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International Group for the Psychology of Mathematics Education

Volume 1

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INTRODUCTION

We want to thank the following sponsors:

The Australian Government Department of Education, Science and Training, for a generous donation.

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The University of Melbourne, for making facilities available.

Texas Instruments for a generous donation.

The City of Melbourne for support for the Conference Dinner.

WELCOME TO PME29: LEARNERS AND LEARNING ENVIRONMENTS

We are delighted to welcome you to the 29th Annual Conference of the International Group for the Psychology of Mathematics Education, being held in Melbourne, Australia. PME29 is being hosted by the University of Melbourne, and the theme of the conference is *Learners and Learning Environments*. This reflects PME's interest in what it is about learners and the circumstances in which they undertake learning experiences that contributes to the successful learning of mathematics. The talks and papers being presented at the conference will give insight into these important questions. We invite all participants to contribute actively to the discourse and analysis of ideas, so that our understanding is deepened. We also encourage all of you to foster a welcoming and stimulating atmosphere at the conference, that all participants may feel included as members of the PME community. We extend a special "G'day" to those attending their first PME conference. Our hope is that the conference will prove a fruitful learning environment for ourselves as learners.

Many of you will be aware of Australia's simultaneously old and young history. We acknowledge the Wurundjeri people of the Kulin Nations, the traditional custodians of the country on which the university stands. The area around Melbourne and the Yarra River was home to the Wurundjeri people for about 40000 years prior to the arrival of European settlers. In contrast to the thousands of years of indigenous history that contribute to our sense of place and identity, the city of Melbourne is much younger, dating from the 1830s. Its character has been influenced heavily by the gold rushes of the 1850s, and The University of Melbourne dates from this time. Melbourne is now a modern city of about 3.5 million people. Waves of immigration, first from the United Kingdom and Ireland, then post-war refugees from Europe, followed by large numbers of other immigrants, including Italian, Greek, Lebanese and Vietnamese, have given Melbourne a wide diversity of cultures.

The history of mathematics education in Australia is one of growing influence and contribution. The Mathematics Education Research Group of Australasia, whose conference precedes PME this year and whose members have contributed to the PME community over many years, is only a year or so younger than PME itself. Australian mathematics educators hosted PME in Sydney in 1984 and ICME in Adelaide in the same year, and always form a large contingent at international mathematics education conferences. With the conference here in Melbourne, we are grateful to those of you who have made the long journey so often made by Aussies in the opposite direction! We promise to be sympathetic if you are feeling slightly jet-lagged!

The Programme Committee and the Local Organising Committee want to express our thanks for the support we have received from experienced PME people, including previous conference organisers who provided useful information. Chris Breen's quiet wisdom and support have been appreciated, and Joop van Dormolen's encyclopaedic

knowledge of PME has been vital. Their advice, suggestions, encouragement, reminders, and understanding have made life easier for us. Joop's amazing database—a labour of love specially designed to keep track of all the things necessary for a PME conference—has been a wonderful asset and its capacity to do many tasks automatically has helped to reduce the workload of the organisers.

Finally, on a personal note, I would like to thank the many people who have contributed to what I hope will be a very successful conference. The Progam Committee, listed in full later, laboured mightily and with care over many important decisions, including the consideration of all the proposals. The Level 7 maths education folk of the Department of Science and Mathematics Education—Kaye Stacey, Lynda Ball, Vicki Steinle, Gloria Stillman, Anne Briner, and Jill Brown—have provided both tangible contributions and a wonderfully supportive environment in which to tackle this task. Kaye's wisdom and experience have been especially valuable. Ela Lugin, Sandra Papa, Craig McBride, and Stephen Goldstraw, together with others in the Department of Science and Mathematics Education, have provided extensive administrative support. Finally, and most importantly, my thanks to Jill Vincent without whom the conference would never have happened: her attention to detail and capacity to keep track of the important things have been incredible.

Helen Chick, Conference Chair

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

History and Aims of PME

PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsrühe, Germany in 1976. Its former presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gerard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), Carolyn Kieran (Canada), Stephen Lerman (UK) Gilah Leder (Australia), and Rina Hershkowitz (Israel). The present president is Chris Breen (South Africa).

The major goals* of PME are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education.
- To promote and stimulate interdisciplinary research in the aforesaid area with the co-operation of psychologists, mathematicians and mathematics educators.
- To further a deeper understanding into the psychological aspects of teaching and learning mathematics and the implications thereof.

PME Membership and Other Information

Membership is open to people involved in active research consistent with the Group's goals, or professionally interested in the results of such research. Membership is on an annual basis and requires payment of the membership fees (AUD$70) for the year 2005 (January to December). For participants of PME29 Conference the membership fee is included in the Conference Deposit. Others are requested to contact their Regional Contact or the Executive Secretary.

Website of PME

For more information about International Group for the Psychology of Mathematics Education (PME) as an association, history, rules and regulations and future conferences see its home page at http://igpme.org or contact the Executive Secretary.

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* Item 2 of the Constitution of PME, http://igpme.org

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PME PROCEEDINGS OF PREVIOUS CONFERENCES

PME International

The tables indicate the ERIC numbers of PME conference proceedings.

No.	Year	Place	ERIC number
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943
7	1983	Shoresh, Israel	ED241295
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, Netherlands	ED411130 (vol.l), ED411131 (vol.2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128 (vol.l), ED411129 (vol.2)
13	1989	Paris, France	ED411140 (vol.1), ED411141 (vol.2), ED411142 (vol.3)
14	1990	Oaxtepex, Mexico	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
15	1991	Assisi, Italy	ED413162 (vol.1), ED413l63 (vol.2), ED41364 (vol.3)
16	1992	Durham,USA	ED383538
17	1993	Tsukuba, Japan	ED383 536
18	1994	Lisbon,Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (vo1.l), ED411135 (vol.2), ED411136 (vo1.3)
20	1996	Valencia, Spain	ED453070 (vol. 1), ED45307 1 (vol.2), ED453072 (vol.3), ED453073 (vol.4), ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol.1), ED416083 (vol.2), ED4l6084 (vol.3), ED416085 (vol.4)
22	1998	Stellenbosch, South Africa	ED427969 (vol.1), ED427970 (vol.2), ED427971 (vol.3), ED427972 (vol.4)
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	ED452301 (vol. 1), ED452302 (vol.2), ED452303 (vol.3), ED452304 (vol.4)
25	2001	Utrecht, The Netherlands	ED466950
26	2002	Norwich, United Kingdom	ED476065
27	2003	Hawai'i, USA	http://onlinedb.terc.edu
28	2004	Bergen, Norway	

Copies of some previous PME Conference Proceedings are still available for sale. See the PME web site at http://igpme.org/publications/procee.html or contact the Proceedings manager Dr. Peter Gates, PME Proceedings, University of Nottingham, School of Education, Jubilee Campus, Wollaton Road, Nottingham NG8 1 BB, UNITED KINGDOM, Telephone work: +44-115-951-4432; fax: +44-115-846-6600; e-mail: peter.gates@nottingham.ac.uk

PME North American Chapter

No.	Year	Place	ERIC number
2	1980	Berkeley, Califomia (with PME2)	ED250186
3	1981	Minnesota	ED223449
4	1982	Georgia	ED226957
5	1983	Montreal, Canada	ED289688
6	1984	Wisconsin	ED253432
7	1985	Ohio	ED411127
8	1986	Michigan	ED301443
9	1987	Montreal, Canada (with PME11)	ED383532
10	1988	Illinois	ED411126
11	1989	New Jersey	ED411132 (vol.1), ED411133 (vol.2)
12	1990	Oaxtepex, Mexico (with PME14)	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vo1.3)
13	1991	Virginia	ED352274
14	1992	Durham, New Hampshire (with PME16)	ED383538
15	1993	Califomia	ED372917
16	1994	Louisiana	ED383533 (vol.l), ED383534 (vol.2)
17	1995	Ohio	ED389534
18	1996	Panama City, Florida	ED400178
19	1997	Illinois	ED420494 (vol.1), ED420495 (vol.2)
20	1998	Raleigh, North Carolina	ED430775 (vol.l), ED430776 (vol.2)
21	1999	Mexico	ED433998
22	2000	Tucson, Arizona	ED446945
23	2001	Snowbird, Utah	ED476613
24	2002	Athens, Georgia	ED471747
25	2003	Hawai‘i (together with PME27)	

Abstracts from some articles can be inspected on the ERIC web site (http://www.eric.ed.gov/) and on the web site of ZDM/MATHDI (http://www.emis.de/MATH/DI.html). Many proceedings are included in ERIC: type the ERIC number in the search field without spaces or enter other information (author, title, keyword). Some of the contents of the proceedings can be downloaded from this site. MATHDI is the web version of the Zentralblatt fur Didaktik der Mathematik (ZDM, English subtitle: International Reviews on Mathematical Education). For more information on ZDM/MATHDI and its prices or assistance regarding consortia contact Gerhard König, managing editor, fax: (+49) 7247 808 461, e-mail: Gerhard.Koenig@fiz-karlsruhe.de

THE REVIEW PROCESS OF PME28

Research Forums. The Programme Committee and the International Committee accepted the topics and co-ordinators of the Research Forum of PME29 on basis of the submitted proposals, of which all but one were accepted. For each Research Forum the proposed structure, the contents, the contributors and the role of the contributors were reviewed and agreed by the Programme Committee. Some of these proposals were particularly well-prepared and we thank their coordinators for their efforts. The papers from the Research Forums are presented on pages 1-93 to 1-202 of this volume.

Working Sessions and Discussion Groups. The aim of these group activities is to achieve greater exchange of information and ideas related to the Psychology of Mathematics Education. There are two types of activities: Discussion Groups (DG) and Working Sessions (WS). The abstracts were all read and commented on by the Programme Committee, and all were accepted. Our thanks go to the coordinators for preparing such a good selection of topics. The group activities are listed on pages 1-205 to 1-218 of this volume.

Research Reports (RR). The Programme Committee received 187 RR papers for consideration. Each full paper was blind-reviewed by three peer reviewers, and then these reviews were considered by the Programme Committee, a committee composed of members of the international mathematics education community. This group read carefully the reviews and also in some cases the paper itself. The advice from the reviewers was taken into serious consideration and the reviews served as a basis for the decisions made by the Programme Committee. In general if there were three or two recommendations for accept the paper was accepted. Proposals that had just one recommendation for acceptance were looked into more closely before a final decision was made. Of the 187 proposals we received, 130 were accepted, 26 were recommended as Short Oral Communications (SO), and 18 as Poster Presentations (PP). The Research Reports appear in Volumes 2, 3, and 4.

Short Oral Communications (SO) and Poster Presentations (PP). In the case of SO and PP, the Programme Committee reviewed each one-page proposal. A SO proposal, if not accepted, could be recommended for a PP and vice versa. We received 73 SO proposals initially, of which 59 were accepted and 5 were recommended as posters; later an additional 19 SO proposal were resubmitted from RR. We received 33 initial PP proposals, of which 24 were accepted and 2 were recommended as SO; later an additional 6 PP proposals were resubmitted from RR. The Short Oral Communications and Poster Presentations appear in this volume of the proceedings.

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The PME29 Program Committee thanks the following people for their help in the review process:

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MODELING STUDENTS' LEARNING ON MATHEMATICAL PROOF AND REFUTATION

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Based on a national survey and some further studies of mathematical proof and refutation of 7th through 9th graders, this paper will show evidence of the existence of continuity between refuting as a learning strategy and the production of conjectures, and between a more effective teaching strategy and the traditional teaching strategy. A detailed analysis of students' refutation schemes will be presented, and a model of their refuting process will be described based on both their refutation schemes and an expert's thinking process on refutation.

INTRODUCTION

Connecting Teaching with Students' Cognition

Research on students' mathematics cognition usually aims to investigate students' thinking and the strategies used, and further to show what guides students' thinking and why the strategies are used. Information about students' cognition can then naturally be applied to redesigning teaching strategies for enhancing students' learning in mathematics classrooms. Both the students' mathematics cognition and the related teaching modules associated with empirical evidence on its effectiveness are meaningful resources for teachers to learn teaching. Indeed, results of research on students' mathematics cognition proved to be key resources for redesigning teaching modules and reforming curriculum to ensure effective learning (Hart, 1980, 1984; Lin, 1991, 2000; Harel, 2002; Boero et al., 1998, 2002; Duval, 2002).

This paper focuses on investigating teaching and learning strategies to connect students' mathematics cognition for enhancing learning on mathematical proof and refutation. We will analyze cognition on proof and refutation in a specific group of students (about one third of their age population). And, for easy implementation in school practices, we chose the coloring strategy for learning proving, and the refuting strategy for learning conjecturing; both strategies are economic and innovative with new thinking. The evidence of using refuting as a learning strategy to generate innovative conjectures shall be presented.

A Research Program on Argumentation and Mathematics Proof

An ongoing two-staged research program on the development of proof and proving is the main reference in this paper. The first stage (2000~03) studied junior high students' understanding of proof and proving. The second stage (2003~07) is studying teaching and learning of mathematics proof. Three phases were carried out

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 3-18. Melbourne: PME.

during the first stage: instrument development, pilot study, and national survey. Six booklets comprising of algebra and geometry questions for 7th, 8th, and 9th graders were developed for the national sampling survey, and the survey involved 1181 seventh, 1105 eighth, and 1059 ninth graders respectively from 61, 60 and 61 classes in 18 sample schools. Most of the items developed in the English study (Healy & Hoyles, 1998) were adopted and modified based on Taiwan students' responses in the pre-pilot study during the first phase of the first stage. In addition, some new tasks were evolved from our interviews, which enabled the features of students' pre-formal reasoning to come through in both the instrument and coding system.

The second stage, teaching and learning mathematics proof, is comprised of an integrated project and four subprojects focusing on algebra (Lin, et al., 2004), geometry (Cheng & Lin, 2005), reading comprehension of geometry proof (Yang & Lin, 2005), and teaching and learning the validity of conditional statements (Yu Wu et al., 2004). The studies are strongly influenced by the work of many current researchers, such as the classification of student proof scheme (Harel & Sowder, 1998) and its application on teacher education (Harel, 2002), the cognitive analysis of argumentation and mathematical proof (Duval, 1998, 1999, 2002), the framework of proof and proving (Healy & Hoyles, 1998), the complexity of students understanding proving (Balacheff, 1987), the function and value of proof (Hanna, 1996, de Villiers, 1991, Hanna & Jahnke, 1993), and the theoretical validation approach of the Italian school (Garuit, Boero & Lemut, 1998).

ONE MORE STEP TOWARD AN ACCEPTABLE PROOF

The Incomplete Proof Group

When the national survey was administered in December 2002, the 9th graders had just learned formal proof in geometry for three months, while the 7th and 8th graders had not yet learned it. Based on the detailed coding schemes, students' performances on geometry proving were regrouped into four types: acceptable, incomplete, improper and intuitive proof. Students missing one step in their deductive reasoning is a typical incomplete proof. Students reasoning non-deductively or based on incorrect properties or with correct properties that do not satisfy with the given premises are codes of the improper proof. Students reasoning based on visual judgment or authority are typical codes of the intuitive proof.

The terminology "acceptable proof" derived from a statement by Clark and Invanik (1997): "Writing, for both students and researchers, is not just about communicating mathematical subject matter. It is also about communicating with individual readers, including powerful gatekeepers such as examiners, reviewers and editors." We took into account teachers' views for assessing whether a proof was acceptable or not.

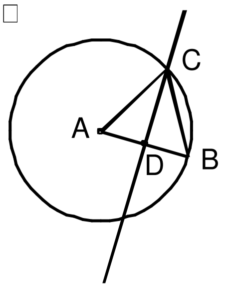
Students in the incomplete proof category were able to recognize some crucial elements for their reasoning (Kuchemann & Hoyles, 2002). They were able to distinguish premises from conclusions in the task setting. Particularly, on the two-step proof items, they were even mindful to check conditions of the theorems applied, i.e., micro reasoning (Duval, 1999.) They were also able to organize statements according to the status, premise, conclusion and theorem into a deductive step. Duval (2002) named such competency as the first level in geometrical proof. The second level is the organization of deductive steps into a proof. From the first step conclusion to the target conclusion, valid deductive reasoning generally moves forward through either successive substitution of intermediary conclusion or coordination of some conclusions. Duval (2002) pointed out that students might have "gaps in the progress of reasoning which makes the attempt of proving failed." This arises either from misunderstanding of the second level organization or from the context of the problem. We shall carefully examine Duval's statement above for the group of students who performed incomplete proofs in the two-step proof tasks.

The data from our national survey showed that one quarter of 9th graders could construct acceptable proofs in a two-step unfamiliar item; approximately one third was able to perform incomplete proofs; and one third did not have any responses at all.

It is obvious that educators would like to focus on this one third of 9th graders who were able to perform incomplete proofs, and to develop a learning strategy for them to fill the gap, i.e., develop one more step toward an acceptable proof. An effective learning strategy should promise that nearly a half of 9th graders will be able to construct a two-step unfamiliar geometry proof.

Incapability of Students with Incomplete Proof Performance

The two-step unfamiliar question used in the survey is as follows.



A is the center of a circle and AB is a radius. C is a point on the circle where the perpendicular bisector of AB crosses the circle. Please prove that triangle ABC is always equilateral.

Two types of incomplete proofs were observed. One type was missing the ending process. Students showed that AC=BC and AC=AB, but did not conclude that the three sides were equal. From a deductive point of view, they were ritually incomplete with the ending process, i.e., if a=b and b=c then a=b=c. Do these students who performed two valid deductive steps still have difficulty in the ending process, a classical syllogism? Or might these students simply be thinking that the two conclusions were too obvious for implying the target conclusion? Should one write this obvious step down? Would this be just an issue in the conventions of mathematical writing? Studies of students' understanding of proof by contradiction (Lin et al., 2002) and mathematical induction (Yu Wu, 2000) showed that senior high students who concluded their proofs without the ending process using either method, very often developed a ritual view about the methods. And the principle of the methods was not understood (Lin et al., 2002). If a teacher considers the two valid deductive steps as an acceptable proof, would the teacher create learning difficulties on mathematical proof for some students? A general question can be asked: How many students who can perform every valid deductive step necessary for a proof task also have difficulty organizing the deductive steps into a proof? Interview data showed that there were students behaving as such.

The other type of incomplete proof was missing one step, either AB=AC or AC=BC. The information "AC is a radius" was implicitly situated within the given premise. This information was invisible for students who did not conclude AB=AC. The property of the perpendicular bisector of a segment seemed unclear for students who did not draw the conclusion AC=BC. Some students of this type might not be aware of the need to derive the equality of all three sides for an isosceles triangle. Thus, the group of students with incomplete proof performance might not be able to:

(1) organize the deductive steps into a proof, or

(2) visualize some implicit information in the given premise, or

(3) recognize a needed mathematics property, or

(4) be aware of all necessary statements/deductive steps.

These four cognitive gaps are due not only to:

(1) misunderstanding of the organization of deductive steps into a proof,

(2) the content of a problem, but also

(3) the context knowledge, and

(4) the epistemic value, i.e., the degree of trust of an individual in a statement, from likely or visually obvious, to a statement becomes necessary (Duval, 2002).

For teaching experiments, one needs to rethink a learning strategy to ensure that students can cross these cognitive gaps.

A Learning Strategy for Promoting One More Deductive Step

Using X as learning strategy for students within their mathematics proof activities is an active research issue. Fifteen paper presentations that dealt with this issue in PME 22~28 are reviewed. The different Xs used in those papers include: arranging the context of proof situations (Garuti et al., PME26) and encouraging interactive discursion to create students' cognitive confliction (Boufi (PME26), Krummheuer (PME24), Douek et al. (PME24), Sackur et al. (PME24), Antonini (PME28)), learning within an ICT environment for conjecturing (Miyazaki (PME24), Gardiner (PME22), Hoyles et al. (PME23), Sanchez (PME27), Hadas (PME22)), emphasizing teachers' questioning as scaffolding (Blanton et al. (PME27), Douek et al. (PME27)), and using metaphors (travel) for setting target goals (Sekiguchi (PME24)). Note that the notation (PME24) indicates the paper appeared in the Proceedings of PME24. We exercised a "thought experiment"(Gravemeijer, 2002) with each of those strategies in addition to typical geometry teaching strategies used in Taiwan secondary mathematics classroom, to match the characterization of the incomplete proof group and enhance them to move one more deductive step. Finally, we chose two strategies that are commonly observed in typical Taiwanese 9th grade geometry classrooms and tested them for helping students achieve one more deductive step. The reading and coloring strategy means that students are asked to read the question, label the mathematical terms, and draw or construct this information on the given figure by color pens. The analytic questioning strategy means that students are asked to reply on what the question asked you to prove, and what conditions in the premise can be useful.

Several phases were conducted in our teaching and learning study:

- *Phase (1):* A three-item diagnostic assessment paper was developed for identifying sample subjects of the focus group. All three items share a common feature with implicitly necessary information.
- *Phase (2):* An instructional interview was conducted on 9 samples individually to examine the effectiveness of implementing both learning strategies simultaneously.
- *Phases (3) and (4):* A small group teaching experiment was carried out to study the effectiveness of only implementing one of the two learning strategies.
- *Phase (5):* A set of learning tasks on geometry proving was developed.

Based on the data resulting from phase (3), we will analyze the function of coloring the mathematical terms in proving. Turning implicit information into explicit information is definitely one function of the strategy. What else happened so that the subjects were able to complete an acceptable proof? It is noteworthy to interpret this with the data collected in the phase (3).

The three items, including the two-step unfamiliar item (G2) used in the national survey, were used in both phases (1) and (2). Nine samples were identified and interviewed. Their performances before the instructional interviews (Pre-I) and after intervening with the reading and coloring strategy (R-C) and analytic questioning strategy (A-C), respectively, during the interviews are presented in Table 1.

The notation (31) denotes the sample who performed an incomplete proof without the ending process due to omission (sample 02) or students' epistemic value that the ending process is unnecessary (sample 05, 06, 09). The notation 31* indicates that sample 01 would not agree with the syllogistic rule "if a=b and b=c then a=b=c" during the interviews, but agreed that "a=b and b=c" are the conditions for an equilateral triangle with sides a, b and c. The behavior of sample 01 on the syllogistic rule reveals one kind of reason for missing the ending process.

Sample	Performance	G1	G2	G3	G1	G2	G3	Sample
01	Pre-I	0	32	0	4	32	32	06
	R-C	4	32	4		32	32	
	A-Q		31*			(31)	(31)	
02	Pre-I	0	(31)	0	0	32	0	07
	R-C	4	4	32	4	32	32	
	A-Q			4		4	4	
03	Pre-I	0	0	32	0	32	4	08
	R-C	4	32	4	4	32		
	A-Q		4		4			
04	Pre-I	32	21	32	0	(31)	32	09
	R-C			4			4	
	A-Q							
05	Pre-I	4	(31)	32				
	R-C			(31)				
	A-Q							

Note: Definition of codes: 4 denotes an acceptable proof; 31 denotes incomplete, missing the ending process; 32 denotes incomplete, missing one deductive step; 21 denotes improper, using an incorrect property; 0 denotes no response.

Table 1: Students' performance with/without the learning strategies R-C and A-Q

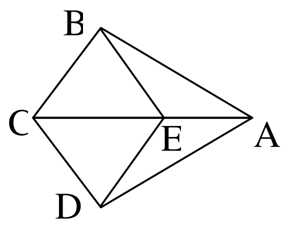
Table 1 shows that among the 24 (27-3) positions of students' performances which need to move towards an acceptable proof, 15 positions were successfully moved before or after the intervening of only the reading and coloring strategy. Since this coloring strategy is procedural in nature, the cognitive demand on learners for using this strategy is much lighter than using the analytic questioning strategy, which demands quite heavy analytical thinking. So, it is worthy to further explore the extent to which the reading and coloring strategy can enhance students' proving performance. Which kind of proof content will be effective by using this strategy? And a further interpretation of the effectiveness also seems interesting. This is the phase (3) study.

Effects of the Coloring Strategy

During the phase (3) study, four two-step unfamiliar new items were developed for 8 new participants. Before intervening with the reading and coloring strategy, out of 32

(8 × 4) performances, 10 were acceptable proofs and 22 were unacceptable, i.e., incomplete, or improper or had no response. Each participant had at least two unacceptable performances. One week later, 8 participants worked on the same items after intervening with the reading and coloring strategy. As a result, 16 out of the 22 unacceptable proofs had progressed to acceptable proofs. However, 4 out of 10 acceptable proofs became unacceptable, in which 3 out of 4 negative effects were coded from the same item 3.

Item 3.

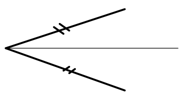


Points A, E, C are collinear,

and △ABC is congruent to △ADC.

Show that: BE=DE

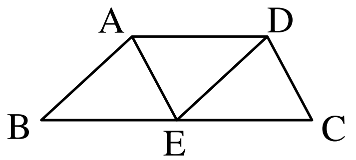
Two students misinterpreted the equality signs labelled on ∠ABC and ∠ADC as ∠CBE =∠CDE. The other student associated the sign around point C, with the angle bisector theorem and applied it improperly. Indeed, colored signs labelling on sub-



figures which cross each other would generate a disturbance that affects visualizers' interpretation on the explicit information transmitted from the sub-figures.

Among the non-effected performance, all six were collected from item 2.

Item 2.



Points B, E, C are collinear,

and △ABE is congruent to △DEC.

Show that: AD//BC

When the equality signs were colored on the six elements, sides and angles of each triangle, the colored signs produced superfluous relations among the elements. Whenever a relation matching his/her target goal was observed by a student, it became active and operational. Students then applied it without justifying deductively. This seemed to be the pattern among those non-effected unsuccessful performances. Analyzing the negative effects and non-effects of the coloring label strategy, a criterion could be used by teachers to restrict the tasks on using the strategy. If a disturbance or superfluous relation from the coloring strategy were intentionally generated onto an item, it may backfire and result in negative effects or non-effects; in this case, the strategy may not be suitable for this item.

Transmission of the Subfigure with Relation to the Theorem Image

In spite of the negative and non-effects of the coloring strategy, we are interested in how the effectiveness (16/22) of the reading and coloring strategy takes place. From neuro-psychological perspectives, “Learning occurs… when transmitter release rate increases make signal transmission from one neuron to the next easier. Hence learning is, in effect, an increase in the number of ‘operative’ connections among neurons” (Lawson, 2003).

Learning was indeed achieved by those subjects who applied the coloring strategy and were able to perform an acceptable proof. How were the operative connections increased among the statements according to specific status and the use of theorems? The necessary theorems existed previously in the subjects’ mental structure, but were inoperative before they applied the coloring strategy. The result of the coloring process revealed subfigures with notable relations that may also correspond to the theorem. If this happens, then learners have increased the relation between the subfigure and the needed theorem. To make it clear, we shall use the term theorem image, similar to the term concept image (Tall & Vinner, 1981), to describe the total cognitive structure that is associated with the theorem, which includes all the mental pictures and associated examples, relations, process and applications. A theorem image is built up over years of learning experiences. It is personal and constantly changing as the individual meets new stimuli. Different stimuli can activate different parts of the theorem image. The stimulus resulting from coloring of mathematical terms in the premise is functioning to lead the transmitter of the revealed subfigure with relation to the corresponding part of his/her theorem image. This leads the effect of the organization of one deductive step.

MAKING DECISIONS ON FALSE CONJECTURES

Some items in each of the six booklets were connected to how students reason to make their decisions on a given false conjecture. Students were asked to make a decision among two (three) choices – agree, disagree, or uncertain (algebraic item) – and then give explanations on their choices. A unity of coding schemes was evolved for both geometry and algebra surveys. The coding schemes were used to analyze the students’ performances. Based on this coding scheme, a model of refuting will be discussed. Firstly, for researchers to make sense of the thinking process in mathematical refutation, an expert was interviewed.

Mr. Counter-Example’s Thinking Process on Refutation

A mathematician, nicknamed Mr. Counter-Example by his peers during his graduate studies, was interviewed to reveal the thinking process of an expert on refutation.

> “Suppose an unfamiliar mathematics proposition is proposed by myself or peers. Reading it and without having much sense with the proposition, the doubtfulness of its truth usually does not arise in my mind. To make sense of the proposition, very often I’ll substitute some individual examples. Then, I will find more and more examples to satisfy the premise. Naturally those examples will be classified according to certain mathematical property. As long as the property is grasped, all kinds of examples will be considered. Finally, a specific kind of example will be identified to counter the conclusion if the proposition is false.”

According to Mr. Counter-Example's description, his refuting process covers five sequential processes:

1. Entry
2. Testing some individual examples point-wisely for sense making
3. Testing with different kinds of examples
4. Organizing all kinds of examples
5. Identifying one (kind of) counterexample when realizing a falsehood

This expert's thinking process on refutation can be inferred to analyze students' reasons on refuting.

On Geometrical False Conjectures

Two conjectures in geometry were adopted from the English study (Healy & Hoyles, 1998):

> "Whatever quadrilateral I draw with corners on a circle, the diagonals will always cross at the center of circle?" (7G1, Geometry)
>
> "Whatever quadrilateral I draw, at least one of diagonals will cut the area of the quadrilateral in half?" (8G1, Geometry)

Three false conjectures were evolved from the interviews carried out during the pilot study phase of the first stage. The following one was included in geometry booklets for both 7th and 8th graders who were the subjects concerned in this section.

> "A quadrilateral, in which one pair of opposite angles are right angles, is a rectangle." (7&8 G5, Geometry)

This coding scheme was evolved according to the performances of the national representative sample and the expert's thinking process on refutation, and is more detailed than the schemes developed in the English study (Hoyles & Kuchemann, 2002), which only focused on high-attainers (top 20~25% of the student population).

On geometrical false conjectures, students either confirmed or refuted it. Comparing the frequency on G5 of 7th and 8th graders' performances, there is no evidence of progress with correct decisions over the year (37% for 8th graders, even more than 26% for 7th graders). Based on the words provided by students who ticked disagree, we classified them into three subcategories: rhetorical argument, correcting the given information, and generating counterexamples. Duval (1999, 2002) classified the relationship between a given statement A and another statement B into two types – the derivation relationship and the justification relationship. For each type, there are two kinds of reasoning that are practiced or required in mathematics teaching and learning. Semantic inference and mathematical proof support the derivation relationship; heuristic argument and rhetorical argument support the justification relationship. In our code scheme, codes c2, c3, c4, g1, g2 are the so-called heuristic arguments that take into account the constraints of the situation in the task. Generally, an argument is considered to be anything that is advanced or used to justify or refute a proposition. This can be the statement of a fact, the result of an experiment, or even simply an example, a definition, the recall of a rule, a mutually held belief or else the presentation of a contradiction (Duval, 1999). Reasons relative to the person spoken to or beliefs of the interlocutor are the rhetorical arguments. Therefore, code d4 is a rhetorical argument, and d3 is a heuristic argument.

False Conjectures if P then Q				
Code	Frequency (%)			
	7G1	7G5	8G1	8G5
Confirmation	44	26	31	37
d_0 – Misunderstanding the given information	1	2	2	1
d_1 – Much ado about nothing	23	5	12	9
d_2 – Confirm Q with incorrect reason	9	3	8	6
d_3 – Giving P’ s.t. P’→Q	3	12	3	17
d_4 – Authority	0.1		0.2	1.1
Refutation	52	67	68	59
Rhetorical argument	8	8	17	11
Correcting the given information	15	51	12	33
c_0 – Criticizing the given information	9	13	3	5
c_1 – Non-example	3	3	5	9
c_2 – Providing alternative Q		32		16
c_3 – Characterizing Q s.t. P’ $\rightarrow Q$	2	3	3	2
c_4 – Empirical decision	0.3		0.5	0.1
Generating (a) counterexample(s)	24	4	34	11
g_0 – Do not believe it is always true	3	1	5	3
g_1 – Giving the possibility of a counterexample	5	0.6	13	4
g_2 – Giving the way of generating a counterexample	4	0.3	4	1
g_3 – Explicit, clear counterexample	12	2	10	3
g_4 – Counterexample with mathematical proof		0.1	0.9	0.1

Note: Non-responses are not included

Table 2: 7th and 8th graders Code Frequencies on items G1 and G5 (N7=1146, N8=1050)

Our coding scheme with code frequencies cover three out of four kinds of reasoning practiced by our 7th and 8th graders on refuting false conjectures: rhetorical argument, heuristic argument and mathematical proof (clear counterexample counts). The relatively high frequency of code c2 in 7G5 was contributed by students who reasoned that under the assumption, a quadrilateral can be either a square or a rectangle. This reason reflects the prevalence of students who misunderstand the inclusion relationship between squares and rectangles. Putting the number of students with codes c2, c3, c4, g1 and g2 together, and computing its frequency, we found that 11% and 36% of 7th graders and 20% and 24% of 8th graders were able to make a heuristic arguments for refuting G1 and G5, respectively.

On Algebraic False Conjectures

Three false conjectures in algebra survey for 7th and 8th graders were chosen for discussion.

A3 "If the sum of two whole numbers is even, their product is odd?" (Both 7th and 8th graders, adopted from Küchemann & Holyes, 2002.)

A6b "The sum of a multiple of 3 and a multiple of 6 must be a multiple of 6?" (8th graders)

The data (3,6,6) in A6b was replaced by (3,6,9) in A6c for 8th graders, and respectively by (2,4,4) and (2,4,6) in A6b and A6c for 7th graders. Students' works on algebraic false conjectures were analyzed with this code scheme: "g3: explicit, clear counter example, can be distinguished into three subcodes," "g31: counterexample without reason," "g32: both supporting and counterexamples," and "g33: counterexample with analytic reasons," which often is a rule for generating a specific counterexample. Referring to the expert's thinking process on refutation, both processes (2) and (5) will be coded by g31. Thus, without words, code g31 could result from primitive or advanced thinking.

Instead of presenting the national survey data, we'll present a brief description of the students' words to model their refutation schemes on algebra. On confirmation: (1) "I believe that only true statements will be presented in my learning" (code d_1); (2) "I consider it correct, because its familiar format is akin to statements in textbooks" (code d_4); (3) "I had supporting examples, e.g., 3+6=9 and 3×2+6×2=18, they are multiples of 9" (A6c) (code d_3). On uncertain responses: (1) "I am not certain because the multiple is not given," students interpreted the term multiple in "a multiple of 3" as specific numbers, a misconception (code r_1); (2) "I had both supporting and counterexamples," in ordinary language, this statement is uncertain (code g_{32}). On refutation performances: (1) "The statement is so elegant, I must have learned it before. But, I did not. So it can't be always correct" (code g_0); (2) Simply adding a negation without reasons (code r_1). Beyond the above beliefs and rhetorical arguments, the students' refutation schemes are coded by g1, g2, g31, g32, g33 and g4. Their thinking process then is similar to certain points in the expert's thinking process.

Refuting Generates Conjectures

When students gave their explanations for refuting, many gave heuristic arguments and explicit counterexamples with reasons, and we observed that some of these students had even produced relations, known properties evidences, general rules, etc. Buying the notion of "Cognitive Unity of Theorems" from the Italian school (Garuti et al., 1998; Boero, 2002), instead of the concerns of the possible continuity between some aspects of the conjecturing process and some aspects of the proving process, we would like to investigate the possible production of conjectures derived from the aspect of the refuting process.

The activity of refuting in mathematics is considered an economic way of helping students to develop competency in critical thinking. Competency of critical analyses has been recognized as a deficit in Taiwan education and is now emphasized in the school curriculum (Ministry of Education, 2003). Two refuting-conjecture tasks in algebra and geometry respectively were developed for the investigation. Each task is comprised of several items. The first item is making decisions on relatively easy false conjectures that aim to motivate students to be aware that the task is on refuting. The second item is given some false conjecture used in the national survey for refuting. The third and fourth items ask students to produce one conjecture and more conjectures, based on their refuting processes.

All nine 7th graders who participated in the investigation with the algebra task produced meaningful conjectures. Three of them even produced a general rule for a whole number m that is divisible by the linear combination of whole numbers $ax+by$.

Seventy-five 9th graders from two classes were asked to participate in the geometry task investigation. The four false conjectures used in the tasks were 7G1 (denotes item G1 in the 7th grade survey), 8G1, 8G5, 9G6, respectively. According to the code of frequencies of refutation schemes, 76%, 73%, 53%, and 60% of their performances were in the category "generating counterexamples" with respect to those false conjectures 7G1, 8G1, 8G5, and 9G6 respectively. The conjectures produced by this group are presented in Table 3.

%	7G1	8G1	8G5	9G6
Thm.	33	20	52	7
New statement	17	8	7	1
Innovation	5	33	8	56
Total	55	61	67	64

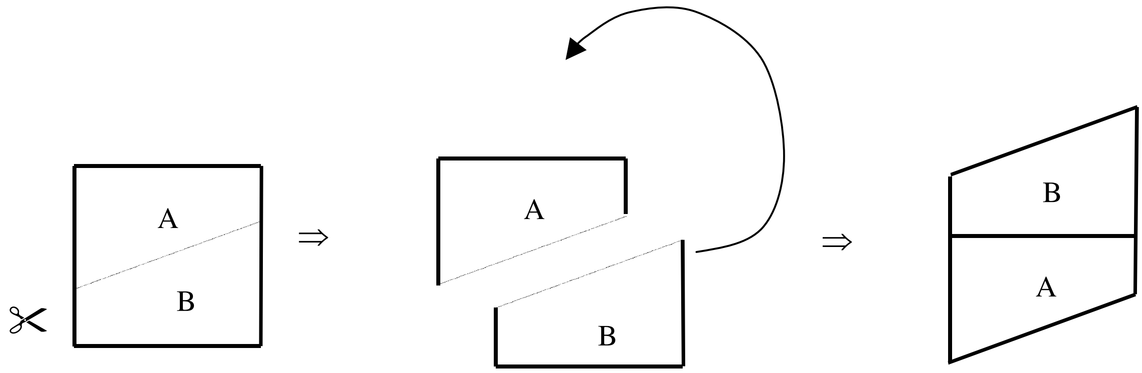
Note: Thm. denotes the conjecture is a theorem. New Statement denotes the conjecture is a new writing of learned properties. Innovation denotes the conjecture is an innovative one.

Table 3: Frequency (%) of different type of conjectures. N=75, 9th graders

Table 3 shows that the success rate for producing correct conjectures on these four tasks was approximately 60% or more. Different frequencies of each type of conjectures imply that 8G1 and 9G6 are excellent for creating brand new conjectures by 9th graders. The item 9G6 is quoted here.

9G6.

A square is cut along the dotted line, then inverted. Is the resulting figure a rhombus?



The conjectures produced by students were further distinguished into "correlating" or "not correlating" to their explanations for refuting.

The relatively high percentages in Table 4 show the continuity of the refuting process and conjecturing process. This claims that refuting is an effective learning strategy for generating conjectures. To create innovative conjectures, the content in the given false conjecture needs to be well-designed, and 9G6 is a good example.

	7G1	8G1	8G5	9G6
T1	40	57	38	69

Table 4: The percentages of conjectures that correlate to refuting

Boero (2002) reported that the Italian school has identified four kinds of inferences, intervening in conjecturing processes: (1) inference based on induction, (2) inference based on abduction, (3) inference based on a temporal section of an exploration process, and (4) inference based on a temporal expansion of regularity. Reading students' productions in the refuting-conjecture tasks, we observed that false conjectures in numbers 7A3 and 8A6 can enhance the generation of conjectures that are inferences based on induction, abduction (e.g., a narrative) and even deduction (e.g., 3h+6k=3(h+2k)); the task with figure dissection 9G6 can generate conjectures that are inferences based on a temporal section of an exploration process (the dissection), and tasks with 7G1 and 8G1 are relatively effective on generating conjectures that are based on the expansion of regularity (such as new statements of some properties). The following excerpt is from 9G6.

> If a line cuts a rectangle along the pair of longer sides into two parts so that the cross segment is equal to the longer side, then the two parts can be inverted to form a rhombus.

This conjecture is produced in association with sequential operations on a rectangle.

CONCLUSION

Based on our study, there is evidence showing the existence of continuity in different aspects of mathematics education. In the mathematics learning aspect, a rather high percentage of students were able to produce correct conjectures when working on refuting-conjecture tasks; this shows the existence of continuity between the refuting process and the production of truth statements. For some students, this continuity can even extend to their proving process. Indeed, some students have already provided counterexamples with analytic or mathematical proofs to refute false conjectures. In the mathematics teaching aspect, the effectiveness of the reading and coloring strategy on geometrical two-step proving shows that teachers can keep their traditional teaching approach, in which they can encourage students to label meaningful information within the given premise and conclusion and then seek linkages between the premise and the conclusion. Without disturbing their approach but suggesting students to use color pens for labelling, teachers can enhance students' proving competencies. This demonstrates continuity between a more effective teaching strategy and the traditional teaching strategy. In the aspect of research in mathematics education, there is continuity between the investigating processes by educators in mathematics education research and by mathematicians in mathematics proving. The six phases of mathematicians in proving identified by Boero (1999) is indeed shared by mathematics educators in their studies, such as the study presented in this paper. Formulating on-going investigating issues is always considered to be connected with reflections on previous phases.

Carrying out more testing on the effectiveness of the refuting-conjecture tasks will create an equilibrated set of conjecturing tasks suitable for activating different types of inferences.

Several phases of research in mathematics education presented in this paper are rather traditional, such as (1) Identifying 1/5~1/3 of students in their age population, whose mathematics understanding are more likely to be enhanced. (2) Characterizing those students' competencies. (3) Carrying out an experimental study with a redesigned learning strategy that connects to the characteristics of their cognition.

This approach can frame local (geological and societal) education issues in the wider context of collaborative international studies, for the purpose of improving mutual education. The experience seems to be a very healthy and effective approach.

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TRAVELLING THE ROAD TO EXPERTISE: A LONGITUDINAL STUDY OF LEARNING

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A longitudinal study of students' developing understanding of decimal notation has been conducted by testing over 3000 students in Grades 4 to 10 up to 7 times. A pencil-and-paper test based on a carefully designed set of decimal comparison items enabled students' responses to be classified into 11 codes and tracked over time. The paper reports on how students' ideas changed across the grades, which ways of thinking were most prevalent, the most persistent and which were most likely to lead to expertise. Interestingly the answers were different for primary and secondary students. Estimates are also given of the proportion of students affected by particular ways of thinking during schooling. The conclusion shows how a careful mapping can be useful and draws out features of the learning environment that affect learning.

In this presentation, we will travel on a metaphorical seven year journey with over 3000 students. As they progress from Grades 4 to 10, learning mathematics in their usual classrooms, we will think of these students as travelling along a road where the destination is to understand the meaning of decimals. The noun "decimal" means a number written in base ten numeration with a visible decimal point or decimal comma. It may be of finite or infinite length. Different students take different routes to this destination, and we will follow these different routes through the territory that is the understanding of decimal numbers and numeration. Of course, the students are simultaneously travelling to many other mathematical and non-mathematical destinations, but our information enables us to follow just one of these journeys. The benefit in following one journey derives from the knowledge that we gain of their paths on this journey, how to help them reach the destination securely and also from being able to generalise this knowledge to understanding their likely paths on their other mathematical journeys.

Our travelling companions: the students

In preparation for our journey, we need to find out about our travelling companions, the transport that is available to them, how we will map their progress, the nature of their destination and the territory through which they travel. Our travelling companions are 3204 Australian students from 12 schools in Melbourne. The schools and teachers volunteered their classes for the study. The youngest students were in Grade 4, the grade when most schools are just beginning to teach about decimals. The oldest students were in Grade 10, two or three years after teachers generally expect their students to have fully developed understanding of decimals. The data is from a cohort study, which tracked individual students for up to 4 years, testing them with the same test each semester (i.e. twice per year). Students entered the study at

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 19-36. Melbourne: PME.

any grade between Grade 4 and 10, and continued to be tested until they left Grade 10, or until they left the schools or classes in the study, or until the end of the data collection phase of the study. In total, the 3204 students completed 9862 tests, and when allowing for absences from class on the testing days, the tests were an average of 8.3 months apart. The schools come from a representative range of socio-economic backgrounds, and were chosen in six geographical groups so that many students could be tracked across the primary-secondary divide. Nearly 60% of the 1079 students who were first tested in primary school (i.e. elementary school, Grades 4 to 6) were also tested in secondary school. More than 600 students completed 5, 6 or 7 tests during the study. The detailed quantitative analyses of the test results presented in this paper are taken from the PhD thesis of Vicki Steinle (2004), whose careful and imaginative contribution to our joint work on students' understanding of decimals is acknowledged with gratitude and admiration.

The transport: their teaching

The transport available to the students along this journey is principally the teaching of decimals that was provided at their schools. In the absence of a prescriptive national curriculum or recommended textbooks in these schools, teaching approaches are selected by teachers. This variety makes it difficult to give a comprehensive picture. Instruction will generally begin by introducing one place decimals as an alternative notation for tenths (e.g. 0.4 is 4 tenths, 1.8 is one plus 8 tenths) in Grades 3 or 4. Dienes' multibase arithmetic blocks and area models are the most common manipulatives used. In some programs, calculations are done with one place decimals (e.g. 0.24, 4.79) in the early years, followed by calculations with two place decimals treated exclusively later. In secondary school, textbooks very frequently ask that all decimal calculations are rounded to two decimal places. Brousseau (1997) is among the authors who have commented that teaching which works exclusively with decimals of a fixed length is likely to support overgeneralisation of whole number properties. In the course of our wider work on teaching and learning decimals, our team has designed and trialled a range of teaching interventions, including use of novel manipulatives based on a length model (Stacey, Helme, Archer & Condon, 2001b) and we have created a set of computer games using artificial intelligence techniques (Stacey, Sonenberg, Nicholson, Boneh & Steinle, 2003b), but only a very tiny percentage of students from the cohort study were involved in trialling any of these interventions. The teaching that the students received in the longitudinal study can therefore be assumed to be a representative sample of teaching across Melbourne.

The destination: understanding decimal notation

What is the destination for this journey? Students will have arrived at the final destination when they have a full understanding of the meaning of decimal notation. For the purpose of our wider work on teaching and learning about decimals, full understanding means that they should be able to interpret a number such as 17.373 in terms of place value in several ways (as 17 + 3 tenths + 7 hundredths + 3 thousandths or as 17 + 373 thousandths, etc) and to appreciate that it is less than halfway between 17 and 18, close to 17.4 but with an infinite number of numbers between it and 17.4. At this point, it is worth noting that decimal notation, as a mathematical convention, involves a mix of arbitrary facts that have to be learned and deep mathematical principles. It is not *merely* a convention. Some aspects are completely arbitrary, for example identifying the units column by the contiguous placement of a decimal point (or a decimal comma in many countries) or placing the larger place value columns on the left rather than the right. However, the notation also embodies deep mathematics, such as the uniqueness of the decimal expansion, with the consequence that all decimals of the form 2.37xxxx are larger than all decimals of the form 2.36xxxx except that $2.36\dot{9} = 2.37 = 2.370$ etc. It is this property that makes the decimal comparison task so easy for experts. In the sense of Pea (1987), decimal notation is an invented symbolic artefact bearing distributed intelligence.

Early explorers mapping the territory

The description of the territory through which students pass is strongly linked to the way in which their progress can be mapped. This is a basic feature of science: there is a two-way interaction between knowledge of a phenomenon and having instruments to observe it. In mathematics education, knowledge of students' thinking depends on asking good questions, and we only know what the good questions are by understanding students' thinking. In the context of students' understanding of decimals, Swan commented on this phenomenon in 1983:

"It is only by asking the right, probing questions that we discover deep misconceptions, and only by knowing which misconceptions are likely do we know which questions are worth asking", (Swan, 1983, p65).

Cumulative research on students' understanding of decimals has broken this cycle to advantage. The task of comparing decimal numbers (e.g. deciding which of two decimals is larger, or ordering a set) has been used since at least 1928 (Brueckner, 1928) to give clues as to how students interpret decimal notation. Refinements to the items used, especially since 1980, improved the diagnostic potential of the task and provided an increasingly good map of the territory of how students interpret decimal notation. For example, Foxman *et al* (1985), reporting on large scale government monitoring of mathematics in Britain, observed a marked difference in the success rates of apparently similar items given to 15 year old students. Asked to identify the largest in the set of decimals {0.625, 0.5, 0.375, 0.25, 0.125}, the success rate was 61%. Asked to identify the smallest, the success rate was a surprisingly much lower 37%. Note that this paper presents all sets from largest to smallest, not in order presented. Further analysis led to the first confirmation in a large scale study that whilst some students consistently interpret long decimals (e.g. 0.625, 0.125) as *larger* numbers than short decimals (e.g. 0.5), which was well known at the time, a significant group interpret them as *smaller* numbers.

> "Despite the large proportions of pupils giving this type of response very few teachers, advisors, and other educationalists are aware of its existence – the monitoring team were among those unaware of the 'largest is smallest' response at the beginning of the series of surveys." (Foxman *et al*, 1985, p851)

Asking students to identify the smallest from this set of decimals was used again as an item by the international "Trends in Mathematics and Science Study" (TIMSS-R, 1999) Table 1 gives the percentage of the international and Australian students giving each response, alongside Foxman *et al*'s 1985 data. The existence of the same general patterns in the selection of responses across countries and times shows that there is a persistent phenomenon here to be studied. There is also a good fit between the results from the TIMSS-R random Australian sample and a prediction made from the Grade 8 sample of the present longitudinal study (re-calculated from Steinle, 2004, Appendix 4, Table 19), which confirms that the results of the longitudinal study presented in this paper are representative of today's Australian students.

Table 1: Percentage response to the item: *Which of these is the smallest number? {0.625, 0.5, 0.375, 0.25, 0.125}* from TIMSS-R (age 13), APU (age 15) and with prediction from present longitudinal study (Grade 8).

Option	TIMMS-R International	TIMMS-R Australia	Foxman *et al.* APU, age 15	Prediction (Grade 8)
0.125	46%	58%	37%	60%
0.25	4%	4%	3%	2%
0.375	2%	1%	2%	2%
0.5	24%	15%	22%	18%
0.625	24%	22%	34%	17%

Working at a similar time to Foxman *et al*, Sackur-Grisvard and Leonard (1985) demonstrated that examination of the pattern of responses that a student makes to a carefully designed set of comparison or ordering tasks could reveal how the student was interpreting decimal notation reasonably reliably and they documented the prevalence of three "errorful rules" which students commonly use. This provided a rudimentary map of the territory through which students pass on their way to expertise in understanding decimal notation. Sackur-Grisvard and Leonard's test was later simplified by Resnick *et al* (1989) and has been steadily refined by our group to provide an instrument which can map where students are on their journey to expertise. Current researchers, such as Fuglestad (1998), continue to find that decimal comparison tasks provide a useful window into students' thinking and progress.

The territory and the mapping tool

Measuring the progress of a large cohort of students along the journey to understanding decimal notation required a mapping tool that is quick and easy to administer, and yet informative. The version of the instrument used in our longitudinal study is called Decimal Comparison Test 2 (DCT2). It consists of 30 pairs of decimals with one instruction: "circle the larger number in each pair". The pattern of responses (not the score) on 5 item-types (subsets of items with similar mathematical and psychological properties) enables classification of students into 4 "coarse codes" (A, L, S and U) which are further broken down into 11 "fine codes" (A1, A2, L1, etc) to describe likely ways of thinking about decimals. Figure 1 gives one sample item from each item-type in DCT2 and shows how students in 7 of the fine codes answer these items. Students are classified into the coarse codes on the basis on their answers to the first two item-types (shaded in Figure 1) whereas the fine codes use all item-types. In summary, we map where students are on their journey by administering a test that is simple to do, but has a complex design and a complex marking scheme. Details of the sampling, the test and its method of analysis and many results have been described elsewhere; for example, Steinle and Stacey (2003) and Steinle (2004). We can think of the 11 fine codes as the towns that students might visit on the journey, although, as in most adventure stories, these towns are mostly not good places to be. The 4 course codes are like shires; administrative groupings of towns (fine codes) that have some connections.

Comparison Item		A1	A2	L1	L2	S1	S3	U2
4.8	4.63	√	√	×	×	√	√	×
5.736	5.62	√	√	√	√	×	×	×
4.7	4.08	√	√	×	√	√	√	×
4.4502	4.45	√	×	√	√	×	×	×
0.4	0.3	√	√	√	√	√	×	×

Figure 1. Sample items from DCT2 and the responses for the specified codes.

Some of the ways of thinking that lead to these patterns of responses are briefly summarised in Table 2. In the presentation, some of these ways of thinking will be illustrated with case studies from Steinle, Stacey and Chambers (2002). The L behaviour (generally selecting a longer decimal as a larger number) was widely known long before the S behaviour (generally selecting a shorter decimal as a larger number) was documented as reported above. Neither coarse code A nor U students choose on length. Students coded A are correct on straightforward comparisons, and U is a mixed group making other responses. The ways of thinking that lie behind these behaviours (other than U) have been identified by interviews with students, supported by close analysis of response patterns to identify the characteristics of apparently similar items to which groups of students react differently. Behind the codes, there are often several different ways of thinking that result in the same patterns of responses to the DCT2. Later refinements of the test enable some of these different ways of thinking to be separated. Space forbids a full description here.

Table 2: Matching of codes to the ways of thinking

Coarse Code	**Fine Code**	**Brief Description of Ways of Thinking**
A apparent expert	A1	Expert, correct on all items, with or without understanding.
	A2	Correct on items with different initial decimal places. Unsure about 4.4502 /4.45. May only draw analogy with money. May have little understanding of place value, following partial rules.
L longer-is-larger	L1	Interprets decimal part of number as whole number of parts of unspecified size, so that 4.63>4.8 (63 parts is more than 8 parts).
	L2	As L1, but knows the 0 in 4.08 makes decimal part small so that 4.7>4.08. More sophisticated L2 students interpret 0.81 as 81 tenths and 0.081 as 81 hundredths etc resulting in same responses.
S shorter-is-larger	S1	Assumes any number of hundredths larger than any number of thousandths so 5.736 < 5.62 etc. Some place value understanding.
	S3	Interprets decimal part as whole number and draws analogy with reciprocals or negative numbers so 0.3>0.4 like 1/3>1/4 or -3>-4.
U	U2	Can “correctly” order decimals, but reverses answers so that all are incorrect (e.g. may believe decimals less than zero)
	U1	Unclassified – not fitting elsewhere. Mixed or unknown ideas.

How adequate is DCT2 as an instrument to map where students are on their journeys to full understanding? Clearly it has limitations, but it also has many strengths. Its ease of administration made the longitudinal study of a large number of students possible. The test can reliably identify a wide range of student responses, as illustrated in Table 2. Test-retest agreement is high. Even after one semester, when one would expect considerable learning to have occurred, 56% of students re-tested in the same fine code (calculation from data in Steinle 2004, Table 5.17). Where we have interviewed students shortly after testing, they generally exhibit the diagnosed way of thinking in a range of other items probing decimal understanding. There is one important exception. Very frequently, students whom the test diagnoses as expert (A1) are (i) not experts on other decimal tasks and (ii) it is also sometimes the case that they can correctly complete comparison items but do *not* have a strong understanding of decimal notation. For this reason our code for expertise is A1, with A standing for *apparent task expert*. In relation to point (i), our intensive use of one task has highlighted for us that expertise in one task does not necessarily transfer to related tasks without specific teaching. For example, A1 students being expert in the comparison test would be able to order books in a library using the Dewey decimal system. However, they may have little idea of the metric properties of decimals: that 0. 12345 is very much closer to 0.12 than it is to 0.13, for example, and they may not be able to put numbers on a number line. We therefore make no claim that our *apparent task experts* in A1 are expert on other decimal tasks. In relation to point (ii), students with either good or poor understanding can complete DCT2 correctly by following either of the two expert rules (left-to-right digit comparison or adding zeros and comparing as whole numbers e.g. compare 63 and 80 to compare 4.63 and 4.8). DCT2 therefore *over-estimates* the number of experts. As a tool to map students' progress it overestimates the numbers who have arrived at the destination. Its strength is in identifying the nature of erroneous thinking. Some mathematics educators may be inclined to dismiss DCT2 as "just a pencil-and-paper test" and take the position that only an interview can give reliable or deep information about student thinking. I contend that carefully designed instruments in any format with well studied properties, are important for advancing research and improving teaching. Many interviews also miss important features of students' thinking and unwittingly infer mastery of one task from mastery of another.

THE JOURNEYS

Some sample journeys

Table 3 shows the journeys of 9 students in the longitudinal study. It shows that Student 210403026 completed tests each semester from the second semester of Grade 4 to the first semester of Grade 7, and was absent on one testing day in Grade 5. Student 300704112 always tested in the L coarse code, which is an extreme pattern that sadly does not reveal any learning about this topic in two and a half years of school attendance. Student 310401041 completed 7 tests, being diagnosed as either unclassified or in the L coarse code. Student 410401088, however, moved from L behaviour to expertise in Grade 7. Some of the students in Table 3 have been chosen to illustrate how many students persist with similar ways of thinking over several years. The average student showed more variation than these. In addition, there is always the possibility that changes between tests have been missed, since students were tested at most twice per year. Some students show movement in and out of A1.

Table 3: A sample of students' paths through the study

ID	**Grade 4**		**Grade 5**		**Grade 6**		**Grade 7**		**Grade 8**		**Grade 9**		**Grade 10**	
210403026		L1		A1	S3	S5	S1							
300704112							L1	L4	L4	L2	L1			
310401041		L2		L1	U1	U1	L4	U1	U1					
390704012							L1	A1	U1	A1	S3			
400704005							A1	A2	A1	A2	A1			
410401088		L1	L1		L4	L1	L2	A1	A1					
500703003								S1	S5		S3	S3	U1	
500703030								S3	S5		S1	A2		
600703029								A1		U1	A1	A1	A3	

Prevalence by grade: where the students are in each year of the journey

Figures 2, 3a and 3b show the percentage of students who are in each of the codes by grade level. This data is the best estimate available from the longitudinal study (technically, the improved test-focussed prevalence of Steinle (2004)). As expected, the percentage of experts on the test (A1 in Figure 2) grows markedly in the early years, rising steadily until Grade 8. However, at Grade 10, which is regarded as the end of basic education, it is still only at 70% indicating that there are likely to be many adults without a strong understanding of decimal numbers. This observation is reinforced by studies of teacher education students (Stacey *et al*, 2001c) and nurses where "death by decimal" (Lesar, 2002) is a recognised phenomenon. Measuring expertise with the DCT2 over-estimates, we summarise by noting that one quarter of students attain expertise within a year or so of first being introduced to decimals (i.e. in grade 5), a further half of students attain expertise over the next 5 years, leaving a quarter of the school population who are not clear on these ideas by the end of Grade 10.

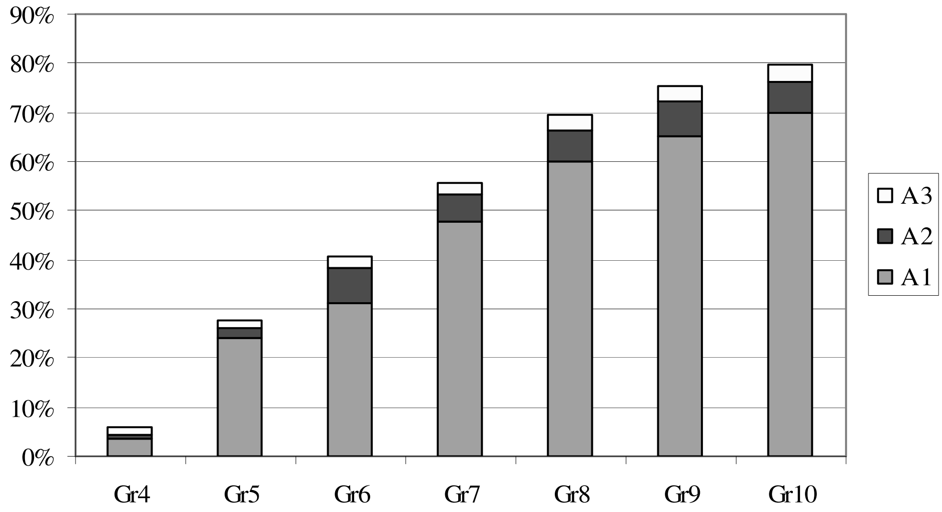


Figure 2: Best estimate of the prevalence of A codes by grade
(from Figure 9.3, Steinle, 2004)

Figure 2 also shows that the percentage of students in the non-expert A group remains (i.e. A2/A3) at about 10% from Grade 6 throughout secondary school, and for reasons related to the test construction, we know this to be an under-estimate. These students operate well on the basic items, but make errors on what could be expected to be the easiest comparisons, such as 4.45 and 4.4502. We believe there are several causes: an over-reliance on money as a model for decimal numbers; over-institutionalisation of the practice of rounding off calculations to two decimal places; and use of partially remembered, poorly understood rules for comparing decimals. A2 and A3 students function well in most circumstances, but may in reality have very little understanding. We have several times overheard teachers describing their A2 students as having "just a few more little things to learn". In fact these students may have almost no understanding of place value.

Figure 3a shows how that the prevalence of L codes drops steadily with grade. As might be expected, the naïve misconception that the digits after the decimal point function like another whole number (so that 4.63 is like 4 and 63 units of unspecified size and 4.8 is 4 plus 8 units of unspecified size), is an initial assumption about decimal numbers, and Foxman *et al* (1985) demonstrated that it is exhibited mainly by low achieving students. The fairly constant percentage of students in category L2 (around 4% up to Grade 9) provides an example of how students' knowledge sometimes grows by just adding new facts to their accumulated knowledge, rather than building a consistent understanding based on fundamental principles. One cause of code L2 is that L1 students simply add an extra piece of information to their pre-existing way of thinking – commonly in this case, the information that a decimal number with a zero in the tenths column is small so that 4.08 < 4.7 even though 8>7.

Figure 3b shows the best estimate of prevalence of the S codes. These codes are less common, but there is no consistent trend for them to decrease: instead about 15% of students in most grades exhibit S behaviour at any one time. The largest group is in code S3, which is again a naïve way of thinking not appreciating place value. That over 10% of Grade 8 students (those in S3) will consistently select 0.3 as smaller than 0.4 is an extraordinary result. Earlier studies had omitted these items from tests, presumably because they were thought to be too easy. We believe that S thinking grows in junior secondary school largely because of interference at a deep psycholinguistic or metaphorical level from new learning about negative numbers, negative powers (e.g. 10^(-6) is a very small number) and more intense treatment of fractions, and a strange conflation of the spatial spread of place value columns with number-lines. These ideas are explained by Stacey, Helme & Steinle (2001a).

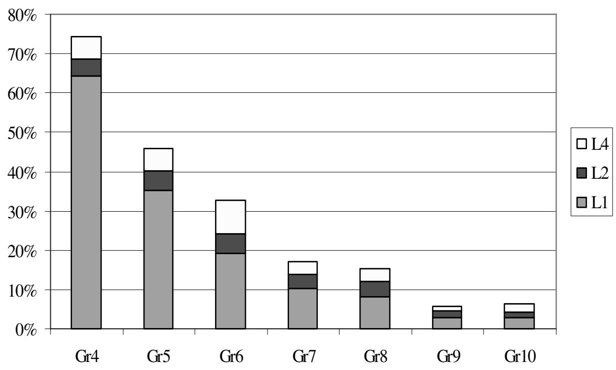


Figure 3a: Prevalence of L codes by grade (from Figure 9.7, Steinle, 2004)

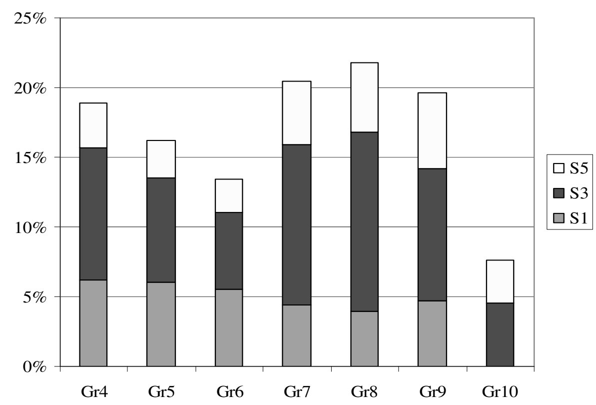


Figure 3b: Prevalence of S codes by grade (from Figure 9.10, Steinle, 2004)

Student-focussed prevalence: how many students visit each town?

The data above have shown the percentage of students testing in various codes – in the journey metaphor, a snapshot of where the individuals are at a particular moment in time. This is one way to answer to the question "how prevalent are these ways of thinking". However, it is also useful to see how many students are affected by these ways of thinking over their schooling, which is analogous to asking how many students visited each town sometime on their journey. Figure 4 shows the percentage of students who tested in each coarse code at some time in primary school, or at some time in secondary school. These percentages add up to more than 100% because students test in several codes. This data in Figure 4 is based on the 333 students in primary school and 682 students in secondary school who had completed at least four tests at that level of schooling. Had any individual been tested more often, he or she may have also tested in other codes. Hence it is evident that the data in Figure 4 are all under-estimates.

This new analysis gives a different picture of the importance of these codes to teaching. For example, less than 25% of students exhibited S behaviour at any one test, but 35% of students were affected during primary school. Similar results are evident for the fine codes, although not presented here. For example, Fig. 3b shows that about 6% of students were in S1 at any one time, but at least 17% of primary and 10% of secondary students were in S1 at some time. As noted above, these are underestimates.

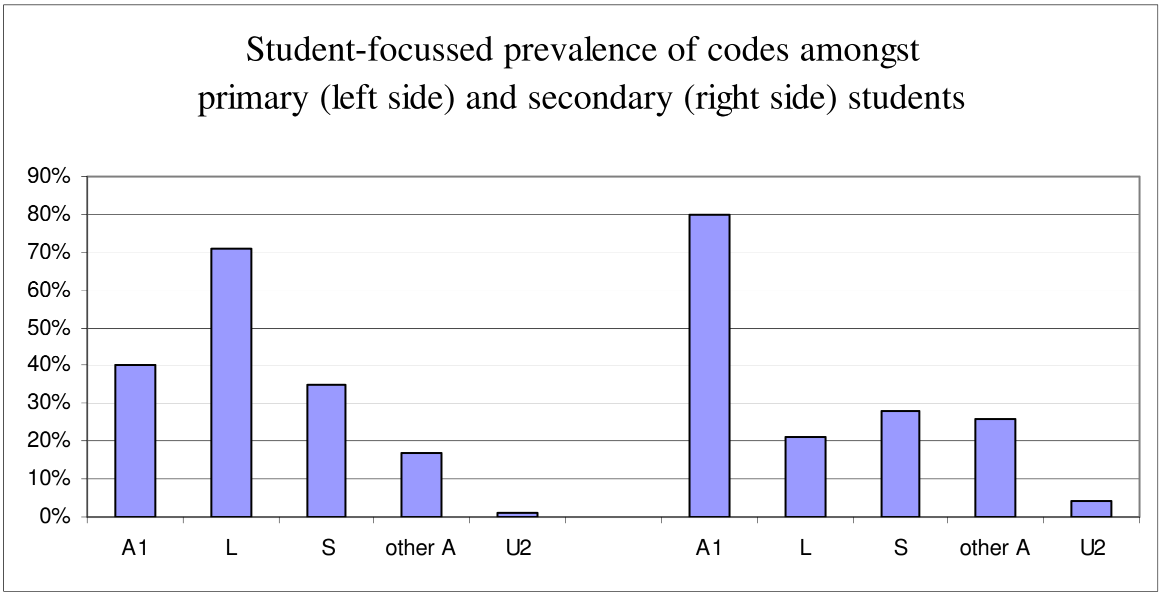


Figure 4: The percentage of students who test in given codes at some stage in primary and secondary school (derived from Steinle, 2004, Ch 9).

Persistence: which towns are hard to leave?

The sections above show where students are at various stages on their journeys. In this section we report on how long they stay at each of the towns on their journey. These towns are not good places to be, but how attractive are they to students? Figure 5a shows that around 40% of students in the L and S codes retested in the same code at the next test (tests averaged 8.3 months apart). The figure also shows that after 4 tests (averaging over two and a half years) still about 1 in 6 students retest in the same code. It is clear from this data that for many students, school instruction has insufficient impact to alter incorrect ideas about decimals.

Fortunately, expertise is even more persistent than misconceptions. On a test following an A1 code, 90% of A1 students rested as A1 and the best estimate from Steinle (2004) is that 80% of A1 students always retest as A1. This means that about 20% of the DCT2 "experts" achieve this status by less than lasting understanding (e.g. by using a rule correctly on one occasion, then forgetting it).

Figure 5b shows an interesting phenomenon. Whereas persistence in the L codes decreases with age (Figure 5b shows L1 as an example), persistence in the S and A2 codes is higher amongst older students. This might be because the instruction that students receive is more successful in changing the naive L ideas than S ideas but it is also likely to be because new learning and classroom practices in secondary school incline students towards keeping S and A2 ideas. The full data analysis shows that this effect occurred in nearly all schools, so it does not depend on specific teaching.

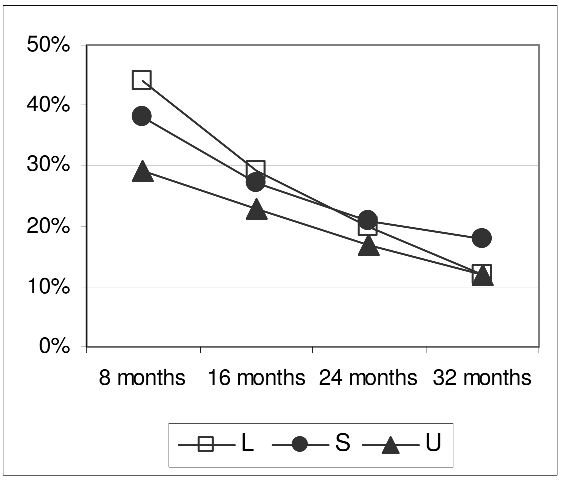


Figure 5a: Persistence in L, S and U codes after 1, 2, 3 or 4 semesters (adapted from Steinle, 2004, Fig. 6.5)

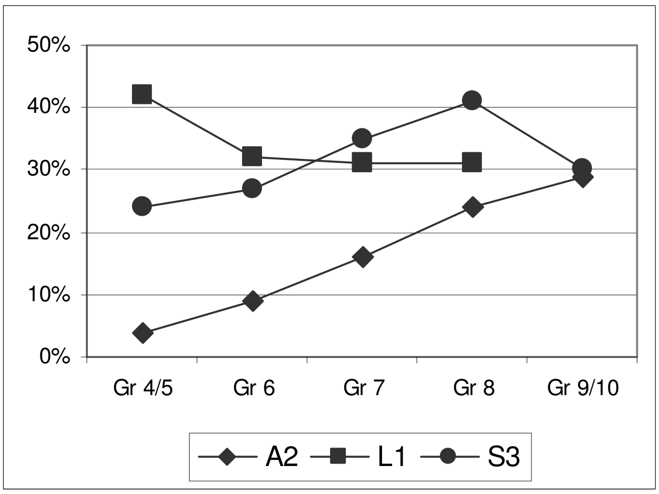


Figure 5b: Persistence in A2, L1, S3 and S5 over one semester by grade of current test (adapted from Steinle, 2004, Fig. 6.1)

Proximity to expertise: which town is the best place to be?

A final question in describing students' journeys is to find which town is the best place to be. In other words, from which non-A1 code is it most likely that a student will become an expert on the next test? Figure 6 shows the best estimates of Steinle (2004) from the longitudinal data. For both primary and secondary students the A codes and the U codes have the highest probabilities. The case of the A codes will be discussed below. The vast majority of students in U ("unclassified") do not respond to DCT2 with a known misconception: they may be trying out several ways of thinking about decimals within one test, or simply be guessing. Figure 5a shows that the U coarse code is the least persistent, and the data in Figure 6 shows that there is a relatively high chance that U students will be expert on the next test. It appears that it is worse to have a definite misconception about decimals than to be inconsistent, using a mix of ideas or guessing. Perhaps these students are more aware that there is something for them to learn and are looking for new ideas.

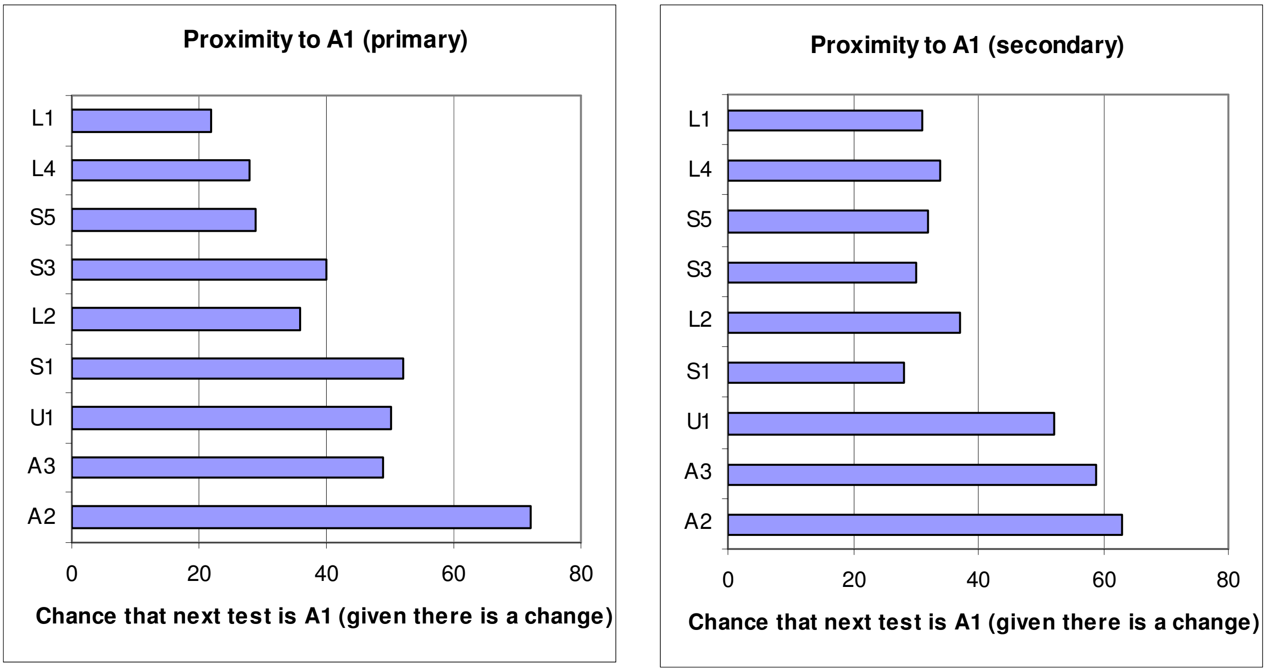


Figure 6: Chance that the next test is A1, given there is a change of code, for primary and secondary cohorts. (Codes ordered according to combined cohort proximity.)

Students in the L codes generally have only a low chance of moving to expertise by the next test. This bears out predictions which would be made on our understanding of the thinking behind the L codes. Since L1 identifies students who generally think of the decimal part of the number as another whole number of parts of indeterminate size, L1 is rightly predicted to be far from expertise. The L2 code (see Table 2) consists of at least two groups: one who graft onto L1 thinking an isolated fact about numbers with a zero in the tenths columns and a more sophisticated group of students with some place value ideas. Is the much greater chance of L2 students becoming expert over L1 students attributable to both or to the more sophisticated thinkers only? This is an example of a question that needs a more refined test than DCT2.

In the above section on persistence, I commented that the S codes behave differently in primary and secondary schools. This is again the case in Figure 6. Whereas primary students in S codes have a better chance than L students to become experts, this is not the case in secondary school. This is *not* because S students are more likely to stay in S, because the analysis has been done by removing from the data set those students who do not change code. Exactly what it is in the secondary school curriculum or learning environment that makes S students who change code more likely to adopt ideas which are not correct, is an open question.

The A codes have very high rates of progression to A1. This is of course good, but there is a caution. As noted above, students who have tested as A1 on one test generally stay as A1 on the next test, but 10% do not (see for example, students 400704005 and 600703029 from Table 3). The A2 and A3 codes are over-represented in these subsequent tests. This indicates to us that some of the A1 students are doing well by following partly understood and remembered versions of either of the two expert rules, possibly so partial as to simply make a decision on the first one or two decimal places (e.g. by analogy with money), truncated or rounded. In a "tricky" case such as the comparison 4.4502/4.45, these partially remembered rules fail. Truncating or rounding to one or two decimal digits gives equal numbers and to carry out the left-to-right digit comparison rule, the 0 digit has to be compared with a blank. Poorly understood and remembered algorithms are likely to fail at this point, resulting in *ad hoc* guessing. As students complete subsequent tests in A1, A2 and A3, moving between them, we see examples of Brown and VanLehn's (1982) "bug migration" phenomenon. There is a gap in students' understanding or in their memorised procedures, and different decisions about how to fill this gap are made on different occasions. Our work with older students (e.g. Stacey *et al*, 2001c) shows that these problems, evident in comparisons such as 4.45/4.4502, remain prevalent beyond Grade 10. The movement between the A codes is evidence that a significant group of the DCT2 "experts" have little place value understanding.

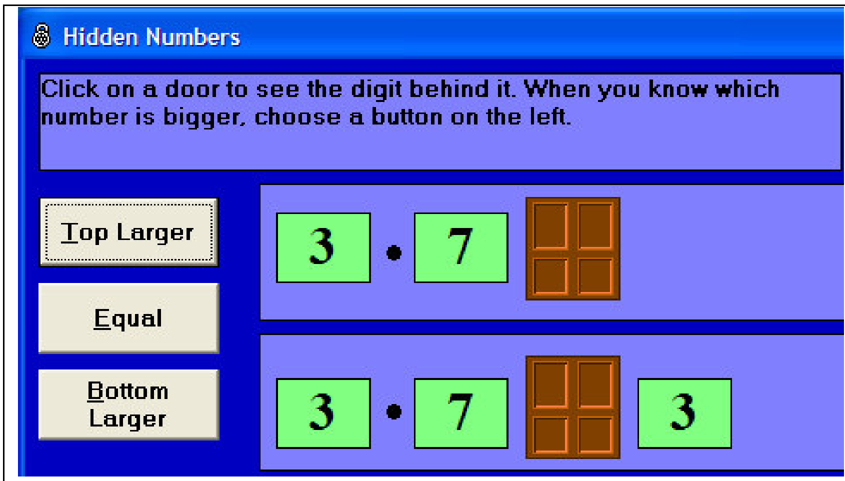
The study of student's thinking especially in the A and S codes has highlighted difficulties associated with zero, both as a *number* and as a *digit*, that need attention throughout schooling (Steinle & Stacey, 2001). Zeros can be visible or invisible and represent the number between positive and negative numbers, or a digit. As a digit, zero operates in three ways numbers; to indicate there are zero components of a given place value, as a place holder to show the value of surrounding digits, and also to indicate the accuracy of measurement (e.g. 12 cm vs 12.0 cm) although the latter interpretation has not been explored in our study. Improved versions of the decimal comparison test, especially for older students, include more items involving zeros in all of these roles, and allow the comparisons to be equal (e.g. 0.8 with 0.80).

HOW IS A DETAILED MAP OF LEARNING USEFUL?

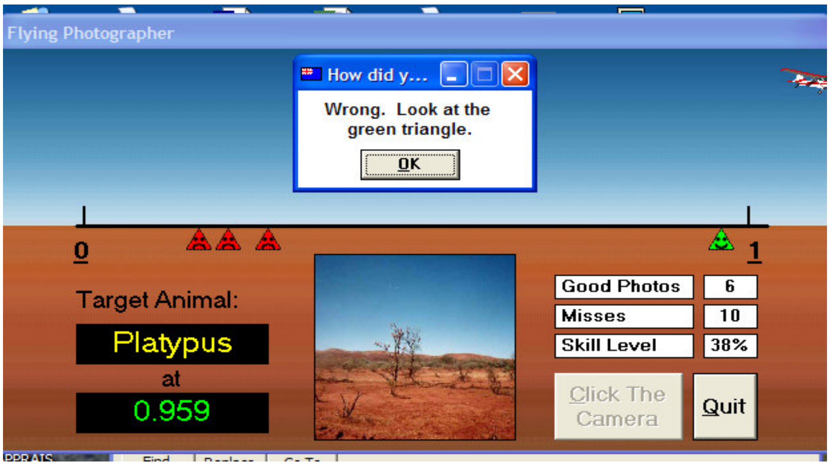
The research work in the 1980s using comparison of decimals identified three "errorful rules". The map of the territory of learning decimals at that stage therefore divided it into four regions (expertise and three others). DCT2 can diagnose students into 12 groups (the 11 of the longitudinal study and one other). As we interviewed students who tested in different codes on DCT2 and examined responses to the sets of items more closely, we came to realise that several ways of thinking lay behind some of our codes (e.g. L2, S3), which opened up the possibility of making further refinements to DCT2 to separate these groups of students. We also discovered other ways of thinking that DCT2 did not properly identify, such as problems with 0. We refined DCT2 to better identify some of these groups. However, the important question which is relevant to all work on children's thinking is how far it is useful to take these refinements. How fine a mapping tool will help students on the journey?

For teaching, it is common for people to say that only the coarsest of diagnoses is useful. The argument is that busy teachers do not have the time to carefully diagnose esoteric misconceptions, and in any case would be unable to provide instruction which responded to the information gained about an individual student's thinking. I agree. Our experience in teachers' professional development indicates that they find some knowledge of the misconceptions that their students might have to be extremely helpful to understand their students, and to plan their instruction to address or avoid misinterpretations. Hence they find that the coarse grained diagnosis available for example from the Quick Test and Zero Test (Steinle et al, 2002) is of practical use.

However, in many countries, we will soon be going beyond the time when real-time classroom diagnosis of students' understanding is the only practical method. The detailed knowledge of student thinking that has been built up from research can be built into an expert system, so that detailed diagnosis can be the province of a computer rather than a teacher. Figure 7 shows two screen shots from computer games which input student responses to a Bayesian net that diagnoses students in real time and identifies the items from which they are most likely to learn. Preliminary trials have been promising (Stacey & Flynn, 2003a). Whereas all students with misconceptions about decimal notation need to learn the fundamentals of decimal place value, instruction can be improved if students experience these fundamental principles through examples that are individually tailored to highlight what they need to learn. Many misconceptions persist because students get a reasonable number of questions correct and attribute wrong answers to "careless errors". This means that the examples through which they are taught need to be targeted to the students' thinking. An expert system can do this (Stacey et al, 2003b).



In *Hidden Numbers*, students pick the relative size of two numbers, revealing digits by opening the doors. This task reveals misconceptions e.g. when students select by length or open doors from the right. An expert system diagnoses thinking and provides tasks for teaching or diagnosis.



The *Flying Photographer* has to photograph animals (e.g. platypus) from an aeroplane, given decimal co-ordinates (e.g. 0.959). This task uses knowledge of relative size, not just order. An expert system tracks responses (e.g. if long decimals are always placed near 1) and selects new items to highlight concepts.

Figure 7. Screen shots from two games which provide diagnostic information to an expert system which can diagnose students and select appropriate tasks.

LESSONS ABOUT LEARNING

An overview of the journey

The longitudinal study has examined students' progress in a specific mathematics topic, which complements other studies that have tracked growth in mathematics as a whole or across a curriculum area. The overall results demonstrate the substantial variation in ages at which expertise is attained, from a quarter of students in Grade 5 to about three quarters in Year 10. The good alignment of data from the longitudinal study and the random sample of TIMSS-R shows that we can confidently recommend that this topic needs attention throughout the grades in most secondary schools. The fact that about 10% of students in every grade of secondary school (fig. 2) are in the non-expert A codes (A2 and A3) shows that many students can deal apparently expertly with "ordinary" decimals, which conceals from their teachers and probably from themselves, their lack of understanding of fundamental decimal principles.

Moreover, the fact that many students retain the same misconception over long periods of time (e.g. about 20% in the coarse codes over 2 years, and around 30% in some fine codes over 6 months) demonstrates that much school instruction does not make an impact on the thinking of many students. Our study of proximity to expertise provides empirical support for the notion that it is harder to shake the ideas of students who have a specific misconception than of those who do not; again this points to the need for instruction that helps students realise that there is something for them to learn, in a topic which they may feel they have dealt with over several years.

One important innovation of this study is to look not just at the prevalence of a way of thinking at one time, but to provide estimates of how many students are affected in their schooling, which provides a different view of the practical importance of phenomena.

How the learning environment affects the paths students take

Another important result of this study is that in the different learning environments of primary and secondary school, students are affected differently by various misconceptions. For example, the S misconceptions in primary school are relatively quickly overcome, being not very persistent and with high probability of preceding testing as an expert, but this is not the case in secondary school.

The very careful study of the responses to DCT2 and later comparison tests has revealed a wide range of students' thinking about decimals. As demonstrated in earlier studies, some students (e.g. L1) make naïve interpretations, overgeneralising whole number or fraction knowledge. Others simply add to a naïve interpretation some additional information (e.g. some L2, and see below). We have proposed that some false associations, such as linking numbers with whole number part of 0 with negative numbers, arise from deep psychological processes (Stacey et al, 2001a). Other students (e.g. some A2) seem to rely only on partially remembered rules, without any definite conceptual framework. We explain the rise in the prevalence and persistence of S and non-expert A codes in the secondary school mainly through reinforcement from new classroom practices, such as rounding to two decimal places and interference from new learning (e.g. work with negative numbers). This shows that other topics in the mathematics curriculum, and probably also other subjects, affect the ideas that students develop and the paths that they take among them.

Learning principles or collecting facts

Although understanding decimal notation may appear a very limited task, just a tiny aspect of a small part of mathematics, full understanding requires mastery of a complex web of relationships between basic ideas. From the perspective of the mathematician, there are a few fundamental principles and many facts are logically derived from them. From the point of view of many learners, however, there are a large number of facts to be learned with only weak links between them. This is demonstrated by the significant size of codes such as A2 (e.g. with secondary students confident only with tenths, without having made the generalisation of successive decimation). Teaching weakly linked facts rather than principles is inherent in some popular approaches, such as teaching one-place decimals first, then two-place decimals the next year, without exposing what we call the “endless base ten chain”. Artificially high success in class comes by avoiding tasks which require understanding the generalisation and principles, and concentrating on tasks with predictable surface features (e.g. Brousseau, 1997; Sackur-Grisvard *et al*, 1985).

For mathematics educators, the challenge of mapping how students think about mathematical topics is made considerably harder by the high prevalence of the collected facts approach. As the case of decimal numeration illustrates, we have tended to base studies of students’ thinking around interpretations of principles, but we must also check whether that current theories apply to students and teachers who are oriented to the collected facts view, and to investigating how best to help this significant part of the school population.

Tracing the journeys of students from Grade 4 to Grade 10 has revealed many new features of how students’ understanding of decimals develops, sometimes progressing quickly and well, but for many students and occasionally for long periods of time, not moving in productive directions at all. The many side-trips that students make on this journey point to the complexity of the learning task, but also to the need for improved learning experiences to assist them to make the journey to expertise more directly.

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IDENTITY THAT MAKES A DIFFERENCE: SUBSTANTIAL LEARNING AS CLOSING THE GAP BETWEEN ACTUAL AND DESIGNATED IDENTITIES

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In the attempt to account for striking differences between learning activities of immigrant mathematics students from the former Soviet Union and of their native Israeli classmates, we introduce the notions of actual *and* designated identities. *These identities are subsequently presented as important factors that mold learning and influence its effectiveness. Since designated identities may be seen as personalized, "customized" versions of people's cultural heritages, ours is the story of the wider culture making its way into individual learning processes.*

> [For me,] school mathematics was … something that one cannot escape and must try to be done with as quickly as possible… The numbers did not scare me; rather the scary part was my complete lack of interest in them… All that I remember now is my constant effort to match formulas with exam questions.

This quote from a retrospective account of a successful university student[1] is unlikely to surprise a person who knows a thing or two about mathematics learning and teaching. We are all only too familiar with this kind of unhappy reminiscences. Much less common are reports about mathematics-related experiences of interest and joy, such as the one provided by another high-school graduate:

> Mathematics lessons were my favorites. If they were difficult, I saw them as a challenge, as a puzzle to cope with. I was ready to invest time and effort in solving special bonus problems.

What is it that makes some students learn mathematics willingly and with interest while leaving many of their peers indifferent, if not openly resistant? How does this difference influence the learning practices of the student? These questions are certainly not new. They have been fueling mathematics education research ever since its inception. The study to be presented in this talk is a result of yet another attempt to come to grips with the long-standing quandaries.

[1] This and the following excerpt are taken from autobiographical accounts of students who participated in university courses given by the first author in the Education Department at the University of Haifa.

Our research project was occasioned by the recent massive immigration from the former Soviet Union to Israel.[2] More specifically, it was triggered by a spontaneous, yet-to-be-tested observation that a disproportionately large portion of this particular group of immigrants could pride itself with impressive results in mathematics, and not just in school, but also in national and international mathematical competitions.[3] We began asking ourselves whether there was anything unique about the immigrant students' mathematics learning and if there was, how this uniqueness could be accounted for. The conjecture we wished to test while launching our investigation was that dissimilarities in learning processes, rather than being a simple outcome of cognitive differences between individual learners, are a mixed product of individual and collective doing. Such differences, we believed, are often reflective of differing sociocultural histories of the learners.

In what follows, we try to substantiate this hypothesis on the basis of our findings. We begin with detailed examples of the two types of learning, the *ritualized* and the *substantial*, signaled by the students' testimonies quoted above. In our study, both kinds of learning have been found in one class consisting of native Israelis and immigrant mathematics students. The dissimilarities in learning paralleled the difference in the students' sociocultural background. In the attempt to understand how sociocultural factors made their way into the learners' individual activities, we introduce the notions of *actual* and *designated identities* which then serve as the "missing link" between culture and learning.

TWO TYPES OF LEARNING: SUBSTANTIAL AND RITUALIZED

Example to think with: NewComers and OldTimers as mathematics learners

The study began in fall of 1998 and focused on one 11th grade class that followed an advanced mathematics program. 9 out of the 19 students were *NewComers* – recent immigrants from big cities in the former Soviet Union such as Moscow, Kiev and Tbilisi. The rest were native Israelis, whom we call *OldTimers*. All of the students came from well-educated families. The second author, a one-time immigrant from the Soviet Union, served as the teacher. In the course of the entire school year all classroom processes were meticulously observed and documented. Numerous interviews with the students, with their parents and with other teachers constituted additional data.

[2] According to the leading Israeli newspaper *Haaretz*, "Approximately 200 thousand children immigrated to Israel in 11 years, most of them from the former Soviet Union; they constitute 15% of the Israeli youth"(31.08.2001).

[3] This conjecture should not be misread as saying that the immigrants from the former Soviet Union are generally highly successful in mathematics. This said, "[t]here are [immigrant] children who arrive at the highest places in international competitions in mathematics and physics and thanks to them, Israel climbed from 24th to 13th place in the 1995 international championship" (*Haaretz,* 2 August 1996).

The salience of the differences between the learning processes of the two groups exceeded our expectations. In this article we present only a tiny vignette from this extensive research project (the full report can be found in Prusak 2003). It must be stressed, however, that the striking intra-group homogeneity and the significant inter-group difference reported on these pages is representative of all our results, whatever the particular aspect of learning considered in the analyses.

The sub-study in question focused on independent learning. Our story begins in the tenth week of the school year, on the day when the class got the unusual homework assignment: After having learned trigonometry for two months and, in particular, after being introduced to the theorem known as *law of sines*, the learners were asked to study the new subject, *law of cosines and its applications*, with the help of a textbook. To guide their independent learning, the teacher proposed a work plan, which was presented as a series of questions to be answered in the course of the study*: (1) How can the law of cosines be presented in words?(2) How can it be formulated in the language of algebra? (3) How can it be proved? (4) What is its importance?* The teacher advised that the students write their answers to the questions once they were sure they understood the subject.

The first difference between the two groups has shown when, a few days later, the teacher asked to see the notes made by the learners as a part of their homework assignment. This request surprised some students. After all, the teacher did not request the written answers, she had only recommended them as potentially helpful. And yet, whereas only 4 out of the 9 NewComers had anything written to show, the OldTimers, with no exception, were able to come up with the kind of notes the teacher was asking for. The two groups differed further in the nature of the available record. As a rule, the OldTimers' answers to the teacher's questions were simply the relevant passages copied from the textbook. Of the four NewComers who did make notes, only one answered all four questions, whereas the sole focus of the other three sets of records was the proof of the cosine law (question 3 in the work plan.) Two of these proofs were quite unlike anything that could be found in other students' notebooks, so it was clear that these were students' reconstructions rather than quotes from the book.

Impressed by this visible disparity, the teacher asked whether anybody in the class felt a need for an additional explanation. This time, there was no difference between the OldTimers and NewComers: All the students felt that the topic has been understood. In spite of this, the teacher declared her wish to probe a bit further. She asked the class to formulate the law of cosines and to prove it in writing. The request was accompanied by a blackboard drawing of a triangle, marked with letters different from those that appeared in the textbook. The following passage from the teacher's journal presents students' reaction to the previously unannounced test:

> Several OldTimers started complaining: "We learned at home with the letters A, B, C and we got used to them"… The Newcomers did not show any sign of surprise. All of them, even Boris, usually the slowest, finished quickly.

Type of response	**Number of responses**	
	OldTimers	**NewComers**
Full proof, textbook version	1	6
Full proof, modified version	-	2
Partial, erroneous proof	1	-
No proof	8	1

Table 1: Students' responses to the request to prove the law of cosines

As shown in Table 1, the results attained by the two groups could hardly be more dissimilar: While all NewComers but one succeeded in the task, only one of the OldTimers was able to produce a reasonable proof. Moreover, two of the NewComers came up with their own versions of the proof, the type of response that is usually taken as the most persuasive evidence of understanding.

OldTimers (translated from Hebrew)	**NewComers** (translated from Russian)
Ada, *who* did not *succeed in reproducing the proof:* I read the chapter in the book and tired to understand When I felt I understood, I copied the proof to the notebook	***Sonya,*** *who succeeded in reproducing the proof:* I read the proof a number of times, trying to remember and making notes on a separate page. I reproduced the proof without writing and I wrote the proof from memory with the book closed. I compared the proof to the one in the book. I then read and tried to understand the examples [of application] in the book
Liora, *who* did not *succeed in reproducing the proof:* Copied the verbal formulation [of the cosine law], drew a triangle *in the head* [the student's own emphasis], read the verbal presentation and translated to letters in the head. Compared the formula to the one in the book and copied into the notebook. Read the proof and understood what they did. Solved the problems with the help of the formula. In case [I] could not do it, read the solved example.	***Misha,*** *who succeeded in reproducing the proof:* I began by translating [to Russian] of all the words in the theoretical text that were unclear to me. I read the theorem again until I understood its proof. When I was sure I understood the theorem, I drew a triangle with vertices marked differently than in the book and I wrote the new proof without looking into the book. After I finished, I checked the correctness of the proof with the help of the book. I read and understood the solved examples [of problems] in the book and began solving the homework problems.

Table 2: Representative responses to the question
How did you learn? Describe the process in some detail.

Once they completed their proofs, the students were asked to describe in writing the steps they performed while implementing the homework assignment. The

NewComers were allowed to respond in Russian. The English version of representative answers can be found in Table 2. The two columns give rise to two strikingly different pictures of the learning process: Whereas the OldTimers satisfied themselves with reading the book and answering the teacher's four questions by copying the relevant passages from the book, the NewComers intertwined reading the textbook exposition with their own independent attempts to formulate and prove the theorem.

We may now sum up and say that the OldTimers and NewComers differed in a consistent manner both in the way they learned and in the results attained. The learning process of the NewComers was clearly associated with their greater success on the test. The fact that the sequence of steps performed by the only OldTimer who managed to produce a correct proof was closer to that of NewComers than to that of OldTimers confirms this latter claim: There seems to be a tight correspondence, perhaps even a causal relationship, between the way NewComers learned and the effectiveness of their learning.

DEFINING SUBSTANTIAL AND RITUALIZED LEARNING

The first thing that strikes the eye in our data is that NewComers' and OldTimers' actions seem to have been directed at different recipients. The fact that the OldTimers implemented all the tasks required by the teacher apparently without asking themselves why they were performing these particular steps shows that, for these learners, the teacher was the ultimate addressee. NewComers, unlike OldTimers, did not perform all the prescribed tasks, and if they did, they did not leave any written records, evidently not being bothered about showing their work to the teacher. Thus, whatever these latter students did at home, they did it for themselves, according to their own assessment of its importance. In this activity, they were their own judges, and we have grounds to suspect that in this role, some of them were more exacting than anybody else, including the teacher.

Activities that have different addressees are usually perceived as having different goals. Clearly, in the eyes the OldTimers the process of learning was the end in itself, whereas the only thing that really counted for the NewComers was a certain product of the process, one that could be trusted to outlast the activity itself. In other words, the NewComers wanted the learning-induced change to be robust and durable. The desired lasting transformation can best be described in terms borrowed from what Harré & Gillet (1995) call *discursive psychology* and what was named *communicational approach to cognition* by other writers (Sfard 2001, Sfard & Lavi 2005; Ben Yehuda et al. 2005). According to the basic tenet of this approach, *thinking can be usefully conceptualized as a form of communication*, with this latter term signifying interaction that does not have to be audible, verbal, synchronic or directed at others. Within this framework, school *learning becomes the activity of changing one's discursive ways in a certain well defined manner*. In particular, learning to think mathematically is tantamount to being initiated into a special form of discourse, known as mathematical. Armed with this conceptual apparatus we may now say that for the NewComers, learning was the activity of introducing a lasting change into their own discursive activity, whereas for OldTimers it meant an episodic, ritualized participation in a discourse initiated by others.

We decided to call the two types of learning *substantial* and *ritualized*, respectively. In ritualized learning the learner engages in the mathematical discourse only in response to other person's request and for this other person's sake. In contrast, substantial learning may be defined as one that results in turning the new discourse from its initial status of a *discourse-for-others* into a *discourse-for-oneself*, that is, into a discourse in which this person is likely to engage spontaneously while solving problems and trying to answer self-posed questions.[4] This special kind of learning has a lasting effect on one's communication with oneself, that is, on this person's thinking.

The NewComer' strenuous effort toward substantial learning, noticed in the learning episode reported above, could be observed all along our extensive study, whatever the aspects of learning considered in its different segments. This effort was clear whether we were watching the students simplifying a complex algebraic expression, proving a trigonometric identity or trying to collaborate with others in solving a non-standard problem. On these diverse occasions, the NewComers' wish to turn the new discourse into a communication with themselves was evidenced also by their constant backtracking and self-examination, by their conspicuous preference for individual work, by their care for the appropriateness of their mathematical expression, and more generally, by their insistence on following all those rules of communication which they considered as genuinely 'mathematical'.

DEFINING IDENTITY[5]

Why talk about identity?

The striking dissimilarities between the OldTimers' and NewComers' learning called for explanation. Although we had a basis on which to claim the existence of some systematic differences in the teaching practices in the former Soviet Union and in Israel, these differences did not seem to tell the whole story. A teaching approach might have been responsible for the NewComers' acquaintance with certain techniques, but this fact, *per se*, did not account for the students' *willingness* to use these methods. We felt that to complete the explanation, we needed to clarify why the

[4] The term *discourse-for-oneself* is close to Vygotsky's idea of *speech-for-oneself*, introduced to denote a stage in the development of children's language (see e.g. Vygotsky 1987, p.71). Our terms also brings to mind the Bakhtinian distinction between *authoritative discourse*, a discourse that "binds us, quite independently of any power it might have to persuade us internally"; and *internally persuasive* discourse, one that is "tightly woven with 'one's own world.' (Bakhtin, 1981, pp. 110-111.)

[5] For a more extensive presentation of the topic see Sfard & Prusak 2005.

participants of our study were among those students who actually took advantage of the learning opportunities created by their teachers.

Yet another obvious explanation for the effectiveness of the NewComers' learning was that their immigrant status amplified their need for success.[6] Although certainly true, this account did not seem to be telling the whole story since it did not explain why school mathematics was singled out by the immigrant participants of our study as the medium through which to exercise their pursuit of excellence. Indeed, no other immigrant population, of which Israel has always had many, displayed a comparable propensity for mathematics. We decided to turn to the notion of identity, viewing it as a conceptual link between the collective and the individual.

Although the term "identity" is not new, it is only quite recently that it began drawing attention of educators at large, and of researchers in mathematics education in particular (see e.g., Boaler & Greeno, 2000; Nasir & Saxe, 2003; Cobb, 2004; Roth, 2004). Its new prominence is reflective of the general sociocultural turn in human sciences. The related time-honored notions of *personality*, *character*, and *nature,* being irrevocably tainted with connotations of natural givens and biological determinants, are ill-suited to the sociocultural project. In contrast, *identity*, which is thought of as man-made and as constantly created and re-created in interactions with others (Holland & Lave, 2003), seems just perfect for the task. Together with the acceptance of identity as the pivotal notion of the new research discourse comes the declaration about humans as active agents who play decisive roles in determining the dynamics of social life and in shaping individual activities.

We believe that the notion of identity is a perfect candidate for the role of "the missing link" in the researchers' story of the complex dialectic between learning and its sociocultural context. However, we also believe that this notion cannot become truly useful unless it is provided with an operational definition.

Defining identity

Its current popularity notwithstanding, the term 'identity' is usually employed without being operatively defined. The few defining attempts that can be found in the literature appear to be a promising beginning, but not much more than that. Gee (2001), who declares that "Being recognized as a *certain 'kind of person*' in a given context" (p. 99) is what he means by 'identity' also relates this notion to "the person's own narrativization" (p. 111), that is, to stories a person tells about herself. The motif of "person's own narrativization" recurs in the description proposed by Holland et al. (1998), even if formulated in different terms:

[6] As observed by Ogbu (1992), the status of minority is a doubly-edged sword. As shown by empirical findings, belonging to minority may, in some cases, motivate hard work and eventual success, whereas in some others it would have an opposite effect. Immigrants, whom Ogbu calls "voluntary minorities" as opposed to those whose minority status was imposed rather than chosen, are more likely than the others to belong to this former group.

> People tell others who they are, but even more importantly, they tell themselves and they try to act as though they are who they say they are. These self-understandings, especially those with strong emotional resonance for the teller, are what we refer to as identities. (p. 3)

If we said that these two descriptions are "promising beginnings" rather than fully satisfactory definitions, this is because of one feature that they have in common: They rely on the expression "who one is" or its equivalents. Unfortunately, neither Gee nor Holland and her colleagues make it clear how one can decide about "who" or "what kind of person" a given individual is. This said, their descriptions have an important insight to offer: By foregrounding "person's own narrativizations" and "telling who one is," these definitions link the notion of identity to the activity of communication. In an attempt to arrive at a more operational definition of identity we decided to build on the idea of identifying as communicational practice, thereby rejecting the notion of identities as extra-discursive entities which we merely "represent" or "describe" while talking.

In concert with the vision of identifying as a discursive activity, we suggest that identities may be defined as collections of stories about persons or, more specifically, as those narratives about individuals that are *reifying*, *endorsable* and *significant*. The reifying quality comes with the use of verbs such as *be*, *have* or *can* rather than *do*, and with the adverbs *always*, *never*, *usually*, etc. that stress repetitiveness of actions. A story about a person counts as *endorsable* if the identity-builder is likely to say, when asked, that it faithfully reflects the state of affairs in the world. A narrative is regarded as *significant* if any change in it is likely to affect the storyteller's feelings about the identified person. The most significant stories are often those that imply one's memberships in, or exclusions from, various communities.

As a narrative, every identifying story may be represented by the triple ${}_{B}A_{C}$, where A is the identified person, B is the author and C the recipient. Within this rendering it becomes clear that multiple identities exist for any person. Stories about a given individual may be quite different one from another, sometimes even contradictory. Although unified by a family resemblance, they depend both in their details and in their general purport on who is telling the story and for whom this story is meant. What a person endorses as true about herself may be not what others see enacted. To ensure that this last point never disappears from our eyes, we denote the different identities with names that indicate the relation between the hero of the story, the storyteller, and the recipient: ${}_{A}A_{C}$, a story told by the identified person herself, will be called A's *first-person* identity (1^{st} P); ${}_{B}A_{A}$, a story told to its main character, will be named *second-person* identity (2^{nd} P); finally, ${}_{B}A_{C}$, a story told by a third party to a third party, will be referred to as *third-person identity* (3^{rd} P). Among all these, there is one special identity that comprises the reifying, endorsable, significant 1^{st} P stories the storyteller addresses to herself (${}_{A}A_{A}$). It is this last type of stories that is usually intended when the word *identity* is used unassisted by additional specifications. Being a part of our ongoing conversation with ourselves, the first-person self-told identities are likely to have the most immediate impact upon our actions.

With the narrative definition, human agency and the dynamic nature of identity are brought to the fore, whereas most of the disadvantages of the traditional discourses on “personality”, “nature” or “character” seem to disappear. The focus of the researcher’s attention is now on things said by identifiers and no essentialist claims are made about narratives as mere “windows” to an intangible, indefinable entity. As stories, identities are human-made and not God-given, they have authors and recipients, they are collectively shaped even if individually told, and they can change according to the authors’ and recipient’ perceptions and needs. As discursive constructs, they are also reasonably accessible and investigable.[7]

Toward a theory of (narrative defined) identity

Since questions about identity can now be translated into queries about the dynamics of narratives, and since this latter phenomenon is amenable to empirical study, the narrative definition may be expected to catalyze a rich theory of identity. Much can now be said about identities simply by drawing on what is known about human communication and on how narratives interact one with another. Let us present some initial, analytically derived thoughts on how identities come into being and develop.

Actual and designated identities. The reifying, significant narratives about a person can be split into two subsets: *actual identity*, consisting of stories about the actual state of affairs, and *designated identity*, composed of narratives presenting a state of affairs which, for one reason or another, is *expected* to be the case, if not now then in the future. Actual identities are usually told in present tense and are formulated as factual assertions. Statements such as *I am a good driver*, *I have an average IQ*, *I am army officer* are representative examples. Designated identities are stories believed to have the potential to become a part of one’s actual identity. They can be recognized by their use of the future tense or of words that express wish, commitment, obligation or necessity, such as should, *ought*, *have to*, *must*, *want*, *can/cannot*, etc. Narratives such as *I want to be a doctor* or *I have to be a better person* are typical of designated identities.

The scenarios that constitute designated identities are not necessarily desired, but are always perceived as binding. One may expect to “become a certain type of person,”

[7] For all these obvious advantages, one may claim that “reducing” identity to narratives undermines its potential as a sense-making tool. Story is a text, the critic would say, and identity is also, maybe even predominantly, an experience (see e.g. Wenger, 1998). Although we agree that identities originate in daily activities and in the “experience of engagement”, we also posit that it would be a category mistake to claim that these characteristics disqualify our narrative rendering of identity. Indeed, it is our *vision* of our own or other people’s experiences, and not these experiences as such, that constitutes identities. Rather than viewing identities as entities residing in the world itself, our narrative definition presents them as *discursive counterparts* of one’s lived experiences.

that is, to have some stories applicable to oneself, for various reasons: because the person thinks that what these stories are telling is good for her, because these are the kinds of stories that seem appropriate for a person of her sociocultural origins or just because they present the kind of future she is designated to have according to others, in particular to those in the position of authority and power. More often than not, however, designated identities are not a matter of a deliberate rational choice. A person may be led to endorse certain narratives about herself without realizing that these are "just stories" and that they have alternatives.

Designated identities give direction to one's actions and influence one's deeds to a great extent, sometimes in ways that escape any rationalization. For every person, some kinds of stories have more impact than some others. *Critical* stories are those core elements, which, if changed, would make one feel as if one's whole identity changed: The person's 'sense of identity' would be shaken and she would lose her ability to tell in the immediate, decisive manner which stories about her are endorsable and which are not. A perceived persistent gap between actual and designated identities, especially if it involves critical elements, is likely to generate a sense of unhappiness.

Where do designated identities come from? The role of significant narrators. Being a narrative, the designated identity, although probably more inert and less context-dependent than actual identities, is neither inborn nor entirely immutable. Like any other story, it is created from narratives that are floating around. One individual cannot count as the sole author even of those stories that sound as if nobody has told them before.

To put it differently, identities are products of discursive diffusion – of our tendency to recycle strips of things said by others even if we are unaware of these texts' origins. Paraphrasing Mikhail Bakhtin, we may say that any narrative reveals to us stories of others.[8] Identities coming from different narrators and being addressed at different audiences are in a constant interaction and feed one into another. These stories would not be effective in their relation-shaping task if not for their power to contribute to the addressees' own narratives about themselves and about others. Thus, the people to whom our stories are told, as well as those who tell stories about us, may be tacit co-authors of our own designated identities. Either by animating other speakers or by converting their stories about us to the first person, we incorporate our 2nd and 3rd person identities into our self-addressed designated identities.

Another important sources of one's own identity are stories about others. There are many possible reasons for turning such narratives into first person and incorporating them into one's own designated identity. Thus, for example, the identity-builder may be attracted either to the heroes of these narratives or to their authors. Another reason may be one's conviction about being "made" in the image of a certain person (e.g., of

[8] Bakhtin (1999) spoke about utterances and words rather than stories.

socially deprived parents, alcoholic father or academically successful mother) and "doomed" to a similar life. Whether a story told by somebody else does or does not make it into one's own designated identity depends, among other things, on how significant the storyteller is in the eyes of the identified person. *Significant narrators,* the owners of the most influential voices, are carriers of those cultural messages that will have the greatest impact on one's actions.

Learning as closing the gap between actual and designated identities. It is now not unreasonable to conjecture that identities are crucial to learning. With their tendency to act as self-fulfilling prophecies, identities are likely to play a critical role in determining whether the process of learning will end with what counts as success or with what is regarded as failure.

These days, in our times of incessant change, when the pervasive fluidity of most social memberships and of identities themselves is a constant source of fears and insecurities, the role of learning in shaping identities may be greater than ever. Learning is our primary means for making reality in the image of fantasies. The object of learning may be the craft of cooking, the art of appearing in media or the skill of solving mathematical problems, depending on what counts as critical to one's identity. Whatever the case, learning is often the only hope for those who wish to close a critical gap between their actual and designated identities.

IDENTITY AS AN INTERFACE BETWEEN CULTURE AND LEARNING

The designated identities of NewComers and of OldTimers

Let us go back to our study on NewComers and OldTimers learning mathematics together and show how our conceptual apparatus helps us in answering the question about cultural embeddeddness of learning. Below we argue that designated identities of the OldTimers and of NewComers were the channel through which these students' cultural background was making its way into their mathematical learning.

To map NewComers' and OldTimers' designated identities, we listened to their stories about themselves told to their teacher on various occasions. True, what we really needed were *self-addressed* stories of the type ${}_{A}A_{A}$ rather than ${}_{A}A_{Teacher}$, because this former type of narrative was more likely to interact significantly with one's actions. This preference notwithstanding, we were confident that the teacher-addressed designated identities would prove informative, especially if they displayed diversity paralleling the observed differences in learning. Further, we made certain deductions regarding the NewComers' and OldTimers' expectations from themselves on the basis of their self-referential remarks, of their comments about others (e.g. the teacher of fellow students), and of our own observations on the ways they acted. As a background, we used interviews with the students' parents and with other teachers. What was found with the help of this multifarious evidence displayed intra-group uniformity and inter-group differences comparable in their salience to those observed previously in the context of the students' learning.

	OldTimers	NewComers
Future plans ("What do you want to do in future?")	• [What I want to do] changes, because I change • For me, the only important thing is to be happy, and I don't have any particular profession in mind.	• In Russia I knew all the time that I'll follow in my brother's footsteps and learn computers. • From the earliest childhood I dreamt to be a medical doctor, like my mother.
The reasons for learning mathematics ("I learn mathematics because…)	• matriculation certificate with advanced mathematics will help me to get to the university, especially if the grade is high • I have to pass matriculation examination if I want to achieve anything in life. • it is obligatory	• I need knowledge and good education, and I love learning. • mathematics is my favorite school subject • I need to be a "full-fledged human being" and I want to feel I did something in life. • for me learning mathematics means creativity • mathematics is important and I like it very much

Table 3: Elements of OldTimers' and NewComers' designated identities

As can be seen from the students' responses to the question "What do you want to do in future?" presented in Table 3, probably the most obvious critical element of the NewComers' vision of themselves in the future was their professional career. Their tendency to identify themselves mainly by their designated professions stood in stark contrast to the OldTimers' declarations on their need "to be happy" and the latter interviewees' adamant refusal to specify any concrete plans for the future. The professions desired by the NewComers (e.g., computer scientist, medical doctor, engineer) were all related to mathematics, and this appeared to account for these students' special mathematical proclivity. And yet, there seemed to be more to these students' inclination toward mathematics than just the wish to promote their professional prospects. According to the NewComers' frequent remarks, the special attraction of mathematics was in the fact that its rules could be seen as universal rather than specific to a particular place or culture. While explaining why they chose to learn advanced mathematics (see students' completions of the sentence "I learn mathematics because…" in Table 3), the NewComers spoke about the knowledge of mathematics as a necessary condition for her becoming "a fully-fledged human being." We have thus reason to claim that mathematical fluency as such, and not just anything that could be gained through it, constituted the critical element in the NewComers' 1st P designated identities. In contrast OldTimers, in explaining their choice of advanced mathematics course, stressed the fact that matriculating in this subject with high grades would largely increase their chances for being accepted to the university. In other words, if OldTimers were attracted to mathematics it was mainly, perhaps exclusively, because of its role as a gatekeeper.

To sum up, the NewComers' designated identities portrayed their heroes as exemplars of what the immigrant students themselves described as "the complete humans," with this last term implied to have a timeless, universal, generally accepted meaning, and with mathematical fluency being indispensable for the completeness. In contrast, the OldTimers expected to have their future life shaped by their own wishes and needs, which, at this point in time, were seen as fluid and, in the longer run, unforeseeable. This also points to a distinct meta-level difference between the two groups: Whereas the NewComers saw their highly prescriptive designated identities as given and apparently immutable, just like the mathematics they wanted to master, the OldTimers' expected their 1st P identities to evolve with the world in tandem.

In accord with our expectations, all this seemed to account, at least in part, for our former findings about the difference between OldTimers' and NewComers' learning. The NewComers needed mathematical fluency in order to close the critical gap between their actual and designated identities. For the OldTimers, this fluency was something to be shown upon request, like an entrance ticket that could be thrown away after use and that had no value of its own. Since mathematical skills did not constitute a critical element of the OldTimers' designated identities, these skills' absence or insufficiency did not create any substantial learning-fuelling tension.

On the cultural roots of designated identities

Where does the disparity between NewComers' and OldTimers' designated identities come from? was the last question we had to address in order to complete our story of designated identity as a link between learning and its sociocultural setting. More specifically, we needed to account for the fact that mathematical fluency constituted the critical element of the NewComers' designated identities but did not seem to play this role in the identities of OldTimers.

The first thing to say in this context is that given the NewComers' immigrant status, their being well versed in mathematics appeared of a redemptive value: The universality of mathematical skills was likely to constitute an antidote to these students' sense of local exclusion. To put it in terms of identity, we conjecture that whereas NewComers were bound to identify themselves as outsiders to their local environment, mathematical prowess was one of those properties that compensated them with the more prestigious, place-independent status of "people of education and culture."

Clearly, the idea that education at large, and the fluency in mathematics in particular, might counterbalance the less advantageous elements of their identity was not the young NewComers' original invention. In general, what the participants of our study expected for themselves was not unlike what their parents and grandparents wished for them. This is what transpired in both groups from the students' assertions about the full accord between their own and their parents' expectations, and from their remarks about the parents' impact on their choices (see sample responses to the question about the parents' expectations in Table 4). This said, there was an important difference between our two populations. Unlike in the case of NewComers, the OldTimers' parents were described as willingly limiting the area of their influence and leaving most decisions in the young people's own hands. We also found it quite telling that parents were rarely mentioned in the OldTimers' autobiographical testimonies, whereas the NewComers' accounts were replete with statements on the elders' authority and with explicit and implicit assertions on the parents' all-important role in their children's lives. Obviously, the OldTimers' parents' stories about their children's future were not as prescriptive as those of the NewComers, nor was the influence of these stories equally significant.

OldTimers	**NewComers**
• My parents want for me what I want myself. They want me to do what I want. • What is good for me – that's what they want for me. I also think that they find my plans appropriate. • My parents want me to be happy, so it is not so important for them what I'm going to do. • They want me to be what I want to be.	• My mother wants me to get good education. The process of learning itself, this is what is important to her. But a good matriculation certificate too, of course. She also wants me to study in the university. • I chose studying computers because my parents "pushed" in this direction. • My parents know best what's good for me. • For me, my grandma is the greatest authority • My mother tells me that if I meet an obstacle, I'll fail because of my laziness. I am lazy.

Table 4: Students' responses to the question about the parents' expectations regarding their children's future

Narratives about education as a universal social lever and about knowledge of mathematics as one of the most important ingredients of education evidently constituted a vital part of the NewComers' cultural tradition. In their native countries, their families belonged to the Jewish minority. According to what we were told both by the students and by their parents, these families had typically identified themselves as locally excluded but globally "at home" thanks to their fine education. Their sense of only partial attachment to the ambient community was likely the reason for the young people's relative closeness to their families. In the interviews, both the parents and the children sounded fully reconciled with their status of local outsiders. Proud of their cultural background and convinced about its universal value, they seemed to consider this kind of exclusion as the inevitable price for, and thus a sign of, the more prestigious, more global cultural membership. It seems, therefore, that the NewComers' identities as local outsiders destined to overcome the exclusion with the help of place-independent cultural assets such as mathematics were shaped by their parents' and grandparents' stories prior to the students' immigration to Israel.

Since significant narrators can count as voices of community, all these findings corroborate the claim that designated identities are products of collective storytelling – of both deliberate molding by others and of incontrollable diffusion of narratives that run in families and in communities. This assertion completes our empirical instantiation of the claim on designated identity as “a pivot between the social and the individual” aspects of learning (Wenger, 1998, p. 145).

CONCLUDING REMARKS

In this study, the narrative-defined notion of identity allowed us to get an insight into the mechanism through which the wider community, with its distinct cultural-discursive traditions, impinges on its members’ mathematics learning. On this occasion, we presented substantial learning as an activity propelled by the tension between actual and designated identities. Let us conclude this talk with two comments on practical and methodological implications of this study.

First, although our account may sound as a praise of the NewComers’ learning, there is, in fact, no side-taking in this report. Even if the NewComers’ practices can count as somehow superior to those of the OldTimers in that they proved more effective in attaining the official goals of school instruction, we are well aware that the goals themselves may be a subject to critical reappraisal. In addition, the price to be paid for this type of learning practice may, for some students, be too high to be worthy. Although carefully crafted stories about one’s “destiny” may sometimes work wonders, they are also likely to backfire when the burden of too ambitious, too tightly designated, or just ill-adjusted identities becomes unbearable.

Second, while constructing the conceptual framework supposed to help us in justifying the claim about the cultural embeddeddness of mathematics learning, we switched from the talk about identity as a “thing in the world” to the discourse in which this term refers to a type of narrative. The difference between these two renderings is subtle. The kinds of data the narratively-minded researcher analyzes in her studies is the same as everybody else’s: these are stories people tell about themselves or about others to their friends, teachers, parents, and observers. The only distinctive feature of the narrative approach is that rather than treat the stories as windows to some other entity that stays the same when “the stories themselves” change, the adherent of the narrative perspective is interested in the stories as such, accepting them for what they appear to be: Words that are taken seriously and shape one’s actions. Mapping the intricate relations between different kinds of narratives and fathoming the complex interplay between stories told and deeds performed was the sole focus of this study. By taking a close look at the narratives’ movement between one generation to another and between the level of community to that of an individual and back, we hoped to be able to account for both the uniformity and the diversity typical of human ways of acting.

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CO-CONSTRUCTING ARTEFACTS AND KNOWLEDGE IN NET-BASED TEAMS: IMPLICATIONS FOR THE DESIGN OF COLLABORATIVE LEARNING ENVIRONMENTS

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Computer-based learning environments for science and mathematics education support predominantly individual learning; from first generation drill and practice programs to today's advanced, knowledge-based tutorial systems, one learner interacting with one computer has been the typical setting. Mathematics educators, however, increasingly appreciate the value of collaborative learning and include team-learning activities in their lessons. In this presentation, drawing on our research in science and design areas, an overview is provided of the approaches and lessons learned regarding computer-supported collaborative learning and a number of design guidelines for computer-supported collaborative learning environments are suggested. Since equations and graphs are so important in mathematics, particular attention is paid to the role of external representations (and their co-construction) for computer-mediated collaboration.

APPROCHES TO FOSTER COLLABORATIVE LEARNING

Why foster collaboration? There are two arguments for supporting individuals as well as groups in cooperative behavior. First, cooperative behavior and, thus, collaborative learning leads to better performance of students compared to individual or competitive learning (Barron & Sears, 2002; Johnson & Johnson, 2004). Second, individuals in a group do not automatically cooperate and act as a group. A huge amount of contributions is dedicated to enhance collaborative learning in computer-mediated and residential cooperative learning. Johnson and Johnson (2004) distinguish four different basic types of cooperative learning: formal cooperative learning, informal cooperative learning, cooperative base groups and academic controversy. Mostly, formal and informal cooperative learning are addressed by methods fostering collaborative behavior. In some cases, the different types of cooperative learning represent several steps in the progress of a group (e.g., a group starts with informal cooperative learning, establishes formal cooperative learning afterwards and, finally, builds a cooperative base group). While informal cooperative learning according to the definition of Johnson and Johnson (2004) is restricted to short time intervals, most programs and assistance focus on the enhancement of formal cooperative learning.

Numerous methods of assisting learners in small group formal cooperative learning have been proposed. Some approaches are on the level of instructional design demanding specific cooperation patterns such as Group Jigsaw, Reciprocal Teaching

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or Problem-Based Learning. Other approaches are direct teaching of cooperative behavior, modeling, or scripting (e.g. Rummell et al., 2002). Especially for groups that are beginning a "collaborative episode" (i.e., there are no or little experiences in cooperative learning and the building of social relationships is at its beginning) such direct intervention is appropriated in order to avoid frustration and to reduce cognitive load. Even more experienced learners may benefit form assistance in cooperation: Especially in groups with many degrees of freedom related to cooperation and task fulfillment little or poor interaction is reported (e.g. Cohen, 1994).

The problem of poor peer interaction is well known in residential collaborative learning, but with the use of typed text-based computer-mediated communication this problem is likely to be increased. It is much more difficult to establish, perform and maintain basic cognitive mechanisms like turn-taking or grounding. But also and in particular social mechanisms like building positive interrelationships, establishing a group identity etc. are afflicted. Major causes for these difficulties derive from a lack of external cues as described in models of cues-filtered out and canal reduction.

Recent research in CMC-based (computer-mediated communication) collaborative learning has contributed a variety of technological/instructional approaches and solutions to overcome these problems. Especially scripting of collaboration (as a scaffolding mechanism) has gained attention in order to enhance turn-taking (Pfister & Mühlpfordt, 2002; Reiserer, Ertl & Mandl, 2002), design rationale (Buckingham-Shum, 1997) or reflection (Diehl, Ranney & Schank, 2001). Reiser (2002) differentiates between two basic mechanisms of these scaffolding techniques: Providing structure and problem orientation. Structured communication is one method to guide learners in the sense of an optimized behavioral model (e.g. problem solving heuristics) or a coordinated exchange between several learners. Furthermore, attention of learners can be drawn to relevant aspects or elements of a collaborative problem-solving process. Thus, scaffolding and scripting can avoid irrelevant or distracting tasks, strategies and processes.

Scripting as a scaffolding mechanism, however, is not always beneficial. Learner guidance in problem solving can also limit the degrees of learners' freedom. Reiser (2002, p. 263) states: "However, given the importance of connecting students' problem solving work to disciplinary content, skills, and strategies, it may also be important to provoke issues in students, veering them off the course of non-reflective work, and forcing them to confront key disciplinary ideas in their solutions to problems." In addition, when structuring interaction and discourse for learners, we always run the risk of interrupting spontaneous discourse. Scripting implies external guidance on sequence or categorization of contributions, but it is very difficult to identify discourse and patterns that are generally appropriate and effective.

In our recent research, we tried to avoid such a drastic and direct intervention that limits learner control by providing an inflexible structure. Instead of pre-structuring, we pursue what we call a "post-hoc structuring", i.e., we take the data derived form interactions (and additional variables assessed from learners) and re-use them for scaffolding. This way we avoid direct interference with the communication process, provide authentic material (based on learners' own contributions) and, hopefully, help students to become more self-efficient. Furthermore, this approach provides learners with accurate information about their current status within a group and group's progress and also with information on possible further directions that can optimize group functions (e.g., communication, group-members' interrelationships and learning or problem-solving outcomes). Before we have a closer look at our methods of collaboration management, a study is presented that analyses a discourse structuring approach.

SCAFFOLDING

In this study, we[1] analysed a scaffolding approach that is typical for what Reiser (2002) coined "providing structure". In this case, structure is provided on how student can communicate with each other. In particular, we looked at three levels of structuring (electronic) communication: Unstructured – a chat tool was provided to groups of (three) students; Simple-Structure: A graphical argumentation schema was provided on a shared whiteboard with four types of "nodes" (claim, pro- and contra-argument, sub-claim; Full-Structure: in this condition, seven node types had to be used (question, pro-and contra argument, idea, decision, fact, and miscellaneous, see Fig. 1) following the IBIS notational conventions (see Buckingham-Shum, 1996).

We ran an experiment with three conditions (Chat, Simple-Structure, Full-Structure) and 5 groups of 3 participants in each condition. Participants had to develop collaboratively an argument for a "wicked" environmental issue, the benefits and risks of transporting oil on sea with tankers. Our expectation was that the higher the degree of argument structure, the better the quality of the arguments a group will produce. In order to evaluate the quality of the arguments, we used the coding scheme of Newman and colleagues (Newman, Johnson, Webb & Cochrane, 1997) that has been developed to assess the quality of arguments exchanged in computer-mediated communication. This method yields a "critical thinking index" which varies between 0.0 and 1.0, with values close to 1.0 indicating higher argument quality.

Argument quality did indeed increase as a function of scaffolding through argument structuring, with a significant differences between all three conditions. It is worth noting, however, that increasing the structure led to a decrease in the frequency of arguments.

[1] Oliver Orth helped with the experimentation and data analysis.

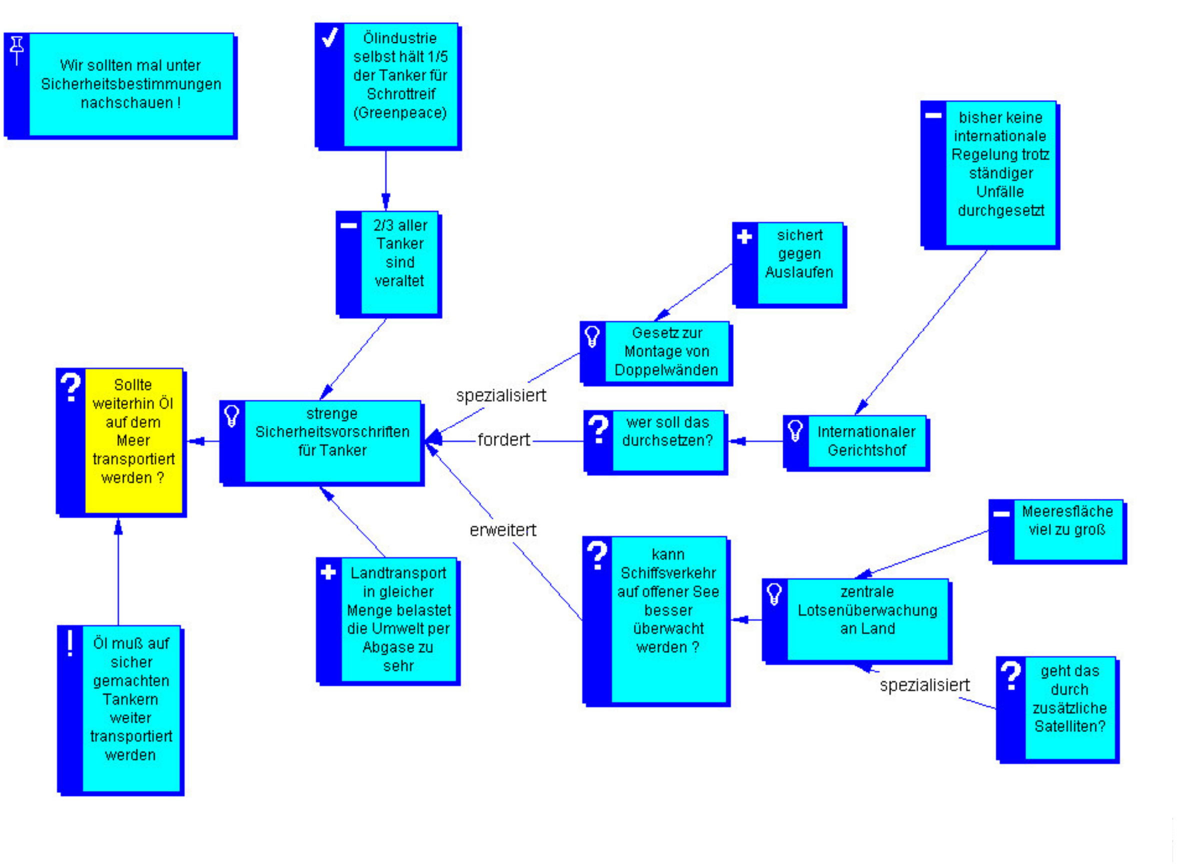

Figure 1: Argument graph using the full-structure (IBIS) notation

FEEDBACK AND GUIDANCE

Substantial research has been dedicated to find support mechanisms for online collaborators. Many authors discuss possibilities of scaffolding by structuring computer-mediated communication (e.g. Dobson & McCracken, 1997; Jonassen & Remidez, 2002; Reiser, 2002). Common to all these approaches is the provision of a structure for discourse and/or problem-solving. Instead of pre-structuring we pursue a way of post-hoc structuring interaction in online learning groups.

CMC itself provides the basis for this approach. During computer-mediated communication, all data can easily be stored and re-used for feedback purposes. In addition, software interfaces designed for CSCL (computer-supported collaborative learning) allow collecting individual quantitative data that can be used for further calculations in real time. Both data sources combined can easily be used to analyze individuals' as well as groups' performance automatically. In this way online learning groups provide the basis for feedback on their process without further interventions.

For instance, Barros and Verdejo (2000) describe an approach to provide feedback of group characteristics and individual behavior during computer-supported collaborative work based on a set of attributes that are computed out of data derived from learners' interactions. Their automatic feedback gives a qualitative description of a mediated group activity concerning three perspectives: a group's performance in reference to other groups, each member in reference to other members of the group, and the group by itself. Their approach allows extracting relevant information from online collaboration at different levels of abstraction. Although this approach seems to be very advantageous for enhancing online collaborators, Barros and Verdejo (2000) give no empirical evidence for the effectiveness of their asynchronous system. Jermann (2002, 2004) describes another possibility of providing feedback based on interaction data. He provides feedback on quantitative contribution behavior as well as learner-interaction during a synchronous problem solving task (controlling a traffic sign system). In an experiment, Jermann compared a group that received feedback about each individual learner's behavior. Another experimental group received feedback about the whole groups' success. He could show that a detailed feedback containing each individual's data enhanced learners' use of meta-cognitive strategies regarding problem-solving as well as discourse.

Our research group follows this line of feedback research. We[2] conducted studies to examine feedback effects on online collaborators during CSCL. One purpose of these investigations is to provide post-hoc scaffolding for subsequent problem solving. Another purpose is to use CMC, extract data from discourses and to provide abstracted views as a substitute for missing communication cues. In particular we investigated how the interaction in and the performance of small problem-based learning groups that cooperate via internet technologies in a highly self-organized fashion can be supported by means of interaction feedback as well as problem-solving feedback. Since the possibility of tracking and maintaining processes of participation and interaction is one of the advantages of online collaboration, ephemeral events can be turned into histories of potential use for the groups. We chose two ways to analyze how such group histories can be used for learning purposes. First, parameters of interaction like participation behavior, learners' motivation (self-ratings) and amount of contributions were recorded and fed back in an aggregated manner as an additional information resource for the group. This data could thus be used in order to structure and plan group coordination and group well-being. Second, we tracked group members' problem solving behavior during design tasks and provided feedback by means of problem-solving protocols. These protocols can be used to enhance a group's problem solving process for further tasks. Two studies testing our methodology in a synchronous and an asynchronous setting, respectively, are described next.

[2] The research reported in this section has been conducted in cooperation with Joerg Zumbach.

Automatic feedback in synchronous distributed Problem-Based Learning

The first laboratory experiment (Zumbach, Muehlenbrock, Jansen, Reimann & Hoppe, 2002) was designed as an exploratory study to test specific feedback techniques and their influence in an online collaboration learning environment.

For this purpose we designed a dPBL-learning environment. In a sample of 18 students of the University of Heidelberg we evaluated six groups of three members each. All students worked together synchronously via a computer network solving an information design problem. Each group was collaborating for about 2,5 hours (synchronously in one session). The task was to design a hypertext course for a fictitious company. All necessary task materials were provided online. In addition, all learning resources related to online information design were accessible as hypertext.

As a communication platform, the software EasyDiscussing was specifically developed for this experiment in cooperation with the COLLIDE-research group at Duisburg University, Germany. This Java-tool makes it possible to display a shared workspace to the whole group that can be modified by each member simultaneously. It contains drag-and-drop functions, thematic annotation cards like "text" (for general comments or statements), "idea", “pro” and “con” to structure the discussion, and it offers a chat opportunity as well (see Figure 2). All parameters are recorded in so-called "action protocols" and analyzed either directly or after the study. This makes it possible to check certain argumentative structures that become obvious during the course work, and also opens up the possibility to provide feedback based on the data produced.

Feedback parameters were gained in the following way: every 20 minutes students were asked about their motivation and their emotional state on a five item ordinal scale (parameters relating to the well-being function: “How motivated are you to work on the problem?” and “How do you feel actually?”). These were displayed to the whole group by means of dynamic diagrams (see Figure 3), showing each group member's motivation and emotional state with the help of a line graph. As a quantitative parameter supporting the production function two diagrams showed each group member's absolute and relative amount of contributions.

In order to test feedback effects we divided the groups into experimental groups that received feedback and into control groups which did not receive any feedback. Both groups had to do a pre- and post knowledge test, a test about attitudes towards cooperative learning (Neber, 1994), as well as some questions about their current motivation and emotional state. Besides our plan to test the techniques of how to provide feedback, we assumed that the experimental groups would be more productive since they were given parameters that would enable them to fulfill their well-being and production functions more easily, they. That means, they were assumed to contribute more ideas in an equally distributed manner, and show a greater amount of reflection, as far as interaction patterns were concerned, as opposed to the control groups.

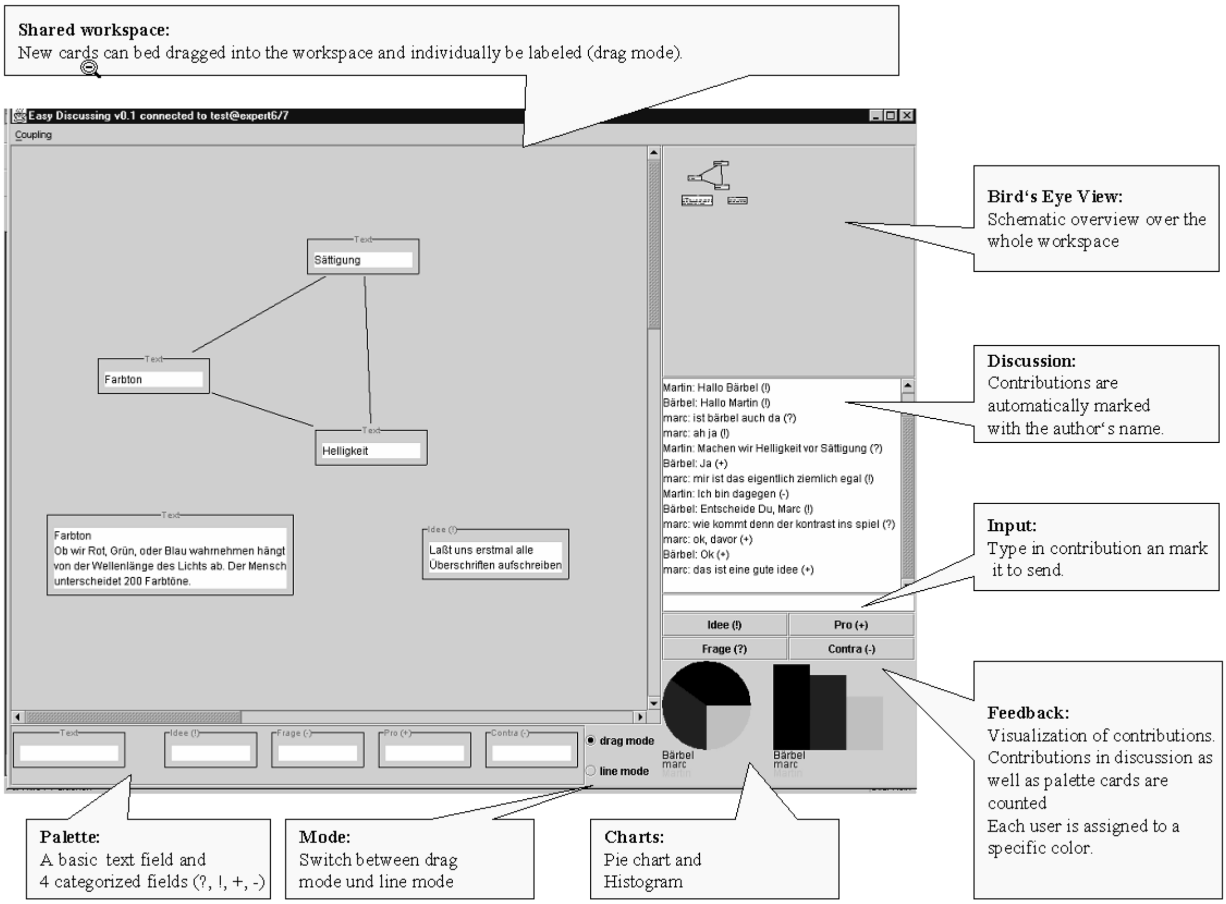

Figure 2. The design of the communication platform EasyDiscussing

The results of subjects' performance in the pre-test revealed no significant differences concerning domain knowledge. There were also no differences between both groups in post-test performance. Both groups mastered the post-test significantly better than the pre-test. There was no significant interaction between both tests and groups. We also found no significant differences regarding subjects' emotional data. The groups also showed no differences in pre- and posttests regarding motivation except a significant interaction between groups and time of measurement. While subjects in the control condition without feedback did not show differences in motivation, experimental groups had an increase from pretest to posttest. A closer look for interaction patterns in subjects' discussions revealed a significant difference in the number of dyadic interactions in groups that received feedback on their contributions.

Overall, the effects of this study indicate that some processes in computer-supported collaboration can be influenced in a positive manner by means of a steady tracking of parameters outside the task itself and immediate feedback of these to a group. Although intervention time in this experiment was short, we found positive influence of motivational feedback as well as feedback on contributions: communication patterns showed more interactive behavior for subjects of the experimental group. As a consequence of these effects, which indicate that our mechanisms have a positive influence on groups' production-function as well as group well-being, we decided to examine these feedback strategies further. For that purpose we arranged a long-time intervention study containing the same kind of visual feedback.

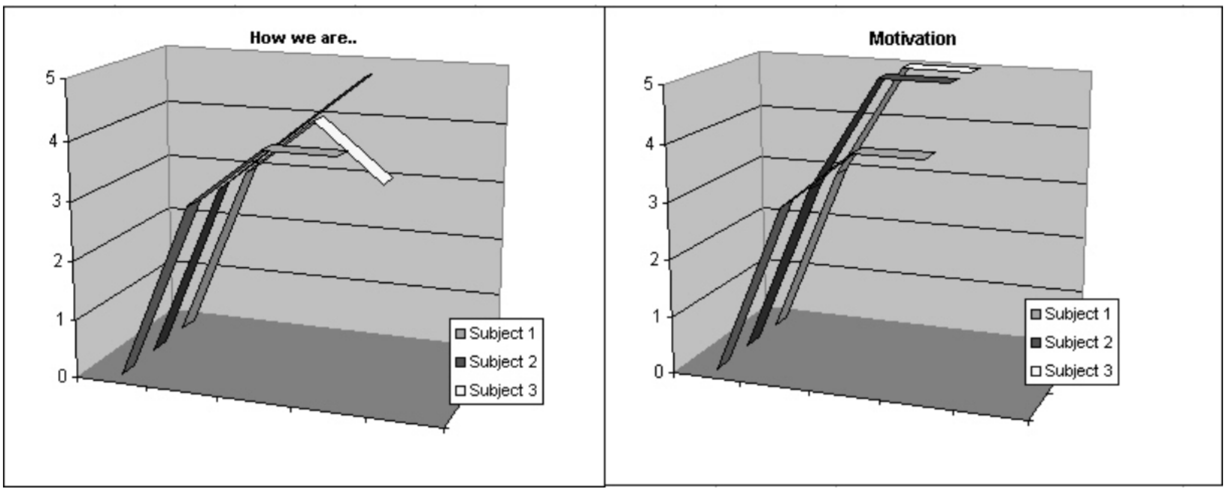

Figure 3. Feedback on emotion and motivation

Investigating the role of feedback mechanisms in long-time online learning

Our main objective in this study was to test different treatment conditions concerning feedback with groups that collaborated solely through an asynchronous communication platform over a period of four months. In this study we examined groups from three to five members – 33 participants overall. These groups participated in a problem-based course about Instructional Design that was conceived a mixture of PBL and Learning-By-Design. Learners were required to design several online courses for a fictitious company. These tasks have been presented as problems within a cover story. Each problem had to be solved over periods of two weeks (i.e. an Instructional Design solution had to be presented for the problem). As in study one, all materials were accessible online and, additionally, tutors were available during the whole course to support the students if questions emerged. At the end of each task, the groups presented their results to other groups. The asynchronous communication facility was based on a Lotus Notes® platform merging tools that can manage documents with automatic display possibilities for interaction parameters and problem-solving protocols (see Figure 4).

All documents as well as attachments were accessible over the collaboration platform. Meta-information showed when a document was created and who created it, so that interaction patterns became obvious and could be recorded. With the same technique of diagrams as in the former study, motivational and quantitative production parameters can be fed back to the user, referred to as *interaction histories*. Students' problem-solving behavior, however, had to be analyzed by the tutors themselves and had to be provided as text documents (*design histories*) in the group's workspace. Invisible for the students, a detailed action protocol was recorded in the background and was available later for analysis.

The groups were randomly assigned to one of four treatment conditions: with interaction history only, with design-history only, with both histories and without any feedback histories, i.e. a 2x2 design with the factors interaction history and design history. Several quantitative and qualitative measures to assess motivation, interaction, problem solving, and learning effects were collected before, during and after the experimental phase on different scales such as the student curriculum satisfaction inventory (Dods, 1997) or an adapted version of the critical thinking scale (Newman et al., 1997). We tried to answer the following question: What kind of influence does the administration of feedback in form of design and interaction histories, as well as their different combinations, have on students' learning? Generally, we assumed that groups with any form of histories would perform better than those without, especially as far as the motivational and emotional aspects supporting the well-being function and the production aspects supporting the production function of a group are concerned.

The results show encouraging outcomes in favor of the application of feedback within the group process. Groups that were shown design histories on their workspaces present significantly better results in knowledge tests, created qualitatively better products in the end, had produced more contributions to the task, and expressed a higher degree of reflection concerning the groups' organization and coordination. At the same time, the presence of interaction histories influenced the group members' emotional attitude towards the curriculum and enhanced their motivation for the task. Slight influences of the interaction history’s visualization regarding number of contributions were also found on the production-function: Learners receiving this feedback produced more contribution than their counterparts without feedback. So far, it seems reasonable to conclude that the different kinds of feedback influence different aspects of group behavior. Whereas feedback in form of design histories seem to influence a group's production function according to McGrath's (1991) conception of group functions, feedback in form of interaction histories seems to have an effect also on the production-function, but mainly on the group's well-being function

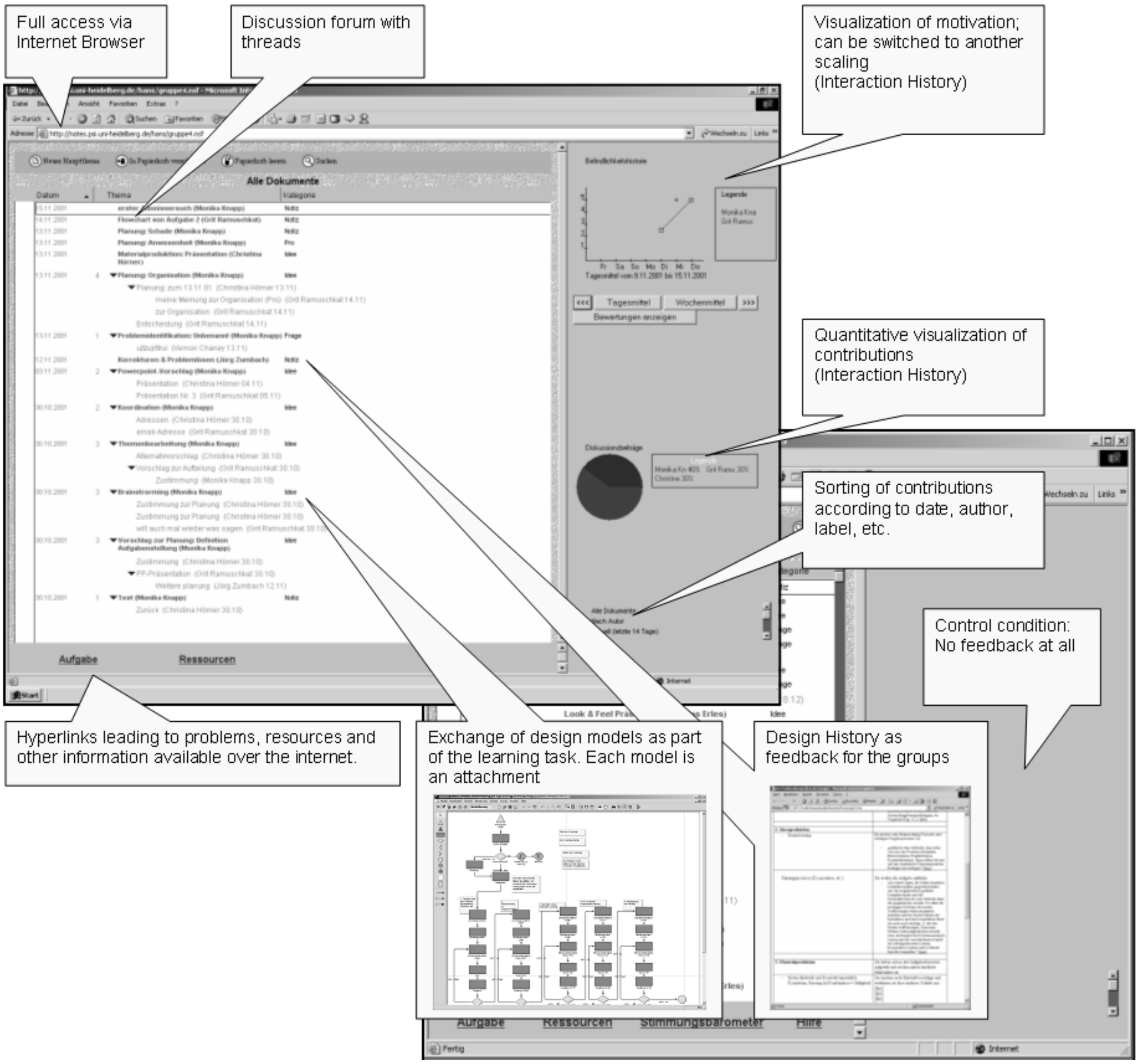

Figure 4. Asynchronous collaboration platform with feedback mechanisms.

TOWARDS ADAPTIVE VISUALISATION SUPPORT

In authentic, long term group work, it is the norm that people make use of a rich, diverse collection of communication systems, such as chat, discussion forums, and video conferencing. It is also typical that they make use of a range of tools and representational notations within one medium including, for example, written text and diagrams. We (Reimann, Kay, Yacef & Goodyear, in press) believe it is critical to begin to explore group support systems that can operate in the context of such media richness, exploiting the potentially huge amounts of data that could be available. We are particularly interested in three classes of learning that could occur in such situations:

- Learning to solve problems in a domain more effectively;
- Learning about the team, its members, and effective ways of cooperating and collaborating;
- Learning to use communication media and representational notations that match the demands of the tasks at hand, including tasks of member and collaboration management.

A number of researchers in the field of Computer-Supported Learning (CSCL) have begun to address this issue of collaboration management. Managing on-line collaboration by means of intelligent support can take a number of forms: mirroring, metacognitive and advice tools (Jermann, Soller & Muehlenbrock, 2001). They all require the ability to trace the interaction between the team members at some level of detail. We are building upon this work and intend to extend it into two directions: Firstly, in addition to supporting member interaction directly with feedback and/or advice systems, there is a need for learners to develop skills in choosing the right communication medium and tool for the situation at hand. Approaches to collaboration management that rely on a single communication medium, and/or on strongly restricted notational systems used for communicating (Conklin, 1993; Kuminek & Pilkington, 2001) need to be extended, because groups typically do not accept such limitations over longer stretches of time (Buckingham Shum, 1997). Having the choice among various communication and representation systems, however, adds to the demands groups face: they now have to deal with the additional issues of task-to-media fit (Daft & Lengel, 1984) and task-to-representation fit (Suthers, 2001). Secondly, we address human-computer interface issues extensively; not only because the management of task and interaction information distributed across various communication media raises serious attention and cognitive load issues, but also because of the social signals that come with using certain media (Robert & Dennis, 2005) and which have not been reflected sufficiently in research on computer-supported learning. We suggest an approach where the shared interface can be adapted to the needs of the work on the task as well as to the needs of interaction and member management. In the absence of a conclusive research base to derive advice from, our short term goal is to create an environment where such phenomena can be studied under controlled conditions and to experiment with various ways of visualizing information for groups and facilitators/moderators.

Adaptive Collaboration Visualisation

There has already been some work towards adaptive systems to provide advice on collaborative learning, for example (Constantino-Gonzalez, Suthers & Escamilla, 2003). There has also been recognition of the importance of social parameters, such as participation patterns (Barros & Verdejo, 2000). We will explore the use of adaptive information presentation using visualisations of the collaboration. These seem particularly promising because they are easier to implement than advice systems and no normative model of collaboration is required.

What to record. We are working on finding research-based answers to three questions around the process: (1) What to record about the learners' performance; (2) How to aggregate and then analyse the traced information; (3) What and how to visualize the results from step 2, in a manner that is adapted to the group's needs. With respect to question (1), we propose to capture *all* task- and group-related exchanges available, regardless of whether these involve the whole group, sub-groups, or individual members. Since we expect to be able to motivate the group members to help monitor their own interactions, we will be able to encourage the use of tools that we have set up to capture a rich record of interactions.

How to aggregate. An immediate effect of this is that we have to deal with large amounts of information. This must be analysed and summarised. Our approach with respect to question (2) is to collect the full set of available, un-interpreted data and then to perform a series of analyses to create both individual learner models and collective group models. We will use machine learning and data mining techniques (association rules, classification and clustering techniques such as hierarchic clustering, k-means, decision trees and data visualisation in particular) to identify patterns in groups' performance and relate those to outcome measures such as the quality of the groups' decision models and participants' satisfaction with the group process. Data mining and machine learning techniques have been successfully used for user modelling and, to a lesser extent, in education contexts. In particular, mining data based upon learners' interactions with a learning environment is promising (Bull, Brna, & Pain, 1995a).

Since a user model captures the system's beliefs about the learner's knowledge, beliefs, preferences and other attributes, it has the potential to play an important role in providing external representations of the individual and group learner models relevant to the group interaction and learning. There has been a growing appreciation of this possibility, with learner models being shared with learners in order to support reflection (Bull et al., 1995a; Bull, Brna, & Pain, 1995b, Crawford & Kay, 1993; Kay, 1995) and to help learners work collaboratively (Bull & Broady, 1997). The challenges in this project are to mine the available data sources to support the construction of a student model (Kay & Thomas, 1995), to provide natural interfaces that enable learners to see and understand the externalised form of that model (Uther & Kay, 2003), to explicitly contribute to it and, finally, but most importantly, to improve our understanding of the ways that this externalised user model can support learning and as well as the operation of the group.

What and how to visualize. Once relevant information is identified, the challenge remains how to communicate this back to the group (question 3). While the question of information visualisation has been researched before, including our own work (Uther & Kay, 2003; Zumbach & Reimann, 2003), research has so far been mainly limited to analysing individual displays of task and participation parameters (Jermann, 2004). The overall configuration of information displays – the interface elements that make up the shared work space – has been assumed as being static. We propose to dynamically adapt not only the content of individual information displays, but the *overall configuration* of information displays. For instance, when the group has to work on complex information together, social information should be reduced (in the absence of conflicts or member problems) so that all the cognitive resources can go into task information processing. Similarly, if interaction problems require attention, then the task information should temporarily be reduced and social information should be displayed with greater salience and detail. If both the task representation(s) and the social information representation(s) are properly adapted, then it should be feasible to provide suitable tradeoffs between the cognitive effort for the core task versus that for processing group and member information.

We also propose to differentiate more systematically between ‘person awareness’ and ‘team awareness’. For instance, the video/audio display of a user – as a “rich” medium (Daft & Lengel, 1984) – primarily provides information about an *individual* group member. It does not depict information about the team as such. The user lists that are part of most chat tools, however, are a rudimentary *team awareness* component – showing who is currently “in” the group activity. Visualisations can, and probably should, play a much stronger role in supporting team awareness. For instance, Erikson and Kellogg (2000) make a number of suggestions on how to visualize social configurations of team members in digital spaces such as chat rooms.

Our current prototype collaboration environment comprises various synchronous and asynchronous communication and information representation tools, including a “digital table” that allows for co-located teamwork. We are experimenting with a number of computational approaches to aggregate collaboration information and identify psychologically and pedagogically meaningful patterns and trajectories. We are also developing means for visualising information relevant for task-, team-, and person-awareness. Building on these, we will experiment with ways to dynamically modify the respective information displays to make the overall interface adaptive to situational parameters (cognitive load, social conflicts, member problems) and to group members’ preferences and individual needs.

CONCLUSIONS

In this paper, we have mainly looked at factors that apply to all forms of distributed collaborative learning, and have in particular dealt with issues that result from a lack of social awareness. While net-based group learning offers exciting opportunities to foster communication and reflection, one should not ignore the psychological challenges that arise from loosing face-to-face contact. In our recent work, we are also devoting increasing attention to the management of the user interface since adding all kinds of meta-information (helpful for reflection) to an already crowded screen space raises serious usability issues.

More would need to be said about the function of shared external representations, such as the symbols that appear on a shared whiteboard. Such shared representations do not only serve as a representation of shared knowledge, and thus play an pivotal role for grounding, they also help the group members to co-ordinate their work and to drive the agenda. The relation between such representations and the actions taken by group members need more attention in future research.

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PLENARY PANEL

WHAT DO STUDIES LIKE PISA MEAN TO THE MATHEMATICS EDUCATION COMMUNITY?

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In a real sense, PISA 2003 has touched the mathematics education community by stealth rather than by storm. Although PISA brings "baggage" commonly associated with international assessments, it takes some refreshing perspectives especially in the way that it envisions and assesses mathematical literacy. In this panel discussion we *focus on some of the issues associated with PISA: scrutiny of student performance, construct and consequential validity, what makes items difficult for students and the potential impact of PISA on mathematics education research. In selecting these issues we merely begin the debate and open the way for your participation.*

WHAT IS PISA?

The *Programme for International Student Assessment* ([PISA], OECD, 2005) is an international standardized assessment in reading literacy, mathematical literacy, problem-solving literacy and scientific literacy. It started in 1997 when OECD countries began to collaborate in monitoring the outcomes of education and, in particular, assessed the performance of 15-year-old school students according to an agreed framework. Tests have typically been administered to 4,500-10,000 students in each country. The first assessment in 2000 which focused mainly on reading literacy surveyed students in 43 countries while the second assessment in 2003 involved 41 countries and focused mainly on mathematics and problem solving. The third assessment in 2006 will largely emphasize scientific literacy and is expected to include participants from 58 countries. In this panel discussion we will concentrate on PISA 2003 and those aspects of it that deal with mathematical literacy.

THE PISA MATHEMATICAL LITERACY ASSESSMENT

In describing their approach to assessing mathematical performance, PISA documents (e.g., OECD, 2004a) highlight the need for citizens to enjoy personal fulfilment, employment, and full participation in society. Consequently they require that "all adults–not just those aspiring to a scientific career–be mathematically, scientifically, and technologically literate" (p. 37). This key emphasis is manifest in the PISA definition of mathematical literacy: " …an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned, and reflective citizen" (OECD, p. 37; see also Kieran, plenary panel papers).

Reflecting this view of mathematical literacy, PISA documents (e.g., OECD, 2004a) note that real-life problems, for which mathematical knowledge may be useful,

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 71-74. Melbourne: PME.

seldom appear in the familiar forms characteristic of "school mathematics." The PISA position in assessing mathematics was therefore designed "to encourage an approach to teaching and learning mathematics that gives strong emphasis to the processes associated with confronting problems in real-world contexts, making these problems amenable to mathematical treatment, using the relevant mathematical knowledge to solve problems, and evaluating the solution in the original problem context" (OECD, 2004a, 38). In essence, mathematical literacy in the PISA sense places a high priority on mathematical problem-solving and even more sharply on *mathematical modelling*.

Although PISA's devotion to mathematical modelling has my unequivocal support, my experience tells me that it is not easy to incorporate effective mathematical modelling problems in a test that has fairly rigid time constraints. In addition, although the term mathematical modelling is relatively new in school mathematics (Swetz & Hartzler, 1991), there are instances of mathematical modelling even in the notorious public examinations of more than 50 years ago. I well remember the following problem in an examination that I took in 1953. It seems to me that it is a genuine modelling problem and it was certainly not a text book problem or a problem that anyone of that era had practised. Moreover, the fact that less than 10% percent of the 15 to 16-year-old students taking the examination solved the problem is both déjà vu and prophetic for those setting the directions for the PISA enterprise.

> In a hemispherical bowl of radius 8 inches with its plane section horizontal stands water to a depth of 3 inches. Through what maximum angle can the bowl be tilted without spilling the water? Give your answer to the nearest degree (University of Queensland, 1953)

Accordingly, even though members of our panel valued the PISA emphasis on real-world problems and mathematical modelling, there was no shortage of issues to debate. In particular, there were issues about the framework, the validity of the assessment, the construction of items, the measurement processes, the conclusions and the interpretations especially interpretations that cast the findings into the realm of an international "league table". Consequently, we faced a problem in selecting which issues to examine. Let me presage the papers of the other panellists by providing an entrée of the issues that reverberated over our internet highways.

WHAT ISSUES DOES PISA RAISE FOR MATHEMATICS EUDCATION?

As the conference theme was *learners and learning* we questioned whether PISA assessment really was designed to support a real-world approach to mathematics teaching and learning. We also raised questions about whether student performance in the PISA assessments mirrored student performance in other mathematics education research on learning and teaching. Although appropriate data was not easily accessible, we wondered what the PISA study told us about patterns of classroom activity in different cultures. Yoshinori Shimizu (plenary panel papers) did examine this from a cultural perspective by scrutinizing Japanese students' responses to some PISA items.

Issues associated with item validity, item authenticity, and item difficulty were consistently part of our discussions. The "triangular park problem" (see Williams, plenary panel papers) was hotly debated and members of the team even spent considerable time looking for triangular parks or car parks. This was part of our conversation on *real world* or *authentic assessment* and this issue is taken up further by Julian Williams under the broader topic of construct validity. Carolyn Kieran (see plenary panel papers) takes up the issue of "what makes items difficult for students?" She observes that the *difficulty levels* of some PISA items are problematic and raises doubts about how much we know about what students find difficult in certain mathematical tasks.

The politics of international assessment studies like PISA (OECD, 2004a) and *Trends in International Mathematics and Science Study* ([TIMSS], Mullis et al., 2004) were high on our debate list. Not only do these debates raise highly volatile issues and national recriminations, they also generate profound questions for those countries that are doing well and for those who are not. In addition to issues that focus specifically on the international league, assessment studies like PISA produce a range of related debates about factors such as gender, ethnicity, socio-economic status, systemic characteristics, approaches to learning, student characteristics and attitudes, and of course fiscal support (OECD, 2004b). Julian Williams (see plenary panel papers) tackles a number of these political issues especially those related to accountability: managing targets, dealing with league tables, and performance-related reviews.

There was considerable interest in discussing the impact of international assessment studies on mathematics education research. At the forefront of such issues is the question: What does PISA say to researchers interested in assessment research? Yoshinori Shimizu (see plenary panel papers) will talk about this more specifically as he refers to the benefits that can be gleaned by researchers through an examination of PISA's and TIMSS's theoretical frameworks, methodologies, and findings. For example, he notes that the detailed item scales and maps in PISA will enable researchers to perform a secondary analysis of students' thinking and accordingly gain a deeper understanding of learners and learning. Michael Neubrand (see plenary panel papers) also looks at the potential of PISA to stimulate research in mathematics education. He focuses on the structure of mathematical achievement especially in the way that PISA conceptualizes achievement through the aegis of a *mathematical literacy* framework. This gives rise to an interesting dialogue with respect to both individual and systemic (collective) competencies in mathematics and how they can be measured. There are of course other important questions such as "What do studies like PISA say to mathematics education researchers about methodological issues such as qualitative versus quantitative research?" Although this particular question is not directly addressed, the panel refers frequently to methodological issues and as such issues a challenge to the participants for further engagement and debate.

CONCLUDING COMMENTS

I believe that this panel discussion is most timely as I am not convinced that mathematics educators are as cognizant as they might be about the impact of the burgeoning industry that encompasses international studies like PISA (OECD, 2003) and TIMSS (Mullis et al., 2004). Although the build up and dissemination of PISA has been slow to take root in the mathematics education research community, the findings have certainly not gone unnoticed by national and state governments, educational systems, business leaders and parent groups. They know where their nation or their state came in the "league stakes" but they have little understanding of the intent and limitations of such studies. Accordingly, an important aim of this panel is to encourage mathematics education researchers to be more proactive not only in publicly illuminating and auditing research like PISA but also in identifying ways in which PISA can connect with and stimulate their own research. In the words of Sfard (2004, p. 6) we should exploit these special times in mathematics education:

> Confronting the broadly publicized, often disappointing, results of the international measurements of students' achievements, people from different countries started wondering about the possibility of systematic, research-based improvements in mathematics education

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FROM A PROFILE TO THE SCRUTINY OF STUDENT PERFORMANCE: EXPORING THE RESEARCH POSSIBILITIES OFFERED BY THE INTERNATIONAL ACHIEVEMENT STUDIES

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The recent release of two large-scale international comparative studies of students' achievement in mathematics, the OECD-PISA2003 and the TIMSS2003, has the potential to influence educational policy and practice. A careful examination of their findings, theoretical frameworks, and methodologies provides mathematics education researchers with opportunities for exploring research possibilities of learners and learning.

BEYOND THE COMPETITIVE EMPHASIS IN REPORTS

The release of results of the OECD-PISA2003 (Programme for International Student Assessment, OECD, 2004) and the TIMSS2003 (Trends in International Mathematics and Science Study, Mullis, et al., 2004) in December 2004 received huge publicity through the media in Japan. The purposes of international studies such as PISA and TIMSS include providing policy makers with information about the educational system. Policy makers, whose primary interest is in such information like their own country's relative rank among participating countries, welcome a simple profile of student performance. Also, there is a close match between the objectives of PISA, in particular, and the broad economic and labour market policies of host countries. The match naturally invites a lot of public talk on the results of the study with both competitive and evaluative emphasis. This was the case in Japan.

There was one additional large-scale study in 2003 of student performance in mathematics in Japan. In the National Survey of the Implementation of the Curriculum, which has also been released recently (NIER, 2005), the students from grades 5 through 9 (N>450,000) worked on items that are closely aligned with the specific objectives and content of in Japanese mathematics curriculum. TIMSS2003 sought to derive achievement measures based on the common mathematical content as elaborated with specific objectives, whereas PISA2003 was explicitly intended to measure how well 15-years-olds can apply what they have learned in school within real-world contexts. The recent release of these studies should shed light on the new insight into learners and learning from multiple perspectives.

The large-scale studies, conducted internationally or domestically, provide a profile of a population of students from their own perspectives. We need to go beyond competitive emphasis in the reports of such studies to understand more about the profile of students' performance and to explore the possibilities of further research that such studies provide.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 75-78. Melbourne: PME.

In this short article, a few released items of PISA2003 are drawn upon to propose that a careful examination of the findings, the theoretical framework and the methodology used as well, provides mathematics education researchers with opportunities to examine further research questions that might be formulated and addressed.

THE SCRUTINIES NEEDED

One of the distinct characteristics of the PISA2003, having mathematics as the major domain in the recent cycle of the project, is the way in which the results of student performance are described and reported. The mathematics results are reported on four scales relating to the overarching ideas, as well as on an overall mathematics scale. The characteristics of the items as represented in the map, which shows the correspondence between the item and the scale, provide the basis for a substantive interpretation of performance at different levels on the scale.

We can now take a closer look at the profile of students' response to the released items. Even the results of a few released items from PISA2003 suggest possibilities for conducting a secondary analysis and further research studies in order to develop deeper understanding of learners and learning. In particular, such items, or overarching ideas, as follows raise questions for Japanese mathematics educators, in particular, and mathematics education researcher, in general, to consider.

An Illuminating Example: SKATEBOARD

One of the items on which Japanese student performance looks differently from that of their counterparts elsewhere is in Question 1 of the item called SKATEBOARD (OECD, 2004, p.76). This short constructed response item asks the students to find the minimum and the maximum price for self-assembled skateboards using the price list of products given in the stimulus. The item is situated in a *personal* context, belongs to the *quantity* content area, and classified in the *reproduction* competency cluster. The results show that the item has a difficulty of 464 score points when the students answer the question by giving either the minimum or the maximum, which locates it at Level 2 proficiency. On the quantity scale, 74% of all students across the OECD community can perform tasks at least at Level 2. The full credit response has a difficulty of 496 score points, which places it at Level 3 proficiency. On the quantity scale, 53% of all students across the OECD community can perform tasks at least at Level 3.

When we look into the data on the students' response rate in each country, a different picture appears. Japan's mean score was significantly lower than the OECD average for the item (See Table 1) and the pattern in the percentages for students' responses look different from their counterparts in other countries.

Of note among the numbers in Table 1 is the lower percentage of correct responses from Japanese students than from their counterparts, as well as the higher no response rate. Students can find the minimum price by simply adding lower numbers for each part of the skateboard and the maximum price by adding larger numbers.

Country	Full Credit	Partial Credit	No Response	Correct
Australia	74.1	9.3	1.8	78.7
Canada	74.9	9.1	2.0	79.4
Germany	71.7	11.5	5.2	77.5
Japan	54.5	8.0	10.6	58.5
OECD Average	66.7	10.6	4.7	72.0

Table 1: The percentage of students' response for SKATEBOARD, Question1 (An excerpt from National Institute for Educational Policy Research, 2004, p. 102.)

The results suggest that some students, Japanese students, in this case, may be weak in handling multiple numbers where some judgment is required, assuming that they have little trouble in the execution of the addition procedure. We need an explanation with scientific evidence for the results.

Another Example: NUMBER CUBES

Another example comes from the result of the item called NUMBER CUBES (OECD, 2004, p.54). This item asks students to judge whether the rule for making a dice (that the total number of dots on two opposite faces is always seven) applies or not with the given four different shapes to be folded together to form a cube. The item is situated in a *personal* context, belongs to the *space and shape* content area, and classified in the *connection* competency cluster. The results show that the item has a difficulty of 503 score points, which places it at Level 3 proficiency. On the space and shape scale, 51% of all students across the OECD community can perform tasks at least at Level 3.

Country	Students' Choice of Correct Judgments					
	Four (Full)	Three	Two	One	None	No Res.
Australia	68.6	14.1	7.2	6.4	2.4	1.2
Canada	69.6	14.0	7.3	6.3	2.1	0.6
Germany	69.0	13.9	7.3	5.6	2.3	1.9
Japan	83.3	8.9	4.2	2.0	0.9	0.7
OECD Average	63.0	16.0	8.9	7.2	2.7	2.3

Table 2: The percentage of students' response for NUMBER CUBES (An excerpt from National Institute for Educational Policy Research, 2004, p. 108.)

The result shows that Japan's mean score was significantly higher than the OECD average as well as being higher than other participating countries (See Table 2). Also, the pattern of students' choice is slightly different from other countries.

In order to complete the item correctly, we need to interpret the two dimensional object back and forth by "folding" it to make the four planes of the cube mentally as a three-dimensional shape. The item requires the encoding and spatial interpretation of two-dimensional objects. Why did a group of students, once again Japanese students, perform well on this particular item? Does the result suggest that those students have a cultural practice with number cubes, or Origami, inside and outside schools? A further exploration is needed to explain the similarities and differences in students' responses among participating countries.

There are other insights offered by the recent international studies. The TIMSS2003 collected information about teacher characteristics and about mathematics curricula. The PISA2003 also collected a substantial amount of background information through the student questionnaire and the school questionnaire. These data on contextual variables as well as performance data related to the cognitive test domain give us rich descriptions of the learning environments of the learners.

As was mentioned above, the recent release of the two large-scale international achievement studies provides mathematics education researchers with opportunities for exploring research possibilities in relation to learners and learning. While we need to examine the results from each study carefully, we also need to synthesize the results from different perspectives as a coherent body of description of the reality of the learners.

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THE PISA-STUDY: CHALLENGE AND IMPETUS TO RESEARCH IN MATHEMATICS EDUCATION

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Beyond the results, a large scale study like PISA may also stimulate the area of research in mathematics education. Since an empirical study needs a sound conceptualization of the field - "mathematical literacy" in the case of PISA - mathematics education research and development may benefit from the structures of mathematical achievement defined for PISA. Further research can build upon the work done in PISA.

PISA, the "Programme for International Student Assessment" (OECD, 2001, 2004) came into the public focus mainly for the results and the prospective consequences to be drawn: "All stakeholders – parents, students, those who teach and run education systems as well as the general public – need to be informed on how well their education systems prepare students for life" (OECD, 2004, p 3). However, the PISA study deserves interest also from the point of view of research in mathematics education. This perspective is inherent to PISA: The PISA-report "considers a series of key questions. What is meant by 'mathematical literacy'? In what ways is this different from other ways of thinking about mathematical knowledge and skills? Why is it useful to think of mathematical competencies in this way, and how can the results be interpreted?" (OECD, 2004, p 36)

This paper draws attention to some of the impulses and challenges to mathematics education research coming from the PISA studies. We recognize both, the international study, and the national option in Germany which was based on an extended framework and included additional components.

SYSTEM RELATED DIAGNOSIS OF MATHEMATICAL ACHIEVEMENT

What are the aims of PISA? PISA's main focus is to measure the outcomes of the whole educational systems in the participating countries, and choses, as the most sensible group to investigate, the group of the 15 years olds in the countries. The key question therefore is on the system level: What do we know about the mathematical achievement and its conditions in an *educational system* compared to what one can observe in an international overview?

Apparently, this is not thoroughly in tune to the mainstream of mathematics education research. There are long and ongoing traditions in mathematics education which point to a contrasting aspect: What are an *individual's* thoughts, difficulties, sources, and strategies when learning mathematics? Our common interest is often more on an individual's understanding, or on the misunderstandings in the social communication among the individuals in the classroom. Thus, it does not wonder that international comparisons found and still find critical reactions, going back as far as Hans Freudenthal's fundamental critique in the beginning of comparative studies in mathematics (Freudenthal, 1975).

Contrasting that tradition, the complementary question towards a *systems' efficiency* in mathematics teaching and learning is not less challenging. One has to define appropriate concepts and instruments to answer the question on a basis which incorporates the knowledge mathematics education research has given us so far. In fact, PISA took that challenge serious in a twofold way: The concept "mathematical literacy" forming the basis for testing mathematics achievement is explicitly bound to the mathematics education tradition (OECD, 2003; Neubrand et al., 2001); and vice versa, the PISA test gave rise to further developments of conceptualizing mathematical achievement (Neubrand, 2004). Thus, PISA provides theoretically based, and empirically working conceptualizations of mathematical achievement, which can be seen as an impetus to mathematics education research.

CONCEPTUALIZING MATHEMATICAL ACHIEVEMENT

Sources of the concept "mathematical literacy"

The specific idea of PISA is that the outcomes of an educational system should be measured by the competencies of the students. The key concept is "literacy". Three roots can be traced back: a tradition of pragmatic education (e.g., Bybee, 1997), Freudenthal's conception that "mathematical concepts, structures and ideas have been invented as tools to organise the phenomena of the physical, social and mental world" (Freudenthal, 1983), and considerations on what mathematics competencies are about (Niss, 2003). From there the PISA-framework developed that PISA aims to test the capability of students "to put their mathematical knowledge to functional use in a multitude of different situations" (OECD, 2003).

Conceptualizing „mathematical literacy" in the international PISA study

The domain "mathematical literacy" was conceptualized and related to the test items (problems) in the international PISA study by three components (Fig. 1).

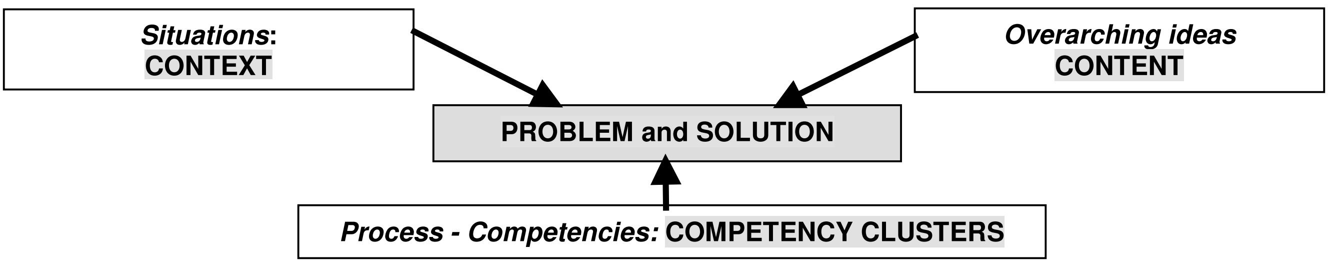

Figure 1. Components of mathematical problems as conceptualized by the international PISA framework (OECD, 2003, p. 30).

It is one of the major impetuses (and challenges) to mathematics education research that (and if) a list of mathematical competencies, accumulated in the Competency Clusters ("Reproduction" - "Connections" - "Reflection"), may hold as "key characteristics" (OECD, 2004, Annex A6) to construct an appropriate instrument to test mathematical achievement. In 2004 PISA reported countries' achievement differentiated by the content-dimension, and it will be a matter of further research to clear how far the competencies itself are present in the countries.

Conceptualizing mathematical achievement in the German national PISA option

Even stronger than PISA-international, the German national option capitalizes that an achievement test like PISA should map mathematics as comprehensively as possible. Therefore, typical ways of thinking and knowing in mathematics should be present in the test items. This model of the test tasks formed the basis (Fig. 2):

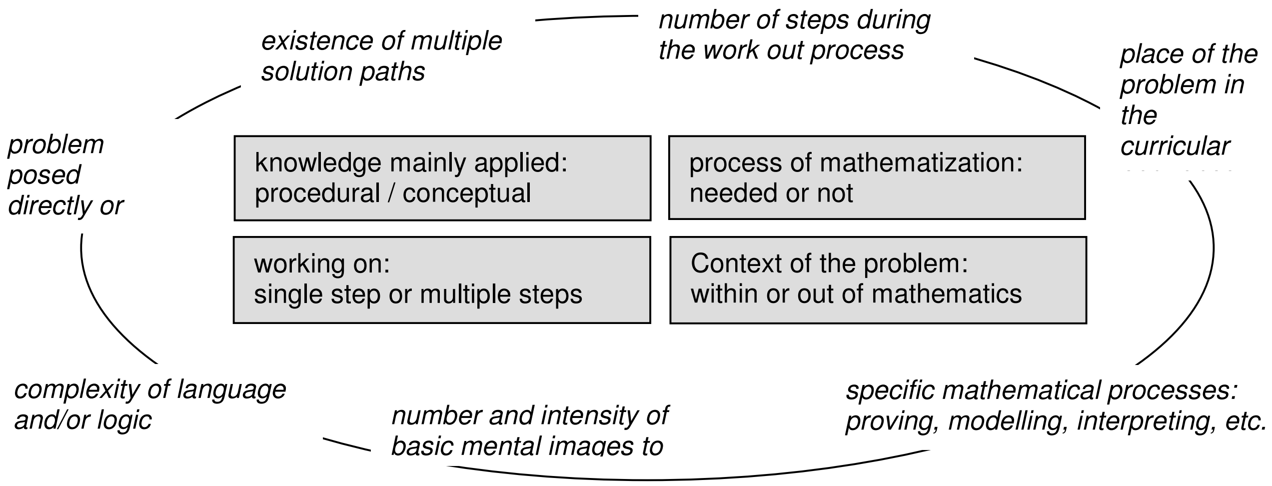

Figure 2. The model of a mathematical problem used in PISA-Germany: The core, and examples of characteristic features (Neubrand, 2004)

With the four basic features (the "core") mathematical achievement can be structured by three "types of mathematical activities" (J. & M. Neubrand in Neubrand, 2004): (i) employing only techniques, (ii) modeling and problem solving activities using mathematical tools and procedures, (iii) modeling and problem solving activities calling for connections and using mathematical conceptions. From the cognitive and the mathematical point of view the three classes realize the full range of mathematical thinking, since one recognizes technical performance, and the essential modes of thinking, i.e., procedural vs. conceptual thinking (Hiebert, 1986).

ANALYTIC RESULTS OF PISA

The defined structures of mathematical achievement express themselves also in the data. But clearly, there remains a lot to do for further research.

International test: Countries show differences in content areas

Not surprisingly, test data show differences among countries in the performance on the defined content areas, the "overarching ideas" (OI). While the students in some countries behave quite uniformly over the content (e.g., Finland, Belgium), in some countries considerable differences appear. For example, Japan shows strengths in the OIs "change and relationships" and "space and shape"; and (relative) weaknesses in "quantity" and "uncertainty". Germany shows weakness in the geometry and stochastics items. Results like these give hints what fields of mathematics should earn greater emphasis in curriculum and teaching. (See OECD, 2004 for details.)

Difficulty of a problem: A question of various features

Analyses done after PISA-2000 in Germany revealed some insight into the processes which make the solution of an item more difficult. However, as said in the beginning, due to the nature of the data, one can get information on mathematical learning and thinking in the whole, and not information of an individual's ways of thinking. Nevertheless, there are interesting results to obtain.

(a) *Not* the same features make a problem difficult in any of the three "types of mathematical activities" (J. & M. Neubrand in Neubrand, 2004). As a consequence, mathematic teaching cannot restrict itself to only a limited scope of mathematics.

(b) There is a competency specific to mathematics, that influences the difficulty of problems, even of those problems which call for modeling processes: the capability to use formalization as a tool (Cohors-Fresenborg & al. in Neubrand, 2004).

(c) Different didactical traditions and ways of teaching lead to different "inner structures" of mathematical achievement, made visible by different performance in the types of mathematical activities (J. & M. Neubrand in Neubrand, 2004).

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SOME RESULTS FROM THE PISA 2003 INTERNATIONAL ASSESSMENT OF MATHEMATICS LEARNING: WHAT MAKES ITEMS DIFFICULT FOR STUDENTS?

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With the announcement of the 2003 PISA results in December 2004, we can now take a closer look at the released items and at how the 15-year-olds of the PISA assessment fared. A brief examination of item difficulty within the "change and relationship" scale suggests that we still know little about what it is that students find difficult in certain mathematical tasks.

MATHEMATICAL LITERACY IN PISA

The PISA concept of mathematical literacy is concerned with "the capacity of students to analyse, reason, and communicate effectively as they pose, solve and interpret mathematical problems in a variety of situations involving quantitative, spatial, probabilistic or other mathematical concepts" (OECD, 2004, p. 37). More precisely, mathematical literacy is defined as "an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen." The objective of the PISA 2003 assessment was "to obtain measures of the extent to which students presented with problems that are mainly set in real-world situations can activate their mathematical knowledge and competencies to solve such problems successfully" (OECD, 2004, p. 57).

HOW MATHEMATICAL LITERACY WAS MEASURED

Students' mathematics knowledge and skills were assessed according to three dimensions: mathematical content, the processes involved, and the situations in which problems are posed. Four content areas were assessed: shape and space, change and relationships, quantity, and uncertainty – roughly corresponding to geometry, algebra, arithmetic, and statistics and probability. The various processes assessed included: thinking and reasoning; argumentation; communication; modeling; problem posing and solving; representation; and using symbolic, formal, and technical language and operations. The competencies involved in these processes were clustered into the reproduction, connections, and reflection clusters. The situations assessed were of four types: personal, educational or occupational, public, and scientific. Assessment items were presented in a variety of formats from multiple choice to open-constructed responses.

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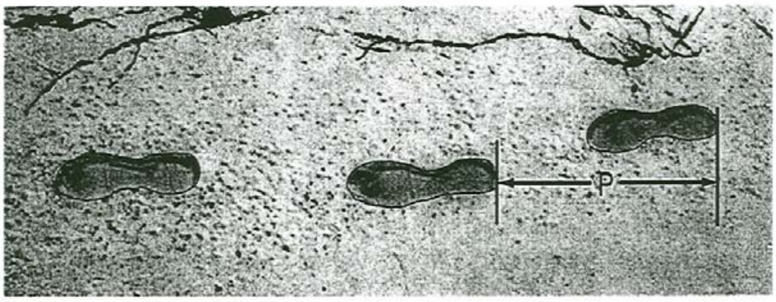
The PISA 2003 mathematics assessment set out to compare levels of student performance in each of the four content areas, with each area forming the basis of a separate scale. Each assessment item was associated with a point score on the scale according to its difficulty and each student was also assigned a point score on the same scale representing his or her estimated ability. Student scores in mathematics were grouped into six proficiency levels, representing groups of tasks of ascending difficulty, with Level 6 as the highest. The mathematics results are reported on four scales relating to the content areas mentioned above. As will be seen, an examination of item-difficulty within these scales reveals some surprises that, in turn, suggest that we, as researchers, may not really know what makes some mathematical tasks more difficult than others for students.

ITEM DIFFICULTIES FOR SAMPLE ITEMS FROM THE CHANGE AND RELATIONSHIP CONTENT AREA: THE WALKING UNIT

The Walking unit (OECD, 2004, p. 64) begins as follows:

WALKING



The picture shows the footprints of a man walking. The pacelength P is the distance between the rear of two consecutive footprints.

For men, the formula, $\frac{n}{P} = 140$, *gives an approximate relationship between n and P where:*

n = number of steps per minute, and
P = pacelength in metres.

Items 4 and 5 from this unit, along with the respective item difficulties and discussion of the competency demands, are presented in Figure 1. The level of difficulty ascribed to Item 4 is difficult to fathom: 611, which places it at Level 5 proficiency – a level at which only 15 % of OECD area students are considered likely to succeed. Yet, the item requires simply substituting n by 70 in the given formula $n/p = 140$, and then dividing 70 by 140. Its difficulty would seem closer to a Level 2 proficiency item, which according to the OECD report typically involves the “interpretation of a simple text that describes a simple algorithm and the application of that algorithm” (p. 69) – a task that 73% of OECD area students would be likely to solve. While students might attempt to solve the equation $70/p = 140$ by a cross-multiplication technique, they could also think about the task in terms of proportion ($70/p=140/1$, i.e., 70 is to 140 as p is to 1) or arithmetically in terms of division (70 divided by what number yields 140?).

WALKING

QUESTION 5

Bernard knows his pacelength is 0.80 metres. The formula applies to Bernard's walking.

Calculate Bernard's walking speed in metres per minute and in kilometres per hour. Show your working out.

Score 3 (723)

Answers which indicate correctly metres/minute (89.6) and km/hour (5.4). Errors due to rounding are acceptable.

Score 2 (666)

Answers which are incorrect or incomplete because:

- They were not multiplied by 0.80 to convert from steps per minute to metres per minute.
- They correctly showed the speed in metres per minute (89.6 metres per minute) but the conversion to kilometres per hour was incorrect or missing.
- They were based on the correct method (explicitly shown) but with other minor calculation error(s).
- They indicated only 5.4 km/hr, but not 89.6 metres per minute (intermediate calculations not shown).

Score 1 (605)

Answers which give $n = 140 \times .80 = 112$ but no further working out is shown or incorrect working out from this point.

This open-constructed response item is situated in a personal context. The coding guide for this item provides for full credit, and two levels of partial credit. The item is about the relationship between the number of steps per minute and pacelength. It follows that it fits the change and relationships content area. The mathematical routine needed to solve the problem successfully is substitution in a simple formula (algebra), and carrying out a non-routine calculation. To solve the problem, students first calculate the number of steps per minute when the pace-length is given (0.8 m). This requires substitution into and manipulation of the expression: n/0.8 = 140 leading to: n = 140 x 0.8 which is 112 steps per minute. The next question asks for the speed in m/minute which involves converting the number of steps to a distance in metres: 112 x 0.80 = 89.6 metres; so his speed is 89.6 m/minute. The final step is to transform this speed into km/h - a more commonly used unit of speed. This involves relationships among units for conversions which is part of the measurement domain. Solving the problem also requires decoding and interpreting basic symbolic language, and handling expressions containing symbols and formulae. The problem, therefore, is rather a complex one involving formal algebraic expression and performing a sequence of different but connected calculations that need understanding of transforming formulas and units of measures. The lower level partial credit part of this item belongs to the connections competency cluster and with a difficulty of 605 score points it illustrates the top part of Level 4. The higher level of partial credit illustrates the upper part of Level 5, with a difficulty of 666 score points. Students who score the higher level of partial credit are able to go beyond finding the number of steps per minute, making progress towards converting this into the more standard units of speed asked for. However, their responses are either not entirely complete or not fully correct. Full credit for this item illustrates the upper part of Level 6, as it has a difficulty of 723 score points. Students who score full credit are able to complete the conversions and provide a correct answer in both of the requested units.

QUESTION 4

If the formula applies to Heiko's walking and Heiko takes 70 steps per minute, what is Heiko's pacelength? Show your work.

Score 1 (611)

Answers which indicate p = 0.5 m or p = 50 cm or p = ½ (unit not required).

This open-constructed response item is situated in a personal context. It has a difficulty of 611 score points, just 4 points beyond the boundary with Level 4. Everyone has seen his or her own footsteps printed in the sand at some moment in life, most likely without realising what kind of relations exist in the way these patterns are formed, although many students will have an intuitive feeling that if the pace-length increases, the number of steps per minute will decrease, other things equal. To reflect on and realise the embedded mathematics in such daily phenomena is part of acquiring mathematical literacy. The item is about this relationship: number of steps per minute and pacelength. It follows that it fits the change and relationships content area. The mathematical content could be described as belonging clearly to algebra. Students need to solve the problem successfully by substitution in a simple formula and carrying out a routine calculation: if n/p = 140, and n = 70, what is the value of p? The students need to carry out the actual calculation in order to get full credit. The competencies needed involve reproduction of practised knowledge, the performance of routine procedures, application of standard technical skills, manipulation of expressions containing symbols and formulae in standard form, and carrying out computations. Therefore the item belongs to the reproduction competency cluster. The item requires problem solving by making use of a formal algebraic expression. With this combination of competencies, and the real-world setting that students must handle, it illustrates Level 5, at the lower end.

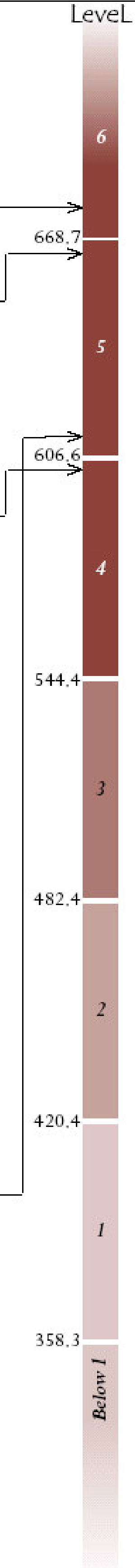


Figure 1. Items 4 and 5 of the Walking unit (OECD, 2004, p. 65)

Curiously, a response earning a partial score of 2 on the seemingly much more difficult Item 5 – at least more difficult from an *a priori* perspective – places it at Level 5 as well, albeit nearer the upper boundary of Level 5. But, it is not clear why a response that is deemed incomplete (and receives a score of 2) because the "112 steps per minute was not multiplied by .80 to convert it into metres per minute" – a conceptual demand that is at the core of Item 5 – is considered superior to the response "n = 140 x .80 = 112," which appropriately receives a partial score of 1. Notwithstanding the argument that could be made for both of these responses" to Item 5 receiving the same score of 1, the main issue concerns the conceptual demands that are inherent in Item 5, but which are lacking in Item 4. Why do students find Item 4 just about as difficult as Item 5?

While some might claim that the procedural demands of Item 4 (with the unknown in the position of denominator) explain to a certain extent why the difficulty level is 611, results from past research studies of equation-solving errors suggest that the difficulty level of this item should not be so high. For example, Carry, Lewis, and Bernard (1980) reported the following success rates for the solving of the given equations among students who covered a range from strong to very weak in algebra skills (e.g., 82%: 9(x+40) = 5(x+40); 76%: 1/3 = 1/x + 1/7; 76% 5/10 = (x-10)/(x+5)). In another study involving classes of 6^{th} to 8^{th} grade students, younger than those tested within PISA, Levin (1999) reported that 30% of the students correctly answered the following question by setting up and solving a proportion using cross multiplication (5/9=2/n): "On a certain map, the scale indicates that 5 cm represents the actual distance of 9 miles. Suppose the distance between two cities on this map measures 2 cm. Explain how you would fine the actual distance between the two cities." The equation was not unlike the one involved in Item 4; moreover, the students had to generate it themselves from the problem situation. One can only conclude that if the PISA results for this item and related symbolic representation items represent a trend with respect to students' abilities to handle rather simple symbolic forms, it is indeed a disturbing one. While Nathan and Koedinger (2000) noted that students find symbolically-presented problems more difficult than story problems and word-equation problems, the PISA results suggest that the discrepancy may be much greater than that reported by these researchers.

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THE FOUNDATION AND SPECTACLE OF [THE LEANING TOWER OF] PISA

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I raise questions about the construct and consequential validity of international studies such as PISA, and about PISA itself. I suggest a fault line runs through the construct 'mathematical literacy', but more importantly, through mathematics education generally, distinguishing 'Realistic' mathematics and 'Authentic' mathematics. I then ask questions about the political consequences of PISA in an audit culture in which targets beget processes. The aim to influence policy is identified with perceptible shifts in PISA discourse. As an instrument in the global education market, with its theft of critical theorists' rhetorical resources, is PISA re-invigorating the spectacle of international league tables?

INTRODUCTION

When I was a boy I visited Pisa and was very impressed by the leaning tower. I recall imagining that one could walk up the tower by spiralling up the outside, and was slightly disappointed by the reality. Later I learned that the inclination of the tower was annually increasing, and engineers feared that it would eventually fall over: they planned to strengthen the foundations to stop this, but did not straighten it. The tower has become a global spectacle, even featuring in jokes etc. (what did Big Ben say to the leaning tower of Pisa? I've got the time if you've got the inclination). The tower of Pisa became globally spectacular because of its dodgy foundations, not despite them.

I aim to raise questions about the validity of PISA (capitals now). First, I examine the construct validity of the foundation of PISA, 'mathematical literacy'; second, I address the consequential validity of PISA, its political consequences, as spectacle.

CONSTRUCT VALIDITY: THE FOUNDATION OF 'MATHEMATICAL LITERACY'

A confession: I find some of the items in PISA seductive, especially some of the Problem Solving items. In one the student is asked to diagnose a faulty bicycle pump, in another they are asked to evaluate some information on various drugs and select an appropriate pain-killer for 13 year old George, an asthmatic child with a sprained ankle. At face value, these represent a kind of functional 'literacy'. Turning to the mathematical literacy item used to explain the notion of mathematical modelling and mathematisation, one finds the park problem: where should a street-light be placed to illuminate a park? The park is mathematised as a triangle, the area lit is a circle, and the solution is the triangle's circumcentre (as long as the park is not obtuse-angled, explains PISA, 2003, p26).

I may be obtuse, but … our parks in English towns are usually locked at night, not lit. Perhaps they mean a car park? But … how many triangular car parks have you seen? I looked around and noticed that the lights were often on the perimeter of the park, which is in turn usually made up of rectangular blocks. For obvious reasons one might expect car parks to be rectangular, especially in modern countries where road systems are grid based. Perhaps one would find them in towns where road networks crystallised on the basis of clusters of medieval villages, like Chester or York? Both these towns are a long way from Manchester, so this prompted me to email my co-presenter from Japan and… he found one! (But where was the lighting?)…

Does the validity of Euclid really lie in such considerations? How has this come to be? I fantasise: Euclid, on a trip to visit the leaning tower, finds a triangular car park and noticing the light at the midpoint of one side… "Eureka: the circumcentre of a right-angled car park lies at the mid-point of the hypotenuse."

But Realistic Mathematics Education (RME) does not require that mathematics be authentic in this 'real' sense: only that the situation is realistic for the entry of the student into a world that begs to be mathematised. The validity test for RME then is (i) mathematical, rather than 'real' functionality, and (ii) empirical (i.e., do the students experience the problem in an intuitive way). Many of the PISA items appear to have this quality, at least to some degree.

I suggest that Realistic mathematics is primarily embedded in a scholastic, pedagogical activity system and is essentially embedded in the students' imaginary, experiential world: the object of activity is, in the end, to learn mathematics. On the other hand, I suggest Authentic mathematics is used as an instrument within an Activity System whose object is not essentially to learn mathematics, but to achieve some 'real' objective in a world outside mathematics. To become Authentically functional is to break out of the scholastic straitjacket and requires what Engestrom (e.g., Engestrom, 1987) called 'expansive' activity: at the very least, the class that 'plans a party' has to really have the party.

I prefer to think of this distinction as a fault line deep underneath the surface of the concept of 'mathematical literacy', rather than a dichotomy as such. Does this line undercut the mathematics education literature too?

And where is PISA? I'd say some of the best tasks are Realistic, but never quite Authentic (you would hope George's 15 year old literate elder sibling would think to ask a good pharmacist before deciding which painkiller to buy his asthmatic younger brother, wouldn't you? Sorry, 'code 0: no credit'). Could they be?

DISCOURSE AND SPECTACLE OF PISA: POLITICAL CONSEQUENCES

PISA has a political aim, that is, it seeks to influence policy. Thus on the one side, we have mathematics-literacy tasks, and the identification of learning outcomes for students. But on the other, we have summative statistics that 'count' for policy. This entails an interesting discursive shift. Initially, PISA (e.g., 2005) suggest that correlations display ‘associations’ that cannot be assumed to be ‘causal’, but later these associations become ‘influences’ that policy makers might find ‘interesting’. What is the difference for policy, i.e. what is the political difference between an influence and a cause? I see from the dictionary (OED) that an influence is in its original usage an astrological one, and later became political: it is essentially the exertion of an action whose mechanism is ‘unseen’ except in its effects.

This is significant because it determines to some extent the ‘consequences’ of PISA. How can policy makers be expected to read PISA’s results on the influence of SES or softer variables such as ‘school climate’ on learning outcomes? We see from the PISA-2000 study, for instance PISA (2005), that school climate explains significant variation in outcomes, but not that school climate is a possible ‘associate’ of high learning outcomes, and in Gill et al. (2002), associations with school background become ‘attributable’ to school background (p xvi).

Michael Power, who calls himself a professor of critical accountancy, has described the discourse of performativity in our audit culture (i.e., that of managing targets, league tables, performance-related reviews, etc.) as a Foucaultian discourse of (mis-) trust (Power, 1999). He and others have pointed to the way measurement constructs become targets and begin to dominate processes: thus as I write Prime Minister Tony Blair is felled by an angry electorate in debate on TV. He is accused of being responsible for the fact that in some doctors’ surgeries patients are not allowed to book an appointment to see their doctor more than 2 days ahead. Why? Because the government had introduced a performance target for the percentage of patients that have to wait more than 2 days. In vain he protests that this was not his intention! How will PISA measures be used, and what will be their unintended consequences?

Stronach (1999) in ‘Shouting theatre in a crowded fire’ construes the international tests and league table performance as a global spectacle, with ‘pupil warriors’ doing their sums for Britain. There’s England in the Premier league, 3 up on old rivals Germany, there’s a cluster of Confucian Pacific rim teams in the lead, but here comes Finland from nowhere suddenly challenging them. Is it social democracy or Nokia that ensures the team’s strength?

The association between PISA/TIMSS league tables and football competitions, the Olympics, horse races etc. is too strong to be denied, and ‘England’ in the tables becomes metonymically the nation and its education system per se, competing in the game with the rest of the world. One forgets that in fact the order of the names in the table are mostly not statistically significant, of course. What else is a table of scores actually for except to emphasise the ordinal at the expense of the complexity of the underlying data/reality? (That is intended to be a mathematically literate observation, if you like.)

The tabloid/redtop press are masters of this spectacle, but we all become implicated: government funding for research (at least in the UK) is increasingly predicated on 'making a difference' to learning outcomes in practice, and hence fulfilling political demands to become 'world class'. But how can world class be judged, except by international competition and league tables, and hence comparative measurement?

With what consequence? Is there no going back? Has the spectacle seduced our rationality? Pisa will always be the place with the leaning tower. While PISA challenges TIMSS by engaging with some 'literacy' rhetoric drawn from critical theory, the source of much that seems seductive in it, one reading of this move might be, as Gee et al. (1996) and others have suggested with 'fast capitalism', that the system steals critical theorists' rhetorical resources and emerges all the stronger for it.

So, where next? Could an expanded Authentic mathematics assessment emerge to confront the Realistic PISA, and in whose interest might that be?

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(see www.education.man.ac.uk/lta/pme/PISA)

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Afterword

Is the metaphoric association of Pisa and PISA – their foundations and their glorious spectacles – valid? If the consequence is that one is inclined to believe that there is a fault underlying 'mathematical literacy', I suggest yes. If one is led to think that this fault is implicated in the faux-spectacle of PISA, perhaps: the argument is that the act of global assessment becomes false *by virtue* of its becoming a political spectacle.

[Acknowledgements: to Google.com for suggesting the Pisa=PISA metaphor, and Ian Stronach for the introduction to this notion of spectacles.]

RESEARCH FORUMS

RF01 *The significance of task design in mathematics education: Examples from proportional reasoning*

Coordinators: Janet Ainley & Dave Pratt

RF02 *Gesture and the construction of mathematical meaning*

Coordinators: Ferdinando Arzarello & Laurie Edwards

RF03 *A progression of early number concepts*

Coordinators: Kath Hart & Ann Gervasoni

RF04 *Theories Of Mathematics Education*

Coordinators: Lyn English & Bharath Sriraman

RF01: THE SIGNIFICANCE OF TASK DESIGN IN MATHEMATICS EDUCATION: EXAMPLES FROM PROPORTIONAL REASONING

Co-ordinators: Janet Ainley and Dave Pratt

Institute of Education, University of Warwick, UK

In the context of the overall focus of PME29 on *Learners and Learning Environments*, we have chosen the topic of pedagogical task design for this Research Forum. We see task design as a crucial element of the learning environment, and wish to explore further the role that it plays for learners. The overarching question for this Research Forum is: Why is task design significant?

To make progress on this question, we raise two issues: how does the task design impact on student learning? How does the agenda of the researcher or teacher shape the task design? More specifically we ask: how does the nature of the task influence the activity of students? What is important for mathematics educators in designing a task?

In order to work on these questions, both in the preparations for the Forum, and within the sessions at the conference, we have chosen to take a specific topic within the curriculum, that of *proportional reasoning*, and to invite the contributors to the Forum to work on designing tasks for the learning and teaching of proportion for pupils of around 11-12 years old.

The contributors

There are four groups of researchers contributing to this Forum, all of whom work on aspects of task design from different perspectives.

Dirk De Bock, Wim Van Dooren and Lieven Verschaffel explore features of the use of words problems in a number of mathematical areas, and have focussed on the ability to discriminate proportional and non-proportional situations.

Koeno Gravemeijer, Frans van Galen and Ronald Keijzer use design heuristics from Realistic Mathematics Education (guided reinvention through progressive mathematization, didactical phenomenology, and emergent modeling) in an approach which also draws on design research.

Alex Friedlander and Abraham Arcavi have many years experience within the Compumath project, which is developing a technology-based curriculum and studying the effects on pupils' learning.

Janet Ainley and Dave Pratt have developed an approach to task design based on creating tasks which are purposeful for pupils within the classroom environment.

We hope that our understanding of task design will be enhanced by making explicit reflections on these differing perspectives in the context of specific examples of tasks and their use by pupils.

The design brief for the contributors

Each of the teams of contributors was asked to design a task which focussed on proportional reasoning. The task had be suitable for pupils aged about 11-12 years, and it also had to be a 'stand alone' task, which could be tackled within one lesson. This condition was a significant constraint for some of the contributors, who would normally design tasks as part of a sequence. Contributors were asked to prepare their task in a form that could be presented to pupils, and were also asked to provide teachers' notes.

Each of the tasks has been trialled with pairs of pupils and the papers by each of the contributing teams which follow this introduction draw on this data to illustrate the discussion of the principles which underpinned their task designs.

Dirk, Wim and Lieven's task

This task focuses on similarities and differences in a set of word problems, some of which require proportional reasoning, while others have a similar format, but are not, in fact, proportional problems.

Yesterday, Mrs. Jones made some word problems to use in the math lessons. But they got all mixed up! Can you help Mrs. Jones to put some order in the word problems? Look at the problems very carefully and try to make groups of problems that belong together.

A Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 5 rounds, Kim has run 15 rounds. When Ellen has run 30 rounds, how many has Kim run?

B Mama put 3 towels on the clothesline. After 12 hours they were dry. The neighbour put 6 towels on the clothesline. How long did it take them to dry?

C Mama buys 2 trays of apples. She then has 8 apples.Grandma buys 10 trays of apples. How many apples does she have?

D John runs a bakery. He uses 10 kg of flour to make 13 kg of bread. How much bread can he make if he uses 23 kg of flour?

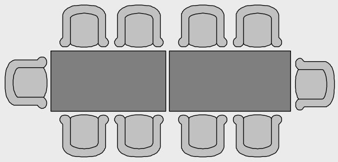
E The locomotive of a train is 12 m long. If there are 4 carriages connected to the locomotive, the train is 52 m long. If there were 8 carriages connected to the locomotive, how long would the train be?

F Today, Bert becomes 2 years old and Lies becomes 6 years old. When Bert is 12 years old, how old will Lies be?

G A group of 5 musicians plays a piece of music in 10 minutes. Another group of 35 musicians will play the same piece. How long will it take this group to play it?

H Yesterday, a boat arrived at the port of Rotterdam, containing 326 "Nissan Patrol" cars. The total weight of these cars was 521 tons. Tomorrow, a new boat will arrive, containing 732 "Nissan Patrol" cars. What will be the total weight of these cars?

I In the hallway of our school, 2 tables stand in a line. 10 chairs fit around them. Now the teacher puts 6 tables in a line. How many chairs fit around these tables?



J In the shop, 4 packs of pencils cost 8 euro. The teacher wants to buy a pack for every pupil. He needs 24 packs. How much must he pay?

Now answer the following questions:

- Write here the different groups of problems. (Use the letters on the sheets)
- Why did you make the groups in that way?
- Can you think of a different way to put the problems in groups? Explain that as well.

Koeno, Frans and Ronald's task

This task is based around the story of Monica and Kim making a cycle trip from Corby to Cambridge. Various resources such as a map of the route, photographs and background information (the reason for the trip, the weather conditions) are provided.

After cycling for 1 hour 30 minutes, the girls reach a village called Catworth where there is a signpost showing 18 miles from Corby and 30 miles to Cambridge. "Okay", Monica says, "this is going well."

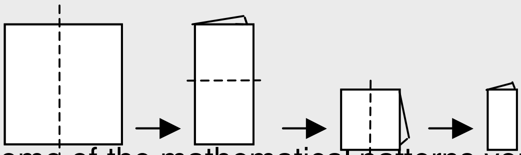
1. Could you tell why she might say this?
2. How much time has it taken them to get to Catworth? And what is the distance they have covered?
   So what can you say about the speed of Monica and Kim? You can use the table to judge their speed.

Cycling at a slow speed:	8 miles per hour
Cycling at a normal speed:	12 miles per hour
Cycling at a fast speed:	18 miles per hour

3. In the table, the speeds of various kinds of cyclist are given. However, if you want to compare the speeds of cyclist who are not riding the same road on the same day, conditions might be different.
   Could you mention the things that have to be taken into account, if we were to measure the speed of a cyclist.
4. After a short stop, Monica and Kim are moving on. They get on the road from Catworth to Cambridge, a distance of 30 miles. At about what time do you think they will arrive in Cambridge?
5. Of course, you cannot be absolutely sure about how long it will take them.
   Could you mention some reasons why you cannot be sure? Still, to make a sensible guess, it might be helpful to know how much time she would need if she were to keep up the same speed.
6. How much time would the ride to Cambridge take if they were to keep up the same average speed as before?

Alex and Abraham's task

This task is based around the practical activity of folding a 32x32 square piece of paper, as shown below. There are then a series of questions to address, some of which use a spreadsheet. In the pupils' materials some guidance for using the spreadsheet is included, which has been omitted here.



2. Describe some of the mathematical patterns you notice as you fold the shapes.
3. ***Predict***: What is the pattern of change in the perimeter, as you fold the shapes?

4.a. Write on the drawings the ***dimensions*** and the ***perimeter*** of the first four shapes in the sequence.

b. ***Collect your data in a spreadsheet table*** that shows the dimensions and the perimeter of the first ten shapes in the sequence.

5. ***Draw a graph*** to show the perimeter of the first ten squares and rectangles in the sequence.
6. Look for ***patterns*** that describe the change in the perimeter, as the square is folded. Explain the connection between your patterns and the folding shapes.

7.a. The teacher asked: *By how many length units does the perimeter get shorter at each folding?* Daniel replied: *At each folding the perimeter gets shortened by the same length.* Do you agree with Daniel?

b. ***Collect data*** that may help you to answer the teacher's question.

c. Do you see any ***patterns*** in the collected data? Explain the connection between your patterns and the folding shapes.

d. Did you change your initial opinion about Daniel's answer? Explain why you did or did not.

8.a. The teacher asked: *By what ration does the perimeter get smaller at each folding?* Daniel answered: *At each folding the perimeter of the new shape is half the perimeter of the previous one.* Do you agree with Daniel?

(b, c and d as for question 7)

9.a. Find pairs of shapes that have a **perimeter ratio of one half**.

b. Give a "rule of thumb" for finding such pairs.

c. Convince a fried why your rule always works.

Janet and Dave's task

For this task pupils have measuring tapes, a spreadsheet. Each group also has a different item of dolls' house furniture.

Children in a primary school want to make a 'dolls' house classroom'. Use the piece of furniture you have been given to work out what size they should make some other objects for their classroom.

DIFFERENT PERSPECTIVES ON TASK DESIGN

The four tasks presented here offer significant differences in the kind of activity that pupils may be engaged in when working on them, but they also arise from different approaches to task design. These are explored and elaborated within the individual papers, but we also draw attention here to one issue which may be discussed within the Forum sessions: the role of the teacher.

Gravemeijer, van Galen and Keijzer emphasise the central role which they see the teacher as playing when a class is working on the task in guiding discussion to focus on mathematical issues and the development of tools to support proportional reasoning. De Bock, Van Dooren and Verschaffel have designed a task which it appears pupils may work on independently, but they also acknowledge the potential role of the teacher in encouraging whole class discussion around the task. Friedlander and Arcavi have constructed a task made up of a sequence of questions, which balances structured questions with more open invitations to make conjectures. Some of the questions are based on hypothetical conversations between the teacher and a pupil, and clearly offer support for pupils to work independently, or for an inexperienced teacher to use the materials. Ainley and Pratt's task is stated very briefly. There is clearly a crucial role for the teacher, who would need an understanding of the approach, in leading discussion to explore and develop the task, but the authors also contrast the activity of pupils who need to rely on continuing support from the teacher, and those for whom the task itself determines the direction of their activity.

NOT EVERYTHING IS PROPORTIONAL: TASK DESIGN AND SMALL-SCALE EXPERIMENT

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INTRODUCTION

Proportional (or linear) reasoning is a major tool for human beings in many cultures to interpret real world phenomena (Post, Behr, & Lesh, 1988; Spinillo & Bryant, 1999), even when the phenomena are not linear 'stricto sensu'. Therefore, not surprisingly, proportional reasoning constitutes one of the major topics in school mathematics from the lower grades of the elementary school to the lower grades of secondary school. From Grades 2 and 3 onwards children learn to multiply and divide and to apply these operations in simple word problems like "1 pineapple costs 2 euro.

How much do 4 pineapples cost?", which are predecessors of proportional reasoning tasks. During Grade 4 and afterwards, proportional reasoning skills are further developed. From this age on, students are frequently confronted with proportionality problems, most often stated in a so-called missing-value structure such as: "12 eggs cost 2 euro. What is the price of 60 eggs?", and are trained to set up and solve the corresponding proportion $12/60 = 2/x$ for the unknown value of x. However, in the last decade, mathematics educators formulated two main deficiencies of this current school practice for teaching and learning proportionality.

First, because almost all proportional tasks students encounter at school are formulated in a missing-value format – and at the same time, non-proportional tasks are very rarely stated in this format – students tend to develop a strong association between this problem format on the one hand and proportionality as a mathematical model on the other hand. Recently, De Bock (2002) provided empirical evidence for that claim. In a series of exploratory studies in one specific mathematical domain, namely, problems about the relations between the linear measurements and the area or volume of similarly enlarged or reduced geometrical figures (such as the dolls' house context in Janet and Dave's task), it was shown that 12-16-year old students have an almost irresistible tendency to improperly apply direct proportional reasoning to length-area or length-volume relationships, especially when the problems are stated in a missing-value format. Changing the problem formulation by transforming the problems into a "comparison format" proved to be a substantial help for many students to overcome the trap of inappropriate proportional reasoning in this domain. This study – together with analogous findings by other researchers – suggests that teachers should at least bring more variation in proportionality tasks and especially take care that these tasks are not *always* formulated in a missing-value format.

Second, as reflected in the *Standards 2000* (National Council of Teachers of Mathematics, 2000, p. 217), "facility with proportionality involves much more than setting two ratios equal and solving for the missing term. It involves recognising quantities that are related proportionally and using numbers, tables, graphs, and equations to think about the quantities and their relationship". In the same respect, Schwartz and Moore (1998, p. 475) explicitly stated that "when proportions are placed in an empirical context, people do not only need to consider at least four distinct quantities and their potential relationships, they also need to decide which quantitative relationships are relevant." The example they gave relates to mixing 1 oz. of orange concentrate and 2 oz. of water, compared to mixing 2 oz. of orange concentrate and 4 oz. of water. If the question is which mixture will taste stronger, the ratios should indeed be compared, but if the question is which mixture will make more, a ratio comparison is of course inappropriate. The claim for the unwarranted application of proportionality was made even stronger by Cramer, Post and Currier (1993, p. 160). They argued that "we cannot define a proportional reasoner simply as one who knows how to set up and solve a proportion".

For the design of a task, we focussed on students' ability to discriminate between proportional and (different types of) non-proportional situations.

DESIGN OF A TASK

Inspiration for the task design was found in a recent study by Van Dooren, De Bock, Hessels, Janssens and Verschaffel (2005). These researchers studied how students' tendency to overgeneralise the proportional model develops in relation to their learning experiences and their emerging reasoning skills. For that purpose, they presented 1062 students from Grade 2 to 8 with a test containing 8 word problems: 2 proportional ones (for which a proportional solution was correct) and 6 non-proportional ones (2 additive, 2 affine and 2 constant). The following are examples of the non-proportional items:

- Additive problem: "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 5 rounds, Kim has run 15 round. When Ellen has run 30 rounds, how many has Kim run?" (correct answer: 40, proportional answer: 90)
- Affine problem: "The locomotive of a train is 12 m long. If there are 4 carriages connected to the locomotive, the train is 52 m long. How long is the train if there are 8 carriages connected to the locomotive?" (correct answer: 92 m, proportional answer: 104 m)
- Constant problem: "Mama put 3 towels on the clothesline. After 12 hours they were dry. Grandma put 6 towels on the clothesline. How long did it take them to get dry?" (correct answer: 12 hours, proportional answer: 24 hours)

The results showed that many 2nd graders already could solve simple variants of proportional word problems, but the firm skills to conduct proportional calculations (i.e. to solve proportional word problems) were acquired between 3rd and 6th grade. With respect to the non-proportional items, more than one third of all answers contained an erroneous application of the proportional model. The tendency to over rely on proportionality developed in parallel with the ability to solve proportional word problems: it was noticeable already in 2nd grade, but increased considerably in subsequent years, with a peak in 5th grade where more than half of the answers to non-proportional items were proportional errors. After this peak, the number of proportional errors gradually decreased, but they did not disappear completely: in 8th grade still more than one fifth of the answers contained a proportional error. There were some remarkable differences according to the mathematical model underlying the non-proportional problems: One would expect that the word problems with a "constant" model (like the "clothesline" problem mentioned above) were the easiest ones in the test (since there was no need for calculations), but these problems got the highest rate of proportional errors (up to 80% in 5th grade). For some word problems (like the additive "runners" item), the performances even decreased (with 30%) from 2nd to 6th grade. The authors concluded that, throughout primary school, students not only acquire skills to calculate proportions and solve proportional problems. The proportionality scheme becomes so prominent in students' minds that they also begin to transfer it to settings where it is neither relevant nor valid.

For the task that we designed, we worked with the same kind of word problems (4 proportional ones, labelled with the letters C, D, H and J) and 6 non-proportional ones, namely 2 additive, 2 affine and 2 constant, respectively labelled with the letters A and F, E and I, and B and G). The exact formulation of the different problems is given in the introductory section of this research forum. To avoid confusion, we didn't include problems for which the proportional model gives a more or less good approximation, but one can discuss its accuracy on the basis of realistic constraints (such as it is the case in the task of Koeno, Frans and Ronald). Although all ten problems in our task have an exact numerical answer, the task that we gave the students was not to calculate a numerical answer, but to group the problems in at least two different categories and to explain the motivation for their grouping. To allow at least one other way of grouping than the one based on the underlying mathematical model, two of the proportional problems (D and H) were given with a non-integer internal ration, while all other problems were based on easy, natural ratios.

To clearly explain and illustrate the nature of the task (and, at the same time, to show its open-ended character), we first confronted the participants with 13 cardboard figures (stars, triangles and circles) in three different colours (grey, black and white). Two fictitious students, Tommy and Ann, were asked to help their teacher, Mrs. Jones, to classify these figures. Tommy suggested grouping all figures with the same shape (i.e., a grouping based on a "mathematical" criterion), while Ann proposed to bring together the figures with the same colour (i.e. a grouping based on a "non-mathematical" criterion). Then, it was stated that Mrs. Jones made a series of 10 word problems to use in the math lessons (labelled with the letters A to J), but again, they got all mixed up. Students were asked to do as Tommy and Ann had done and to help Mrs. Jones to classify the word problems. More concretely, they were invited to "look very carefully at the problems and to try to make groups of problems that belong together". After that, they had to answer the following questions:

- Why did you make the groups in that way?
- Ann and Tommy did something different when they made groups of the figures. Can you think of a different way to put the problems in groups? Explain that as well.

A SMALL-SCALE EXPERIMENT

The task was given to four students (aged 11 years): Alice, Freya, Hans and Jonas. The researcher first introduced the task and checked pupils' understanding of the instructions. Then, for about 20 minutes, the children were allowed to read the problems and sort them into groups. As each finished, the researcher directed the pupils to record their reasoning, and then to find other groupings.

Alice worked for about 14 minutes to find a first grouping in three categories: group 1 (A and F, the two additive problems) because "they sound similar", group 2 (B and G, the two constant problems) because "it is all like 'how long will it take this person to do this?' and stuff like that", and group 3 with the six remaining problems (the four proportional and the two affine problems). Alice's grouping is based on the underlying mathematical model of the problem, although she was unable to articulate this criterion. In her grouping, she made no distinction between the "pure" proportional problems and the affine problems (which, in fact, ask for a combination of multiplication and addition). After the researcher insisted, Alice came with a second (rather superficial) grouping into two categories (discriminating the problems with "how" and the problems with "what" in the problem statement).

Freya needed about 14 minutes to find a first grouping into three categories: group 1 (H), group 2 (B, C, D, E, F, I and J) and group 3 (A and G). She explained her criterion as follows: "I made the groups due to the operation you have to do to work out the answer. E.g. in group 2, you have to do multiplication to find the answer, and in group 3, you have to divide to find the answer". Clearly, Freya's actual grouping was not based on the criterion she formulated. Being invited by the researcher to find other ways of grouping, Freya proposed a second grouping in three categories: group 1 (A, B, C and F), group 2 (D, E, G, I and J) and group 3 (H) and gave the explanation "I sorted my groups in this way by how easy, moderate or hard the questions were to work out".

Hans who worked for about 19 minutes before coming up with a first grouping also proposed three categories: group 1 (C, D and I), group 2 (A, B and E) and group 3 (F, G, H and I), explaining the motivation for his grouping as follows: "because group 1 is 'times question', group 2 is questions you divide by and group 3 are add and multiply". We cannot see any rationale in Hans' grouping, nor a link between his actual grouping and the explanation he gave for it. After the researcher directed Hans to find a second set of groupings, Hans came with a categorization in four distinct groups: group 1 (A, E and H), group 2 (B and C), group 3 (I and J) and group 4 (D, F and G), but, once more, his justification remained unclear for the researcher.

John, who worked for about 17 minutes, found a classification into two different groups: group 1 (C, E, F, G, H and J) and group 2 (A, B, D and I). He rather superficially explained the motivation for his grouping as follows: "I made these groups because I think it was the most common way and I managed to make them into two groups without any left over". After directed to find a second grouping, John proposed four categories: group 1 (E), group 2 (A, C, I and J), group 3 (B, G and F) and group 4 (D and H). He now explained: "I put them into groups of weight, time and number (respectively groups 2, 3 and 5) and I could not find a group for the 'train and locomotive' one (problem E)" (which is not in line with John's actual grouping).

CONCLUDING REMARKS

The scale of the experiment was very small, so one can hardly infer definite conclusions from it. We observed that the four participating students showed great difficulties in making and motivating classifications of the ten word problems. They mainly looked for linguistic or other superficial differences between the problem formulations and not for an underlying mathematical structure. Possible explanations refer to the nature of the task and the type of problems that we used.

With respect to the task, one can argue that, these students were unfamiliar with classification tasks. Typically, students are expected to "solve" mathematical problems, i.e., to give numerical answers (most often based on the numbers given in the problem formulation), and not to classify problems. Moreover, in retrospect, we think the task was also rather difficult or too "abstract" for 11-year old students. A possible alternative approach meeting more or less the same goals would have been to ask students to combine different problem statements with correct and incorrect (proportional or non-proportional) solution strategies provided by the teacher or experimenter.

With respect to the problems we used, one can argue, in line with Ainley (2000) and several other authors, that the "word-problem" format is inadequate or insufficient to meaningfully contextualise mathematics in the mathematics classroom. Several authors (e.g. Reusser & Stebler, 1997) showed the beneficial effect of meaningful, authentic tasks also for problems where students inappropriately tend to apply linear methods. In this respect, Van Dooren, De Bock, Janssens and Verschaffel (2005) recently showed that students' problem-solving behavior strongly improves when non-linear problems are embedded in a meaningful, authentic context and students are invited to perform an authentic action with concrete materials (i.e. when students are invited to cover a dollhouse floor with "real" tiles instead of calculating this number of tiles in a word-problem context).

Notwithstanding these limitations and shortcomings and the rather disappointing results of our small-scale experiment, the various reactions of the four participating students also suggest that that this type of task design can be a rich starting point for significant classroom discussions on mathematical modelling: which operation is needed in a given problem situation?

DESIGNING INSTRUCTION ON PROPORTIONAL REASONING WITH AVERAGE SPEED

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Instructional design in Realistic Mathematics Education aims at both fostering student reasoning, and at putting instructional tasks in a perspective of long-term learning processes. We try to illustrate this with a task on reasoning about average speed.

TASK DESIGN

There is a long history of instructional design, within which instructional tasks were designed with a primary focus on behavioral objectives. Central instructional design strategies were task analysis and the construction of learning hierarchies. Lessons would be planned on the basis of well-defined prerequisites and precise lesson goals. Teachers were expected to evaluate each lesson by assessing whether those goals were reached at the end of the lesson.

Today this type of instruction is criticized as being 'instructionist' or as reflecting a 'transmission model' of teaching. In contrast to teachers *instructing*, the emphasis is now on students *constructing*. Following Cobb (1994) we may argue that constructivism—as an epistemology—does not have direct implications for teaching, as "the constructivist maxim about learning may be taken to imply that students construct their ways of knowing in even the most authoritarian of instructional situations" (Cobb, 1994, 4). Still, constructivism may inspire one to consider how we can influence the construction processes of the students. One of the results of such considerations is a shift in attention from behavioral objectives to the mental activities of the students. In this respect, we may refer to Simon's (1995) notion of a hypothetical learning trajectory. We may notice the flexibility and the situatedness of this concept. A teacher will design a hypothetical learning trajectory for the students in his or her classroom, given where the students are at this moment, while taking into account goals and teaching practices. Moreover the teacher will adjust the hypothetical learning trajectory on the basis of his or her interpretation of how the students act and reason. This puts the notion of task design in a different perspective. What the task entails is not fixed, as tasks are interactively constituted in the classroom. When we expect teachers to orient themselves on the mental activities of the students, and consider those in relation to the intended end goals, we might argue that teachers should be supported in making these considerations.

In the Netherlands we constructed an instructional design strategy, which is aimed at developing prototypical instructional sequences and local instructional theories that are to offer teachers a framework of reference for constructing their own hypothetical learning trajectories. This strategy is based on what is called design research and on the use of three design heuristics from realistic mathematics education (RME), namely, guided reinvention through progressive mathematization, didactical phenomenology, and emergent modeling. In the following paragraphs we explain this in more detail.

Design research can be thought of as a combination of design and research aimed at developing both a sequence of instructional activities and a local instructional theory. A classroom teaching experiment forms the core element of this type of research (Gravemeijer, 1998). This consists of an interactive and cumulative process of designing and revising instructional activities. To this end, the designer conducts anticipatory thought experiments by envisioning both how proposed instructional activities might be realized in the classroom, and what students might learn as they engage in them. These instructional activities are tried out in the classroom. Then, new instructional activities are designed or redesigned on the basis of analyses of the actual learning processes. At the end of a cumulative process of designing and revising instructional activities, an improved version of the instructional sequence is constructed. After some design experiments, the rationale for the instructional sequence eventually acquires the status of a local instructional theory.

The other core element of our instructional design strategy is the use of the three design heuristics that characterize the domain-specific instruction theory of RME. This educational theory originated in the Netherlands inspired by Freudenthal's idea of mathematics as an activity of organizing or mathematizing. The first heuristic has to do with Freudenthal's (1973) idea that students should be given the opportunity to experience a process similar to the process by which mathematics was invented, and is called guided reinvention through progressive mathematization. According to this heuristic, the designer takes both the history of mathematics and the students' informal solution procedures as sources of inspiration (Streefland, 1990), and tries to formulate a provisional, potentially revisable learning route along which a process of collective reinvention (or progressive mathematization) might be supported.

The second heuristic concerns the phenomenology of mathematics, and asks for a didactical phenomenological analysis. The developer looks at present-day applications in order to find the phenomena and tasks that may create the need for students to develop the mathematical concept or tool we are aiming for. The goal of a phenomenological investigation is, in short, to find problem situations that may give rise to situation-specific solutions that can be taken as the basis for vertical mathematization.

In the instructional design we are reporting in this paper, the focus is on the emergent modeling heuristic (Gravemijer, 1999). Models in RME are related to the activity of modeling. This may involve making drawings, diagrams, or tables, or it can involve developing informal notations or using conventional mathematical notations. It is important that these notations have the context situation of the problem as starting point and are developed by the students as they attempt to come to grips with the problem and find ways to solve it. The conjecture is that the emergence of the model is reflexively related to the construction of some new mathematical reality by the students, which may be labeled as more formal mathematics. Initially, the models refer to concrete or paradigmatic situations, which are experientially real for the students, and are therefore to be understood as context-specific models. On this level, the model should allow for informal strategies that correspond with situated solution strategies. As the student gathers more experience with similar problems, the model gets a more object-like character, becoming gradually more important as a base for mathematical reasoning than as a way of representing a contextual problem. The *model of* informal mathematical activity becomes a *model for* more formal mathematical reasoning.

THE DESIGN TASK: (UN)JUSTIFIED PROPORTIONAL REASONING

In the context of the research forum, we were asked to design a single task on proportional reasoning, while also addressing the issue of unjustified proportional reasoning. We chose a task on speed. Reasoning about speed in everyday-life situations asks students to coordinate pure proportional reasoning with realistic considerations on what may distort the proportionality in actual reality. The task we designed was a problem about two girls who make a bicycle trip. After 1 1/2 hour they pass a signpost telling them that they have already cycled a distance of 30 kilometers, and they still have 45 kilometers to go. In the story one of them comments: ‘This is going well’, and the question the students have to answer is why she would say so. There were five more questions, but, in a sense, the first one covers them all; the other questions discuss the relevant points more explicitly. The remark ‘This is going well’ is expected to raise a discussion about questions like:

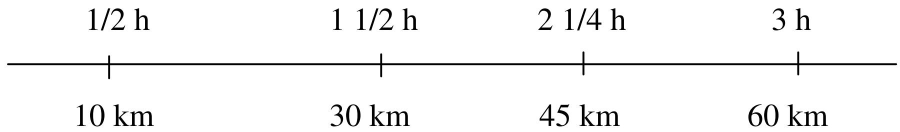
- Is 30 kilometers in one hour and a half an achievement one would be happy with? What would have been their speed, in terms of kilometers per hour, and would that be fast, or slow?
- The girl might be happy because she sees that they have done a big part of their trip already. So what is the relation between the 30 kilometers and the distance the girls still have to cycle? Would it be possible to estimate how much time they need for the rest of their trip?
- Will a calculation lead to an exact prediction, or are there other factors to take into account?

Note that the numbers were chosen carefully as to make easy computations. The task was tested both in the Netherlands and in the UK; the English version was about a trip from Corby to Cambridge, with 18 miles done and 30 miles still to go. Note also that there are various ways to calculate the time needed for the second part of the trip. Students can compare 30 km and 45 km and conclude that the second part will take 1 1/2 time as long, they might see that 30 km in 1 1/2 hour gives 10 km in half an hour and reason from this, or they might calculate the average speed in km per hour.

The student activities that we anticipate are threefold:

- The students will (start to) reason proportionally in the context of speed.
- The students' explanations will allow the teacher to start a discussion about how to record proportional reasoning on paper. This could be a lead in to a discussion about the use of models like the double number line or the ratio table.
- The students will realize that proportional reasoning does not predict the arrival time in a precise manner, but do realize that calculations are a useful tool in making estimations.

Models for proportional reasoning, and therefore also for reasoning with average speed, are the double number line and the ratio table. They both offer a systematic way of writing down the relation between distance and time. On the double number line the position of points is meaningful, whereas the columns of the ratio table can be in any order. Both models can function as a tool, allowing one to break down complicated calculations into intermediate steps.



time	1 1/2 h	1/2 h	2 hs	2 hs 15 min
distance	30 km	10 km	40 km	45 km

In our view students should be stimulated to reinvent these models; they should not be offered as a ready-made products. This does not mean that students are expected to reinvent the exact way numbers are written in rows and columns in the ratio table, but they should be stimulated to think about systematic forms of notations, and thereby learn to appreciate the 'official' ratio table as one of the possible forms.

Following the emergent modeling perspective, the students' activity with double number line and ratio table will be grounded initially in thinking about its contextual meaning. Doubling in the ratio table, for example, will be justified by thinking of traveling twice as long. Later the ratio table may be used for reasoning with linear relations. As we argued elsewhere, students may eventually start to use the ratio table in a semi-algorithmic manner to execute multiplications, without necessarily having to think of possible contextual meanings of the numbers involved (Gravemeijer, Boswinkel, Galen, & Heuvel-Panhuizen, 2004).

SOME FINDINGS

The task was tested twice, once with a small group of four students in England and once in a class with 10 to 12 year old students in the Netherlands. In the experiment in England the teacher introduces the problem by focusing heavily on exploring the situation and the circumstances that influence the time one needs to cycle from Corby to Cambridge. The situation is meaningful enough for the students to bring forward many aspects that could influence the cycling time. They mention that the time to travel the whole distance could be influenced by the weather, the hills alongside the route, the breaks the girls take, etcetera. In this setting the students developed ideas on how much time it takes to cycle the whole tour, but the numbers they bring forward are mostly guesses. They agree that it should take the children at least two hours to ride the 30 miles from the road sign to Cambridge. Only two students replace their guesses about the time needed to cycle from Corby to Cambridge by calculations and schemes.

The Dutch experiment also starts with an exploration of the context. As the students here are more familiar with a bike a means of transport, they easily bring forward what should be done if one undertakes a tour as mentioned in the task. When next the students receive the worksheet with the map and the road sign, they find little problem in interpreting the situation. The teacher here, like her English colleague, discusses one of the girls saying ‘This is going well’, when they arrive at the road sign.

In the Dutch version of the task in took the children one and a half hours to cover the first 30 kilometer. At that point there is still 45 kilometer to go. The students formulate several arguments why 30 kilometer in one and a half hour is quite a distance for such a short time.

The teacher frequently asks the students to explain their ideas. Therefore the discussion focuses more and more on mathematical arguments. One of the students for example claims that he cycles 3 kilometers in a quarter of an hour. He argues that in that speed it takes one and a half hours to cover 18 kilometers. 30 kilometer in one and a half hour therefore is fast cycling.

Unlike her English colleague, the Dutch teacher at certain points redirected the discussion to the use of mathematical arguments. The Dutch students therefore all reasoned in terms of ratios to calculate the arrival time. Moreover, the arrival time is next discussed in terms of the context, where the students decide to add about an hour for breaks, flat tires and weather conditions.

We were in the fortunate position to thus find two settings where the teachers both choose a different manner to guide the students. This enabled us to analyze the teacher’s role and to test (in this specific context) our ideas on this. We noticed that the Dutch students did not have any problem with putting their calculations into perspective. They could easily compute how much time would be needed for the next 45 km, but it was also obvious to them that such calculations only give you a first approximation. In the English experiment the students were aware that one could only estimate the arrival time, but the setting did not stimulate them to further mathematize the problem.

CONCLUSION

In Realistic Mathematics Education instructional design concerns series of tasks, embedded in a local instruction theory. This local instruction theory enables the teacher to adapt the task to the abilities and interests of the students, while maintaining the original end goals. The task we designed should be viewed from this perspective. In an educational setting it would not be an isolated task, but part of a longer learning route. Goals of such a learning route would be:

- Students learn to reason proportionally.
- They develop tools for proportional reasoning, tools that can also be used for calculations, like the double number line and the ratio table.
- At the same time, however, they learn to see the relativity of their calculations; when making predictions other factors in the context may have to be taken into consideration.

When our task was tested, the emphasis was on the third goal. Within a longer learning route, however, the challenge would be more to help students develop the right tools for proportional reasoning. Among other things, these tools would help children to discriminate between situations where proportional reasoning is, and is not justified. RME describes this process of developing mathematical tools as emergent modeling.

In the test situations there was no discussion, or only a limited discussion about tools like the double number line and the ratio table. Within design cycles of testing and revising this could lead to the decision to make certain changes, in this case, for example, to change the numbers in such a way that students would not be able to do the calculations in their heads. But even when an activity, after some revisions, has found its definite form, success cannot be guaranteed, of course. This underscores the central role of the teacher in supporting the learning process. The teacher should be capable to make changes, like asking certain questions, focusing the discussion on certain topics, and so on. An essential condition to establish this is, that the teacher knows and understands the local instruction theory behind the activities.

FOLDING PERIMETERS: DESIGNER CONCERNS AND STUDENT SOLUTIONS

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In this paper we first describe some of the concerns and approaches that have influenced the process of designing the *Folding Perimeters* activity. Then, we will present several selected episodes from the actual solutions produced by two pairs of

12-year-old, higher ability students, in view of the design concerns that were encountered in the development of this activity.

TASK CHARACTERISTICS

Folding Perimeters was designed as the last and most advanced activity in a learning series on ratio and proportion. This section describes the main characteristics of the activity, and some considerations that led to its present design.

Context. In this activity, students investigate the perimeters of an alternating sequence of squares and rectangles, during a process of repeated folding-in-two (Fig. 1). The use of context enables a constructivist path of learning (Hershkowitz et al., 2002). When students start with a problem situation such as the above, they can rely on their acquaintance with its non-mathematical components and on their ability to observe, to experiment and to act on the situation itself. As indicated by Ainley and Pratt in this collection of papers, the characteristics of a task may also contribute to provide a sense of purpose and ownership. Moreover, a problem situation can also contribute to students' understanding of the need for constructing appropriate tools and concepts, first investigating the problem at an intuitive level and later on, analysing the newly formed tools and concepts in a more extended and mathematically formal manner. Tourniaire and Pulos (1985), in reviewing the research on proportional reasoning, concluded that context plays a crucial role in student performance and that use of a wide variety of contexts is needed in the teaching of this domain. In our case, we considered the context of paper folding to be simple and familiar, on the one hand, and to be rich in mathematical opportunities on the other hand.

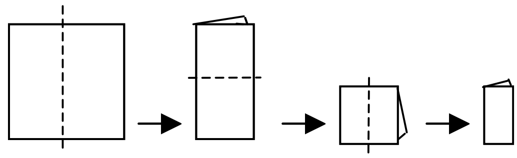

Figure 1. Context of *Folding Perimeters*

Mathematical content. The activity integrates various mathematical domains - for example, geometry (squares, rectangles, perimeters, opposite sides, measurement), arithmetic (numerical tables, operations, difference, ratio), and algebra (*Excel* formulas and pattern generalizations). The mathematical content is stated clearly throughout the activity, and is one of the factors that determine the sequence of tasks. The first three tasks in the activity require a more geometrical and visual investigation, there is a task that relates to the differences between the perimeters of two adjacent shapes, and the last two tasks focus respectively on the perimeter ratios of two adjacent, and of every other shape. However, some other tasks in the activity are less directive with regard to content or solution strategy open. More specifically, these tasks require students to find any patterns of perimeter change and justify them. Similarly to Dirk, Wim and Lieven’s task, the patterns of change in our activity do not constitute a classical and straightforward application of the idea of proportionality, common in many textbooks.

Multiple representations. The presentation of mathematical concepts and operations in various representations is central in investigative activities (Friedlander & Tabach, 2001a). One of our reasons for using spreadsheets as a mathematical tool is their ability to simultaneously support work on various representations, and to present the algebraic representation as an efficient and meaningful means of constructing data. In our activity, students are specifically required to present perimeters and perimeter changes in actual paper, in drawing, in numerical tables, as algebraic formulas, in bar diagrams, and in verbal descriptions. Some of the tasks focus on the construction and use of a specific representation, whereas others leave this issue open to the students. Figure 2 presents a numerical and graphical representation of the data and some of the results obtained by the observed students, regarding the alternating sequence of shapes in the activity. Some of the algebraic formulas used by the observed students will be discussed in the next section.

	A	B	C	D	E	F
1	fold #	side	side	perim.	differ.	ratio
2	0	32	32	128	32	3/4
3	1	32	16	96	32	2/3
4	2	16	16	64	16	3/4
5	3	16	8	48	16	2/3
6	4	8	8	32	8	3/4
7	5	8	4	24	8	2/3
8	6	4	4	16	4	3/4
9	7	4	2	12	4	2/3
10	8	2	2	8	2	3/4
11	9	2	1	6	2	2/3
12	10	1	1	4		

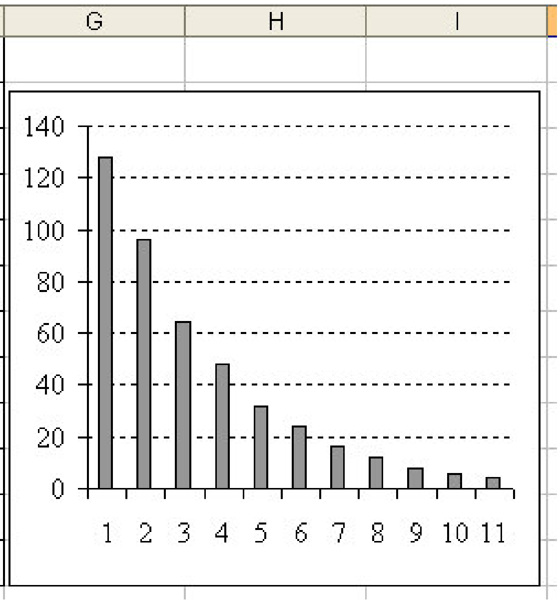

Figure 2. Spreadsheet representation of data and results in *Folding Perimeters*.

Task sequencing. Investigative activities (including *Folding Perimeters*) frequently follow a flow pattern that is in many ways similar to the PCAIC investigative cycle (*pose*, *collect*, *analyze*, *interpret, and communicate*) proposed by Kader & Perry (1994). This cycle is adapted from the domains of data investigation and scientific research, and is inductive in nature. First, specific cases are collected, organized, and analyzed, and then general patterns are formed and conclusions are drawn, interpreted and applied.

Generalization of patterns. Many activities associated with generalization – including ours, assume that the process of pattern generalization is inductive and based on a limited number of cases. In the next step, the discovered pattern is explained and justified (Friedlander et al., 1989). This flow pattern is frequently used in the design of generalization tasks. In our activity, this sequence of tasks is applied in several cycles, with regard to any patterns of perimeter change, then regarding the difference, and finally regarding the ratio of perimeters of two consecutive shapes.

Level of task openness. The process of task design is based on a constant state of tension that exists between the design of unstructured open tasks that do not require that the problem posed be solved by a specific method, a certain representation or an implicitly given sequence of steps, as opposed to a structured approach that poses specific requests with regard to the variables mentioned above. The open approach reflects the designers' striving to develop problem solving skills, to develop creative mathematical thinking, to provide opportunities for students to actually experience investigation, and to achieve a meaningful construction of knowledge. The structured approach enables students to pursue a more predictable and planned agenda in the domains of mathematical content and the processes of problem solving. The activity discussed here addresses this issue by presenting a sequence of tasks of both kinds. Open tasks require students to identify any properties of the presented sequence of shapes, make predictions, and then look for patterns that describe the change in perimeter. Tasks that are more directive require the student to collect data for the first ten shapes in the sequence, organize it in a spreadsheet table, present it as a diagram, investigate patterns of perimeter change by considering first the difference and then the ratio between pairs of adjacent shapes, and of shapes placed in the sequence at a distance of two steps. One may argue that leading students through a sequence of tasks, rather than presenting only a problem situation and a "big question", decreases in itself the extent of freedom in student work. We suggest, "walking a fine line" between opening and closing a task by directing students to some extent through a sequence of leading questions, within an open problem situation. This approach to task design supports a convergence towards a meaningful progress in the students' solution, without curtailing their sense of ownership of the task (in the sense of Ainley and Pratt in this collection of papers). Such a sense of ownership stems from the opportunity to observe, experiment and act on a "realistic" situation, and not necessarily from the task's degrees of freedom.

Verbalization. Requests for descriptions of patterns, explanations, discussions of another (fictitious) student's solutions and reports of results are included in this, as well as many other activities. These requests are the result of designers' desire to develop communication and documentation skills, to make students consider verbal descriptions as mathematical representation, and to change the stereotypic view of mathematics as the exclusive domain of numerical and algebraic symbols only.

Use of spreadsheets. Our experience of students working in a spreadsheet environment shows that spreadsheets can serve as a powerful tool, and allow for some of the design heuristics proposed by Gravemeijer and his colleagues in this collection of papers. They support students' processes of creating emergent models and their "vertical mathematization" of the problem situation. The use of this technological tool to support and promote processes of generalization and algebraic thinking has been amply discussed in terms of theory and investigated empirically (for design considerations in spreadsheet activities, see for example, Hershkowitz et al., 2002; Friedlander & Tabach, 2001b). Because of space limitations, we will only briefly list the following considerations that led the designers to use spreadsheets in this particular activity:

- they serve as a powerful tool for data collection, organization and representation,
- they provide continuous and non-judgmental feedback throughout the solution process,
- they present the concept of proportion dynamically, as a sequence of constant ratios obtained by applying the same rule to numerous pairs of numbers or quantities,
- they enable the analysis of an extended collection of data,
- they emphasize the meta-cognitive skills of monitoring and interpreting results,
- they promote algebraic thinking and present algebraic formulas as a useful and meaningful tool.

STUDENT SOLUTIONS

As previously mentioned, two pairs of students (referred here by the initials of their first names as MS and MG) were observed by one of the authors as they worked on the *Folding Perimeter* activity, during a period of about 80 minutes for each pair. For the purpose of this paper, we will not distinguish between the two members of a pair, and will refer to each pair as an entity. The students had previous experience in using *Excel* in mathematical investigations, but had not pursued the learning sequence of ratio and proportion that included our activity. The interviewer's interventions were minimal and limited to occasional requests to clarify answers or to start working on the next item. The latter case included dealing with "unproductive" paths of solution – defined by Sutherland et al. (2004) as cases of "construction of idiosyncratic knowledge that is at odds with intended learning", and require the teacher's intervention in regular classroom situations. A systematic analysis of student work, according to the eight designer concerns described in the previous section is not possible, because of the space limitation.

In general, the students followed the prescribed sequence of tasks and solved them in a mathematically rich and resourceful manner. However, we will focus here on some differences between the observed students' solution processes and the designers' plans and predictions.

Contrary to our expectations (see the comments on task sequencing and generalization of patterns in the previous section), both pairs reached, at the initial stage of predictions, generalizations that were "scheduled" by the designers to be reached only later on, and on the basis of the collected data. By examining their folded paper square and the drawing of the folding process (Fig. 1), the students considered visual and global aspects regarding the sides that were "lost" through folding, and made the following predictions:

MG: It [the perimeter] gets smaller by the length of the side that gets halved.

MS: In my opinion it [the perimeter] will be 3/4. The vertical lines will stay and the horizontal lines lose one half and one half – and that's a whole side. [After Interviewer asks "And what happens from the second to the third shape?"] It comes out 4/6 because we are left with 4 out of 6 halves [of the longer sides of the rectangle].

Both pairs produced general patterns at a very early stage of the activity - MG is reasoning additively, by looking at differences, whereas MS is thinking proportionally, by considering ratios. The issue of interest for designers and/or researchers is that the processes of pattern generalization can follow two routes:

- inductive generalization based on the collection and analysis of data (as followed by the sequence of tasks in this activity),
- deductive generalization based on a global analysis of the problem situation, and on general reasoning (as followed by the two pairs of students).

We assume that both the students' mathematical ability and task design (e.g., the representation used in the initial description of the problem situation) affect the choice of the route.

The use of spreadsheets was also a source of unexpected developments. The observed students did not encounter any technical difficulties with regard to the handling of the tool. They read, understood, and performed the computer-related instructions, and were familiar with the *Excel* syntax for writing formulas. However, the following three episodes observed during the students' work indicate that the spreadsheet's intrinsic properties can provide opportunities for higher-level thinking, and help both the student and the teacher detect and relate to conceptual difficulties.

a) MS: They construct the spreadsheet table for the first ten shapes (see Fig. 2). They write in the first line of the perimeter column (for the perimeter of the original square) the formula =4*B2 and in the next line (for the perimeter of the rectangle produced by the first folding) =2*B3+2*C3.

"But we can't drag down [two formulas]…Then let's change this [the first formula] into this [the second]". They rewrite the formula for the square as =2*B2+2*C2 and drag it down.

b) MG: They write for the length of sides (see Fig. 2) a formula (pattern) indicating the halving of the above-situated cell, and drag it down cell by cell – one cell at a time, hoping that this method would produce the desired sequence of pairs of identical numbers.

c) MG: They construct the column for the difference of adjacent perimeters (see Fig. 2) by writing in the first line the formula =D2-D3 and dragging it down to the last line. As a result, the last number shows an uncharacteristic increment in the difference sequence (….8, 8, 4, 4, 2, 2, 4) – a result of the difference of the last perimeter (4) and the next empty cell that is interpreted by Excel as zero. They notice the outcome, retype the same pattern (=D3-D4) in the second line and again drag it down to the last line –obtaining of course the same results as before.

In episode (a), work on *Excel* provided an opportunity to perform a higher-level analysis for students without any background in formal algebra: they compared two algebraic expressions and identified one (4B) as a particular case of the other (2B+2C, when B=C). However, episodes (b) and (c) showed that the observed pair of students thought that changing the place or the physical handling of a pattern expressed as an algebraic formula will change its essence.

CONCLUSIONS

The considerations related to the design of the *Folding Perimeter* activity are closely connected to a wide variety of theories and research findings on student cognition, and on the use of technological tools for teaching mathematics. Our experience in implementing many similarly structured investigative activities indicates that they provide opportunities for meaningful learning of mathematical concepts.

We also described here several episodes of student work on a particular activity to show that differences between a designer's planned actions and student work should be expected. Whether, and if so how, these differences should influence the design of this particular activity or the principles of task design remains an open question.

THE DOLLS' HOUSE CLASSROOM

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The design of our task uses the framework of *purpose and utility* (Ainley & Pratt, 2002, Ainley et al., forthcoming). *Purpose* reflects our concern to create tasks which are meaningful for pupils. One strand of research on which we draw is that of mathematics in out-of-school contexts (e.g., Nunes et al., 1993) which has highlighted the contrast between the levels of engagement of learners in mathematical activities in and out of school. In a PME plenary, Schliemann (1995) claimed '*we need school situations that are as challenging and relevant for school children as getting the correct amount of change is for the street seller and his customers. And such situations may be very different from everyday situations.*' (p. 57). We argue that setting school tasks in the context of 'real world' situations, for example through the use of word problems, is not sufficient to make them meaningful for pupils. Indeed, there is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide a purpose that makes sense to pupils (see for example Ainley, 2000; Cooper & Dunne, 2000).

We see the *purposeful* nature of the activity as a key feature of out-of-school contexts which can be brought into the classroom through the creation of well designed tasks. Drawing partly on constructionism (Harel & Papert, 1991), we define a purposeful task as one which has a meaningful outcome for the learner in terms of an actual or virtual product, the solution of an engaging problem, or an argument or justification for a point of view (Ainley & Pratt, 2002; Ainley et al., forthcoming). This feature of purpose for the learner, *within the classroom environment*, is a key principle informing our pedagogic task design.

The purpose of a task, as perceived by the learner, may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a 'real world' context. The purpose of a task is not the 'target knowledge' within a didactical situation in Brousseau's (1997) sense. Indeed it may be completely unconnected with the target knowledge. However, the purpose creates the necessity for the learner to use the target knowledge in order to complete the task, whether this involves using existing knowledge in a particular way, or constructing new meanings through working on the task. Movement towards satisfactory completion of the task provides feedback about the learner's progress, rather than this being judged solely by the teacher (Ainley et al., forthcoming). Harel (1998) proposes the 'necessity principle', which addresses the issue of creating the need to learn particular things in a different way. In Harel's terms an 'intellectual need' for a mathematical concept should be created before embarking on the teaching of the concept. However, intellectual need and purpose clearly differ, since intellectual need is related specifically to a mathematical concept, while the purpose of a task is not explicitly mathematical, but relates to the outcome of the specific task. The necessity principle perhaps relates more closely to the second construct within our framework: utility.

UTILITY

Understanding the *utility* of a mathematical idea is defined as knowing how, when and why that idea is useful. A purposeful task creates the need to use a particular mathematical idea in order reach the conclusion of the task. Because the mathematics is being used in a purposeful way, pupils have the opportunity not just to understand concepts and procedures, but also to appreciate how and why the mathematics is useful. This parallels closely the way in which mathematical ideas are learnt in out-of-school settings. In contrast, within school mathematics ideas are frequently learnt in contexts where they are divorced from aspects of utility, which may lead to significantly impoverished learning. Utility thus has some similarity to Harel's 'intellectual need'. However, Harel sees intellectual need as providing the motivation for learning a concept, whereas utility, why and how the concept is useful, is seen as an intrinsic, but frequently unacknowledged, facet of the concept itself.

THE DOLLS' HOUSE CLASSROOM TASK

The Dolls' House Classroom task focuses on scaling, which is a key idea in proportional reasoning. The outcome of the task is a set of instructions for another group of children to make items for the dolls' house classroom. The purposeful nature of the task would, of course, be increased if the pupils were involved in the actual manufacture of the product. We developed the idea for this task from the work of a primary school class who used a similar approach to building scenery for a play based on the Nutcracker ballet. There was a need to make the scenery large enough for the people to appear the size of rats.

At the beginning of the task, each group of pupils is given an item from a dolls' house which corresponds to something they will have in their own classroom (e.g., a chair, a table, a door, a computer). The activity of comparing this with its full-size equivalent will involve measuring and discussion, as pupils decide on which are the most important measurements to use. For example, although the particular design of chairs may vary, the height of the seat above the ground remains fairly constant.

Once they have arrived at a pair of measurements for the full-size and dolls' house items, they enter the most crucial part of the task: deciding how the use these in order to scale other measurements. The role of the spreadsheet is important here in allowing pupils to experiment with different ways of using the measurements, and applying them to other items which they decide to include. It is important that there is an opportunity here for the pupils to make decisions about which other classroom items they will use, as this adds to their ownership of the task. We note here a close affinity with Friedlander and Arcavi, who set out in this collection of papers some of the reasons why they also adopted spreadsheets.

The above considerations reflect our practical research and teaching experience as well as our theoretical perspective. In order to illustrate some of the characteristic features of such a design approach in action, we gave the dolls' house task to two pairs of eleven year old students (one pairs of boys and one of girls). It turned out that the girls needed considerably more support than the boys from the teacher/researcher. Interestingly, this had the effect of closing down the task for the girls, who followed a much more one-dimensional route through the problem, staying close to the suggestions of the teacher. In contrast the boys were more adventurous in their approach and were able to exploit the opportunities that the task offered. This contrast acts as a useful reminder that the notions of purpose and utility are design imperatives, which act as potentials for the students but how those potentials are realised will vary according to a range of personal attributes (knowledge, confidence and so on) brought to the situation by the children and the structuring resources of the setting, including *inter alia* the approach of the teacher. (Indeed, we note that all authors in this collection of papers found to a greater or smaller extent that there were discrepancies between the learning trajectory that they had envisaged and that which ensued in practice. We make further comment on this at the end of this section.) As a result of this contrast between the boys and the girls, we focus below more on the activity of the boys, which better illustrates the implications of designing for purpose and utility.

PURPOSE AND UTILITY IN ACTION

We were struck by the relationship between the boys' construction of purpose and utility and how the interplay between the two evolved during the 40 minute session. Initially the boys tried to relate the task to their own experiences. One boy told the teacher about how his grandfather used to make dolls' furniture. The other talked about scaling in maps in response to the teacher's mentioning of the term *scale factor*. From an early stage, the boys questioned the nature of the task that they had been set. (Figures in brackets indicate time elapsed in minutes.)

> [6:06] Is this real? Are a Year 6 class really going to do this?

The researcher admitted that this was not actually going to happen.

> [6:35] Why can't they just buy the dolls' house?

What do we make of these questions? Are they challenges that suggest the boys are resisting the invitation of the teacher to engage with the problem? If so, it would be hard to explain the subsequent activity, which was marked by the boys' considerable intent and persistence. Rather, we believe that these questions indicate a process in which the boys were beginning to take ownership of the task, They were, in our opinion, delimiting the task, asking where are its boundaries with reality, recognising that is was important to appreciate the true nature of the task as this would later inform their strategies for its solution.

The task itself continued to act as the arbitrator of the activity (in contrast, the girls required the teacher to direct their activity throughout the session). At one point one of the boys encouraged his partner to move on.

> [17:14] You can't just keep doing the table; we've got to do something else.

The boys recognised that there was an implication in the task to build a range of artefacts. It was not necessary to ask the teacher what they should do next.

At times, the boys were even prepared to follow the path indicated to them by the task rather than that suggested by the teacher. Thus, at one point, the teacher asked how the boys would find the height of the little shelf for the dolls' house.

> [13:40] Before we do that, won't we have to do the width of this table first?

When students take ownership of a task, the levels of engagement can be very high; it is our belief that the opportunity to make choices is influential in helping students to make a problem their own. Furthermore, a well-designed task will also enable students to follow up their own personal conjectures when they try to make sense of the task. Such personal conjectures might be seen by other researchers as misconceptions but our stance recognises the need, from the design point of view, for students to be given the opportunity to test out for explanatory power their own meanings, in this case for proportion. Thus the boys' spreadsheet shows several different attempts at ratio. In one set of cells, they divided the height of the real table by that of the supplied dolls' table (68.5 / 4.3 = 15.93). But when it came to the width of the table, they divided the dolls' table by the real table (5.5 / 134.2 = 0.040983607). In another part of the spreadsheet, they divided the real shelf width by the real table width (75.5 / 134.2 = 0.562593). Each of these calculations has possible utility for their task but whether any particular approach has explanatory power depends on exactly how the boys wanted to use the result and what sense they could make of the feedback. The nature of the task allowed them to explore all three routes, rather than following a route defined prescriptively by the teacher.

Such explorations enabled the boys to construct meanings for the divisions being carried out on the spreadsheet. The spreadsheet handled the calculations, allowing the boys to focus on whether the ratio was actually useful to them in their task. Even so, the technical demands of deciding what to divide by what could become so absorbing that the context could be temporarily forgotten.

> [13:20] So, this table [pause] the height of this table divided by the height of that table [pause] I've forgotten how this is going to help!

Nevertheless, the boys recognised that there was a purpose to this technical effort and they were eventually able to reconstruct the reason behind that work. We see this statement and the subsequent activity as evidence that the boys were indeed linking the purpose of the task to a utility for comparing dimensions. The measurements enabled them to derive a scale factor, which could be used to calculate the dimensions of imaginary objects. The utility emphasises how the scale factor might be useful, admittedly in a situated narrative, rather than the technical aspects of how to calculate a scale factor.

This utility was planned. However, when we design for purpose and utility, there is a strong likelihood of other utilities emerging in unpredictable ways. In well-designed tasks there should be a richness of possibilities. When we listened to the recording of the boys working on this task, we were able to identify unplanned opportunities to focus on a utility for rounding. Thus, consider again the occasion when the boys divided the width of the dolls' table by the real table to obtain 0.040983607.

> [17:40] How do you shorten that down?

The boys intuitively knew that it would be useful to reduce the length of the decimal. However, they did not know the technicalities of how to do this. Had the teacher been available at that point, there may have been an opportunity to focus on rounding in the context of making numbers more manageable. In the event the boys moved away from this calculation and considered an alternative approach. Nearly ten minutes later [26:50], another rounding opportunity appeared. On this occasion the numbers were easier and so the boys were able to round manually 8.0665 to 8.1.

Another illustration of the richness of such tasks occurred when the boys were considering the area of the tables.

[13:50] We have to find the area of that *(referring to the dolls' table)* and then the area of one of these tables and then combine the area of…

One of the most difficult ideas in secondary level work on proportion is the notion of an area scale factor and how it relates to a linear scale factor. There was potential here for the students to explore the utility of area scale factors.

FINAL COMMENTS

We advocate stressing in task design how mathematical concepts might be useful in particular situations. Such utility does not imply real world relevance. The dolls' house task is somewhat contrived if judged against such a criterion. Nevertheless, the boys took ownership of the task, partly because they were able to make choices of their own and partly because they were able to construct their own narrative for the task. As the activity evolved, the emphasis on making sense of the task itself by relating it to personal experiences and testing its boundaries transformed into creating solution strategies, guided by the purpose of task. In their efforts to construct meanings for the feedback from the spreadsheet, the boys constructed a utility for scale factor. At the same time, there was a richness in the task that is typical in our experience of tasks designed according to the constructs of purpose and utility. This richness manifested itself in the way that the boys followed numerous paths and stumbled into situations that offered potential for engagement with other mathematical utilities.

We note with interest that all the authors in this collection of papers appear to have attempted to include some aspect of purpose or utility in their task designs, without of course seeing what they did in precisely those terms. Word problems in themselves can appear dry, even hackneyed, but in Dirk, Wim and Lieven's task, the problem was transformed. The children had to work on the word problems at a meta level, deciding which problems were like which others. As De Bock, Van Dooren and Verschaffel subsequently observed, the task proved to be rather challenging but we too have seen in the past that this type of transformation can imbue a sense of purpose to the task for many children. In Koeno, Frans and Ronald's task, there was an attempt to connect children's thinking to their experiences of journeys. The approach seemed to offer the children the opportunity to construct a utility for proportion in relation to planning such journeys. In Alex and Abraham's task, we saw the potential for practical activity, which might even have been opened up further by considering other aspects of paper folding that can lead to other interesting proportions.

Finally, and almost as a cautionary tale, we remind you (and ourselves) that the girls working on our own task went down a much narrower predictable pathway than did the boys. One level of response to this result is simply to argue that no task can offer rich pathways for all children. On the other hand, perhaps there are lessons to be learned, not just from the boys' work, but also from that of the girls. Gravemeijer, van Galen and Keijzer have explained how they see the demands of this research forum as at variance to some extent with their normal activity. The principle of progressive mathematization, utilised by designers in the Realistic Mathematics Education school, is not one that sits easily with designing a single task in one shot. We too see task design in terms of design research and, in that spirit, would interpret all these efforts at task design as “bootstrapping” or first exploratory attempts.

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RF02: GESTURE AND THE CONSTRUCTION OF MATHEMATICAL MEANING

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The role of gestures in mathematical thinking and learning is examined from the perspectives of cognitive science, psychology, semiotics and linguistics. Data from situations involving both children and adults, addressing mathematical topics including graphing, geometry, and fractions, are presented in the context of new theoretical frameworks and proposals for the analysis of gesture, language, signs and artefacts.

INTRODUCTION

Recent research in mathematics education has highlighted the significance of the body and, specifically, perceptuo-motor activities in the process of mathematics teaching and learning (Lakoff & Núñez, 2000; Nemirovsky *et al*., 1998). The analysis of the role of the body in cognition takes place within a wide multi-disciplinary effort, involving neuroscience, cognitive science, experimental psychology, linguistics, semiotics and philosophy. These disciplines offer complementary tools and constructs to those who wish to investigate the complex interactions among language, gesture, bodily action, signs and symbols in the learning and teaching of mathematics. The goal of the Research Forum is to examine the role that gesture plays in the construction of mathematical meanings. More specifically, we are concerned with the following questions:

- How can we describe the phenomenology of gestures in mathematics learning (e.g.: What kind of gestures are there? Is the classification created by McNeill (1992) adequate for mathematical gestures?)
- How does gesture function in the processes of learning mathematical concepts?
- Can gesture provide evidence about how mathematical ideas are conceptualized?
- Are gestures context-dependent? In particular, how do they change when students interact with artifacts?
- Which theoretical frameworks are suitable for analysing gestures in mathematics learning taking into account work on gesture carried out within disciplines outside of mathematics education?
- What consequences of the research on gesture can be drawn for mathematics students, teachers, and prospective teachers?

The analysis of gesture, both within and outside of mathematics education, takes place within the broader framework of recent work in embodied cognition and cognitive linguistics. As applied by Lakoff and Núñez (2000), this framework holds that human bodily experience, as well as unconscious mechanisms like conceptual metaphors and blends, are essential in the genesis of mathematical thought. In this view, mathematics is a specific powerful and stable product of human imagination, with its origins in human bodily experience. As noted by Seitz (2000, emphasis in the original), "In effect it appears that we *think* kinesically too [….] and has been postulated [….] that the body is central to mathematical understanding (Lakoff & Nunez, 1997), that speech and gesture form parallel computational system (Mc Neill, 1985, 1989, 1992)." In a similar vein, R. Nemirovsky (2003) has emphasized the role of perceptuo-motor action in the processes of knowing:

> While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action based on how we have learned and used the subject itself". [As a consequence,] "the understanding of a mathematical concepts rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending of the context. (p. 108)

Furthermore, attention is now being paid to the ways in which multivariate registers are involved in how mathematical knowing is built up in the classroom. This point is illustrated by Roth (2001) as follows:

> Humans make use not just of one communicative medium, language, but also of three mediums concurrently: language, gesture, and the semiotic resources in the perceptual environment. (p. 9)

This attention to the body does not negate the fact that mathematics and other forms of human knowledge are "inseparable from symbolic tools" and that it is "impossible to put cognition apart from social, cultural, and historical factors": in fact cognition becomes a "culturally shaped phenomenon" (Sfard & McClain, 2002, p. 156).

The embodied approach to mathematical knowing, the multivariate registers according to which it is built up, and the intertwining of symbolic tools and cognition within a cultural perspective are the basis of our frame for analysing gestures, signs and artefacts. The existing research on those specific components finds a natural integration in such a frame.

GESTURES VIEWED WITHIN PSYCHOLOGY

Within a psychological perspective, we begin with the seminal work of McNeill (1992), who stated that, "gestures, together with language, help constitute thought" (p. 245). McNeill (1992) classified gestures in different categories: *deictic* gestures (pointing to existing or virtual objects); *metaphoric* gestures (the content represents an abstract idea without physical form); *iconic* gestures (bearing a relation of resemblance to the semantic content of speech); *beat* gestures (simple repeated gestures used for emphasis). Since his study, much research has analysed how gesture and language work together and influence each other. Alibali, Kita and Young (2000) develop McNeill's view that gesture plays a role in cognition, not just in communication, in the Information Packaging Hypothesis (IPH):

> Gesture is involved in the conceptual planning of the messages, helps speakers to "package" spatial information into verbalisable units, by exploring alternative ways of encoding and organising spatial and perceptual information…gesture plays a role in speech production because it plays a role in the process of conceptualisation (p. 594-5)

According to the IPH, the production of representational gestures helps speakers organise spatio-motoric information into packages suitable for speaking. Spatio-motoric thinking (constitutive of representational gestures) provides an alternative informational organisation that is not readily accessible to analytic thinking (constitutive of speaking organisation). Analytic thinking is normally employed when people have to organise information for speech production, since, as McNeill points out, speech is linear and segmented (composed of smaller units). On the other hand, spatio-motoric thinking is instantaneous, global and synthetic (not analyzable into smaller meaningful units). This kind of thinking, and the gestures that arise from it, is normally employed when people interact with the physical environment, using the body (interactions with an object, locomotion, imitating somebody else's action, etc.). It is also found when people refer to virtual objects and locations (for instance, pointing to the left when speaking of an absent friend mentioned earlier in the conversation) and in visual imagery.

Within this framework, gesture is not simply an epiphenomenon of speech or thought; gesture can contribute to creating ideas:

> According to McNeill, thought begins as an image that is idiosyncratic. When we speak, this image is transformed into a linguistic and gestural form. ... The speaker realizes his or her meaning only at the final moment of synthesis, when the linear-segmented and analyzed representations characteristic of speech are joined with the global-synthetic and holistic representations characteristic of gesture. The synthesis does not exist as a single mental representation for the speaker until the two types of representations are joined. The communicative act is consequently itself an act of thought. ... It is in this sense that gesture shapes thought. (Goldin-Meadow, 2003, p. 178)

Another important aspect of the analysis of gesture concerns the relationship between the content of the speech and the gesture. We can speak of a *gesture-speech match* (M) if the entire information expressed in gesture is also conveyed by speech. If not, that is, if different information is conveyed in speech and gesture, we have a *gesture-speech mismatch* (Goldin-Meadow, 2003). This information is not necessarily conflicting but possibly complementary, and may signal a readiness to learn or reach a new stage of development (Alibali, Kita & Young, 2000; Goldin-Meadow, 2003). According to Goldin-Meadow, mismatch is "associated with a propensity to learn" (p. 49), "appears to be a stepping-stone on the way toward mastery of a task" (p. 51); and may place "two different strategies [for solving a problem] side by side within a single utterance" highlighting "the fact that different approaches to the problem are possible" (p. 126). In general gesture-speech mismatch reflects "the simultaneous activation of two ideas" (p. 176).

GESTURES VIEWED WITHIN SEMIOTICS

The fact that gestures are signs was pointed out many years ago by Vygotsky, who wrote:

> A gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture. (Vygotsky, 1997, p. 133)

Semiotics is a useful tool to analyse gestures, provided that a wider frame, which takes into account their cultural and embodied aspects as well, is considered. An analysis of this kind has been carefully developed by Radford, who introduces the notion of *semiotic means of objectification* (Radford, 2003a):

> The point is that processes of knowledge production are embedded in systems of activity that include other physical and sensual means of objectification than writing (like tools and speech) and that give a corporeal and tangible form to knowledge as well....These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call *semiotic means of objectification*. (p. 41)

Gestures can be important components of semiotic means of objectifications, whether used when communicating directly with others, or to highlight aspects of artefacts and symbolic representations of mathematical concepts.

Psychologists now distinguish between *linguistic* and *extralinguistic* modes of expression, describing the former as the communicative use of a sign *system*, the latter as the communicative use of a *set* of signs (Bara & Tirassa, 1999). When students are learning the signs of mathematics, they often use both their linguistic and extralinguistic competence to understand them; e.g. they use gestures and other signs as semiotic means of objectification. Of course, in all these means of objectification both modalities (linguistic and extralinguistic) are present, with different strengths and in different ways depending on the dynamics of the situation.

SUMMARY OF THE RESEARCH FORUM

The papers of the Research Forum address the main questions and themes summarized above. F. Arzarello *et al.* present an example involving geometric visualization to illustrate a new theoretical framework for analysing gesture and speech in mathematics learning environments. M.G. Bartolini Bussi analyses the genetic links between artefacts and gestures in pupils (9 years old) who use real and virtual artefacts. L. Edwards utilizes data from adult students discussing fractions to argue that the original narrative-based classification of gestures should be adjusted and modified for analysing gestures in mathematical discourse. R. Nemirovsky and F. Ferrara approach gestures from the point of view of perceptuo-motor thinking, showing the connections between parallel strands of bodily activities, in a microanalysis of gestures and eye motions during a graphing activity. L. Radford explains the role of semiotics in analysing gestures as means of semiotic objectification, illustrating his framework with data from modeling activities.

SHAPING A MULTI-DIMENSIONAL ANALYSIS OF SIGNS

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INTRODUCTION AND BACKGROUND

Recently the analysis of gestures and their role in the construction of meanings has become relevant not only in psychology, but also in mathematics education. Gestures are considered in relation with speech, and with the whole environment where mathematical meanings grow: context, artefacts, social interaction, discussion, etc. Mathematics, as an abstract matter, has to come to terms with our need for seeing, touching, and manipulating. It requires perceivable signs and so the environment is crucial in the teaching-learning process.

In this paper, we elaborate on two different ways to look at the cognitive processes of students when they communicate and reason during a mathematical activity. We propose a theoretical frame shaped by the encounter of certain perspectives, developed in the disciplines of mathematics education, psychology, neuroscience, and semiotics. In particular, the theoretical notions we use here are the following: from psychology, the *Information Packaging Hypothesis* (Alibali, Kita & Young, 2000); from semiotics, the idea of *semiotic means of objectification* (Radford, 2003a) and that concerning the different functions of signs, i.e. *iconic, indexical* and *symbolic* (Peirce, 1955; Radford, 2003a), and from psycho-linguistics, the distinction between *linguistic* and *extra-linguistic* modes of expression (Bara & Tirassa, 1999). Let us sketch them here for our purpose; a more detailed account is given in the introduction of the present research forum.

In psychological research, the Information Packaging Hypothesis (IPH) describes the way that gesture may be involved in the conceptual planning of the messages, by considering alternative "packagings" of spatial and visual information, so that this information can be verbalized in speech (Alibali, Kita and Young, 2000). Within the similar perspective that gestures play an active role not only in speaking, but also in thinking, *gesture-speech matches* and *mismatches* are defined (Goldin-Meadow, 2003). A match occurs when all the information conveyed by a gesture is also expressed in the uttered speech; a mismatch happens in all the other cases. Mismatches are the most interesting since they indicate a readiness for learning, conceptual change or incipient mastery of a task. But gestures are also significant from the side of semiotics if seen as *signs*. Vygotsky (1997) already pointed out that "a gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture" (p. 133). Nevertheless, semiotics is useful to analyse gestures only if does not forget their cultural and embodied aspects. Such a direction has been followed in mathematics education by Radford (2003a) with the introduction of the so-called semiotic means of objectification. These semiotic means are constituted by different types of signs, e.g. gestures, words, drawings, and so on. They have been introduced to give an account of the way students come to generalise numeric-geometric patterns in algebra. Different kinds of generalisation have been detected. Among them is the so-called *contextual generalisation*, which still refers heavily to the subject's actions in time and space, within a precise context, even if he/she is using signs that could have a generalising meaning. In contextual generalisation, signs have a two-fold semiotic nature: they are becoming symbols but are still indexes. These terms come from Pierce (1955) and Radford (2003a). An *index* gives an indication or a hint of the object: e.g. an image of the Golden Gate, which makes you think of the city of San Francisco. A *symbol* is a sign that contains a rule in an abstract way: e.g. an algebraic formula. As relevant in communication (in thinking as well) gestures can be considered with respect to linguistic and extra-linguistic modes of expression. The former is characterised as the communicative use of a sign *system*, the latter as the communicative use of a *set* of signs: "linguistic communication is the communicative use of a symbol system. Language is compositional, that is, it is made up of constituents rather than parts... Extra-linguistic communication is the communicative use of an open set of symbols. That is, it is not compositional: it is made up of parts, not of constituents. This brings to crucial differences from language..." (Bara & Tirassa, 1999; p. 5). In communicative acts the two modes co-exist. Students who learn the signs of mathematics, often rely on both their linguistic and extra-linguistic competences to understand them: for example, they use gestures and words as semiotic means of objectification. Typically, gestures are extra-linguistic modes of communication, whereas speech is on the linguistic side.

A NEW FRAMEWORK: THE PARALLEL AND SERIAL ANALYSIS

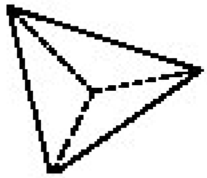

Figure 1

We show a brief example from the activity of some 8th grade students involved in approaching a geometrical problem. They have been asked to describe the geometric solid formed when two square pyramids are placed side by side (with one pair of base sides touching). The solution, which must be visualized by the students, is a tetrahedron seen from an unusual point of view.

Consider the following utterances by Gustavo, and one of his concomitant gestures:

Gustavo: Yeah, it is a solid, made of two triangles placed with the bases below, which are those starting in this way and going up, and two triangles with the bases above that are those going in this way [see Fig. 2].

We can analyse data like these in a double way, using what we call *parallel* and *serial* analysis. Both analyses take into consideration the dynamics of what we think of as the major components of processes of objectification: not only speech and gestures (respectively ***s*** and ***g*** in Fig. 3), but also written words and mathematical signs (respectively, ***w*** and ***x*** in Fig. 3). The latter, even if not directly part of the communication acts, are a product of them, and often arise from gestures and words used by the involved subjects (Gallese, 2003; Sfard & McClain, 2002).

Figure 2

The components of objectification processes may develop according to two types of dynamics. We call the first dynamics *Parallel Process of Objectification* (***PPO***); it results when (some of) the different components are seen as a group of processes synchronically developing (e.g. when one talks and gestures simultaneously). They can match or mismatch with each other in the way they are encoding information.

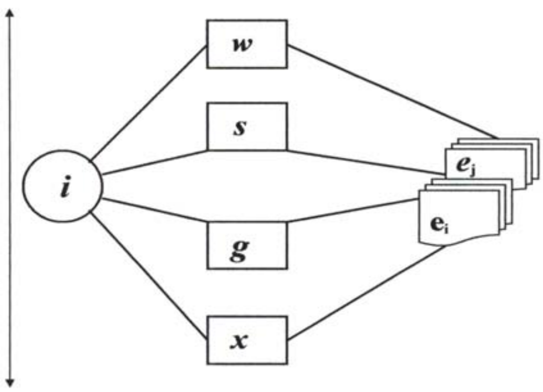

Figure 3: The *PPO*

In such a case, we are interested in a parallel analysis of the components (see the vertical arrow in Fig. 3), which focuses on the mutual relationships among them, where all components refer to the same source ***i*** and possibly to different encoding e_i's. The main elements of a parallel process of objectification are: (i) the idea of semiotic means of objectification; (ii) the Information Packaging Hypothesis; (iii) Match and Mismatch (Goldin-Meadow, 2003).

We call *Serial Process of Objectification* (***SPO***) a second type of dynamics, which results when two different components are spread over time and happen in different moments, as steps of a unique process. An example is given by a sign produced as a frozen gesture (Vygotsky, 1997), or by a gesture embodying some features of a previous sign. In this case, we are interested in a serial analysis (see the horizontal arrow in Fig. 4) focusing on the subsequent transitions from different sources ***i*** to different encoding e_i's.

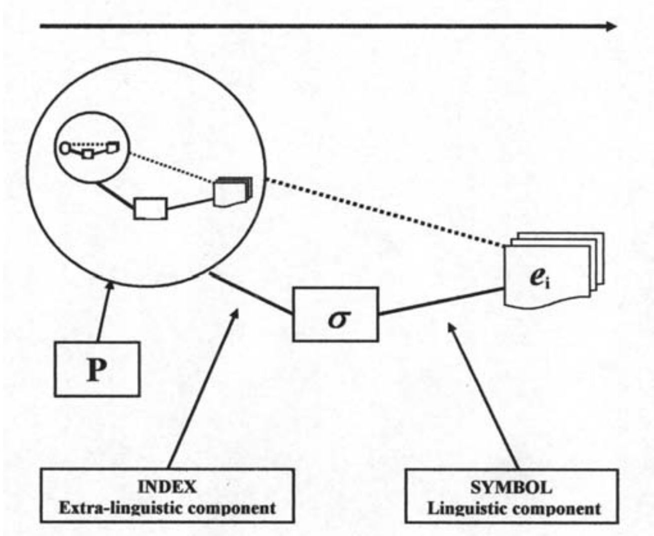

Figure 4: The *SPO*

The Serial Process of Objectification is shown in Fig. 4. Its main elements are again: (i) the semiotic means of objectification; and (ii) the Information Packaging Hypothesis. But there are also two other elements: (iv) the indexical-symbolic functions of signs; and (v) the linguistic and extra-linguistic modes of communicative acts. A serial process of objectification happens when one (or more) serial (or parallel) process(es) **P**, represented in the circle of Fig. 4, is (are) the grounding for the genesis of a new sign (indicated by **σ**).

For technical reasons, just one component appears in the circle, but there could be more. The sign **σ** is the pivot of the process; it can be any kind of sign: a drawing, a word, a gesture, a mathematical sign, etc. It is generated by the previous process(es) P and produces an encoding of **P**. The relationships between **σ** and **P** are mainly extra-linguistic, whereas the relationships between **σ** and $\boldsymbol{e_i}$ ***are*** mostly linguistic. In other terms, the sign **σ** has an indexical function with respect to P, but it has also a fresh symbolic function with respect to the encoding $\boldsymbol{e_i}$. Thus, the ***SPO*** could be the basis for a new serial process, and so on, in an ongoing series of nested generalisations. Examples of ***SPOs*** are given by the learning of speech in kids or by that of reading written texts in young pupils. Mathematical examples are the processes undertaken by students who are learning Algebra or some other chunks of mathematical ideographic language, from Arithmetic to Calculus.

Generally both types of dynamics, ***PPO*** and ***SPO***, can support the genesis of signs. As a consequence, each process of objectification may be analysed from both points of view, that is, as a parallel process and as a serial process. We call *parallel* and *serial* the two resulting types of *analysis*. Let us go back to the initial example that we can now interpret through the two analytical lenses. The parallel analysis points out the conflict between the two pieces of Gustavo's theoretical knowledge concerning the 2D and 3D figures. The serial analysis shows that Gustavo's gestures are mediating the transition from the 2D features of the triangles to the 3D ones of the solid. After this episode, the experiment goes on and culminates with the acknowledgement by students of the tetrahedron as a "triangular pyramid". Parallel and serial analysis allow us to focus properly on the dynamics of what is happening. As such they are useful tools of investigation. In fact, parallel analysis reveals itself as a tool suitable for identifying conflicts, even before they appear to block or slow students' activities. On the other hand, the serial analysis represents a tool suitable for focusing on the dynamics through which the subjects try to overcome obstacles met in their activities.

Acknowledgments: Research program supported by MIUR and by the Università di Torino and the Università di Modena e Reggio Emilia (COFIN03 n.2003011072).

WORKING WITH ARTEFACTS: THE POTENTIAL OF GESTURES AS GENERALIZATION DEVICES

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INTRODUCTION

We shall summarize some findings of two studies (Bartolini et al., 1999; Bartolini et al. in press) concerning primary school. In the former we have studied the genesis of a germ theory of the functioning of gears. In the latter we have studied the construction of the meaning of painting as the intersection between the picture plane and the visual pyramid. The studies have been carried out in a Vygotskian framework that has been gradually enriched with contributions of other authors. As a result, classroom activity has been designed and orchestrated by the teacher in order to foster the parallel development of different semiotic means (language, gestures, drawing), which form a dynamic system (Stetsenko, 1995, p. 150).

In both studies, concrete artefacts came into play. Wartofsky's distinction between primary, secondary and tertiary artefacts proved to be useful (1979). *Primary artefacts* are "those directly used" and *secondary artefacts* are "those used in the preservation and transmission of the acquired skills or modes of action". Technical tools correspond to primary artefact whereas psychological tools are the individual counterparts of secondary artefacts. *Tertiary artefacts* are objects described by rules and conventions and not strictly connected to practice (e. g. mathematical theories, within which the models constructed as secondary artefacts are organised).

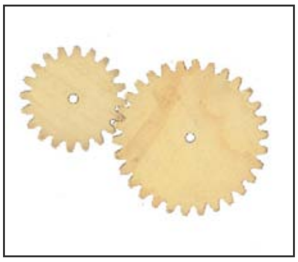
WHEN THE ARTEFACT IS A GEAR.

The role of gestures when concrete tools are into play is obviously very large. Wartofsky himself emphasizes mimicry, among the different representations used to preserve and transmit the modes of action. Gestures are essential to use the artefact, as 'a machine is a device that incorporates not only a tool but also one or more gestures' (Leroi-Gourham, 1943). We found that, from 2nd grade on, when the teacher

[1] Abridged version of a study (in preparation) carried out together with Maria Alessandra Mariotti, and Franca Ferri, within the National project Problems about the teaching and learning of mathematics: meanings models, theories (PRIN_COFIN 03 2003011072).

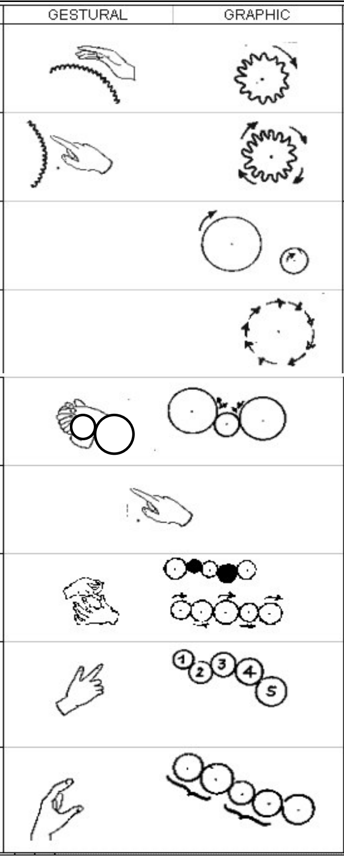
designs suitable activities aiming at constructing a germ theory of the functioning of gears and supports pupils' work, there is a parallel and intertwined development of three different semiotic means: gesture – drawing - speech (in oral and written forms): the development is towards the appropriation of the meaning of motion direction, represented by a sign ('arrow') with an appropriate syntax, that also allows students to solve difficult problems concerning trains of any number of gears.

Our findings are summarized in Table 1, adapted from (Bartolini et al., 1999, p. 79) which relates the findings of that study to issues discussed in this forum.



The *primary artefacts* are given, in this case, by tools with gears and toothed wheels inside. In the figure, a pair of toothed wheels is represented (courtesy of R. Nemirovsky, TERC). To start the gear a gesture is needed: it creates an action scheme that 'enables students to tackle virtually any particular case successfully' (*factual generalization,* Radford, 2003a, p. 47).

Table 1. *From gesturing to signs (Bartolini et al. 1999, p. 79)*



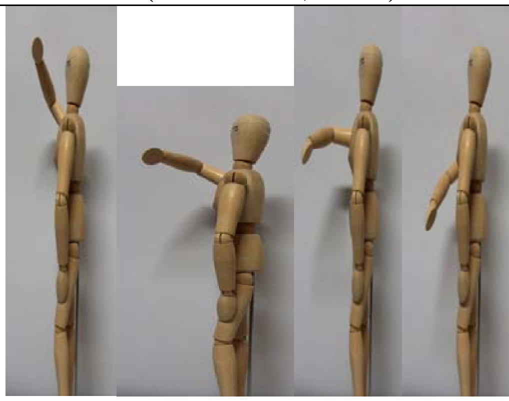
n	GESTURAL	GRAPHIC	VERBAL	*Wartofsky*	*Edwards / McNeill*	*Radford*
1			push this wheel this way this wheel goes this way	PRIMARY Gesture on a primary artefact to turn the wheel as a whole or pushing a point.	Iconic physical	*Factual generalization*
2			this tooth goes this way		Iconic physical	*Factual generalization*
3			clockwise anticlockwise	Construction / appropriation of secondary artefacts	No gesture	*Contextual generalization*
4			left - right up - down		No gesture	*Contextual generalization*
5			wheels turn in pairs	Gesture to represent a primary artefact (secondary)	Iconic physical	*Contextual generalization*
5'			this wheel pushes that wheel	Construction / appropriation of secondary artefacts	Iconic physical	*Contextual generalization*
6			white-black A-B thumb-index this - the other way	*Towards tertiary artefacts* Gesture to represent a mathematical model	metaphorical	*Contextual generalization*
7			one-two-three-four-...		metaphorical	*Symbolic generalization*
8			wheels are paired		metaphorical	*Contextual generalization*

When young pupils (e.g., 2nd grade ones) are asked to represent this experience by drawing, they spontaneously introduce the sign 'arrow' (a semiotic mean of objectification) that seems to objectify on paper the gesture of the hand. Later the sense of the sign changes together with the parallel evolution of drawing and speech. In Table 1 we have related our findings with those of other authors.

When the artefact is a sentence evoking a concrete artefact

In a 4th grade classroom (Bartolini et al. in press), a complex activity about perspective drawing has been started. The first step has been the exploration and the interpretation of an artefact (Dürer's glass) built in wood, metal and Plexiglas, where one observes through the eyehole the perspective drawing of the skeleton of a cube put behind the glass. Some months later, at the beginning of the 5th grade, when the concrete artefact is no longer in the classroom, a very short sentence from L. B. Alberti (De Pictura, 1540) is given to interpret in classroom discussion: "*Thus painting will be nothing more than intersection of the visual pyramid*". Gestures are very important in the interpretation: gestures mime planes and lines and constitute a fundamental support to imagine a pyramid.

<table>
<tr><td colspan="2">Table 2
"Thus painting will be nothing more than intersection of the visual pyramid"
L.B.Alberti (De Pictura, 1540).</td></tr>
<tr><td>▸ You have to imagine it. I understood this, if you saw it near the object you obtain a large image; if you saw it near the eye you get a smaller image. [With gestures, many children saw the visual pyramid].

▸ If you go down straight, because with our hands we form a kind of plane parallel to the one of the objects [With his hands he traces two parallel planes in space]. In this way you certainly obtain a figure which is exactly the same as the base of the pyramid, but smaller.
1a</td><td>
1b</td></tr>
<tr><td>2b

[…] A visual pyramid is a kind of pyramid 'made by you', that is the pyramid helps you to see what you see in different ways, in fact, as I have drawn, it makes you see the sun in several ways. I have drawn that drawing, because it clarifies how a visual pyramid is and also how it must be shaped. I have enjoyed making the sun, bigger and bigger, because it makes one understand much. Anna's eye is open and the other is closed, it is not visible but if you notice there is her arm pointing close to the other side of her face to close the other eye.</td><td>2a

</td></tr>
</table>

The pupils do not seem troubled by this imaginary context, as the following exchange shows:

> Luca: How can you possibly saw the visual pyramid, which is a solid that does not exist?
>
> Alessandro B.: Exactly how you imagine it. If you see it because you imagine it, you can saw it as well. You have to work with the mind.

Three months after this discussion, the pupils are asked to comment individually, in writing (using also drawing if they wish), about the same sentence by Alberti (Maschietto & Bartolini, submitted). In Table 2 some exemplary protocols from the above activities are presented: 1a. The transcript (with comments) of an oral exchange between two pupils in classroom discussion; 1b. The simulation of gesture by means of a dummy; 2a. A drawing produced to explain Alberti's sentence; 2b. An excerpt of the written text, added as a commentary of the sentence and of the drawing.

The right way to produce the gesture ('straight down' i.e. vertically) is verbally explained immediately by the second speaker. This way of cutting an 'imaginary' pyramid in the air becomes a shared action scheme in the classroom, repeatedly used by the pupils and by teacher as well. The gesture works in any position (*contextual generalization*, Radford 2003a). Three months later most pupils prove to have internalized the meaning of the visual pyramid and produce meaningful drawings. In the one reported here there is another instance of *contextual generalization*, which concerns the possibility of tilting any 'imaginary' picture plane in non-vertical position. We know from the history of perspective that this was not a trivial problem.

DISCUSSION

Wartofsky's elaboration of artefacts refers to 'external' objects. He discusses the secondary artefacts as follows:

> Such representations […] are not 'in the mind', as mental entities. They are the products of direct outward action, the transformations of natural materials, or the disposition or arrangement of bodily actions […].

In the classroom pupils construct/appropriate these cultural products by means of social activity carried out together with their peers under the teacher's guidance. We have shown in two cases concerning spatial experience with concrete artefacts how internalization of social activity, is realised by semiotic means of objectification (Radford, 2003a) that are used in parallel and intertwined with each other.

THE ROLE OF GESTURES IN MATHEMATICAL DISCOURSE: REMEMBERING AND PROBLEM SOLVING

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The purpose of this analysis is to examine the role of gestures within the context of a particular setting involving mathematical discourse, specifically, an interview where students were asked to describe how they learned certain mathematical concepts and to explain how they solved problems involving fractions. The overall goal of the study was to examine both the form and function of gestures within a context of mathematical communication and problem solving, and to begin to develop an analytic framework appropriate to understanding gesturing within the domain of mathematics.

Previous research has examined the role of gesture in a number of different mathematical contexts, including learning to count (Alibali & diRusso, 1999; Graham, 1999), classroom communication (Goldin-Meadow, Kim & Singer, 1999), ratio and proportion (Abrahamson, 2003), motion and graphing (Nemirovsky, Tierney & Wright, 1998; Radford, Demers, Guzmán. & Cerulli, 2003, Robutti & Arzarello, 2003), and collaborative problem solving (Reynolds & Reeve, 2002; also see Roth, 2001, for a review of research on gesture in mathematics and science). Gesture is defined as "movements of the arms and hands ... closely synchronized with the flow of speech" (McNeill, 1992, p. 11). In contrast with speech, which is linear, segmented and composed of smaller units, gesture is global and synthetic; it can express meanings as a whole and one gesture can convey a complex of meanings (McNeill, 1992). Gesture can be seen as an important bridge between imagery and speech, and may be seen as a nexus bringing together action, imagery, memory, speech and mathematical problem solving. The investigation of gesture in mathematics takes place within a theoretical context that sees cognition as an embodied phenomenon, and that examines how both evolutionary constraints and individual bodily experience provide a foundation for the distinctive ways that humans think, act, and speak about mathematics (Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999).

The data for the study comprise a set of gestures displayed by twelve adult female students while talking about their memories of learning fractions, and during and after solving problems involving fractions. The participants were prospective elementary school teachers, and the interviews were carried out in pairs. A corpus of more than 80 gestures was collected. The majority of the gestures were displayed in response to questions asking the students to recall how they first learned about fractions.

Figure 1: "I think we did, like, just <u>a stick or a rod</u>…"

These gestures generally fell into four categories, representing an extension of McNeill's original typography of gestures into iconics and metaphorics:

(1) Iconic gestures referring to physical manipulatives or actions (e.g., "a stick or rod" or "cutting a pie")

(2) Iconic gestures referring to inscribed representations of physical manipulatives (e.g., "a pie chart")

(3) Iconic gestures referring to specific written algorithms (Figure 1b)

(4) Metaphoric gestures (referring to an abstract idea or action, e.g. Figure 2)

In Figure 1a, the student describes a manipulative (possibly fraction bars), and goes on to talk about "dividing it again and again," moving her right hand in a chopping gesture toward the right to indicate the iteration of this division. This chopping motion can also be categorized as an iconic gesture referring to a physical action.

Figure 1b shows an example of a student displaying an "iconic-symbolic" gesture: gestures that refer not to a concrete object but to a remembered written inscription for an algorithm or mathematical symbol; that is, an "algorithm in the air" (Edwards, 2003). The importance of written algorithms for mathematics, and for students memories of learning mathematics, would seem to require this expansion of the typology of gestures that McNeill originally developed to analyze narrative discourse.

Figure 2 shows a part of a gesture made by a student responding to a question about how she would introduce fractions to children. The gesture began with the two hands close together, with whole hands slightly curled and facing each other, and ended with the hands opening out and moving to the right. These somewhat vague metaphorical gestures about generic mathematical operations contrast sharply with the very precise iconic-symbolic gestures used when describing specific arithmetic algorithms with fractions.

Figure 2: “Like the different formulas”

In addition to gestures displayed in response to the interviewer’s questions, one student displayed a complex sequence of gestures associated with a description of how she solved a problem involving comparing two fractions. She and her partner had worked out which was larger, 3/4 or 4/5, and the student was explaining her solution after the fact. The student’s spoken words are below (underlining indicating words synchronized with a gesture):

S2: Well, I mean it’s like I’m thinking if I had a pie and I had 5 people versus 4 people then,[R: Ah.] you know, we’re each kinda getting less of a piece [R: Ah.] because there’s a fifth piece we have to like, put out to the other four people.

The four gestures corresponding to the underlined words or phrases consisted of (1) pointing with right index finger to right temple (“thinking”); (2) moving the first two fingers of the right hand from right to left at chest height (“less”); (3) a diagonal chopping motion with the whole right hand at face height (“fifth piece”); which continues into a (4) circular movement of the whole hand in front of and parallel to the face and chest (“put out to the other four”). This use of gesture did not seem to be a static illustration of remembered objects or inscriptions, as some of the other gestures were. Instead, the sequence of gestures was fully synchronized with the description of the problem solution, and may have played a facilitating role in solving the problem. The first gesture would be described as an emblem (a conventionalized gesture for “thinking” by pointing to the temple), but the other four gestures highlighted important aspects of the solution: the relative size of the fractions; i.e., the denominators (“getting less of a piece”), the number of pieces, i.e., the numerators (“a fifth piece”) and a sharing operation (“put out to the other four people”).

The current study elicited a wide variety of gestures, primarily associated with students’ memories of learning fractions, but also occasionally in connection with current problem solving and reasoning. In either context, the gestures were not simple illustrations, but reflected important aspects of the materials and representations present while the students were learning. These findings are similar to those in a study of bodily motion and graphing, in which the authors stated, “The way students describe functions shows deep traces of their actions and interactions with instruments and representations. Such traces are not complementary to the concept but are an essential component of its meaning" (Robutti & Arzarello, 2003, p. 113).

The analysis of the gesturing in mathematical contexts has provoked a re-examination of the categories developed by McNeill for describing gestures elicited in association with narrative descriptions. The initial analysis of the fraction data stimulated a division of McNeill's category of iconic gestures into two sub-categories: iconic-physical and iconic-symbolic. However, the nature of mathematics as a discipline may require an even more refined categorization of gestures. This is because while in everyday life, concrete objects do not "refer" to anything beyond themselves, in mathematics teaching, many concrete objects have been designed to "represent" more abstract mathematical objects. So when a student gestures in a circle when talking about fractions, she may be referring simply to the plastic fraction pieces she remembers from elementary school, or she may be thinking about those pieces in regards to a particular fraction or operation. Furthermore, outside of mathematics, written symbols are not usually manipulated as if they were objects. Thus, descriptions and analyses of gesture in mathematics should take into account these features of mathematical practice and discourse. Furthermore, the analysis of gesture may help to illuminate the relationships and developmental path among physical actions, speech, internalized imagery, written symbols, and mathematical abstractions.

CONNECTING TALK, GESTURE, AND EYE MOTION FOR THE MICROANALYSIS OF MATHEMATICS LEARNING

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INTRODUCTION AND BACKGROUND

In the last years deep changes have characterised the study of thinking and learning based on ongoing research in neuroscience, psychology, and cognitive science. These changes were supported by the availability of new technologies, which allow for a fine-grained recording of human activity. Different areas of cognition (such as language, vision, motor control, reasoning), which in the past were considered largely autonomous, have started to be studied as integrated and working in unison. This trend entails that research can get a wider and more detailed viewpoint to analyse thinking and learning processes. Examples come from the psychological research on gestures since the '80 (see Kita, 2003) and from vision science (e.g., Tanenhaus *et al.*, 1995). These emerging studies are generating new insights on the nature of thinking in educational research and the study of mathematics learning. For instance, Nemirovsky (2003) argues that “thinking is not a process that takes place “behind” or “underneath” bodily activity, but it is the bodily activities themselves”. Within this viewpoint, even “the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending on the circumstances” (ibid.). The integrated study of bodily activity calls for a type of analysis, which is sometimes called “microgenetic analysis”; that is, a detailed examination of the genesis of ideas and approaches by a subject over short periods of time (minutes or seconds), while they are occurring. Microanalytic studies can document variability and actual processes of local change. Furthermore, the advent of digital video and other tools (portable eye trackers are an example), made microanalysis practical and more widespread.

EYE MOTION AND PERCEPTION

Perception and motor control (main constitutive aspects of thinking) are inextricably related in eye motion. Contrary to common belief, the eyes do not take whole snapshots of the surroundings onto our brains. Studies in eye motion provide evidence for Gibson’s (2002/1972) thesis that visual perception is not an all-at-once photographic process of image-taking from the retina to the brain but a “process of exploration in time” (p. 84). Since “perception is *not* supposed to occur in the brain but to arise in the retino-neuro-muscular system as an activity of the whole system” (ibid.; p. 79), eye motion is crucial for such a process. Our study focuses on a type of eye motion, the saccadic one, consisting of rapid transitions (“saccades”) between “fixations”. A fixation is a point in the field of view around which the eyes stay on a relatively long period of time, commonly in the range of tenths of a second. The exploration in time results in some repeated cycles or trajectories formed by the successive fixations, the so-called scanpaths (Norton & Stark, 1971). The scanpaths clearly depend by the circumstances, are idiosyncratic to the individual seeing, and reflect the questions one has in mind. As a consequence, our eyes are constantly and actively traversing the surroundings. They do not record the environment, but they interrogate it, as Yarbus (1967) pointed out in the case of subjects looking at paintings. Other researchers have studied eye motion in context as a means to analyse the strategies different subjects activate when involved in a mathematical activity Some studies (Epelboim and Suppes, 2001) show that eye motion is central not only to seeing what is out there, as it were, but also for imagining things that are not present in the field of view. Therefore given that imagination and visualisation are essential for mathematical understanding, eye motion can be an important tool to reveal thoughts in catching a solution or grasping a meaning.

We will examine the coordination of talk, gesture, and eye motion, moment-by-moment, for a subject interviewed on graphs of motion. In our example, graphs describe a motion story read and interpreted by the subject, who wears an eye tracker recording his eye motions while a second camera films his gestures.

AN EXPLORATIVE EXAMPLE

The example briefly considered is based on an exploratory interview we conducted with a graduate student wearing a state-of-the-art portable eye tracker. The battery-operated eye tracker was carried within a small backpack connected to a head-mounted pair of miniature cameras (for the image of the scene, and for eye motion on the scene: see Fig. 1*a*, where at any time the cross represents the fixation). An external camera recorded gestures and hand motion (Fig. 1*b*). The interview included a "Motion Story" telling the imaginary motion of a person:

Figure 1

> I was quietly walking to the bus stop. I looked back and saw that the bus was fast approaching the stop. Then I ran toward the bus stop. However, the bus went by me and did not stop. I slowed down and kept walking toward the bus stop to wait for the next one. But, I forgot to put a letter in the mailbox, which is placed just a few metres behind where I was. So, I walked quickly toward the mailbox and I posted my letter. As soon as I realized that the next bus was coming, I ran back and I waited for it at the bus stop.

The interviewee (L) was asked to draw on a whiteboard a graph of position vs. time relative to the story and then the corresponding velocity vs. time and acceleration vs. time graphs. The ensuing conversation was about the characteristics of these graphs, maxima and minima, etc. Our analysis strives to trace the process of graph construction over time. For reasons of space we can just sketch the dynamics. At first, L is looking in the story for information to use for drawing the position vs. time graph. His eyes go back and forth from the right side (see Fig. 2*b*) where he has to draw, to the story placed on the left side (Fig. 2*a*). Fixations are located in the written text on places useful to gather important information to be translated in pivotal points of the graph. After L determines the points in time, he draws straight lines connecting them.

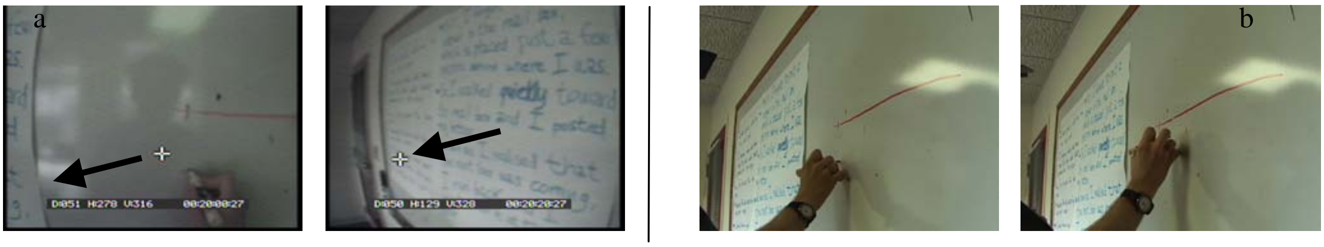

Figure 2

For example at time 3.55.09, L focuses in the story (Fig. 3*a*) on the speed feature (fixation on "quickly") of the piece of the graph he is starting to trace (Fig. 3*b*).

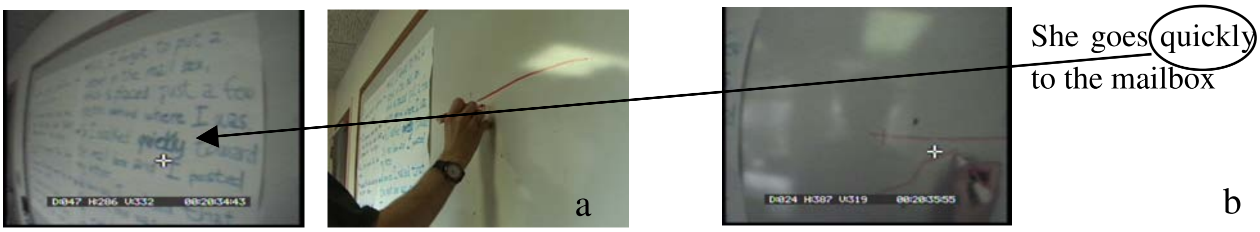

Figure 3

The resulting graph is shown in Fig. 4, where the position of the bus stop is set by L as the zero for the distance axis. Then a second phase started, in which the drawing is checked in relation to the story. The hand is kept still on a graphical element as to not lose the reference in the drawing, while the fixation goes to the text at the corresponding moment (Fig. 5). Then the eye comes back on the graph to traverse, together with the hand (Fig. 6), the motion started at that moment; moreover, L joined this description with his utterance ("She [the character of the story] ran back").

Figure 4

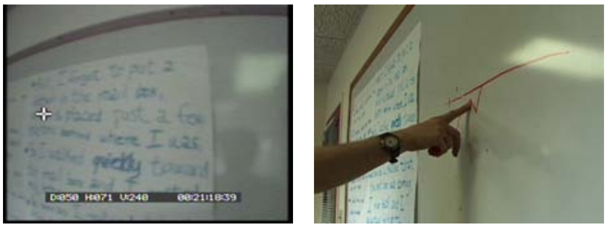

Figure 5

Figure 6: She ran back

In an ensuing phase L gathers from the distance vs. time graph information needed to draw the velocity vs. time graph. L's eyes and hands moved to relate the two graphs, their relations, and the physical quantities related to motion (a sequence of fixations and gestures is shown in Fig. 7).

Figure 7: L draws the velocity vs. time graph

Then a question by the interviewer (F in the following) marks the beginning of a reflection on the shape of the two graphs:

F: So, you suppose that in these three time intervals [hand pointing to the three pieces at the same height on the velocity vs. time graph] she has the same velocity?

To answer L goes back to the story. Then his eyes go from velocity vs. time to the distance vs. time to check the relations between the graphs and the motion described in the motion story; checking leads L to erase and redraw part of the graph (Fig. 8).

Figure 8: checking relations between graphs and motion

The dialogue between L and F developed further as L justified his changes or choices for the drawing, in trying to assess whether the pieces of distance vs. time indicated by F have the same slope:

L: I mean, I guess, I gave the word quickly the same magnitude basically as the running, so…

F: So, that's the reason because on this graph this part and this part have the same slope [hand pointing to the two pieces on the whiteboard]

L: Yeah.

F: That's the reason. What about these two parts? [hand pointing to the other pieces of the graph with same slope]

L: Those are the same, I think, because… although I guess maybe I'm not so good in drawing. I guess this one [L is pointing to the first segment] could be a little faster than this one… 'cause it says quietly walking [L is pointing to the second segment]… quietly walking versus walking

There seems to be three major functions of L's fixations: *locating*, e.g. when L needs essential information in the story, or when he has to choose where to draw a critical point; *checking*, e.g. when he goes back and forth from one source of information (say, the story) to another (say, the graph) to make sure they cohere; *directing*, e.g. when the eye helps the hand to get the (approximately) correct height of the critical points for the velocity vs. time graph (later for the acceleration vs. time graph). Furthermore, although each completed graph is in some sense a static object, L's eye motion shows that at any given time he is focusing on a very particular aspect, either coordinating with elements of the written story or of another graph. Each visual focusing appears to always have a question motivating it (e.g. should it be steeper? longer? Are these two the same speed?). Each graphical segment has to comply with numerous demands (consistency with the time interval, steeper than another one, etc.) and often his drawing of a segment complies with one or some of them but not with all of them. L goes through an iterative process of repair and re-drawing. As he draws and redraws he also becomes increasingly familiar with the motion story, needing less direct consulting of the text. Examining every single fixation as an effort to address a certain question is significant to a microanalysis of the situation. The sense of the whole for a graph (or a narrative) emerges gradually out of repeated focusing on particular events and shapes. In this sense, knowing how to graph a distance vs. time graph, or deriving velocity and acceleration from it, entails an intuitive sense of what to look at and how to look at it over time, in order to address ongoing questions.

WHY DO GESTURES MATTER? GESTURES AS SEMIOTIC MEANS OF OBJECTIFICATION

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One of the most intriguing aspects of gestures is that in such varied contexts as face-to-face communication, talking over the phone, and even thinking alone, we all make gestures but we still do not know why. Explanatory models have been proposed by neuro-psychology, information process theories, etc. Our problem here is narrower. We are interested in understanding the role of gestures in the mathematics classroom. However, before going further, we should ask: why do gestures matter? Contemporary forms of knowledge representation are challenging the cognitive primacy with which the written tradition has been endowed since the emergence of printing in the 15th century. The audio and kinesthetic dimensions of oral communication of the pre-print era –dimensions that were replaced by the visual and linear order of the written text– are nowadays viewed with a revived and rejuvenated cognitive interest. Current studies on gestures and perceptual-motor activity belong to this stream.

Now, the way in which each one of us, as mathematics educators, may understand the role of gestures is naturally linked to the theoretical framework underpinning our research. From the semiotic-cultural approach that I have been advocating (Radford, 1998, 2003b), gestures are part of those means that allow the students to objectify knowledge -that is, to become aware of conceptual aspects that, because of their own generality, cannot be fully indicated in the realm of the concrete. In a previous article I have called those means *semiotic means of objectification* (Radford, 2003a). In addition to gestures, they include signs, graphs, formulas, tables, drawings, words, calculators, rules, and so on.

Our answer to the question: “Why do gestures matter?” can then be formulated as follows. Gestures matter because, in learning settings, they fulfill an important function: they are important elements in the students’ processes of knowledge objectification. Gestures help the students to make their intentions apparent, to notice abstract mathematical relationships and to become aware of conceptual aspects of mathematical objects.

However, considered in isolation, gestures have -generally speaking- a limited objectifying scope. We have tried again and again the following experiment: we have turned off the volume of many of the hundreds of hours of our video-taped lessons and, even though we *see* the students making gestures and carrying out actions, our understanding of the interaction is very limited. The same can be said of other semiotic systems. Thus, we have also turned off the image and, even though we *hear* the discussion, our understanding of the interaction is again very restricted. We have also stopped both the sound and the image and limited ourselves to *reading* what the students *wrote*, and the result has been as poor as in the previous cases. The reason behind the poor understanding of the students' interaction that results from isolating one or more semiotic systems present in learning is that knowledge objectification is a multi-semiotic mediated activity. It unfolds in a dialectical interplay of diverse semiotic systems. Each semiotic system has a range of possibilities and limitations to express meaning. The conceptuality of mathematical objects cannot be reduced to one of them, not at least in the course of learning, for mathematical meaning is forged out of the interplay of various semiotic systems.

SEMIOTIC NODES

The theoretical construct of *semiotic node* (Radford *et al*. 2003) is an attempt to theorize the interplay of semiotic systems in knowledge objectification. A *semiotic node* is a piece of the students' semiotic activity where action and diverse signs (e.g. gesture, word, formula) work together to achieve knowledge objectification. Since knowledge objectification is a process of becoming aware of certain conceptual states of affairs, semiotic nodes are associated with the progressive course of becoming conscious of something. They are associated with layers of objectification.

Let us illustrate these ideas through a story-problem given to a Grade 10 class. In the story-problem two children, Mireille and Nicolas, walk in opposite directions, as shown in Figure 1. The students were asked to sketch a graph of the relationship between the elapsed time and the remaining distance between the children.

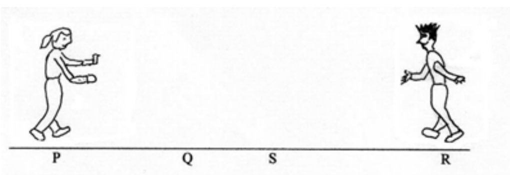

Figure 1. Mireille walks from P to Q. Nicolas walks from R to S

Supported by the students' previous experience, one of the Grade 10 students, Claudine, proposed a compelling -although incorrect- argument: the graph, she suggested, is something like an "S". Ron did not agree, but could not counter Claudine's argument. He claimed that the graph should be something like a decreasing curve, although the details were still unclear for him. In an attempt to better understand the details, he deployed a series of arguments and gestures that were intended not only for his group-mates but for him as well. In Fig. 2 there is an excerpt of the discussion.

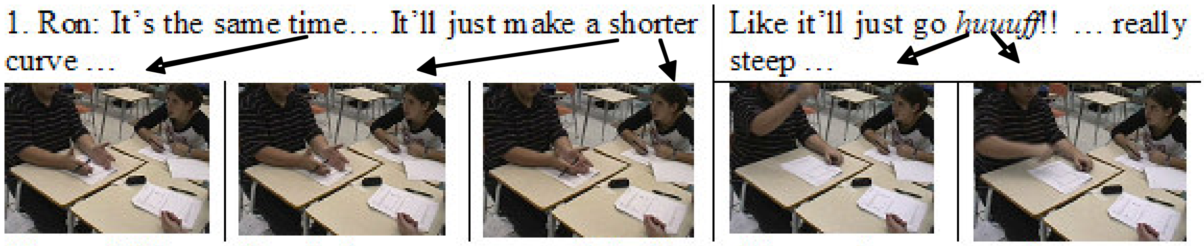

Figure 2. Pictures 1 to 5. Some gestures made by Ron while uttering sentence 1.

To objectify the relationship between distance and time, in the first picture, Ron put his hands one on each one of the students of the story-problem as drawn in the activity sheet. Insofar as the hands stand for something else, they become signs. But in opposition to written signs, which are unavoidably confined to the limits of the paper, hands can move in time and space. Capitalizing on this possibility, to make *apparent* the fact that the distance decreased, Ron moved his hands in opposite directions (pictures 2 and 3). In pictures 4 and 5 he made a vertical gesture sketching the graph time vs. distance, right after have finished the sentence. Three seconds later, remarking that Claudine was not convinced, he started his explanation again. Uttering the first sentence led him to better understand the mathematical relationship, so in the second attempt he was able to produce a more coherent discourse and to better co-ordinate gesture and word. Here, he reached a clearer layer of knowledge objectification.

Pictures 6 to 8 show gestures similar to those in Figure 2, except that now they are made in the air and Ron talks in the first person. In pictures 9 and 10 a familiar situation is invoked (the motion of two trucks). There is, however, another more fundamental aspect that has to be stressed. While in sentence 1, time remained essentially implicit (it was mentioned to emphasize the fact that the children started walking at the same time), in sentence 2, time became an explicit object of reference. Time, however, was not indicated through gestures. It was indicated with words. Even if both are semiotic means of objectification, gestures and words dealt with different aspects of the students' mathematical experience.

In each of the previous cases, the different co-ordination of words and gestures constitutes a distinct semiotic node reflecting different layers of knowledge objectification. One of the research problems that my collaborators and I are currently investigating is related to the theoretical and practical characterization of layers of knowledge objectification. As we saw, gestures play an important role therein. But this role, we suggest, can only be understood if gestures are examined in the larger context of the dialectical interplay of the diverse semiotic systems mobilized by teachers and students in the classroom.

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GESTURES, SIGNS AND MATHEMATISATION

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Where to start: To summarise, criticise, synthesise? The topic of ‘gesture’ seems so vast, and yet we know (especially with regard to mathematisation) so little. Reading these four papers for the first time, they seem like four ships crossing a huge ocean, moving in different directions, occasionally signalling each other using semaphore!

A SUMMARY: CONTEXT

None of these papers is about gesture alone. All see gesture as part of an integrated communication system with language and, in this case, mathematics. Edwards even defines gesture, after McNeill, in this way, i.e. the gesticulation accompanying speech. Two of these papers are about externalisation in the Vygotskyan sense (Arzarello et al. and Bussi & Maschietto are explicit about this reference) when children are involved in group problem solving. This is also true implicitly of Edwards’ students’ who gesture as they talk about their previous mathematical work, though her primary reference to theory is in that of Embodied Cognition.

But Ferrara & Nemirovsky’s study situates gesture in a more complex setting where seeing (active ‘interrogating’ with the eye-brain-muscle) is integrated with externalising actions involving gestures, and actually graph-drawing (despite the others’ papers’ reference to Vygotsky’s remark to the effect that gesture gives birth to writing/script, the quote seemed even more apt here!) I highlight the context of gesture, because it influences function and hence categorisation systems.

CLASSIFICATION OF GESTURES/GESTICULATIONS

There is a 2000-year history to the development of classifications of gestures (see Kendon, 2004). Edwards builds her corpus of gestures in the mathematics education context, and this inevitably extends and refines that of McNeill (1992, extended in 2000). Her recognition of context is important: the different functions of gesture in mathematics education imply the need for multiple corpora, each perhaps with its own, albeit related, classification systems.

McNeill’s context of interest was mostly that of narrative/narrators, and he was particularly influenced by the significance of ‘imagistic’ functions of gesture in relation to the emergence of language in an utterance (the so-called growth point, where the gesture precedes the linguistic formulation).

Such an approach has obvious relevance for the emergence of mathematics in children’s talk, such as when the child points to figures before articulating (Radford, 2003a, p 46, Episode 1,1, the video clip is not downloadable):

Josh: It’s always the next. Look! [and pointing to the figures with the pencil he says the following] 1 plus 2, 2 plus 3 […]

McNeill’s notion that gestures are associated with ‘internal’, intra-mental images, and their linguistic ‘parallels’ associated more with the external, inter-mental social/socio-cultural ‘verbal’ representation, is an interesting one for mathematisation (e.g., in Arzarello et al.). The idea here is that the ‘sign’ constituted by a gesture with its linguistic parallel constitutes a unity of internal with external elements, and that conflicts between these elements represent contradictions, and hence opportunities for realignment, or learning. The gesture-and-word unit offers a reflection of Vygotsky’s thought-and-word (or thought-and-utterance) unit of analysis.

So Edwards takes, applies and extends McNeill ‘imagistic’ categories (iconic and metaphoric) to mathematics contexts. This is a good start, and I immediately want to extend this formulation to include McNeill’s non-imagistic gesture categories: I think I see ‘beats’ (Radford speaks of ‘rhythm’) in the gestures used by children to indicate number patterns in ‘factual generalisation’, as in the rhythmic articulation and pointing-beating of the “1 plus 2”, “2+3” etc.

In my own work, I have stressed the significance of deictics in mathematical communication: pointing and waving when associated, or better fused, with models signify mathematics (e.g., Williams & Wake, 2003; Misailidou & Williams, 2003).

In coordination with a model (such as a graph in Roth’s original examples) deictic gestures can signify mathematical objects before they are named, and when the points/segments of a drawing, model or graph have multiple significations, we have an ambiguous moment in communication that can perhaps hold just the right tension in communication.

Beyond gesticulation, there are yet other categories of gesture that mathematics education should consider: ‘Cohesives' and ‘Butterworths' will perhaps emerge or even dominate corpora involving problem solving and proving for instance.

And, to extend further, do the students’ graphing gestures, in Ferrara & Nemirovsky, belong to a different category system, somewhere near the ‘conventional language’ end of the gesturing spectrum (where Kendon and McNeill put sign-languages)?

SEMIOTICS, GENERALISATION AND GESTURE

Arzarello et al. and Bussi & Maschietto inscribe gesture, in part, within Radford’s cultural semiotic theory of ‘semiotic objectification’. Radford’s classification of factual, contextual and symbolic generalisation draws on Peircean categories and conceptions of sign: the index, icon, and symbol, but these are not to be too superficially identified with deictic, iconic, and metaphoric or symbolic gestures.

When a gesture, possibly integrated with parallel action/utterance, is used to denote another object, it constitutes a sign (hence Radford’s term: semiotic objectification). In such a case the gesture can be indexical, iconic, and/or symbolic in Peirce’s (but not McNeill’s) sense. (Peirce, 1955). This now provides a semiotic classification of gestures-in-context that Radford used to analyse significant differences in meanings, such as when the meaning of a formal algebraic expression is indexical for the children but symbolic for the teacher (marking a contradiction between contextual and symbolic generalisation).

I think this difference between McNeill and Peirce/Radford (Wartofsky is another story) explains my concerns with classification systems being equated in Bussi & Maschietto, table 1: a classification system works best if it associated with a particular theoretical scheme. The table thus begs us to examine the relation between the underlying frameworks: Embodied cognition/cognitive linguistics, linguistics, cultural semiotics, that the category systems 'indicate'. (And then there is Wartofsky.)

At this point I would like to consider the disjuncture between the imagistic gestures, or gesticulations in Arzarello et al., Bussi & Maschietto, and Edwards with the gestures and eye foci of the graph-drawing students of Ferrara & Nemirovsky. The gestures of a graph drawer are less strongly bound to the linguistic parallel; but they form a unit of signification with the graph itself, as when the gestures of an operator working a machine form an action because of the mediation of the machine.

In addition, graph drawing has more 'conventional' and 'symbolic' reference rather than iconic, and operate more at the conscious level (in this data anyway, these operations on the graph have not yet descended with practice into the subconscious). In the context of cultural semiotics, this distinction between conscious-unconscious in action-operation suggests an activity theory perspective (Leont'ev, 1981; Williams & Wake, under review) might provide an analytical framework for bringing the two elements together.

It seems there is plenty of empirical and theoretical work to be done still.

BUILDING INTELLECTUAL INFRASTRUCTURE TO EXPOSE AND UNDERSTAND EVER-INCREASING COMPLEXITY

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From the abstract brain-in-a-vat, to the brain neurologically instantiated in a head, to a brain interacting with symbolic tools, to a brain embodied in a walking, talking, gesturing body, to a brain situated in a culture-imbued crowd , … we confront ever increasing complexity in phenomena. Ever more of what was invisible or ignored becomes visible and subject to study, what was excluded becomes included. As so clearly pointed out by Nemirovsky, the subtle new phenomena of gesture, bodily action and perception, eye-movement, and so on, are inevitably and intimately connected with the larger phenomena of thinking, learning, acting and speaking. Indeed, these newly studied phenomena seem, in many cases, to be what the gross phenomena are made of.

With the increasingly complexity comes pressure to expand our repertoire of techniques, conceptual frameworks, and perspectives, our intellectual infrastructure.

Each Forum paper reflects a sophisticated response to the new phenomena being exposed, and each reflects the process of building new intellectual infrastructure intending to expose and make sense of these subtle new phenomena. To a significant extent, the value of the papers resides in the intellectual infrastructure that they are making available to the field of Mathematics Education, a contribution that extends well beyond the particulars of the specific studies reported.

DISTINGUISHING FORMS OF GENERALIZATION AND ASSOCIATED SEMIOSIS

Arzarello, Ferrara, Robutti, Paola, & Sabena develop two means of analysis of the processes of semiotically-based objectification, Parallel and Serial, and, most importantly for our purposes, a way of accounting for the *grounded genesis of a new sign*, which in turn includes Radford's notion of contextual generalization. This account is very similar to one developed by Kaput, et al. (in press). However, the latter make a distinction between contextual generalization and the lifting out of repeated actions as the following example illustrates.

Consider a situation where students have been working with open number sentences such as 8+_=13 or perhaps using a literal, $8 + x = 13$. After solving and discussing some number of these kinds of sentences, it is noticed that the answer always seems to be of the form 13 – 8, that is, in verbal terms, "you subtract the left-hand number from the right-hand number to get the answer." The students can be thought of as being in the process of building a rule, a generalization that applies to a parallel set of additive number sentences written in a number-sentence symbol system. This is an example of the grounded genesis of a new sign, where children's intermediate step could be in form of the verbal version of the rule as given. Mathematically, it is a generalization over a subset of the expressions writable in the number sentence system. At some point, as the result of a combination of discussion and perhaps the teacher-led cataloging and recording of cases, the rule gets extended to cover cases where the "unknown" is in the first position, as in "_ + 6 = 15." But now, in order to ensure that the rule covers all such cases and will extend to more cases in the future, the teacher suggests that they think of it as "subtracting the same number from both sides (of the equation)." While it need not be written in what we would recognize as algebraic form, this new verbally described operation on the number-sentence objects is another, and major, contribution to building a new symbol system which consists of expressions of generalizations about *actions on number sentences*. It is a distinct representation of general actions, and as such is part of a new *operative* symbol system being "lifted out" of in order to serve as a new, more general way of thinking about and operating on the number sentence objects.

This is a critically important kind of symbolization in mathematics, but it is a different *kind* of move, I believe, from contextual generalization. Whereas the previously described move involved expressing variation across *statements*, the new one expresses *actions* on the inscription-objects of the initial symbol system. Indeed, the number-sentence statements themselves are likely to be products of such a lifting-out-of-actions. Further, some of the lifted actions based in arithmetic can be represented directly in terms of the structure of the system, such as the distributive law of multiplication over addition in the usual number systems, which allows the substitution of a * (b + c) by a * b + a * c or vice-versa. The action is an equivalence-preserving substitution, which has parallels in the other basic properties of operations as well as substitution actions such as factoring and expanding polynomials that are built directly on them. I expect that the Research Forum will help us unify these different forms of semiosis.

GESTURE, SEMIOSIS AND DELIBERATE GENERALIZATION

I hope that we can jointly address the matter of those acts of communication and sense-making that are driven by deliberate generalization vs. those that are driven by more immediate acts of communication as described in the papers by Arzarello and the paper by Bartolini Bussi & Maschietto. A similar issue can be raised in the study by Ferrara & Nemirovsky, who examine a particular, highly concrete act of representation. Given the essential role of argument and expression in generalization, and the fact that younger learners need to use natural language and other naturally occurring forms of expression, my sense is that we have much to learn about generalization and hence the development of algebraic thinking, from studies of gesture and talk – including intonation.

My sense is that the purposively integrative style embodied in Radford's notion of *semiotic node* holds great promise in deepening our understanding of how speech, gesture and the many different systems of signs interact, particularly if we adopt his perspective that knowledge objectification is almost always, particularly in education, a multi-modal, semiotically mediated phenomenon. His prime example is of particular interest to me because we have used such tasks in a technological context, where the motions of two objects approaching each other, for example, can be created on a computer screen through almost-free-hand drawn graphs produced by students. The interaction between the particular and the general becomes even more pronounced. Indeed, our work also involves activities similar to that used by Ferrara & Nemirovsky, but where the students' graphs can be re-enacted dynamically. Furthermore, these kinds of constructions can be done in a wirelessly connected classroom where different students can systematically contribute different parts of the same graph in the context of a classroom discussion by sending to a shared public display a graph segment produced on their own hand-held device. Or they can import a physical motion that then, as it is relayed (and not merely graphed) interacts in specifiable ways on a public screen with someone else's imported and reenacted motion. In this case, the semiotic acts become highly public and social, and the need for theoretical constructs such as those offered by Radford becomes more acute than ever before.

THE ISSUE OF GENERALITY OF FINDINGS

Edwards' taxonomy of gestures reveals subtleties that any long-term account of gesture in mathematics education would seem to include. Clearly, we need to examine cases of all sorts, from people describing mathematics that they already know, to people learning mathematics, to people teaching mathematics, to people using mathematics in modeling and problem solving, and, most importantly, we need to vary the kinds of mathematics involved, including mathematics centered on generalization vs. mathematics centered on visualization or computation. Taxonomy, of course, helps generate theory, which informs the structuring of the taxonomy. Of particular interest is the use of gesture in the context of technology use, especially because certain actions in a technological environment amount to tracing gestures – as when one drags a hotspot in a dynamic mathematics system, especially a geometric one such as Cabri or Sketchpad. All such actions amount to gestures captured within a mathematically defined system, so the design and use of such systems is an arena for the immediate application of research in gesture.

The eye-tracking microanalytic work by Ferrara and Nemirovsky, pioneering as it is, raises all sorts of questions and tempts all sorts of hypotheses. While more intrusive eye tracking work has been used for many years in areas that involve traditional character-string symbol systems, including arithmetic and algebra, as well as geometry as they cite, the contexts that Ferrara and Nemirovsky investigate are extremely rich, both visually and in mathematical content. In keeping with an underlying theme of the Forum, the authors stress the functional unity of eye motion, kinesthetic experience, and thought. It will be especially interesting to see how differences in eye-tracking patterns relate to prior experience. For example, how would a novice learner of motion-graph interpretation differ from one who is very experienced, or how would the patterns change if the motion were more regular and perhaps algebraically definable? In this case, the graph might, in fact be seen in a more gestalt-like manner.

I will close by briefly offering yet another perspective on the core issues being explored, the perspective of evolutionary psychology, in particular, the highly integrative, culturally oriented approach developed by Merlin Donald (1991, 2001). Donald's analysis of the physical, "mimetic" roots of reference helps explain the intricately intertwined role of physical gesture in thought and communication and, more broadly, the physical-social embodiment of thought and language. Space limitations prevent further exploration of Donald's more recent work on the co-evolution of human consciousness and culture (2001) that helps provide a rationale for Radford's strongly cultural approach that deliberately takes into account layers of objectification that integrate the many forms of symbolic expression and the major modalities (action, speech, writing/drawing) in which they can be instantiated.

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RF03: A PROGRESSION OF EARLY NUMBER CONCEPTS

Kathleen Hart

The purpose of the research forum is to describe the research evidence available concerning a Progression in Early Number Concepts. Children around the world are taught some Arithmetic as soon as they start school. The content matter may be dictated by the school or teacher, a book or very often by the curriculum laid down by the government. The word 'curriculum' may describe the range of experience to which the child is introduced when first attending school. For the purpose of this forum we limit discussion to the Number Syllabus for grades 1 to 4. The speakers involved have some evidence of what appears hard and what easy for young children in different parts of the world. The aim is to consider what gives success for the majority of children not what is possible for a talented few.

Participants are urged to bring a copy of the syllabus [grades 1 to 4] from their own country and evidence from their own research with young children or national surveys carried out on the child population. The intention is not to compare performance among countries but to judge the progression of difficulty of concepts through pupils' success or failure. It is likely that there is a great deal in common.

The allocation of time for the forum is three hours and we want to end with some suggestions of what we know and the identification of areas about which we have little or no information. The following activities are planned:

1. In the first session to study and discuss what is required by the published syllabuses of various countries. In many countries these lists of topics form the base on which the efficiency of schools and teachers are judged. Inspectors and evaluators use the syllabus to judge what is happening in schools. How are these lists drawn up?

The syllabuses we have may have a lot in common. They may make assumptions on the relative difficulty of ideas. Do any of them alert the teacher to a great leap in intellectual demand? Is there an assumption that the great majority of pupils will succeed. Is success measured in terms of mastery of most/all of the content or is a pass mark assigned which admits to success in only 30-40 % of the topics?

2. Talks by invited researchers who have investigated the learning of Number with young children, the steps of increasing difficulty and the pitfalls.

3. Participants are encouraged to add their own evidence.

4. We have planned a debate on the idea of 'achievability' [does everything in the syllabus have to be achievable by the pupils?] with a proposer and opponent, speakers from the floor and a vote. There is however only half an hour available for this activity.

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The questions asked in this research forum are:

- From the accumulated evidence can we suggest a progression of Early Number Concepts that seem to achievable by children in even the most basic of learning circumstances?
- Can we identify from the available evidence parts of Arithmetic which cause problems ?
- Can we provide some help for the teachers concerning these 'bottlenecks'?
- Can we formulate some research questions which could add more evidence?

USING GROWTH POINTS TO DESCRIBE PATHWAYS FOR YOUNG CHILDREN'S NUMBER LEARNING

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One important outcome of the Early Numeracy Research Project was the development of a framework of growth points to describe young children's number learning. This paper provides a brief overview of the development and use of these growth points.

INTRODUCTION

The Early Numeracy Research Project ([ENRP], Clarke, 1999) was a three-year project initiated in 1999 by the then Victorian Department of Education, Employment and Training (DEET). The aim was to enhance the mathematical learning of young children (5-year-olds to 8-year-olds) through increasing the professional knowledge of their teachers. The project was conducted in 35 matched samples of trial and reference schools that were representative of the broader population across the state. It could be expected, therefore, that any underlying dimension of achievement, like most human characteristics, would approximate a normal distribution (Rowley, Horne et al., 2001). This was an underlying assumption of the data analysis undertaken throughout the ENRP.

GROWTH POINTS FOR DESCRIBING MATHEMATICAL LEARNING

A basic premise of the ENRP was that knowledge about children's mathematical understanding and development is needed for teachers to plan effective learning experiences for their students. To increase teacher's knowledge of children's mathematical development, the ENRP research team developed a framework of growth points to:

- describe the development of children's mathematical knowledge and understanding in the first three years of school, through highlighting important ideas in early mathematics understanding in a form and language that was useful for teachers;
- reflect the findings of relevant Australian and international research in mathematics education, building on the work of successful projects such as Count Me in Too (Bobis & Gould, 1999);
- reflect the structure of mathematics;
- form the basis of mathematics curriculum planning and teaching; and
- identify those students who may benefit from additional assistance or intervention.

As the impetus for the ENRP was a desire to improve young children's mathematics learning, in order to document any improvement, it was necessary to develop quantitative measures of children's growth. It was considered that a framework of key growth points in numeracy learning could fulfill this requirement. Further, the framework of growth points enabled the identification and description of any improvements in children's mathematical knowledge and understanding, where it existed, by tracking children's progress through the growth points. Trial school students' growth could then be compared to that of students in the reference schools.

In developing the framework of growth points, the project team studied available research on key "stages" or "levels" in young children's mathematics learning (Bobis, 1996; Boulton-Lewis, 1996; Fuson, 1992b; Mulligan & Mitchelmore, 1996; Pearn & Merrifield, 1998; Wright, 1998) as well as frameworks developed by other authors and groups to describe learning. A major influence on the project design was the New South Wales Department of Education initiative Count Me In Too (Bobis & Gould, 1999; New South Wales Department of Education and Training, 1998) that developed a learning framework in number (Wright, 1998) that was based on prior research and, in particular, on the stages in the construction of the number sequence (Steffe et al., 1988; Steffe et al., 1983). The *Count Me In Too Project* used an interview designed to measure children's learning against the framework of stages. It was decided to use a similar approach for the ENRP, but to expand the content of the interview to include domains in measurement and space, and to extend the range of tasks so that is was possible to measure the mathematical growth of all children in the first three years of school.

Following the review of available research, the ENRP team developed a framework of growth points for Number (incorporating the domains of Counting, Place value, Addition and Subtraction Strategies, and Multiplication and Division Strategies), Measurement (incorporating the domains of Length, Mass and Time), and Space (incorporating the domains of Properties of Shape, and Visualisation and Orientation). Within each mathematical domain, growth points were stated with brief descriptors in each case. There are typically five or six growth points in each domain (see Appendix 1, at the end of the Forum papers), and each growth-point was assigned a numeral so that the growth points reached by each child could be entered into a database and analysed. For example, the six growth points for the Counting domain are:

1. *Rote counting*
   Rote counts the number sequence to at least 20, but is unable to reliably count a collection of that size.
2. *Counting collections*
   Confidently counts a collection of around 20 objects.
3. *Counting by 1s (forward/backward, including variable starting points; before/after)*
   Counts forwards and backwards from various starting points between 1 and 100; knows numbers before and after a given number.
4. *Counting from 0 by 2s, 5s, and 10s*
   Can count from 0 by 2s, 5s, and 10s to a given target.
5. *Counting from x (where x>0) by 2s, 5s, and 10s*
   Can count from x by 2s, 5s, and 10s to a given target beginning at variable starting points.
6. *Extending and Applying*
   Can count from a non-zero starting point by any single digit number, and can apply counting skills in practical tasks.

Each growth point represents substantial expansion in mathematical knowledge, and it is acknowledged that much learning takes place between them. In discussions with teachers, the research team described growth points as key "stepping stones" along paths to mathematical understanding. They provide a kind of conceptual landscape upon which mathematical learning occurs (Rowley, Gervasoni et al., 2001). As with any journey, it is not claimed that every student passes all growth points along the way. Indeed, (Wright, 1998) cautioned that "it is insufficient to think that all children's early arithmetical knowledge develops along a common developmental path" (p. 702). Also, the growth points should not be regarded as necessarily discrete. As with Wright's (1998) framework, the extent of the overlap is likely to vary widely across young children. However, the order of the growth points provides a guide to the possible trajectory (Cobb & McClain, 1999) of children's learning. In a similar way to that described by Owens & Gould (1999) in the *Count Me In Too* project: "the order is more or less the order in which strategies are likely to emerge and be used by children" (p. 4).

So that the stability of the growth point scale could be determined, test-retest correlations over one school year and for a 12 month period were calculated. The correlations for March to November ranged from 0.48 to 0.71 in the trial group and from 0.43 to 0.68 in the reference group (Rowley, Horne et al., 2001). With the addition of the summer break, twelve-month test-retest correlations dropped slightly, as would be expected. Over such a long period of time, when children are developing at a great rate, this represents a high level of stability, in that the relative order amongst the children is preserved quite well, although, as the data showed, considerable growth took place (Rowley, Horne et al., 2001).

The framework of growth points formed the structure for the creation of the assessment items used in the ENRP Assessment Interview. Both the interview and the framework of growth points were refined throughout the first two years of the project in response to data collected from more than 20,000 assessment interviews with children participating in the project. The assessment interviews provided teachers with insights about children's mathematical knowledge that otherwise may not have been forthcoming. Further, teachers were able to use this information to plan instruction that would provide students with the best possible opportunities to extend their mathematical understanding. These themes were also present in responses to a survey asking trial school teachers to explain how their teaching had changed as a result of their involvement in the ENRP (Clarke et al., 2002).

The longitudinal nature of the ENRP and the detailed information collected about individual children's mathematical knowledge meant that the data could be analysed to identify particular issues related to mathematical learning. For example, the complexity of the teaching process was highlighted by the spread of growth points within any particular grade level. For Grade 2 children in 2000, the spread in the Counting domain was from Growth Point 1 to Growth Point 6. It is clear that in providing effective learning experiences for children, teachers needed to cater for a wide range of abilities. This is important knowledge for teachers, and implies that the curriculum in which the children engage needs to be broad enough to cater for the differences. This type of professional knowledge also makes it possible for teachers to transform the curriculum and the mathematics instruction they provide. However, while the aim is for all teachers to be so empowered, the reality is that it is difficult for teachers to cater for all children's learning needs in the classroom. This is why alternative learning opportunities are beneficial for some children.

CONCLUSION

The ENRP framework of growth-points, the professional knowledge gained through the ENRP assessment interview and the professional development program, and the analysis of ENRP data about children's mathematical learning provided teachers with many insights about effective mathematics assessment, learning and teaching. This culminated in teachers being more confident that they were meeting the instructional needs of children, and more assured about the curriculum decisions they made.

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NUMBER ATTAINMENT IN SRI LANKAN PRIMARY SCHOOLS

Kathleen Hart

From 1998 to 2003 the Primary Mathematics Project *was operative in Sri Lanka. Part of the project was a longitudinal survey with a number of cohorts of children. Here only the progress in Number is quoted and only one cohort is considered. Other data are available. For the purpose of the forum the data are used to identify what in the syllabus for Number appears to be available to all the pupils and what concepts cause difficulty.*

Sri Lanka is an island off the southern tip of India having an area of some 66 000 square kilometres. The population is composed of Sinhala,Tamils, and Muslims. About 74% are Sinhala who are predominantly Buddhist , about 18% are Tamil and are predominantly Hindu, the 7% who are Muslim speak mainly Tamil. A civil war has continued for 20 years, waged mainly in the north but with sporadic bombings in the cities and resulting in many refugees in the east of the country.

The country has very nearly universal primary education. There is a school within walking distance of each village and the pupils are provided with school uniforms and learning materials by the government. The literacy rate on the island is one of the highest in Asia [87% in 1986] but repeated surveys have shown that mathematics attainment is low. The *Primary Mathematics Project*, funded by DfID of Great Britain and the Sri Lankan government , from which these data are produced, worked in schools all over the island but had limited access to the north because of the war. Part of the project was the National Basic Mathematics Survey [NBMS] designed to provide information on which reforms could be based. Here we report only those aspects of NBMS which concern mathematics attainment. In 1998, a total of 7400 children in grades 3, 5 and 7 were tested with written papers and a smaller sample from grades 1 and 2 were interviewed. The papers were designed to match the curriculum and to cater for what was emphasised in the school textbooks. A group of 30 teachers studied them , tried the questions in schools and revised items. The papers were produced in Tamil and Sinhala. These teachers became the evaluation team and carried out the testing in the nationwide school sample. The emphasis was on the child completing as much of the test as possible so members of the evaluation team were told to read items to pupils who appeared to have trouble reading them and to allow about an hour for completion. The report of the survey appeared in 1999 (Hart & Yahampath, 1999).

In 1999 a longitudinal study was started, taking three regions of the country and following a sample from schools of the four types found in the state education system, both Tamil and Sinhala speaking and with both boys and girls. Over two hundred children from each of grades 3 and 5, at this time, were tested in consecutive years until 2002. The pupils who were first and second graders in 1999 were tested each year until they were in grade five. The data from these youngest cohorts are reported here. In 1999 we took five children from the first grade and five from the second in each of six schools, in three towns. Tasks which matched contents of the class syllabus and which employed manipulatives and symbols were used. Each child was interviewed by a teacher from another school who had been trained on the tasks. An audio tape of the interview and notes from two observers provided the data.

COHORT ONE

The 87 first grade pupils interviewed in 1999 had only been in school for five months. The syllabus indicated what was considered suitable at this stage and so the tasks were chosen to reflect this. Sorting tasks, the use of vocabulary for 'front', middle' and 'behind' were included but here we will concentrate on Number. A form of the classic Piagetian conservation task was used with questions such as 'Are there the same number?' referring to two piles of objects and then a displacement of one set was made to see if the child changed his/her opinion. Under half the sample responded correctly [47, 42, 40 per cent.] Another task was the recognition of symbols for 1,2,and 3. A card with the symbol was shown and the child asked 'Give me that number of toys'.' Read the card for me'.

Ninety five percent could read '1' and 78% could give the correct amount of toys. For the number '2' this was reduced to 85% and 55% and for '3' the results were 70% and 56%. Given a card with '3' written on it but only two toys with it, 55% could rectify the situation. When asked to count beads [16], 50% could do it correctly, with a further 20% completing part of the count.

We did not interview this group of pupils for another 17 months, towards the end of their second year in school but another group of first graders were interviewed towards the end of their first year in 2000. They were from the same schools. The Piagetian conservation task was more successful, 64, 50 and 58 per cent but it is clear that this task cannot be assumed to be within the grasp of the great majority of the children. However matching groups of objects to the symbols'1', '2', '3', was achieved by 100, 93 and 97% of the children and 90% could correct the number of toys to give '3' In this group 90% could accurately give seven objects, matching the symbol.. The range of objects which could be counted was also extended, so that 88% could count up to 16. However when ,as the syllabus suggested, the children were asked to add 3 and 4 [written on cards] only 67% could do it. Forty five percent counted on their fingers to add these two numbers.

When we tested cohort one towards the end of their second year in school, they were again interviewed on tasks which reflected the class syllabus. By now over 90% of the group [the sample was reduced to 79 from 87] could read number symbols of 1 to 9, say which number was smaller and identify that the cards for 5, 7 and 9 were missing from a sequence of cards. Given a set of dominoes they could total the number of dots on two touching sections, that is provided with objects to count they could provide a total over ten.

In 2000 the interviewers added some questions on subtraction, since it was at the end of the second grade. 'Eight birds were in the tree and three flew away, how many were left?'. Eighty percent had this correct and 95 % when the question was repeated with '8 flew away'.

All the questions given to grades 1 and 2 reported so far were given orally. The syllabus does contain some written computations so the following were given to the pupils, written on paper. The percentage success is shown below in Table 1.

5 +3 ——	7 +8 ——	4 +4 ——	2+4=……	6+6=……	
86%	62%	91%	80%	68%	Success rate
5 -2 ——	9 -2 ——	7 -7 ——	8-4……	3-3……	
72%	61%	58%	58%	54%	Success rate

Table 1. Written Computations Year Two .[2000]

The questions are now too difficult for nearly half the pupils so the syllabus seems to be ahead of the children.

THIRD GRADE. COHORT 1

Towards the end of the third grade the same cohort of children were asked questions pertaining to the syllabus. By now the expectation is that pupils are writing computations in their books and there is a third grade textbook. The tests, given in November, had some questions given orally and a test paper which had printed questions but which the evaluator could read to the child if needed [there were only five in a group]. The oral questions were about Number, Shape and Money and very similar to those asked in Year 2. For Number there was a further question about the number which comes before and after ' 7.' On this latter there was success at the 85% level and on the earlier questions success was at over 95%. The second year work tested here had been consolidated. When it came to the regular third grade questions on the test paper the mean score for the paper was 39%. Failure has arrived.

By the end of third grade the pupils are expected to deal with two and three digit numbers, do addition and subtraction algorithms including decomposition. cope with multiplication of two 2 digit numbers and even shade one half of a diagram. The only question which had a facility of over 85% was completing a sequence of numbers from the five times table. About half the pupils could correctly identify the number of hundreds, tens and ones given a three digit number. The two digit algorithms were adequately completed only if it was single digit work involved, that is no regrouping of tens. This is shown in Table 2 below.

75 -32 _____	81 -25 _____	39 +18 _____	305 +217 _____	
72%	41%	45%	36%	Success rate

Table 2 . Two Digit Algorithms. Grade 3

Cohort One was tested again in grades four and five. Other cohorts were followed and it was obvious that although performance was mixed, those who performed badly or even at a 'middle' level in grade 3 never achieved great success later. Grade 3 seems a very great hurdle. According to the teachers of these children 'place value' is a problem and certainly the algorithms quoted above become not just difficult but very difficult when decomposition is involved.

In the forum we will look at these and other data and try to sequence what is in the syllabus so that the difficulties become more obvious to a teacher. The aim is not to throw out what teachers, certainly in this sample, feel is the mathematics they want or intend to teach but to offer information which might provide a better chance of success. All participants are encouraged to bring data and also the Number Curriculum taught in the first four years of their primary schools.

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MATHEMATICS RECOVERY: FRAMEWORKS TO ASSIST STUDENTS' CONSTRUCTION OF ARITHMETICAL KNOWLEDGE.

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Mathematics Recovery was the outcome of a three-year research and development project at Southern Cross University in northern New South Wales, conducted in 1992-5. The project received major funding from the Australian Research Council and major contributions in the form of teacher time, from regional government and Catholic school systems. Over the 3-year period, the project involved working in 18 schools with 20 teachers and approximately 200 participating first-grade students (Wright, 2000).

MR can be regarded as consisting of two distinct but interrelated components. One component concerns an elaborated body of theory and practice for working with students, that is, teaching early number knowledge (Wright et al., 2000; & Wright et al., 2002). The second component concerns distinctive ways of working with teachers, that is, providing effective, long-term professional development in order to enable teachers to learn about working with students (Wright, 2000, pp.140-4).

The theoretical origins of MR are in the research program of Les Steffe, a professor in mathematics education, at the University of Georgia in the United States. In the 1970s and 1980s, Steffe's research focused almost exclusively on early number learning (e.g. Steffe & Cobb, 1988; Steffe, 1992). The goal of this research is to develop psychological models to explain and predict students' mathematical learning and development. Of particular interest in this approach, is the strategies – for which Steffe uses the Piagetian label of 'schemes', that the student uses in situations that are problematic for the student, and how these schemes develop and are re-organised over the course of an extended teaching cycle, as observed in teaching sessions mainly, but also in pre- and post- interview-based assessments.

Steffe's research and Mathematics Recovery have as their basic orientation, von Glasersfeld's theory of cognitive constructivism – an epistemological theory that has been developed and explicated over the last 30 or more years, (e.g. von Glasersfeld, 1978; 1995). Von Glasersfeld's theory is a theory about knowing – how humans come to know, rather than for example, an approach to teaching.

Assessment in Mathematics Recovery involves a one-on-one interview, in which the student is presented with groups of tasks, where each group relates to a particular aspect of early number learning. The assessment has two broad purposes. First, it should provide a rich, detailed description of the student's current knowledge of early number. Second, the assessment should lead to determination of levels on the relevant tables in the framework of assessment and learning (Wright et al., 2000)

One of the key elements of the MR program is its framework for assessment and learning – usually referred to as the Learning Framework in Number. One important function of the framework is to enable summary profiling of students' current knowledge. The profiling is based on six aspects of number early number knowledge referred to as a model. Each model contains a progression of up to six levels indicating the development of students' knowledge on that particular aspect of early number learning. Taken together, the models can be regarded as laying out a multi-faceted progression of students' knowledge and learning in early number, and in this sense the models are analogous to a framework (Wright et al., 2002, e.g. p. 77).

The view in MR is that models consisting of progressions of levels of student knowledge constitute one important part of a learning framework. A comprehensive learning framework should also contain: (a) descriptions of assessment tasks that relate closely to the levels on each of the models, and thus enable determination of the student's level; (b) descriptions of other assessment tasks which might not relate directly to the models but nevertheless, have the potential to provide important information about early number knowledge; (c) comprehensive descriptions of the likely responses of students to the all assessment tasks; and (d) descriptions of other aspects of early number knowledge considered to be relevant to students' overall learning of early number. A framework as just described can rightly be regarded as a comprehensive framework for assessment and learning.

The Learning Framework in Number (LFIN) is regarded as a rich description of the students' early number knowledge. This includes, but is not limited to, the strategies that student uses to solve what adults might regard as simple number tasks (additive, subtractive). While it is important to document students early arithmetical strategies, it is not sufficient to describe students' knowledge merely in terms of the currently available strategies. As well, there are important aspects of students' knowledge not simply described in terms of strategies used to solve problems. These aspects include for example, facility with spoken and heard number words, and ability to identify (name) numerals.

The six aspects of the framework are described in terms of a progression of levels. These are: (a) strategies for counting and solving simple addition and subtraction tasks; (b) very early place value knowledge, that is, ability to reason in terms of tens and ones; (c) facility with forward number word sequences; (d) facility with backward number word sequences; (e) facility with numeral identification; and (f) early knowledge of multiplication and division. Other aspects of the framework relate to: (a) combining and partitioning small numbers without counting; (b) using five and ten as reference points in numerical reasoning; (c) use of finger patterns in numerical contexts; (d) relating number to spatial patterns; and (e) relating number to temporal sequences. While each aspect can be considered from a distinct perspective, it is also important to focus on the inter-relationships of the aspects.

MR assessment tells the teacher 'where the student is' and the learning framework indicates 'where to take the student', but teachers don't necessarily have the time to design and develop specific instructional procedures. In the period 1999-2000, Wright and colleagues developed an explicit framework for instruction. Thus the instructional settings and activities used in earlier versions of MR were incorporated into an instructional framework (usually referred to as the Instructional Framework for Early Number – IFEN). The instructional framework differs in form from the learning framework because its purpose is different. Nevertheless it is informed by and strongly linked to the learning framework (Wright et al., 2002). The framework sets out a progression of key teaching topics which are organized into three strands as follows:

- Counting — instruction to progressively develop use of counting by ones, to solve arithmetical tasks.
- Grouping — instruction to develop arithmetical strategies other than counting by ones.
- Number words and numerals — instruction to develop facility with FNWSs, BNWSs and a range of aspects related to numerals.

Each of the three strands spans a common set of five phases of instruction. Each key topic contains on average, six instructional procedures. Each instructional procedure includes explicit descriptions of the teachers' words and actions, as well as descriptions of the instructional setting (materials, instructional resources), and notes on purpose, teaching and students' responses. Finally, each instructional procedure typically is linked to a level in one or more of the models (aspects) of the learning framework. Thus the teacher is not only provided with exemplary instructional procedures suited to any particular student but is forearmed with detailed knowledge of ways the student is likely to respond to each instructional procedure.

Recent research (Wright, 1998; 2002), highlights the relative complexities of students' early number knowledge, and the usefulness of close observation and assessment in enabling detailed understanding of students' arithmetical knowledge and strategies. Critical to the efforts of teachers to address students' learning difficulties in mathematics are elaborated exemplars of theory-based practice directed at addressing mathematics learning difficulties.

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Appendix 1
ENRP Number Growth Points (Preparatory – Year 2)

Counting Growth Points

0. Not apparent.
   Not yet able to state the sequence of number names to 20.
1. Rote counting
   Rote counts the number sequence to at least 20, but is not yet able to reliably count a collection of that size.
2. Counting collections
   Confidently counts a collection of around 20 objects.
3. Counting by 1s (forward/backward, including variable starting points; before/after)
   Counts forwards and backwards from various starting points between 1 and 100; knows numbers before and after a given number.
4. Counting from 0 by 2s, 5s, and 10s
   Can count from 0 by 2s, 5s, and 10s to a given target.
5. Counting from x (where x >0) by 2s, 5s, and 10s
   Given a non-zero starting point, can count by 2s, 5s, and 10s to a given target.
6. Extending and applying counting skills
   Can count from a non-zero starting point by any single digit number, and can apply counting skills in practical tasks.

Place Value Growth Points

0. Not apparent
   Not yet able to read, write, interpret and order single digit numbers.
1. Reading, writing, interpreting, and ordering single digit numbers
   Can read, write, interpret and order single digit numbers.
2. Reading, writing, interpreting, and ordering two-digit numbers
   Can read, write, interpret and order two-digit numbers.
3. Reading, writing, interpreting, and ordering three-digit numbers
   Can read, write, interpret and order three-digit numbers.
4. Reading, writing, interpreting, and ordering numbers beyond 1000
   Can read, write, interpret and order numbers beyond 1000.
5. Extending and applying place value knowledge
   Can extend and apply knowledge of place value in solving problems.

Strategies for Addition & Subtraction Growth Points

0. Not apparent
   Not yet able to combine and count two collections of objects.
1. Count all (two collections)
   Counts all to find the total of two collections.
2. Count on
   Counts on from one number to find the total of two collections.
3. Count back/count down to/count up from
   Given a subtraction situation, chooses appropriately from strategies including count back, count down to and count up from.
4. Basic strategies (doubles, commutativity, adding 10, tens facts, other known facts)
   Given an addition or subtraction problem, strategies such as doubles, commutativity, adding 10, tens facts, and other known facts are evident.
5. Derived strategies (near doubles, adding 9, build to next ten, fact families, intuitive strategies)
   Given an addition or subtraction problem, strategies such as near doubles, adding 9, build to next ten, fact families and intuitive strategies are evident.
6. Extending and applying addition and subtraction using basic, derived and intuitive strategies
   Given a range of tasks (including multi-digit numbers), can solve them mentally, using the appropriate strategies and a clear understanding of key concepts.

Strategies for Multiplication & Division Growth Points

0. Not apparent
   Not yet able to create and count the total of several small groups.
1. Counting group items as ones
   To find the total in a multiple group situation, refers to individual items only.
2. Modelling multiplication and division (all objects perceived)
   Models all objects to solve multiplicative and sharing situations.
3. Abstracting multiplication and division
   Solves multiplication and division problems where objects are not all modelled or perceived.
4. Basic, derived and intuitive strategies for multiplication
   Can solve a range of multiplication problems using strategies such as commutativity, skip counting and building up from known facts.
5. Basic, derived and intuitive strategies for division
   Can solve a range of division problems using strategies such as fact families and building up from known facts.
6. Extending and applying multiplication and division
   Can solve a range of multiplication and division problems (including multi-digit numbers) in practical contexts.

RF04: THEORIES OF MATHEMATICS EDUCATION

Lyn English
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The purpose of this Forum is to stimulate critical debate in the area of theory use and theory development, and to consider future directions for the advancement of our discipline. The Forum opens with a discussion of why theories are essential to the work of mathematics educators and addresses possible reasons for why some researchers either ignore or misunderstand/misuse theory in their work. Other issues to be addressed include the social turn in mathematics education, an evolutionary perspective on the nature of human cognition, the use of theory to advance our understanding of student cognitive development, and models and modelling perspectives. The final paper takes a critical survey of European mathematics didactics traditions, particularly those in Germany and compares these to historical trends in other parts of the world.

INTRODUCTION

Our conception and preference for a particular mathematics education theory invariably influences our choice of research questions as well as our theoretical framework in mathematics education research. Although we have made significant advances in mathematics education research, our field has been criticized in recent years for its lack of focus, its diverging theoretical perspectives, and a continued identity crisis (Steen, 1999). At the dawn of this new millennium, the time seems ripe for our community to take stock of the multiple and widely diverging mathematical theories, and chart possible courses for the future. In particular, we need to consider the important role of theory building in mathematics education research.

Issues for consideration include:

1. What is the role of theory in mathematics education research?
2. How does Stokes (1997) model of research in science apply to research in mathematics education?
3. What are the currently accepted and widely used learning theories in mathematics education research? Why have they gained eminence?
4. What is happening with constructivist theories of learning?
5. Embodied cognition has appeared on the scene in recent years. What are the implications for mathematics education research, teaching, and learning?
6. Theories of models and modelling have received considerable attention in the field in recent years. What is the impact of these theories on mathematics research, teaching, and learning?

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7. Is there a relationship between researchers' beliefs about the nature of mathematics and their preference for a particular theory?
8. How do theories used in European mathematics didactics traditions compare with those used in other regions of the world? Do European traditions reveal distinct theoretical trends?

There are several plausible explanations for the presence of multiple theories of mathematical learning, including the diverging, epistemological perspectives about what constitutes mathematical knowledge. Another possible explanation is that mathematics education, unlike "pure" disciplines in the sciences, is heavily influenced by cultural, social, and political forces (e.g., D'Ambrosio, 1999; Secada, 1995; Skovsmose & Valero, 2002). As Lerman indicates in his paper, the switch to research on the social dimensions of mathematical learning towards the end of the 1980s resulted in theories that emphasized a view of mathematics as a social product. Social constructivism, which draws on the seminal work of Vygotsky and Wittgenstein (Ernest, 1994) has been a dominant research paradigm ever since. On the other hand, cognitively oriented theories have emphasized the mental structures that constitute and underlie mathematical learning, how these structures develop, and the extent to which school mathematics curricula should capture the essence of workplace mathematics (e.g., see Stevens, 2000).

Stokes (1997) suggested a new way of thinking about research efforts in science, one that moves away from the linear one-dimensional continuum of "basic, to applied, to applied development, to technology transfer." Although this one-dimensional linear approach has been effective, Stokes argued that it is too narrow and does not effectively describe what happens in scientific research. In Pasteur's Quadrant, Stokes proposed a 2-dimensional model, which he claimed offered a completely different conception of research efforts in science. If one superimposes the Cartesian co-ordinate system on Stokes' model, the Y -axis represents "pure" research (such as the work of theoretical physicists) and the X-axis represents "applied" research" (such as the work of inventors). The area between the two axes is called "Pasteur's Quadrant" because it is a combination (or an amalgam) of the two approaches. If we apply Stokes' model to mathematics education research, we need to clearly delineate what is on the Y-axis of Pasteur's quadrant if we are to call our field a science. Frank Lester elaborates further on this issue in the opening paper of this Forum. Steve Lerman extends the discussion initiated in Lester's contribution on the pivotal, albeit misunderstood role of theories in mathematics education, and presents theoretical frameworks most frequently used in PME papers during the 1990-2001 time period. Lerman's analysis reveals that a wide variety of theories are used by PME researchers with a distinct preference for social theories over cognitive theories. An interesting avenue for discussion is whether the particular social theories used in this time period reveal a distinct geographic distribution, and if so why? Luis Moreno-Armella presents an evolutionary perspective on the nature of human cognition, particularly the evolution of representations, which he aptly terms pre-theory, as it serves as a foundation for the present discussion. John Pegg and David Tall compare neo-Piagetian theories in order to use the similarities and differences among theories to address fundamental questions in learning. Lyn English and Richard Lesh present a models and modeling perspective which innovatively combines the theories of Piaget and Vygotsky to pragmatically address the development and real life use of knowledge via model construction. The Forum concludes with a review by Günter Törner and Bharath Sriraman on European theories of mathematics education, with a focus on German traditions. Eight major tendencies are highlighted in 100 years of mathematics education history in Germany; these tendencies reflect trends that have occurred internationally.

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THE PLACE OF THEORY IN MATHEMATICS EDUCATION RESEARCH

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As most, if not all, of you know, the current emphasis in the United States being placed on so called *scientific research* in education, is driven in large part by political forces. Much of the public and some of the professional conversation has begun with an assumption that the purpose of research is to determine “what works,” and the discourse has focused largely on matters of research design and methods. One consequence has been the rekindling of attention to experimental designs and quantitative methods that had faded from prominence in education research over the past two decades or so. Far less prominent in recent discussions about educational research has been the place of theory.

Scholars in other social science disciplines (e.g., anthropology, psychology, sociology) often justify their research investigations on grounds of developing understanding by building or testing theories. In contrast, the current infatuation in the U.S. with "what works" seems to leave education researchers with less latitude to conduct studies to advance theoretical goals. It is time for a serious examination of the role that theory should play in the formulation of problems, in the design and methods employed, and in the interpretation of findings in education research. In this brief presentation, I speculate about why so many researchers seem to misunderstand or misuse theory and suggest how we might think about the goals of research that might help eliminate this misunderstanding and misuse.

Why is so much of our research atheoretical?

Mathematics education research is an interesting and important area for such an examination. Although math ed research was aptly characterized less than 15 years ago by Kilpatrick (1992) and others as largely *atheoretical*, a perusal of recent articles in major MER journals reveals that theory is alive and well: indeed, Silver and Herbst (2004) have noted that expressions such as "theory-based," "theoretical framework," and "theorizing" are common. In fact, they muse, manuscripts are often rejected for being atheoretical. The same is true of proposals submitted for PME meetings. However, the concerns raised decades ago persist; too often researchers ignore, misunderstand, or misuse theory in their work.

We are our own worst enemies

In my mind there are two basic problems that must be dealt with if we are to expect theory to play a more prominent role in our research. The first has to do with the widespread misunderstanding of what it means to adopt a theoretical stance toward our work. The second is that some researchers, while acknowledging the importance of theory, do not feel qualified to engage in serious theory-based work. I attribute both of these problems to: (a) the failure of our graduate programs to properly equip novice researchers with adequate preparation in theory, and (b) the failure of our research journals to insist that authors of research reports offer serious theory-based explanations of their findings.

Writing about the state of U.S. doctoral programs, Hiebert, Kilpatrick, and Lindquist (2001) suggest that mathematics education is a complex system and that improving the process of preparing doctoral students means improving the entire system, not merely changing individual features of it. They insist that "the absence of system-wide standards for doctoral programs [in mathematics education] is, perhaps, the most serious challenge facing systemic improvement efforts. . . . Indeed, participants in the system have grown accustomed to creating their own standards at each local site" (p. 155). One consequence of the absence of commonly accepted standards is that there is a very wide range of requirements of different programs. At one end of the continuum of requirements are a few programs that focus on the preparation of researchers. At the other end are those programs that require little or no research training beyond taking a research methods course or two. In general, with few exceptions, doctoral programs are replete with courses and experiences in research methodology, but woefully lacking in courses and experiences that provide students with solid theoretical underpinnings for future research. Without solid understanding of the role of theory in conceptualizing and conducting research, there is little chance that the next generation of mathematics education researchers will have a greater appreciation for theory than is currently the case. Put another way, we must do a better job of cultivating a predilection for theory within the mathematics education research community.

During my term as editor of the *Journal for Research in Mathematics Education* in the early1990s, I found the failure of authors of research reports to pay serious attention to explaining the results of their studies one of the most serious shortcomings. A simple example from the expert-novice problem solver research illustrates what I mean. It is not enough simply to report: Experts *do X when they solve problems and novices do Y.* Were the researcher guided by theory, a natural question would be to ask WHY? Having some theoretical perspective guiding the research provides a framework within which to attempt to answer *Why* questions. Without a theoretical orientation, the researcher can speculate at best or offer no explanation at all.

Many mathematics educators hold misconceptions about the role of theory

Time constraints prevent me from providing a detailed discussion of what I see as the most common misconceptions about theory, so I will simply list four and say a few words about them.

1. Theory-based explanation given by "decree" rather than evidence. Some researchers (e.g., Eisenhart, 1991) insist that educational theorists prefer to address and explain the results of their research by "theoretical decree" rather than with solid evidence to support their claims. That is to say, there is a belief among some researchers that theorists make their data fit their theory.

2. Data have to "travel." Sociologist and ethnographer, John Van Maanen (1988), has observed that data collected under the auspices of a theory has to "travel" in the sense that (in his view) data too often must be stripped of context and local meaning in order to serve the theory.

3. Standard for discourse not helpful in day-to-day practice. Related to the previous concern, is the observation that researchers tend to use a theory to set a standard for scholarly discourse that is not functional outside the academic discipline. Conclusions produced by the logic of theoretical discourse too often are not at all helpful in day-to-day practice. Researchers don't speak to practitioners! The theory is irrelevant to the experience of practitioners (cf., Lester & Wiliam, 2002).

4. No triangulation. Sociologist, Norman Denzin (1978) has discussed the importance of theoretical triangulation, by which he means the process of compiling currently relevant theoretical perspectives and practitioner explanations, assessing their strengths, weaknesses, and appropriateness, and using some subset of these perspectives and explanations as the focus of empirical investigation. By using a single theoretical perspective to frame one's research, such triangulation does not happen.

There is no doubt that rigid, uncritical adherence to a theoretical perspective can lead to these sorts of offenses. However, I know of no good researchers who are guilty of such crimes. Instead, more compelling arguments can be marshaled in support of using theory.

Why theory is essential

Again, time constraints for this presentation prevent me from elaborating on the reasons why theory should play an indispensable role in our research. Let me mention a few of the most evident. (In the following brief discussion I borrow heavily from an important paper written about 15 years ago by Andy diSessa [1991])

1. There are no data without theory. We have all heard the claim, "The data speak for themselves!" Dylan Wiliam and I have argued elsewhere that actually data have nothing to say. Whether or not a set of data can count as evidence of something is determined by the researcher's assumptions and beliefs as well as the context in which it was gathered (Lester & Wiliam, 2000). One important aspect of a researcher's beliefs is the theoretical perspective he or she is using; this perspective makes it possible to make sense of a set of data.

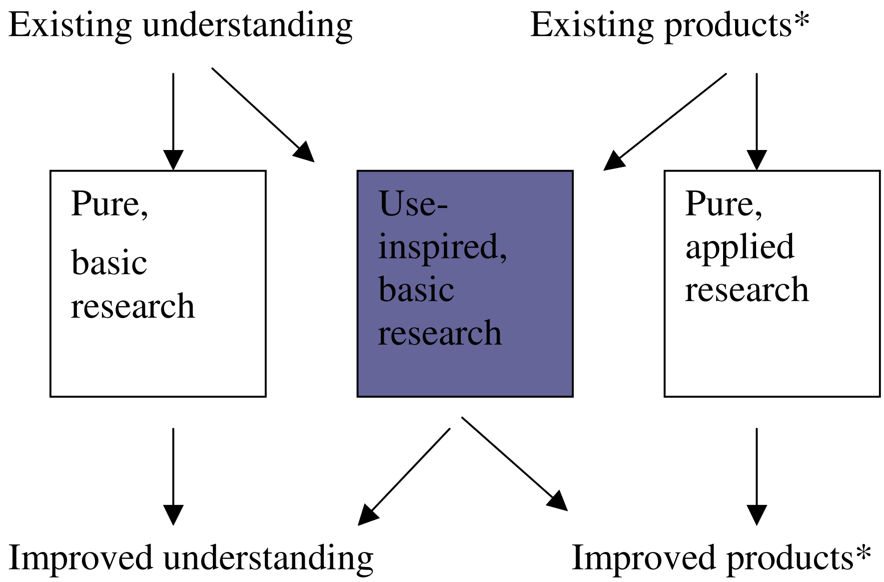
2. Good theory transcends common sense. In the paper mentioned above, diSessa (1991) argues that theoretical advancement is the linchpin in spurring practical progress. He notes that, sure, you don't need theory for many everyday problems—purely empirical approaches often are enough. But often things aren't so easy. Deep understanding that comes from concern for theory building is often essential to deal with truly important problems.

3. Need for deep understanding, not just "for this" understanding. Related to the above, is the need we have to deeply understand some things—the important, big questions (e.g., What does it mean to be intelligent? What does it mean to understand something?)—not simply find solutions to immediate problems and dilemmas. Theory helps us develop deep understanding. (I say more about understanding in the next section.)

A different way to think about the goals of research and the place of theory

In his book, *Pasteur's Quadrant: Basic Science and Technological Innovation,* Donald Stokes (1997) presents a new way to think about scientific and technological research and their purposes. Stokes begins with a detailed discussion of the history of development of the current U.S. policy for supporting advanced scientific study (I suspect similar policies exist in other industrialized countries). He notes that from the beginning of the development of this policy shortly after World War II there has been an inherent tension between the pursuit of *fundamental understanding* and *considerations of use*. This tension is manifest in the, often radical, separation between basic and applied science. He argues that prior to the latter part of the 19th Century, scientific research was conducted largely in pursuit of deep understanding of the world. But, the rise of microbiology in the late 19th Century brought with it a concern for putting scientific understanding to practical use. He illustrates this concern with the work of Louis Pasteur. Of course, Pasteur working in his laboratory wanted to understand the process of disease at the most basic level, but he wanted that understanding to be applicable to dealing with silk worms, anthrax in sheep, cholera in chickens, spoilage in milk, and rabies in people. The work of Pasteur suggests that one could not understand his science without knowing the extent to which he had considerations of use in mind as well as fundamental understanding. Stokes proposed a model for thinking about scientific research that blends the two motives: the *quest for fundamental understanding* and *considerations of use*.

Adapting Stokes's model to educational research in general, and mathematics education research in particular, I have come up with a slightly different model (see Figure 1). In the final section of this short paper, I describe the relationship between my model and the place of theory in mathematics education research.



* "Products" include such things as instructional materials, professional development programs, and district educational policies.

Figure 1. Adaptation of Stokes's model to educational research

A *bricolage* approach to theory in mathematics education research

Even if there is no need to make a case for the importance of theory in our research, there is a need to suggest how researchers, especially novices, can deal with the almost mystifying range of theories and theoretical perspectives that are being used. In a chapter to appear in a forthcoming handbook of research in mathematics education, Cobb (in press) considers how mathematics education researchers might cope with the multiple and frequently conflicting theoretical perspectives that currently exist. He observes:

The theoretical perspectives currently on offer include radical constructivism, sociocultural theory, symbolic interactionism, distributed cognition, information-processing psychology, situated cognition, critical theory, critical race theory, and discourse theory. To add to the mix, experimental psychology has emerged with a renewed vigor in the last few years. . . . In the face of this sometimes bewildering array of theoretical alternatives, the issue . . . is that of how we might make and justify our decision to adopt one theoretical perspective rather than another.[1]

Cobb goes on to question the repeated (mostly unsuccessful) attempts that have been made in mathematics education to derive instructional prescriptions directly from background theoretical perspectives. He insists that it is more productive to compare and contrast various theoretical perspectives in terms of the manner in which they orient and constrain the types of questions that are asked about the learning and teaching of mathematics, the nature of the phenomena that are investigated, and the forms of knowledge that are produced. To his recommendation, I would add that comparing and contrasting various perspectives would have the added benefit of both enhancing our understanding of important phenomena and increasing the usefulness of our investigations (c.f., Lester & Wiliam, 2002).

I propose to view the theoretical perspectives we adopt for our research as sources of ideas that we can appropriate and modify for our purposes as mathematics educators. This process of developing tools for our research is quite similar to that of instructional design as described by Gravemeijer (1994). He suggests that instructional design resembles the thinking process characterized by the French word *bricolage,* a notion borrowed from Claude Levi–Strauss. A *bricoleur* is a handyman who invents pragmatic solutions in practical situations and is adept at using whatever is available. Similarly, I suggest, as do Cobb and Gravemeijer, that rather than adhering to one particular theoretical perspective, we act as *bricoleurs* by adapting ideas from a range of theoretical sources to suit our goals—goals that should aim not only to deepen our fundamental understanding of mathematics learning and teaching, but also to aid us in providing practical wisdom about problems practitioners care about. If we begin to pay serious attention to these goals, the problem of theory is likely to be resolved.

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[1] Because Cobb's paper is currently in draft form and is not yet publicly available, no page numbers are provided.

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THEORIES OF MATHEMATICS EDUCATION: A PROBLEM OF PLURALITY?

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Today, in many countries around the world, constraints on the funding of Universities are leading to restrictions on educational research. In some countries national policy is also placing constraints on the kinds of research that will be funded (e.g. the effects of *No Child Left Behind* policy in the USA). At the same time we see research in mathematics education proliferating, not just in quantity but also, as in the concerns of this Research Forum, in the range of theories that are drawn upon in our research. In my contribution I want to ask: is this surprising, or unusual, and is it necessarily a hindrance to the effectiveness of educational research in mathematics?

In discussing this, I would argue that we need a specific language that enables an analysis of intellectual fields and their growth, a language that will not be provided by mathematics or by psychology. I will draw on some of the later work of the sociologist of education, Basil Bernstein, in particular his 1999 paper on research discourses (Bernstein, 1999). Following that, I will make some remarks about the use of theory.

A Language of Research Fields

Bernstein draws on two notions: hierarchy and verticality. Discourses are described as hierarchical where knowledge in the domain is a process of gradual distancing, or abstraction, from everyday concepts. Hierarchical discourses require an apprenticeship; they position people as initiated or apprenticed. Clearly academic and indeed school mathematics are examples of hierarchical discourses. Research (Cooper & Dunne, 2000) shows that setting mathematics tasks in everyday contexts can mislead some students, namely those from low socio-economic background, into privileging the everyday context and the meanings carried in them over the abstract or esoteric meanings of the discourse of academic mathematics.

His second notion, verticality, describes the extent to which a discourse grows by the progressive integration of previous theories, what he calls a vertical knowledge structure, or by the insertion of a new discourse alongside existing discourses and, to some extent, incommensurable with them. He calls these horizontal knowledge structures. Bernstein offers science as an example of a vertical knowledge structure and, interestingly, both mathematics and education (and sociology) as examples of horizontal knowledge structures. He uses a further distinction that enables us to separate mathematics from education: the former has a strong grammar, the latter a weak grammar, that is, with a conceptual syntax not capable of generating unambiguous empirical descriptions. Both are examples of hierarchical discourses in that one needs to learn the language of linear algebra or string theory just as one needs to learn the language of radical constructivism or embodied cognition. It will be obvious that linear algebra and string theory have much tighter and specific concepts and hierarchies of concepts than radical constructivism or embodied cognition. Adler and Davis (forthcoming) point out that a major obstacle in the development of accepted knowledge in mathematics for teaching may well be the strength of the grammar of the former and the weakness of the latter. Where we can specify accepted knowledge in mathematics, knowledge about teaching is always disputed.

As a horizontal knowledge structure, then, it is typical that mathematics education knowledge will grow both within discourses and by the insertion of new discourses in parallel with existing ones. Thus we can find many examples in the literature of work that elaborates the functioning of the process of reflective abstraction, as an instance of the development of knowledge within a discourse. But the entry of Vygotsky's work into the field in the mid-1980s (Lerman, 2000) with concepts that differed from Piaget's did not lead to the replacement of Piaget's theory (as the proposal of the existence of oxygen replaced the phlogiston theory). Nor did it lead to the incorporation of Piaget's theory into an expanded theory (as in the case of non-Euclidean geometries). Indeed it seems absurd to think that either of these would occur precisely because we are dealing with a social science, that is, we are in the business of interpretation of human behaviour. Whilst all research, including scientific research, is a process of interpretation, in the social sciences, such as education, there is a double hermeneutic (Giddens, 1976) since the 'objects' whose behaviour we are interpreting are themselves trying to make sense of the world.

Education, then, is a social science, not a science. Sociologists of scientific knowledge (Kuhn, Latour) might well argue that science is more of a social science than most of us imagine, but social sciences certainly grow both by hierarchical development but especially by the insertion of new theoretical discourses alongside existing ones. Constructivism grows, and its adherents continue to produce novel and important work; models and modelling may be new to the field but already there are novel and important findings emerging from that orientation.

I referred above to the incommensurability, in principle, of these parallel discourses. Where a constructivist might interpret a classroom transcript in terms of the possible knowledge construction of the individual participants, viewing the researcher's account as itself a construction (Steffe & Thompson, 2000), someone using socio-cultural theory might draw on notions of a zone of proximal development. Constructivists might find that describing learning as an induction into mathematics, as taking on board concepts that are on the intersubjective plane, incoherent in terms of the theory they are using (and a similar description of the reverse can of course be given). In this sense, these parallel discourses are incommensurable.

There is an apparent contradiction between the final sentences of the last two paragraphs. If I am claiming that there is important work emerging in different discourses of mathematics education research, but I also claim that discourses are incommensurable, within which discourse am I positioning myself to write these sentences? Is there a meta-discourse of mathematics education in which we can look across these theories? I will make some remarks about this position in the next section.

Theories in Use in Mathematics Education

First I will make some remarks drawn from a recent research project on the use of theories in mathematics education. Briefly we (Tsatsaroni, Lerman & Xu, 2003) examined a systematic sample of the research publications of the mathematics education research community between 1990 and 2001, using a tool that categorised research in many ways. I will only refer here to our findings concerning how researchers use theories in their work as published in PME Proceedings.

Our analysis showed that just over 85% of all papers in the proceedings had an orientation towards the empirical, with a further 5% moving from the theoretical to the empirical, and this has changed little over the years. A little more than three-quarters are explicit about the theories they are using in the research reported in the article. Again this has not varied across the years. The theories that are used have changed, however. We can notice an expanding range of theories being used and an increase in the use of social theories, based on the explicit references of authors, in some cases by referring to a named authority. These fields or names represent theories used, not the frequency of their occurrence in papers.

Year	Theoretical fields other than educational psychology and/or mathematics
1990	Brousseau
1991	Philosophy of mathematics
1992	Vygotsky
1993	Vygotsky
1994	Brousseau, Chevellard, Poststructuralism
1995	Embodied cognition, Educational research
1996	Vygotsky, Situated cognition, Philosophy of mathematics
1997	Situated cognition, Vygotsky, Philosophy of mathematics
1998	Situated cognition, Vygotsky, Philosophy of mathematics
1999	Socio-historical practice
2000	Chevellard
2001	Semiotics, Bourdieu, Vygotsky, Philosophy

Table 1: Theoretical fields

We might suggest that there is a connection here with creating identities, making a unique space from which to speak in novel ways, but we would need another study to substantiate and instantiate this claim.

We can say that there has been a substantial increase in the number of fields from 1994, although it is too early to say whether this trend will continue, as 1999 and 2000 showed a dropping off. What is clear is that the range of intellectual resources, including sociology, philosophy, semiotics, anthropology, etc., is very broad.

In our analysis of how authors have used theories we have looked at whether, after the research, they have revisited the theory and modified it, expressed dissatisfaction with the theory, or expressed support for the theory as it stands. Alternatively, authors may not revisit the theory at all, content to apply it in their study. We have found that more than three-quarters fall into this last category, just over 10% revisit and support the theory, whilst four percent propose modifications. Two authors in our sample ended by opposing theory. This pattern has not changed over the years. Further findings can be found in Tsatsaroni, Lerman and Xu (2003).

The development and application of an analytical tool in a systematic way, paying attention to the need to make explicit and open to inspection the ways in which decisions on placing articles in one category or another, enables one to make all sorts of evidence-based claims. In particular, I would argue that one can observe and record development within discourses and the development of new parallel discourses because of the adoption of a sociological discourse as a language for describing the internal structure of our intellectual field, mathematics education research.

Conclusion

Finally, I will comment on concerns about the effectiveness of educational research in a time of multiple and sometimes competing paradigms, described here as discourses. ‘Effectiveness’ is a problematic notion, although one that certainly figures highly in current discourses of accountability. It arises because by its nature education is a research field with a face towards theory and a face towards practice. This contrasts with fields such as psychology in which theories and findings can be applied, but practice is not part of the characteristic of research in that field. Research in education, in contrast, draws its problems from practice and expects its outcomes to have applicability or at least significance in practice. Medicine and computing are similar intellectual fields in this respect.

However, what constitutes knowledge is accepted or rejected by the criteria of the social field of mathematics education research. Typically, we might say necessarily, research has to take a step away from practice to be able to say something about it. Taking the results of research into the classroom calls for a process of recontextualisation, a shift from one practice into another in which a selection must take place, allowing the play of ideology. To look for a simple criterion for acceptable research in terms of ‘effectiveness’ is to enter into a complex set of issues.

Indeed 'effectiveness' itself presupposes aims and goals for, in our case, mathematics education. To ignore the complexity is to lose the possibility of critique and hence I am not surprised by the multiplicity of theories in our field and the debates about their relative merits, nor do I see it as a hindrance.

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THE ARTICULATION OF SYMBOL AND MEDIATION IN MATHEMATICS EDUCATION

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I describe some basic elements of a *pre-theory* of Mathematics Education. Our field is at the crossroad of a science, mathematics, and a community of practice, education. The interests of this community include the people whose learning takes place at schools and the corresponding intellectual offer from the institutional sides. But as soon as we enter the space of mathematics, we discover a different discipline from the natural sciences. It is the strictly *symbolic nature* of mathematics that makes a big difference and gives to mathematics education, as a research field, its characteristic features that distinguishe it from similar endeavours with respect to other scientific fields, such as biology for instance. I am not implying, of course, that there is no abstraction or concept development involved in those other fields.

More recently, the presence of computers has introduced a new way of looking at symbols and mathematical cognition and has offered the potentiality to re-shape the goals of our whole research field. The urgency to take care of teaching and learning from the research activities has resulted in practices without corresponding theories. Again, I must make clear I am not dismissing the considerable and important results this community has produced. I simply want to underline that institutional pressures can result more frequently than desirable, in losing track of research goals. Perhaps this is a motive to re-consider the need to enter a more organized level of reflection in our community. There is nothing bad in having the chance to look at educational phenomena from different viewpoints but it is better if we can generate a synergy between those viewpoints that, eventually, has as its output a new and stronger theory. Nevertheless the tension between the local and the global also comes to existence here. Being an interested observer and modest participant in the field, I have come to think that only local explanations are possible in our field. *Local theories* might be the answer to the plethora of explanations we encounter around us. But even if local, a mathematics education theory must be developed from scaffolding that eventually crystallizes in the theory. In our case, part of that scaffolding is constituted by mathematics itself, and by a community of practice, as already mentioned.

What sort of machine is the human brain, that it can give birth to mathematics? – an old question that Stanislas Dehaene has aptly posed anew in his book *The Number Sense* (1997). This is the kind of question that, in the long run, must be answered in order to improve the understanding of our field. Nevertheless, trying to answer it will demand an interdisciplinary and longitudinal effort. At the end of the day, we will need to understand why we are able to create symbolic worlds (mathematics, for instance) and why our minds are essentially incomplete outside the co-development with material and symbolic technologies. Our symbolic and mediated nature comes to the front as soon as we try to characterize our intellectual nature. Evolution and culture have left its traits in our cognition, in particular, in our capacity to duplicate the world at the level of symbols.

Diverging epistemological perspectives about what constitutes mathematical knowledge modulate multiple conceptions of learning and the present theories of what constitutes mathematical education as a research discipline. Today, however, there is substantial evidence that the encounter between the conscious mind and distributed cultural systems has altered human cognition and has changed the tools with which we think. The origins of writing and how writing as a technology changed cognition is key from this perspective (Ong, 1988). These examples suggest the importance of studying the evolution of mathematical systems of representation as a vehicle to develop a proper epistemological perspective for mathematics education.

Human evolution is coextensive with tool development. In a certain sense, human evolution has been an *artificial* process as tools were always designed with the explicit purpose of transforming the environment. And so, since about 1.5 millions years ago, our ancestor Homo Erectus designed the first stone tools and took profit from his/her voluntary memory and gesture capacities (Donald, 2001) to evolve a pervasive technology used to consolidate their early social structures. The increasing complexity of tools demanded optimal coherence in the use of memory and in the transmission, by means of articulate gestures, of the building techniques. We witness here what is perhaps the first example of deliberate teaching. Voluntary memory enabled our ancestors to engender a mental template of their tools. Templates lived in their minds, resulting from activity, granting an objective existence as abstract objects even before they were *extracted* from the stone. Thus, tool production was not only important for plain survival, but also for broadening the mental world of our ancestors –introducing a higher level of objectivity.

The actions of our ancestors were producing a *symbolic* version of the world: A world of intentions and anticipations they could imagine and *crystallize* in their tools. What their tools meant was the same as what they *intended* to do with them. They could *refer* to their tools to *indicate* their *shared* intentions and, after becoming familiar with those tools, they were looked as *crystallized* images of all the activity that was embedded in them.

We suggest that the synchronic analysis of our relationship with technology, no matter how deep, hides profound meanings of this relationship that coheres with the co-evolution of man and his tools. It is then, unavoidable, to revisit our technological past if we want to have an understanding of the present. Let us present a substantial example.

Arithmetic: Ancient Counting Technologies

Evidence of the construction of one–to–one correspondences between arbitrary collections of concrete objects and a *model set* (a template) can already be found between 40000 and 10000 B.C. For instance, hunter-gatherers used bones with marks (tallies). In 1937, a wolf bone dated to about 30000 B.C. was found in Moravia (Flegg, 1983). This reckoning technique (using a one-to-one correspondence) reflects a deeply rooted trait of human cognition. Having a set of stone bits or the marks on a bone as a *modeling set* constitutes, up to our knowledge, the oldest counting technique humans have designed. The modeling set plays, in all cases, an instrumental role for the whole process. In fact, something is crystallized by marking a bone: The *intentional* activity of finding the size of a set of hunted pieces, for instance, or as some authors have argued, the intentional activity of computing time.

The modeling set of marks, plays a role similar to the role played by a stone tool as both mediate an activity, finding the size, and both crystallize that activity. Between 10000 and 8000, B.C. in Mesopotamia, people used sets of pebbles (clay bits) as modeling sets. This technique was inherently limited. If, for instance, we had a collection of twenty pebbles as modeling set then, it would be possible to estimate the size of collections of twenty or less elements. Nevertheless, to deal with larger collections (for instance, of a hundred or more elements), we would need increasingly larger models with evident problems of manipulation and maintenance. And so, the embodiment of the one-to-one technique in the set of pebbles inhibits the extension of it to further realms of experience. It is very plausible that being conscious of these difficulties, humans looked for alternative strategies that led them to the brink of a new technique: the idea that emerged was to replace the elements of the model set with clay pieces of diverse shapes and sizes, *whose numerical value were* *conventional*. Each piece *compacted* the information of a whole former set of simple pebbles —according to its shape and size. The pieces of clay can be seen as embodiments of pre-mathematical symbols. Yet, they lacked rules of transformation that allowed them to constitute a genuine mathematical system.

Much later, the consolidation of the urbanization process (about 4000 B.C.) demanded, accordingly, more complex symbol systems. In fact, the history of complex arithmetic signifiers is almost determined by the occurrence of bullae. These clay envelopes appeared around 3500-3200 B.C. The need to record commercial and astronomic data led to the creation of symbol systems among which mathematical systems seem to be one of the first. The counters that represented different amounts and sorts —according to shape, size, and number— of commodities were put into a bulla which was later sealed. And so, to secure the information contained in a bulla, the shapes of the counters were printed on the bulla outer surface. Along with the merchandise, producers would send a bulla with the counters inside, describing the goods sent. When receiving the shipment, the merchant could verify the integrity of it.

A counter in a bulla *represents* a *contextual* number — for example, the number of sheep in a herd; not an abstract number: there is five of something, but never *just five*. The shape of the counter is impressed in the outer surface of the bulla. The mark on the surface of the bulla *indicates* the counter inside. That is, the mark on the surface keeps an *indexical relation* with the counter inside as its referent. And the counter inside has a *conventional* meaning with respect to amounts and commodities. It must have been evident, after a while, that *counters inside were no longer needed;* impressing them in the outside of the bulla was enough. That decision altered the semiotic status of those external inscriptions. Afterward, instead of impressing the counters against the clay, scribes began using sharp styluses that served *to draw* on the clay the shapes of former counters. From this moment on, the symbolic expression of numerical quantities acquired an infra-structural support that, at its time, led to a new epistemological stage of society. Yet the semiotic contextual constraints, made evident by the simultaneous presence of diverse numerical systems, was an epistemological barrier for the mathematical evolution of the *numerical ideographs*. Eventually, the collection of numerical (and contextual) systems was replaced by one system (Goldstein, La naissance du nombre en Mesopotamie. La Recherche, *L'Univers des Nombres* (hors de serie),1999). That system was the sexagesimal system that also incorporated a new symbolic technique: numerical value according to position. In other words, it was a *positional* system. There is still an obstacle to have a complete numerical system: the presence of zero that is of primordial importance in a positional system to eliminate representational ambiguities. For instance, without zero, how can we distinguish between 12 and 102? We would still need to look for the help of context.

Mathematical objects result from a sequence of crystallization processes that, at a certain level of evolution, has an ostensible social and cultural dimension. As the levels of reference are hierarchical the crystallization process is a kind of recursive process that allows us to state:

Mathematical symbols co-evolve with their mathematical referents and the induced semiotic objectivity makes possible for them to be taken as shared in a community of practice.

In what follows, we should try to articulate some reflections regarding the presence of the computational technologies in mathematical thinking. It is interesting to notice that even if the new technologies are not yet fully integrated within the mathematical universe, their presence will eventually erode the mathematical way of thinking. The blending of mathematical symbol and computers has given way to an *internal mathematical universe* that works as the reference fields to the mathematical signifiers living in the screens of computers. This takes abstraction a large step further.

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USING THEORY TO ADVANCE OUR UNDERSTANDINGS OF STUDENT COGNITIVE DEVELOPMENT

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INTRODUCTION

Over recent years, various theories have arisen to explain and predict cognitive development in mathematics education. Our focus is to raise the debate beyond a simple comparison of detail in different theories to move to use the similarities and differences among theories to address fundamental questions in learning. In particular, a focus of research on fundamental learning cycles provides an empirical basis from which important questions concerning the learning of mathematics can and should be addressed.To assist us with this focus we identify two kinds of theory of cognitive growth:

- **global theories of long-term growth** of the individual, such as the stage-theory of Piaget (e.g., Piaget & Garcia, 1983).
- **local theories of conceptual growth** such as the action-process-object-schema theory of Dubinsky (Czarnocha et al., 1999) or the unistructural-multistructural-relational-extended abstract sequence of SOLO Model (**S**tructure of **O**bserved **L**earning **O**utcomes, Biggs & Collis, 1982, 1991; Pegg, 2003).

Some theories (such as that of Piaget, the SOLO Model, or more broadly, the enactive-iconic-symbolic theory of Bruner, 1966) incorporate both aspects. Others, such as the embodied theory of Lakoff and Nunez (2000) or the situated learning of Lave and Wenger (1990) paint in broader brush-strokes, featuring the underlying biological or social structures involved. A range of global longitudinal theories each begin with physical interaction with the world and, through the use of language and symbols, become increasingly abstract. Table 1 shows four of these theoretical developments.

Piaget Stages	van Hiele Levels (Hoffer,1981)		SOLO Modes	Bruner Modes
Sensori Motor	I	Recognition	Sensori Motor	Enactive
Preoperational	II	Analysis	Ikonic	Iconic
Concrete Operational	III	Ordering	Concrete	Symbolic
Formal Operational	IV	Deduction	Symbolic	
	V	Rigour	Formal	
			Post-formal	

Table 1: Global stages of cognitive development

What stands out from such 'global' perspectives is the gradual biological development of the individual, growing from dependence on sensory perception through physical interaction and on, through the use of language and symbols, to increasingly sophisticated modes of thought. SOLO offers a valuable viewpoint as it explicitly nests each mode within the next, so that an increasing repertoire of more sophisticated modes of operation become available to the learner. At the same time, all modes attained remain available to be used as appropriate. As we go on to discuss fundamental cycles in conceptual learning, we therefore need to take account of the development of modes of thinking available to the individual.

LOCAL CYCLES

Our current focus is on 'local' theories, formulated within a 'global' framework whereby the cycle of learning in a specific conceptual area is related to the overall cognitive structures available to the individual. A recurring theme identified in these theories is a fundamental cycle of growth in the learning of specific concepts, which we frame within broader global theories of individual cognitive growth.

One formulation is found in SOLO. This framework can be considered under the broad descriptor of neo-Piagetian models. It evolved as reaction to observed inadequacies in Piaget's formulations and shares much in common with the ideas of such theorists as Case, Fischer, and Halford.

In particular, SOLO focuses attention upon students' responses rather than their level of thinking or stage of development. It arose, in part, because of the substantial décalage problem associated with Piaget's work when applied to the school learning context, and the identification of a consistency in the structure of responses from large numbers of students across a variety of learning environments in a number of subject and topic areas. While SOLO has its roots in Piaget's epistomelogical tradition, it is based strongly on information-processing theories and the importance of working memory capacity. In addition, familiarity with content and context invariably plays an influential role in determining the response category.

At the 'local' focus SOLO comprises a recurring cycle of three levels referred to as *unistructural, multistructural, and relational* (a UMR cycle). The application of SOLO takes a multiple-cycle form of at least two UMR cycles in each mode where the R level response in one cycle evolves to a new U level response in the next cycle. This not only provides a basis to explore how basic concepts are acquired, but it also provides us with a description of how students react to reality as it presents itself to them. The second cycle then offers the type of development that is most evident and a major focus of primary and secondary education.

Another formulation concerns various theories of process-object encapsulation, in which processes become interiorised and then conceived as mental concepts, which has been variously described as *action*, *process*, *object* (Dubinsky), *interiorization*, *condensation*, *reification* (Sfard) or *procedure*, *process*, *concept* (Gray & Tall).

Theories of 'process-object encapsulation' were formulated at the outset to describe a sequence of cognitive growth. Each of these theories, founded essentially on the ideas of Piaget, saw cognitive growth through actions on existing objects that become interiorized into processes and then encapsulated as mental objects.

Dubinsky described this cycle as part of his APOS theory (action-process-object-schema), although he later asserted that objects could also be formed by encapsulation of schemas as well as encapsulation of processes. Sfard (1991) proposed an operational growth through a cycle she termed interiorization-condensation-reification, which she complemented by a 'structural' growth that focuses on the properties of the reified objects formed in an operational cycle.

Gray and Tall (1994) focused more on the role of *symbols* acting as a pivot, switching from a process (such as addition of two numbers, say 3+4) to a concept (the sum 3+4, which is 7). The entity formed by a symbol and its pivotal link to *pro*cess or con*cept* they named a *procept*. They observed that the growth of procepts occurred often (but not always) through a sequence that they termed procedure-process-procept. In this model a procedure is a sequence of steps carried out by the individual, a process is where a number of procedures (≥0) giving the same input-output are regarded as the same process, and the symbol shared by both becomes process or concept.

The various process-object theories have a spectrum of development from process to object. The process-object theories of Dubinsky and Sfard were mainly based on experiences of students doing more advanced mathematical thinking in late secondary school and at university. For this reason their emphasis is on formal development rather than on earlier acquired forms of thinking such as associated with Piaget's sensori-motor or pre-operational stages. Note too that Sfard's first state is referred to as an 'interiorized process', which is the same name given in Dubinsky's second, however, both see the same main components of the second stage:– that the process is seen as a whole without needing to perform the individual steps.

We now turn to the cycles of development that occur within a range of different theories. These have been developed for differing purposes. The SOLO Model, for instance, is concerned with assessment of performance through observed learning outcomes. Other theories, such as those of Davis (1984), Dubinsky (Czarnocha et al., 1999), Sfard (1991), and Gray and Tall (1994) are concerned with the sequence in which the concepts are constructed by the individual).

SOLO of Biggs & Collis	Davis	APOS of Dubinsky	Gray & Tall
 Unistructural Multistructural Relational Unistructural	 Procedure (VMS) Integrated Process Entity	 Action Process Object Schema	[Base Objects] Procedure Process Procept

Table 2: Local cycles of cognitive development

As can be seen from table 2, there are strong family resemblances between these cycles of development. Note that Davis used the term 'visually moderated sequence' for a step-by-step procedure. Although a deeper analysis of the work of individual authors will reveal discrepancies in detail, there are also insights that arise as a result of comparing one theory with another as assembled in table 3.

<table>
<tr><td>SOLO</td><td>Davis</td><td>APOS</td><td>Gray & Tall</td></tr>
<tr><td rowspan="2">Unistructural</td><td rowspan="3">VMS
Procedure</td><td></td><td>Base Object(s)</td></tr>
<tr><td rowspan="2">Action</td><td rowspan="2">Procedure
[Multi-Procedure]</td></tr>
<tr><td>Multistructural</td></tr>
<tr><td>Relational</td><td>Process</td><td>Process</td><td>Process</td></tr>
<tr><td>Unistructural
(Extended
Abstract)</td><td>Entity</td><td>Object
Schema</td><td>Procept</td></tr>
</table>

Table 3: The fundamental cycle of conceptual construction

CONCLUSION

Our purpose in this brief paper is not so much to attempt to produce a unified theory incorporating these perspectives. Instead, it is to advocate an approach that seeks to understand the meanings implicit in each broad theory and to see where each may shed light on the other, leading to theoretical correspondences and dissonances.

While at first glance there may appear to be irreconcilable differences between the theoretical stances (e.g., van Hiele is concerned with underlying thinking skills and SOLO with observable behaviours), a closer examination can reveal there is much to consider. A synthesis provides a fresh perspective in considering student growth in understanding.

A primary goal of teaching should be to stimulate cognitive development in students. Such development as described by these fundamental learning cycles is not inevitable. Ways to stimulate growth, to assist with the reorganisation of earlier levels need to be explored. Important questions about strategies appropriate for different levels or even if it is true that *all* students pass through all levels in sequence. Research into such questions is sparse. Nevertheless, the notion of fundamental cycles of learning does provide intriguing potential for research.

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TRENDS IN THE EVOLUTION OF MODELS & MODELING PERSPECTIVES ON MATHEMATICAL LEARNING AND PROBLEM SOLVING

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Models and modeling (M&M) research often investigates the nature of understandings and abilities that are needed in order for students to be able to use what they have (presumably) learned in the classroom in "real life" situations beyond school. Nonetheless, M&M perspectives evolved out of research on concept development more than research on problem solving; and, rather than being preoccupied with the kind of word problems emphasized in textbooks and standardized tests, we focus on (simulations of) problem solving "in the wild." Also, we give special attention to the fact that, in a technology-based *age of information*, significant changes are occurring in the kinds of "mathematical thinking" that is coming to be needed in the everyday lives of ordinary people in the 21st century – as well as in the lives of productive people in future-oriented fields that are heavy users of mathematics, science, and technology.

In modern knowledge economies, systems – ranging from communication systems to economic or accounting systems - are among the most important "things" that impact the lives of ordinary people. Some of these systems occur naturally, while others are created by humans. But, in any case, mathematics is useful for making (or making sense of) such systems precisely because mathematics is the study of structure. That is, it is the study of systemic properties of structurally interesting systems.

In future-oriented fields that range from design sciences to life sciences, industry advisors to university programs consistently emphasize that:

> The kind of people we most want to hire are those who are proficient at (a) making sense of complex systems, (b) working within teams of diverse specialists, (c) adapting rapidly to a variety of rapidly evolving conceptual tools, (d) working on multi-staged projects that require planning and collaboration among many levels and types of participants, and (e) developing sharable and re-useable conceptual tools that usually need to draw on a variety of disciplines – and textbook topic areas.

Both of the preceding trends shift attention beyond *mathematics as computation* toward *mathematics as conceptualization, description, and explanation*. But, they also raise the following kinds of questions that lie at the heart of M&M research in mathematics education.

- What is the nature of the most important classes of problem-solving situations where mathematics, science, and technology are needed for success in real life situations beyond school?
- What mathematical constructs or conceptual systems provide the best foundations for success in these situations?
- What does it mean to "understand" these constructs and conceptual systems?
- How do these understandings develop?
- What kinds of experiences facilitate (or retard) development?
- How can people be identified whose exceptional abilities do not fit the narrow and shallow band of abilities emphasized on standardized tests – or even school work?

Related questions are: (a) Why do students who have histories of getting A's on tests and coursework often do not do well beyond school? (b) What is the relationship between the learning of "basic skills" and a variety of different kinds of deeper or higher-order understandings or abilities? (c) Why do problem solving situations that involve collaborators and conceptual tools tend to create as many conceptual difficulties as they eliminate? (d) In what ways is "mathematical thinking" becoming more multi-media - and more contextualized (in the sense that knowledge and abilities are organized around experience as much as around abstractions, and in the sense that relevant ways of thinking usually need to draw on ways for thinking that seldom fall within the scope of a single discipline or textbook topic area). (e) How can instruction and assessment be changed to reflect the fact that, when you recognize the importance of a broader range of understandings and abilities, a broader range of people often emerge as having exceptional potential?

M&M perspectives assume that such questions should be investigated through research, not simply resolved though political processes - such as those that are emphasized when "blue ribbon" panels of experts develop curriculum standards for teaching or testing. Furthermore, we believe that such questions are not likely to be answered through content-independent investigations about *how people learn* or *how people solve problems*, and they are only indirectly about the nature (and/or the development) of humans - or the functioning of human brains. This is because they are about the nature of mathematical and scientific knowledge, and they are about the ways this knowledge is useful in "real life" situations. So, researchers with broad and deep expertise in mathematics and science should play significant roles in collaborating with experts in the learning and cognitive sciences.

Theoretical perspectives for M&M research trace their lineage to modern descendents of Piaget and Vygotsky - but also (and just as significantly) to *American Pragmatists* such as William James, Charles Sanders Peirce, Oliver Wendell Holmes, George Herbert Mead, and John Dewey. And partly for this reason, M&M perspectives reflect "blue collar" approaches to research. That is, we focus on the development of knowledge (and conceptual tools) to inform "real life" decision-making issues – where (a) the criteria for success are not contained within any preconceived theory, (b) productive ways of thinking usually need to draw on more than a single theory, and (c) useful knowledge usually needs to be expressed in the context of conceptual tools that are powerful (for some specific purpose), sharable (with other people), and re-useable (beyond the context in which they were developed). Thus, M&M research often focuses on model-development rather than proceeding too quickly to theory development and hypothesis testing; and, before rushing ahead to try to teach or test various mathematical concepts, processes, beliefs, habits of mind, or components of a productive problem solving personae, we conduct developmental investigations about the nature of what it means to "understand" them.

One way that mathematics educators have investigated questions about what is needed for success beyond school is by observing people "thinking mathematically" in everyday situations. Sometimes, such studies compare "experts" with "novices" who are working in fields such as engineering, agriculture, medicine, or business management - where "mathematical thinking" often is critical for success. Such ethnographic investigations often have been exceedingly productive and illuminating. Nonetheless, from the perspectives of M&M research, they also tend to have some significant shortcomings. For example, we must be skeptical of observations which depend heavily on preconceived notions about <u>where</u> to observe (in grocery stores? carpentry shops? car dealerships? engineering firms? Internet cafés?), <u>whom</u> to observe (street vendors? shoppers? farmers? cooks? engineers? baseball fans?), <u>when</u> to observe (when they're estimating sizes? calculating with numbers? minimizing routes? describing, explaining, or predicting the behaviors of complex systems?), and <u>what</u> to count as "mathematical thinking" (e.g., planning, monitoring, assessing, explaining, justifying steps during multi-step projects, or deciding what information to collect about specific decision-making issues). Consequently, in simple observational studies, close examinations of underlying assumptions often expose unwarranted prejudices about what it means to "think mathematically" - and about the nature of "real life" situations in which mathematics is useful.

A second way to investigate *what's needed for success beyond school* is to use *multi-tier design experiments* (Lesh, 2002) in which (a) students develop models for making senses of mathematical problem solving situations, (b) teachers develop models for creating (and making sense of) students' modeling activities, and (c) researchers develop models for creating (or making sense of) interactions among students, teachers, and relevant learning environments. We sometimes refer to such studies as *evolving expert studies* (Lesh, Kelly & Yoon, in press) because the final products that are produced tend to represent significant extensions or revisions in the thinking of each of the participants who were involved. Such methodologies respect the opinions of diverse groups of stake holders whose opinions should be considered. On the one hand, nobody is considered to have privileged access to the truth – including, in particular, the researchers. All participants (from students to teachers to researchers) are considered to be in the model development business; and, similar principles are assumed to apply to "scientific inquiry" at all levels. So, everybody's ways of thinking are subjected to examination and possible revision.

For the preceding kind of *three-tiered design experiments*, each tier can be thought of as *a longitudinal development study in a conceptually enriched environment*. That is, a goal is to go beyond studies of <u>typical</u> development in <u>natural</u> environments to also focus on <u>induced</u> development within <u>carefully controlled</u> environments. Finally, because the goal of M&M research is to investigate the nature and development of constructs or conceptual systems (rather than investigating and making claims students per se), we often investigate how understandings evolve in the thinking of "problem solvers" who are in fact teams (or other learning communities) rather than being isolated individuals. So, we often compare individuals with groups in somewhat the same manner that other styles of research might compare experts and novices, or gifted students and average ability students.

Investigations from an M&M perspective have led to the growing realization that, in a technology-based age of information, even the everyday lives of ordinary people are increasingly impacted by systems that are complex, dynamic, and continually adapting; and, this is even more true for people in fields that are heavy users of mathematics and technology. Such fields include design sciences such as engineering or architecture, social sciences such as economics or business management, or life sciences such as new hyphenated fields involving bio-technologies or nano-technologies. In such fields, many of the systems that are most important to understand and explain are dynamic (living), self-organizing, and continually adapting.

M&M research is showing that it is possible for average ability students to develop powerful models for describing complex systems that depend on only new uses of elementary mathematical concepts that are accessible to middle school students. However, when we ask *What kind of mathematical understandings and abilities should students master?* attention should shift beyond asking *What kind of computations can they execute correctly?* to also ask *What kind of situations can they describe productively?* ... This observation is the heart of M&M perspectives on learning and problem solving.

Traditionally, problem solving in mathematics education has been defined as getting from givens to goals when the path is not obvious. But, according to M&M perspectives, goal directed activities only become problematic when the "problem solver" (which may consist of more than an isolated individual) needs to develop a more productive way of thinking about the situation (given, goals, and possible solution processes). So, solutions to non-trivial problems tend to involve a series of modeling cycles in which current ways of thinking are iteratively expressed, tested, and revised; and, each modeling cycle tends to involve somewhat different interpretations of givens, goals, and possible solution steps.

Results from M&M research make it clear that average ability students are indeed capable of developing powerful mathematical models and that the constructs and conceptual systems that underlie these models often are more sophisticated than anything that anybody has tried to teach the relevant students in school.

However, the most significant conceptual developments tend to occur when students are challenged to repeatedly express, test, and revise their own current ways thinking - not because they were guided along a narrow conceptual trajectory toward (idealized versions of) their teachers ways of thinking (Lesh & Yoon, 2004). That is, development looks less like progress along a path; and, it looks more like an inverted genetic inheritance tree - where great grandchildren trace their evolution from multiple lineages which develop simultaneously and interactively.

In general, when knowledge develops through modeling processes, the knowledge and conceptual tools that develop are instances of situated cognition. Models are always molded and shaped by the situations in which they are created or modified; and, the understandings that evolve are organized around experience as much as around abstractions. Yet, the models and underlying conceptual systems that evolve often represent generalizable ways of thinking. That is, they are not simply situation-specific knowledge which does not transfer. This is because models (and other conceptual tools) are seldom worthwhile to develop unless they are intended to by powerful (for a specific purpose in a specific situation), re-useable (in other situations), and sharable (with other people).

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ISSUES AND TENDENCIES IN GERMAN MATHEMATICS-DIDACTICS: AN EPOCHAL PERSPECTIVE

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It is a positive sign that an international discussion on theories of mathematics education is taking place especially in the wake of TIMMS and PISA. It is laudable of PME to take the initiative to closely examine specific geographic trends in mathematics education research in comparison with trends that are concurrently occurring (or occurred) elsewhere (as reported in English et al., 2002; Schoenfeld, 1999, 2002). In doing so we can reflect and hypothesize on why certain trends seem to re-occur, sometimes invariantly across time and geographic location. Numerous reviews about the state of German mathematics didactics are available in German (see [1], Hefendehl et al., 2004; Vollrath et al., 2004). However there are no extant attempts to trace and analyze the last hundred years of "mathematics didactic" trends in Germany <u>in comparison to</u> what is happening internationally. This is our modest attempt to fill this void.

Some preliminary remarks on terminology and history: It has become standard practice for researchers writing in English to use the term "Mathematikdidaktik" when referring to mathematics education in Germany. However, there is no real comprehensive English equivalent for the term "Mathematikdidaktik". Neither "didactics" nor "math-education" describes the full flavor and the historical nuances associated with this German word. Even the adjective "German" is imprecise since educational research approaches in Germany splintered in the aftermath of World War II, with different philosophical schools of thought developing in the former East (GDR) and the west (FRG) on research priorities for university educators, until the reunification which occurred in 1990. Currently the 16 states in Germany reveal a rich heterogeneity in the landscape of mathematics teaching, teacher training and research methods, which manifests itself to insiders who microscopically examine the TIMSS- and PISA-results. However the reasons for this heterogeneity remain a mystery to outsiders. Given the page limits we outline in macroscopic terms the historical reasons for this heterogeneity. In doing so we do not differentiate explicitly between the alignment (or misalignment!) of theories preferred by university educators in comparison to practices of mathematics instruction in schools. The mutual dependencies between the two is certainly an interesting research question which brings into focus the system wide effectiveness (or ineffectiveness) of educational research (see for example Burkhardt & Schoenfeld, 2003).

1. **The Pedagogical tradition of mathematics teaching-Mathematics as Educational Value:** Reflections on the processes of mathematics teaching and learning have been a long-standing tradition in Germany. The early proponents of these theories of teaching and learning are recognizable names even for current

researchers. Chief among these early theorists was Adam Reise "the arithmetician" who stressed hand computation as a foundational learning process in mathematics. This emphasis is found in the pedagogical classics of the 19th century written by Johann Friedrich Herbart (1776-1841), Hugo Gaudig (1860-1923), Georg Kerschensteiner (1854-1932) (see Jahnke, 1990; Führer, 1997; Huster, 1981). The influence of this approach echoed itself until the 1960's in the so-called didactics of mathematics teaching in elementary schools to serve as a learning pre-requisite for mathematics in the secondary schools.

2. **Mathematician-Initiators of traditions in didactics research (20th Century*)*:** In the early part of the previous century, mathematicians like Felix Klein (1849-1925) and Hans Freudenthal (1905-1990) (who was incidentally of German origin) became interested in the complexities of teaching and learning processes for mathematics in schools. The occasionally invoked words "Erlangen program" and "mathematization" are the present day legacy of the contributions of Klein and Freudenthal to mathematics education. Klein characterized geometry (and the teaching of it) by focussing on the related group of symmetries to investigate mathematical objects left invariant under this group. The present day emphasis of using functions (or functional thinking) as the conceptual building block for the teaching and learning of algebra and geometry, is reminiscent of a pre-existing (100 year old) Meraner Program. During this time period one also finds a growing mention in studying the psychological development of school children and its relationship to the principles of arithmetic (Behnke, 1950). This trend was instrumental in the shaping of German mathematics curricula in the 20th century with the goal being to expose students to mathematical analysis at the higher levels. The most notable international development in this time period was the founding of the ICMI in 1908, presided by Felix Klein. One of the founding goals of ICMI was to publish mathematics education books, which were accessible to both teachers and their students. We see this as one of the first attempts to "elementarize" (or simplify) higher level mathematics by basing it on a sound scientific (psychological) foundation. Mathematics educators like Lietzmann (1919) claimed that "didactic" principles were needed in tandem with content to offer methodological support to teachers. This approach mutated over the course of the next 50 years well into the 1970's. The over-arching metaphor for mathematics education researchers during this time period was to be a gardener, one who maintains a small mathematical garden analogous to ongoing research in a particular area of mathematics. The focus of research was on analyzing specific content and using this as a basis to elaborate on instructional design (Reichel 1995, Steiner, 1982). This approach is no longer in vogue and is instrumental in creating a schism between mathematicians and "mathematics-didakters," partly analogous to the math wars in the United States.

3. "Genetic" Mathematics Instruction: Ineffectual Visionary Bridges (1960 – 1990): The word "genetic" was used to exemplify an approach to mathematics instruction to prevent the danger of mathematics taught completely via procedures

(Lenné, 1969). Several theorists stressed that mathematics instruction should be focussed on the "genetic" or a natural construction of mathematical objects. This can be viewed as an earlier form of constructivism. This approach to mathematics education did not gather momentum. The word "genetisch" occurs frequently in the didactics research literature until the 1990's.

4. The New Math (1960 – 1975): Parallel to the new math movement occurring in post-Sputnik United States, an analogous reform movement took place in Germany (mostly in the West, but partly adopted by the East, see [1]). A superficial inspection seems to point to a realization of Klein's dream of teaching and learning mathematics by exposing students to its structure. This reform took on the dynamic of polarizing scientists (mathematicians) to work in and with teacher training, the resulting outcome being a lasting influence on mathematics instruction during this time period. Unlike the United States teachers were able to implement a structural approach to mathematics in the classroom. This can be attributed to the fact that during this time period there was no social upheaval in Germany, unlike the U.S where the press for social reform in the classroom (equity and individualized instruction) interfered with this approach to mathematics education. The fact that German "new math" did not survive the tide of time indicates that there was difficulty in implementing it effectively.

5. The birth of didactics as a research discipline (1975): While the new mathematics movement was subject to a host of criticisms, one positive outcome was the founding of the Gesellschaft für Didaktik der Mathematik (German Mathematics Didactics Society), which stresses that mathematics didactics was a science whose concern was to rest the mathematical thinking and learning on a sound theoretical (and empirically verifiable foundation). This was a radical step search for mathematics education research in Germany, one that consciously attempted to move away from the view of a math educator as a part-time mathematician (recall Klein's garden). Needless to say, we could easily write an entire book if we wanted to spell out the ensuing controversy over the definition of this new research discipline in Germany (see Bigalke, 1974; Dress, 1974; Freudenthal, 1974; Griesel, 1974, Laugwitz, 1974; Leuders, 2003; Otte, 1974; Tietz, 1974 Wittmann, 1974; 1992). However, the point to be taken from the founding of this society and a new scientific specialty is that the very debate we have undertaken here, that is, to globally define theories of mathematics education has in fact many localized manifestations such as in Germany.

6. Mathematical Teaching and Learning- A Socialistic and an Individualistic Process (1980 – today): One of the consequences of founding a new discipline of science was the creation of new theories to better explain the phenomenon of mathematical learning. The progress in cognitive science in tandem with interdisciplinary work with social scientists led to the creation of "partial" paradigms about how learning occurs. Bauersfeld's (1988,1995) views of mathematics and mathematical learning as a socio-cultural process within which the individual operates can be viewed as one of the major contributions to theories of mathematics education.

7. An Orientation Crisis - The Conundrums posed by new Technology (1975 – today): Weigand's (1995) work poses the rhetorical question as to whether mathematics instruction is undergoing yet another crisis. The advent of new technologies opened up a new realm of unimagined possibilities for the learner, as well as researchable topics for mathematics educators. The field of mathematics education in Germany oriented itself to address the issues of teaching and learning mathematics with the influx of technology. However the implications of redefining mathematics education, particularly the "*hows*" of mathematics teaching and learning in the face of new technology poses the conundrum of the need to continually re-orient the field, as technology continually evolves (see Noss / Hoyles (1995) for an ongoing global discussion).

8. TIMMS and PISA -The Anti-Climax (1997 – today): The results of TIMMS and PISA brought these seven aforementioned "tendencies" to a collision with mathematics educators and teachers feeling under-appreciated in the wake of the poor results. These assessments also brought mathematicians and politicians back into the debate for framing major policies, which would affect the future of mathematics education in Germany. Mathematics education is now in the midst of new crisis because the results of these assessments painted German educational standing in a poor global light. A detailed statistically sieved inspection of the results indicated that poor scores could be related to factors other than flaws in the mathematics curriculum, and/or its teaching and learning, that is to socioeconomic and cultural variables in a changing modern German society. Thus mathematics education in Germany would now have to adapt to the forces and trends creating havoc in other regions of the globe (see Burton, 2003; Steen, 2001).

Conclusions

Epochal viewpoints: The eight major tendencies that we have highlighted in the 100 years of mathematics education history in Germany reflect trends that have occurred internationally. Each epoch is characterized by an underlying metaphor that shaped the accepted theories of that time period. Felix Klein's view of a mathematics educator was that of a mathematician-gardener tending to all aspects of a specialized domain within mathematics, including its teaching and learning. This shifted to a focus on the structure of modern mathematics itself and partly to the teacher as a "transmitter" of structural mathematics in the 1960's during the New Math period. This was followed by an epoch where the science of mathematics education and the student (finally!) came into focus and brought forth attempts to delineate theories for this new science such as Bauersfeld's socio-cultural theories. New technologies shifted the focus of theories to accommodate how learning occurs in the human-machine interface. Finally TIMMS and PISA brought into focus assessment issues along with societal and political variables that are changing conceptions of mathematics education as we speak. In a sense we have come full circle because we still haven't defined what mathematics didactics is. However, in the search through history for the answer, we have understood the epochal nuances of this interesting term. Perhaps it is time we finally defined it!

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CONCLUDING POINTS

The diversity in the perspectives presented in the six contributions parallel conundrums recently elicited by Tommy Dreyfus at the 4th European Congress in Mathematics Education (Spain, February 2005). In his concluding report about the working group on mathematics education theories, Dreyfus stated that although theories were a vital aspect of mathematics education, they were much too wide of a topic. However the field can take solace from the fact that although contradictions exist, there are also connections and degrees of complementarities among theories. The coordinators of this particular Forum have reached a similar conclusion. Many of the points we make here echo the recommendations of Tommy Dreyfus. Although it is impossible to fully integrate theories, it is certainly possible to bring together researchers from different theoretical backgrounds to consider a given set of data or phenomena and examine the similarities and differences in the ensuing analysis and conclusions. The interaction of different theories can also be studied by applying them to the same empirical study and examining similarities and differences in conclusions. Last but not least, although it is impossible to expect everybody to use the mathematics education “language,” a more modest undertaking would be to encourage researchers to understand one or more perspectives different from their own. This will ensure that the discussion continues as well as creates opportunities for researchers to study fruitful interactions of seemingly different theories. We consider such work vital to help move the field forward.

DISCUSSION GROUPS

DG01: MATHEMATICS AND GENDER: SHOULD THE WORLD STILL CARE?

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Helen Forgasz, Monash University, Australia

In 2001, Gilah Leder discussed in her keynote address at PME in Utrecht that

> gender equity concerns have attracted considerable research attention by (mathematics) educators in many countries, and that over time the body of work on gender and mathematics education has increasingly reflected a greater diversity of inquiry methods used to examine and unpack critical factors. Research reports presented at PME contain only limited evidence of these trends (p. 1-41).

Our goal for this discussion group is to take up the challenge implicit in Gilah Leder's talk and provide a venue for overt attention to this issue within PME. And while attention to issues of equity has shifted its focus away from gender in some countries, gender remains a salient variable of study as evidenced at ICME 10.

Activities

We will begin with brief introductions and a short reading to stimulate discussion. Depending on the size of the group, we may break into small groups to discuss critical questions such as those posed below or others that emerge from the participants. Small and large group discussions will be synthesized into key ideas for continued discussion, possible joint research, or future action.

What are critical issues in your country related to gender?

What is the interaction of gender with other factors such as socioeconomic status, race, or ethnicity? We have been discussing the need for doing research that integrates issues of gender, race, ethnicity, and social class for a number of years and it is still an extant agenda item.

Which groups (or sub-groups) of boys and/or girls may be advantaged or disadvantaged in their mathematics learning?

Who has influence at the state and/or national level on the mathematics curriculum and/or the assessment program? Is gender a factor here?

What does a researcher do when gender is no longer on the agenda? How does one access resources to support questions of continued importance?

What methodological approaches and theoretical framework(s) would enable us to investigate difficult and unresolved issues concerning gender?

DG02: ABSTRACTION IN MATHEMATICS LEARNING

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At a forum in PME-26 (Boero et al., 2002), three approaches to the study of abstraction in mathematics learning were presented. Papers based on one of these, the RBC (Recognizing, Building-With, Constructing) model, have been presented at PME every year since 2001 (see references below).

At PME-28 last year, another view of abstraction was presented by Mitchelmore & White (2004), who argued for a reconsideration of the role of empirical abstraction in the learning of fundamental mathematical ideas.

Short, informal discussions at PME-28 began to explore the similarities and differences between the empirical abstraction model and the RBC model. The aim of the proposed discussion group is to continue and widen the interaction process, with the aim of refining both models and identifying their respective ranges of application. The ultimate aim is to improve learning through the design of learning environments that enable more students to abstract more mathematics.

It will be assumed that participants are already familiar with empirical abstraction and the RBC model. Each session will then focus on the learning of a particular topic (one elementary, one more advanced). In small groups, participants will explore how the two models could help to interpret student-teacher interactions in sample interview protocols. General discussion will then draw inferences about the robustness of each model.

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DG03: RESEARCH BY TEACHERS, RESEARCH WITH TEACHERS

Coordinators: Jarmila Novotná, Charles University, Prague, Czech Republic
Agatha Lebethe, Mathematics Education Primary Programme, South Africa
Gershon Rosen, Western Galilee Regional Comprehensive School for Science and Arts, Israel
Vicki Zack, St. George's School, Montreal, Quebec, Canada

This Discussion Group was introduced at PME 28 as the follow-up from the Plenary Panel *Teachers who navigate between their research and their practice* held at PME 27/PME-NA 25 in Hawai'i in 2003. We invite all who are interested in practitioner research, especially teachers who are (or wish to be) researchers in schools as well as university people who would like to do collaborative research with teachers in schools. Teachers who do research in their classrooms deal intimately with the focal interest of our PME 29 conference, namely *Learners and learning environments*. We submit that as teachers study closely what is going on in their classrooms, they may well come to better understand the mathematics and the children's thinking, and this may in turn affect their practice (Novotná, Lebethe, Rosen & Zack, 2003, p. 85-89).

For the discussions we propose to use as points of departure several points which were raised during the sessions of DG 4 at PME 28 and during follow-up informal discussions:

- The continuum from a reflective practitioner to a teacher-researcher.
- Encouraging increased involvement of teachers in researching their own practice; importance of well-defined responsibility and tasks for teachers and researchers who are engaged in joint research.
- Ways to engage pre-service students in reflective practice and research.
- Sharing of models of research methodology used in pre- and in-service teacher education programs, and seeing whether, and if yes how, they might apply to teacher research.

The questions will provide a general framework for the two Discussion Group sessions. The considerations will be based on a particular example and developed in various directions towards a more general perspective.

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DG04: THOUGHT AND LANGUAGE IN THE CONTEXT OF MATHEMATICS EDUCATION

Coordinators: Jorge Tarcísio da Rocha Falcão (Universidade Federal de Pernambuco, Department of psychology-Brazil), Steve Lerman (London South Bank University, Department of Education – UK), Cristina Frade (Universidade Federal de Minas Gerais – Centro Pedagógico – Brazil), Luciano Meira (Universidade Federal de Pernambuco, Department of psychology-Brazil).

The group will focus on some approaches concerning the theoretical conceptualization of thought and language. The contributions of some authors will necessarily (but not exclusively) be discussed (see references). We will also try to discuss the nature of mathematical activity in this theoretical context. Aims include:

1. To discuss some theoretical contributions concerning the relationship between thought and language in the context of mathematical activities at school.
2. To examine some empirical data (videorecords) concerning students' mathematical activity.
3. To establish connections between points 1 and 2, in order to improve both research and educational practice in mathematical education.

GUIDING QUESTIONS

a) Are thought and language different processes? If so, what are their specific characteristics and developmental pathways? To which extent these processes can be (or cannot be) investigated as detached or independent?

b) What are the relevant consequences of this discussion to mathematical education?

b.1.) In which extent is mathematical competence a discursive competence?

b.2.) What is the theoretical status of non-explicit pragmatic abilities of illiterate mathematical users (e.g., carpenters dealing with geometrical concepts of area and/or perimeter, third world "children of the streets" dealing with money in real business-context)?

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DG05: TOWARDS NEW PERSPECTIVES AND NEW METHODOLOGIES FOR THE USE OF TECHNOLOGY IN MATHEMATICS EDUCATION

Bibi Lins, UNICSUL; Victor Giraldo, UFRJ; Luiz Mariano Carvalho, UERJ; Laurie Edwards, Saint Mary's College

At PME 28 we started the discussion group aiming to initiate a dialogue that moves away from current methods and frameworks to new perspectives and new methodologies for considering the use of technology in mathematical education. Three general questions led the discussion:

1. What perspectives are used to investigate the use of technology in Mathematics Education in different countries?
2. How would new perspectives allow us to re/think the role of users of technology?
3. What new methodologies would enable us to investigate difficult issues concerning teaching and learning situations in microworlds environment?

The first session went as freely as possible for encouraging the participants to speak about their own work, own perspectives and views about Technology: its use and the role of its users. There were about 20 participants who vivid engaged in the discussion while listening to each other's views. We spent most of the session on this discussion, leaving the last five minutes to decide what "we" would be doing about the second session. The "conversation" was very fruitful for all participants as a way of knowing where each of us come from in terms of perspectives and methodologies. This session served as a background to what this discussion group could come to be and what direction it could take.

In the second session, Bibi Lins was asked to present some of the known approaches about Technology and introduced the approach of treating Technology as Text and users as readers from an Anti-Essentialist viewpoint (Lins 2002, Woolgar 1997) to be discussed within the group. The discussion was about four different approaches to Technology: technological determinism, social shaping, actor-network and technology as text.

As it came to be a quite stimulating discussion, the coordinators were strongly asked to carry on the discussion group to the PME 29 and gradually to build up what "we" would like to do and to take from it.

Some of the participants, from Australia, had suggested inviting some school teachers to come along to make it even more interesting and to have the opportunity of sharing the teachers' views about Technology apart from the researchers and mathematics educators'.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 209. Melbourne: PME.

INDIGENOUS COMMUNITIES AND MATHEMATICS EDUCATION: RESEARCH ISSUES AND FINDINGS

Annette R Baturo — Queensland University of Technology, Australia

Miriam Amit — Ben-Gurion University of the Negev, Israel

Hsiu-fei Sophie Lee — National Taitung University, Taiwan

This new Discussion Group grew from a paper delivered at PME28 which focused on the issues surrounding research in mathematics education in rural and remote Queensland Indigenous communities. The discussion that followed indicated that researchers in Indigenous mathematics education in other countries are also challenged by the need to develop teaching and learning practices that will better redress the culturally-shared under-performance of Indigenous students when compared with non-Indigenous students' performance. Another major issue to emerge was the ethics of Indigenous research being undertaken by non-Indigenous researchers and the subsequent validity of findings.

The aim of the Discussion Group is to build a community of PME members from around the world who have researched Indigenous mathematics education (or who would like to undertake research in the field but are unsure of the protocols involved) in order to enhance mathematics outcomes and refine research methodologies appropriate for Indigenous communities. The two sessions will provide an opportunity for the coordinators and other researchers to outline their research and the findings that appear to be emerging from these studies. However, the major focus of the sessions will be to examine implicit assumptions that may be unwitting barriers to research outcomes that are beneficial to Indigenous communities. For example: *Do Indigenous people share many researchers' imperatives with regard to the efficacy of high mathematics performance?*

Research in Indigenous mathematics education has complexities that go beyond that of mainstream mathematics education. Smith (1999) argues that much past research has served colonial oppression rather than empowered Indigenous communities. She argues that research, particularly by non-Indigenous researchers, should focus on improving the capacity and life chances of Indigenous peoples. Such research should be community driven, collaboratively planned, executed and analysed – that is, involve real power-sharing between the researcher and the researched. This Discussion Group would like to address questions as to how such as: *Who are the Indigenous? Is this a pejorative label? How can research findings be transformed to practice?* It is hoped that this Discussion Group can collectively plan a way to move forward with respect to further research in Indigenous mathematics education both within and across countries.

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2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 210. Melbourne: PME.

DG07: FACILITATING TEACHER CHANGE

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University of Turku, Finland and La Trobe University, Australia

The intention of both pre-service and in-service teacher education as well as that of many interventions in schools is to promote some kind of change in teachers. This change can be an increase in knowledge and skill, but also it can be changes in the (student) teachers' emotional disposition, beliefs or classroom actions. Various case studies suggest that it is possible to influence knowledge, attitudes and/or practices of (student) teachers and many educators have developed their own techniques for changing (student) teachers. This discussion group will consider the nature of such changes and processes for measuring and reporting on such changes.

We can distinguish, for example, the following types of approach to facilitating teacher change:

- Professional development, where the initiative for, and the direction of, change comes from teachers and the educators' task is to facilitate this process.
- A 'therapeutic' approach, where the intention is to facilitate (student) teachers in addressing mathematics anxiety or other attitudes, or their beliefs about the nature of mathematics, the ways people learn mathematics, or the ways mathematics can be taught.
- Structural change, where the aim is to consider the school structures in order to enable more sustained development in the community.

There are several practical problems in facilitating such changes, especially if changes require a radical conceptual change (e.g. in teaching philosophy) or a change in psychologically central parts of the affective domain (e.g. identity). There are also ethical questions about the appropriateness of imposing a change that has not been initiated by the (student) teachers themselves. There are also methodological considerations about ways of measuring and reporting on changes, recognising that self report, especially after some intervention, may be unreliable.

We invite people to share their own experiences of and views about facilitating and researching teacher change.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 211. Melbourne: PME.

DG08: EMBODIMENT IN MATHEMATICS: METAPHORS AND GESTURES

Laurie Edwards, St. Mary's College of California, USA
Chris Rasmussen, San Diego State University, USA
Ornella Robutti, Univerity of Torino, Italy
Janete Bolite Frant, PUC, Sao Paolo, Brazil

The purpose of the Working Session is to continue the study of the role of cognitive processes in mathematical learning, thinking, teaching and communication, deepening our understanding about meaning production in mathematics education by focusing on theories of embodiment, gesture and language. Starting from the framework that considers cognition to be grounded in physical experience, the Working Session will examine how processes such as conceptual metaphor and conceptual blends, drawn from the field of cognitive linguistics, contribute to the construction of mathematical ideas (Lakoff & Núñez, 2000; Fauconnier & Turner, 2002). The session will also take semiotic and psychological views on language and gesture and their roles in teaching, learning and thinking about mathematics (McNeill, 1992, 2000; Goldin-Meadow, 2003).

Depending on the interests of the participants, the Session will consider questions including the following:

- How do gestures relate to speech during social interaction?
- How are gestures meaningful in teaching situations?
- When does gesture reveal thoughts that are not expressed in speech?
- What are the relationships among conceptual metaphors and blends, gesture and language?
- How can cognitive linguistics and semiotics help in understanding mathematics learning and improving mathematics teaching?

The structure will include an introductory review of basic concepts, followed by sharing of data or problems to be jointly analyzed within smaller groups, concluding with a discussion of progress made in understanding and synthesizing the topics of the session.

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DG09: DEVELOPING ALGEBRA REASONING IN THE EARLY GRADES (K-8): THE EARLY ALGEBRA WORKING GROUP

Coordinators: Elizabeth Warren & Tom Cooper

Australian Catholic University & Queensland University of Technology

The Early Algebra Discussion Group's focus is on investigating and describing what we construe as the possible geneses of algebraic reasoning in young children, and in developing and investigating ways to enhance that reasoning through innovative instruction, applications of appropriate technology and professional development for teachers. The EADG was formed in response to a call form the International Commission on Mathematical Instruction (ICMI) to hold a study conference on "The Future of the Teaching and Learning of Algebra" in December, 2001 in Melbourne. Following that initial conference, the group has conducted working session at PME 27/PME-25 meeting in Hawaii, 2003 and PME28 meeting in Bergen, 2004.

PLANNED ACTIVITIES FOR PME29

We plan to hold two 90 minutes sessions. While research into children's capacity for early algebraic thinking began almost four decades ago, it has, until recently, had little impact on the mainstream research, which in the area of algebraic thinking was largely focused on the introduction of algebra in secondary or middle school.

The first session reviews the research that has occurred in this area in the last 3 years. Researchers in Early Algebraic Reasoning will present a brief summary of their research together with examples of different approaches for fostering algebraic reasoning, the key transitions in developing understanding for both teachers and young children, and the cognitive obstacles that both teachers and young children experience. Participants will be encouraged to engage in discussions about

1. What constitutes algebraic reasoning in the elementary classroom? What do we know about what young students can do algebraically?
2. What do we know about Teacher's Knowledge with regard to early algebraic reasoning?
3. What do we know about how early algebra impacts on student learning in secondary mathematics? What needs further research?

The second session specially focuses on research with respect to patterning. There appears to be very limited literature on patterning per se. But commonly researchers have used patterning ability as an indicator of readiness for other mathematical ideas or as a precursor to reasoning. The following questions will be used to guide the discussion:

1. How does an ability to pattern support mathematical understanding?
2. What research has specifically occurred in patterning per se? What needs further research?

WORKING SESSIONS

WS01: TEACHING AND LEARNING MATHEMATICS IN MULTILINGUAL CLASSROOMS

Mamokgethi Setati, University of the Witwatersrand, South Africa

Anjum Halai, Aga Khan University, Pakistan

Richard Barwell, University of Bristol, UK

Multilingualism is a widespread feature of mathematics classrooms around the world. In particular, for many learners the main language used in their mathematics lessons is a language they are in the process of learning. Research on mathematics education in such classrooms has generally argued that learners' home languages should play a role as learners develop proficiency in the main classroom language. What does this mean for the selection and design of tasks for use in multilingual classrooms? What kinds of tasks are relevant for use in multilingual mathematics classrooms in which learners learn mathematics in a language that is not their home language? Selection and design of tasks for learners to work on is an important activity that all teachers engage in every day. The tasks that learners work on structure their experiences of mathematics and are central in their mathematical development. The aim of this working group is to develop possible criteria for the selection and design of tasks that are appropriate for use in multilingual mathematics classrooms.

ACTIVITIES

The two working sessions will be devoted to sharing, designing, doing, refining and critiquing tasks for use in multilingual mathematics classrooms, as well as developing possible criteria for the selection and design of such tasks. In the first session, we invite participants to work on selected mathematics tasks. We will then invite participants to reflect on the appropriacy of the tasks for learners who learn mathematics in a language that is not their home language.

In the second session, we invite participants to modify selected items or design tasks or activities for a mathematics class from a multilingual context with which they are familiar. We will then reflect on the appropriacy of the tasks for multilingual learners. From these discussions we will develop possible criteria for the selection and design of tasks that are suitable for learners in multilingual mathematics classrooms.

WS02: EXAMINING THESES

Kath Hart
University of Nottingham

Anne Berit Fuglestad
Agder University College

Many members of PME are involved in the supervision of students studying for higher degrees. Additionally they act as examiners of the theses that are usually needed for successful completion. We have had a Discussion Group on the topic 'Examining Theses' for a few years. In these discussions we have heard of situations in various universities and the advice that is given to examiners. We have started to compile a book list of recommended texts and we have aired opinions on what are legitimate comments for an examiner to make. The expectations of students have been particularly interesting.

We now wish to use these two working sessions to (a) discuss, design and write an article for Educational Studies in Mathematics on the topic and (b) produce some guidelines which might help students and new examiners.

SHORT ORAL COMMUNICATIONS

MATHEMATICAL KNOWLEDGE CONSTRUCTION: RECOGNIZING STUDENTS' STRUGGLE

Roselainy Abdul Rahman, Yudariah Mohd. Yusof

Universiti Teknologi Malaysia (UTM)

J. H. Mason

Open University, UK

We would like to share some findings of an on-going research on improving students' mathematical learning through reforming classroom practice. The sample of the study is a group of engineering undergraduates at UTM studying Calculus of multivariable functions. The mathematics curriculum is mainly taught as service subjects. It is our belief that the most important attribute that students can carry over to their core study area is the awareness of mathematical processes and problem solving skills. Our presentation will describe the struggle students displayed in trying to make sense and understand the mathematics taught. We could see that part of the struggle is due to existing difficulties the students had such as in working with multiple representations, coordinating procedures, and difficulties in recalling prior mathematical knowledge (Tall & Razali, 1993; Mohd Yusof, Y. & Tall, 1994; Khyasuddeen et al., 1995). We will discuss how we supported our students in enhancing their mathematical understanding through making the mathematical processes and thinking explicit. We had adapted and extended existing mathematical activities and tasks to invoke students' use of their own mathematical powers, assist them in developing these powers further as well acknowledge and address their struggle and difficulties. The pedagogical strategies that we used were devised based on the work of Mason, Burton and Stacey (1985) and Watson and Mason (1998). Excerpts of some students' experience and work will be shown.

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FIGURAL INTERPRETATION OF STRAIGHT LINES THROUGH IDENTIFICATION, CONSTRUCTION AND DESCRIPTION FOCUSSED ON SLOPE AND y-INTERCEPT FEATURES

Dra. Claudia Acuña, DME Cinvestav-IPN

To investigate how high school students use the figural elements, we proposed some tasks of construction of straight lines, to identify of equal or different straight lines in a coordinate system, and to explain conditions under which two different straight lines would be equal. To solve the tasks student need the ability to recognise figural landmarks on the graph, that is, the figural shape of slope is visually associated to the position between two straight lines, this is, how the straight line goes down or up from left to right related with x-axis; meanwhile the y-intercept is a point shared by the given straight line and y-axis. These figural shapes are enough to decide at first glance if two straight lines are the same, but is not enough to recognize slope and y-intercept to accept them as a visual criteria, tasks such as construction, identification or explanation of figural conditions. Our practices show that there are some additional ideas that imposed a different way to see the equality among straight lines. Some additional problems are:

- The Euclidean idea about free straight lines without restrictions can be an obstacle in the interpretation of the figural aspects of the analytical straight line.
- Other kind of obstacles are related to the treatment of the graph as a drawing or as a figure. In this case, students add irrelevant properties taken from the present representation that they even treat them like physical objects.
- Although Gestalt relation is an important aspect of the shape on the plane, students frequently omit it, that is, they joint both figures (grounded and form) in only one; for example, a straight line on the plane looks like a triangle when we join the straight line and axes.
- Most students prefer a good gestalt to graphical composition: although many students used slope and y-intercept as criteria to decide if two straight lines are equal, they base their explanations and constructions on prototypes.

Three different tasks give us three points of observation about figural activity of the straight line. The results of our observation were: About construction the students can achieve a good figural criterion at a global level that allows them to easily obtain an adequate representation. The identification requires focusing on the figural criteria in order to do the adequate election; in this case, the slope and the y-intercept are useful as validation tools for the figural aspect of graphics. Finally, in the task related to the explanation of changes on a straight line that match another one, the figural criterion should be exhibited as a validation tool, but in many cases students avoid this figural criterion in the explanation of changes, and get involved in almost empirical processes based on a natural language, thus, this task has more problems for obtaining right solutions.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 222. Melbourne: PME.

THINKING MATHEMATICALLY: PERSONAL JOURNEY IN THE MODELING OF A CLINICAL WASTE INCINERATION PROCESS

Sabariah Baharun, Yudariah Mohd Yusof, Tahir Ahmad & Khairil Annuar Arshad

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This paper will share a part of a mathematical journey of the first author in an on-going mathematical research. The main objectives of the research were to model the clinical waste incineration process mathematically and to find suitable solutions to this mathematical model. In this presentation, we will discuss an episode of the experience that we thought to be the most challenging and crucial phase in the problem solving process.

The research work was motivated by the results of the Königsberg bridge problem solved by Euler, solutions of linear equations by Kirchhoff and changes in differential calculus considered by Cayley (Harary, 1969). Comparing and contrasting these applications among others has initiated us to hypothesize that the relationship could be presented as a graph. Thus, the research work begins with the construction of a graphical model to represent the flow of the variables in the incinerator plant (Sabariah et al. 2002a; 2002b). However, the graphical model was found to be an inadequate representation of the phenomenon due to the dynamical nature of the process. It was here that the challenge begins. We adopted Mason, Burton & Stacey's (1982) thinking strategies in our mathematical journey. Reflecting and extending the problem together with long periods of mulling for new insights has led us to continue with the problem solving. Appropriate rubrics, questions and prompts that were used to trigger and to reveal the thinking processes that had helped us to proceed and to get out of the 'STUCK!' situation will be highlighted. Some of the mathematical outcome will also be illustrated.

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THE FRUITFUL SYNERGY OF PAPER & PENCIL AND CABRI GÉOMÈTRE: A CASE STUDY

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It is widely reported in literature that the introduction of new technologies may change the way of teaching and learning mathematics. Along this stream, several studies deal with the potentialities of new technologies and show how the use of computational environments in teaching can improve students' understanding of mathematics.

This paper focuses on the interaction between two different learning environments (paper & pencil and Cabri) as emerged by the analysis of a teaching experiment carried out in junior secondary school. In the experiment we planned the alternation and integration of the two environments, as classroom culture and conditions allow. A fruitful synergy of the environments emerges along the whole research study, particularly in an episode concerning the production and validation of a conjecture. The activity of two students is presented and discussed.

It is interesting to reflect on the dynamics that take place in the environments and on the role played by drawings and measures. We focus on drawings and measures since they are the source of the production and validation of the conjecture. Furthermore, they represent two "actors" with different characteristics in the two environments: in paper & pencil, the drawing is static, and in Cabri the figure is dynamic; as regards measure, in paper & pencil the process may encompass mistakes, whereas Cabri gives immediate access to a series of measures.

In the first phase of the teaching experiment (i.e. activity in paper & pencil), students' perception, supported by the drawing in paper & pencil, causes an ascending process (formulation of a conjecture); numbers (and calculation) support the descending process, showing that the conjecture cannot be validated. In the second phase (i.e. activity with Cabri), number at first guide the dragging, after they support the production of the right conjecture (ascending process). During this phase, the measures in Cabri allow the students to grasp the relationship existing between the areas of two squares, as required by the problem. Measures also give a first validation of the conjecture (descending process).

We observe that figures and numbers have different value and status according to the environment where action is set; they also have a different role in producing and supporting a conjecture. The study of this role is an interesting issue, worth of development in further research.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 224. Melbourne: PME.

DEVELOPMENT OF A RATIONALE IN A US TEXT AND IN SINGAPORE'S SCHOOL MATHEMATICS TEXTS

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The presentation of several standard procedures and formulas in a US 6th grade mathematics textbook and in the texts used in schools in Singapore were examined to determine whether foundational assumptions and definitions were clear and consistent, whether a rationale for the procedure or formula was developed in the text, and how many and what types of problems or activities were provided to help students develop understanding of a rationale for the procedure or formula. Major differences were found between the US text and the texts used in Singapore.

Current reform efforts in mathematics education focus on sense-making, reasoning, and proof. Since curriculum materials are an important factor in classroom instruction, recent research aims to investigate the opportunities that curriculum materials provide for children to engage in reasoning and proving (Stylianides & Silver, 2004). One function of reasoning and proof is to provide a rationale for the procedures and formulas used in solving mathematics problems. Results from the TIMSS 1999 Video Study (Hiebert et al., 2003) indicate that the development of a rationale may be weaker in mathematics lessons in the US than in countries in which children scored higher on the TIMSS assessment.

This study investigated the treatment of several standard procedures and formulas in a traditional US 6th grade text and in the texts used in Singapore. In the US 6th grade textbook, foundational assumptions and definitions were sometimes not clearly specified or were not consistent, unlike in the Singaporean texts. For example, three different definitions were implicitly used in discussing the meaning of fraction multiplication in the US text, but none were tied to each other in any way. Rationales for procedures and formulas were sometimes not given in the US text but were always found in the Singaporean texts. When they were given, rationales were not as fully developed in the US text as in the Singaporean texts.

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HOW CHILDREN SOLVE DIVISION PROBLEMS AND DEAL WITH REMAINDERS

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This study is part of a greater project that investigated factors that affect division problem solving. The project involved 128 children (aged from 8 to 14 years old) who were asked to solve 16 division problems that varied in meaning for division (partition and quota), in symbolic representation used (tokens, pencil and paper, oral representation or use of calculator), and in size of the remainder (small or large).

The research is based on the same theoretical references and similar methodological approach of previous studies (Borba & Nunes, 2002; Selva, 1995). It is an experimental study based on the theory of conceptual fields (Vergnaud, 1982).

It was analysed how children attending two Brazilian state schools (mean age: 11 years and 11 months) solved division problems with remainder by using pencil and paper. It was observed the representations used, the success in the usage of these symbolic representations and in dealing with the remainder of the division problems.

Most of the children (88%) were able to solve correctly the problems posed and more than half (69%) used the conventional division algorithm. The remaining students used heuristics or drawings (pictorial or similar in form to the objects mentioned in the problems) and most of these were also successful in finding the quotient and the remainder. However, only in 18% of the problems were the remainders treated correctly, splitting remainders in partition problems and increasing the quotient in quota problems, in order to exhaust the total quantities mentioned in the problems.

Division problems need to be discussed thoroughly with children for them to present meaningful answers. Division must be taught related to the study of rational numbers so children can understand what must be done to the remainder of division problems.

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STUDENT MATHEMATICAL TALK: A CASE STUDY IN ALGEBRA AND PHYSICS

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The aim of the study is to investigate the ways of communicating used by students in mathematics and physics. Language is a critical and potentially overlooked component of classroom culture and the teaching and learning of mathematics. By attending to the ways we talk, we can come to understand what we think and believe. Therefore, discourse, observable dynamic acts of communication in social settings, reflects one's thinking and beliefs about the content of that discourse (Sfard, 2001).

The study design is ethnographic and qualitative. A case study methodology was employed. Data collected include daily observations of algebra and physics classes, observation of groups of students working mathematics and physics, and group student interviews. The length of the study was one semester. Participants were students in one algebra class with an enrollment of 31 students and an introductory physics class with an enrollment of 27 students. The same teacher taught both class sections. The main characteristics of students' talk in the two classes were identified by an iterative analysis process.

Language genres, "talking science" and "talking mathematics" have been identified by several researchers (Chapman, 1997; Lemke, 1982). Student talk, therefore, was categorized in to one of two main types: algebra talk and physics talk. Comparisons were then made of these student talk characteristics.

In algebra talk, student utterances were of two types: tutoring and group problem solving. In physics the students were more likely to be working together to solve the problem and their questions were focused on the underlying concepts and the mathematical calculations. In both student physics talk and student algebra talk, the students' utterances were short, often incomplete, and co-constructed among the group.

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ETHICAL CONSIDERATIONS IN A MATHEMATICS TEACHER EDUCATION CLASSROOM.

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This short oral will be based on research conducted with a group of students taking a one-year postgraduate course to qualify them to teach mathematics in secondary schools. The two key theoretical underpinnings of the work focus on the issues of listening and the setting up of a community of practice. The issue of listening draws on the work of Davis (1996), who focuses his main attention on developing a different form of listening from the common types of evaluative and interpretive listening – that of hermeneutic listening. Cotton (2002) draws on Lave and Wenger (1991) to examine schools and classrooms as communities of practice. In this way, students in mathematics classrooms engage with each other in practice and develop a sense of self in relation to that community of practice. For some students there is a greater synergy and sense of belonging as they fit in with the group and the teacher's expectations of the class, whereas for others, there is a sense of rejection and little sense of identity within the communities of practice. For those students for whom there is little sense of belonging and a lack of sense of identity, there is greater danger of exclusion from that community of practice.

This report focuses on one particular activity where the class was split randomly into five groups, each containing three members. After introducing the three different levels of listening (evaluative, interpretive and hermeneutic as outlined above) in order to emphasise the importance of listening as a tool for working with others, the student teachers were given the task to work on the Painted Cubes problem as a learning community. They were told that the group's focus should be on the process they developed as they tackled the problem rather than the solution obtained. As an assignment they were asked 'to describe the way in which their community came together and the contributions that the various members made to the experience'. The paper discusses some of the important results and raises the question as to the extent to which mathematics teacher educators need to raise issues of ethical know-how as they emerge in teaching sessions with pre-service mathematics teaching students.

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CECI N'EST PAS UN "CIRCLE"

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There is a famous painting by Magritte. It depicts a smoker's pipe with the caption "Ceci n'est pas une pipe". The joke was that it was not a pipe; it was a painting of a pipe. The painting has fuelled many discussions about the attachment of signifiers to signifieds: how exactly do symbols and words represent an object? As soon as we enter the domain of language we inevitably move to some sort of ideological frame that, in turn, brings with it a host of filters that condition our understanding of the material we are examining. This paper is concerned with the perception of mathematical concepts and seeks to explore some of the linguistic filters and socio-cultural factors which influence human understanding of such concepts.

The study (presented in full elsewhere, Bradford and Brown, 2005; Atkinson, Brown and England, in press) reports on a teacher's practitioner research which took place at different times within successive modes of immersion in linguistic domains, that she sought to observe, understand, participate within (or resist) and transform through her participation. She recorded successive perspectives on successive actions in her work in a Ugandan school with a focus on how the term "circle" was seen from alternative cultural perspectives. Yet in the research process it was the writing generated by her that provided anchorage to her thoughts, but only in the limited sort of way in which the word "circle" served as an anchor for more mathematically oriented discourse. The word itself was more stable than the way it held meaning. Similarly, the writings simultaneously sought to explain the past and shape the future, but in the meantime provided orientation and a conceptual space for examining how the mathematical terms were being used. Yet each component of this writing was constantly in the process of having its status amongst its neighbours unsettled. The teacher was involved in the production of stories that had a limited shelf life as "stories in their own right". The reflective writings and the mathematical words they contained were historically and ideologically defined entities. The passage of time, however, provided the distance necessary to see the previous frame as being outside of oneself. And of how it had reflexively encapsulated the teacher, the learner and the mathematical objects that they had sought to share.

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DEVELOPMENT AND EVALUATION OF A CONCEPT FOR PROBLEM-SOLVING AND SELF-GUIDED LEARNING IN MATHS LESSONS

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Knowing how to solve problems is what should be 'kept' from maths lessons, it is part of what constitutes the general educational value of math lessons. Problem-solving contributes to an adequate image of mathematics and covers both a specialist and a cross-curricular component.

From 2002 to 2004 we worked on a material-based teaching concept which is intended for teachers as a guideline on how to learn problem-solving in conjunction with self-guidance. Central part of the concept to integrate problem-solving is to get the pupils used to structured proceeding in the mathematical problem-solving process and to help them find individual problem-solving strategies. Problem-solving elements and strategies of self-regulation were then punctually integrated into successive learning phases of all subjects treated in the maths lessons.

For the development of a teachers training programme a multi-step approach over three project phases, each with another main emphasis, was selected:

At first a comprehensive and practicable teaching concept for the systematic integration of the training contents and intention into the regular maths lessons had to be found, including the development of evaluation tools for the use in subsequent project phases. The Repertory Grid technique was adapted for the registration of subjective ideas on maths problems of the teachers.

The next step was to establish a training programme for teachers in the first training phase. This project was run in a university course at the Technical University Darmstadt and evaluated. The training programmes were then tested in the second phase of the teacher training.

In order to integrate the developed teaching concept, which proved to be acceptable and practicable, in regular maths lessons, concepts for continual teacher training with different support systems have been tested in the present project phase since June 2004.

Evaluation measures were used in each of the three project phases to analyse the variables "acceptance/identification with the concept" and "skills in the application of the concept" as adopted by the teachers.

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MATHEMATICAL PROBLEM-SOLVING IN A SPREADSHEET ENVIRONMENT: IN WHAT WAYS MIGHT STUDENT DISCOURSE INFLUENCE UNDERSTANDING.

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This study is concerned with how investigating mathematical activities in a spreadsheet environment might filter understanding, with particular consideration of variance in discourse. What is the nature of the learning process in this environment, and how might this particular pedagogical medium shape children's approach to the activities?

A group of twenty ten-year-old children developed individual and collaborative approaches to problem solving. Part of the facilitation of this process was their engagement in investigating mathematical situations with spreadsheets. While there was an initial instructional process within a mathematical problem-solving context, one aim of the study was the use of the spreadsheet as an investigative tool, and the implications for understanding this evoked.

Central to the study is the place of discourse. This also involves theoretical perspectives such as phenomenology and its relationship with mathematics education (e.g. Brown, 2001), and the social-constructivist viewpoint (e.g. Cobb, 1994). Hence an ethnographic, interpretive methodology underpinned the research. The children were observed, their conversations recorded, and they were interviewed, both in groups and individually. Comparisons with pencil and paper methods to investigating were also analysed, and attitudinal surveys undertaken. This data provided some triangulation, and enabled a more fulsome picture to emerge.

Preliminary analysis showed interesting insights into the way the participants familiarised themselves with the investigations, and how investigating with a spreadsheet led to particular discourse, and approaches to investigation. Typically, participants proceeded by entering formulas to generate organised tables of data. These structured tables often led, through discussion, to the resetting of investigational sub-goals, and further exploration. Participants also commented that the seemingly unlimited space and the speed of response to inputted data were other aspects that affected their approach. How this might shape the children's understanding, and if distinctive, what might get lost, are areas of on-going examination.

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THE VALUE OF PLAY IN MATHEMATICS LEARNING IN THE MIDDLE YEARS

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The value of play has been well established in the early years of schooling, however in the years that follow, a transmission approach where the learner is 'drilled' in mathematical concepts and processes often dominates the curriculum. Mathematical play provides an alternative to the transmission model by recognizing the need for pedagogy where sensory-motor experiences, metalanguage and metacognition are employed to support learners in the transition from concrete to abstract. This requires an epistemology grounded in the constructivist approach with open-ended inquiry.

The study focuses on two main goals: to describe activities that constitute 'playful learning' in the middle years and to analyse and explain the elements of play that enhance student engagement in learning and contribute to deep conceptual development. The focus of this research is to understand from a student's perspective the value of play activities in enhancing mathematical understanding.

Within the literature on mathematical play, a clear definition of 'play' is difficult to find. Explanations and exemplars of mathematical play focus on the objects of the learning context, encompass an awareness of interactive cognitive engagement and address the links between affect and learning. Mathematical play is interactive and involves social discourse and domain specific communication. The perception of play activities as pre-abstract is in fact a misrepresentation of the application of sensory-motor stimuli and cuing using visual and kinaesthetic representation.

As the intent of the research was to document and analyse students' reflections on the value of play, a retroductive approach was adopted. The research was a case study of a single primary level class, where students had already been engaged in 'play-based learning'. The 27 students in the class ranged from 9 years to 12 years. Students were observed over a ten week period. At the end of each weekly cycle of activities, students were engaged in class conferences, where they described 'play' and made comments or written reflections on how well the activities supported their learning.

The results of the study indicate that the students believe play activities in mathematics engage all students at their level of understanding. Play activities can be uni-conceptual or multi-conceptual requiring students to make links to other mathematical concepts. They are challenging and diverse, not repetitive. Both 'cognitive conflict' and 'cognitive challenge' were identified through the study as features of play activities that enhance mathematical understanding and application. Play activities allow a continuum from concrete to abstract that engages all students. The study noted that interactive play created a supportive environment in which there was no failure. The responses from the students in the study were overwhelmingly in favour of play activities as an effective learning context.

THE POTENTIAL OF CAS TO PROMOTE CHANGES IN TEACHERS' CONCEPTIONS AND PRACTICES

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This presentation reports a four year study on the potential of computer algebra systems as a vehicle to promote changes in lower secondary school mathematics teachers' conceptions and practices. In 1999, the Mexican Ministry of Education equipped one hundred lower secondary schools spread out in the country with TI92 calculators, such that each student in the school had individual access to the machine in the mathematics classroom at least twice a week.

Method: The categories proposed by Franke et al (1997) were used. These categories allow us to distinguish four levels of teachers' performance and provided a referent to follow up the evolution of the teachers throughout the field work. At the beginning of the project the teachers answered an initial questionnaire inquiring about their previous teaching experience, their teaching method(s) and their professional background. In order to complement these data, an individual interview was administered each year of the project to 30 teachers chosen out from 800 taking part in the study. The teachers were accompanied during three years by professional instructors who worked with them in their respective schools four hours on Friday and Saturday every six weeks. The training program was outlined by teaching materials especially designed for this project and focused on using the calculator as a cognitive tool and discussing ways to define the teacher's and students' role in the classroom (http://sec21.ilce.edu.mx/matematicas/calculadoras/). All work sessions with the teachers were videotaped and used as a data source for the present study.

Results: The chart below summarizes the changes that were registered in the 30 teachers throughout the three years of the study. The evidence provided by the field work strongly indicates that these changes occurred due to the use of CAS.

	Initial profile	1st year end	2nd year end	3rd year end
Level 1	25	22	4	2
Level 2	5	7	21	8
Level 3	0	1	5	10
Level 4	0	0	0	10

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PRE-SERVICE TEACHERS' SELF-EFFICACY TOWARD ELEMENTARY MATHEMATICS AND SCIENCE

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Beginning with research in the 1970s, teacher efficacy was first conceptualized as teachers' general capacity to influence student performance. Since then, the concept of teacher's sense of efficacy has developed continuously and currently is discussed relevant to Albert Bandura's (1977) theory of self-efficacy, indicating the significance of teachers' beliefs in their own capabilities in relation to the effects of student learning and achievement. Thus, this study was to compare the difference of their sense of efficacy, including two cognitive dimensions, personal teaching efficacy and teaching outcome expectancy, after receiving various long-term programs of teacher training. Two teachers' self-efficacy belief instruments were used for data collection from 340 senior students in ten departments of National Taichung Teachers College, Taiwan.

Both pre-service teachers' sense of efficacy toward mathematics and science were significantly different among these ten programs. Further, both groups of students from Department of Mathematics Education and Department of Science Education had more confidence in their own teaching abilities than other students who did not specialize in either mathematics or science, as well as in providing efficient teaching in the classroom. Moreover, statistically significant relationships were found between efficacy ratings toward mathematics and science as well as all subscales.

In summary, the traditional teacher preparation program designs, oriented in cultivating elementary generalists, are inadequate for accomplishing the requirement of having qualified teachers in every classroom and for every subject area. As students have diverse needs and distinct characteristics, it is truly essential that specialized teachers exist for every subject area in every school. To enhance prospective teachers' sense of efficacy toward mathematics and science, all faculty members of teacher preparation programs should rethink the program design, the curriculum structure, and the content provided and the pedagogy used in preparing them to teach mathematics and science. Further, even though more preparations in these two subject areas are no guarantee of higher quality of pre-service teachers and their better understandings of their subjects, insufficient preparation will definitely result in inadequacy of content and pedagogical knowledge and teaching skills in mathematics and science. This inadequacy will surely have a great influence on the quality of future teachers and the performance of their students and should be the core of ongoing educational reforms.

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STUDENTS AND SOFTWARE: TALES OF ANXIETY, SONGS OF SUPPORT

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This report offers insights into the views of students facing their first experience of using professional scientific software (MATLAB) for doing and learning mathematics. The data was captured from samples of a population of 508 undergraduate students at different stages of their technology-integrated learning experience in two early undergraduate Algebra & Calculus courses. The findings illuminate and quantify the range of reactions. Examples are offered of the voices of the majority who chorused songs of support for the use of software for learning and doing mathematics in advance of the initiative (more than three quarters of those entering the courses). In contrast, equally important tales of anxiety expressed by the vulnerable minority who felt negative about the prospect of using computer software for learning are reported.

BACKGROUND AND SUMMARY OF FINDINGS

This study forms part of an investigation of learning and attitudes in a technology-enriched early undergraduate learning environment over the years 2001 to 2004. The majority of students (76% of those surveyed early in 2003 and 2004) expressed positive beliefs and attitudes about using software for learning mathematics in their responses to open questions on entry to the course. Typical examples of these "voices" are presented. Affective responses hinged on perceptions of the use of computers as enjoyable, novel and fun, and some students were clearly excited about the technology intervention. Cognitive responses hinged on the time-saving benefits of computer power and efficiency, and opportunities for deeper learning and investigation. The belief that learning to use professional software would be of later value in their studies and careers was clearly a strong and important motivation.

On the other hand, students' espoused fears and concerns focused on personal feelings of inadequacy when using computers, and beliefs that computers do not aid learning. Some were very anxious about their lack of experience and confidence with computers: two reported negative prior experiences. Students' attitudes to the intervention were generally not closely related to their liking for mathematics: in fact, the most negative technology attitudes came from students who said they like mathematics. Conversely, the positive computer software attitudes of a potentially vulnerable group of students who were not enthusiastic about learning mathematics, made it clear that computer interventions of this kind have the potential to motivate levels of engagement in learning tasks. These early base-line beliefs are an important and reassuring finding for any technology learning intervention that seeks to harness powerful computer software for the learning of mathematics.

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COMPOSITION AND DECOMPOSITION OF 2-DIMENSIONAL FIGURES DEMONSTRATED BY PRESERVICE TEACHERS

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International and national studies highlight the vital role of teachers' content knowledge in mathematics learning. These studies concur that many practicing teachers in USA elementary and middle schools have impoverished conceptual understanding of many of the (Algebraic) mathematical concepts and processes they are required to teach. This study investigates prospective teachers' Geometry content knowledge with specific focus on the process of composing and decomposing two-dimensional figures. For this study, this process will be referred to as composition.

Composition appears infrequently in many elementary and middle school curricula yet teachers of geometry are expected to possess conceptual understanding of composition in order to provide students with meaningful opportunities to transform, combine and subdivide geometric figures. Despite the curricula, the understanding of composition is a vital component of content knowledge for teachers of geometry at all levels as it contributes to such vital skills as perceptual constancy, position in space-perception, visual discrimination, perception of spatial relationships and figure-ground perception. To site just a few examples of its importance in the school curriculum, the knowledge of composition facilitates working with area, with congruence, and with angle computations of polygons where the ability to recognize that a polygon can be decomposed into triangles is critical.

This study examines the conceptual understanding of composition by 125 preservice teachers who were enrolled in a college on the East coast of the USA in the Fall of 2004. While results from surveys and interviews showed that many preservice teacher could successfully compose and decompose figures, over half had severely limited ability to recognize alternative methods of composition. Since these students are enrolled in a geometry course designed for preservice teachers, they will again be surveyed in the Spring 2005 semester. Final and comparative results with suggestions for improved pedagogy will be reported.

The research was funded in part by The Council of Deans at The College of New Jersey

WHAT'S IN A NAME? ANONYMITY OF INPUT IN NEXT-GENERATION CLASSROOM NETWORKS

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This study looks at student use of anonymity of input with next-generation classroom networks in a pre-calculus class. Results show that the more activity input can be considered right/wrong, the more students want to submit anonymously.

Next generation classroom networks are poised to become a significant presence in schools. In contrast with current networks which connect students to the internet and outside information, these networks harness and share the knowledge within classrooms, sharing and aggregating data among all the members. Common to all of the next-generation classroom networks is the feature of anonymity of input to the public display. Yet, no research has been done to show if anonymity is an important design element in network-supported learning and, if so, what about anonymity is significant. This project looks at anonymity of input across a series of activities in a pre-calculus classroom seeking to answer the question: Does activity type influence students' use of anonymity?

Next-generation classroom networks allow for a positive view of anonymity. Anonymity opens the information that has been displayed to the whole class for interpretation. With a range of mathematical responses collected from the class displayed, students can talk about any one of the answers as if it was theirs. Or, as if it was someone else's. Once information has been submitted to the public display, a student can assume any of the identities and advocate it as if it were there own. In this way, anonymity opens up the classroom allowing students to try on new roles.

Early analysis of project data shows that students attune most closely to the ability of their answer or participation to be considered incorrect when deciding whether to show or hide their identity in the display space. Activities ranged across submitting responses to homework questions, controlling a point in a scatter plot, networked Sim-Calc lessons, and HubNet (networked NetLogo) simulations. Students were most likely to hide their names if their input could be seen as "wrong". In this way the activity design had a strong impact on students' use of anonymity. Additionally, female students were more likely to need to be confident of the correctness of an answer, before choosing to display their name. Finally, there were two negative impacts to classroom interactions from tying names to responses in the display space. First, the teacher's use of the class' responses became directive rather than inclusive. Instead of opening up the discussion of responses to all students, the teacher called on the student who submitted the response. Second, students no longer felt free to critique the responses in the display space. With names associated, discussing responses become "personal" and no longer just about the ideas.

STUDENTS' USE OF DIFFERENT REPRESENTATIONS IN PROBLEM SOLVING AT HIGH SCHOOL LEVEL

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This study reports how high school students used different forms of representations during problem solving and some of their related conceptual difficulties associated with the use of these representations. The focus students solved nine problems which involved the use of algebraic thinking in geometry. One important aspect of algebraic thinking is the use of different forms of representations which include: verbal, numerical, graphical, and symbolic representations. The chosen problems involved one or more of these different forms of representations.

Theoretical perspectives from Goldin (2002) and Dreyfus (1991) were used in the study. Goldin (2002) has claimed that individual representations belong to a *representational system*. He has postulated two types of representations: internal and external. Dreyfus (1991) proposed a theory about representation of concepts which complements Goldin's theory described above. Dreyfus claimed that to represent a concept means to generate an instance, specimen, example, or image of it. A symbolic representation is externally written or spoken, usually with the aim of making communication about the concept easier, whereas a mental representation refers to the internal schemata or frames of reference which a person uses to interact with the external world. For Dreyfus, learning processes consist of four stages: (a) using a single representation, (b) using more than one representation, (c) making links between parallel representations, and (d) integrating representations and flexibly switching between them.

The results demonstrate that students could work separately with each of these four forms of representations. However, they had difficulties switching from one form of representation to another and making links between parallel representations of the same concept. The algebraic form of representation took precedence over the verbal form in some cases, such as the description of the Pythagorean Theorem. Sometimes not understanding a particular term caused some representation problems for the students. In other situations, the students overlooked the use of a diagrammatic representation in finding a solution and this led them to incorrect solutions

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AUTOMATISED ERRORS: A HAZARD FOR STUDENTS WITH MATHEMATICAL LEARNING DIFFICULTIES

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While researchers have demonstrated the benefits of automatising number facts for the carrying out of problem solving and complex algorithms (e.g., Cumming & Elkins, 1999), there has been relatively little research on the problems caused by automatised fact errors. This may be a critical pedagogical issue for students with learning difficulties in mathematics, who are characterised by pronounced difficulties in mastering basic facts (Ginsburg, 1997), by a high level of errors on retrieved facts, and by a distinctive pattern of counting errors (Geary, 2004).

Recently, however, after confirming Barouillet's discovery that students may substitute counting string associations to one of the addends in a basic fact combination (Barouillet et al., 1997), Geary (2004) has proposed that difficulties in inhibiting the retrieval of irrelevant associations may be an underlying cause of difficulties in mastering arithmetic facts. Furthermore, Hopkins and Lawson (2004) demonstrated that the variable response times noted for retrieval of facts by students with mathematical learning difficulties may in part be caused by increased response times for trials which immediately follow trials where students have made errors.

This presentation will demonstrate the hazard of recurring errors for students with learning difficulties by presenting data from sessions focussed on teaching the ten facts to a 9 year old student with a significant mathematical learning difficulty. Conversely, by paying active attention to his errors, the student was able to successfully teach himself the nine times tables to the point of mastery.

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TEACHING GEOMETRY IN TWO SECONDARY CLASSROOMS IN IRAN, USING ETHNOMATHEMATICS APPROACH

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In a study that was conducted in the 2003-2004 school year, a mathematics instruction was designed for teaching geometry in grades 10 & 11 (2^{nd} & 3^{rd} year high school) using the ethnomathematics approach. The purpose of this study was two folded; the first was to investigate the ways in which, this notion could be used in a mathematics classroom in Iran- taking into account that the system of education is extremely centralized, and teachers are responsible for every page of every textbook they are teaching. The second was to study the effect of ethnomathematics teaching on students' perception about mathematics, as well as their understanding of the geometric concepts of high school.

The data for the study were collected through two geometry classes in a girl's high school, in Ghom in which, the second author was teaching geometry in both classes (all schools in Iran are segregated). The data constituted of teacher's observations and her field notes, students' reflective comments about the instruction that they received, and students' responses to two questionnaires about their perceptions about mathematics. The data were analyzed using Bunks (1994) framework of ethnomathematics approach.

The results of the study showed that, even in a highly centralized system, ethnomathematics approach could be taken to teach geometry. This was especially important for Iranian students, since, they expressed their great appreciation for their cultural and scientific heritage, and the contribution that they made to the development of mathematics at the local and global level. Furthermore, students became interested to do more inquiry about the role of mathematics in traditional artwork and handicrafts, including tiling, painting, and woodworks in Iran. The ethnomathematics approach helped students to change their perception about mathematics, as well as a better understanding of geometric concepts.

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TEACHING STRATEGIES TO SUPPORT YOUNG CHILDREN'S MATHEMATICAL EXPLANATIONS

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The research reported here forms part of a small-scale international collaborative study, *Talking Across Cultures*[1], investigating children's mathematical explanations during mathematics lessons in Australia, Hungary and Japan, during the first year of school.

In this report, we identify strategies used by an expert Year 1 Japanese teacher to support young children's mathematical explanations in a lesson based on identifying children's solutions for the subtraction problem 14 – 8. These strategies, which included interweaving the concrete with the abstract, public and permanent recording of explanations, giving children ownership of ideas, and promoting high level written explanations, are examined briefly from the perspective of their cultural, pedagogical and traditional bases, to establish their pertinence to other educational settings.

As Clarke (2002) argues in his discussion of the problematic nature of international comparative research, the purpose of studying international classroom practices is not merely to mimic them, but rather to support reflection on our own practice. Thus it is important to distinguish between those classroom practices that are specifically cultural, those that are based on deliberate pedagogical decisions, and those that are the unintended consequence of other actions and decisions.

A high quality Australian lesson, also based on subtraction, was video-recorded as part of this study. The contrast between the teachers' strategies used in the Japanese and Australian subtraction lessons, suggests that there are aspects of each that could be profitably explored in the other country. However our analysis also suggests that there may be significant barriers to adopting practices from different cultures. For example, while Australian teachers want to give students ownership of ideas, this is very difficult to do when there is no tradition of identifying, recording and attributing these ideas to individual children for later use.

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[1] *Talking Across Cultures* was funded by the Deakin University *Quality Learning Research Priority Area*. The project team is Susie Groves, Brian Doig (Deakin University), Toshiakira Fujii, Yoshinori Shimizu (Tokyo Gakugei University) and Julianna Szendrei (Eötvös Loránd University, Budapest.

HOW DO WE PROVIDE TASKS FOR CHILDREN TO EXPLORE THE DYNAMIC RELATIONSHIPS BETWEEN SHAPES?

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We consider to what extent van Hiele's levels of geometrical understanding can be used at the classroom level and raise the issue of appropriate tasks for children to engage with in order to challenge and stimulate their understanding of geometric definitions.

Within a design research methodology (Cobb *et al.,* 2003) we considered how task design can begin to meet the needs of children who are struggling with geometric definitions. van Hiele (1986) offers an overview of children's geometrical understanding. However we think of van Hiele's levels as belonging to a family of macrolevel theories and our focus is much more humble. We focus on the classroom, where we see a child whose knowledge is in a state of flux and under constant pressure from outside influences. Of those many structuring resources in the classroom setting on which we are now focussed, we ask, "What is the contribution of the task to this complex and excitingly unsmooth dynamic?"

It was through the classroom-based use of a product (which in this case was a design for a task for nine- and ten-year olds) within an iterative process that children's definition of quadrilaterals was explored with the aim that we would first be able to abstract principles related to the design of a task about geometric definitions and subsequently propose more generic principles for task design.

It became clear from analysis of the data that there were several sources of confusion about the nature of geometric definitions. These included the identification of instance versus class, the attributes of the shapes, the inclusive nature of definitions, and the definitions themselves.

In light of our findings, we will present the principles that will underpin our next task and explain how they are being operationalised through the *Constructionist* (Harel & Papert, 1991) tenet that technology facilitates the construction of knowledge through use of that knowledge (see the *Power Principle* in Papert, 1996).

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PARTICIPATION, PERFORMANCE & STAGE FRIGHT: KEYS TO CONFIDENT LEARNING AND TEACHING IN MATHEMATICS?

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This session will be an exploration of both theorisations of what is often named 'identity' and of what it means to be confident in learning and teaching maths. Some models of self-image and individuality can produce restricted understandings of the experience of many learners and teachers of mathematics (Henriques et al. 1984). I will discuss the notion of 'subjectivity' and consider ways in which it offers a better analytical frame. This research was started in the PME 27 discussion group on the interface between psychological and sociological paradigms for mathematics education research (Gates et al. 2003).

I present these explorations in a mix of textual commentary and a patchwork of vignettes from my research experience. These are brought into juxtaposition, using 'what is to hand' to create something new; in a form of bricolage (e.g., Levi-Strauss 1966). This is intended to evoke connections and parallels that are concealed by more traditional modes and to offer an account of how a re-examination of practices operates and new meanings are formed in education research. This bricolage uses data and reflections from particular research projects that I have undertaken. These were 'to hand'. The first of these is an analysis of teacher guidance video material to explore the discursive practices of teachers and children in exemplar mathematics lessons (Hardy, 2004). In a second project, exploring the effects of whole-class interactive teaching, I worked with practicing teachers and pre-service student teachers. I have also borrowed incidents related in interviews with teachers and made connections with other mathematics education research. I will outline how, through this analytical tactic, constructs of a 'good learner' or a 'good teacher' of mathematics can be shaken up and discuss new understandings that are generated. The key theme of being willing and able to participate in mathematics classrooms in ways that are seen as valid was highlighted through this research and will be offered for discussion.

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AN EMERGENT MODEL FOR RATE OF CHANGE

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It has long been a concern that students develop only procedural competence with differentiation and seem to lack a deeper understanding.(Orton, 1984; Stump, 2001). The aim of this study was to explore the possibility of using the simulation software JavaMathWorlds to develop a 'model of' (Gravemeijer, 1999) a rate of change as motion leading to an emergent model for any context.

JavaMathWorlds, a product of the successful SimCalc project (HREF1), simulates the motion, of a lift or characters walking, and provides links to the numeric, graphical and symbolic representations of motion. The lift animation was used, as a starting point to investigating rate of change in a motion context, because it draws a strong connection between floors in the building and scale on the vertical axis. This encouraged the forging of stronger links between an experientially real situation and its graphical model thus supporting the development of a mathematical 'model of' the motion. Of particular interest was the notion that experience with problem solving, in a motion context only, is sufficient for the development of a transferable 'model for' rate of change regardless of the context.

The study involved year 9 (15 year old) students from two classes at an Australian secondary school. Both teachers used material consisting of four lessons introducing the software and posing problems for students to solve. It was hoped that the cognitive residue of the instructional sequence would be a more complete concept image for rate of change than is usual for students of this age and stage.

Data collected include students' pre and post test scripts, notes based on conversations with teachers and transcribed student interviews. Pre and post tests used consist of both motion and non-motion questions probing students' understanding of the concept of rate of change across multiple representations of the context.

Findings indicate that, for many students, this technology enriched learning environment, based on motion alone, does facilitate the development of an emergent model for rate of change.

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THE ROLE OF ACTIVITIES IN TEACHING EARLY ALGEBRA

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The mathematics curriculum in New Zealand, like many similar documents around the world, emphasises the use of meaningful contexts and practical activities (Ministry of Education, 1992). Some justifications for this are that the mathematics encountered in real life is always in context and the mathematics taught in schools should prepare students for this. Also the use of contexts and activities is likely to motivate students and promote mathematical understanding (de Lange, 1996). However little is known about how students in New Zealand high schools make use of activities when learning algebra.

This work, which forms part of a doctoral thesis, is a qualitative study of learning in a Year 9 class, which monitored four students during twenty-seven consecutive lessons. The data set consisted of videotapes of the lessons, students' written work, stimulated recall interviews and field notes. The teaching programme made extensive use of cooperative group activities.

All four students were engaged in the activities, and enjoyed them. For the two less numerate students, the activities gave them only a vague idea of the purposes of algebra, but for the more numerate students, the activities allowed them to write equations for situations and have a purpose in solving them. However the activities did not directly facilitate the students to develop an understanding of formal solution processes. A possible reason for this is that the students did not usually make use of the contexts when solving equations, working at the symbolic level instead. The students' use of activities when learning to solve equations was very different to when they were learning to operate on integers. During the work on addition and subtraction of integers the physical activity provided a metaphor for the intended mathematical activity, allowing meanings to be constructed through mental and verbal juxtapositions. However none of the activities used in the study provided a metaphor for the formal method of solving equations. The few examples of keeping the context in mind when solving equations could be regarded as metaphors for solving equations by the strategy of *guess and check*. This study reinforces de Lange's (1996) claim that only some contexts are useful for the development of concepts.

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COORDINATED ANALYSES OF TEACHER AND STUDENT KNOWLEDGE ENGAGED DURING FRACTION INSTRUCTION

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Adults and children often understand joint activity in different ways, but little research has investigated consequences of such differences for classroom teaching and learning. In fact, teachers' knowledge and teaching has been a separate sub-field of educational research from students' cognition and learning. The present study reports a coordinated analysis of one U.S. sixth-grade teacher's and her students' understandings of lessons that used linear and area models to develop fraction multiplication. The main research questions were (1) what conceptual structures did the teacher and her students have available for interpreting the tasks contained in the instructional materials and (2) how were students' opportunities to learn shaped by the ways in which they and their teacher engaged those structures. The theoretical perspective on classroom instruction was informed by Cohen and Ball (2001), who argued that instruction is shaped fundamentally by interactions among teachers, students, and content as mediated by instructional materials. The theoretical perspective on cognitive structures in the domain of fractions was informed by Steffe's (e.g., 2003) recent work examining how students can construct understandings of fractions using their understandings of whole numbers as they work with linear and area models. Data for the present study came from videotapes of (1) Ms. Archer's instruction every day in one class over a period of 6 weeks; (2) concurrent, weekly interviews with four pairs of students from the same class during which students worked tasks like those in the lessons and interpreted video excerpts from Ms. Archer's related instruction; and (3) concurrent, weekly interviews during which Ms. Archer interpreted the instructional materials, explained her pedagogical decisions, and interpreted the same video excerpts from lessons and further video excerpts from her students' interviews. The method for inferring conceptual structures involved fine-grained analysis of talk, hand gesture, and drawing as captured in the videotapes. Results indicate that Ms. Archer and her students evidenced a range of conceptual structures relevant for using linear and area models to understand fraction multiplication but did not consistently engage those structures during the lessons. As a result of under utilizing their available cognitive resources, Ms. Archer and her students often misunderstood one another and, as a result, constrained the opportunities to learn.

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DIFFERENCES IN LEARNING GEOMETRY AMONG HIGH AND LOW SPATIAL ABILITY PRESERVICE MATHEMATICS TEACHERS

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For many years now the International Association for the Evaluation of Educational Achievement (IEA) has conducted several international comparative studies of the mathematics and science performance of students around the world. One of the most recent, TIMSS-R, was conducted in 1999, and continued to show that United States (US) students' mathematical achievement lagged behind that of several other countries. More specifically, in the geometry content area, United States student achievement was in the bottom third of all countries tested. When geometry scores are examined, Japanese students performed at the top with a score of 575. The international average score was 487 and the US scored 473 in the geometry content area. Among the 38 countries, 26 countries outperformed the US in the geometry content area (Mullis et al., 2000). One question to consider is why is US student performance in geometry so low when compared with their peers in other countries?

While investigating the learning process, one has to consider the role a teacher plays during instruction. A primary consideration has to be the content-knowledge a teacher brings to teaching. The objective of this study was to investigate and characterize the geometric thinking of four preservice middle mathematics teachers while considering spatial ability levels. Specifically the study was guided by the following research question: *what differences, if any, exist between preservice middle and secondary mathematics teachers with different spatial ability levels and their understanding of geometry?* This report is a part of larger study, focusing on four contrasting cases in terms of their spatial ability levels. The study used the van Hiele model to provide a description of geometric thought. Participants were chosen using the Purdue Visualization of Rotations test (Bodner & Guay, 1997) from among a pool of preservice middle and secondary mathematics teachers (n=26) at a major research university. Using the Mayberry (1981) protocol four participants' van Hiele levels were identified at the beginning and end of an informal geometry course for mathematics education majors. Field notes were kept for all class sessions and student work was reviewed. Results indicated that students with a high spatial score had larger gains in van Hiele levels than preservice teachers with low spatial ability scores.

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MATHEMATICAL KNOWLEDGE FOR TEACHING PROBABILITY IN SECONDARY SCHOOLS

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The paper will report on an ongoing study that is attempting to identify and describe the mathematical knowledge teachers need to know and know how to use in order to teach probability well to secondary school students. Research questions are (i) what mathematical knowledge is evident in curriculum documents? (ii) what mathematical knowledge(s) do teachers draw on while teaching probability?

Shulman (1986) argues that teaching entails more than knowing the subject matter, it requires "pedagogical content knowledge" which "goes beyond knowledge of the subject matter *per se* to the dimension subject matter knowledge for teaching" (1986, p9). Adler (2004), Ball *et. al.* (2001), Brodie (2004), among others, emphasise that mathematics teachers need 'mathematical knowledge for teaching' (MKFT) which includes knowing how to do mathematics as well as how to use the mathematics in practice (teaching). Therefore, MKFT can only be identified and described by studying practice.

The theoretical framework that underpins the study is that MKFT is situated in the practice of teaching (Adler, 2004). Therefore, and also from literature, a study of MKFT entails an analysis of the curriculum in both (i) documentation and (ii) practice. The paper focuses on MKFT in practice and reports on findings from observations of Grade 8 probability lessons in one township school in Johannesburg, South Africa. The main source of data is videotapes of the lessons. Each lesson will be broken down into episodes of 'what the teacher was doing'. Within each episode, attempts will be made to describe the knowledge sources the teacher draws on.

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STUDENT TEACHERS' VIEWS ON LEARNING THROUGH INTERACTIVE AND REFLECTIVE METHODS

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INTRODUCTION

First Year mathematics student teachers at Tonota College of Education in Botswana participated in a study on the preparation of preservice teachers on the Integration of Assessment and Instruction (Kesianye, 2002). The study employed teaching and learning methods designed to create an environment where student teachers interacted with others and reflected on their learning. Data was collected through "free writing reports", lesson diaries and structured questionnaires. Qualitative types of analysis were employed to interpret data. Student teachers' views about the learning environment created in the study are the substance of this presentation.

DISCUSSION

Research indicates that student-teachers arrive into teacher education programmes with certain conceptions of teaching, some of which may be vague and difficult to articulate, and which appear resistant to substantial change, as observed by Haggarty (1995). However, Amit et al. (1999) suggest that teachers should be provided with experiences where these conceptions are challenged and that they be given opportunities to reflect on and rethink their conceptions. The findings reflected that student teachers made critical and constructive observations about their learning experiences. They articulated their learning progress and made suggestions for improvement. Their reactions indicated a deeper understanding of the purposes and practicalities of employing these methods, which changed previously held conceptions of teaching.

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LOW-ACHIEVEMENT STUDENTS' RATIONALE ABOUT MATHEMATICS LEARNING

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Adopting the perspective of activity theory, Mellin-Olsen argues the case for the significance of a student's rationale for engaging in classroom activity. He identifies two rationales for learning. These are the S-rationale (Socially significant) and the I-rationale (instrumental). And then, Simon adds the P-rationale (practice) and the N-rationale (no rationale). This study investigates which of these four rationales low-achievement students have. This study also intends to find out what teaching method is best for low-achievement students.

The subjects are from the lower 5% first year students of each class of an urban high school. First, I had an interview with their teacher. And then, I had interviews with students in December. The interview was conducted once per student, and when more information was needed, I conducted additional interviews (about 3 students). It took an hour or an hour and a half hour at half-structured interview.

The result showed one of those students as having N-rationale while the others were either of I-rationale or S-rationale. In contrast to my suspicion, though they are low-achievement students, they know that they need mathematics and want to study mathematics. But compared with primary and middle school students, high school students need to be provided with more complementary measures. They have not ever talked and consulted with a mathematics teacher, classroom teacher, parents, nor another person about mathematics learning. So we have to provide a more careful concern for low-achievement students.

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CHARACTERISTICS OF EARLY ELEMENTARY STUDENTS' MATHEMATICAL SYMBOLIZING PROCESSES

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Mathematical symbolizing is an important part of mathematics learning. But many students have difficulties in symbolizing mathematical ideas formally. If students had had experiences inventing their own mathematical symbols and developing them to conventional ones natural way, i.e., learning mathematical symbols via expressive approaches (Gravemeijer et al., 2000), they could understand and use formal mathematical symbols meaningfully. These experiences are especially valuable for students who meet mathematical symbols for the first time.

Hence, there are needs to investigate how early elementary school students can and should experience meaningful mathematical symbolizing. The purpose of this study was to analyze students' mathematical symbolizing processes and characteristics of theses.

We carried out teaching experiments that promoted meaningful mathematical symbolizing among eight first graders. And then we analyzed students' symbolizing processes and characteristics of expressive approaches to mathematical symbols in early elementary students.

As a result, we could places mathematical symbolizing processes developed in the teaching experiments under five categories. And we extracted and discussed several characteristics of early elementary students' meaningful mathematical symbolizing processes.

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TEMPORAL ORDER IN THE FORMAL LIMIT DEFINITION

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Mathematically a definition determines how one may structure a valid mathematical argument and proof. This study describes a difficulty advanced calculus students have with the formal definition of proof and how this might affect their proof writing. Moore (1994) suggests that students' lack of understanding of both the role of definitions in proving and the meaning of a formal definition are an integral part of their struggles with proving. Davis and Vinner (1986) found calculus students have a naïve conceptualization of the limit concept which impedes their understanding of the formal limit definition. Inherent in this definition is an underlying process. Davis and Vinner call this the *temporal order;* i.e. first given an ε, then a δ must be found which makes the implication which follows true, hence δ is a function of ε. We offer examples of students' lack of understanding the role of the temporal order process in the limit and we comment on the implications of this difficulty on their proof writing.

This research report compares data between semester long workshops for freshmen calculus students and juniors in an advanced calculus course. Students were asked to determine if the limit of a particular function exists and then to produce a proof to justify their answer. Student discourse was coded for the students' use of language associated with understanding of the process in the epsilon-delta definition.

This report focuses on the students' evaluating the limit as $x \to 0$ of $s(x)$, where the function $s(x) = x + 1$ for $x \in \boldsymbol{Q}$ and $s(x) = 1$ for $x \notin \boldsymbol{Q}$. The students were focused on how to prove the limit exists. However, they began looking for an ε, rather than looking for a δ determined by the given ε. In fact, Molly stated, "If we choose $\varepsilon = \delta$, we've got it." No one in the group objected to Molly's goal. Kelly clarified the suggestion, "If we could pick a δ such that $[|x - a| < \delta]$ is true then we can pick the same value for ε." Notice the group did not see δ as a function of ε. Molly explained, "First we resolve δ, then we go on to resolve ε."

These students knew some aspects of the formal definition of limit. In fact, the group eventually produced a correct proof including a given statement for ε and defining δ as a function of ε. This suggests that at some level they have learned an appropriate structure for a proof that a limit exists, but do not grasp the underlying temporal order of the formal definition. This implies they likely do not see the role that the structure of the definition plays in determining the structure of an appropriate proof.

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RESEARCH ON THE PROCESS OF UNDERSTANDING MATHEMATICS: WAYS OF FINDING THE SUM OF THE MEASURE OF INTERIOR ANGLES IN A CONVEX POLYGON

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This is a part of the series of research on the process of understanding mathematics based on the "two-axes process model" that consists of two axes, i.e. the vertical axis implying levels of understanding such as mathematical entities, relations of them, and general relations, and the horizontal axis implying three learning stages of intuitive, reflective, and analytic at each level (Koyama, 2000, 2003, 2004). The purpose of this research is to closely examine the 40 fifth-graders' process of understanding the sum of the measure of interior angles in a convex polygon in a classroom at the national elementary school attached to Hiroshima University.

In order to improve their understanding of the sum, with a classroom teacher, we planned the teaching unit of "The Sum of the Measure of Interior Angles" and in total of 8 forty-five minutes' classes were allocated for the unit in the light of "two-axes process model". Throughout the classes we encouraged students to think the sum in a various and logical/mathematical way. The data collected in the observation and videotape-record during the classes were analysed qualitatively to see the change of students' thinking and the dialectic process of individual and social constructions through discussion among them with their teacher in the classroom.

As a result, the teaching unit starting from the tessellating congruent triangles to the finding/explaining ways for the sum of the measure of interior angles in quadrilaterals, pentagons, and hexagons could improve the students' mathematical understanding and logical thinking. Especially, their whole classroom discussion on the various way of finding the sum in a hexagon was effective for the students to share with and reflect on their ideas leading to the general formula for the sum in a convex polygon with n sides.

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MATHEMATICS: COPING WITH LEARNER SUCCESS OR FAILURE

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Despite the valiant efforts of community organizations, dedicated teachers and others, success in mathematics has eluded the majority of learners in South Africa. This is a growing concern for mathematicians and parents.

The National Department of Education has embarked on a national science and technology strategy instituted to promote and popularize mathematics. This is done at a time when fewer learners are choosing to study mathematics at high school level. Also there has been a marked decrease in the number of learners studying mathematics at Higher Grade, the "grade" of study needed for further studies in scientific fields of study.

South African is on the brink of curriculum transformation in the high school years. This will be implemented in 2006. The question on many minds is whether this will "turn the tide" in the mathematics classrooms.

THIS RESEARCH

This research looks at the challenges facing South African learners and teachers in the South African Mathematics classroom. This study is undertaken through interviews with learners who have experienced repeated failure in doing mathematics despite their honest attempts to do well. These learners are taken from both rural and urban schools in our city, Durban in South Africa. This testimony does not do much in my attempts as subject advisor to promote the subject amongst learners who do not want to hear us speak of mathematics.

In addition interviews are undertaken with mathematics teachers who often have given up hope in some instances to reach out to learners who often time have lost interest in the subject. My engagement with teachers suggests that they are often frustrated with poor results. This research will attempt to itemize some reasons why children perform poorly in mathematics.

Improving mathematics results or popularizing the subject is a challenge worldwide. It is hoped that the research will give us a new perspective to learner difficulties, and teacher initiatives to improve mathematics results.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 254. Melbourne: PME.

AN ANALYSIS OF TEACHER-STUDENTS INTERACTION IN KOREAN ELEMENTARY MATHEMATICS CLASSROOMS

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Recently mathematics educators have made an effort to change teacher-centered instruction to student-centered. However, many teachers still have difficulties with their instructional changes. These difficulties come not merely from the complexity of instruction itself but also from the lack of understanding what is constituted of student-centered and what kind of classroom culture needs to be established.

The purpose of this study was to provide useful information for instructional improvement by analyzing various teacher-students interaction in reform-oriented mathematics classrooms. As an exploratory, qualitative, comparative case study, 4 classrooms were selected in which teachers attempted to implement student-centered instruction. As a total of 34 mathematics lessons of fraction were videotaped and individual interviews were conducted with the teachers. A theoretical framework of data analysis resulted mainly from Hufferd-Ackles, Fuson, & Sherin (2004), in conjunction with Pang (2000) and Wood (2003). Specifically, teacher-students interaction in each classroom was identified by 4 levels in questioning, explaining, and the source of mathematical ideas, respectively.

Detailed analyses of classroom episodes showed what kinds of interaction were fostered. As for questioning, teachers asked for reasons rather than simply answers, but the degrees of students' questions were different. As for explaining, the quality of students' justification varied depending on the teacher's role of listening. As for the source of mathematical ideas, students came up with multiple ideas but they tended to focus on the diversity of representations rather than solution methods. Whether students discussed similarities and differences of their various ideas was different across classrooms. The differences in three components set forth implications of what aspects of teaching and learning need to be focused for a real instructional change.

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STUDENTS' GRAPHICAL UNDERSTANDING IN AN INQUIRY-ORIENTED DIFFERENTIAL EQUATION COURSE: IMPLICATION FOR PRE-SERVICE MATHEMATICS TEACHER EDUCATION

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Although graphical understanding has been emphasized in teaching and learning of mathematics, research shows that school mathematics curriculum is not enough to facilitate the development of students' graphical understanding (Dyke & White, 2004; Knuth, 2000). From this perspective, the inquiry-oriented differential equations course (IODE) was designed to emphasize the integration of multiple mathematical methods such as graphical, numerical, and qualitative methods as well as analytic method. The research measures the impact of the IODE by investigating students' graphical understanding and sought for implications for teacher education. For comparative analysis, pre- and post-tests, made up of problems requiring graphical understanding, were given to two groups of students of a university in Seoul, Korea. Since IODE was a course offered in a university pre-service program, the experimental group consisted of 36 pre-service students in the department of mathematics education. The control group consisted of 30 students who enrolled in a traditional differential equations course based on lecture (TRADE) in the mathematics department of the university. Following are the results of the analysis:

The IODE group got statistically higher mean score than the TRADE group did.

The IODE group tended to use the graphical/qualitative method, while the TRADE tended to use the analytical method only.

The IODE group was significantly better than the TRADE group at the problems requiring the connection between an equation and a graph. In particular, while the rate of NO ANSWER to this type of problems was very high in the TRADE group, it was very low in the IODE group.

These results show that the IODE contributed positively to develop the students' graphical understanding, specifically, their abilities to use graphs for problem solving, to interpret the meaning of a graph, and to grasp the connection between a graphical method and other methods. Moreover, this research implies an alternative model to the current pre-service mathematics teacher education programs which instruct mathematical knowledge and pedagogical contents separately. By combining subject matter knowledge and pedagogical content, the IODE not only transformed the quality of the students' graphical understanding but also provide opportunity for the pre-service students to reflect on how to teach to develop graphical understanding in their future teaching career.

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TYPES OF VISUAL MISPERCEPTION IN MATHEMATICS

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A *misperception* is the act of perceiving via a single sensory modality (e.g. seeing in mathematics or hearing in music) something that is different from reality or an imagined reality (e.g. visualising a rotated shape in maths).

Visual misperceptions seriously inhibit students' efforts to learn. The authors have found misperceptions occurring not only with school students but also with pre-service and practicing teachers (Lamb, Leong & Malone 2002). Teachers who perceive in their mind's eye something that is different from reality are likely to mis-teach, and students who misperceive will experience learning problems also.

This ongoing study of 720 Year 8 Australian school students has revealed three different types of misperception occurring in very simple mathematical tasks. During three separate tests, over 40% of the participants misperceived at least once.

The topic selected for the study was linear transformations, a topic that lends itself to displaying the misperception phenomenon (Kuchemann, 1982; Sherris, 2003; Edwards, 2003). The students in our study had been taught about reflections and rotations approximately two years beforehand.

The negative effects of misperceptions on learning have been largely unappreciated, and are usually misdiagnosed to be the result of student errors or misconceptions (Shaw, Durden & Baker, 1998). The study has revealed how misperceiving students can be identified, and how using manipulatives and specially designed software, some misperceptions can be corrected.

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DEVELOPING STUDENTS' HIGH-ORDER THINKING SKILLS THROUGH INCREASING STUDENT-STUDENT INTERACTION IN THE PRIMARY CLASSROOM

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Mathematics curriculum reform at the primary level in Hong Kong has been a concern of primary mathematics teachers since the implementation of the curriculum guide (CDC, 2000) in 2002. This study investigates mathematics learning and teaching in the primary classroom with a particular focus on the changing role of the teacher as a facilitator who helps students develop high-order thinking skills through using mathematical tasks and classroom discussion (as specified in the 2002 policy document "Learning to Learn Key Learning Area Mathematics Education"). Recent research suggests that the majority of teachers in HK largely still use the textbook in a routine, "chalk & talk" mode as the main material for the introduction and consolidation of mathematical concepts by students (Wong, N.Y., Lam, C.C., Leung, F.K.S., Mok, I.A.C. & Wong, P.K.M., 1999). These teachers' classrooms are dominated by traditional teaching practices. Furthermore the rare teacher training provided towards the implementation of the new curriculum seems to have had minimal influence on teachers' philosophies and beliefs about the learning and teaching of mathematics (CDC, 2000). The study reported here was set up with the intention to encourage more student-student interaction in the classroom, to enhance students' thinking and communication skills and to use diversified learning activities and tools (including mathematical tasks & information technology) for improving learning and teaching. In collaboration with two teachers in two elementary schools, learning materials were developed for selected topics. Trial lessons were conducted in these teachers' schools over the last three years. Data was collected in the form of audio-taped interviews with teachers and groups of students, video-taped classroom observations, field notes, documents and students' annotated work. Preliminary analysis suggests that the new learning environment contributed significantly in the development of students' high-order thinking skills; that by using the mathematical tasks, it created a new platform for students to learn in a collaborative mode; and that increased opportunity for communication meant that students, even those usually reticent in the classroom, could freely put forward their ideas and suggestions.

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SUPPORTING TEACHERS ON TEACHING FRACTION EQUIVALENCE BY USING RESEARCH-BASED DATA

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The emphasis on anticipating students' learning process and integrated it into teaching is developed on current reform in mathematics teaching. The success of CGI is a case (Fennema et al., 1996). Simon's model indicates that the teacher's knowledge evolves simultaneously with the growth in the students' knowledge (Simon, 1995). The continual change of hypothetical learning trajectory as a result of the increment of understanding students' thinking can be as an indictor of the process of a teacher constructing knowledge. This study offered the teachers with the key ideas suggested in the literatures of research on children learning fraction equivalence [FE] (Post, 1992). How a teacher learned the results of the research on fraction equivalence and integrated them into classroom teaching is the focus of the study.

Six teachers participated in the study, while only one teacher, Jing-Jing, teaching in fifth grade, was report here. The data collected in the study included: videotapes of five classes, audiotapes of three weekly meetings, teacher's reflective journals, students' pre- and post-test of FE, and students' responses to the assessment tasks.

The processed of constructing pedagogical knowledge of FE were characterized as: suspecting the instructional sequence scheduled in textbook, conjecturing and justifying hypothetical learning trajectory, and reflecting on FE teaching. Two suspects relevant to instructional sequence of FE Jing-Jing addressed included suspecting the priority of continuous model in FE teaching and suspecting the activities of generating FE multiplying denominator and numerator by a nonzero number. The hypothetical learning trajectory was supported by the following arguments: 1/2 as a reference point strategy of ordering fractions, FE first developed in discrete model and followed by continuous model, and naming a fraction in more than one way by various ways of packing. The study found that the teacher elaborated and refined her pedagogical knowledge of FE on the basis of the objectives scheduled in the textbook was resulted from research-based data of students' learning fraction provided in the teacher professional development program.

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PUPILS' TOOLS FOR COMMUNICATING META KNOWLEDGE

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How do young pupils with learning difficulties and their teachers understand the special education provision? Is it possible that differences in understanding creates a less positive learning context for pupils with special educational needs, and contributes to, or reinforces, their difficulties?

These questions emerge from a study of how teachers, pupils and parents understand the classroom context. The phenomenon of understanding belongs to the micro context of a person. The understanding constitutes in term an individual rationale for action (Lindén 2002). The participants understanding will differ from each other because of the different part they play in the special educational arena. They have different obligations and expectations regarding the provision. In the classroom situation the pupils are supposed to acquire new knowledge. At the same time the pupils acquire knowledge about knowledge, about learning itself. This is what Bate son (1972) calls meta-knowledge. The meta-knowledge represents a person's understanding of the situation. If the pupils do not understand what learning is about, if the subject taught in school does not belong to the pupils' field of interest, the goal for learning is not part of the pupils understanding. The reason for learning is not present. In this perspective it is interesting to search for an explanation to some of the learning problems developed in school.

The presentation will focus on and discuss the pupil's tools for communicating their understanding as they appear in the study.

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MATHEMATICS EDUCATION, CULTURE AND NEW TECHNOLOGIES

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Dr. Iara R. B. Guazzelli Rosângela M. C. Bonici
UNICSUL - Brazil

This talk discusses some theoretical issues related to culture and meaning production for technology that come from previous research done by Guazzelli (2002, 2004) and Lins (2002, 2004) during their doctoral studies. For this reason, methodological issues will be narrowed discussed here. The three-year research study is about a collaborative work with 13 Brazilian secondary state school teachers. Two are Master students of the Graduate Program in Science and Mathematics Education at University of Cruzeiro do Sul (UNICSUL). Mathematics Education can play the role with respect to the anthropological dimension showed in the culture. In terms of theoretical perspective, it is our intention within the research study to exploit the four poles proposed by Morin (1977, 1981, 2000): the day-by-day experience, knowledge, code and patterns in a way of elucidating the meaning production to technologies, which expresses the way of being of that community. D'Ambrosio (2003) calls our attention to a new conception of mathematical education in favour to the development of complex thinking, one of the most important change nowadays; it relates to getting rid of linear thinking and integrating qualitative and quantitative dimensions in a richer and more complex synthesis. Another change concerns the use of technologies. Their uses were mostly linked to the sense of power, control and economical growth. Education, in particular Mathematics Education, can take the challenge of joining complex thinking to the use of technologies and appropriate them as mediation to a new cultural perspective. We believe that by carrying out this research study will give room for discussing new theoretical, methodological and epistemological perspectives to the use of technologies in Mathematics Education.

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INVESTIGATION OF STUDENTS' VIEWS OF MATHEMATICS IN A HISTORICAL APPROACH CALCULUS COURSE

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The role of college students' views of mathematics in the leaning of advanced mathematics has been cited in several research literatures (e.g., Carlson, 1999; Kloosterman & Stage, 1991), endorsing Schoenfeld's claim that developing a mathematical point of view is a potential indicator of strategies or approaches students adopt while engaging in mathematical tasks. On the other hand, several documents also call for awareness to enhance students' understanding of the nature of mathematics (AAAS, 1990; NCTM, 2000). To this end, one of the general goals for all students is learning to value mathematics, to focus attention on the need for student awareness of the interaction between mathematics and the historical situations from which it has developed (NCTM, 2000).

The purpose of this study was to investigate how Taiwanese college students develop their mathematical point of views in a historical approach calculus course. At the beginning of the semester, by means of administering all students an open-ended questionnaire, conducting follow-up interviews to a random sample, the study attempted to examine students' initial conceptions of mathematics. During the subsequent academic semester, the students experienced a calculus in which the sequence was structured in historical order and historical problems plays a central role serving to lead students to search for solutions and compare diverse thinking mode of mathematicians in history. Near the end of the semester, all participants answered the identical questionnaire and the same students were interviewed to pinpoint what shift their views on mathematics had undergone. It was found that participants initially tended to hold an instrumentalist view of mathematics, yet were more likely to value logical processes in doing mathematics afterward and leaning toward a conservative attitude toward certainty of mathematical knowledge. Their focus seemingly shifted from mathematics as a product to mathematics as a process.

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CONJECTURE ACTIVITIES FOR COMPREHENDING STATISTICS TERMS THROUGH SPECULATIONS ON THE FUNCTIONS OF FICTITIOUS SPECTROMETERS

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Mathematics teachers are supposed to design meaningful tasks to motivate students' interest and to enhance students' communication and reasoning. Under various contexts, if meaningful tasks are designed by mathematics teachers for students to work out, then students would benefit more from those contexts of problem solving. For example, in the unit for learning three statistics terms, i.e. *median*, *mode*, and *range*, the authors provided opportunities for students to formulate, verify, and modify their presumptive rules rather than directly told them the rules to find those three terms. Such a learning process might result in better student performance.

The purpose of this study was to describe students' problem solving performance when they were initiated to make conjectures for comprehending three statistics terms. To give students the opportunities to formulate, verify, and modify their conjectures, three terms first were temporarily replaced by three spectrometers in the instructional activities. Students then tried to conjecture the functions of the three spectrometers and thus to give names for the three spectrometers according their intuitive perceptions.

During the conjecture activities, as developing computational ability was not the focus of the study, the authors encouraged students to utilize calculators to find their answers, so as to reduce their mistakes. There were three stages in the conjecture activities. First, the data and fictitious spectrometers were introduced to students. Second, based on the analyses the relationships between data and answer, students identified the functions of the three spectrometers. Finally, students were asked to name the three spectrometers according to their functions. After the three stages, students were encouraged to discuss the applications of the median, mode and range.

The findings of this study were as follows: 1) Students could comprehend statistics rules from inductive speculation. 2) Students could find the functions of three spectrometers from the process of conjecturing, verifying and modifying. 3) Students could provide intuitive terms for the functions of the three spectrometers 4) Students could make sense of three terms from the process of giving names. 5) Students modify those names successfully through peer discussion.

TURNING MATHEMATICAL PROCESSES INTO OBJECTS

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Conceptualising mathematical processes as abstract objects (commonly referred to as 'reification' in the literature) is considered to be an essential mathematical activity. According to several authors, it is an ability that is difficult to acquire. In everyday life and language people have no difficulty thinking about processes and actions as things (e.g., "a rise in interest rates; "consumption of junk food"; "global warming"). In the field of mathematics education, thinking about processes as things is considered to be a major difficulty at all levels of algebra learning. It is well known that beginners in algebra are surprised to find that an expression (e.g., $x + 5$) can denote an operation to be carried out (i.e., a process) and also denote the result of that process - a mathematical object. However this initial obstacle is overcome with appropriate teaching. In contrast, the cognitive adjustment required to conceptualise a process itself - not its result - as an object is considered to be difficult for many students at all levels of mathematics and perhaps impossible for some. I have found no reports of empirical studies that explain how evidence for this difficulty has been obtained, how widespread it is, and how it may be overcome. Nevertheless there are various theories about the cognitive actions and structures that may be involved.

It is interesting to note that turning processes into objects is a major focus of literacy teaching in the middle grades of schooling. Mastering grammatical features of language enables actions to be described as things and new concepts to be developed. Many students who are able to write about "what I did" and "what happened" in the primary grades need much instruction and practice in the middle grades as they learn to describe these events as things. When processes become things (expressed by noun phrases instead of verbs) they can be reflected on, generalised, placed in causal relationships with other things, and discussed; they can take part in other processes. For example, "the temperature of planet Earth is increasing" becomes "global warming", which can now be placed in relationships with other things such as "the melting of Antarctic ice sheets" and "extinction of species". Similarly, mathematical processes can be conceptualised as things that can be used in chains of deductive reasoning, placed in relation to other things, and take part in other processes.

In ordinary language, people learn how to turn processes into things. Why should they not learn, with similar success, how to turn mathematical processes into things? With this question in mind, I discuss speculations, opinions and theories in the literature on how reification of mathematical processes takes place.

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THE BENEFICIAL AND PITFALL ROLE OF THE SPOKEN LANGUAGE IN THE INFORMAL DEFINITION OF STATISTICAL CONCEPTS

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The formal definitions of mathematical concepts constitute of a symbolic language which is independent of any spoken language. Yet, since these definitions in their symbolic format are difficult to teach and understand, we use the spoken language to define them non-formally. In case of the Hebrew language, we use various words to describe mathematical concepts. Some of these words have the same meaning in the everyday use as in the mathematics such as ‘average (mean)’; some of them have a different meaning such as ‘mode’, and some have an opposite meaning such as ‘significance level’.

Researchers explored various aspects regarding the understanding of statistical concepts (Falk, 1986) such as how technology can help students understand, integrate, and apply fundamental statistical concepts (Chance et al., 2000). In the current study we examine students' difficulties while defining statistical concepts informally and how they are influenced by the everyday meaning of the same words.

A questionnaire which included several statistical concepts and everyday expressions bearing the same meaning as the statistical concepts was given to second year college students that had already studied probability and statistics. The students were asked to write down a definition to each concept in their own words (informally) and to add an example of its use.

Categorization of the students’ definitions revealed the following: using the meaning of each word separately; confusing between related concepts; using the concept within its definition; using the word's stem or tone; bringing the mathematical notation of the concept and others.

In the presentation, we will provide our full categorization and examples for each one of the revealed category and also bring possible explanations regarding the influence of the spoken language and the everyday use of words on the informal formalization of statistical concepts.

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A DIDACTIC PROPOSAL FOR SUPPORTING PUPILS' PROPORTIONAL REASONING

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This communication aims to propose ideas concerning the teaching of the topic of 'ratio' and proportion' in primary school. Results from teaching sessions focused on 'ratio' tasks are the focus of the presentation. These sessions were part of a case study concerning the teaching of 'ratio' in a primary school classroom consisting of 29 pupils (aged 10-11).

A diagnostic test for ratio and proportion was used, prior to the teaching sessions, for exposing the pupils' strategies and errors in varying 'ratio' items (Misailidou and Williams, 2003). Selected problems from that test (Misailidou and Williams, 2004) were used as central tasks for the subsequent teaching sessions. The role of discussion and of the generation of arguments was considered crucial in aiding the pupils' proportional reasoning. Thus, 'tools' for facilitating the pupils' arguments in discussing the 'ratio tasks' were designed and used.

During this communication, some representative episodes from the teaching sessions will be outlined; the tools that have been used for facilitating discussion and the resulting pupils' argumentation will be presented and discussed.

It is argued that the main characteristic of an effective didactic proposal for supporting pupils' proportional reasoning is the exchange of arguments in discussion. Such an exchange can only be productive when supported by tools specifically designed for the task under discussion.

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AN INVESTIGATION ON PROOFS EDUCATION IN KOREA

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In this article, we investigate the various attempts in didactical transposition by teachers and the difficulties which students have in learning proofs. Finally, we suggest implications for improving proofs education.

THEORETICAL FRAMEWORK

The results and discussion that follow in this article arise from the "didactic transposition theory" originally described in the works of Chevallard (1988; Kang, 1990), and "quasi-empiricism" originally described in the work of Lakatos (1976).

METHODOLOGY

The proofs-classes analysed and discussed in this article are 30 classes of 8-grade. I used the participating observation method to analyze the features of proofs-classes. In addition, we informally interviewed two teachers and some students by using descriptive questions.

RESULTS AND DISCUSSIONS

- The Various Attempts in Didactical Transposition by Teachers
- The Weak Points in Teaching Proofs
- The Difficulties in Learning Proofs

THE IMPLICATIONS FOR PROOFS-EDUCATION

The first implications for teaching proofs on the basis of the results of analysis in this study in that we should teach proofs as a dynamic reasoning activity that unifies the analytical thought and the synthetical thought. Second, we should make students guess the conclusion by themselves by giving the assumption alone instead of giving both assumption and conclusion, then make students perform the proofs in order to justify the truth of their own conclusion.

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IMPROVING SPATIAL REPRESENTATIONS IN EARLY CHILDHOOD

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In this paper, a teaching intervention programme aiming at improving spatial reasoning in early childhood is presented. An initial study (Oikonomou, 2005), which investigated the development of spatial representations in pre-school children (4, 5 – 6 years old) showed that, despite important individual differences, the ways in which the subjects represented and handled spatial situations evolved: from the exploitation of either a holistic or partial approach in the beginning to the control of elaborated one and two dimensions situations. These results contributed to a better understanding of the development of children's spatial representations, as detected and discussed by Case and Okamoto (1996) and Newcombe and Huttenlocher (2000) and more in accordance to Siegler's than to Piaget's approaches (Siegler, 1998).

Based on these findings, a teaching intervention programme was designed aiming at the investigation of the possibility to help children to improve the ways they represent and handle spatial situations and hence the knowledge and the abilities required.

The 52 pre-schoolers who participated in the experiment were pre-tested and classified in different groups according to their performances. The teaching intervention included group activities, where the task was related to the reproduction of material or to graphical configurations. The spatial relations involved were relative positions or locations in space, locations according to a reference system, colinearity, horizontality, perpendicularity. The children were post-tested a month after the end of the intervention and their performances was compared to that of a control group.

The comparison of the children's performance in the diagnostic and the evaluative tests showed that the improvement of the experimental group was very significant (54%, with an effect size=1,24), whereas that of the control group was modest (28%). The improvement concerned all the dimensions of the test, thus showing that, working with appropriate tasks, the children of the sample could ameliorate their spatial reasoning.

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THE DEVELOPMENT OF PATTERNING IN EARLY CHILDHOOD

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It is well recognised that patterning is fundamental to the abstraction of mathematical ideas and relationships and the development of mathematical reasoning. (English, 2004; Mulligan, Prescott & Mitchelmore, 2004). Educators in the early years can promote the development of patterns and relations, by helping children think beyond the 'play' situation, building on everyday mathematics but incorporating traditional strands of the mathematics curriculum (Ginsburg, 2002). This study raises two key research questions: Is there a link between a child's ability to pattern and their development of pre-algebraic and reasoning skills? Can an intervention program focused on identification and application of patterns, show long term benefits for children's overall mathematical development?

This project tracks the development of 53 young children's pre-algebra (patterning) skills from preschool to the second year of formal schooling. Case-studies of two matched preschools ('intervention' and 'non-intervention') examined the influence of a mathematics intervention promoting children's patterning over a 6 month period. Individual task based interviews were conducted at three intervals over an 18 month period. Tasks comprised construction of towers using blocks, subitising dot patterns, arrays, grids, patterns in the formation of borders and hopscotch, as well as numerical sequences.

Children who performed poorly on patterning tasks at all interview points were identified as low achievers on other numeracy assessments. Children from the intervention program consistently showed a greater level of improvement in patterning tasks than the non-intervention sample at the end of the pre-school year. This was sustained at follow up interviews one year later. Survey and interview data from participating teachers highlighted their lack of confidence and awareness of the importance of patterning in mathematical reasoning and understanding.

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THE DEVELOPMENT AND PILOTING OF A SIX-MONTH PRE-ITE MATHEMATICS ENHANCEMENT COURSE

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In order to expand the pool of potential secondary mathematics teachers, a Mathematics Enhancement Course [MEC] was designed and run. This has already enabled graduates with a broad range of first degree subjects to access initial teacher education courses. Our pilot MEC led to some insights into [a] what priorities are needed in such courses, and [b] what fresh perspectives such graduates can bring to the subject and to their subsequent teacher preparation course. Our key research question was whether a course based upon consistent attempts to develop profound mathematical understanding could succeed over an intensive six-months.

THE DESIGN OF MEC

The fundamental design concept was to follow the lead given by Ma [1999] encapsulated by the maxim *'Know how, but also know why'*. Profound understanding of fundamental mathematics was built into our approach to course design and in-built network of connections between its taught units; it also formed the basis for our assessment strategies.

Responses and persistence of MEC students

The piloting was very closely monitored by the funding government agency and by their appointed evaluator. Apart from the natural 'goldfish bowl' effect, this added to the range of data collected on student responses. Some students found the whole idea of pursuing profound understanding quite counter to their prior experience and cultural assumptions. These students were the most 'at risk' and despite a very strong group support ethic fostered among the students, four of the original 25 did not complete the six-month course. Responses of those who completed the course were very positive, and 20 moved directly on to a one-year teacher preparation route.

Initial findings

Half way through the teacher preparation route, 19 are still continuing, one having intermitted but not withdrawn. The indications at this stage from all observers are that the MEC has been successful. This short presentation will draw out some key issues.

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AUTHENTIC ASSESSMENT IN MATHEMATICS CLASSROOM: A PARTICIPATORY ACTION RESEARCH

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Authenticity is an important element of new model of assessment. Some defined authentic assessment as a synonym for performance assessment, while others argue that authentic assessment put a special emphasis on the realistic value of the task and context. The definition of authentic assessment used in this study is an assessment requiring students to use the same competencies, or combinations of knowledge, skills, and attitudes that need to apply in the criterion situation in professional life.

The purpose of this study was to select and develop procedure for multiple authentic assessment tools and techniques (such as: task, work project, open-approach question, observation, interview, and self-evaluation) involved in real life or authentic tasks and contexts of mathematics in geometric content area for grade seven students. The study was also described in which university instructors served as the participatory researcher to provide collaborative research with school mathematics teachers. Participatory action research process and characteristics was derived including a series of six phases. They were phase 1: forming collaboration; phase 2: problem identification for action research; phase 3: data collection and analysis; phase 4: data synthesis and generation of recommendation; phase 5: design of data-driven action/intervention; and phase 6: evaluation of intervention.

Among the salient finding identified were (a) to ensure the effectiveness of authentic assessment should be linked to authentic instruction and learning activities, (b) regarding impact on student's learning, varying multiple assessments and criterion situations should be related to meaningful real-life situations, and (c) scoring rubrics of content knowledge, skills, and attitudes preferred venues of communication assessment criteria and results to both students and teachers. Methods for developing and refinement of authentic assessment procedure were described.

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THE CANDY TASK GOES TO SOUTH AFRICA: REASONING ABOUT VARIATION IN A PRACTICAL CONTEXT

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Lack of recognition of the role that variability plays in sampling prompted research into developing the 'candy' task (Reading & Shaughnessy, 2004), various forms of which have been given to students (Grades 4 to 12) in a number of countries. Responses were categorized according to *Centre* and *Spread* (Shaughnessy et al., 1999) exhibited when students predicted samples. Of interest was how the reasoning about variation in a sampling situation by students from the Republic of South Africa (RSA) compared with that of students in the 'other' countries where the task had already been implemented. A sample of 100 students, from primary (Grade 6) and secondary (Grades 8 and 10) schools, were given a detailed *Demonstrated Questionnaire* (a compromise between a questionnaire and an interview) with questions in three different formats, *List, Choice* and *Range*. Students were offered the opportunity to alter responses after viewing demonstrations of the sampling.

Differences between performances RSA students and those from 'other' countries were significant, with more *correct* (in relation to both *Centre* and *Spread*) responses for RSA students in all three question types. For *Centre*, RSA students exhibited a similar number of incorrect (poorly centred) responses but showed a distinct trend to be too low, compared to too high for the 'others'. For *Spread*, the RSA students exhibited more incorrect responses but these were similar to the 'other' responses in terms of incorrect variation expectations as too narrow or wide. While Grade 6 students performed much better than the 'others', there was a lack of comparable improvement across Grades 8 and 10. Most RSA students with correct responses chose not to alter them when given the chance; of those who did change, Grade 6 and 8 students mostly produced better responses while many Grade 10 students reduced the quality of their response ('other' data for changed responses was not available for comparison). Teachers already acknowledge the importance of personal experiences that students bring to learning situations. Further investigation is needed to determine what learning environment factors in the RSA could contribute to performance differences on such sampling tasks, influencing students' reasoning about variation.

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SOME CHARACTERISTICS OF MENTAL REPRESENTATIONS OF THE INTEGRAL CONCEPT – AN EMPIRICAL STUDY TO REVEAL IMAGES AND DEFINITIONS

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This work deals with students' mental representations of the integral concept. By means of empirical data, students' conceptual images and corresponding solving competence for this specific mathematical concept were analysed on the theoretical basis of the concept image and concept definition model. This dual confrontation is the core of Tall's and Vinner's theory (1981) which emphasizes the interaction of intuitively and heuristically influenced images of a mathematical concept and its formal definition. Our empirical study (Rösken, 2004) shall demonstrate the fruitfulness of this approach and is based on the following questions: Which concept images and which concept definitions exist in the conceptual field of the definite integral? Which incoherence is there between concept image and concept definition? Could this be the reason for misconceptions? On the background of the theory described, students' concept images and concept definitions of the integral concept are established by combined survey methods. First, following Tall and Rasslan (2002), we assume that the concept image will become clear by working with the corresponding questions. Furthermore, for the concept image the students have to create a structured mind map to develop a suitable description of the cognitive structures via a graphical representation. The representations of the concept definition are examined by query of the definition line.

The results of the empirical study show that the students developed varied concept images of the definite integral. The mind maps revealed that the majority of students knew all aspects relevant to the concept. As expected, the definition line was represented rather weakly. Part of the students had major problems with the concept image tasks. The answers concerning the concept definition already showed that the geometric interpretation of the integral as an area was reflected as only one limited aspect.

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PRE-SERVICE TEACHERS' MATHEMATICS SUBJECT KNOWLEDGE: ADMISSIONS TESTING AND LEARNING PROFILES

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We will report the development of an assessment instrument that provides a profile of the mathematical ability of pre-service students for each strand of the curriculum: Number, Measurement, Space and Shape, Chance and Data, Algebra, and Reasoning and Proof. We will describe the test developmental cycle, our research analyses and test validation involving a sample of 430 pre-service students in the first year of their training. We will report our evidence for the predictive power of the test for course achievement.

Our test was not only summative but also provided opportunities for diagnostic assessment. Errors and misconceptions were collected and analysed for all items. We will report the patterns of errors of these adult learners. We will also outline how course teachers can use such errors to support their students' learning.

Students seeking admission to primary teacher education courses (both undergraduate and post-graduate) come with a variety of mathematics backgrounds. Since they will be required to teach mathematics, their mathematical attainment level is of importance in admission decisions. In Australia the range of mathematical credentials of students seeking admission to teacher education courses makes informed selection difficult. Additionally, a single achievement grade provides no detail of student areas of strength or weakness. Evidence of mathematical attainment thus is weak.

We sought to strengthen that evidence. We suggest that better 'tools' can be used to find potentially strong teachers who may not have taken traditional routes in the school curriculum. Our research makes a contribution to knowledge by providing fine-grained detail about pre-service teacher subject knowledge in mathematics, including current attainment, patterns of errors and misconceptions and predictors of course success.

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MATHEMATICS TEACHERS' PEDAGOGICAL IDENTITIES UNDER CONSTRUCTION: A STUDY

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It is well argued today that for any educational reform to succeed, it should aim at constructing new pedagogical identities, that is, new 'forms of consciousness' in teachers and learners. These identities emerge as reflections of differing discursive acts (Bernstein, 2000).

Most of today's mathematics education reforms tend to see mathematics as a fallibilistic discipline, its learning as meaningful in its own right and also to life and its teaching in socio-constructivist terms. In this context, a challenging way of pursuing the formation of corresponding pedagogical identities by teachers would be through discourse and reflection within the context of a community of practice (Lave & Wenger, 1991). In such a perspective, teachers would come to see themselves as being joined with colleagues to discuss teaching practice, develop consensus on alternative ways to promote students' mathematical thinking and support each other through difficult points in the change process.

The work presented here reports on an attempt towards this direction. Its purpose was to introduce alternative mathematics teaching approaches to secondary schools with culturally mixed students' classes in the north-eastern part of Greece. The 15 rather traditionally educated and teaching teachers involved in the two years study were asked to exploit a package of mathematical activities in their classrooms. The constitution of this package was based on offering opportunities for autonomous learning to students, on utilising their everyday experiences and on interacting in small groups. In parallel, the teachers participated in regular meetings with colleagues and mathematics educators at school and also at prefecture level to share dilemmas, failures, convictions, beliefs, etc. In these meetings, mathematics educators systematically used the accumulative knowledge of the field to feed in the teachers' specific actions and understandings of the learning environment under construction. The results showed that such an approach has the potential to develop a powerful and robust sense of teacher identity by making explicit, deconstructing and problematising his/her personal theories through reflection and discourse.

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MIDDLE SCHOOL MATHEMATICS AND THE DEVELOPMENT OF MULTIPLICATION CONCEPTUAL KNOWLEDGE

Rebecca Seah and George Booker

The need to think mathematically has become essential for students (Booker, 1998) in a new millennium 'awash in numbers' and 'drenched with data' (Steen, 2001). Many middle school students are increasingly disengaged from school in general and mathematics in particular and are not gaining basic mathematical ideas. In particular, a lack of multiplicative thinking/reasoning ability appears to be a major cause of students' difficulties with further mathematics (Thomas & Mulligan, 1999). Multiplication is part of a larger context of what Vergnaud (1994) termed a 'multiplicative conceptual field' – a bulk of situations and concepts that involve multiplication and division. An ability to engage in multiplicative thinking requires a clear conceptual understanding and full knowledge of mathematical processes and the relationships between them. Many high school teachers assume that such basic, fundamental ideas are taught in primary school and tend to focus solely on higher mathematics learning and abstract reasoning irrespective of their students' readiness.

Two Year 8 classes in a socio-economically disadvantaged area were examined to ascertain the level of mathematical understanding and investigate classroom interactions that promote mathematics learning. Class A emphasised collaborative teaching and 'open dialogue' to construct mathematical ideas. Class B employed a traditional 'initiation-reply-evaluation' approach (Richards, 1991) where the teacher taught for the first 10 - 20 minutes then asked students to work on a set task individually. Results indicated that the most students' knowledge of multiplication was restricted to procedural rather than conceptual understanding. Class A students demonstrated a deeper degree of conceptual knowledge than Class B.

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A REVISIT ON PROBLEM SOLVING

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A problem is traditionally considered as a set of criteria and tasks to be accomplished. In the issues of problem solving, the focus has been centered on problem understanding and strategy planning, which also indicate the mark of two different stages. Most models on the problem solving implicitly suggest that the picture of a problem is quite robust and will maintain its appearance throughout the solving procedure. However, the fact is that solvers usually tackle the problem with misunderstanding and hence fail to finish the job.

The objective of this study is to explore, by using qualitative methodology, the key points where a solver is assuming a good understanding or realizes his mistake on a problem. By analyzing the extent of a problem, recognized by a solver, in the course of problem solving, we try to provide a different perspective on describing a problem and problem reading. Twenty samples have been selected out of different grades, abilities and experiences. The interview data is collected right after samples have finished their work on a set of word problems.

The finding shows that the extent of a problem is not robust. Solvers are frequently bringing in not only useful but also irrelevant or false material in reading the statement of the problem or in the process of problem solving. A solver may find his conception on the problem mistaken when he has new findings which do not agree with the previous ones. Sometimes a solver may come up with a wrong answer serving as the best candidate to meet all the requirements asked by the problem. And the solver can find no hindrance when carrying out the checking procedures due to an incorrect realization of the problem. According to the findings of this study we propose that a problem, in a solver's mind, should be considered as an organic object and its extent will grow or mutate along the course of the problem solving. The problem reading is then better considered as a cognitive activity in realizing every single piece of the material sensed by the solver from the very beginning to the end of the problem solving. Surprisingly, the problem solving efficiency can be greatly improved merely by reminding the solver to keep reevaluating his understanding about all the data in his mind.

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PROBING INDIGENOUS STUDENTS' UNDERSTANDING OF WESTERN MATHEMATICS

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The Supporting Indigenous Students' Achievement in Numeracy (SISAN) Project 2003-2004 was aimed at researching the impact of authentic (rich) assessment tasks on the numeracy outcomes of middle years Indigenous students in a targeted group of remote schools in the Northern Territory[1]. The project involved trialling and evaluating a range of tasks aimed at identifying starting points for numeracy teaching. Initial results suggested that while the rich tasks helped identify 'what works' and highlighted important areas of learning need more generally, for example, number sense and mathematical reasoning, much more work was needed to develop these tasks to the point where they could be used more widely to support remote Indigenous student numeracy learning.

As a consequence, a small number of more focused tasks were introduced which provided a broader range of response modes and allowed teachers to identify learning needs more specifically. Originally developed to support pre-service mathematics teacher education at RMIT University, the Probe Tasks, as they were referred to, were chosen because they require relatively low levels of student literacy and focus on key number ideas and strategies, the area broadly identified by student responses to the rich assessment tasks. Participating teachers typically reported that as student responses to the Probe Tasks were more readily observed, interpreted, and matched to expected levels of performance, they felt more confident about identifying and responding to student learning needs in a targeted way, and as a consequence, more likely to have a positive impact on student numeracy learning. This was particularly the case for the Indigenous teacher assistants and secondary-trained teachers with a non-mathematics background. This suggests that the Probe Tasks, and the associated Probe Task Advice developed to support the work of teachers in this instance, offer a useful means of building remote teacher's pedagogical content knowledge for teaching mathematics.

This presentation will illustrate the Probe Tasks and explore the implications of the teachers' responses to the use of the tasks in relation to improving the levels of Indigenous student numeracy in remote communities.

1 Funding for this project was provided by the Australian Government Department of Education, Science and Technology under the *National Literacy and Numeracy Strategies and Projects Programme*. The views expressed here are those of the author and do not necessarily reflect those of the Australian Government Department of Education, Science and Technology or the NT Department of Employment, Education and Training.

ASSESSING BEGINNING PRE-SERVICE TEACHER KNOWLEDGE: AN EARLY INTERVENTION STRATEGY

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The purpose of this preliminary study is to investigate a way of assessing beginning preservice teachers' knowledge and philosophies of teaching and how that assessment could potentially be used as an early intervention strategy for knowledge development in teacher education programs.

Much of the research in teacher education has divided teacher knowledge into two separate categories: subject matter knowledge and pedagogical knowledge (Ball & Bass, 2000). Learning to teach requires the development of both subject matter knowledge and pedagogical knowledge. Shulman (1986) described pedagogical content knowledge as the knowledge teachers need to teach a particular subject. Grossman (1990) further addressed four knowledge components of pedagogical content knowledge: knowledge of instructional strategies, curricula, how students learn, and why teaching a particular subject is important. This study addresses relationships between pre-service teachers' conceptions of teaching mathematics and knowledge of instructional strategies. It is conjectured that these relationships can address the development of knowledge of how students learn.

The researchers acted as co-teachers in a 14-week undergraduate methods course for students preparing to become middle or secondary school mathematics teachers. Each week, students were required to develop four different representations based on introducing concepts related to proportional reasoning. Students were also required to write a personal philosophy of teaching for a teaching portfolio. The representations and philosophies of three students were compared and were found to be conflicting. All three pre-service teachers expressed a desire to engage students and teach them the utility of mathematics. Yet, two relied on formalized mathematics throughout their representations. This information can be used to address pre-service teachers' emerging perspectives of teaching mathematics.

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THE IMPEDIMENTS TO FORMULATING GENERALIZATIONS

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In this paper, the psychological impediments to the process of generalization are explored within the context of classroom experiments. Extant descriptions of the process of generalization are (1) the abstraction of similarities from disparate problem situations Mitchelmore, 1993); (2) *uniframing*, i.e., casting out cases that don't fit a general concept or definition emerging from numerous cases (Lakatos, 1976; Sriraman, 2004), among many others. These descriptions consist of some similar elements, yet the Gestalt (or the whole) remains elusive. One reason why the Gestalt remains elusive is that the literature describes components of "*what is*" generalization. There is a lack of studies that describes components of "*what is not*" generalization, which would greatly add to the existing literature and fill the missing elements in extant descriptions. To study the impediments to generalization a teaching experiment was conducted with 14 year old students at a rural American high school in which students were asked to solve 5 non-routine combinatorial problems characterized by a common principle (namely the Dirichlet principle) over a 3-month period. Since the research was concerned with exploring factors constituting individual student's psychology of generalization all student work was non-collaborative. Data collected via journal writings and clinical interviews was analyzed using the constant comparative method of Glaser & Strauss (1977) to sieve out factors that impeded students from formulating generalizations. As a result, similar "impeding" focussing factors fell under the categories of "*repeated use of similar examples*", "*focus on superficial (numerical features) of representation*" and "*focus on context*". The study reveals that these focussing phenomenon play an important role in how and what students abstract from a given problem situation and often impede the formulation of generalizations. One implication of these findings is that the theoretical properties of focussing phenomenon need to be further studied by constructing different classes of problems in which the complexity is varied gradually via the use of a complexity load metric. This will allow researchers to document the finer perturbations/variations within focussing phenomena that lead to false generalizations

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STUDENT NUMERACY: A STUDY OF THE NUMERACY DEVELOPMENT OF TEACHING UNDERGRADUATES

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Teacher education is increasingly coming under pressure to be accountable to government for the quality of the graduates going out into the schools. This is at a time when University funding in Australia has been effectively cut back so that even provision of the essential professional experience component of course is under scrutiny because of costs. The literacy and numeracy competencies of teacher graduate teachers are being analysed and questioned. Teachers were once held responsible for lack of literacy and numeracy of students, now it is the University Teacher Education programmes that are being blamed for 'lack of standards' in schools. So the political stance seems to be that 'teachers are not being taught how to teach reading, writing and arithmetic'! In Australia, Institutes of Teaching (Vic) or Teachers (NSW) have been established in each state and are responsible for endorsing courses and programs of teacher preparation, as well as for managing teacher accreditation against determined teaching standards for beginning teachers. The Victorian Institute of Teaching (VIT) requires beginning teachers to build a portfolio containing evidence that teaching standards for registration have been met during a period of induction. The study described here is significant as it addresses one aspect of this public concern, that is, the numeracy competence of undergraduate primary school teachers.

This study aimed to identify the relationship between teacher education students' prior numeracy skills and knowledge and to track the development of numeracy competence during their undergraduate course. All students who entered the BEd(Primary and EC) and BSocSc(Psychology)/BTeach(Prim) courses in 2001 completed an ACER Mathematics Competency test at the beginning of their studies. Ten students from the Primary course were randomly selected and the progress of each of these students was tracked over the passage of their four year course through case studies.

This paper will report the findings from the data collected from the students who have graduated in 2005 and demonstrate growth in mathematical understanding, classroom competence as mathematics teachers and increasing confidence with mathematics as a learning/teaching area in the primary school.

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INTERACTION BETWEEN TEACHING NORMS AND LEARNING NORMS FROM PROFESSIONAL COMMUNITY AND CLASSROOM COMMUNITIES

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This study was designed to support teachers on developing teaching norms based on classroom communities in which students are willing to engage in discourse. A collaborative team consisted of the researcher and four first-grade teachers. The professional community intended to generate norms of acceptable and appropriate teaching based on teachers' observation of their students' learning mathematics in classroom.

The study was based on the theoretical perspectives of cognitive development and sociology in order to examine both teaching and learning in professional community and classroom communities.

The teaching norms being developed in the study was an important notion of analyzing the teachers' pedagogical reasoning in their classroom communities (Tsai, 2004). The teaching norms were not predominated by a criteria set by outside of the professional community. Instead, the teaching norms were generated from each teacher's classroom community and were then continually evolved by the professional community between the interactions of the teachers and the researcher based on some events of classroom communities of the teachers.

As we have shown, the teachers established the teaching norms through their negotiations in the professional community were actively restructured personal beliefs and values and then resulted in their increasing ability of autonomous teaching. The result indicated that teaching norms and learning norms were mutually interactive. It is not only developing teachers' practical teaching, but also improving children's learning. Therefore, one way of improving teacher' pedagogical reasoning and students' learning with understanding simultaneously was to create a professional community for developing teachers' teaching norms in which were based on the development of students' learning norms and contributed to the development of students' learning norms.

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"ERRONEOUS TASKS": PROSPECTIVE TEACHERS' SOLUTIONS AND DIDACTICAL VIEWS

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There is a wide call to have mathematics classes in which students make conjectures, explain their reasoning, validate their assertions, discuss and question their own thinking and the thinking of others, and argue about the mathematical validity of various statements (e.g., NCTM, 2000). One of the issues addressed in the literature is the need to provide the students with rich mathematical environments. However, teachers' readiness to adapt innovative instructional practices is related to their beliefs about mathematics instruction. In our study, we developed materials aimed at encouraging the participants to discuss different solutions to mathematical tasks in light of related rules and definitions. The tasks are designed to stimulate the discussion of questions like: "Why is this so?", "Can we justify this?" "Is this explanation acceptable?"

In this paper we address secondary-school mathematics prospective teachers' solutions to an "erroneous task" (i.e., a task that includes contradicting data) and their disposition towards the presentation of such tasks, in high school classes.

Prospective teachers were presented with an "erroneous task", in which they were asked to determine whether it is possible to draw the graph of a function that satisfies four types of given: (a) $f:R \rightarrow R$, (b) 5 points $(x_i; f(x_i))$, (c) the range where $f'(x)$ and $f''(x)$ are positive, zero or negative, (d) the asymptotes. If the participants' answer was "yes", they had to draw the graph, and if it was "no", they had to justify their position. Then, the prospective teachers were asked whether they would use this task in their classes.

Our results show that the participants knew that it was impossible to draw the graph, and in their justifications they correctly pointed to different options of contradicting given. However, while they mentioned that by this task they gained extra insight into the relationship between the various components of the function and the related graph, their position regarding the presentation of such tasks in classrooms, varied. Several participants unconditionally supported the presentation by addressing the mathematical potential or the instructional merits of the task. Some rejected the presentation and others specified conditions to be fulfilled in order to allow the presentations of such tasks. The conditions related to the students that would benefit (e.g., only advanced students) and to the adequate timing for presenting such tasks (e.g., as an introductory activity).

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BLIND STUDENTS' PERSPECTIVE ON LEARNING THE GEOMETRIC CONCEPTS IN TAIWAN

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This study focused on gaining perspective into the blind student's experience with mathematical concepts addressed included the ideas of parallel, perpendicular, angle and slope. The main purpose of this study collected data on the learning and understanding of aforementioned geometric concepts: parallel and perpendicular. The following questions will be addressed in the study: (1) What are experiences of blind students in learning geometry and mathematical concepts?(2)What experiences do blind students share?(3) What obstacles to learning mathematics are encountered?(4) How does learner come to an understanding of the concept?(5)What types of learning experiences are most beneficial?(6)How are the concepts related to their "every day lives"?(6)What do they think might another learner better understanding the concept? Data collection was conducted primarily through semi-structured interviews. Although there will be a basic outline for the interview. The results of this study provide evidence to support the importance of grounding the learning of mathematical concepts in everyday practical experience, especially for the blind student.

The results of this study provide evidence to support the importance of grounding the learning of mathematical concepts in everyday practical experience, especially for the blind student. The blind student's comprehension of certain geometric concepts is based primarily on her or his application of these concepts to real life situations. The more personal and practical the experience, along with the use of sensorimotor activities and embodied processes, the stronger the comprehension of the mathematical concept. Therefore, accommodations must be made to provide meaningful experiences within the formal educational setting that make use of experiential, kinaesthetic and auditory learning if the results of formal instruction in geometry are to be effective and useful to the blind learner.

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MATHEMATICAL MODELLING AS LEARNING ACTIVITIES

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The purpose of this paper is to elaborate upon mathematics modelling as learning activities using dynamic geometric software for designing mathematics learning system. This learning system consists of a computer environment to display the properties of objects with multiple dynamic linked representations. The major benefit in learning how to do mathematical modelling for students is to encourage them developing a particular way of reflecting and acting on mathematics through making connection between mathematics and real world. We will discuss some examples to show how the learning activities be used in mathematical modelling instruction.

Mathematical modelling is a process of using mathematical language to describe, communicate, express, and think about the real world. The mathematical modelling process is important in both research and learning, however, the process has been applied in the research field, while, students seldom have experience when learning. Research in cognitive science and cognitive development has made it possible to progressively move to new levels of thinking about educational environments that promote learning. Students should be given opportunities to practice mathematical modelling for translating, interpreting, organizing and verifying the real problem to conjecture and generalize their finding. In this paper, we present a theoretical framework for designing and implementing a learning system with multiple dynamic linked representations based on reflection on action. It consists of a learning environment in which students control and operate the objects by means of mathematical modelling.

A prototype computer-based environment for students making mathematical modelling has developed with dynamic geometric software. The experiment of mathematical modelling as learning activities has shown following impacts on students' mathematics learning.

The computer can be an effective tool for students thinking their own way to solve the problems. Students increase their metaconceptual awareness that mathematics is not just the product of mathematicians work, but continually evolves in response to challenge both internal and external representations. Students realized that it is not necessary to spend much time on tedious calculations and memorizations in mathematics learning. Instead, they noted that it is useful for explore mathematics by the use of computer simulation.

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INNOVATIVE TEACHING APPROACHES IN DIFFERENT COUNTRIES

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Graham Littler, University of Derby

This paper presents the results of the second year of the project IIATM "*Implementation of Innovative Approaches to the Teaching of Mathematics*". The project is realized by the collaboration of four European Universities (Charles University of Prague, Aristotle University of Thessaloniki, Cassell University and University of Derby) and aims to bring together teachers and teacher-trainers from different European countries. Teachers, across national boundaries, experience common constructive teaching approaches in their classroom, recording and exchanging experiences. International research shows that the shift from a familiar instructional practice to an innovative approach is not easily accomplished (Fennema & Neslon, 1997). Teachers find it very difficult to change from their established transmissive ways of teaching, even if the curricula and their trainers propose very interesting and creative tasks (Desforges & Cockburn, 1987). Research also shows that providing teachers with experiences where their own practices are challenged and opportunities to reflect on and rethink about them, has the potential to facilitate new insights and understandings of the teaching process (Aichele & Caste, 1994). The IIATM project allows teachers to try common activities in culturally different classrooms and encourages them to exchange ideas and experiences, comparing the use of constructive teaching strategies.

During the first year, the groups of researches and teachers established in each institution, developed tasks concerning various mathematical topics (Geometry and Polygons, Functional Thinking, Patterns leading to Algebra, First Arithmetic Concepts) (Tzekaki & Littler, 2004). During the second year, the groups exchanged the tasks and evaluated their use in different environments.

The collection of illustrations of the constructivist approaches with common tasks, including the analysis of classroom experiences from teachers' outcomes, as well as comments, discussion and an overview of remarks gained from this evaluation in two countries (Greece and Czech Republic) will be developed in this presentation. *(The Project is funded by the European Commission's Socrates/Comenius).*

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EQUITY AND TECHNOLOGY: TEACHERS' VOICES

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Teachers use technology in mathematics to enhance the classroom ambience, assist tinkering, facilitate routine processes, and to accentuate features of mathematics (Ruthven & Hennessy, 2002). However there is some evidence to suggest that the use of technology may be accentuate cultural inequalities (Vale, Forgasz & Horne, 2004). Furthermore, innovations in the use of ICT in schooling, including those involving mathematics, have not targeted students from socially or culturally disadvantaged backgrounds (Kozma, 2003).

Teachers' understanding of diversity and equity is varied and related to their school setting (Quiroz & Secada, 2003). According to the literature, equity involves equal access, equal treatment, fairness and a commitment to achieving equal outcomes and the characteristics of equitable classrooms include: connectedness, collaboration, support, intellectual quality, and respect for difference. However achieving it is a complex task for teachers working in socially disadvantaged schools. In this presentation I will report on a current study that is exploring teachers' understanding of diversity and equity and how this relates to their practice regarding the use of technology in their junior secondary mathematics classes.

Teachers from socially disadvantaged secondary schools in Melbourne were participants in this study. Twelve teachers who use technology regularly in junior secondary mathematics and who gave priority to success for all students in their classrooms were selected. In this first phase of the study, the teachers have been interviewed about the meaning of equity and how they used technology in mathematics. Preliminary analysis of these data will be presented.

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THE STATE AND IMPACT OF GEOMETRY PRE-SERVICE PREPARATION – POSSIBLE LESSONS FROM SOUTH AFRICA

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The responsibility of pre-service preparation programs includes making evident the complexity of teaching and adequately preparing teachers for their roles as mediators of learning, interpreters and designers of learning programs and materials. This paper summarizes a two year study investigating the state of prospective teachers' (PTs') (n=254), teachers' (n=18) and students' knowledge (n=103) of Grade 7 geometry (using the van Hiele theory (1986) and acquisition scales of Gutierrez, Jaime & Fortuny (1991)). Three foci direct this study, the impact of different pre-service preparation time frames (3 years versus 4 years) on PTs' geometry content knowledge being the first. The second investigates the possible relationship between teachers' content knowledge and the students' learning gain (measured by the same van Hiele-based geometry questionnaire). The third focus is the effect of teaching experience on both the teachers' own level and degree of geometry acquisition as well as the resulting (self-reported) classroom practice. Results indicate that both teachers and PTs (irrespective of preparation time frame) fail to reach the level of geometric thinking and degree of acquisition expected (van Hiele Level 3- the level teachers are expected to teach). Only one of the four participating grade 7 classes made a practical significant learning gain on the informal deductive level (van Hiele Level 3). There seems to exist a possible relationship between the learning gain made by students and the teachers' pre-service education and years of teaching experience. Results further show that PTs exit school with higher geometrical acquisition than after three years of mathematics content and methodology training or after four years of methodology training. This shocking revelation could indicate that pre-service preparation programs had no significant impact by either maintaining or positively impacting on the already attained thought levels. One conclusion is that PTs and teachers are not adequately in control of the grade 7 Geometry subject matter they have to teach which has implications for classroom teaching and learning. The results have serious implications for pre- and in-service training and suggestions on the features of an improved program are made.

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EXAMINING TASK-DRIVEN PEDAGOGIES OF MATHEMATICS

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Task-oriented pedagogies predominate in mathematics classrooms around the world. These pedagogies are premised on the belief that children's mathematical learning develops best through teachers' careful management of task selection, allocation, administration, and assessment. This practice must be examined in light of international declarations advancing children's right to participate in decisions affecting their lives, and recognising the benefits of learner-negotiated pedagogies.

Recent mathematics research has focussed on classroom norms as a significant variable in children's learning, and particularly on how mathematical tasks enhance children's thinking, reasoning and working mathematically. Children have seldom been viewed as critical collaborators in this process. The following responses from child interviews gathered over three years of research in 33 primary classrooms in New Zealand (Walls, 2003) typify children's experiences of learning mathematics:

Jared: The teacher says, "Go and get your maths books out", and she writes stuff on the board for maths. (Late Year 3)

Georgina: We get into our [ability] groups and do the worksheet. (Mid Year 4)

These statements speak powerfully about everyday classroom cultures in which mathematics and its learners are shaped by teacher-selected mathematical tasks.

Changes in ethical and legal discourse in support of children's participatory rights as global citizens, oblige us to re-examine current pedagogies of mathematics. Pollard (1997) for example, advocates for *negotiated curriculum:* "…rather than reflect the judgments of the teacher alone, it builds on the interests and enthusiasms of the class…Children rarely fail to rise to the occasion if they are treated seriously" (p. 182). Similarly, in an explication of the articles of the 1989 UN Convention on the Rights of the Child, a recent UNICEF report says, "we see…children actively involved in decision-making at all levels and in planning, implementing, monitoring and evaluating all matters affecting the rights of the child" (UNICEF, 2002, p.11).

If we are to honour children's right to significant agency in their own learning journeys, mathematics educators must now consider partnership with young learners as a necessary evolution from adult centred, task-driven pedagogies of mathematics.

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INCLUSIVE MATHEMATICS: CATERING FOR THE 'LEARNING-DIFFICULTIES' STUDENT

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Current mathematics education reform efforts require a commitment from schools to offer inclusive learning opportunities that will maximise student potential. At the forefront of this requirement are questions concerning the optimal student learning arrangements for students with learning difficulties. Policy makers recognise that for reform intent to be operationalised in the classroom there needs to be a school–wide approach to the organisational aspects of inclusive teaching and learning. Given our commitment to the mathematical development of students with learning difficulties, we wanted to examine the approaches taken by schools to cater for the mathematics experience of students with learning difficulties/learning disabilities (LD).

In the study we were particularly interested in the approach to mathematics teaching and learning of LD students undertaken by schools to see if it matched anecdotal evidence. Anecdotal evidence suggests that the LD classroom teacher is not always the person who wants to work with LD students. The classroom experience of LD mathematics classes, the story goes, is about keeping the students occupied and busy, and, above all, out of trouble. Achievement and effort grades assigned to these students are often restricted to C, D or E, regardless of student effort and content knowledge. Furthermore, anecdotal evidence suggests that LD students have poor motivation, their self-esteem is very low, and their post-school options are limited.

We wanted to determine how the mathematics learning needs of LD students were being met. Barash and Mandel (2004) report on the introduction of a programme for seventh grade LD students, developed and taught by pre-service teachers. We wanted to know what types of programmes were established for LD students in New Zealand schools. We were interested in exploring how mathematics classes are organised for LD students, how LD students were selected for mathematics groups and by whom, how their learning was assessed and who teaches them.

In the presentation we will report on the findings from our survey research with 74 schools. The research elicited information on a wide range of school issues concerning LD students.

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CHILDREN'S NOTATION ON EARLY NUMBER COMPUTATION

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Mental computational strategies have taken centre stage in a range of numeracy programmes world-wide. However, despite the move away from traditional written algorithms notation remains a critical component of students' mathematical development. Reform practices encourage students to be active participants in inquiry processes—as part of the communication process student need to develop notational systems to describe their own mathematical activity. Students' emerging ways of symbolizing and notating provide a vehicle for communication, representation, reflection and argumentation (McClain & Cobb, 1999). Within New Zealand, numeracy curriculum support material (Ministry of Education, 2004) indicates that informal jottings of students are to be encouraged as a "way to capture their mental process" (p. 8) so that their ideas can be shared with others. Notation models provided include the empty number line and annotated ten frames.

This paper reports on a teaching experiment focused on the Year 5/6 students' development of notational schemes within a unit of addition and subtraction. In particular, the research was interested in determining how expectations for using notation to record mathematical thinking could be more firmly established within the classroom and how notational practices might best support students' sense-making practices by providing a reference point within group and class discussions.

While the study provided evidence that teacher focused use of notational schemes can effectively support the norms of explanation and justification, it also highlighted a range of dilemmas for the teacher. These included timing of the introduction of notational schemes for some children, the tension between individual children's idiosyncratic notational schemes and more formalised teacher notation, and the potential for notational schemes to act as a barrier by over-riding children's informal thinking. The findings illuminated the complexity of the classroom and the challenges that the teacher faces in attempts to place children's thinking at the centre of her decision making.

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TEACHER BEHAVIORS AND THEIR CONTRIBUTION TO THE GROWTH OF MATHEMATICAL UNDERSTANDING

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This research focuses on teacher behaviors and the impact of those behaviors on students' mathematical activity and understanding over the course of several months. We do this by documenting, coding, and analyzing video-data, reflections and other artifacts collected from an eighth grade inner city mathematics class. The classroom visits were part of a professional development project in which the teacher/researcher (first author), who is a mathematics education researcher at a local university, routinely met with local teachers in their classroom and at a course (that the first and second author taught), where they discussed key ideas relating to classroom implementation, the development of mathematical ideas, and other relevant issues.

Our conceptualization of teacher behaviors extends the literature on teacher questions and behaviors (eg., Schorr, Firestone & Monfils, 2003), based on the first author's experience in a variety of teachers' classrooms and inductive analysis of the data in the present study. Some examples of identified teacher behaviors include the teacher: showing evidence of listening to a student's idea; highlighting or placing a high value on a student's idea; encouraging a student to link representations to each other.

This can have important implications for teacher development. For example, we noticed that as the teacher encouraged students to look at the relationship between and amongst their own representations, students were able to link these representations to each other, which contributed to their move to an outer layer of understanding within the Pirie/Kieren model (Pirie & Kieren, 1994). By looking more closely at the relationship between teacher behaviors, student behaviors, and ultimately linking this to the Pirie/Kieren theory for the growth of understanding, we can gain deeper insight into the chain of events that unfold in classrooms. This, in turn, will allow us to obtain a richer understanding of the complexities of teaching in an urban environment, and has the potential to contribute to the research base relating to teacher development and student learning.

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TEACHERS' BELIEFS OF THE NATURE OF MATHEMATICS: EFFECTS ON PROMOTION OF MATHEMATICAL LITERACY

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Various studies have shown that what teachers consider to be optimal ways of teaching mathematics and mathematical literacy is influenced by their beliefs about the nature of mathematics. It is therefore advantageous to determine teachers' conceptions of the nature of mathematics before developing curriculum interventions. In this study various methods were employed to stimulate teachers to both reflect on their beliefs and to make them explicit. A Likert-scale questionnaire was administered to 339 in-service teachers in urban and rural areas of the Eastern Cape, South Africa. A sample of ninety-five of these teachers completed a questionnaire based on videotapes of lessons recorded during the TIMSS (1995) study that they had viewed. These teachers also ranked their own teaching on a continuum ranging from traditional to constructivist approaches and provided explanations for their ranking. A further sub-sample of thirty-six teachers participated in individual interviews, which explored their perceptions of the nature of mathematics and their own teaching practice. In order to investigate whether these beliefs are mirrored in practice, four teachers were videotaped in their classrooms. The data generated by these videos support the findings of similar studies, i.e. that teachers' beliefs of the nature of mathematics are often not reflected in their practice. This has far-reaching implications for the implementation of compulsory mathematical literacy to grades 10, 11 and 12 in South Africa, as the mode of delivery is envisaged to be through contextual problem solving.

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ACTION RESEARCH ON INTEGRATING BRAIN-BASED EDUCATIONAL THEORY IN MATHEMATICS TEACHER PREPARATION PROGRAM

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At the end of the 20^{th} century, the United States implemented the educational philosophy and principles based on the research evidences of human brain and learning. Educators emphasized that the more understandings of our brain, the more effective curriculum and instruction can be designed by teachers (Chang, Wu, & Gentry, 2005). Thus, the evidences and reflections from brain research influenced educational theories and provided scientific characters for all levels. In Taiwan, the brain-based educational theory is still a new phrase that needs to be introduced and implemented extensively and intensively. Thus, the purpose of this action research, using a mixed method approach along with non-sequential and concurrent triangulation strategies, is to apply it into the real classrooms within the mathematics teacher preparation program and examine the processing changes of pre-service teachers who do not specialize in mathematics education in order to assist them for the future teaching.

In order to face the various myths and misunderstandings of brain's development and confront teaching and learning problems of elementary mathematics education, the better way is to go back to the teacher education and devote extra efforts to train the pre-service teachers. Accordingly, researchers re-designed the course of "teaching elementary school mathematics" by shifting the curriculum and instruction design associated with the brain-based educational theory and the nature of mathematics. By working with 66 pre-service teachers closely in National Hsin Chu Teachers College, Taiwan and providing more integrated contents, hands-on activities, and opportunities of thinking, data were collected qualitatively and quantitatively with pre- and post-tests, observations, and reflections. Results indicated that their self-efficacy ratings toward mathematics increased significantly, as well as rasing their interests and reconstructing confidences in learning and teaching mathematics in the elementary classrooms. Reflections and recommendations were also valuable for revising the course design.

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USING WRITING TO EXPLORE HOW JUNIOR HIGH SCHOOL GIFTED STUDENTS CONSTRUCT MODEL IN PROBLEM SOLVING

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This study explored modeling of junior high school gifted student by thirty 7th-grade students as they participated in model eliciting activity and writing of learning journal in problem solving. This article describes development of student's ability of description and modeling through team cooperative discussion, verify, presentation and comment in inquiry oriented teaching environment, and using Discourse analysis to team cooperative verifying process, evaluation and reflection that are excerpted from student learning journal.

INTRODUCTION

This study uses writing-to-learn strategy in mathematics classes, engage thirty 7th-grade gifted student to participate in model eliciting activity and inquiry oriented teaching environment. The forms of students' journals writing are used in this study involves logs, journals and expository writing (Strackbein and Tillman, 1987). Theoretically, "Modeling cycles" (Lesh and Doerr, 2003) , "Three modes of inference making employed in sense-making activities" and "A past instance of semiosis can become the object of new semiosis" (Kehle and Lester, 2003) are used in this study as foundation of writing text analysis and model eliciting activities.

METHODOLOGY

Essentially, researcher as teachers adopted inquiry-oriented teaching strategy, and guided students writing journal with inquiry of problem-solving task in this class. In practice, there are nine problem tasks in this study. We audio taped teaching and interview sessions, and adopted Discourse analysis (Gee, 1999) to organize, analyse, classify, and consolidate the data which included writing text, then determine themes.

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AN ALTERNATIVE MODEL ON GIFTED EDUCATION

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Due to the limitation on the availability of the resources, students have to pass certain examinations to be able to enter the gifted programs. The precious gems characterizing giftedness, such as creativity and originality, are mostly destroyed in preparing for the exams. To avoid this disaster, we address an alternative teaching model suitable simultaneously both for gifted identification and development.

In this eighteen-month study, the samples consist of seventy fifth graders in Taiwan, with above-average academic achievement. The design of the instruction is based on the philosophy that learning is an opportunity for finding personal intrinsic ability instead of mere knowledge acquisition. Inspired by this philosophy, the instructor plays the role as a supporter, not an examiner, of students' idea. For identification, the learning behaviour scale (LBS) is chosen according to the longitudinal nature of the implementation and the high validity as an indicator of the academic achievement. The principles of the instruction and the factors of LBS are analyzed, compared, and matched. This remains mostly absent in the previous literatures on the topics related to LBS. To test the domain specific ability, the factor concerning the creativity on mathematics is added, an aspect that has long been ignored in LBS.

The results of this study are:

1. Hatched by the sophisticated knowledge from the instructor, students' raw, but exclusive, idea can provide a different view that is more interesting than the traditional way. Students' eminent creativity, demonstrated in treating a new task by devising novel schemes, is nurtured rather than taught.
2. A teaching model to serve both for identification and development is possible on the group with above-average academic achievement.
3. LBS scores do provide a good correlation with the mathematics achievement.
4. The frequency of presenting student's personal thinking has high correlation with student's creativity. This factor yields a powerful facet on gifted identification.

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LEARNING MATHEMATICAL DISCOVERY IN A CLASSROOM: DIFFERENT FORMS, CHARACTERISTICS AND PERSPECTIVES. A CASE STUDY

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Nowadays the phenomenon of mathematical discovery, its mechanism and mental processes remain into the educational research limelight (Burton, 1999; Tall, 1980). Indeed, the concept of mathematical discovery has quite many common features with learning process for being considered together. We understand *learning mathematical discovery in a classroom* as a short-term active learning process aimed at the development of students' abilities to assimilate new knowledge through the use and interpretation of their existing knowledge structures with the help of a teacher or with considerable autonomy and only teacher's control of the direction of inquiry activities within the topic studied. The main question of our research was the following: How could students' inquiry work in a classroom be modified to simulate mathematicians' practice and what were the ways of evaluation of students' work in such activities? We tried to answer this question in the context of using three different forms of students' inquiry work in a classroom. We took the position that Active Fund of Knowledge of a Student (AFKS, Yevdokimov, 2003) was the most relevant structure to introduce new characteristics for studying this process. For quantitative evaluation of student's conscious involvement in the process of learning mathematical discovery in a classroom we considered an index I of using own AFKS by every student, i.e. I served as indicator how much AFKS was involved in doing each task. We studied the character (logical or non-logical) of using AFKS within learning mathematical discovery in a classroom. Analysing the data received we found out that we can construct a set of key problems with indicated in advance quantitative scale of using extra-logical processes for students' inquiry activities in learning mathematics. Thus, we can distinguish and regulate the illumination stage of learning mathematical discovery, we can adapt it to the needs of classroom activities or to the thinking process of a certain student involved in these activities.

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STUDENTS' VIEWS ABOUT COMMUNICATING MATHEMATICALLY WITH THEIR PEERS AND TEACHERS

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This paper reports on the responses of approximately 180 nine- to eleven-year-olds during individual interviews. The children were asked a range of questions designed to explore their perspectives on mathematics learning, including questions about the importance of working out problems mentally, of getting answers correct, and whether they thought that there was only one or several different ways of working out an answer. They were then asked the following questions and the reasons for their responses:

> Do you think it is important for you to know how other people get their answers? Is it important for you to be able to explain to other people how you worked our your answer? What about your teacher - is it important to be able to explain your thinking to your teacher?

Idea	**Yes**	**No**	**Not Sure**	**Total**
Knowing others' strategies is important	38.8	47.9	13.3	176
Explaining thinking to others is important	55.4	27.1	17.5	178
Explaining thinking to teachers is important	84.6	7.7	7.7	169

Table 1: Percentage of students who thought particular ideas were important

Almost all students concurred with the idea that explaining one's thinking to one's teacher is important. A wide range of reasons was given for agreement with this idea. Some reasons were related to teachers' actions, such as assessing students' understanding, making decisions about grouping students, helping students with their learning, reporting to parents. Other reasons were more to do with students' concerns, such as "proving" that they had worked out answers for themselves. A few students thought that explaining their thinking to their teacher could help that teacher with his/her own mathematics learning. The verbatim quotes from individual children provide insights into the children's unique perspectives on their mathematics learning, and underline the importance of taking children's views into account (Cook-Sather, 2002; Young-Loveridge & Taylor, in press).

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POSTER PRESENTATIONS

WORKING WITH MATHEMATICAL MODELS IN CAS

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CENTRAL BOX WITH THE KEY QUESTIONS

According to experiences gained by the participants in the development project 'World Class Math & Science' computer use in upper secondary school mathematics has certain potentials. The key question in the Ph.D. project is:

-How could these potentials be identified, captured and conceptualised?

Initial inquiries and studies led to the hypothesis: Introduction of the new construct of a conceptual tool denoted 'flexibility' is a suitable conceptualisation of the potentials.

The suitability of this conceptual tool was evaluated on the background of the Ph.D. project's objective of changes. A subproject introducing the modelling approach to differential equations was chosen for inquiry of the questions:

- Is 'flexibility' a supportive construct for articulation of experiences from teaching and learning within a modelling approach? For realisation of the learning potentials of students' concept formation within this approach?

SURROUNDING TEXT BOXES

Setting: The World Class Math and Science project. Laptops in math and science for upper secondary school.

Flexibility: Background, foundations and definition. Also in http://www.icme-10.dk/index.html

Changes: 1) At curriculum level: Change from a structural point of view on differential equations to a dynamic, modelling viewpoint. 2) In the use of models and modelling: From a functional perspective of 'applied math' to inclusion of a concept formation perspective. 3) At the level of teachers' professional development: Articulation of the teachers' tacit knowledge.

Methodology. Interpretative approach, using a teaching experiment design with classroom observations etc., analysed from an 'emergent perspective' (Kelly&Lesh)

Example of data analysis. Excerpt from transcription of video recordings of students work in a small group, followed by analysis of the episode. Interpretation in terms of flexibility of how an 'emergent model' was established and negotiated

Conclusion

The notion of flexibility is useful to structure the analysis and put some potential of computer use and of modelling perspectives in focus of attention.

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Gravemeijer et al. (2002). *Symbolising, modeling and tool use in mathematics education*. Kluwer.

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A NON-STANDARD MATHEMATICS PROGRAM FOR K-12 TEACHERS

Patricia Baggett — New Mexico State University

Andrzej Ehrenfeucht — University of Colorado

At New Mexico State University in Las Cruces, New Mexico, USA, we offer a program in mathematics, started in 1995, for practicing and future teachers which attempts to provide mathematical knowledge that is at the same time both modern and useful. It significantly changes the mathematical content of teachers' mathematical education. It leans toward concrete applications and design and creation of artifacts, and uses calculator technology from the earliest grades. The program is not connected to any specific curriculum.

We offer six one-semester courses (and a seventh in August 2005) covering topics that teachers from kindergarten through high school can use. Each course has a central focus and can be taken at the graduate level (by practicing teachers) and at the undergraduate level (by students who are future teachers). The foci are: Arithmetic and Geometry (mainly for elementary teachers), Algebra with geometry and Use of technology (mainly for middle school teachers), and Mathematics with science, Algebra with geometry II, and Calculus with hands-on applications (mainly for high school teachers). Undergraduates act as apprentices to practicing teachers and are required to make at least ten hours of visits to their classrooms, where they observe, co-teach, and teach under their mentors' supervision. Teachers and future teachers often teach lessons that they studied in the university class to pupils, adapting them to their particular grade level. In the university class, writing is the central method used in assessing students' learning. We collect writings of teachers and undergraduates, and evaluate their understanding of the material, and how they taught the lessons in classrooms. We gather recalls of pupils who were taught the lessons, and artifacts that they created. So we can see what has been learned at several levels.

Participants consistently evaluate the program as relevant and interesting. We know that many alumnae and alumni who are now practicing teachers still use the lesson plans that they originally studied in these courses. We evaluate the effectiveness of individual lessons and courses, but not of the program as a whole.

In the poster we address three aspects of our program: What we are attempting to teach and why, and how it is being done. We will include examples of specific lesson plans, show samples of the work of pupils from different grades, and discuss evaluations of the lessons and courses.

Many lessons used in the courses, current syllabi, and a more complete description of our program, are at http://math.nmsu.edu/breakingaway.

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SUPPORTING THE DEVELOPMENT OF MIDDLE SCHOOL MATHEMATICS TEACHERS'EVOLVING MODELS FOR THE TEACHING OF ALGEBRA

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This study investigates the evolving instructional models in the daily practice of middle school teachers as they design, test, and revise reflection tools to guide their teaching of algebraic thinking and modelling.

Many middle school mathematics teachers equate the teaching of algebra with demonstrating procedures for symbol manipulation, simplifying algebraic expressions and solving and graphing linear, quadratic and more complex equations. In the US, most students' first experiences with algebra are in a traditional algebra course offered at the 7^{th}, 8^{th} or 9^{th} year. Rather than traditional symbol manipulation instruction, students at all levels should have opportunities to model a wide variety of phenomena mathematically, to represent, explore, and understand quantifiable relationships in multiple ways. In order for this learning change to take place in classrooms, teachers' instructional models of teaching must change (Doerr & Lesh, 2003). This study investigates those teaching models as they evolve in the daily practice of middle school teachers as they design, use, and revise reflection tools to guide their teaching of algebraic thinking and modelling at the middle school level.

The ideas offered in this poster presentation are preliminary results from a research project in progress. The aim of this study is to document and articulate the change and growth of teachers as they use their classroom practice as a learning environment for their teaching. It adopts a design experiment method (Brown, 1992) in which the participating teachers are designing, implementing and revising reflection tools for analysing their practice as they design learning environments for their students to learn a "new" algebra.

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EVALUATION OF SUGGESTED ITEMS IN PORTUGUESE MATHEMATICS TEXTBOOKS

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Textbooks play an important role in mathematics education. Very often, teachers rely on textbooks to implement mathematics curriculum which influences students' achievement. Thus, analysis of textbooks may help to understand student's mathematics performance. In mathematical textbooks, the suggested items reflect situations where students can potentially be actively involved in the learning process. In order to enhance student's mathematical literacy, tasks should be designed to trigger the use of different cognitive processes. This study focuses on the analysis of mathematical items and purports to be a contribution to the analysis of textbooks.

In this research study, the proposed items of two textbooks were analysed according to the OECD/PISA framework. They were 9th grade mathematics manuals reasonably popular among teachers. Three components were analysed: context – the part of the student's world in which the tasks are placed; mathematical content – mathematics "major domains", and competencies – mathematical processes that need to be activated to solve a real problem through the use of mathematics. The cognitive activities that competencies encompass are grouped into three competencies clusters: (1) Reproduction; (2) Connection; (3) Reflection.

From the 344 items analysed, 61% of them did not present a context. This means that de majority of the items did not provide a situation that could be a part of students life. With respect to mathematical content, the items are mainly included in the "Quantity" and "Space and Shape" categories. Data analysis also showed that 81.4% of the analysed items only require competencies encompassed in the reproduction cluster. These are items that lead students, predominantly, to select routine procedures and/or apply standard algorithms. Also, they mainly involve mainly familiar contexts, clearly defined questions and require only direct reasoning and literal interpretation of the results.

In conclusion, the analysis showed that in most cases items suggested in manuals do not have a real-world context and only lead to the reproduction of practiced knowledge. This type of problems does not give the opportunity to perceive mathematics as a way of understanding. Instead they lead to believe that doing and knowing mathematics means memorizing and applying a sequence of algorithms/rules correctly.

THE INVESTIGATION OF CONCEPTUAL CHANGE AND ARGUMENTATION IN MATHEMATICAL LEARNING

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Following the conceptual transition of learning and teaching, the object of knowledge construction have to be developed by students through their teachers. Therefore, the target of mathematical learning is to emphasize understanding of mathematical knowledge rather than repetition from memory. This paper reports on the performance of three students in their first-grade of senior high school on tasks about integral number, involving questions on divisibility.

This paper was a qualitative research project. The first purpose of this study was to use Posner's (1982) conceptual change model (CCM) to inquire how the three students make others' conceptual ecology become unbalanced by their dialogues and to bring their conceptual change under the cooperative learning context. The second purpose of this study was to use the framework proposed by Toulmin (1958) to examine the three students' argumentative performances. The collected information included the videos, coding data recording the process of the three students' learning, and the individual student's papers.

The main results were: Firstly, The three students would change their conceptual framework after their conceptual ecology became unbalanced through communicating, thinking and reasoning with each other. Secondly, the approach of the three students' argumentation included visual experienced argumentation, using examples argumentation and formal theory argumentation.

This highlights that the teacher can and should construct a learning context in which students can think, participate in mathematically valid argumentation, and develop meaningful mathematical learning.

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TEACHING TIME BY PICTURE BOOKS FOR CHILDREN IN MATHEMATICS CLASS

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The experience of time is by no means strange to children. However, length, weight, area, etc can be taught by suitable physical objects but time could not be. Since the current belief in mathematics teaching stresses connecting real life (NCTM, 2000) and horizontal mathematization (Freudental, 1991) reasonable to provide some concrete situations in teaching time. This is conformed with the ides of the Realistic Mathematics Education (RME), anything that can help children to image, to development a model of some thing up to a model for something else, is good.

Monroe, Orme, and Erickson (2002) said that there are general or highly specific situation to help learners build time concept in children literature. For example, Willians told how the Shelans worked in cotton field from sunrise till sunset in (Working Cotton) to develop time vocabulary, time quantity and the order of events. The researcher led a group of teachers to search out picture books to teach time for different grades.

We collected sixteen pictures books, ten of them were published in Chinese translation. The title of twelve books contained time terms such as *Sunshine*, *Spring is here*, *Tuesday*, …etc. We analysed each book and listed themes associated with time concept. For example, *Clocks and more clocks* is suitable for low grades to discuss the order of events, how to tell time and to sense the flowing of time. *The Grouchy Ladybug* is suitable for low and middle grades to discuss the order of events, how to telling time, am-pm, what is a day, and the periodicity of day. *All in a day* is suitable for middle and high grades to discuss 24 o'clock, what is a day, the periodicity of day, and the time zone and lapse. In the design of a teaching plan, we use the picture books in the three ways, to induce interesting, to develop concept and to extend or apply.

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THE OVER-RELIANCE ON LINEARITY: A STUDY ON ITS MANIFESTATIONS IN POPULAR PRESS

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At several places, the practical and research-oriented literature on mathematics education (and occasionally also the literature on science education) mentions students' tendency to illicitly rely on linearity in non-linear situations. Recently, numerous manifestations of students' overuse on linearity in diverse mathematical domains and at various educational levels were re-analysed by De Bock, Van Dooren, Janssens and Verschaffel (2004) in order to unravel the psychological and educational factors that are at the roots of the occurrence and persistence of this phenomenon. As a result, these authors found three (complementary) explanatory elements for students' overuse on linearity, namely (1) students' experiences in the mathematics classroom, (2) the intuitive, heuristic nature of the linear model, and (3) elements related to the specific mathematical problem situation in which the linear error occurs.

This poster shows the results of an ongoing study on the overuse of linearity in newspapers and popular magazines. Different manifestations are discussed and related to the explanatory factors unravelled by De Bock et al. (2004). Moreover, these manifestations are classified and commented from the perspective of the authors' intentions while consciously or unconsciously overusing linearity. This led us to three different categories: (1) manifestations clearly intended to mislead and manipulate the reader, (2) authors' deliberate choices to justifiably simplify a non-linear situation for his or her audience, and (3) examples in which the author was clearly unaware of the problematic use of linearity in the given situation.

After having illustrated and categorized different manifestations of the overuse of linearity, we discuss the usefulness of misleading (linear) representations in popular press for mathematics education. Is it desirable and feasible to design learning activities based on misleading or partial (linear) representations that regularly appear in newspapers and magazines? Can we learn students to disguise this type of information and can it contribute to educate them to become critical citizens? To what extent this is a more general educational goal or a specific goal for mathematics education?

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SOFTWARE FOR THE DEVELOPMENT OF MULTIPLICATIVE REASONING

Dmitri Droujkov — Natural Math, LLC
Maria Droujkova — North Carolina State University

Interactive multimedia tools can help children make images, mathematize actions, link formal and informal representations, and notice properties of systems. Software can support the growth of mathematical reasoning from qualitative, intuitive grounding.

We research and develop a suite of programs helping young children work in multiplicative environments and see the underlying algebraic structures (Carraher, Schliemann, & Brizuela, 2000). To support various learning actions, suite parts provide different levels of openness and direction.

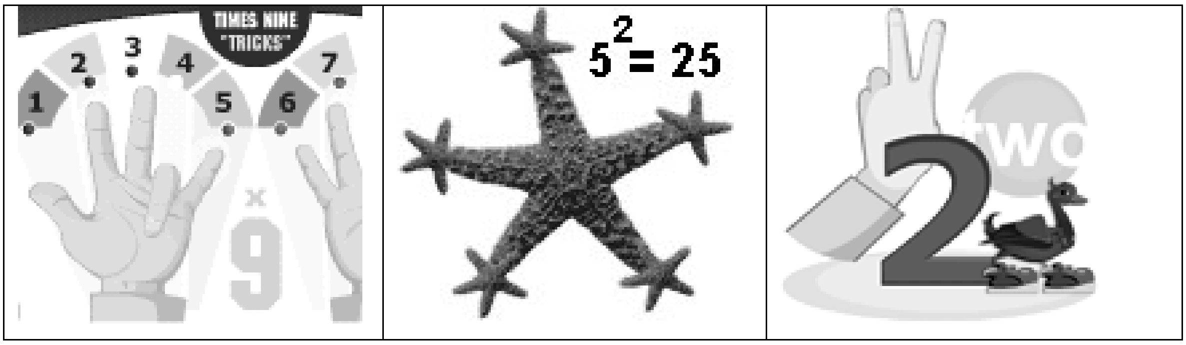

Figure 1: Screenshots of the software components

Theme playgrounds establish common mathematical actions, such as “finger calculator” tricks or creation of combination tables
Translation puzzles link different formal and informal representations and help children develop a mathematical language shared with others
Dynamic illustrations support interactive “eye openers” and grounding
Design worlds allow children to create their own representations
Problem solving tasks help with classical and novel multiplicative problems
The software helps children coordinate qualitative and quantitative worlds (Droujkova, 2004) in each context, providing qualitative grounding for mathematical reasoning.

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TABLES AND YOUNG CHILDREN'S ALGEBRAIC AND MULTIPLICATIVE REASONING

Maria Droujkova

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Far beyond the humble role of a storage device, the table can be a powerful tool for young children's conceptual learning. Using tables in qualitative, additive and multiplicative worlds, children develop algebraic and multiplicative ideas such as covariation, binary operation, distribution, or commutativity.

This study focuses on children age four to seven working with table representations. Children start learning the row-column structure from the qualitative operation of *combining* features, such as eyes and mouths in simple face drawings. They move to iconic representations of quantities and *counting* operations, and to symbolic representations of numbers with *additive* (Brizuela & Lara-Roth, 2002) and *multiplicative* operations (Figure 1).

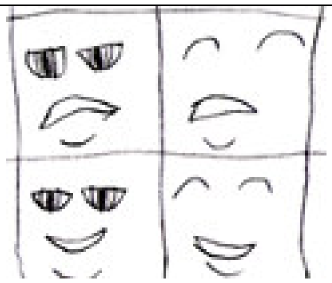	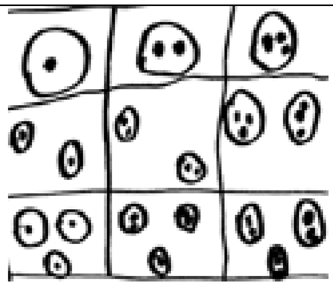	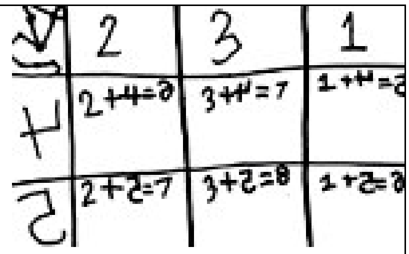
Combine mouths & eyes	*Count* dots & circles	*Add* numbers

Figure 1: Combining, counting, and adding operations in tables

Several issues with children's use of tables came up in the study. Children prefer to see features appear in each cell, rather than to use row and column labels. Children either work with a binary operation between co-varying row-column features, or with a unary operation on columns, varying the operation by rows. These two ways of thinking lead to significantly different table actions and reasoning. Children can transfer the table structure and actions between qualitative, additive and multiplicative worlds (Droujkova, 2004). Educators can help young children develop table reasoning qualitatively using established everyday ideas and transfer it to quantitative operations.

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STUDENTS' USE OF ICT TOOLS IN MATHEMATICS AND REASONS FOR THEIR CHOICES.

Anne Berit Fuglestad

Agder University College

This poster reports from a three year development and research project with mathematics classes in year 8 – 10. The aim was, in accordance with the curriculum guidelines (KUF, 1999), to develop and evaluate students' competence to chose appropriate ICT tool for a specific mathematical problem (Fuglestad, 2004). The project was situated within a social constructivist perspective of learning aiming to develop an ICT rich learning environments with opportunities for students' choices and discussions. In project meetings with the teachers every term some ideas and material for teaching were provided, and an important part was to report and discuss experiences, features of the ICT tools and further developments of ideas.

In a two weeks working period in the final part of the project the students were given a collection of 12 tasks to work on. The tasks were designed to give options for ICT use, with variation in levels and degree of openness; some had a clear question and others presented just an open situation and students had to set their own tasks. The students chose what tasks to work on, and what tools to use: mental calculation, a calculator, paper and pencil, ICT tools or a combination. They could work alone or in pairs and discuss their solutions. The work in the classes was observed, and partly audio and videotaped. Data was also collected in a questionnaire.

One or two weeks later the students were given a questionnaire connected to their experiences in the work, what tools they chose to use and why. They answered questions about tasks they had worked on, what they liked and did not like and for some new tasks they were asked to read and judge what they think were appropriate.

The results revealed that many students liked challenges and difficult tasks and disliked the same again and again. On the other hand some liked easy tasks, and overall students liked tasks they could master. The students gave reasonable answers concerning their choices of tools, for about 18 % their reasons were clearly related to features of the software, whereas for 46 – 60% less informative reasons were given.

The poster will display answers from the questionnaire and a selection of students' solutions to tasks and how the results relate to their choice of ICT tools.

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TEACHER ORIENTATIONS TO EQUIPMENT USE IN ELEMENTARY MATHEMATICS CLASSROOMS

Joanna Higgins

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Three orientations to equipment use in the classroom are examined in terms of the extent to which each supports students' thinking and discussion of mathematical ideas. The responsibility for action and the group configuration change across the three orientations of procedural, conceptual and dialogical. The comparison draws on excerpts from interviews and observations in three classrooms participating in the New Zealand Numeracy Development Project.

A COMPARATIVE ANALYSIS

The New Zealand Numeracy Development Project has emphasised the use of equipment through the introduction of a teaching model. The Teaching Model (Ministry of Education, 2004) drawing on the work of Pirie and Kieren (1989) represents levels of abstraction in representations of mathematical ideas. The model suggests that a process of working from using materials to using number properties be followed when encountering new mathematical ideas.

This poster uses a table format to compare three orientations to equipment use and illustrates each orientation with material drawn from interviews, classroom observations and project artifacts. The analysis of each orientation draws on activity theory (McDonald, Le, Higgins, & Podmore, 2004) to examine the claim that the Numeracy Development Project has shifted teachers' use of equipment from a focus on physical action on the equipment in a procedural orientation, to equipment used as a tool for thinking in a conceptual orientation, to equipment mediating discussion in a dialogical orientation.

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PROCESS OF CHANGE OF TEACHING ON RATIO AND PROPORTION BY MAKING AWARE OF A KNOWLEDGE ACQUISITION MODEL: CASE STUDY

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Teaching and learning of ratio and proportion is a big issue in mathematics education because of its relevance in daily life but also for learning science and advanced mathematics. However, as shown by the results of several international achievement tests and nation-wide tests, the percentages of correct answers by Japanese children in ratio and proportion are not high, even though they often score highly on calculations. An assumption of this study is that this problem requires investigation and modification of everyday teaching practice in the mathematics classroom. In actual lessons, although teachers take account of correct instruction of textbook terms or notations, they do not necessarily recognize the relation between children's learning of them and the development of their proportional reasoning.

In this study, through a collaborative effort with a teacher in preparing, implementing, reflecting and revising lessons on ratio and proportion based on a model "mechanism of internalization of mathematical notations by learner" (Hino, 2002), opportunities are provided for the teacher with thinking about the relationship between teaching of terms, notations, calculations and/or formulas in the textbook on the one hand, and developing pupils' proportional reasoning on the other hand. The purpose of the study is to investigate the thinking process of the teacher when he faced a challenge in making aware of the model in his teaching.

In collecting data, we developed lesson plans on three content units on ratio and proportion based on the model. Over three months, the lessons conducted by the teacher were observed and behaviors of focused pupils were examined intensively. The teacher was informed of the results of the observation as early as possible. After every lesson, the teacher also reflected on his teaching and made a brief report about observations of children's thinking and notations. We also had time for a weekly discussion. The teacher was asked to say freely about his conflict, questions, worries, etc., and also creative ideas and inventions. In the poster, the teacher's thinking process is illustrated together with some episodes. An important theme is the emergence of a jointly-created perspective "transformation of pupils in the classroom." The teacher became interested in the pupil changes reported by the researcher. The perspective provided a situation of discussion between the researcher and the teacher and created ideas of teaching. Furthermore, a proposal of letting the pupils draw figures attracted the teacher's attention to overcome his worries, which also contributed to deepening the discussion between us.

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PRE-SERVICE MATH TEACHERS' BELIEFS IN TAIWAN

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In Taiwan, to meet the educational reform of nine-years curriculum, it is urgent to change primary math teachers' instructional values and beliefs. There has been quite extensive research on this in Australia and the United States, but there has been less work in Taiwan and no work with pre-service teachers. There are two purposes proposed for the current work. The first is to develop the instrument into a stable measurement tool for considering teachers' self-beliefs in instructional approach. Second, because students need to spend three years to finish the primary teachers programs and they are from different learning background, it is necessary to explore whether students' grade levels, the variation in the departments, and the teaching background make differences in their teaching values.

I used two studies as a basis for my research, Clarke (1997) and Ross, McDougall, Hogaboam-Gray and LeSage (2003) and designed an instrument based on their frameworks with some items revised to meet the needs of instructional contexts in Taiwan. The questionnaire instrument considered the scope of the curriculum design, preparing open-ended activities, asking students to have multiple solutions, the use of discovery process to construct student's math knowledge, the role of math teachers as leaders, the use of manipulatives, student-student interaction, students' assessment, active teaching, and levitating student's confidence. A five-point Likert scale was used, from strongly agree to strongly disagree. Participants were pre-service teachers in the three-year primary school programs at National Taiwan University of Arts (NTUA) and 50 pre-service teachers were sampled for each grade. They completed the questionnaires, and the data were analysed using factor analysis and three-way analysis of variance.

Statistically significant differences among the groups of students involved in the courses were found for a number of items. Results depended on mathematical background and teaching experience. Specifically, students who take the math instruction course tend to help children find the answer, use different ways to solve problems, and connect other subjects to math and be able to prepare the math lessons. Those who have teaching experiences are more likely to use supplementary materials, use constructive approach, and lead students explain the answer.

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A PROCEDURAL MODEL FOR THE SOLUTION OF WORD PROBLEMS IN MATHEMATICS

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In solving word problems in Mathematics one must create a bridge between mathematical language, which demands seeing the various mathematical components, and the natural language which itself demands a textual literacy. Identifying the components in the text depends on the meta-language awareness of the place of form, word or sentence in the text and especially awareness to symbols and syntax. Bridging between the natural language and the mathematical language needs a model that will connect the semantic situation and the mathematical form. This bridge will direct mental activity in finding possible solutions before a further deeper analysis of the problem (Greer, 1997). The literature available states that creating a model can be done in two different ways: translating the verbal situation into mathematical concepts (Polya cited in Reusser & Stebler, 1997) or alternatively organizing the mathematical content unit (Freudenthal, 1991). Our suggested procedural model shows how one can combine these two different ways.

We will demonstrate examples of mathematical word problems in which the solution depends on the transfer from a verbal situation to a mathematical form. We are suggesting a ten-stage model, which connects the verbal and mathematical languages. This model suggests an interactive multi stage process allowing decoding of the verbal and mathematical text in order to find the meanings of the word problem. This process of giving meaning according to the model suggested is one of creating a "textual world" based on the schema of the reader. This is formed by using a repetitive interactive action based on the following stages:

Decoding graphic symbols.
Understanding the obvious content.
Understanding the semantics of the problem.
Understanding the mathematical situation.
Making a correspondence between these two situations.
Matching the schema of the text and the schema of the reader.
Posing ideas for solutions.
Sieving out unsuitable solutions.
Making a mathematical representation.
Finding a solution which can be checked.

We will bring examples of using this model in solving word problems for the upper classes of primary school, high schools and teacher training. We will show that a process of stages using comprehensible schema with simple word problems will enable the pupil to confront more complex verbal problems.

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THE 5TH- 11TH GRADE STUDENTS' INFORMAL KNOWLEDGE OF SAMPLE AND SMAPLING

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This paper investigated how well 5th- 11th grade 235 students recognize the concept of sample and sampling.

In the Korean curriculum, students learn the concept of sample, sampling and other concepts related to sample and sampling, when they have reached the 11th grade of high school. But before the 11th grade, they have an activity about data collection, data analysis and the formulation of conclusion. We then investigated and analyzed the informal knowledge of students before they receive formal instructions. The informal knowledge of students is very useful for later learning of statistics.

For this inquiry, I modified the content of MIC[1], the related concept of sample and sampling, and designed questions to inquire students' about informal understanding. The results enabled the identification of the maximum response rate for each question that each student agreed or disagreed with. In particular, it didn't agree with how students consider the characteristic of population in the process of sampling, and the students agreed on a sampling process without considering the characteristic of the population or the components that consist the population.

It showed that 5th grade students didn't investigate the data connected with sampling, and didn't understand the validity of sample survey process. While, 6th grade students equally understood sample size, sampling process, the reliance of data acquired through sample survey that applied to the source of judgment. But in details, it revealed that student had a misconception, or stayed at a subjective judgment level. The significant point is that many high school students didn't adequately understood a sample size with sampling.

Though statistics instruction has traditionally been delayed until upper secondary education, this inquiry convinced us that this delay is unnecessary as the Jacobs' result.

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[1] Mathematics in Context(MiC) is a comprehensive curriculum for the middle grades. The National Science Foundation funded the National Center for Research in Mathematical Sciences Education at the University of Wisconsin-Madison to develop and field-test the materials from 1991 through 1996.

FOSTERING TEACHERS' ETHNOMATHEMATICAL LEARNING AND TRAINING: HOW DONE IN FACT AND WHAT CAN BE LEARNED ABOUT?

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This poster presentation will report wherewith learning mathematics from a cultural perspective, or learning Ethnomathematics interpreted in college classroom environment, where student teachers, Jewish and Bedouin (an ethnic group of Arab background) alike, who came together in 'History of Mathematics' course, explored, learned and debated some basic-activities of mathematics in the cultural group they come from. The research, conducted during the course, was an attempt to examine the mathematical-socio-cultural dialogue on mathematics education that develops following the learning process. Further, it was an attempt to expose, from the teacher's perspective, the values that can emerge from introducing subjects identified with Ethnomathematics into the teachers' education. The methodological framework was based on Grounded Theory approach, which uses a comparative method for data analysis, when the data sources include lesson protocols, lesson plans, feedback-questionnaires and open interviews.

Ethnomathematics comprises a combination of the ethno, signifying the socio-cultural context, and mathematics, interpreted as corpora of knowledge derived from practices (D'Ambrosio 1985). Hence, etnomathematical training can direct teachers toward understanding that exposure to mathematics from practices helps to create a learning environment encouraging the links to the real social world (Katsap, 2004). Therefore, it is advisable to include Ethnomathematics in the pre-service mathematics education programs, where teachers are obliged to learn instructional skills, accommodate different backgrounds and understand that mathematics' values are a contribution by all (Shirley, 2001). The program of Ethnomathematics teaching in the course was designed in accordance with each culture, Jewish and Bedouin, and two mathematical themes, geometry patterns and time calculation, chosen as mathematical background, were applied to seven topics. Data samples of unique demonstrations made during the course will be provided at the poster.

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STUDENTS' MISCONCEPTION OF NEGATIVE NUMBERS: UNDERSTANDING OF CONCRETE, NUMBER LINE, AND FORMAL MODEL

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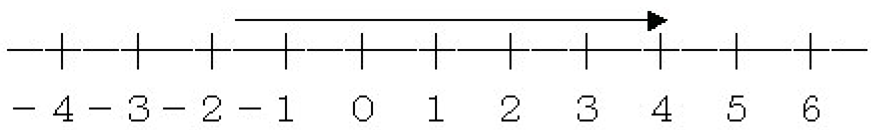
In Japan, it is difficult for many students to understand the operation with negative numbers. The previous researches (cf. Bruno and Martinon, 1996; Lytle, 1994) have been not enough to show why students have a misconception about negative numbers.

The purpose of this paper is to investigate why they have a misconception about negative numbers. Their understanding of negative numbers are analysed with regard to (1) concrete model (the east-west direction model), (2) number line model, and (3) formal model. And 129 students in seventh Grade were given some questionnaire tests. As a result, there became clear reason why they have a misconception about negative numbers.

(1) They keep on the conception formed through the informal experience. In calculating problem ((-1)-(-2)), they answered "-3" by doing (-1)+(-2). Because they said that result of operation would be less than (-1) if they subtract (-2) from (-1).

(2) They apply the mistake rule to relate the result of operation with the models. When some students were asked to represented the operation ((-2) ×(-3)) by using the arrow on the number line, they wrote as follows;



(3) They interpret the results of operation by the property involved references model. When they were asked to interpret the operation((-5)-(-3)) by concrete model, they said that "At first man walk at 5km to the west direction, and next at 3km, and now stay the west point from the start point at 8km". Because they said that they conjectured the operation (5+3=8) as "At first man walk at 5km to the east direction, and next at 3km, and now stay the east point from the start point at 8km".

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THE EFFECTS OF MATHEMATICS PROGRAM FOR GIRLS BASED ON FEMINIST PEDAGOGY

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The purpose of this research is to develop a mathematics program based on the feminist pedagogy (Jacobs, 1994; Warren, 1989) and to analyze its effects. 21 female students participated in this mathematics program for 3 weeks. All the participants finished the 9th grade to translate to the 10th grade. The goals of this mathematics program are to entice young women to study mathematics and to convince their mathematical competence. Based on the feminist pedagogy, the program encouraged the participants to construct mathematics through social interaction based hand-on activities connected to experientially real contexts for girls.

The effect of this mathematics program was analyzed in mixed methods. We have collected video recordings of all class session, which were transcribe for discourse analysis. Tests were given to the students at the beginning and the end of the program in order to investigate comparatively the effect of the program on the students' conceptual understanding of function and data analysis. In addition, surveys and interviews were provided to inquire the students' affective change. Worksheets and reflective journals were collected to supplement the result of the data analysis.

The data analysis supported the significant impact of the program in the improvement of the students' conceptual understanding and affect toward mathematics. Specifically, the analysis of classroom discourse and tests showed that the students' mathematical reasoning has changed from analytic to holistic and from linear to nonlinear. This change is considered to reflect the development of the students' willingness to approach mathematics in diverse ways, which is one of the characteristics of good problem solver. Moreover, the analysis of interview and survey showed that the students became to realize their mathematical competence and the importance of social skills in doing mathematics through their participation in this program. These positive results suggest that further research is of essence to develop an inclusive instructional model for mathematical empowerment of female students.

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ARGUMENTATION AND GEOMETRIC PROOF CONSTRUCTION ON A DYNAMIC GEOMETRY ENVIRONMENT

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The proof is an important teaching object at secondary school, whose several functions that takes in mathematics education allow it to be the validation mean for the generated knowledge but also be one mean to communication, to discovering, to exploration and to explanation. However, its learning has several difficulties related with different aspects such its conception or meaning, the difference in Geometry between drawings and figures, and the relation between proof and argumentation.

To study this situation, we have planned a research project to study the arguments generated at geometrical proof's development in one secondary school at México, under a dynamic geometry environment (with Cabri-Géomètre), on the field of triangle and quadrilateral geometry, and considering some theoretical arguments that seems to us relevant, like the Cognitive Unit of Theorems (Boero et al., 1996) and the differences about proof's meaning among different institutions in Godino and collaborators' sense (see Godino & Batanero, 1994).

In this project is proposed that proof's meaning in scholastic institution is linked with argumentative actions in which is looked the conviction of individual and other people that some mathematical fact occurs, and that argumentation has a deductive structure.

We used activities with triangles and quadrilaterals for a teaching experiment, and we noted students' behaviours which show that figural and conceptual components (Fischbein, 1993) have not harmony, and appeared too confusions in the objects' meanings. Furthermore, the presence of argumentative justifications of observed properties and the apparent lack of a "natural" need to justify through mathematical proofs (deductions) in this educational level might lead us to re-expound the proof's meaning at educational context, both by teachers and by students, although this meaning must take as reference that of mathematicians' institution.

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WHAT'S WRONG WITH THIS SOLUTION PROCEDURE? ASKING CHILDREN TO IDENTIFY INCORRECT SOLUTIONS IN DIVISION-WITH-REMAINDER (DWR) PROBLEMS

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Research studies in psychology and in mathematical education show that children make different kinds of mistakes when solving division-with-remainder (DWR) problems (e.g., Silver, Shapiro & Deutsch, 1993 and Squire & Bryant, 2002). As important as knowing children's difficulties is to examine whether they are able to identify mistakes when faced with incorrect procedures of solving division-with-remainder (DWR) problems. The present study aimed to investigate this aspect in 100 low-income Brazilian children who presented difficulties in solving this kind of problems at school. Half of these children formed an experimental group and the other half was the control group. Children in the experimental group individually received specific intervention involving the solution of division-with-remainder (DWR) problems (materials were made available), in which the examiner presented situations that required the child to (i) understand the effect of increasing/diminishing the divisor over the dividend; (ii) understand the inverse relations between the number of parts and the size of the parts in a division problem, and (iii) analyze correct and incorrect processes of solution. All the children were submitted to a pre-test and a post-test, both consisting of six incorrect procedures of resolution related to the same kind of mistakes that children usually make. In each situation two procedures of resolution were shown: one incorrect and another one correct. The children were asked to identify which of the two procedures of resolution was incorrect and to explain the nature of the mistake identified. The data were analyzed according to the number of correct responses and according to the explanations given. No significant differences were found between groups in the pre-test. However, in the post-test children in the experimental group were significantly more successful than those in the control group. These children performed significantly better in the post-test than in the pre-test. The main conclusion was that the intervention helped the children to identify and analyze the types of incorrect procedures of resolution, as well as to develop a metacognitive ability related to problem solving. This ability is crucial for the learning of mathematics.

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INTERVIEWING FOUNDATION PHASE TEACHERS TO ASSESS THEIR KNOWLEDGE ABOUT THE DEVELOPMENT OF CHILDREN'S EARLY NUMBER STRATEGIES

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This poster addresses the tool that was used in the COCA (Count One Count All) project for assessing the teachers and the results thereof.

The purpose of the tool is a baseline assessment of the teachers' knowledge of early number strategies. After a two-year professional development programme, the tool will be used again to assess the efficiency of the intervention.

The professional development programme is connected to the Learning Pathway for Numeracy (LPN) that is being developed in the COCA project. This project is a SANPAD funded project carried out by the University of Cape Town (UCT) in collaboration with the Freudenthal Institute (FI), the Schools Development Unit (SDU) and the Cape Peninsula University of Technology (CPUT).

The data collection tool is a structured interview in which the teachers have to inform the interviewer about their knowledge of solving operations with numbers up to 100. The teacher is presented with a number of slips containing learner strategies and has to arrange them in an instructional sequence according to their classroom experience. Apart from some background information about the COCA project, the poster will show the tool that was used and a selection of the data that was collected with it. In addition to the results presented in text form on the poster, photographs and video clips will be shown on a laptop.

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A STUDY OF DEVELOPING PRACTICAL REASONING

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Problem solving and reasoning are two of the five process standards (NCTM, 2000). They are two important skills for students to cope with the real world. According to the results of TIMSS 2003, 4th graders in Taiwan did not do well on the reasoning problems; only 43.3% students passed. As a result, the researcher was drawn to study this phenomenon. This study, one of several researcher's projects via internet discussion board funded by National Science Council in Taiwan (e.g., Ma, 2004), will investigate and analyse the development of students' practical reasoning.

The participants in this study were 24 fifth graders from Taichung County, Taiwan, who had basic computer skills, used the internet regularly, and had computer and internet access at home. The participants were divided into 6 groups. Each group was given a theme, which included hiking, culture, food, historic spot, sightseeing, and picnic, and then were asked to plan a trip according to their themes. The main problems for these 5th graders to solve, for example, were: What do we need to do before our trip? How do we plan our budget? Participants had conversations on an internet discussion board, in order to preserve the problem-solving and reasoning processes. Each group worked on the project by communicating and exchanging ideas with others. The teacher applied the five-step heuristics (i.e., focus, analyse, resolve, validate, reflect), claimed by Krulik and Rudnick (1993), as the instruction program to guide the students to develop practical reasoning. In addition, she monitored the interactions among participants, and also kept them on track via the same discussion board. This activity lasted from October, 2003 to June, 2004.

Based on this study: (a) The researcher gained insights about how students generated practical reasoning, and applied the five-step heuristics of the reasoning with sub-skills. (b) These five heuristics were used back and forth when students settled on a situation through thematic approach. (c) The teacher played a critical role in this study, guiding students and helping groups to focus on the sub-topics related to their themes. By participating in this study, students applied their mathematical skills and knowledge to problem solving and reasoning for daily real-life situation.

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DEVELOPING IDEAS: A CASE STUDY ON TEACHING 'RATIO' IN SECONDARY SCHOOL

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This poster provides results from a case study concerning the development of ideas for a more effective teaching of 'ratio' in secondary classrooms. Such ideas developed in three stages. The first stage involved teaching suggestions that were generated from the author's study of problem solving in small groups of pupils. The result of that study was a 'cultural teaching device', i.e., a combination of a challenging task context, a pictorial model and a related collection of arguments and teaching interventions: this device has been found to aid the pupils' proportional reasoning (Misailidou & Williams, 2004).

The second stage of development involved communicating the 'teaching device' to a 'teachers' inquiry group' ('TIG'): this was a group consisting of secondary mathematics teachers and researchers who met and worked together with the aim of developing effective teaching practice. After discussing and reflecting on the author's proposal, Alan, a teacher and member of the group decided to teach 'ratio' in his class. Thus, the third stage of development involved Alan's implementation of the teaching suggestions in his class. Alan, adopted the general principles of the teaching device but its particular aspects were 'transformed' to suit Alan's teaching style and the needs of his class: the task context was altered and the pictorial model was substituted by tabular arrangements.

This poster presents a 'model' of the development of effective teaching on ratio: the cultural teaching device as originally proposed by the author, the transformations through the TIG and Alan's particular needs and the final teaching device that was implemented in Alan's class. It is argued that such a 'model' is necessary for the successful implementation of a research proposal in a normal classroom.

Acknowledgement: The project was funded by ESRC (Award R42200034284).

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ANALYSES OF US AND JAPANESE STUDENTS' CORRECT AND INCORRECT RESPONSES: CASE OF RATIONAL NUMBERS

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International research has documented that US students lack solid understandings of rational numbers in comparisons to their peers in high performing nations such as Japan. As part of a study of US and Japanese students' and teachers' conceptual understandings of rational numbers, the present study examined students' solutions to part-whole, proportion and ratio problems. We were particularly interested in students' correct and incorrect responses that may help us uncover their rational number understandings. Data were collected from 183 fourth graders in Japan and 91 fourth graders in the US. Effort was made to recruit students so that achievement levels were comparable between the two nations. In each nation, students worked on a paper-and-pencil test that included multiplication and division problems, part-whole problems and word problems about proportions and ratios.

As expected, no national differences were found on the overall performance between the US and Japan, F (1, 274) = 1.00. The general patterns of performance were remarkably similar. On most problems, students' correct answers were expressed in one way. For two of the proportion problems, however, Japanese students came up with multiple ways to express correct answers. To figure out how many cups of water is needed to make a soup for 6 people when the recipe for 8 people calls for 2 cups of water, Japanese students responded with 1 1/2, 1 2/4, 3/2, and 8/6 cups in addition to the standard 1.5 cups. For the problem of the amount of cream for 6 people when the recipe for 8 calls for 1/2 cup, we saw .75/2 among Japanese responses. As for incorrect responses, we found that more US than Japanese students solved the soup problem by simply multiplying the recipe for 8 by 6 to find the amount needed for 6 people. On the ratio problems in which students were asked to determine how much food to give to fish according to their relative size, we found that more US than Japanese students ignored the ratio given but instead focused on the relative size (e.g., bigger and smaller) to arrive at an answer. We have recently collected additional data to examine if these cross-national characteristics can be replicated.

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MATHEMATICAL ACTIVITIES AND CONNECTIONS IN KOREAN ELEMENTARY MATHEMATICS

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In recent international comparisons Korean students have consistently demonstrated superior mathematics achievement not only in mathematical skills but also in problem solving (e.g., OECD, 2004). This draws attention to mathematics education in Korea (Grow-Maienza, Beal, & Randolph, 2003). A textbook is a strong determinant of what students have an opportunity to learn and what they do learn because all Korean elementary schools use the same mathematics textbooks and, more importantly, almost all teachers use them as their main instructional resources.

The most recently developed seventh curriculum and concomitant textbooks have a level-based differentiated structure and emphasize students' active learning activities in order to promote their mathematical power. The textbooks intend to provide students with a lot of opportunities to nurture their own self-directed learning and to improve their problem solving ability. This resulted from the repeated reflection that previous textbooks were rather skill-oriented and fragmentary in conjunction with the expository method of instruction.

Given this background, the poster presents main characteristics of current elementary mathematics textbooks along with some representative examples. The characteristics include encouraging students to participate in concrete mathematical activities, proposing key questions of stimulating mathematical reasoning or thinking, reflecting mathematical connections, and assessing students' performance in a play or game format.

With regard to each characteristic, this poster first presents some background information and rationales in brief. It then shows examples from the textbooks so as to highlight key features, followed by an elaboration on the examples. The topics of examples vary such as subtraction with base-10-blocks, rotation of a semicircle, calculation of a decimal divided by a fraction, and figuring out divisors. As for mathematical connections, in particular, this poster displays how the addition and subtraction of fractions with different denominators at a fifth grade level are based on other related concepts and operations at the previous grade levels.

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ONLINE INSTRUCTION FOR EQUATION SOLVING

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Mathematical learning theory has been used as the basis of an interactive visual program with frequent feedback for learning equation solving. This was then used to investigate its effects on the approach students chose to solve linear equations.

BACKGROUND

Engineering and science students use many mathematical formulae which need to be transposed. A good understanding of equation solving is needed and as students often find it difficult, online instructional software was developed. Mathematics requires a particular type of thinking: the ability to see both the global strategic view of a problem as well as the detailed view of each step. This type of thinking, in the context of mathematics, has been called versatile thinking (Thomas, 1995).

SOFTWARE

An important feature of the software design was to separate the strategic and detailed thinking and allow students to develop strategies for equation solving without performing the detail of each step. This was achieved by providing frequent feedback of the type recommended by Tedick (1998). A sequence of screen shots of the software will be used on the poster to show its features.

METHOD AND ANALYSIS

A pre-test/post-test trial used adult students with the software recording their choices. This sequence of information showed the approach students chose to solve equations at each stage and this will be displayed on the poster.

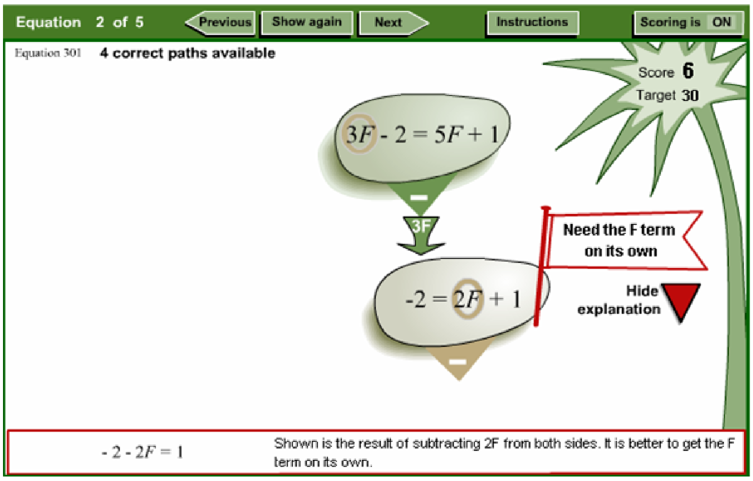

Figure 1: Screen shot of software

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USING THE CALCULATOR TO UNDERSTAND REMAINDERS OF DIVISIONS AND DECIMAL NUMBERS[1]

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This study investigated the understanding of the meaning of remainders of divisions and their decimal representations through an intervention in which problems solved with pencil and paper were compared with the results obtained in the calculator. Vergnaud (1987) stressed the importance of using different symbolic representations whilst teaching, however, some forms have been given priority over others and very few usage of multiple representations has been observed at school (Selva, 1998).

Children aged 9 and 10 (n=18) and aged 11 and 12 (n=14) of a Brazilian state school took part in the study that involved a pre-test, an intervention and a post-test. In the intervention the children were assigned to one of two conditions: Group 1 – initially solved the problems with pencil and paper and then with the calculator; Group 2 – solved the problems initially with the calculator and then with pencil and paper.

It was observed that the mean score in the pre-test of the 9 and 10-year-olds was 53.71%. In the pos-test the mean score of the children from the paper-calculator group was 85.19% and of the calculator-paper group was 59.53%. The mean score of the 11 and 12-year-olds in the pre-test was 64.29%. In the post-test the mean score of the paper-calculator group was 90.48% and for the calculator-paper group it was 95.29%. Thus, the intervention was effective for both groups of 11 and 12-year-olds but for the 9 and 10-year-olds significant improvement was observed only in the condition in which the children used the calculator after solving the problems with pencil and paper. Possibly it was easier for these children to relate both representations (drawings or algorithms in paper and decimals in the calculator) when they first used a more familiar representation and reflected about the result obtained.

It was concluded that understanding decimal representation is not always straightforward but children can benefit from teaching conditions that promote the relations between this representation and more familiar ones. Interventions like the one proposed in this study can lead children to raise hypotheses about decimals and should be considered by teachers that teach mathematics.

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GENERATIVE ACTIVITIES AND FUNCTION-BASED ALGEBRA

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This poster will display the materials and results of a semester long intervention with two Algebra 1 teachers. Researchers worked with teachers to create function-based generative algebraic activities which would engage all students and foster greater understanding of the key concepts of equivalence, equality and linear equations.

The multiple-strands based approach to curricula promoted by the NCTM has not impacted the role of the single-strand Algebra I course as gatekeeper in the US educational system. If anything, Algebra I is more central at many levels. In secondary mathematics, improving outcomes in Algebra I is, perhaps, the single most strongly felt need at nearly every level in the national educational system. Traditional approaches to improving outcomes (e.g., doubling class time) have had only minimal success. The No Child Left Behind legislation requires that we “raise the bar” of performance for all students and do so in a way that also closes the gaps in performance identified by disaggregating testing, enrolment and graduation outcomes.

We have very good evidence pointing to the effectiveness of function-based algebra at a small scale (Brawner, 2001) and at a very large scale (NCES, 2003). We need solid mid-level results that point specifically and in a detailed way to the effectiveness of function-based algebra. With the requirement that local, state and national adoptions need to be scientifically based, it is vital that research speak directly to this requirement. Additionally the research must be optimized to speak to issues of raising the standards of performance for all students. Generative activities utilize student created artefacts as the core for instruction. For example, using next-generation classroom networks, a possible generative activity is to have all students submit a point whose Y value is twice the X. This group created, set of points, displayed to the class, becomes the focus of discussion.

In collaboration with the Texas Educational Agency and Texas Instruments a semester long intervention was done in a rural high school, just outside of a major southwest city. The study used a Solomon 4 research design with 250 students in both treatment and control groups. Throughout the semester, researchers worked with teachers to create and implement a series of function-based, generative activities utilizing next-generation classroom networks.

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TEACHING STATISTICS WITH CONSTRUCTIVIST- BASE LEARNING METHOD TO DESCRIBE STUDENT ATTITUDES TOWARD STATISTICS

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In this study, the researcher examined the effect of a semester-long, constructivist-based learning approach method on student attitudes toward statistics in an introductory statistics course. A major goal of an introductory statistics class is to teach students to think critically, using the fundamental concepts of statistics. Students should be able to organize and summarize data, draw inferences from such summaries, and incorporate such summaries and inferences into reports. Constructivist-based learning techniques promote learning through small group work experiences and involvement in learning activities other than just listening. These can include projects that require class participation through hands-on experiments or demonstrations that illustrate lecture material.

The author investigated whether students who engaged in constructivist-based learning environment performed more positive attitudes toward statistics. In order to answer research question, a t-test was used to compare the average scores on four subscales of the Survey of Attitudes Toward Statistics (SATS) performance of pretest and posttest. The t-test results indicated that there was a statistically significant difference between the SATS mean score of the pretest and posttest, at the 0.05 significance level. Using $\alpha = 0.05$ as the pre-study determined level of testing, there was sufficient evidence to reject the null hypothesis regarding differences in the measure of the average scores on the SATS at the beginning and the end for the students learning statistics with constructive learning approach.

The findings reported in this article show that students who were exposed to constructivist-based learning approach in an introductory statistics class gained positive attitudes. These results suggest that such constructivist-based learning techniques may be useful for enhancing learning. Constructivist-based learning methods may also offer alternative learning opportunities for students who do not fully grasp course material in the traditional lecture format. Constructivist-based learning approach provides students with the opportunity to apply theory to real-life situations and bring concepts and theories to life, thereby enhancing student learning.

STUDENT BEHAVIORS THAT CONTRIBUTE TO THE GROWTH OF MATHEMATICAL UNDERSTANDING

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The Pirie-Kieren model for the growth of understanding (Pirie & Kieren, 1994) provides a framework for analyzing student growth in understanding, via a number of layers through which students move, both forward and backward. The Pirie-Kieren model for the growth of understanding was conceived as a dynamic model, in which student movement between layers is a critical feature. Yet in the model itself there is no indication of the force, or motive, that impels a student to move from one layer to another. This poster documents a model for a motive that stimulates moves through the layers that form the structural base of the Pirie-Kieren model. These student behaviors involve a change in the learner's focus of attention (e.g., Warner, Alcock, Coppolo & Davis, 2003) and include the ability of the learner to: interpret his/her own or someone else's idea (through explaining, questioning and/or using it; reorganizing and/or building on it); use multiple representations for the same idea; link representations to each other; connect contexts; raise hypothetical situations based on an existing problem (such as a "What if" scenario).

In this poster, an example of three sixth grade students' movement through the layers is used to identify how the behaviors relate both to each other and to the overall process of understanding (movement through the layers in the Pirie-Kieren model). A summary of the percentages of observed student behaviors that were associated with a move to a particular layer in the Pirie-Kieren model are also displayed.

Results indicate that for each of the three students, certain behaviors, including student questioning, explaining, re-explaining and using of one's own or others' ideas, in the inner layers of understanding, appeared to stimulate a move to setting up hypothetical situations, connecting contexts and linking representations to each other, which is associated with moves to outer layers of understanding. In the outer layers of understanding, setting up hypothetical situations also appeared to stimulate a move to connecting contexts and linking representations, which is associated with moves to the outer-most layers of understanding.

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ASSESSMENT OF SPATIAL TASKS OF GRADE 4-6 STUDENTS[1]

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The importance of geometry and the development of spatial abilities for literacy, especially mathematical literacy, now is an excepted fact. Different authors concur that spatial perception does not consist of a single skill or ability (Tartre 1984; Del Grande 1987, 1990). Del Grande (1987: 127; 1990:14) describes seven spatial abilities: eye-motor coordination, figure ground perception, perceptual constancy, position-in-space perception, perception of spatial relationships, visual discrimination and visual memory. These seven spatial abilities can be grouped into two major categories, i.e., spatial orientation and spatial visualization (Tartre 1990: 217). Teachers are not aware of the fact that researchers consider proper and effective spatial development of the young learner more complex than number development (Bryant 1992: 7; Van Niekerk 1997; Wessels 1989), therefore spatial development is often neglected in the teaching and learning of geometry.

An instrument consisting of 23 spatial tasks was developed to study learner responses to spatial problems in order to determine the present developmental state of spatial abilities and the efficacy of current teaching strategies used in a government school in South Africa. The degree of difficulty of the spatial tasks was determined using the Wattanawaha's classification system (Clements 1983:16) using four independent properties to classify spatial problems. Assessment rubrics for each task were set coding answers according to a nominal scale ranging from 0 to 1 or 2, depending on the requirements for the task. Examples of the coding of difficulty of the tasks, assessment rubrics and the coding of learner responses will be given, as well as preliminary findings of the research.

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DIDACTICAL ANALYSIS OF LEARNING ACTIVITIES ON DECIMALS FOR INDONESIAN PRESERVICE TEACHERS

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Research (Stacey et al., 2001) has indicated that misconceptions and difficulties in understanding decimals persist with preservice teachers. It is posited that many Indonesian preservice teachers will also have limited conceptual knowledge on decimals as the approach of teaching and learning decimals is dominant with the exposure of whole numbers rules in particular when dealing with addition and subtraction of decimals.

A set of learning activities on decimal notation adapted using 'theory guided bricolage' approach (Gravemeijer, 1994, 1998) for Indonesian preservice teachers will be presented in a poster. The basic principles of *Realistic Mathematics Education* (RME) are employed in designing the learning activities in line with the current reform effort called "PMRI" to improve mathematics education in primary school since late 2001.

The concrete model based on length is employed in measurement context to promote an understanding of the repeated decimating process and give meaningful interpretation of place value in decimal numbers. This is in line with the didactical phenomenology principle where the application serves as a possible source of learning. The guided reinvention principle of RME is exercised in the activities where the preservice teachers use the concrete model in a measurement context and to observe the decimal relationships between different pieces of the model. Activities of constructing a decimal number in different ways using the concrete models are devised to explore the additive and multiplicative structures of decimals.

Didactical analysis will be presented in the poster to point out the possible contribution of the learning activities and how the learning tasks are related to basic tenets of RME.

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WORKSHOP ON DESIGNING "SCHOOL-BASED" MATHEMATICS INSTRUCTIONAL MODULES

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The main purpose of this workshop was to assist pre-service elementary teachers' in instructional design through providing them with opportunities for designing "school-based" mathematics teaching modules. The reason for the focus on "school-based" was the current curriculum reform in Taiwan. Theoretical foundations including "scaffolding theorem", "constructivism-based teaching strategies" and "the settings arrangement of cooperative learning" was applied in this ten-week workshop, researcher guided pre-service teachers to develop "school-based" mathematics instructional modules step by step. There were 41 undergraduates at 3rd and 4th grades in a national university of southern Taiwan participating in this workshop. Students were initially divided into 8 groups according to their choice, 4-6 students in a group. The issue of "school-based curriculum development" was emphasized by offering pre-service teachers many practice opportunities of instructional design. As for the design of "school-based" teaching module, every group firstly chose an elementary school as the basis of developing school-based curriculum. By collecting information from Internet, libraries, or interview with the elementary school, group members could understand the background, feature, and resource of this school. After coordinating with the mathematic teaching materials in elementary schools, the draft of "school-based" teaching module was developed. The researcher guided pre-service teachers to reflect, to examine and to revise their designs by cooperation, sharing, and discussion within and between teams. Eight "school-based" mathematics instructional modules were developed (see Table 1). From the process of developing instructional modules of pre-service teachers, the researcher frequently reflected on the contents and methods of teacher-preparing, and tried to find the important components and appropriate way to prepare pre-service teachers' in mathematics instructional design. The results showed that through a series of stages, "preparing stage, base stage, practicing stage, sharing stage, integration stage", it was useful for helping pre-service teachers to develop school-based teaching modules and enhancing their professional development in mathematics instructional design.

Table1. The list of the school-based instructional module design

Topic of the module	Mathematics-related concepts involved	Grade
Welcome to Ming-Hsiung	Multiplication and division; length; capacity; bar chart; three-dimensional pictures.	3
"An-Ping" vs. "Ping-An"	Addition and subtraction; multiple; length; time; the "space"-related concepts.	3
A visit in the frontline.	Addition, subtraction and multiplication; divide equally; multiple; length; weight; time; statistical table; bar chart.	3
A nice trip in "Peng-Hu"	Time; weight; two and three dimensional shapes; direction; bar charts.	5
A legend in Dong-Dan	Integer; average; addition and subtraction; time; hour; direction; perpendicular and parallelism; a cube in the shape of a rectangle; a prism in the shape of a triangle.	5
A trip in Bei-Dou	Sale; direction; bar chart; line chart.	4 & 5
The beautiful scenery in Ken-Ding	Length; time; scale; bar chart.	6
A "Green Island" melody	Length; capacity; square measure; volume; direction; scale; bar-chart.	6

CHILDREN'S "EVERYDAY CONCEPTS OF FRACTIONS" BASED ON VYGOSTKY'S THEORY: BEFORE AND AFTER FRACTION LESSONS

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Vygotsky (1934/1987) categorized concepts into two types: everyday concepts and scientific concepts. Everyday concepts are not based on a system but in rich daily contexts, thus sometimes children use them incorrectly from mathematical view points. In contrast, scientific concepts, (henceforth mathematical concepts) are based on "formal, logical, and decontextualized structures" (Kozulin, 1990). In accordance with Vygotsky's theory, Yoshida (2000; 2004) suggests these two concepts are finally sublated, i.e. everyday and mathematical concepts 1) conflict with each other through formal learning, 2) are lifted to higher levels respectively, and 3) are preserved as a unified concept, or sublated concepts.

It is difficult for children to understand fractions and worldwide researchers have struggled with more effective ways of learning/ teaching fractions. Based on the theory above, first, it is important to identify children's everyday concepts of fractions, and then it becomes possible to consider better learning/ teaching situations of fractions, taking the children's everyday concepts into consideration. I conducted pre- and post- questionnaire survey, before and after fraction lessons, on February 16, 2001 through an all-at-once style of exam in a classroom, and on March 15-18, 2001 as homework, respectively, targeting about 40 third graders making up one classroom in Japan. It was the first time for them to take fraction lessons formally in school at that time. (The lessons are discussed in detail in Yoshida (2004).

This poster will show the questionnaire entries with figures and children's answers with illustrations presented in the questionnaires, chart the questionnaire data, and identify what kind of "everyday concepts of fractions" children have and how the everyday concepts of fractions develop or do not develop through the lessons.

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