviewer: Can you do this series of questions. (*15×2=, 15×4=, 15×8=*)

Howard: 30, 60, 120. (*Writing down the answers immediately*.)

Interviewer: Very good! Could you tell me how you can do that so quickly?

Howard: Just because 15 times 2 is 30, 4 is double of 2, thus 30 times 2 is 60. And next is the same, 60 times 2 is 120.

Table 2 shows the simple statistic of Howard and Jane's performance in the pre- and post-tests and the conduction of calculator-assisted instruction activities.

Table 2. Howard and Jane's performance and the conduction of instruction

	Pre-test				**Instruction Activity**				**Post-test**			
	Q1	Q2	Q3	Q4	A1	A2	A3	A4	Q1	Q2	Q3	Q4
Howard	×	○	×	×	˅	—	˅	˅	×	—	○	○
Jane	×	○	×	×	˅	—	˅	˅	×	—	○	○

DISCUSSION AND CONCLUSION

One of the results that both Howard and Jane failed Question 1 (corresponding to "range game") in the post-test was beyond our expectations. It was clear to notice that they both had developed the ability to find a specific number as a referent to make estimation gradually in the "range game" activity. After our discussion which

also included two senior primary mathematics teachers, we considered that the problem seemed to be the contexts of the word problems of Question 1 in the questionnaire. For fifth grade pupils, word problems should be the most difficult question style. We also noticed that both Howard and Jane were struggling hard to mathematise the problem context into a mathematics problem, but unfortunately they were stuck in the procedure of mathematisation and unable to apply the ability of finding a referent to make estimation.

The research results suggest that the calculator-assisted instruction programme could provide the pupil, especially the low achiever in arithmetic, an alternative, non-counting dependent procedure to develop number sense. This kind of findings seems echo Gray and Tall's "procept theory" (Gray and Tall, 1994). We probably could extend the concept of "procept" in some way, which was originally applied to a symbol, say "+", and the process of "addition" and the concept of "sum". Number sense can be considered as a general procept, it contains the corresponding computation as the process and some specific number sense as the concept. Teachers who appreciate computational ability too much but neglect the development of the concept of number sense will make the pupils unable to construct the whole "procept" of number sense. In addition, pupils who are low achievers in arithmetic could also develop the concept of number sense through using a calculator. However, it should be stressed that computational ability is also important for developing the whole procept of number sense since "procept" contains "process" and "concept".

Although, based on the research results, all positive indications suggest that calculator-assisted instruction could enhance the two research subjects' learning number sense, the calculator is definitely not a panacea. The teacher plays a very crucial role in the calculator-assisted instruction activities. Including the well-designed activities, the teacher needs to foster the skill of posing probing questions to guide the student to concentrate on thinking the mathematics problem.

Acknowledgements

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FOURTH-GRADE STUDENTS' PERFORMANCE ON GRAPHICAL LANGUAGES IN MATHEMATICS

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This study investigated the performance of 217 fourth-grade students (9 or 10 years) on a 36-item test that comprised items from six distinct graphical languages (e.g., maps) that are commonly used to convey mathematical information. The results of the study revealed that: fourth graders have difficulty decoding a variety of graphics; some graphical languages are more difficult for students to access than others; only one of the graphic languages revealed gender differences; and there were positive correlations in performance between all pairings of graphic languages. The implications of this study include the need to support the development of students' ability to decode graphics beyond activities usually investigated in mathematics curricula.

The importance of representation in teaching, learning and understanding mathematics is widely acknowledged (e.g., Cucuo & Curcio, 2001). However, although our society utilises a vast array of "information graphics" (e.g., graphs, diagrams, charts, tables, maps) for the management, communication, and analysis of information (Harris, 1996), there has been scant attention to the interrelationship between numeracy and representation (Pugalee, 1999). This relationship involves the ability to decode mathematical information from graphics and encode mathematical information into graphics (Baker, Corbett, & Koedinger, 2001). This paper focuses on primary students' ability to decode the embedded mathematical information in graphics. These students need to become "code-breakers" in order to access the mathematical information in graphics that are employed in tasks, texts, tests, software, and other everyday situations.

INFORMATION GRAPHICS

Information graphics convey quantitative, ordinal and nominal information through a range of perceptual elements (Mackinlay, 1999). These elements are position, length, angle, slope, area, volume, density, colour saturation, colour hue, texture, connection, containment, and shape (Cleveland & McGill, 1984). Although there are thousands of graphics in use, they can be categorised into six "graphic languages" that link the perceptual elements via particular encoding techniques (Mackinlay, 1999) (see Table 1). An example item for each of these graphic languages is included in Appendix A.

Students' performance in decoding information graphics is likely to be influenced by their age and the particular types of graphics in use, and their visual-spatial abilities. Each of these influences is briefly discussed.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 265-272. Melbourne: PME.

Graphical Languages	Examples	Encoding Technique
Axis Languages	Horizontal and vertical axes	A single-position encodes information by the placement of a mark on an axis.
Opposed-position Languages	Line chart, bar chart, plot chart	Information is encoded by a marked set that is positioned between two axes.
Retinal-list Languages	Graphics featuring colour, shape, size, saturation, texture, orientation	Retinal properties are used to encode information. These marks are not dependent on position.
Map Languages	Road map, topographic map	Information is encoded through the spatial location of the marks.
Connection Languages	Tree, acyclic graph, network	Information is encoded by a set of node objects with a set of link objects.
Miscellaneous Languages	Pie chart, Venn diagram	Information is encoded with additional graphical techniques (e.g., angle, containment).

Table 1: Overview of six Graphical Languages.

Age and Types of Graphics

Some students find some graphics difficult to decode. For example, on the National Assessment of Educational Progress [NAEP], many fourth graders had difficulty reasoning from a bar graph (National Center for Education Statistics [NCES], n.d., 1992-4M7-03, 1992-8M7-03) and using a scale (NCES, n.d., 2003-4M6-19, 2003-8M6-18 (see Table 2). These students' success on the scale item was no better than chance accuracy (1 out of 4, 25%). Although eighth graders outperformed fourth graders' on both these items, many older students' also had difficulty with graphics (see Table 2). The performance differences between the bar graph item (Opposed-position) and the scale item (Axis) at each grade level indicate variance in the difficulty level of particular graphical languages. Differences in the relative difficulties of some graphics and increased success with age were also reported by Wainer (1980), however, that study investigated limited graphics.

A plausible reason for differences in students' success on particular items and at different grade levels is their knowledge of the embedded mathematics content. However, the particular graphic used to represent information is a major factor in students' success. Baker et al. (2001) reported substantial variance in eighth- and ninth-grade students' ability to interpret informationally equivalent graphics with students' comparative success rates of 95% on a histogram, 56% on a scatterplot, and 17% on a stem-and-leaf plot. They argued that this performance variance was due to students' transfer of knowledge about bar graphs to the other three graphics, and that

although histograms and scatterplots share surface features with bar graphs, stem-and-leaf plot vary at the surface level from bar graphs.

NAEP Item Description	Graphical Languages	Grade 4	Grade 8	Performance Difference
Reason from a bar graph	Opposed-position	49% (n = 1622)	62% (n = 1759)	13%
Use a scale to find a distance between two points	Axis	24% (n = 36764)	39% (n = 30578)	15%

Table 2: Two graphic items from NAEPs Assessment for Grades 4 and 8.

Visual-Spatial Abilities

Decoding information graphics involves the interpretation of information presented in a visual-spatial format. Hence, it draws on spatial ability, which is a composite of abilities rather than a unitary construct and includes mental rotation, spatial perception, and spatial visualisation (Voyer, Voyer, & Bryden, 1995). Some students may be predisposed to high or low performance on decoding tasks. For example, students who have high spatial ability may decode graphics with relative ease due to their enhanced ability to process visual information (e.g., Raven, 1998). By contrast, students who have visual perception or processing problems may experience particular difficulties decoding graphics (e.g., Zangemeister & Steihl, 1995). Although gender and spatial ability has been the subject of much research, a meta-analysis revealed that the only gender difference in under 13-year-old students is limited to performance on mental rotation tests, which favour males (Voyer et al., 1995). Mental rotation tasks focus on orientation, which is included in the Retinal-list languages (see Table 1).

METHODOLOGY

This investigation is part of a 4-year longitudinal study that is designed to enhance our understanding of the development of primary students' ability to decode information graphics that represent mathematical information. Here, we report on the first stage of our study, which was:

1. To document fourth-grade students' knowledge of particular graphical languages in mathematics;
2. To establish whether there are gender differences in students' decoding performance in relation to the six graphical languages; and
3. To determine the relationships between decoding performance across the six graphical languages.

The Instrument

The Graphical Languages in Mathematics [GLIM] Test is a 36-item test that was developed to determine students' decoding performance for each of the six graphical languages (see Appendix A for an example of each of the six graphical languages items.). A bank of 58 items was variously trialled with primary-aged children (N = 796) in order to select items that: (a) varied in complexity; (b) required substantial levels of graphical interpretation; and (c) conformed to reliability and validity measures. The items were selected from state and national year-level mathematics tests that have been administered to students in their final three years of primary school or to similarly aged students (e.g., QSCC, 2000).

The Participants

The participants (n = 217) were randomly selected from six primary schools in a large rural city in Australia. Fourth-grade students (aged 9 or 10) were selected for investigation because other aspects of this study will monitor these students' decoding performance over the last three years of their primary schooling.

RESULTS AND DISCUSSION

The first aim of the investigation was to document the participants' knowledge of particular graphical languages in mathematics. As there were six items in every category, "6" is the maximum score. As shown on Table 3, the fourth-grade students were more successful in completing the Miscellaneous [$\overline{X}$ = 4.24, S.D.= 1.39] and Map [$\overline{X}$ = 4.06, S.D.= 1.29] languages than the other four categories of graphics. Interestingly, these two types of graphical languages are explicitly taught in key learning areas outside the mathematics syllabus. By contrast, the mean score for the Opposed-position category was considerably lower (15% less than the Miscellaneous category) despite the concentration of activities involving line charts, bar graphs and histograms in the curriculum and in state numeracy tests (e.g., QSCC, 2000). These results indicate differences in the difficulty level of various graphical languages for fourth-grade students. Although some of this variance could be attributed to content, the particular graphical languages used are likely to be a major factor influencing students' performance (Baker et al., 2001).

The second aim of this investigation was to establish whether there were gender differences in students' decoding performance in relation to the six graphical languages. The mean scores for the male fourth-grade students were higher than that of the female students in all six categories. T-tests were conducted to determine whether there were statistically significant differences between the performances of males and females across the six graphical language categories. There were no statistically significant differences across the gender variable for five out of the six categories [*Opposed-position* (t = .001, p = .98); *Retinal-list* (t = .84, p = .36); *Map* (t = 2.71, p = .10); *Connection* (t = 1.28, p = .26); and *Miscellaneous* (t = .15, p = .69)]. There was, however, a statistically significant difference between the mean scores of

the male and the female students in relation to the Axis graphical languages [*Axis* ($t = 12.2$, $p \leq .001$)]. A gender difference for Axis languages was unanticipated because no mental rotation was required and the students were under 13 (Voyer et al., 1995). However, these results are consistent with Hannula's (2003) findings of gender differences on a number line task (Axis) in favour of boys for fifth-grade (n = 1154) and seventh-grade Finnish students (n = 1525). Hannula's explanation that gender differences appeared to occur on tasks that were more difficult for students is inadequate for our study because Axis items were the third easiest of the six graphical languages (see Table 3).

Graphical Languages	Total	Male (n = 115)	Female (n = 102)
Miscellaneous	4.24 *(1.39)*	4.27 *(1.39)*	4.20 *(1.38)*
Map	4.06 *(1.29)*	4.20 *(1.28)*	3.91 *(1.29)*
Axis	3.79 *(1.19)*	4.04 *(1.13)*	3.50 *(1.20)*
Opposed-position	3.26 *(1.71)*	3.27 *(1.30)*	3.26 *(1.02)*
Connection	3.13 *(1.28)*	3.32 *(1.32)*	3.03 *(1.23)*
Retinal-list	2.95 *(1.38)*	3.03 *(1.41)*	2.86 *(1.34)*

Table 3: Means (and Standard Deviations) of the six Graphical Languages.

The third aim of the investigation was to determine relationships between students' decoding performance across the six graphical languages. The six graphical languages were all positively correlated with each other (see Table 4). With the exception of the Axis-Opposed-position correlation ($r = .15$, $p \leq .05$), all correlations were statistically significant at a $p \leq .01$ level. Nevertheless, even the strongest relationships [Connections-Maps ($r = .39$, $p \leq .01$) and Miscellaneous-Connections ($r = .41$, $p \leq .01$)] were only moderately correlated with each other despite the strong statistically significance. The Connections-Map and Miscellaneous-Connections correlations accounted for approximately 16% of the variance. The Miscellaneous and Connection categories had the strongest correlations with the other graphical language (with all correlations ≤ 0.27).

By contrast, the correlations between the Retinal-list languages category and other language categories were somewhat weaker (most correlations less than 0.30). In most cases, the Retinal-list items required the participants to consider graphical features including shape, size, saturation, texture, and orientation. Thus, decoding these graphics required an understanding of the use of perceptual elements to convey mathematical information. The weakest correlation was between the Axis and Opposed-position languages. Although both of these languages use axes, they differ substantially at a structural level with information encoded in only one dimension in Axis languages and in two dimensions in Opposed-positional languages (Mackinlay, 1999).

Graphical Languages	Axis	Opposed-position	Retinal-list	Map	Connection
Axis					
Opposed-position	.15*				
Retinal-list	.26**	.19**			
Map	.33**	.24**	.29**		
Connection	.32**	.31**	.27**	.39**	
Miscellaneous	.37**	.34**	.32**	.31**	.41**

* p = .05 level ** p = .01 level

Table 4: Correlations among the six Graphical Languages.

CONCLUSIONS AND IMPLICATIONS

The ability to decode information graphics is fundamental to numeracy. However, the results of this study revealed that many fourth-grade students have difficulty decoding the graphics used in each of the graphical languages and that some languages are more difficult for students than others. It is also noteworthy that boys outperformed girls in each of the graphical languages (with a significant gender difference for Retinal-list languages). There were positive correlations among the six graphical languages although most significant relationships were only moderately correlated.

The results of this study indicate five educational implications.

1. Teachers need to be proactive in supporting the development of students' ability to decode information graphics. However, the provision of appropriate support may be challenging for teachers because they have difficulty identifying which types of graphics are easier or harder for students (Baker et al., 2001).
2. Due to gender differences in performances on Retinal-list languages, girls should receive strategic support in activities incorporating these languages.
3. Learning opportunities should be broad and include graphical languages that are typically used outside formal mathematics contexts (i.e., Maps, Connections, Miscellaneous, Retinal-list) in addition to those explicitly incorporated into the mathematics syllabus (Axis, Opposed-position).
4. Where appropriate, explicit links between graphical languages should be made to facilitate cognitive transfer.
5. The informational content of graphics used for instructional purposes should be explicated to ensure that all students have access to embedded information.

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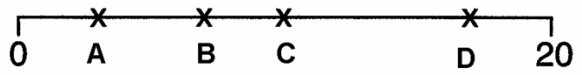
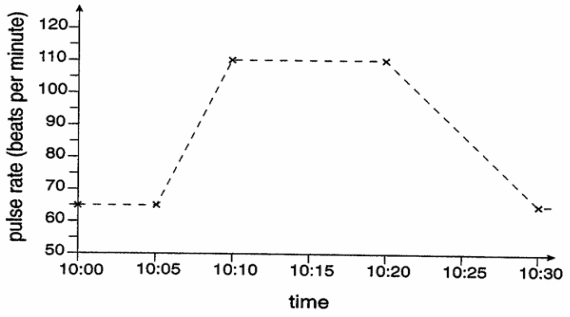
APPENDIX A: EXAMPLES OF SIX GRAPHICAL LANGUAGES ITEMS

Estimate where you think 17 should go on this number line.



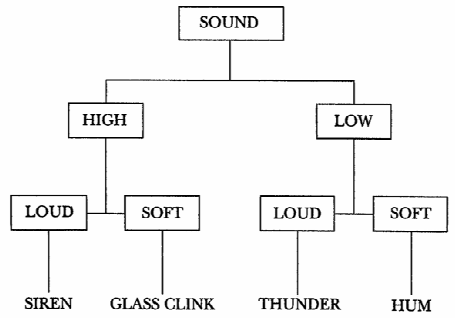
Axis Item (Adapted from QSCC, 2000, p. 11)



This graph shows how Sam's pulse rate changed while she exercised. What is the difference between Sam's lowest and highest pulse rate in beats per minute?

Opposed-position Item (Educational Testing Centre, 1999, p. 8)

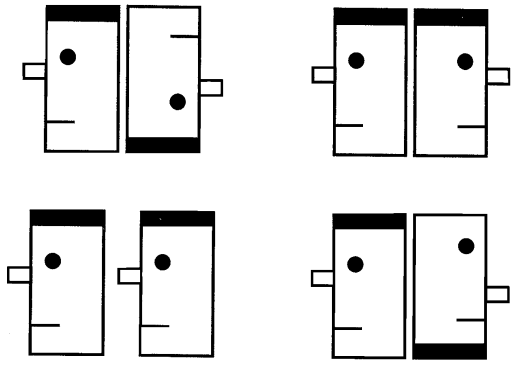
This flowchart shows a way to describe sounds.



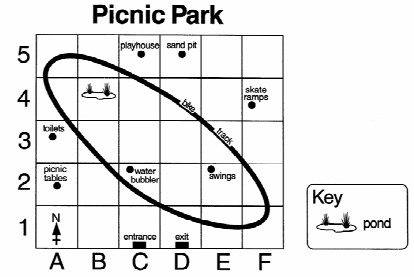
Which of the following describes a 'hum'?

Connection Item (Educational Testing Centre, 2001, p. 2)

Which two faces show a flip?



Retinal-list Item (Adapted from QSCC, 2001a, p. 13)



Deb rides her bike along the bike track. What part of the park won't she ride through?

Map Item (QSCC, 2001b, p. 16)

Which date is 3 weeks before 29 May?

MAY 2003

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Miscellaneous Item (QSCC, 2002, p. 9)

PRESERVICE TEACHERS' CONCEPTIONS OF VARIATION

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This paper reports on common findings from two recent studies of preservice teachers' conceptions of variation, one involving prospective elementary teachers and the other prospective secondary teachers. The studies both found that preservice teachers tend to use different conceptions of variation when the context of a problem changes. In addition, their descriptions of the spread of a distribution were similar, with both studies reporting the teachers using informal terminology in preference to standard descriptions (shape, mean, etc.). Implications for practice are discussed.

INTRODUCTION

Much research is being conducted about how students learn and conceptualize variation (Lee, 2003; Ben-Zvi & Garfield, 2004). Preservice teachers, however, often lack the experience and depth of understanding necessary to engage their students in the kinds of tasks that show promise in promoting the kinds of statistical understanding recommended by the research community. In particular, teachers are encouraged to provide students with authentic, inquiry-based tasks meant to develop children's reasoning about variation and distribution, but little is known about how the teachers themselves reason about variation. The purpose of this paper is to highlight two findings common to each author's doctoral research, with Canada's (2004) focus being on elementary preservice teachers and Makar's (2004) being on secondary preservice teachers. The two findings from these studies provide insight into strengths and barriers that preservice teachers have in their statistical reasoning about variation. One finding concerns their use of non-standard language when describing elements of a distribution, particularly spread. Another finding concerns the relevance of the context in which questions are embedded. The authors' studies build upon and add to previous research on conceptions of variation, most of which is aimed at students in grades 3-12. Research has provided few exploratory results on the conceptions of variation held by teachers or preservice teachers.

PREVIOUS RESEARCH

Much of the recent work on how learners develop notions of variation has come out of work at the middle school level, although researchers often find that teachers without experience with data-handling often have similar notions of randomness and variation to that of students. Teachers' conceptions of variation have previously been studied by only a handful of researchers (e.g., Mickelson & Heaton, 2003; Watson, 2001). In a study by Hammerman and Rubin (2004), inservice teachers used the software Tinkerplots to design meaningful data representations and develop reasoned arguments in a compelling context. Because of the dearth of research on this

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population, we turn to research on students at various levels to gain insight into potential paths to develop teachers' conceptions of variation.

Earlier studies have looked at student understanding of variation through tasks involving data sets and graphs, sampling, and probability situations. In considering the distribution of data sets, Mellissinos (1999) noted that college students had some awareness that only looking at the center would not capture the whole picture. At the school level, researchers found that older students generally had a higher level of understanding of variation than younger students (Torok and Watson, 2000). Watson and Moritz concluded that many students from grades 3-9 did not recognize that smaller samples were "more likely to give an extreme or biased result" (2000, p. 66). Shaughnessy and Ciancetta (2001) found that by making predictions and then conducting simulations, secondary students were able to focus more attention on the variation inherent in the set of outcomes rather than just focus on the expected value for any particular outcome.

THEORETICAL FRAMEWORK AND METHODOLOGY

Two studies, one with prospective primary teachers (Canada, 2004) and the other with prospective secondary teachers (Makar, 2004) were conducted independently in one-term preservice courses at two large public universities in the United States. Both studies were designed to strengthen teachers' conceptual understanding of probability and statistics, and focused on providing the prospective teachers with multiple hands-on experiences, conducting experiments and investigations, and interpreting data in an applied context. Data collected from the studies were primarily qualitative. Pre-post tests were also given to assess conceptual understanding of statistics, particularly reasoning about variation. Subjects were interviewed to gain additional insight into their statistical reasoning as well as to investigate the ways that the teachers articulated how they were "seeing variation". Transcriptions were analysed using a grounded theory approach, allowing common themes to emerge from the teachers' descriptions and actions. The themes fall into a framework for understanding variation which was developed from prior research (Canada, 2004). The framework posits three main *aspects* of understanding variation: expecting, displaying, and interpreting variation. Within the aspect of expecting variation, we found that our results reflected the theme involving distributional reasoning. Within the aspect of displaying variation our results reflected the theme of emphasizing decisions in context, and within the aspect of interpreting variation, our results reflected the themes concerning causes of variation.

Study of Prospective Elementary Teachers

The thirty subjects in this study (24 women, 6 men) were enrolled in a ten-week preservice course at a university in the north-western United States designed to give prospective teachers a hands-on, activity-based mathematics foundation in geometry and probability and statistics. Prior to any instruction and again at the end of the course, subjects took an in-class assessment designed to elicit their understanding on

a range of topics in probability and statistics. Six subjects were selected for additional one-hour interviews outside of class before and after instruction in probability and statistics so that their conceptions of variation could be further explored. A series of activities were conducted in class specifically designed to offer opportunities to investigate and discuss variation. The activities were centered around the three realms of data and graphs, sampling, and probability situations. Take-home surveys were given after each activity.

Study of Prospective Secondary Teachers

The seventeen subjects (14 women, 3 men) in this study, all majors in mathematics or science, were enrolled in a secondary preservice course on assessment at a university in the southern United States. About half of the preservice teachers had previous coursework in statistics or had learned statistics within a mathematics or science course for their major. The study was designed to address the larger research question of how prospective teachers used the concepts of variation and distribution to support their understanding of issues of equity in testing. A subquestion related to the results reported in this paper aimed at uncovering their understanding of the concepts of distribution and variation. The course provided the prospective secondary teachers with opportunities to examine issues of equity through interpreting large-scale, school-level, and classroom-based assessment data using the statistical learning software, Fathom™. Readings in assessment and related issues of equity and high-stakes testing were assigned to deepen their contextual understanding of the data. Rather than be a course about statistics, statistical concepts were learned "as needed" as tools to investigate and gain insight into equity in assessment through the analysis of data. At the end of the course, the prospective teachers chose a topic of interest and conducted an in depth three-week data-based investigation of an issue of equity in assessment and presented their findings both in writing and as a class presentation.

ELEMENTS OF DISTRIBUTIONAL REASONING: NOTIONS OF SPREAD

In tasks involving an evaluation of a data set or a comparison of two data sets, subject responses could be categorized according to the elements of distributional reasoning used: center, range, shape, and spread. While all four elements are critical to a more sophisticated understanding of distribution, the notion of the spread of data around the center of distribution in traditional coursework is often limited to a discussion of the standard deviation or other standard statistical measures. However, just as Torok and Watson conducted a study that "successfully explored students' understanding of variation without ever employing the phrase 'standard deviation'" (2000, p. 166), so too did we find that many of our subjects used non-standard language to convey their sense of variation when reasoning about distributions of data. This section reports examples of non-standard language used by our subjects as they evaluated and compared data sets.

Prospective Elementary Teachers

A task from the post-interview showed weights for 35 different muffins bought from the same bakery, and asked what subjects thought their own (36th) muffin might weigh. The set of data for the 35 muffin weights were shown in both a boxplot and a histogram. Here are three subjects discussing the data and their own expectations:

SP: How much would I expect my muffin to weigh? Well, I'm guessing that it could be anywhere in between, somewhere around where the *bulk of this data* is. [Circling a central part of the histogram and the boxplot]

EM: Looking over here at the histogram, and that it does seem within...112 to 115.5, it seems like that seems to be a *concentration of data*. I'm going to think that it's probably going to be in the interquartile range.

DS: This one [Boxplot] you can see more…real clearly that 50 % are *really clustered* between 112 and a half and 115 and a half.

Note how SP used the phrase "bulk of this data", EM talked about the "concentration of data", and DS considered how the data was "really clustered". In other tasks, subjects referred to data presented in dot plots as being "scattered" or "bunched", which are other examples of non-standard language. All of these terms suggest a relative grouping, and they appeal to the theme of spread.

Prospective Secondary Teachers

Similar non-standard language was articulated by the prospective secondary teachers. An interview conducted at the beginning and end of the study showed the teachers two stacked dot plots of authentic student-level data from a local middle school documenting the improvement of each student from one year to the next on the state-mandated exam. The interviewer asked the prospective teachers to compare the relative improvement on the state-mandated exam for students enrolled in a test-preparation course ("Enrichment") compared to the rest of their peers to determine if the test-preparation course was effective. Results showed that only 53% mentioned the mean at the beginning of the study (82% at the end of the study), despite the fact that the means were marked on the graph shown to them; and those with previous coursework in statistics were no more likely than those without to use the means to compare the groups. Even more (67%) of the prospective teachers talked about the spread or distribution of the data at the beginning of the study (76% at the end of the study), with most articulating these concepts using non-standard descriptions similar to those found in the study with prospective primary teachers. A few excerpts from interviews at the beginning of the study are given below:

Marg: These are more *clustered* [than the other group]. So where's there's little improvement, at least it's consistent.

Brian: It seems pretty *evenly distributed* across the whole scoring range.

June: This is kind of *dispersed off* and this is like, *gathered* in the center.

Similar non-standard descriptions were used at the end of the study:

Rachel: It's more *clumped*, down there in the non-Enrichment.

Marg: [Enrichment] has a much *wider spread*.

Anne: [The Enrichment] are all kind of *scattered out* almost evenly. Whereas [the non-Enrichment] are more *bunched up* together.

EXPECTATIONS OF RANDOMNESS IN DIFFERENT CONTEXTS

Consistent with the findings of Pfannkuch and Brown (1996) and Confrey and Makar (2002), both of our studies revealed that the context of the problem had a strong influence on the prospective teachers' expectation or tolerance for variation. The prospective teachers were more likely to expect random variation when they were trying to explain outcomes *a posteriori* than when they were trying to predict them. In addition, they were more tolerant of variation in problems set in probabilistic settings (e.g. dice, coin flips) and were more likely to explain variation in real-world settings with contextual explanations, particularly before instruction. Experiences with data-handling in both courses showed improved growth and stability across contexts in understanding of variation, a critical area for applying statistical reasoning in everyday contexts. Examples from each study are given below.

Preservice Elementary Teachers

In a pre-interview, subjects were given a pair of problems each involving 60 rolls of a fair die, but situated in different temporal contexts. The first question asked subjects to provide a list for the frequencies of each face of the die which might actually occur in 60 rolls. That is, how many "1s" would they expect, how many "2s", and so on. Care was taken in wording the task so that subjects knew I wanted them to imagine what might happen if they really threw the die 60 times.

RL I'm going to say 10 for every single one.

DC Ok. Why? Why do you think those numbers are reasonable?

RL I think that's {10,10,10…} going to happen. Because rolling any given number is no more likely than rolling any other number. Can it happen? Sure it can happen. I think that's {10,10,10…} going to happen.

After establishing his expectation of no variation for the frequency of faces resulting from 60 tosses, RL had a different idea in the context of the second question. For the second question, subjects were asked to assume the role of a teacher who assigned 60 rolls to students as homework. Four of the students' supposed lists of results were shown, and subjects were asked to identify which lists were authentic and which were fabricated. For example, "Lee's" results were "10,10,10,10,10,10", suggesting that was what he claimed to have really obtained. In reviewing the "Lee's" list, RL said:

RL I don't believe that [Lee] actually got that.

DC And yet on the previous page, you said that that's what you think would really happen? [I turn the page back to look at his own list of all tens]

RL That's…that would be the basis of my expectations. But it would be pretty funny for, in a world of imperfect scientific conditions, to see the likelihood matched so closely. It doesn't seem to account for…

At this point, RL was clearly re-examining his earlier expectations. Putting him in a position to judge results after an event had supposedly already occurred on the second question caused him to go back and change his mind on the first question:

RL We're living in the real world, this is not going to be 10,10,10…[He's changing his own list of "10, 10, 10, 10, 10, 10" to "12, 15, 9, 11, 8, 5"]

DC This [new list] is more like what you think might really happen?

RL Yeah, oh yeah. The reason why I would not go for this [all tens] after all is because you're going to see a range of results. I'm changing my mind [on the first question] because I was considering average but not considering variation. You need to consider variation to get the full picture.

Based on the results of other subjects who responded similarly to RL, the context of the first question (predicting results *a priori*) has a markedly different effect on reasoning about variation than the context of the second question (evaluating results *a posteriori*). Due in part to the cognitive conflict induced by the sequencing of these two die-tossing questions in the pre-interview, subjects such as RL began a shift in expectations that led to a more consistent appreciation for variation after instruction.

Preservice Secondary Teachers

The secondary preservice teacher study examined the influence of thinking about variation in a probabilistic setting versus a real-world setting. Overall, performance on deeply contextual problems on the pretest was consistently among the greatest areas of difficulties for the secondary preservice teachers. Most of them were fairly comfortable thinking probabilistically in dice and coin problems, but tended to think deterministically if the same problem were posed in a real-world context like one might find in the media. One example comes from a pair of problems from the pre-post tests, chosen to assess the teachers' expectation of variation in small samples, with identical structure and data (adapted from Pfannkuch & Brown, 1996). One problem was set in the context of dice whereas the other problem described the number of children born with deformities in various regions in New Zealand. Pfannkuch and Brown noted that even though the problems draw on the same statistical knowledge, the context of dice encourages more tolerance of variation for small subgroups, while less variability is tolerated for real-world contexts. This result was found as well in the secondary preservice teacher study where the prospective teachers had little problem tolerating random variation in the probabilistic setting of dice, with 67% attributing variability of outcomes to randomness. However, they were likely to attribute the variability of children's deformities in the real-world example to contextual factors (83%), for example as evidence of a nearby chemical plant. The ability to tolerate random variation increased significantly after instruction

with 83% attributing variation to randomness in the probabilistic setting and 50% in the real-world setting on the post-test.

SUMMARY AND IMPLICATIONS

One goal of education is to provide students with an understanding of concepts in multiple contexts so that they can apply their understanding to complex problems encountered outside of school. Little is known about how teachers' concept of variation changes across problem contexts, even though teacher knowledge is a critical factor in student understanding. The studies reported here indicate that prospective teachers often hold competing beliefs about random variation when the setting of the problem changed. For example, many subjects considered their own preference for rain, tolerance in waiting for trains, interest in seeking justice for deformed children, or desire for weighty muffins in drawing conclusions. In these cases, the context invited the subject to consider whether or not the amount of variation shown was personally desirable or appropriate. In both studies, however, the preservice teachers' tolerance for random variation stabilized and improved across multiple contexts and settings after instruction.

Both authors were also interested in gaining insight into prospective teachers' understanding of variation. Although the pre-post tests were designed as one measure of their understanding of variation, it did not probe into *how* the preservice teachers would articulate seeing variation. The prospective teachers tended to take note of qualitative attributes of variation more often than quantitative or conventional ones (e.g., shape, center) and their choice of words were very similar in both studies.

This kind of non-standard, informal use of language use needs to be given a greater emphasis in research on statistical reasoning. Describing a distribution as "more clumped in the center" conveys a more distribution-oriented perspective than quoting standard deviation or range. Research on adults' statistical reasoning has often focused on descriptive statistics (e.g., graphical interpretation, measures of center and sometimes spread), or inferential statistics (e.g., sampling distributions, hypothesis testing). These are often the only types of statistical training offered for teachers. We would argue a need for an intermediate level of coursework, located between descriptive statistics and inferential statistics; using the results of these studies, one that develops greater sense of variation and informal inference to promote teachers broader awareness of concepts of distribution and variation. Furthermore, we believe that teachers need to develop understanding and respect for the informal language their students use when describing distributions. There are several reasons for this. For one, teachers need to learn to recognize and value informal language about concepts of variation and spread to better attend to the ways in which their students use this same language. Secondly, although the teachers in this study are using informal language, the concepts they are discussing are far from simplistic and need to be acknowledged and valued as statistical concepts. Thirdly, the scaffolding afforded by using more informal terms, ones that have meaning for the students may

then help to redirect students away from a procedural understanding of statistics and towards a stronger conceptual understanding of variation and distribution. A fourth benefit of using informal language is to broaden students' opportunities for access to statistics, an important consideration for educational equity.

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THE EQUIVALENCE AND ORDERING OF FRACTIONS IN PART-WHOLE AND QUOTIENT SITUATIONS

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This paper describes children's understanding of equivalence, ordering of fractions, and naming of fractions in part-whole and quotient situations. The study involves eighty first-grade children, aged 6 and 7 years from Braga, Portugal. Three questions were addressed: (1) How do children understand the equivalence of fractions in part-whole and quotient situations? (2) How do they master the ordering of fractions in these situations? (3) How do children learn to represent fractions with numbers in these situations? A quantitative analysis showed that the situations in which the concept of fractions is used affected children's; their performance in quotient situations was better than their performance in problems presented in part-whole situations. The strategies used also differed across these two situations.

FRAMEWORK

Research has shown that children bring to school a store of informal knowledge that is relevant to fractions. Pothier and Sawada (1983) have documented that students come to instruction with informal knowledge about partitioning and equivalence; Behr, Wachsmuth, Post and Lesh (1984) have found that children come to instruction with informal knowledge about joining and separating sets and estimating quantities involving fractions; Mack argues that students construct meaning for formal symbols and procedures by building on a conception of fractions that emerges from their informal knowledge (Mack, 1990; 1993); Empson (1999) provided evidence that young children can develop a set of meaningful fraction concepts using their own representations and equal sharing situations as mediating structures. Ball (1993) claims that learning mathematics with understanding entails making connections between informal understandings and more formal mathematical ideas. However, research is needed to explore how students build upon their informal knowledge to improve their understanding of fractions. This includes focusing on which situations can be used in instruction in order to make fractions more meaningful for children.

Several authors have distinguished situations that might offer a fruitful analysis of the concept of fractions. Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Kieren (1988) considers measure, quotient, ratio and operator as mathematical subconstructs of rational number; Mack (2001) proposed a different classification of situations using the term 'partitioning' to cover both part-whole and

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quotient situations. In spite of the differences, part-whole, quotient, measures and operator situations are among the situations identified by all of them. However, there is no unambiguous evidence about whether children behave differently in different situations or not. This paper focuses on the use of fractions in two situations, part-whole and quotient situations, and provides such evidence.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. So, 2/4 in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In a quotient situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). Thus number meanings differ across situations. Do these differences affect children's understanding in the same way that they do in problems with whole numbers?

Fraction knowledge is not a simple extension of whole number knowledge. Regarding the child's conception of natural numbers, Piaget's hypothesis is that number is at the same time both a class and an asymmetrical relation. Extending this analysis of natural numbers to fractions, one has to ask how children come to understand the logic of classes and the system of the asymmetrical relations that define fractions. How do children come to understand that there are classes of equivalent fractions – 1/3, 2/6, 3/9, etc – and that these classes can be ordered – 1/3 > 1/4 > 1/5 etc? (Nunes et al., 2004).

Concerning the ordering of fractions, Nunes et al. (2004) suggest that children have to consider two ideas: (1) that, for the same denominator, the larger the numerator, the larger the fraction, (e.g. 2/4 < 3/4); (2) that, for the same numerator, the larger the denominator, the smaller the fraction, (e.g., 3/2 > 3/4). The first relation is simpler than the second one, because in the second the children have to think of an inverse relation between the denominator and the quantity represented by the fraction. Thus, it is relevant to know under what condition children understand these relations between numerator, denominator and the quantity.

The learning of fractions cannot be dissociated from the words and numbers used to represent them. According to Nunes et al. (2004), in the domain of natural numbers, the fact that two sets are labelled by the same number word – say both sets have six elements – might help children understand the equivalence between two sets. This understanding is probably more difficult with fractions, since equivalent fractions are

designated by different words – one half, two quarters – and also different written signs – 1/2, 2/4 – but they still refer to the same quantity.

In this paper we investigate whether the situation in which the concept of fractions is used influences children's performance in problem solving tasks and their procedures. The study was carried out with first-grade children who had not been taught about fractions in school. Three specific questions were investigated. (1) How do children understand the equivalence of fractions in part-whole and quotient situations? (2) How do they master the ordering of fractions in these situations? (3) How do children learn to represent fractions with numbers in these situations? Following the work of Streefland (1993; 1997) and Nunes et al. (2004), it is hypothesised that children's performance in quotient situations will be better than in part-whole situations and that children's procedures will differ across the situations, because quotient situations can be analysed through correspondences more naturally than part-whole situations. Although some research has dealt with part-whole (Sáenz-Ludlow, 1995) and quotient situations (Streefland, 1993) separately, there have been no comparisons between the two situations in research on children's understanding of fractions. Yet in many countries the traditional teaching practice is to use part-whole situations almost exclusively to introduce the fractional language (Behr, Harel, Post & Lesh, 1992). There seems to be an implicit assumption that this is the easiest situation for learning fractional representation; however, there is no supporting evidence for this assumption.

METHODS

Participants

Portuguese first-grade children (N=80), aged 6 and 7 years, from the city of Braga, in Portugal, were assigned randomly to work in part-whole situations or quotient situations with the restriction that the same number of children in each level was assigned to each condition in each of the schools. All the participants gave informed consent and permission for the study was obtained from their teachers, as required by the Ethics Committee of Oxford Brookes University. The children had not been taught about fractions in school, although the words 'metade' (half) and 'um-quarto' (a quarter) may have been familiar in other social settings. The two participating schools are attended by children from a range of socio-economic backgrounds.

The tasks

The tasks presented to children working both in quotient situations and in part-whole situations were related to: (1) equivalence of fractions; (2) ordering of fractions; and (3) naming fractions. An example of each type of task is presented below (Table 1). The instructions were presented orally; the children worked on booklets which contained drawings that illustrated the situations described. The children were seen individually by an experimenter, a native Portuguese speaker.

Table 1: Types of problem presented to the children

Problem	Situation	Example
Equivalence	Part-whole	Bill and Ann each have a bar of chocolate of the same size; Bill breaks his bar in 2 equal parts and eats 1 of them; Ann breaks hers into 4 equal parts and eats 2 of them. Does Bill eat more, the same, or less than Ann?
	Quotient	Group A, formed by 2 children have to share 1 bar of chocolate fairly; group B, comprising of 4 children have to share 2 chocolates fairly. Do the children in group A eat the same, more, or less than the children in group B?
Ordering	Part-whole	Bill and Ann each have a bar of chocolate the same size; Bill breaks his bar into 2 equal parts and eats 1 of them; Ann breaks hers into 3 equal parts and eats 1 of them. Who eats more, Bill or Ann?
	Quotient	Group A, formed by 2 children has to share 1 bar of chocolate fairly; group B which consists of 3 children has to share 1 chocolate fairly. Who eats more, the children of group A, or the children of group B?
Naming	Part-whole	Children write a half and indicate what the numbers mean. The researcher summarizes the children's description, ensuring that they realize that ½ means "you cut something into two equal parts and take one". The children are then asked to represent a child cutting a chocolate into three parts and taking one: what fraction will s/he get?
	Quotient	Children write a half and indicate what the numbers mean. The researcher summarizes the children's description, ensuring that they realize that ½ means "you share a chocolate between two children". The children are then asked to represent a child sharing a chocolate between three children: what fraction will each get?

Design

Children were randomly assigned to either the part-whole or the quotient situation.

The six equivalence items and the six ordering items were presented in a block in random ordered at the beginning of the session. The naming items were presented in a fixed order at the end of the session. In the naming tasks, the children were taught how to write fractions at the start of the tasks; this teaching covered four examples (1/2, 1/3, 1/4, 1/5) and two further examples of non-unitary fractions, such as 2/3 and

3/7 were given. These were followed by four test items where the children were asked to name the fractions. No feedback was given for any of the test items. The numerical values were controlled for across situations.

RESULTS

Descriptive statistics for the performances on the tasks for each working situation are presented in Table 2.

Table 2 – Proportions of correct answers and (standard deviation) by task

	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Equivalence	0.35(0.25)	0.49 (0.26)	0.09 (0.11)	0.1 (0.08)
Ordering	0.55 (0.35)	0.71 (0.22)	0.24 (0.23)	0.2 (0.14)
Naming	0.88 (0.28)	0.88 (0.24)	0.58 (0.23)	0.6 (0.27)

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and problem solving situation (quotient *vs* part-whole) as between-subjects factor, and tasks (Equivalence, Ordering, Naming) as within-subjects factor. Because the tasks presented to children involved 6 problems of equivalence of fractions, 6 problems of ordering of fractions and 4 problems of name fractions, proportional scores were analysed and an arcsine transformation was applied.

There was a significant main effect of the problem situation, $F(1,76) = 125.69$, $p<0.001$, indicating that children's performance when problems are presented to them in quotient situations is significantly better than when problems are presented in part-whole situations. There was a significant tasks effect, $F(2,152) = 82.37$, $p<0.001$. Simple contrasts showed that the children were significantly better at naming fractions than at ordering fractions, $F(1,76) = 69.93$, $p<0.001$, and significantly better at ordering than at judging the equivalence of fractions $F(1,76)=20.06$. There were no other significant effects.

An analysis of children's procedures allowed the identification of correspondence and partitioning as the most used procedures by children to solve the problems. Correspondence is defined as an association established between the shared parts and each recipient; partitioning is defined as the division of an item into parts. Examples of children's use of correspondence and partitioning are in Figure 1.

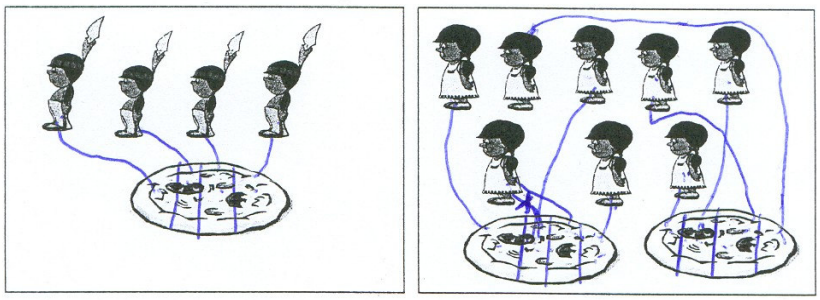

Figure 1 – The use of correspondence and partitioning by a child solving a problem in quotient situation.

Table 3 – Proportion and (standard deviation) of the use of correspondence and partitioning by task and problem situation.

	Quotient				Part-whole	
	Correspondence		Partitioning		Partitioning	
Task	6 yrs	7 yrs	6 yrs	7 yrs	6 yrs	7 yrs
Equivalence	0.24(0.25)	0.33(0.30)	0.53(0.39)	0.6(0.33)	0.35(0.46)	0(0)
Ordering	0.33(0.34)	0.43(0.36)	0.43(0.40)	0.66(0.4)	0.34(0.45)	0(0)
Naming	0.02(0.07)	0.02(0.07)	0.03(0.1)	0.05(0.22)	0.05(0.16)	0(0)

Table 3 shows that children from both age groups used correspondence when problems were presented to them in quotient situations. No children used correspondence in part-whole situations. Partitioning was used in quotient situations more often than in part-whole situations. In quotient situations many children combined correspondence and partitioning to solve the problems (see Figure 1). In part-whole situations partitioning was never combined with correspondence.

DISCUSSION

Children's ability to solve problems of equivalence of fractions and ordering of fractions is better in quotient situations than in part-whole situations. The analysis of the procedures used by children suggests that correspondence play a role in children's understanding of fractions. The use of correspondence helped children to solve problems correctly when working in quotient situations. In opposite to this, when problems were presented to children in part-whole situations the correspondence seemed not to be an acceptable procedure by children. This might be due to the different reasoning expressed by correspondence in part-whole and quotient situations. Nunes et al. (2004) argues that in part-whole situations the values in the fractions refer to quantities of the same nature, expressing a relation between the part and the whole. Here, one-to-many correspondence reasoning expresses the ratio

between the number of parts taken and the number of parts left, which is different from the fraction representation that is based on the number of parts into which the whole was cut and the number of parts taken. In quotient situations the values in the fraction refer to two variables of a different nature; the numerator indicates one quantity - the item being shared, the denominator indicates other quantity – the number of recipients. This explains why quotient situations can be analyzed through correspondences more naturally than part-whole situations. Moreover, the use of correspondence allowed children to use sharing activities, i.e. to use distribution of the items among recipients, (Nunes & Bryant, 1996; Correa, Nunes & Bryant, 1998) which demands the recognition of three number meanings: the number of items to share, the number of equal shared parts and the number of recipients. To solve problems of equivalence and ordering of fractions children have to recognize the direct relation between the number of items being shared and the size of the shared parts, and the inverse relation between the size of the shared parts and the number of recipients. These relations are extremely relevant to the understanding of fractions.

Also when children were asked to represent fractions symbolically, their level of success differed between part-whole and quotient situations. There is also evidence that children's ability to learn name fractions in quotient situations is easier than in part-whole situations. The results suggest that children can construct appropriate meanings for mathematical symbols when problems presented to them are closely matched to problems drawn on their informal knowledge.

CONCLUSION

There is evidence that the situation in which the concept of fraction is used influences children's performance in problem solving tasks. The levels of success in children's performance on equivalence and order of fractions using quotient situations supports the idea that children have some informal knowledge of the logic of fractions, i.e., children have some knowledge of the logic of fractions that was developed in their everyday life, without instruction in school. Children's performance in problems presented in quotient situations is better than their performance in problems presented in part-whole situations. Nevertheless, traditional teaching practices use part-whole and not quotient situations to introduce the concept of fractions. Thus, maybe we should rethink which is the best situation to introduce children to fractions in the classroom.

This study involved tasks in which children were working only in part-whole and quotient situations. We also need to know how children would perform and what strategies they would use to solve fractional problems in other situations.

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GROWTH OF MATHEMATICAL UNDERSTANDING IN A BILINGUAL CONTEXT: ANALYSIS AND IMPLICATIONS

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The emergence of the Pirie-Kieren theory for the dynamical growth of mathematical understanding has inspired studies focusing on monolingual students. However, prior to my research, a study had yet to be undertaken that applied this theoretical model to a bilingual context. The purpose of this paper, therefore, is twofold: one, to illustrate an application of the Pirie-Kieren theory within a bilingual context, as a language for – and a way of observing, and accounting for – the growth of understanding; and, two, to examine within this bilingual context the subtle relationship between language switching and growth of mathematical understanding.[1] In order to pursue these purposes, the work of two Tongan bilingual students is analysed in order to apply the findings toward teaching and learning.

Since 1987, Pirie and Kieren have been more interested in investigating the "process of understanding" as an alternative way of looking at mathematical understanding, as "always under construction." (Kieren, Pirie & Reid, 1994, p. 49) The Pirie-Kieren theory was developed as a theory *for* the growth of mathematical understanding of a specific topic, by a specific person, over time. The analysis in this paper draws upon the Pirie-Kieren theory, a theory previously presented and discussed at a number of PME meetings (Kieren, Pirie & Reid, 1994; Martin & Pirie, 1998; Pirie, Martin & Kieren, 1996).[2] The Pirie-Kieren theory posits eight potential layers of understanding, attainable through either informal or formal actions, for a specific person, and for a specified topic. Beginning with the innermost layer, these layers are: *primitive knowing, image making, image having, property noticing, formalizing, observing, structuring,* and *inventising*. Each layer is embedded in all succeeding layers, and, with the exception of primitive knowing, each contains all previous layers. Hence, *growth of mathematical understanding* is observe as a continuous, back-and-forth movement through these layers of understanding, as the individual reflects on, and re-constructs, his or her current knowledge.

To illustrate the use of the Pirie-Kieren model within a bilingual context, a piece of video data is analysed from a case study with secondary school students in Tonga, located in the South Pacific. The Tongan bilingual context shares many features with other bilingual programs around the world, with the dominant English language being used to teach mathematics at the secondary level. Such dominant languages are considered to have "superior" mathematics vocabularies compared to indigenous

[1] The paper stems from my research that forms part of my PhD thesis on the aforementioned relationship, presently being submitted to the University of British Columbia under the supervision of Professor Susan E. B. Pirie.

[2] Features of the Pirie-Kieren theory were elaborated on in these meetings, and elsewhere (see Pirie & Kieren, 1994). Due to the scope of this paper, I will not elaborate on these features, but will explain each layer *through* the analysis.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 289-296. Melbourne: PME.

languages. Yet, the Tongan students prefer to learn and talk about mathematics in their native language, especially in peer discussions. Largely because of the inadequacy of most indigenous languages in the language of mathematics, and the students' lack of proficiency in the language of instruction, bilingual students of the Tongan-type, as well as teachers, *switch* languages during mathematical discourse. This paper, which uses the Tongan bilingual context to study growth of mathematical understanding, offers an opportunity to appreciate, and encourages further study of, the unique characteristics of bilingual students in the South Pacific islands – a subject and a region traditionally marked by the absence of such useful educational research.

METHOD, SETTING AND TASK

The majority of this brief paper's analysis features a video case study of two Form 3 (13- to 14-year-old) Tongan bilingual students, Alani (A) and Maile (M), working together as a group (without the presence of any external observer). Video recording, widely used in case study research[3] (and most studies using the Pirie-Kieren model), was chosen as the most appropriate means of recording, collecting, and examining growth of mathematical understanding in a small-group setting. Accompanying the collected video data were students' work sheets, along with follow-up interviews about their recorded work. Using the Pirie-Kieren model, I will highlight a "mapping**",** a technique, for tracing the two students' growth of understanding of the chosen topic, "pattern".[4] At the same time, the analysis directs attention to the students' language use, particularly the role of their language switching. Language switching, also known as "code switching", is a unique feature of any bilingual situation, described by Baker (1993) as the way bilingual individuals alternate between two languages, in words, phrases, or sentences.

For the case study in question, Alani and Maile were given the first three diagrams of a continuing sequence in diagrammatical form (**Figure 1**).

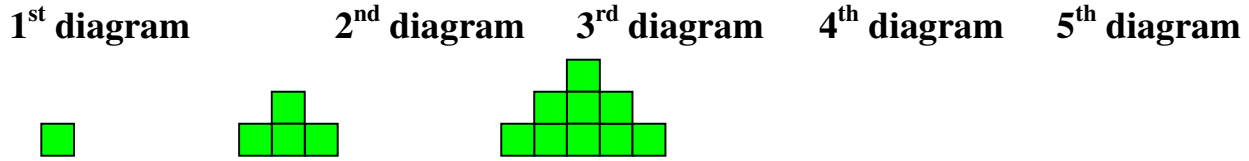

Figure 1: The task's pictorial sequence.

The task was designed for students to create, manipulate, test, and explore their ideas of or about patterns. To do that, the given pictorial sequence was accompanied by a set of questions to guide and validate their mathematical activity. The students were left to construct and draw the 4th and 5th diagrams (Questions 1 and 3), determine the difference between the 3rd and 4th diagrams (Question 2), and then later asked to discuss and predict totals for the 6th, 7th, 17th, and 60th diagrams in the given pictorial

[3] Bottorff (1994) and Pirie (1996) discuss, in-depth, the nature and use of video recordings in qualitative research.
[4] *Tracing* involves examining students' constructed images. An "image" can be any physical or mental representation students may have about the topic, and that is, by its nature, specific, mathematically limiting, and context dependent.

sequence (Questions 5, 6, 9, and 10.) In particular, the students were asked to identify the patterns in the number of square blocks they added each time (Question 7) and the total number of square blocks used in each diagram (Question 8).

DATA ANALYSIS: MAPPING

When Alani is given the task, he reads the first question aloud, then immediately switches languages to work in Tongan [**LS1**] as he points out the total of the first two diagrams, "So it's one there [1st diagram] --- four there [2nd diagram] --- Hold on --- one there; one, two, three, four --- add three to that [1st diagram]?"[5] Alani's actions indicate evidence of **image making** [**G1**], as he first engages directly, through counting, in an activity associated with constructing an image of what he sees, then he reviews his work in an attempt to make sense of it.

In his part, Maile responds to Alani in Tongan: "Do it quickly --- you just draw it [4th diagram] --- you don't have to waste time." Then Maile sketches the 4th diagram and explains in Tongan his pictorial image of the pattern as a set of ascending and descending vertical columns of square blocks. (**Figure 2a**) Starting with four square blocks in the middle, Maile explains, "Just do it like this: four, three, two, one [middle column to left] --- three, two, one [right columns] --- finish!" In response to Maile's constructed image [**G2**], Alani explains in Tongan his constructed image for the pattern along the base layers [**G3**]. He says, "Five there [base of 3rd diag.] --- so it's five there and seven at the bottom there [base of 4th diag.]. Right? Seven there and then five there, and three there. Right? Yes!" (**Figure 2b**) Both students, therefore, move out in their growth of understanding to work at the **image having** layer and are also able to articulate the features of the sequence in Tongan.

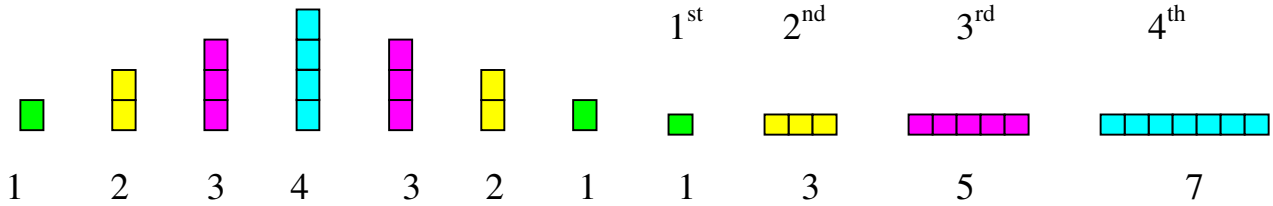

Figure 2a: *Maile's image (columns).* **Figure 2b**: *Alani's image (base layers).*

Next, Alani uses his constructed image to draw the 4th diagram by "stacking" the horizontal layers together, arranged appropriately from the bottom to top in ascending order, meanwhile speaking only in Tongan. [**G4**] (See **Figure 3**)

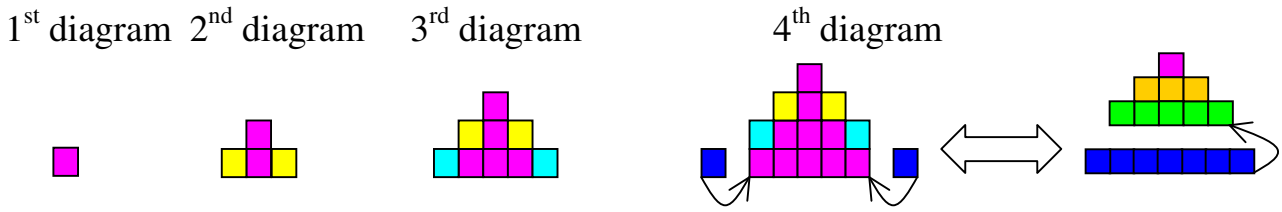

Figure 3: Alani's images for the pattern along the base layers, and at the corners.

[5] The **underlined** words are the English translation of the bilingual students' Tongan discourse. The codes "**LS**" and "**G**" refer to evidences of language switching (or no language switching) and growth of understanding respectively.

Then Alani moves to answering Question 2, and after reading it, Alani **extracts** the key words, "extra square" [**LS2**] by saying "Extra square --- so it's seven there [base of 4th diagram]; and then nine the base there [5th diagram] --- and then eleven the base there [6th diagram]. Alani reflects and articulates on his constructed pictorial image for the pattern along the base layers from the 4th to the 6th diagram (**Figure 2b**), as evidence of his continued working at the **image having** layer [**G5**].

During his work at the image having layer, Alani consciously sees the relationship between the layers, and concludes that the pattern in his pictorial images – the *common difference* – along the base layers *and* between horizontal layers is "Add two extra squares" (see **Figure 3**) [**LS3**]. Alani has moved out to **property noticing** [**G6**] by constructing a context-specific property, based on his knowledge in manipulating and combining aspects of his constructed pictorial images. In this instance, Alani associates the key phrase, "extra square", from the question with the common difference (of two square blocks) between any two consecutive layers. This extracted phrase appears, therefore, to *dictate* the way Alani approaches his constructed images, and at the same time *directs* him toward outer-layer thinking sophistication.

However, Alani continues describing his constructed images in Tongan, then he sketches the base layer of the 3rd diagram and asks Maile, "What is it called that one at the bottom there?" Maile responds, "Base!" [**LS4**]. Still, Alani is not satisfied as he continues to use equivalent English words such as "row" and "step" as metaphors for the extra square blocks being added on both ends of the base layers. However, Maile intervenes and expresses his intention of "thinking" about the task [**Extract 1**[6]**]:**

1	M:	Hold on while I think. You move from that one there while I think myself.
2		You do number three while I think it [*Question 2*] myself.
3	A:	No! Look here: add two to the last block.
4		Which is that [*base of 2nd*]; add two to the last step.
5		--- get then that five there. [*Base of 3rd*]
6		Add two to the last step [*base* of 3rd] --- and get then a seven [*base of 4th*]
7		--- what's an explanation for it? Two extra square blocks ---
8	M:	Oh --- I already know it --- add the prime number.
9		--- Look here: it's three; it's five; it's seven; it's nine ---

Alani continues to work at the **property noticing** layer [**G7**] but still struggles to find the "right" English label for his noticed property, although he seems comfortable expressing it in Tongan [3-7]. He associates the *Tonganized*[7] equivalent words "sitepu" (step) and "poloka" (block) with the phrase "poloka fakamuimuitaha" (last block) [3] and "sitepu faka'osi" (last step) [4]. These "verbal associations" have no effect on his growth of understanding. [**LS5**] But Maile's statement, "Hold on while I think ---" [1], shows *stepping back* and *looking* for any connection within or among his constructed image(s). This stepping-back process allows Maile to notice a

[6] Because of the length of this paper, selected video transcripts represent segments of the entire video recording, but the comments and discussions draw on considerable additional data.
[7] Tonganization is a form of "conventionalisation" in which English words have been borrowed and used in Tongan.

numerical pattern along the base layers when he declares, "I already know it --- add the prime numbers" [8]. This evidence of moving out to **property noticing** [**G8**] is accompanied by a **shift** in language to using the non-equivalent English word, "prime" [**LS6**]. In this situation, Maile's growth of understanding *brings about* the act of language switching using the term, "prime numbers". However, Maile's mathematical meaning for this label becomes apparent, in relation to Alani's constructed image (**Figure 2b**), when Maile explains, "Look here: it's three, it's five, it's seven, it's nine" [**LS7**]. This incident is discussed further in the conclusion.

Meanwhile, Alani answers Question 3 by finding the total number of square blocks in the first four diagrams. He shifts to working only in Tongan as he reflects on his previous constructions. Maile continues to "think" silently while Alani finishes drawing the 4th diagram. Alani then moves back out to discussing, in Tongan, the total for the 5th diagram with Maile (**Question 4**), and together the two students come up with a total of "Ua-nima --- Twenty five square blocks". [**LS8**] As he attempts to draw the 5th diagram, Alani finds himself unable to come up with the diagram. This prompts him to *fold back* from property noticing [**G9**], where he was currently working, to image having [**G10**], and to work with his constructed images for the pattern. Pirie and Kieren (1994) call this action "folding back", because Alani is faced with a challenge that is not immediately solvable, prompting him to return to an inner mode of understanding in order to re-construct and extend his current *inadequate mathematical* understanding. In this situation, language switching is not involved, as he speaks only Tongan. Alani goes back to his image along the base layers (**Figure 2b**), and uses it to build the 5th diagram as a stack of horizontal layers.

While Alani draws the 5th diagram, Maile uses "trial-and-error" method to manipulate random numbers arithmetically in order to match the diagrams' totals, again through thinking aloud in Tongan. Then Maile suddenly notices another property [**Extract 2**]:

10	M:	I already know it. Look here; here it is:
11		[*Points to the relation between diagrams' ordered number and totals*]
12		One by one is one; two by two is four; three by three is nine; four by four
13		--- is sixteen; five by five, twenty-five…seven by seven is forty-nine.
14		How about that? Square number. How about that? Got it!
15		I just thought of it and get it. Five to that --- [*draws the layers of 5th diag.*]
16	M:	So it's seventeen by seventeen --- yeah!
17	A:	One, two, three, four, five, six, seven; and then next is nine.

In this instance, Maile finds a relationship between the diagrams' numbers and their corresponding totals [10-13], again illustrating working with his constructed numerical images at the **property noticing** layer [**G11**]. He identifies this noticed property of the numerical totals as "square numbers" [14] [**LS9**], then quickly applies it to the 17th diagram [16]. He later verifies the total for the 6th diagram (Question 5), as evidence of a constructed, organized scheme for the pattern. The two students continue using Maile's newly found arithmetical rule to determine the total for the 7th diagram (Question 6). In explaining their answer, Maile simply says, "No need to

draw it!" [**LS10**]. Such a declaration suggests Maile's readiness to move outward in his growth of understanding, to **formalizing**. [**G12**]

The two students then move to answering Question 7 about the pattern in the extra number of square blocks. Alani, after reading the question, **shifts** languages again to discussing his generalization of the "extras" (additional square blocks) in Tongan. [**G13**] He explains and *reformulates* his formalized understanding of the pattern as, "Add two to the last step and total them." [**LS11**] But his peer, Maile, reflects on his own prior construction by saying: "You just add the prime number to the last row". Furthermore, Maile associates the mathematical label, "prime numbers", with the set {1, 3, 5, …}, which stands for the number of square blocks along the "last row or the base". [**LS12**] Both students are observed to have moved out to work at the formalizing layer. [**G14**] Maile then goes on to read Questions 9 and 10 in predicting the totals for the 17th and 60th diagrams [**Extract 3**]:

18	M:	Square the seventeen. What number? Seventeen by seventeen.
19	A:	Seven by seven, forty-nine; One by seven --- eleven (*add 7 and carried 4*)
20		One, seven, zero --- Nine, eighty-two --- Two eighty-nine.
21		Do that --- seventeen times seventeen --- just the same ---
22	M:	Square --- (*Writes as* "289. Explanation is, $17^2 = 289$")
23	A:	Square --- square the seventeenth diagram.
24	M:	Ten --- "Can you group predict the 60th diagram?" Sixty by sixty.
25	A:	Sixty times sixty --- six by zero is zero --- zero ["3600, 60^2 or square 60"]
26		six by six is thirty-six --- Three thousands, six hundreds!

In this extract, Alani and Maile are observed to move out to work at the **formalizing** layer [**G14**], through their quick application of the generalized arithmetical rule for calculating the totals for the 17th and 60th diagrams. Each time, the students **shift** back-and-forth between reading the question in English, to calculating the totals in Tongan – the language they feel most comfortable in expressing themselves [**LS13**].

DISCUSSION AND IMPLICATIONS

Figure 4 shows the mapping of Alani and Maile's growth of understanding of "pattern". (Alani's mapping in **bold** line, and Maile's in *thin* line.) Like all Pirie-Kieren mappings[8], each "point" marks a significant incident in the students' growth of understanding. This example, therefore, highlights the power of the Pirie-Kieren theory as an observational tool, not just in monolingual cases, but in bilingual situations as well. As a result, four specific relationships between language switching and growth of mathematical understanding emerged: one, language switching can still occur without growth of mathematical understanding (see discussion on **Extract 1, LS5**); two, growth of mathematical understanding can take place in the absence of language switching (see examples in **G2**, **G3, G11-G12**, and folding back in **G9-**

[8] The tabular format of this mapping reveals very little of the complex nature of growth of understanding as viewed through the Pirie-Kieren theory. But it helps to see clearly the back-and-forth movements discussed in this example. [**Keys:** primitive knowing (**PK**), image making (**IM**), image having (**IH**), property noticing (**PN**) and formalizing (**F**)]

G10); three, language switching can *enable* growth of mathematical understanding (example **LS3/G6**); and four, growth of mathematical understanding can *bring about* language switching (see examples in **LS6/G8** and **LS9/G11**).

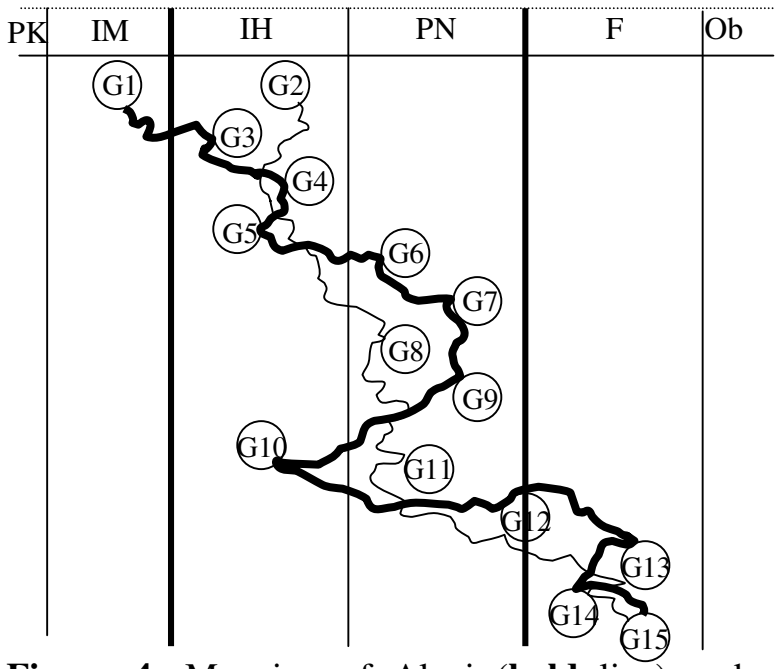

Figure 4: Mapping of Alani (**bold** line) and Maile's growth of understanding of "pattern".

In the first case, the ability of a bilingual learner to express his or her mathematical understanding through language switching comes from his or her underlying *bilingual* language capacity, which, in Pirie-Kieren terms, is a part of the learner's primitive knowing – the prior knowledge he or she brings *into* the task.

But in the second case, the example of Alani and Maile shows two students progressing in their understanding of the topic "pattern" without the necessity or effect of language switching. This finding challenges the assumption that bilingual students will find mathematics harder if the language of instruction is in their second language, and that such students are therefore naturally disadvantaged in mathematics in comparison to monolingual students.

More importantly, Alani and Maile are able to work mainly with mathematical ideas and images to further their growing understanding of "pattern", in spite of their weak English skills. Based on these two students' example, mathematical understanding does rest with the ideas and images, and not with the words (Lakoff & Núñez, 2000).

The case of Alani and Maile also shows they continued to progress in their growth of understanding without needing to use English, instead resorting to their native language while *working* and *thinking* mathematically. Such an observation offers a profound implication for teaching in bilingual situations, because it refutes two naïve assertions: first, that bilingual students' first language is irrelevant to their understanding of mathematics, and second, that indigenous language learning is actually detrimental to a student's education.

Finally, Maile's incorrect *borrowing*[9] of the phrase, "prime numbers", did not deter him from his understanding of the mathematics, or his growing understanding of the topic. This example explains why Tongan-type bilingual students do not necessarily

[9] I described "borrowing" as a form of language switching, involving mixing of *non-equivalent* English words.

have to be good at their second language, in order to be better in mathematics. Teachers, however, often assume students literally misunderstand whenever teachers "hear" wrong mathematical labels, and consequently "believe" they have encountered a lack of mathematical understanding. Thus, teachers ought to pay attention to distinguishing wrong mathematical labels from a lack of mathematical understanding, particularly in bilingual situations where an act of language switching can be easily overheard, identified, and likely *evoke* various instantaneous images.

In conclusion, there is a dilemma within the bilingual mathematics classrooms between "teaching language" versus "teaching mathematics" (Adler, 1998). If, in a bilingual situation, mathematics is the goal of teaching, then mathematical ideas, concepts, and images, have to be the focus of teaching; teachers must also be aware of their appropriate use of the mathematical language, terminology, and convention.

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MOTIVATIONAL BELIEFS, SELF-REGULATED LEARNING AND MATHEMATICAL PROBLEM SOLVING

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Learning is influenced by multiple components of interrelated belief and self-directed strategies. In this study we focus on motivational beliefs (MB) and self-regulated learning (SRL) in the context of mathematical problem solving (MPS). The aim was to search for relationship between 5th and 6th Graders' MB (self-efficacy, task value beliefs and goal orientation) and SRL (use of cognitive, metacognitive and volitional strategies) and between MB and performance in MPS. Analysis of the data from 219 students, using a self-report questionnaire and a paper and pencil test, showed a significant relation between all dimensions of MB and SRL and between self-efficacy, intrinsic goal orientation and performance in MPS. The results draw attention on SRL strategies to guide instruction and scaffolding that enhances MB during MPS.

THEORETICAL BACKGROUND

Motivational beliefs

Recent educational and psychological research highlights the role of multiple affective variables and specifically of motivation towards learning in pursuing educational goals (Boekaerts, 2001). Motivation refers to the forces encouraging a person to engage on a task or to pursue a goal; in the school setting it concerns the reason for which a student works persistently to reach a desirable result (Wolters & Rosenthal, 2000). Although there are many theories of motivation that are relevant to students learning (Seiferd, 2004), the present quest pertains to three notions, namely (a) self-efficacy beliefs, (b) task value beliefs and (c) goal orientations, which are elaborated in Pintrich (1999) and Wolters and Rosenthal (2000).

Self-efficacy has been defined as one's judgment of his ability to plan and execute actions that lead to achieving a specific goal (Bandura, 1986; Tanner & Jones, 2003). In other words, self-efficacy is a self-appraised belief concerning one's competence to succeed in a task. It is supported that high self-efficacy functions as incentive for the pursuing of a goal; on the contrary, low self-efficacy functions as barrier that urges to avoiding the goal (Hamilton & Ghatala, 1994; Seiferd, 2004). For example, students who view themselves as capable to solve mathematical problems will choose to perform that task compared to low efficacious students who might attempt to avoid involvement in the task. Recent research has consistently shown that efficacy beliefs significantly influence academic achievement and they especially constitute the most powerful indicator of the prediction of the performance in mathematics tasks (Gaskill & Murphy, 2004). Moreover, it is reported that high efficacious students are more likely to use SRL strategies than low efficacious students (Tanner & Jones, 2003).

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Task value beliefs refer to the students' evaluation about the value of the task. Eccles (1983; in Pintrich, 1999) has proposed that a student may be motivated towards working on a task if the task itself is important, interesting and useful for him (e.g. help him to cope with high school demands or for his career and life in general). It has been found that task value beliefs are correlated to performance, even though not as strongly as self-efficacy correlates (Pintrich, 1999).

Goal orientation refers to the students' perception of the reasons why to engage in a learning task. Although a number of studies have discussed goal orientation using alternative terms and definitions (Pintrich, 1999), in the present study we focus on intrinsic and extrinsic goal orientation, a classical distinction proposed by Heider as early as 1958 (in Hamilton & Ghatala, 1994). Intrinsic goal orientation concerns the degree to which a student perceives himself to be participating in a task for reasons such as challenge, curiosity and mastery, using self-set standards and self-improvement. Extrinsic goal orientation denotes that a student participates in a task for reasons such as grades, rewards, performance, evaluations by others and competition (Hamilton & Ghatala, 1994). It was found that mastery goals are positively related to performance in general tasks for middle school students, while on the contrary extrinsic goals were negatively related to performance in the same tasks for the same students (Pintrich, 1999).

Besides the relation of self-efficacy, task value and goal orientation to performance, research has especially examined the relation between motivational beliefs and self-regulated learning, which is elaborated in the following section.

Self-regulated learning

In traditional schools, the teacher assumed sole responsibility in the teaching learning process from choosing short and long-term goals to specifying activities, provision of materials, the time allocation, etc. In today's classrooms there is a tendency for a redistribution of learning responsibility between the teacher and the students. This conception leaves much room for the students to become self-regulated learners, i.e., to set goals, select from a repertoire of strategies, and monitor progress toward the goal (Panaoura & Philippou, 2003; Pape, Bell & Yetkin, 2003). Research has examined this new trend in different domains (Rheinberg, Vollmeyer & Rollett, 2000) including mathematics (Schoenfeld, 1992). Mathematics educators adopted the theory of SRL as an important change that has emerged during the last two decades of the 20th century; they expect students to assume control and agency over their own learning and problem solving activities (De Corte, Verschaffel & Op't Eynde, 2000).

Self-regulated learning could be conceptualized in three distinct ways: First, as the learner's ability to use metacognitive strategies or differently, to control cognition. Pintrich, Smith, Garcia and McKeachie (1991) refer to the metacognitive strategies of planning, monitoring and regulating, while Rheinberg et al. (2000) identify them with control strategies such as attention, motivation, emotion and decision control. A second approach views SRL as the learner's ability to use both metacognitive and

cognitive learning strategies (Schoenfeld, 1992). Rehearsal, elaboration, and organizational strategies are identified as important cognitive strategies (Pintrich, 1999) and are related to the students' different learning styles or, differently, to information about the way in which students can learn. Finally, a third view highlights the importance of incorporating motivation, cognitive, and metacognitive components of learning (Tanner & Jones, 2003). Research based on the latter view suggests that there is a connection between motivation and SRL and, more specifically, that the former promotes and sustains the latter variable (Rheinberg et al., 2000). Specifically, empirical evidence showed that high efficacious middle school students, who believe that their course work is interesting, important and useful and adopt a mastery goal orientation, are more likely to engage in various cognitive and metacognitive activities in order to improve their learning and comprehension (Pintrich, 1999; Wolters & Rosenthal, 2000).

In this study we adopt the view that motivational beliefs and self-regulated learning should be studied as parts of an integrated whole, as neither component is alone sufficient to successfully interpret learning outcomes. For example, a student who exhibits a high degree of motivation and puts forward a considerable effort toward a goal may not be able to accomplish his academic targets if he lacks self-regulated learning strategies. On the other hand, a student who possesses a rich repertoire of self-regulatory strategies may lack enough motivation to invest the necessary effort and resources. Moreover, there is a third possibility that high motivated students who are aware of cognitive and metacognitive strategies are unable to use them due to the lack of volitional strategies (Pape et al., 2003).

Although it may seem as a paradox to talk about self-regulatory strategies if the students are unable to use them, Rheinberg et al. (2000) argue, "*…there are students who cannot force themselves to engage in aversive learning activities, even if the consequences of the learning outcome are very important*" (p. 516). The missing element may be not motivation, but volition. Volitional strategies refer to the knowledge and the skills that are necessary to create and support an intention until goal attainment (De Corte et al, 2000). Research in this area has established the importance of volition in both SRL and motivation. Pape et al. (2003) found that after one year of teaching SRL strategies, students' knowledge and awareness of strategies had been increased; yet, their volitional control was too limited to sustain their use of the strategies. Wolters and Rosenthal (2000) examined the relation between motivational beliefs and the following five distinct volitional strategies: self-consequating, environment control, interest enhancement, performance self-task and mastery self-task. They found that the students' MB are related to their use of volitional strategies.

In this paper we argue that volitional strategies should be an integral part of self-regulated learning theory, together with cognitive and metacognitive strategies. In other words we propose a model that, compared to the model offered by Pintrich (1999), includes an additional dimension, namely the volitional strategies, as described in Wolters and Rosenthal (2000). Within this conceptual framework the

aim of this study is to search for relationship between primary students' MB and SRL behaviour (including volitional strategies) with respect to solving mathematical problems and between MB and performance in the same problems.

THE PRESENT STUDY

Pintrich (1999) examined the relationship between motivational beliefs and the use of cognitive and metacognitive strategies in the middle school, the college and university contexts. Wolters and Rosenthal (2000) examined 8th grade students motivational beliefs in relation to their volitional strategies. However, it seems that no research has so far investigated the relation between elementary students' MB on the one hand, and the three components of self-regulatory strategies (cognitive, metacognitive and volitional) used by the same students, on the other hand. Furthermore, the previous researchers focused on the effect of motivational beliefs on self-regulatory strategies. In this study, following Bandura's (1997) notion of "reciprocal determinism" we examine the effect of self-regulatory strategies on motivational beliefs, i.e., we reverse the role of the two variables. The construct motivational beliefs encompasses: Self-efficacy beliefs, task-value beliefs, and goal orientations, while self-regulatory strategies integrate cognitive and metacognitive components (Pintrich, 1999), and the concept of volitional strategies, as used by Wolters & Rosenthal (2000).

Mathematical problem solving was chosen for two main reasons: First, it is considered to be one of the most difficult tasks elementary children have to deal with, since it requires the application of multiple skills (De Corte et al., 2000). Second, as a complex task, problem solving is a potentially rich domain to study SRL, due to demands of cognitive and metacognitive skills (Panaoura & Philippou, 2003).

Specifically, this study sought answers to the following four questions: (1) Is there a significant relation between motivational beliefs (self-efficacy beliefs, task value beliefs, intrinsic and extrinsic goal orientation) and the level of students use of cognitive, metacognitive and volitional strategies, while engaging in problem-solving tasks? (2) Is there a significant relation between motivational beliefs and performance in problem solving? (3) Which components of self-regulated learning can effectively predict students' motivational beliefs? (4) Which components of motivational beliefs can effectively predict the problem solving performance?

METHOD

Data were collected from 219 5th and 6th grade students (108 boys and 111 girls), 110 students were five graders whereas 109 six graders. Students were coming from five different elementary schools, and ten different classrooms.

The students' problem-solving performance was measured through a specially developed test, comprised of six mathematical tasks. Four of them were "routine", whereas the other two were "non-routine" tasks. Two of the routine tasks were one-step problems and the other two were two-step problems in the sense that two

successive operations were required for their solution. These tasks were chosen to cover four types "change", "proportion" "grouping" and "compare" problems, whereas the "non-routine" problems required the "retrogradation" strategy and the "do a table" strategy, respectively.

We used a modified version of the Motivated Strategies for Learning Questionnaire (MSLQ, see Pintrich et al., 1991), a self-report instrument designed to measure students' motivational beliefs and self-regulated learning in classrooms contexts. The motivation subscale consists of 20 items, assessing students' self-efficacy in problem solving (e.g. "I am certain I can understand the most difficult mathematical problems presented in my mathematics classroom"), value beliefs (e.g. "I think I will be able to apply in other courses what I learn in problem solving") and their goals while solving mathematical problems (e.g. "I prefer to solve mathematical problems that really challenge me, so I can learn new things" or "Getting a good grade in problem solving test is the most satisfying thing for me right now"). The self-regulated strategy subscale comprises of 20 items regarding students' use of cognitive strategies (e.g. "I memorize key words to remind me of important concepts of mathematical problems"), metacognitive strategies (e.g. "When trying to solve mathematical problems, I make up questions to help focus my reading") and volitional strategies (e.g. "I tell myself I should keep working to learn as much as I can"). All statements were Likert type with four points (from 1-"I absolutely disagree" through 4-"I absolutely agree"). Students were first given the mathematical problem performance test and as soon as they had finished it they were given the questionnaire for MB and SRL. Pearson correlation and multiple regression analysis of the Statistical Package for the Social Scientists (SPSS) were used for the analysis of the data.

FINDINGS

Table 1 shows the correlation coefficients between each dimension of the motivational beliefs and self-regulated learning strategies. Clearly, all components of self-regulated strategies relate to all components of motivational beliefs. It is noteworthy that the relation between extrinsic goal orientation and cognitive and metacognitive strategies appears slightly lower, while all correlations between volitional strategies and motivational beliefs appear to be at in the range 0.346-0.407, which is quite higher than the level of cognitive and metacognitive strategies. It was further found that students' problem solving performance significantly related with self-efficacy beliefs, and intrinsic goal orientation. No significant correlation was found between problem solving performance and task-value beliefs, or extrinsic goal orientation. It is important to highlight the finding that the strongest correlation appears between problem solving performance and self-efficacy beliefs.

Applying multiple regression analysis with motivational beliefs as the dependent variable and cognitive strategy use (X1), metacognitive strategy use (X2) and volitional strategy use (X3) as the independent variables, the following regression equation was obtained: Motivational Beliefs = 0.412 X3 + 0.281 X1, [R=0.621**,

(F=52.09, p= 0.00), R^2 =0.386**, **p< 0.01]. It is noteworthy that the use of volitional strategies in MPS, constitutes the most powerful indicator of the students' motivational beliefs (Beta=0.412, t=5.433, p=0.00).

	Self-Regulated Learning strategies			Performance in MPS tasks
Motivational beliefs	Cognitive strategies	Metacognitive strategies	Volitional strategies	
Self-efficacy	0.378**	0.366**	0.407**	0.321**
Task-value beliefs	0.432**	0.230**	0.346**	0.132
Intrinsic goal orientation	0.424**	0.282**	0.352**	0.178**
Extrinsic goal orientation	0.170*	0.186**	0.376**	-0.02

p*<0.05, p**<0.01

Table 1: Correlation coefficients between Motivational Beliefs and Self-regulated Learning strategies and between Motivational Beliefs and performance in Mathematical Problem Solving

Multiple regression analysis applied to the data with performance in MPS tasks as the dependent variable, and each of the components of MB, self-efficacy beliefs (X1), task-value beliefs (X2), intrinsic goal orientation (X3) and extrinsic goal orientation (X4), as the independent variables, resulted in the following regression equation: Performance = 0.476 X1, [R=0.320**, (F=21.44, p= 0.00), R^2 =0.102**, **p< 0.01]. Self-efficacy beliefs concerning MPS, constitutes the one and only indicator of the performance in MPS (Beta=0.32, t=4.631, p=0.00).

DISCUSSION

The research reviewed here suggests that the use of self-regulated learning strategies promotes students' motivational beliefs. Concerning the first research question, the findings showed that all three components of SRL are significantly and positively related to all dimensions of MB (Rheinberg et al., 2000). This leads to the conclusion that students who tend to use SRL strategies while solving a mathematical task, are more probable to have increased MB, and vice versa. Analytically, elementary school students who use cognitive, metacognitive and volitional strategies are more likely to feel more efficacious about their ability to do well during MPS procedure. In addition, they are more likely to report higher appreciation of the MPS task, personal interest in the task and utility value of the task for future goals. Students also seem to adopt intrinsic goal orientation, which means that the use of SRL strategies helps them to a concern with learning and mastering the task using self-set standards and self-improvement. Similar results appeared for middle school and college students in

various courses (Pintrich, 1999; Wolters & Rosenthal, 2000). However, a concern for getting grades and pleasing others is not strongly associated with an adaptive pattern of SRL. Similarly, Pintrich (1999) found negative relations for all SRL variables with extrinsic goals, for middle school students. It seems interesting to investigate further the reasons for the appearance of these developmental differences between elementary and middle school students, in the pattern of relations between extrinsic goal orientation and SRL.

As a response to the second research question, it was found that elementary school students, who believe that hold high efficacy beliefs with respect to problem solving and are confident in their skills, are more likely to achieve higher performance in MPS (Gaskill & Murphy, 2004). Intrinsic goal orientation was also positively related to performance in MPS, even though this relation was not as strong as the one of self-efficacy beliefs. The goal or criterion of learning and mastery seems to be a much better motive for higher performance in mathematics than an extrinsic goal. Extrinsic goals were the only motivational variable that showed negative relation to performance. It seems that students who are operating under an extrinsic goal of just getting good grades, pleasing others, or avoiding a punishment do not achieve higher performance, compared to students who adopt intrinsic goal orientations.

Multiple regression analysis, applied for the third research question, showed that volitional strategies were the best predictor of MB. This outcome has been expected since volitional strategies seem to be "conceptually closer" to the idea of motivation. Motivational beliefs refer to the reasons that move a person to work on a task, while volitional strategies concern one's willingness and ability to regulate his motivation and actions. This finding suggests that, the students who have the knowledge and skills to create and support an intention until goal attainment can be predicted as having strong MB (Rheinberg et al., 2000). In other words, elementary school students' use of volitional strategies can lead to greater effort and persistence in MPS tasks. This is in line with Wolters and Rosenthal (2000), about general academic tasks for 8th graders. Furthermore, multiple regression analysis showed that self-efficacy was the sole dimension of motivation that predicted performance in problem-solving. Therefore, self-efficacy is a personal resource that students can draw upon when they are faced with the difficult and time-consuming tasks associated with solving mathematical problems, as Pintrich (1999) stated for academic tasks.

In conclusion, the adoption of motivational beliefs during solving mathematical problems in elementary school is neither easy nor automatic and should not be taken for granted. Many students have little if any motivation to work on a mathematics task or to pursue a goal, while others depend solely on extrinsic motivation. In that, mathematics classroom practices should and can be changed to facilitate adaptive efficacy beliefs about MPS, encourage interest and appreciation of the tasks' value and foster the adoption of mastery goals toward MPS. The results, of the research presented here, draw attention on the possibility to enhance motivational beliefs by

promoting self-regulated learning through instruction of cognitive, metacognitive and especially volitional strategies.

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FRACTIONS IN THE WORKPLACE: FOLDING BACK AND THE GROWTH OF MATHEMATICAL UNDERSTANDING

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This paper presents some initial findings from a multi-year project that is exploring the growth of mathematical understanding in a variety of construction trades training programs. In this paper, we focus on John, an entry-level plumbing trainee. We explore his understandings for fractions and units of imperial measure as he attempts to solve a pipefitting problem. We contend that it cannot be assumed that the images held by adult apprentices for basic mathematical concepts are flexible or deep and that folding back to modify or make new images as needed in particular contexts is an essential element in facilitating the growth of mathematical understanding in workplace training.

MATHEMATICAL UNDERSTANDING AND THE WORKPLACE

Although recent years have seen an increase in the attention paid by researchers to mathematics in the workplace there is still only a limited body of work that considers cognition and understanding in a vocational setting. Research that does exist in the area of adult mathematical thinking within a vocational setting is primarily concerned with the use of informal mathematics (e.g. Noss, Hoyles & Pozzi, 2000) or the place of mathematics within the workplace (e.g. Wedege 2000a,b), rather than with the process of coming to understand mathematical concepts for the individual learner in the workplace. In the larger ongoing study we are looking at the growth of mathematical understanding and numeracy in workplace training, and more specifically at some of the difficulties faced by apprentices as they engage with mathematics in workplace tasks. In this paper we focus on John and explore his understandings for fractional units of length.

MATHEMATICAL UNDERSTANDING AND THE PLACE OF IMAGES

The research reported in this paper is framed by the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994). This theory provides a way to look at, describe and account for developing mathematical understanding as it is observed to occur in action. The Pirie-Kieren theory posits eight layers of understanding together with the cognitive activity of 'folding back' as crucial to the growth of understanding. Two of the inner layers are defined as *Image Making* and *Image Having*, and it is these layers that are relevant to our discussion in this paper.

At Image Making learners are engaging in specific activities aimed at helping them to develop particular ideas and images for a concept. By "images" the theory means any

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ideas the learner may have about the topic, any "mental" representations, not just visual or pictorial ones. Image Making often involves the drawing of diagrams, working through specific examples or playing with numbers. However, it does not have to have an observable physical manifestation, it is the thinking and acting around the concept that is the actual process of making an image.

By the Image Having stage the learners are no longer tied to actual activities, they are now able to carry with them a general *mental* plan for these specific activities and use it accordingly. This frees the mathematical activity of the learner from the need for particular actions or examples. At this layer, the learner *has* an understanding, although this may still be very specific, mathematically limiting and context dependent, which they are able to employ when working on mathematical tasks.

FOLDING BACK AND FRACTIONS

Apprentices in workplace training are often re-learning mathematics that they have already met and have existing images and understandings for. As they engage in mathematical activity during training they may need to re-visit these existing understandings and images, make sense of them again in specific trades situated applications, and if necessary construct new understandings. Within the Pirie-Kieren Theory this process is known as 'folding back' implying that when a learner revisits earlier images and understandings he or she carries with them the demands of the new situation and uses these to inform their new thinking at the inner layer. Different images may be required for the same concept when used in the workplace rather than the school classroom (See Forman and Steen, 2000), or a previously held image may need to be modified or broadened. As Martin, Pirie & Kieren (1994) note "fraction learning involves constructing an ever more elaborate, complex, broad and sophisticated fraction world and developing the capacity to function in more complex and sophisticated ways within it. Such an achievement will prove impossible if the foundations laid by the images the learners hold are not adequate to the task" (p.248).

In the primary school fractions are often taught with reference to parts of a whole circle (usually described as being a pizza or pie) and the image here is quite specific and one based on an area representation. However, when using fractions in the context of measurement, it is more appropriate to see a fraction as a point on the number line – although of course to be able to read a fractional unit of measurement still requires an understanding of the part-whole relationship. This number-line image is particularly important for working with measurements in imperial units, where lengths are stated in fractional units, unlike in metric units where decimals are more commonly utilised. (For example, one would rarely talk about three and seven tenth centimetres, though it is of course mathematically valid and correct). Certainly in the workplace it is the measurement model (or image) for fractions that is likely to prove the more powerful and the one that can allow the development of the kinds of essential skills necessary to function effectively on the job. However, as we shall

illustrate through the case of John, it cannot be assumed that apprentices necessarily have and can use this image.

METHODS AND DATA SOURCES

The larger study, currently underway, is made up of a series of case studies of apprentices training towards qualification in various construction trades in British Columbia, Canada. This paper draws on one of these case studies, and although our conclusions are specific to this case, we would suggest that there are implications that may be relevant to other mathematical concepts and areas of workplace training.

The trainees and their instructor were observed and video-recorded over a number of sessions, as they worked in a classroom and also in a workshop setting. The second author acted as a participant observer in the sessions. The episode on which this paper focuses involved a small group of apprentices in the shop working to calculate the length of a pipe component required for a threaded pipe and fitting assembly to be built to given specifications.

JOHN AND THE PIPEFITTING TASK

The following extracts of transcript occured a few minutes into the session. John is trying to measure out the length of pipe that he needs, using an imperial units tape measure, prior to cutting it. He is talking to Steve, another apprentice in the group. Although he has the correct length of pipe, he thinks that he is incorrect when measuring it. This is because when needing to count in eighths on the tape measure, he actually counts in sixteenths (also marked on the tape), thus resulting in him reading an incorrect measurement from the tape. The researcher points this out to him:

Researcher: Yeah. Now wait a sec. There's a problem. Make sure you're not counting sixteenths.

John: OK.

Steve: Those are sixteenths.

R: OK. This from here to here is a half, right? *(pointing to the interval on the tape measure bounded by 8" and 8 1/2")*

J: Right.

R: From here to here is? *(pointing to the interval on the tape measure bounded by 8" and 8 1/4")*

J: A quarter.

R: A quarter, yeah. From here to here is? *(pointing to the interval on the tape measure bounded by 8" and 8 1/8")*

J: *(pause)* That's, the little, the little *(pointing to the mark on the tape measure indicating 8 1/8")*, that's one eighth.

R: Yeah, like if this is a quarter, an eighth is?

J: I know that's a sixteenth *(pointing to a small vertical line on the tape measure indicating a sixteenth between eighths.)*

R: Half of an eighth. Or, sorry, an eighth is half of a quarter. That's a quarter, then an eighth is from here *(pointing to tape measure)* to here.

S: So every second line is one sixteenth.

J: Right.

R: Every second line. And every single line here is a?

J: Eighth.

S: Thirty second. *(Spoken simultaneously with John)*

R: Every single line?

J: Every line is one is an eighth, is it not? The small line?

S: Every line is a thirty second.

Here we see John and Steve giving different answers to the questions, using their existing images for fractions. However, while both correctly use the language of fractions in this context, it is not clear at any point that John is actually thinking in terms of equal parts of a whole inch, and thus that to measure involves a comparison with the various fractional units that are superimposed upon one another on the tape. Instead, John seems to have and be using an image based on the fact that measuring tapes use different length lines to represent the different fractional divisions of each inch (i.e. very short vertical lines for thirty-seconds, becoming progressively longer towards the half inch), which does not encapsulate an understanding based on a part-whole fractional relationship. John's confusion is apparent at the end of the extract when he incorrectly states that every single line is equivalent to one eighth, whereas Steve seems more comfortable with the fractional scale and gives the correct answer of one thirty-second. At no time does John talk about an inch being divided into a given number of equal sized parts, which is an image that could help him to more easily work with the complex measuring tape. John continues his explanation to the researcher:

John: Each line here *(pointing to the fine makings on the tape measure)*, see how …. differences between the small and the little taller ones, right? I'm trying to make it as, here we go… *(Grabs a piece of paper and holds it over the tape measure so that only the end points of the marking lines are visible above the paper.)* Here we go, ok. See how every line is different here, right?

Researcher: Mmhu.

J: 'Cause one line that's smaller than the others. So…

R: Yeah.

J: That's one sixteenth, a big one is eighth, correct?

It is not clear if John recognises the part-whole relationship of fractions, or is simply still making an image based on the length of lines on the measuring tape scale. Sensing this the researcher introduces a set of ruler scales printed on acetate which can be stacked to illustrate how fractions are represented on a measuring tape. Each acetate rule layer has the inches divided up into a different fractional unit, so the first rule has only inches on it, the next shows half inches, the next has quarter inches and so on. This provides a visual image for how a standard tape measure actually incorporates a number of different fractional units superimposed onto one scale. The dialogue continues:

Researcher: So, lets look at your ruler (*the tape measure*) up against this (*the stack of acetates-teaching tool)*. So I'm going to line up the eighth, lay it down flat.

John: OK.

R: Look at where your eighth.

J: So that's one, two, three, (*counting off lines at 8 1/8, 8 3/8 and 8 5/8 on the acetate ruler*) correct? Am I counting that correct?

The set of acetate rulers are used here to offer John a valuable image making tool, allowing him to physically manipulate a set of representations of the different fractional units and to see how these relate, both to each other and to the standard measuring tape. Using the resource also exposes the problem that John is having in working with these fractional units. When he comes to count three-eighths, he points to one-eighth correctly, but then continues to point to the lines of the same length as this, i.e., three-eighths and five-eighths. He does not count two-eighths or four-eighths, as these are equivalent to one quarter and to one half, and thus on the ruler are represented by lines of different lengths.

We see clearly that John did not have an image for a fraction as a location on his measuring tape i.e. as a point on a number line, and does not see the relationship of the numerator and denominator of a written fraction to the part-whole of an actual inch. The researcher responds:

Researcher: No, you're doing spaces. Count spaces.

John: Ok. So that's one.

R: One, Yeah.

J: Two. *(now counting eighth inch intervals on the acetate eighths ruler)*

R: Two, and at the end of that space is your…

J: Is this right here? *(marks 8 3/8 point on his tape measure.)*

R: Right there.

J: Ok. Ok. So, ⋯ *(Re enacts the stepping process with his pencil tip from 8 inches this time on his tape measure.)* Ok. Didn't see it, here we go. I got yah. *(He explains it back to the researcher.)* Ok, so, this is our one, our

one sixteenth right here, correct? *(pointing to 8 1/8 point on tape measure),* no our one eighth, am I out?

R: Which is it?

J: This one right here, the big one *(pointing to tape measure).*

R: Which is it, eighth or sixteenth?

J: That's eighth.

R: Ok.

J: Ok. This one here is sixteenth *(pointing to 8 1/16 point on tape measure).*

R: Yeah.

J: This one here is thirty two.

R: Thirty-second, yeah.

J: Thirty-second. Am I on the ball with that?

R: Absolutely.

J: Ok.

In this extract the researcher engages John in a directed image making activity, he tells him to count spaces, and John carries out this physical action. In doing this, we suggest that John has folded back to do something physical that will allow him to modify his inappropriate image for the fractional scale. He returns to Image Making, and is trying to make sense of how the ruler is to be correctly read, and <u>why</u>. After counting with his pen on the ruler and reaching eight inches and three-eighths, he re-enacts this stepping process, suggesting that he is now actively making an appropriate image, and is able to count-on in fractions whilst making the correct one-to-one correspondence with the ruler scale. The physical act of working with a manipulative allows him to understand what he is doing, and towards the end of the extract it seems that he now has an image for a fraction as a point on a continuous number line, as printed on the ruler. He is able to confidently identify the other fractional units, though we do not see him count on in these. Of course we would not suggest that the image John has is now a complete one for measuring using imperial units, and recognise that he may need to fold back again and again to continue to develop his understanding. The researcher probes a little more:

Researcher: Because, that's how many of those little spaces would fit in a whole inch.

John: Right. So if we went one.

R: What are we counting now? What kind of fraction?

J: Eight, eight. Right?

R: Ok. Eighths.

J: So that's one eighth from here *(8 inch point)* to here *(8 1/8 inch point on tape measure).* That's two eighths *(pointing to the interval between 8 1/8*

> *inch to 8 1/4 inch)* and that's three eighths *(pointing to the interval between 81/4 inch to 8 3/8 inch)*. Gotyah. Ok. I understand now. Now that I can actually express it and point it out, ok.

Again in this final extract we see John confidently counting on in eighths, and recognising that he has an understanding of what he is doing. It is not totally clear here whether he fully grasps the idea of "how many of those little spaces would fit in a whole inch" that is articulated by the researcher, but would hope that the future exploration of different fractional units will help him to develop his image to be one that he can confidently use and apply whatever he is asked to measure. We suggest that he has a useful and appropriate image for imperial units of measure, based on a deeper understanding of the part-whole relationship of fractions.

CONCLUSIONS

It is beyond the scope of this paper to comment in any depth on the complex role that mathematical understandings and images play in the trades training process, but we suggest that trades educators should expect that their trainees may not come with a useful and easily applied range of images for required mathematical concepts needed in their training. It would seem that offering opportunities for apprentices to fold back and to engage in appropriate image making activities for some mathematical concepts would be an appropriate way to occasion their growth of understanding, and enable the development of more widely applicable skills. This is particularly important for those apprentices who perhaps struggled with school mathematics and come to trades training with a very limited set of mathematical images, and also possibly with a negative attitude to mathematics.

Whilst we recognise that in some ways, returning to "play around" with physical manipulatives might seem to be both a backward step and time consuming, we do believe that there is a need to re-engage with some basic mathematical concepts, but within the new context of the workplace. As Forman and Steen (1995) noted, there is a need in the workplace for "concrete mathematics, built on advanced applications of elementary mathematics rather than on elementary applications of advanced mathematics (p.228). Certainly for John, he was being asked to use relatively elementary mathematical concepts but to use these in problem solving contexts that are very different from those in which the concepts will have been taught or used in school. Of course, we also recognise that not all apprentices may need this re-engagement, but given the wide range of needs that exist in trades training, the provision of an opportunity for folding back when necessary is essential.

We see John folding back to image making in the later extracts, and through working with the set of acetate rules and having an opportunity to play around with these (and the mathematics embedded within them) he does seem to have an appropriate image for fractional units at the end of the session. This kind of activity and accompanying learning tools are invaluable for encouraging mathematical understanding that goes beyond being able to merely read a scale or operate on numbers.

Clearly, the measuring tape is a fundamental part of working in pipe trades, and the ability to use this, and to understand the mathematics that is captured by this tool is essential for a worker. Whilst we acknowledge that such understandings are not likely to be made explicit during every task, the possession of a powerful and flexible set of mathematical images related to this offers something to fold back to, should memory fail, or the need arise to work in a new application. We suggest that in the technical training classroom there is a need to re-visit concepts such as fractions and to go beyond learning merely how to operate on and with these.

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MEANING CONSTRUCTION THROUGH SEMIOTIC MEANS: THE CASE OF THE VISUAL PYRAMID

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This paper presents some elements of our study on the construction of mathematical meanings in terms of development of semiotic systems (gestures, speech in oral and written form, drawings) in a Vygotskian framework with reference to cultural artefacts (Wartofsky). It concerns with a teaching experiment on perspective drawing at primary school (4th-5th grade classes). We analyse the appropriation of an element of the mathematical model of perspective drawing (visual pyramid) through the development of gestures, speech and drawings, starting from a concrete experience with a Dürer's glass to the interpretation of a new artefact as a concrete model of that mathematical object.

INTRODUCTION

In Bartolini et al. (in press), we presented the rationale, design and early findings of a teaching experiment, carried out with 4th – 5th graders, on the mathematical modeling of perspective drawing. It is well known that early theoreticians of perspective conceived "*painting as nothing more than intersection of the visual pyramid*" (Leon Battista Alberti, *De Pictura*, 1540). The appropriation of this conception is a good opportunity for pupils to deepen spatial experience and to construct the meaning of mathematical objects, such as pyramid and intersection, as abstract entities.

THEORETICAL FRAMEWORK

The study was carried out in a Vygotskian framework, which has been gradually enriched with contributions by other authors. In short, the theoretical framework is built around three different poles: (1) the cultural-historical pole, to describe the features of technical and psychological tools; (2) the didactic pole, to describe the way of designing, implementing and analysing processes of semiotic mediation; (3) the cognitive pole, to describe the process of internalisation of interpsychological activity, that creates the plane of individual consciousness (Bartolini et al., 1996, 1999). To deepen the study of relations between technical and psychological tools ("language, various systems for counting, mnemonic techniques, algebraic symbol systems, works of art, writing, schemes, diagrams, maps, and mechanical drawings, all sorts of conventional signs and so on", Vygotskij 1974 – p. 227), we adopted Wartofsy's distinction between primary, secondary and tertiary artefacts (Wartofsky,

[1] This contribution is a part of a study carried out together with Maria Alessandra Mariotti, Franca Ferri, within the National Project PRIN_COFIN 03 2003011072.

1979, p. 200 ff). *Primary artefacts* are "those directly used" and *secondary artefacts* are "those used in the preservation and transmission of the acquired skills or modes of action". Technical tools correspond to primary artefacts, whereas secondary artefacts are representations, i.e. "physical and perceptual embodiments of a mode of action", realized by different semiotic means (e.g. gesture, speech, drawing). *Tertiary artefacts* usually correspond to objects that are described by rules and conventions and are not strictly connected to practice.

In the quoted study (Bartolini et al., in press), two hypotheses were stated and tested: 1) the Hypothesis of Polysemy: "The intrinsic polysemy of the artefact supports the production of the polyphony of voices, in classroom activities"; 2) The Hypothesis of Embodiment: "The concreteness of the artefact fosters the production of gestures, linguistic expressions and drawings; they are elements of complex semiotic systems that evolve in time, using classroom activities. These systems support the transition to secondary and tertiary artefacts". This second hypothesis has been modified slightly taking into account the importance of drawings in our experiment in accordance with the entire Vygotskian tradition (e.g. Stetsenko, 1995). The crucial educational problem is to create conditions where the polysemy of artefacts, which is related to the polyphony of classroom activities, can be internalised by pupils. The aim of this report is to defend the following thesis, which links the two hypotheses in a functional way: *the parallel development of several semiotic systems (gestures, drawings, oral and written language) produces the internalisation of the polysemy; it is led by the teacher, who draws on the concreteness of the artefact (hypothesis of embodiment).* These semiotic systems allow pupils to construct (or to appropriate) the meaning of mathematical objects, starting from an exploratory activity concerning a primary artefact, then secondary artefacts, to the mathematical model. In our study, the mathematical model is a visual pyramid. The basic elements of this model are: the construction of the pyramid (monocular vision and base), the intersection between the pyramid and plane of painting, which gives the prospective image of the chosen object, the similitude of the images obtained by cutting the pyramid with planes parallel to painting. Due to limitations of space, in this paper we only consider the first element; the second one is analysed in (Bartolini & al., in press; Bartolini & Maschietto, in press). To discuss our thesis, we analyse several excerpts of protocols from different steps of the teaching experiment, presented in the following paragraph.

THE TEACHING EXPERIMENT

The teaching experiment, which was composed of several steps (Bartolini et al., in press; Maschietto et al., 2004), began at the end of the 4th grade course (steps 1 and 2 – May 2002) and constituted a part of the mathematical curriculum of the 5th grade course (from October 2002). In this paper, we consider the steps described below.

- Step 1. Exploration of a primary artefact, i.e. a Dürer's glass (Figure 2). This is the simplest perspectograph, composed of an eyehole and a transparent screen, where the artist traces the apparent contour of the object directly. A copy of a it was explored

during a mathematical discussion (Bartolini, 1996). It is made of wood, plexiglas and metal; it has three eyeholes but only by looking through the one in the middle can you see the drawing superimposed on the skeleton of a cube inside (Figure 2). Pupils were asked to use it: each pupil was asked to look through the eyehole and compare the different images he/she could see through the different holes.

- Step 2. Pupils were asked to draw the artefact that was sitting on the teacher's desk.

- Step 3. Interpretation of secondary artefacts. During a mathematical discussion, pupils were asked to interpret some excerpts of texts drawn from ancient treatises on paintings (Piero della Francesca, L.B. Alberti). Among these excerpts, the sentence that introduced the first mathematical model for Dürer's glass, which was no longer available in the classroom, was "Thus painting will be nothing more than intersection of the visual pyramid" by L.B.Alberti (*De Pictura*, 1540).

- Step 8. Individual text on Alberti's visual pyramid: Alberti's sentence on the visual pyramid was proposed for an individual task three months after step 3.

- Step 9. Exploration and use of a new artefact (Figure 1): a model of the visual pyramid was introduced into the classroom. This model is made of wood (poles and glass with hole) and threads forming the edges of a pyramid. Each thread passes through a horizontal plane with holes and is taut: one end is fixed inside the hole and the other is attached to a weight. Another pole with some small holes in is present. A discussion followed.

Figure 1: model

In this paper we analyse the construction of the visual pyramid (vertex, faces and edges) through the development of gestures, speech (in oral or written form) and drawings. At the end, we show how a new artefact, very similar to the explored Dürer's glass in material and design, was instead interpreted by the pupils, as a concrete model of an abstract mathematical object, which highlights its polysemy.

THE ANALYSIS OF PROTOCOLS

The analysis is based on several kinds of data that were collected: individual protocols (texts, drawings); audio-recordings (and sometimes video-recordings) of classroom activities; photos of the pupils at work; and the teacher's and observer's notes.

Speech and gestures (step 1 – Discussion on Dürer's glass)

The focus on the functioning of the model evoked previous experiences with other objects with a monocular vision (e.g. camera, video camera) and allowed the pupils to pay attention to the importance of the point of view, which was the position that implies coincidence between the drawing and the cubic frame.

AleB I think there is only one hole as you can see the perspective better with one eye. If you look at it with two eyes it is different.

Fede Because if I keep my eyes open, and now I place my fist in front of the edge of this ... of this machine, I get confused between my hand and the wooden stick and I don't understand anything. If I place my fist against the edge and close one eye I can see both perfectly.

Figure 2: exploration of artefact

Figure 3: Ange's drawing

Analysis of the discussion shows that the exploration of the perspectograph fosters the production of linguistic expressions and gestures, such as: closed eye/open eye, and index finger (Figure 2) to explore the relationships between the drawing on the glass and a real object (cubic skeleton). They contribute to the construction of simple operational schemes of the artefact. Since they develop in the context of a mathematical discussion (that is, a situation of interaction), they can also be considered the germs of a secondary artefact aimed to the transmission of modes of action.

Drawings (step 2 – Individual drawings of the perspectograph)

In Figure 3, the evoked modes of action are: pupil position with respect to the three holes indicates the correct choice of central ocular (superimposition of the drawing on the glass over the cubic skeleton); the position of the pupil's arm is similar to the position in Figure 2. This drawing also shows the result of the seeing action through the chosen eyehole: the contour of the red cubic skeleton is black, which corresponds to the real view through the central eyehole. Figure 4 contains the identified elements in a different way: the choice of the eyehole (and so the singleness of the point of view) is demonstrated by the sentence "the bigger hole (...)" and corresponding drawing; the result of the seeing action is drawn with a comment indicating the mode of action ("If I look through the eyehole, I see the cube fits together with the contour"). Figure 5 shows the use of a single eye, with the choice of the central hole. Some of these drawings can be interpreted as secondary artefacts, as they present modes of action of the perspectograph. They also contain gestures that appeared in the previous activity.

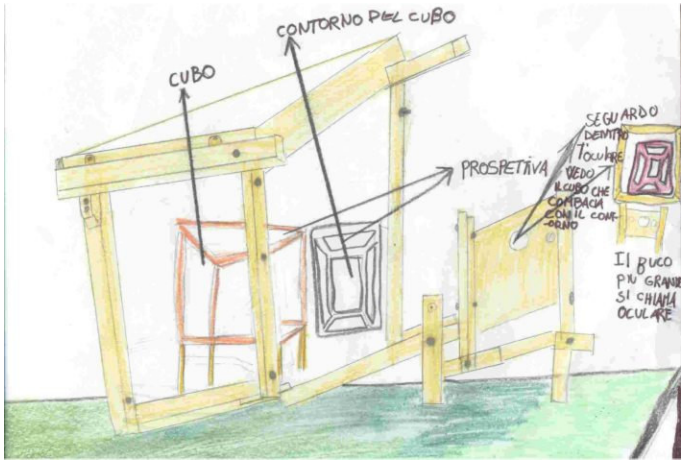


Figure 4: Ales's drawing

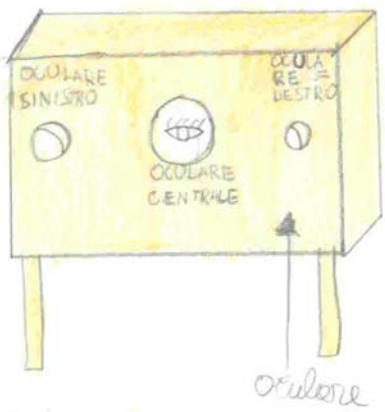


Figure 5: Carl's drawing

Speech and gestures (step 3 – Discussion on Alberti's sentence)

AleB If the base is triangular it has 4 [faces], if the base is square it necessarily has 5. It depends on the base. The one we are talking about has either a square or a rectangular base, because we imagine a painting or a piece of glass and the point of the triangles reaches the eye.

Fede Yes, but Leon Battista Alberti's is not a real solid, it's an imaginary solid which takes shape while you're looking at it. We can't see it we can see it only when we think of it, if we want to see it. For example we can see it now because we have just read it.

Assia Of course it's imaginary, otherwise it would harm you and then it wouldn't even allow you to see.

Voices Can you imagine a solid getting into your eye!

[*Many gestures, funny ones as well! A moment of confusion and jokes about the visual pyramid with participation of the entire class*].

Pupils' statements referred to two elements of the visual pyramid: the arbitrary base and the vertex entering an eye. So, the single eye used for the perspectograph takes on the new role of vertex (into play between real and imaginary object).

Drawings and text (step 8 – Individual comments on Alberti's sentence)

Individual protocols on Alberti's sentence seem to reveal that the visual pyramid had been internalised. In pupils' drawings, gestures that mimed planes and lines during the discussion of step 3 became very precise signs. They bore traces of the single eye arising from both the use of the perspectograph and the appropriation of Alberti's sentence. For the first case, the traces are: looking with one eye closed and the other open (Figures 6 and 7), the gesture of closing one eye was explicitly evoked by the drawing and accompanying comment (Figure 9, where the pupil raised her hand to close her eye). In his drawing (Figure 7), Giac gave an explicit reference to his experience with the real instrument, because he labelled the open eye as an "eyehole". In the second case, the eye was the vertex of the pyramid (Figure 6, 7, 8 and 9), which was a visual one in the pupils' comments; so that eye was the only one considered.

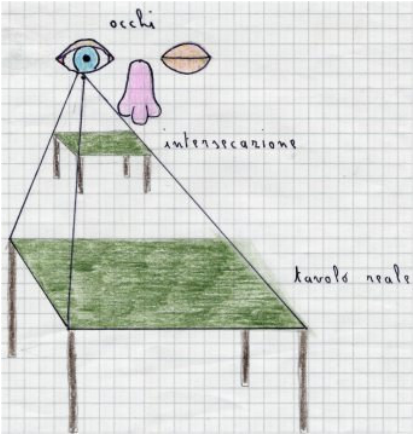


Figure 6: AleB's drawing

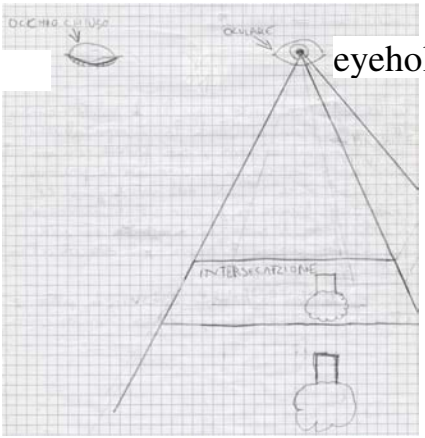
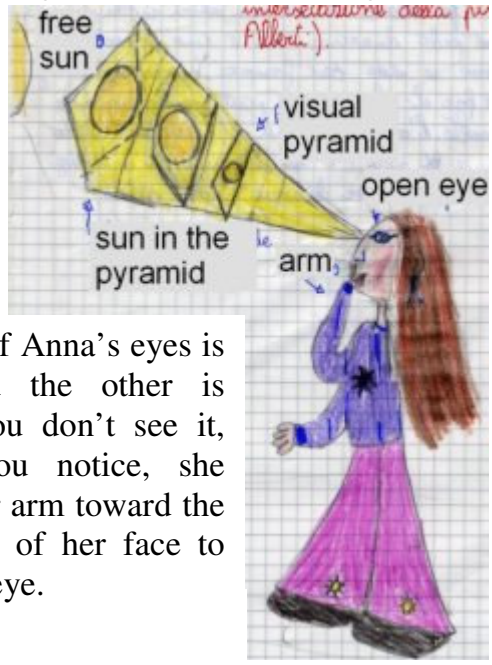


Figure 7: Giac's drawing

Figure 8: Elis's drawing



(...) One of Anna's eyes is open and the other is closed; you don't see it, but if you notice, she moves her arm toward the other side of her face to close her eye.

Figure 9: Ange's drawing and text

Speech and gestures (step 9 – Discussion on a model of the visual pyramid)

At the beginning, the teacher encouraged the pupils to explore the artefact (Figure 1).

AleB	All [the threads] go to the same point.
Teacher	Exactly. Very good, Ale. All [the threads] go to the same point, which must be the vertex ... of what?
Voices	Of the pyramid [*very quietly*] [...]
Voices	To make the intersection.
Teacher	(...) And so, where must the upper vertex of the pyramid be?

[*Silence among the pupils*]

Daniele	The vertex of the pyramid must be [*his arm reaches the glass with the hole*] on the other side of the visual pyramid, so more or less here [*his hand indicates a space beyond the glass*] ... here [...]
Maru	Franca, in this [*he touches the pole with the holes*] without [*he indicates the glass with the hole*] you can see it from different points of ... [...]

Maru	Of view
Elis	Before, we said that it [*upper vertex*] was here (Figure 10) since it was a point [...]
Voices	Imaginary
Marc	The pyramid is ... is continued by the eye when you see through the eyehole, but it is imaginary, it is not seen.
Ale	It is as if those threads [*left hand raised to his eye*] formed your imaginary pyramid.
Anna	To understand the pyramid.
Ale	The one you would make with your eye.

Figure 10: Elis's gesture

Figure 11: seeing with the model

Although the exploration of the object was physical and accompanied by new gestures (referring to edges, faces and vertices of that pyramid built by threads), the pupils seemed to appropriate it as a secondary artefact. They considered it as a model (statements by Marc, Anna and Ale). Gestures related to the process of appropriation of the artefact (for instance in Figure 10, Elis pulled the "virtual" threads in the air, beyond the glass with the hole), determined the constitution of modes of action (Figure 11), which was different from the perspectograph one (Figure 2). In Figure 11 pupil did not put his eye near the hole (as in Figure 2), but he placed it a few centimetres away, where the vertex of the pyramid must be. During this process, the pupils made gestures that were present in the previous activities: for example, during his speech ("It is as if those threads (left hand raised to his eye) formed your imaginary pyramid"), Ale raised his right hand to his eye to indicate the position of the vertex. After these statements, another pupil, Maru, suggested some changes to the model to the teacher: by evoking the perspectograph, he suggested inserting a new glass to obtain a different intersection. This discussion presents two generalisations: 1) changes to the point of view (vertex) and pyramid base (choice of different polygons); 2) intersection between the glass and threads (the glass used in different positions). These facts reinforce the role of the model of this artefact for pupils.

CONCLUDING REMARKS

In this paper, we have considered the relationship between the Hypothesis of Embodiment and the Hypothesis of Polysemy stated in Bartolini et al. (in press). The internalisation of the polysemy is achieved through the parallel development of several semiotic systems. It starts from activities concerning primary artefacts, then secondary ones to mathematical models. During this long process, teacher mediation plays a crucial role. The analysis of pupils' protocols highlights not only the parallel development of different semiotic systems (gestures, speech in oral and written form, drawings), but also their mutual enrichment as time goes by. They are not independent of one another, but complementary. For instance, the text is not just a simple comment on a drawing (and vice versa), but text and drawing contribute to meaning construction. In a lot of protocols, the drawings seem richer than the accompanying written text when present. This fact is consistent with Stetsenko's claim (1995) of the importance of drawing in pupils' evolution. Gestures in the air simulated points (e.g. the vertex), straight lines (e.g. the threads extended in visual rays), planes (e.g. the faces of the pyramid are mimed by hands touching two threads simultaneously). They were connected on one hand to pupils' concrete experiences, on the other hand to the abstract and general mathematical model of the visual pyramid (for other details, Bartolini & Maschietto, in press).

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STUDENTS' MOTIVATIONAL BELIEFS, SELF-REGULATION STRATEGIES AND MATHEMATICS ACHIEVEMENT

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In this study we examine the relationships between motivational beliefs, self-regulation strategies use, and mathematics achievement in Cypriot pre-service teachers. Specifically, we developed a model depicting connections and causal relations among cognitive and affective factors, which was tested on the basis of self-report data collected from 194 pre-service teachers using a modified version of MSLQ and a mathematics achievement test. We found that the data fits the theoretical model very well, meaning that the model explains the structure of the above relationships, with self-efficacy being a strong predictor of mathematics achievement and self-regulation strategies use having a negative effect on achievement.

INTRODUCTION AND THEORETICAL FRAMEWORK

Social cognitive theory (Bandura, 1986) has provided a theoretical basis for the development of a model of self-regulated learning in which personal, contextual and behavioral factors interact in such a way as to give students an opportunity to control their learning. Within this framework, Pintrich (1999) describes self-regulated learning as an active, constructive process whereby learners set goals for their learning plan actions and monitor, regulate and control their cognition, motivation and behavior. These actions are guided and constrained both by their goals and the contextual framework and can mediate the relationships between individuals and the context and their overall achievement (Zimmerman, 2000).

Using a theoretical framework for conceptualizing student motivation, Pintrich and De Groot (1990) proposed that there exist three motivational components that may be linked to the three corresponding dimensions of self-regulated learning, namely: (a) an *expectancy component*, which refers to students' beliefs about their expected success in performing a task, (b) a *value component,* which concerns students' appreciation of and beliefs about the importance of the task for them and (c) an *affective component,* comprised of students' emotional reactions to the task. In line with that, motivational components were found to be significantly linked to students' cognitive engagement and academic performance in the classroom, with mastery goal orientation strongly related to the use of cognitive strategies, self-regulation and self-efficacy (Pintrich & De Groot, 1990).

Self-regulation has been found to be positively correlated to achievement, with highly self-regulated students being more motivated to use planning, organizational, and self-monitoring strategies than low self regulated students (Pintrich & De Groot,

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 321-328. Melbourne: PME.

1990). Pintrich and his colleagues (1994) have articulated a model of student cognition, which argued that students regulate their cognition by using motivational strategies in addition to cognitive and metacognitive strategies. Pintrich and De Groot (1990) found a positive correlation between motivational beliefs and self-regulated learning and furthermore, all affective components were related to academic performance. In line with these findings, Schunk and Zimmerman (1994) reported that there was a positive relationship between self-efficacy and academic achievement and that if students are trained to have higher self-efficacy beliefs their academic performance also improves.

THE PROPOSED MODEL

The proposed model is based on the theoretical assumption that views motivational beliefs as factors that may help to promote and sustain different aspects of self-regulated learning. Moreover, how these affective factors, that are the motivational beliefs and the self-regulation strategies use, may influence mathematics achievement (Pintrich, 1999). In other words, in the context of mathematics learning and teaching, how do self-regulation strategies influence the development of positive motivational beliefs and especially self-efficacy and how self-regulation strategies use and motivational beliefs influence mathematics achievement of pre-service teachers.

The model for self-regulated learning described in the present study comprises of two main components. The first refers to self-regulation strategies use and the second to motivational beliefs. In addition, the self-regulation strategies involve cognitive learning strategies and self-regulatory strategies to control cognition (metacognitive learning strategies) (Garcia & Pintrich, 1994). Cognitive learning strategies consisted of elaboration and organizational strategies. Elaboration strategies include paraphrasing or summarizing the material to be learned, creating analogies, generative note taking and connecting ideas in students' notes. The organizational strategies include behaviours such as selecting the main idea from text, outlining the text or material to be learned, and using a variety of specific techniques for selecting and organizing the ideas in the material (Garcia & Pintrich, 1994).

Students' metacognitive and self-regulatory strategies can have an important influence upon their achievement. Self-regulation would then refer to students' ability to setting goals planning activities, monitoring progress, controlling, and regulating their own cognitive activities and actual behaviour (Pintrich et al., 1993). Planning activities include analysis of the task, choosing strategies and making decisions on specific behaviours. Monitoring stands for comparing progress against goals or standards in order to guide the following actions. For instance, a type of self-regulatory strategy for reading occurs when a student slows the pace when confronted with more difficult or less familiar text (Tanner & Jones, 2003).

In the present study we have focused on three general types of motivational beliefs: (a) self-efficacy beliefs, (b) task value beliefs, and (c) goal orientation. The construct self-efficacy beliefs stands for a student's sense of ability to plan and execute actions

to achieve an academic goal (Bandura, 1997). Self-efficacy also represents students' confidence in their cognitive and learning skills in performing a task. Task value beliefs refer to students' evaluations about the importance and usefulness of the task. Task value beliefs in mathematics could refer to beliefs that mathematics is useful in academic settings (other school subjects) and in finding a good job (Hannula, 2002). Goal orientation is discussed on two dimensions, named mastery goal orientation and extrinsic orientation. A mastery goal orientation refers to a concern with learning and mastering the task using self-set standards and self-improvement. Extrinsic orientation refers to expected reward or avoiding punishment, as the main criterion for investing resources e.g., pleasing teachers or parents (Pintrich, 1999).

Pintrich's research indicated that there are strong relationships between motivational beliefs and self-regulation strategies use. More specifically, in terms of self-efficacy, the findings showed positive correlations between self-efficacy and self-regulated learning (Pintrich & Garcia, 1991). Students who felt more efficacious with respect to a certain task or course were more likely to report using all types of cognitive strategies to succeed in pursuing the task. It has also been reported that self-efficacy was positively related to self-regulatory strategies use and strongly related to academic performance (Pintrich & De Groot, 1990). In addition, task value beliefs were correlated positively with cognitive strategy use including elaboration, and organizational strategy. Task value was also correlated to performance, albeit these relations were not as strong as those for self-efficacy (Pintrich, 1999).

Research in the field of goal orientation resulted in consistent relations between the different goals and self-regulation (Tanner & Jones, 2003). As reported by Pintrich (1999), mastery goal orientation was positively related to the use of cognitive strategies as well as self-regulatory strategies. In addition, mastery goal orientation was positively related to actual performance in the class. On the contrary, extrinsic goal orientation was consistently found to be negatively related to self-regulated learning and performance.

In arguing that motivational beliefs and self-regulation strategies use formulate a model for self-regulated learning, which influences mathematics achievement, we propose the following latent factors; self-efficacy, mastery goal orientation, extrinsic goal orientation and task value beliefs represents the motivational beliefs factors. With regard to self-regulation strategies use, cognitive strategies, a second order factor, is comprising two variables: elaboration and organization strategies. Self-regulation strategies use, represented by a high order latent factor, consisted by cognitive strategies and metacognitive self-regulation strategies.

We hypothesise that self-efficacy and self-regulation strategies use affect directly mathematics achievement, while mastery goal orientation, extrinsic goal orientation and task value affect academic achievement in mathematics via their effects on self-efficacy. We further assume that achievement is related to the cognitive strategies that students use to learn and that motivational beliefs and self-regulation strategies

use constructs influence students' cognitive engagement and achievement in mathematics.

Based on these assumptions, the purpose of the study was to test the prediction of a causal model that explains the impact of self regulated learning, which encompasses students' motivational beliefs and self-regulation strategies use on their achievement in mathematics. More specifically, the present study addresses the following questions: (a) what are the relationships between mathematics achievement and self-regulation strategies use and motivational beliefs? (b) What is the predicting value of the proposed model for self-regulated learning on mathematics achievement? The above-mentioned questions are addressed through the estimation of a theoretically informed causal model in which the hypothesized relationships between the factors (motivational beliefs, self-regulation strategies use and mathematics achievement) are decomposed through the introduction of mediating second order latent factors. The proposed model is estimated using a Structural Equation (MPlus) program.

METHOD

Participants and Measures

The analysis in the present study was based on data collected from 194 sophomore pre-service teachers who attended a course on mathematics and its didactics during the autumn semester in 2004. The 26-item questionnaire used in this study was based on the MSLQ questionnaire (see Pintrich et al., 1993). Responses were recorded on a 7-point scale with extreme points labelled *strongly disagree* (1) and *strongly agree* (7). Achievement in mathematics was measured using students' score in their midterm examination. The questionnaire and the test were distributed on two days during the semester. Using listwise deletion of missing values in drawing the variables that were needed for the analysis, the final sample included 187 students. Of these 18% were males and 82% females.

Variables

The tested model contained both observed (measured) variables and latent constructs. The observed variables were specified as indicators for each of the latent constructs. One of the first order factors was measured by six indicators (Self-Efficacy (SE)), two factors were measured by four indicators (Task Value (TV) and Elaboration (El)) and four of the first order factors were measured by three indicators each (Mastery Goal Orientation (MGO), Extrinsic Goal Orientation (EGO), Organization (Org) and Metacognitive Strategies (MCS)). The Cognitive strategies (CS) latent factor was measured by the two first order factors Elaboration (El) and Organization (Org). Finally, the high order factor Self-Regulation Strategies use (SRSu) was measured by Cognitive Strategies (CS) and Metacognitive Strategies (MCS). Table 1 is a brief description of the latent constructs with specimen items (For a full set of items you can contact the first author of the paper).

RESULTS

In this study, 26 indicators were hypothesized to represent one high order, one second order and nine first-order factors (see Table 1). The parameter estimates were reasonable and all factor loadings were large and statistically significant. Moreover, the loadings of the Elaboration and Organization factors that comprised the Cognitive Strategies factor were also large and statistically significant. Finally, the loadings of the Cognitive and Metacognitive Strategies factors that comprised the high order Self-Regulation Strategies Use factor were also large and statistically significant. The goodness-of-fit index was good in relation to typical standards (e.g., Comparative Fit Index (CFI) for the total sample .923, which indicates a ''good'' fit; X^2 = 450.907, df = 303 and RMSEA = 0.056).

Factors	Items
Mastery Goal Orientation (MGO)	In a class like this, I prefer course material that really challenges me so I can learn new things.
Extrinsic Goal Orientation (EGO)	Getting a good grade in this class is the most satisfying thing for me right now.
Task Value (TV)	Understanding the subject matter of this course is very important to me.
Self-Efficacy (SE)	I am certain I can understand the most difficult material presented in the readings for this course.
Elaboration (Elab)	When reading for this class, I try to relate the material to what I already know.
Organization (Org)	When I study the readings for this course, I outline the material to help me organize my thoughts.
Metacognitive Strategies (MCS)	Before I study new course material thoroughly, I often skim it to see how it is organized.

Table 1: Factors and Specimen Items of the Proposed Model.

Specifically, all indicators loaded strongly and distinctly on each of the latent constructs, with the standardized loadings being all above 0.46. As Figure 1 shows, the loadings of the two first order factors comprising the Cognitive Strategies second order factor were statistically significant and above 0.50. These results indicated that the hypothesized structures can be adequately represented through the one second order factor and the two first order factors.

A standardized solution for the model is presented in Figure 1. The results showed that self-efficacy is a good predictor of mathematics achievement; on the contrary self-regulation strategies use showed a statistically significant negative effect on

mathematics achievement. It was also found that only task value beliefs constitute predictive factor of self-regulation strategies use, while on the contrary mastery and extrinsic goal orientation do not constitute predictive factors, since their loadings to self-regulation strategies use were not statistically significant. The fact that task value beliefs have a high effect on self-regulation strategies use shows that students who believe in the value of mathematics tend to use different cognitive and metacognitive strategies while working on mathematical tasks. The results of the study confirmed that mastery goal orientation, that is, intrinsic goals can predict a student's self-efficacy. Its causal effect on self-efficacy was very high, being .846. This finding indicates that mastery goal orientation is a strong predictive factor of self-efficacy and therefore has an indirect effect on achievement through self-efficacy.

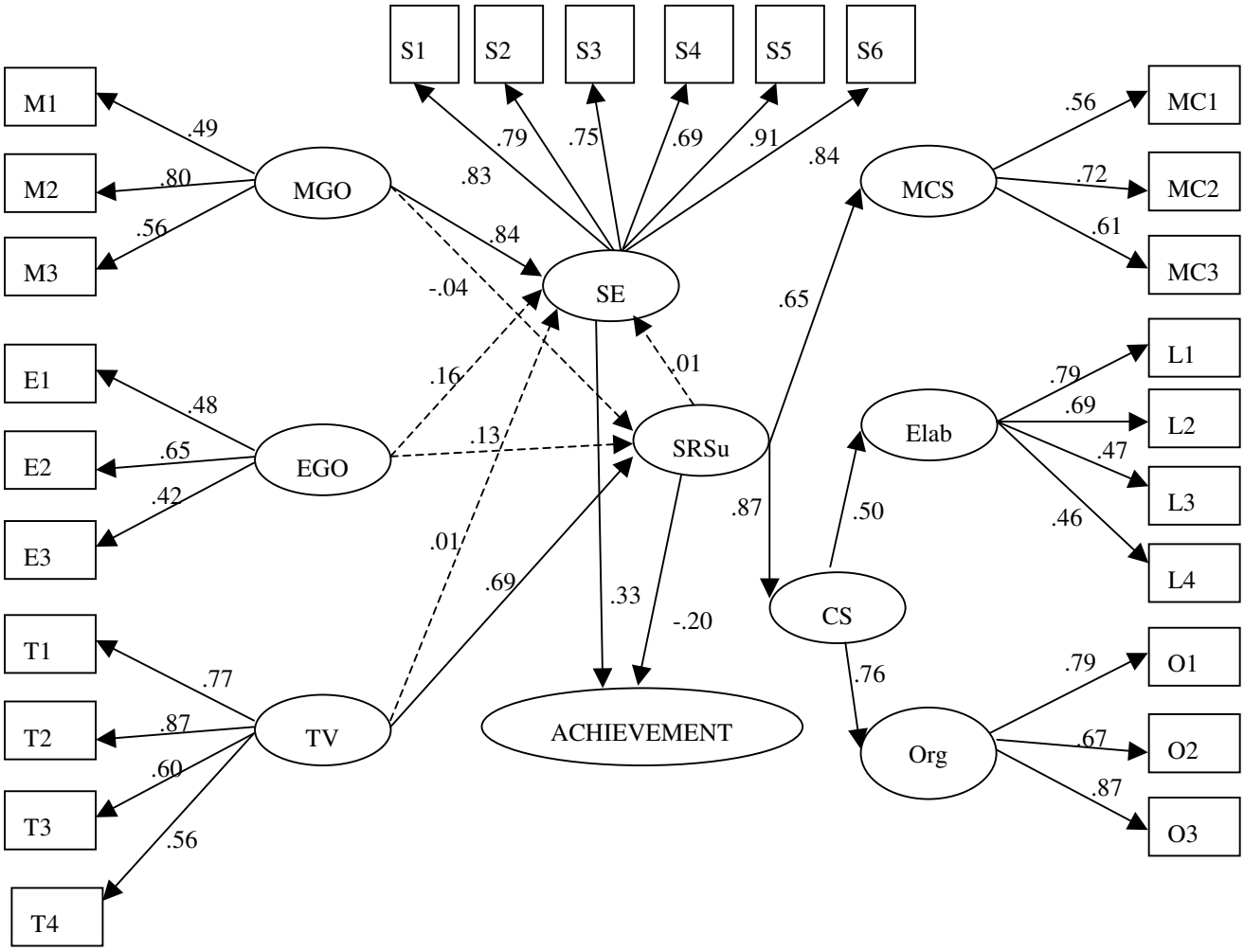

Figure 1: The proposed model with regression coefficients

Note: Regression values between latent factors appear in bold. Not statistically significant regressions appear in dotted line.

The main finding of the present analysis is that the developed model fits very well. This indicates that, first, the factor loadings reflect the relations between the particular indicators and the corresponding latent constructs and, second, the hypothesized constrained- data-based model can adequately explain the structure of

the relationships between mastery goal orientation, self-efficacy, cognitive strategies, metacognitive strategies, and mathematics achievement. The second finding of the study is that self-efficacy is a strong predictor of mathematics achievement, while on the contrary self-regulation strategies use is a moderately negative predictor of achievement in mathematics.

DISCUSSION

The purpose of this study was to examine the relationships between motivational beliefs, self-regulation strategies use and mathematics achievement in Cypriot pre-service teachers. Consistent with our hypothesis and Pintrich and De Groot's findings (1990), we found that self-efficacy was a strong predictor of academic performance in mathematics. It seems evident that students with strong self-efficacy beliefs, mastery goal orientation and high task value, employ different kinds of cognitive and metacognitive learning strategies more actively and they are more sensitive to regulate their motivational beliefs (Pintrich, 1999). This finding is in line with efficacy theory (Bandura, 1986) and suggests that self-efficacy is adaptive and do help to promote and sustain self-regulated learning. However, there is need for more developmental and longitudinal research on how different motivational beliefs can influence self-regulation strategies use and at the same time how the use of various cognitive and metacognitive strategies influence students efficacy for learning, interest in the task, and the goals they adopt in mathematics (Tanner & Jones, 2003).

On the contrary, we found that self-regulation strategies use had a moderate negative effect on mathematics achievement. We particularly expected to replicate the results of studies which found a positive relationship between mathematics achievement and self-regulated strategies use (Pintrich & DeGroot, 1990). A possible reason for this may be the fact that pre-service teachers in Cyprus were found to hold high beliefs regarding their self-regulation strategies use regardless of their ability (Christou, Philippou & Menon, 2001). However, the present study is more specific than previous ones in that it examined a model for self-regulation in the domain of mathematics and explored how motivational beliefs like mastery goal orientation, task value and self-regulation strategies use affect the formation of self-efficacy and therefore mathematics achievement. In a more technical aspect, the contribution of the study can be drawn from the framework in which the analysis was conducted. The fact that the model containing the examined factors was validated using confirmatory factor analysis, ensured that motivational and self-regulation factors were correctly measured and therefore any relations that appeared hold true.

In this context, the results provide a strong case for the role of educators in the formation of pre-service teachers' self-efficacy beliefs both as students and as future teachers. On a practical level, prospective teachers should be well trained self-regulators so as to be able to teach these skills, since instructions to monitor the very early stages of skill acquisition may disturb the learning process (Bandura, 1997). In line with this, Boekaerts (1997) recommends that teachers should be trained to create

effective environments in which students can learn to regulate their learning process and design tasks that help students improve their planning, organizational and metacognitive abilities.

Finally, a possible direction for future research in the area could be the investigation of the relationships between motivational beliefs and self-regulation strategies use in non traditional mathematics classroom settings, where teaching and learning should be designed to improve motivation or self-regulated learning (Pintrich, 1999).

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THE TRANSITION TO POSTGRADUATE STUDY IN MATHEMATICS: A THINKING STYLES PERSPECTIVE

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In this paper we focus on the transition from undergraduate to postgraduate study in mathematics through the lens of the construct 'thinking style' as defined by Sternberg (1997). A cross-sectional study (N=54) was conducted in the Department of Mathematics of a large university in Greece. The data analysis reveals some statistically significant (though far from straightforward) stylistic differences between the undergraduate mathematics students and those who follow a taught graduate mathematics degree: the undergraduates appear to have a stronger preference for details, concreteness and conformity in their thinking than the postgraduates.

INTRODUCTION

The students' problematic transition through the various stages of studying mathematics is an issue that has attracted the interest of several mathematics educators. Students encounter various problems when the level and nature of mathematics changes (e.g., transition to algebra, Kieran, 1991) or when they move to a higher stage of education (e.g., from school to university, Alcock, 2001). Indeed, these issues have been approached from different perspectives including the affective (Daskalogianni & Simpson, 2001), the cognitive (Pinto, 1998) etc.

Although there is a wide body of research looking into the transition from school to university or into transitional issues within school and undergraduate university mathematics, the transition to postgraduate study in mathematics is less well explored. Most of the studies investigating this specific transition are focused on moving to doctoral study (e.g., Duffin & Simpson, 2002), which is particularly interesting for mathematics educators as, among other issues, it involves a dramatic change in the didactical contract (in the sense of Brousseau, 1997).

However, it seems equally important to find out more about the students that choose to follow a *taught* postgraduate degree in mathematics, which may be less influenced by extreme pedagogic changes. The requirements of entering such a degree ensure that most of these students belong to those with higher undergraduate performance. Identifying special characteristics of this population will provide us with further insight into the outcomes of the existing undergraduate education system. Moreover, since the vast majority of the research students in mathematics will have completed a taught postgraduate degree in mathematics, knowing more about the special characteristics of this population will help our understanding of the transition to doctoral study.

The aim of this study is to shed some light on this population from the perspective of cognitive styles: What are the stylistic differences and similarities between

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undergraduate mathematics students and the students who choose to follow a postgraduate taught programme in mathematics?

STYLES, STRATEGIES AND APPROACHES

We wish to contrast two different theoretical constructs within the general notion of an 'approach to study': *style* and *strategy*. Marton & Säljö (1976) describe an approach to study as the way people choose to react in encountering a study situation. These have been classified as 'deep', 'surface' or 'achieving' based on the level of learning they are thought to promote (respectively deep, surface or focused on performance, which leads to a variable level of learning; Biggs, 2001). However, there is some debate about the extent to which approaches to study can be either stable across many different tasks or task specific – that is, the distinction between style and strategy.

The construct of cognitive *style* has been widely researched in psychology (for a review, see Rayner & Riding, 1997). It can be defined as "an individual's characteristic and consistent approach to organising and processing information" (Tennant, cited in Riding, 1997). Although there appear to be various conceptualisations of cognitive styles (for a classification, see Sternberg & Grigorenko, 1997), most of the researchers agree that cognitive style is a construct which is relatively stable over domain and time.

Strategies, however, are employed by the students in order to cope with a specific task (Adey, Fairbrother, Wiliam, Johnson & Jones, 1999). The main difference between style and strategy is that the style describes a general preference, whereas a strategy refers to a specific choice made and, hence, is dependent on several factors e.g. the nature and purpose of the task, time, place etc.

In this study, we have chosen to focus on cognitive styles, as we are interested in characteristics of the population that are relatively stable and not in task-specific behaviour. More specifically, we consider one manifestation of cognitive styles, namely *thinking style*.

THEORETICAL FRAMEWORK

Thinking styles are defined as the "preferred ways of using the ability one has" (Sternberg & Grigorenko, 1997, p.700). While some mathematics educators, who examined graduate study, have concentrated more generally on cognitive styles (e.g., Duffin & Simpson, 2002), focussing more narrowly on Sternberg's thinking styles seems to be more suitable for this study as they derive from a coherent, clearly structured theory: the notion of *mental self-government*. Sternberg (1997) draws parallels between the way that the individuals organise their thinking and the way that society is governed and identifies thirteen thinking styles, grouped in five dimensions: *function*, *forms*, *levels*, *leanings* and *scope* of mental self-government (see Table 1).

These thinking styles, although relatively stable, are considered to be largely shaped by the individual's interaction with the environment and, thus, they are subject to medium to long term change (Sternberg, 1997). Furthermore, thinking styles can be measured by an instrument that has shown its validity and reliability in various studies and countries: the Sternberg-Wagner Thinking Styles Inventory (TSI[1]; Sternberg, 1997).

Dimension	Thinking Styles
Function	**Legislative** (preference for creativity) **Executive** (preference for implementing rules and instructions) **Judicial** (preference for judging)
Forms	**Monarchic** (preference for focusing on only one goal) **Hierarchic** (preference for having multiple objectives sequenced according to their priority) **Oligarchic** (preference of having multiple equally important targets) **Anarchic** (preference for flexibility)
Levels	**Local** (preference for details and the concrete) **Global** (preference for the general and the abstract)
Leanings	**Liberal** (preference for novelty and originality) **Conservative** (preference for conformity)
Scope	**Internal** (preference for working alone) **External** (preference for working in a group)

Table 1: Dimensions of thinking styles (Sternberg, 1997).

We argue that there are two competing views about the influence of undergraduate study on the thinking styles of graduate students. Duffin and Simpson (2002) looked into the transition to doctoral study in mathematics from a cognitive styles perspective. They suggest that, among the styles they identified, the existing undergraduate educational university system, with the rapid delivery of new material, might favour students who have an *alien* style (preference for absorbing new information without any particular short-term search for links with existing knowledge). Sternberg's idea suggests that being in such an environment over the long period of undergraduate study might have some implications on the students' thinking styles. Thus, one conjecture is that the postgraduates, being that part of the population who most successfully survived university, might be more 'executive' or 'local' (Sternberg's terms for styles apparently closest to Duffin and Simpson's 'alien') than the undergraduate population as a whole.

However, Zhang and Sternberg (2001) report significant correlations between approaches to study and thinking styles and Biggs (2001) notes that although most of the undergraduates "become increasingly surface and decreasingly deep in their

[1] Though the TSI dates back to 1992, Sternberg (personal communication, June 10, 2004) suggested using the version of the TSI included in his 1997 book.

orientation to learning […] students with aspirations for graduate study do not show this pattern in their chosen area of study" (*ibid*, p. 91). Bearing in mind the differences between approaches and style, it can be hypothesised that a similar phenomenon might occur with the differences in thinking style. Thus, a second conjecture is that the postgraduates might be more legislative, liberal or judicial than the undergraduate population as a whole, since these styles were found to be correlated to a deep approach to study (e.g., Zhang & Sternberg, 2001).

METHODOLOGY

For this study, we used the Sternberg-Wagner Thinking Styles Inventory (TSI; Sternberg, 1997). This is a self-report, paper-and-pencil test, consisting of 104 seven-scale Likert type items (eight for each style). Each participant's preference for a style is labelled (six labels ranging from 'very low' to 'very high') according to the norms developed by Sternberg's research, which varies according to the participant's gender and education (collegiate and non-collegiate). As a norm referenced test, TSI does raise some cross-cultural validity and reliability issues, since there appears not to be a published norm for Greek university students. However, the cross-cultural validity of the TSI has been generally demonstrated by previous studies (e.g., Zhang, 1999).

TSI was independently translated and back translated from English to Greek by three individuals. The translated TSI ('t-TSI') was piloted and further refined before it was administered. Note that, in this study, we decided not to examine the scope thinking styles dimension (internal-external), as we were not interested in the students' thinking preference, as far as working alone or with others is concerned. All the participants were asked for their written consent and were subsequently informed about their thinking styles profile if they requested it.

It was decided that the participants' scores would be labelled both according to Sternberg's norm ('Sternberg's labels') and according to a norm ('adjusted labels') produced by the data of this population following Sternberg' process[2]. This provides us with two 'lenses': a wide-angled lens which allows us to see the participants against a nominal 'general population' and a tighter lens which allows us to see differences within sub-populations of mathematics students.

THE PARTICIPANTS OF THIS STUDY

This study was conducted in a large Greek university. Overall 54 students participated (Table 2) divided in two equal groups of undergraduate and postgraduate students. The undergraduate group ('BSc') is more heterogeneous consisting of students of various year groups, with varying interests in mathematics (the Greek educational system produces a large number of students entering a mathematics department without this being either their first or second choice).

[2] The Kolmogorov-Smirnov test statistically supports the idea of using the adjusted labels by demonstrating the normality of the data from this sample.

		BSc	MSc	Total
Age	Mean	22.09	24.96	23.6
	Median	21	25	24
	St. Dev.	2.55	1.8	2.63
Male		11	15	26
Female		16	12	28
Total		27	27	54

Table 2: The participants of this study.

On the other hand, the postgraduates group ('MSc') is more ability, age and interest homogeneous: they all have an above average grade for their BSc Mathematics (mean 68.87%, median 68.40%, st. dev. 0.473). In order to minimise the pedagogical effect of the taught graduate programme, all postgraduate students were in the first semester of their two-year MSc.

THE UNDERGRADUATE-POSTGRADUATE CONTRAST

In this experiment, the t-TSI was found to be both internally consistent[3] and the interscale Spearman correlation matrix shows that, in general, this use of the instrument corresponds to the theory[4]. The construct validity of the t-TSI was examined[5] and accorded well with previous studies e.g., Zhang & Sternberg (2001). Overall, the t-TSI shows good cross-cultural validity and reliability.

Recall that, since the aim of this study is to explore the stylistic differences and similarities between the undergraduate and postgraduate students of mathematics, we were able to look at the data through two lenses – the general population norm and the intra-population norm.

Comparing against Sternberg's norms, the Mann-Whitney test, conducted for all the 11 measured thinking styles, revealed statistically significant differences in only one area: the conservative thinking style (z= -3.215, p< .001), where there is a very significant difference. A closer look into the frequencies of the conservative labels shows that the vast majority of the undergraduates (70.37%) have a 'very high' preference for a conservative style of thinking, but so do a modal number of postgraduates (Figure 1a).

However, with the adjusted labelling process, we are able to describe how the participant's score compares to the scores of the other participants, which is at the

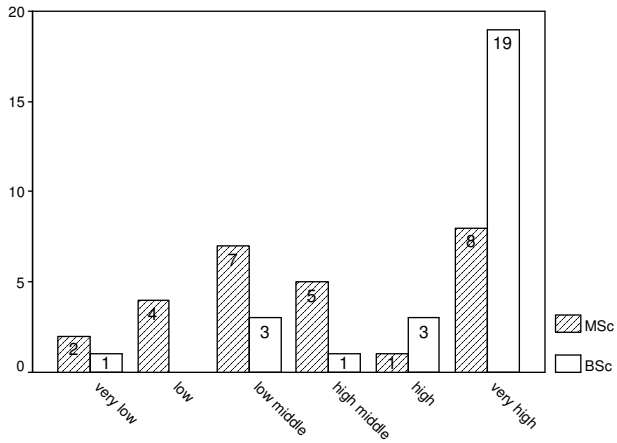

Figure 1a: Sternberg's 'Conservative' Labels.

[3] The alpha coefficients for 8 of the 11 measured thinking styles are above 0.7, which suggests the internal reliability of a test (Muijs, 2004). The three styles that were measured below 0.7 (but over 0.62) were also found to be relatively less consistent in previous studies (e.g., Zhang & Sternberg, 2001).

[4] For example, Sternberg's conservative and liberal were negatively correlated (Spearman's rho= -.309, p= .023).

[5] A principal component factor analysis led to a three factor solution with eigen-values greater than one (accounted for the 61% of variance). The first factor is related to creative and independent thinking (liberal, legislative, anarchic and oligarchic), the second factor is linked to procedural and already tested thinking (executive, conservative, local), and the third factor is related to a methodical and critical thinking (judicial, hierarchic, monarchic and global).

crux of answering the question at the heart of this research.

A look at the adjusted label frequencies (Figure 1b) can help in examining our population in more detail. The adjusted labels enable us to 'zoom in' to the 'very high' grouping found with the Sternberg's norm. The Mann-Whitney test was conducted for all the measured styles. With this more tightly focussed lens, significant differences were found for the conservative style (z= -3.260, p< .001) and for the local style (z= -2.204, p< .05). Thus, with this view the undergraduates are still significantly more conservative than the postgraduates, but they do not accumulate in just one label, but we also notice a difference between the groups for the local style (see Figure 1c).

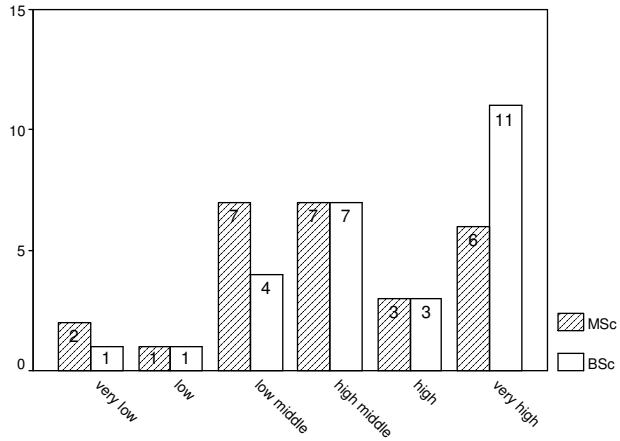

Figure 1b: Adjusted 'Conservative' Labels.

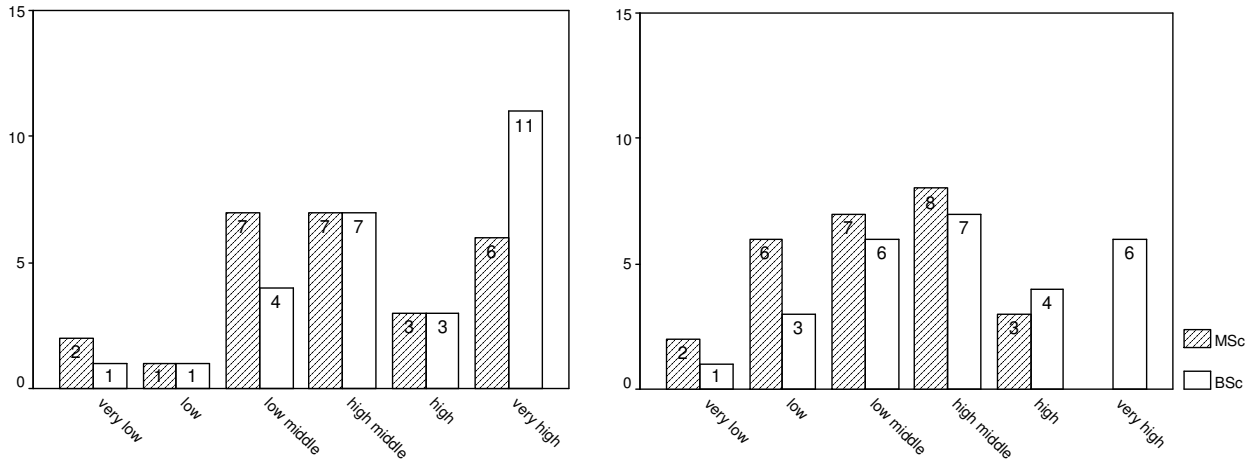

Figure 1c: Sternberg's and Adjusted 'Local' Labels.

Looking at the adjusted labels for both 'conservative' and 'local' styles, we can see that we have two distributions skewed to the opposite edges of the scale, highlighting the difference between the BSc and MSc groups more clearly. Thus, the adjusted labels provide us a finer instrument, complementary to Sternberg's labels, to look *within* our population.

DISCUSSION

These findings indicate that the mathematics students who choose to follow a taught postgraduate degree in Greece have considerably less of a preference for conformity, detail and concreteness in their thinking than the undergraduate students (they are less 'conservative' and less 'local'). Based on the rationale of Duffin and Simpson (2002), we would expect executive and local styles to be *more* prominent in the

chosen postgraduate population. On the contrary, the results of this study are closer to the conjecture derived from Biggs (2001) 'approaches to study' perspective.

We suggest two complementary accounts for this difference. On the one hand, it may be that students who manage to be highly successful in university mathematics do so mainly by developing the *strategies* needed for such an achievement and not by adjusting their *style*. That is, these students realised that in order to be successful in university they have to develop certain strategies which are closer to alien learning without giving up a fundamentally natural style. Thus, the sub-population who study for higher degrees may have underlying styles which focus less on the detail and memory for concrete procedures which might allow one to survive undergraduate study: that is, they are likely to be less 'local' in their thinking styles.

On the other hand, our two sub-populations can be seen to be quite different in other aspects of their relationship to mathematics. Because of the Greek degree system, the undergraduates, as noted, may well be studying a subject which was not their first (or even second) choice. The postgraduates, however, have made a clear subject choice.

The undergraduates are likely to have a particular goal (achievement of a degree) often for pragmatic purposes (to obtain status and a better job). The postgraduates may be more likely to have chosen their route for the sake of interest and be less clear about its pragmatic worth. The decision to leave university on completion of a degree is the majority one, while the decision to remain requires an element of non-conformity. Thus, the population of undergraduates who remain to study higher degrees are likely to be less 'conservative'.

Overall, we conjecture that a double 'filtering process' (involving both the educational system and some self-regulation) might be the reason for the complexity of our findings. That is, although students who are in (and who survive) the undergraduate mathematics educational system may be skewed towards conservative and local styles, it seems that the sub-population of students that choose a taught postgraduate programme may have managed to be successful by choosing appropriate *strategies*, without actually having or developing these *styles*. Moreover, when these students choose their postgraduate direction, they make the less conservative choice, which reflects their style.

Acknowledgements

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