CHILDREN'S CONCEPTUAL UNDERSTANDING OF COUNTING

Hannah Slovin and Barbara J. Dougherty

University of Hawai'i

This paper describes a design research study with ten second-grade students who are part of the Measure Up (MU) research and development project underway at the University of Hawai'i. Students were asked how they counted in multiple bases, specifically how they knew when to go to a new place value and why it was necessary to do so. All ten students showed skillfulness in counting and representing the numbers, but analysis of their responses showed different levels of generalization of method and explanation of underlying ideas.

INTRODUCTION

Educators and the general public continually lament that children are not prepared for the challenge of complex and sophisticated mathematics found in high school mathematics and beyond. To help students attain higher levels of mathematics it is imperative that we reconsider the foundation that children receive in the early grades.

With this in mind, we began a new study called Measure Up (MU) that focuses on establishing a different (and stronger) mathematical foundation from which children can build their understandings in the early grades. One area of early mathematics that students explore from a different perspective is counting.

Theoretical Foundation of Measure Up and Counting

Children in MU begin their mathematical development from the perspective of measurement and algebraic representations. V. V. Davydov (1975), a Russian psychologist, proposed that some of Piaget's findings (1973) suggested limitations on children's learning that could be overcome if instruction (and the mathematical content associated with it) were changed. Vygotsky's distinction between spontaneous and scientific concepts (1978) provided a means by which children could learn more sophisticated mathematics before we may have thought they were developmentally ready. Spontaneous or empirical concepts are developed when children can abstract properties from concrete experiences or instances. Scientific concepts, on the other hand, develop from formal experiences with properties themselves and progress to identifying those properties in concrete instances. As an example, spontaneous concepts progress from natural numbers to whole, rational, irrational, and finally real numbers, in a very specific sequence. Computations and other ideas are taught within each number system and are often not connected across systems. Scientific concepts reverse this idea and focus on real numbers in the larger sense first, with specific cases found in natural, whole, rational, and irrational numbers at the same time. Davydov (1975) conjectured that a general-to-specific approach in the case of the scientific concept was much more conducive to student

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 4 pp 209–216

understanding than using the spontaneous concept approach. This idea can be extended to counting even though counting is thought to be a very specific and concrete action.

Drabkina (1962, (cited in Davydov, 1975)) and Minskaya (1975) noted that counting is not a simple act. It involves defining a unit of measure and using it a whole number of times. Thought of in this way, it is clear that a one-to-one correspondence requires that a child 'see' the unit and think of it as being iterated multiple times.

In one-to-one correspondence counting, children have to first know what unit they are using to count. Typically, they assume that they are counting 'one' thing, but the object(s) being counted could be grouped so that multiple objects constitute a unit. To be competent counters, children also have to understand that objects must be counted with like units. You cannot, for example, count a group of cups and saucers and say 4 when you counted two of them as a cup and saucer set and two as individual cups. A unit has to be carefully defined. This notion of counting is well developed in MU before students move to place value and computations involving multi-digit numbers.

If understanding units is a prerequisite of counting, then what should precede the introduction of number and counting? From the Davydov perspective (1975) instruction should begin by having students identify traits and attributes of objects that can be compared. These comparisons can be described without using numbers—shorter, longer, heavier, lighter, more than, less than, and equal to--and represented with relational statements, like $G > L$, that use letters to represent the quantities being compared. Fundamentally, students realize that if $G > L$, then it is also the case that $L < G$. First-grade students can write these relational statements and understand what they mean, because the statements describe the results of physical actions that the children themselves have performed in the comparison process.

These comparisons, based on Davydov's proposal, can be used to model equivalence properties, addition, and subtraction. For example, if $B > T$, then $T \neq B$, $B \neq T$, and $T < B$. As one first-grader noted, 'if it's an inequality, then you can write four statements. If it's equal, you can only write two.' [Justin, 2003] From these comparisons of two unequal amounts, they can be made equal by performing one of two actions—decrease the greater quantity (subtraction) or increase the lesser quantity (addition). The quantity added or subtracted is the difference between the quantities.

All of these ideas are developed before number is introduced in grade 1. Dealing with a problem in which students must find the relationship between two quantities requires identifying a unit and then defining the relationship between the unit and the larger quantity. These experiences establish the need to clearly identify the unit *before* counting because its definition has an impact on the counting process and result. It is only now that number is introduced.

Number introduction is linked to measurement, as a means of representing the relation between a chosen unit and a larger quantity. Children must identify the unit before a numerical value can be used to represent a quantity. The unit used to measure two or more quantities must be the same if they are to be compared. Children as young as 6 years old can determine the relationship between units of measurement if a quantity is measured first with one unit and then with a different one. For example, if mass *F* is measured with mass-unit *B* first and then by mass-unit *E*, the relationships can be expressed as—

$$\frac{F}{B} = 7 \qquad \frac{F}{E} = 9$$

This indicates to students that mass-unit *B* is larger than mass-unit *E* because it took fewer of the *B* units to measure the quantity.

This idea is then extended to grade 2 where students begin to explore place value. Traditionally, children work with the decimal or base-ten system. Using Davydov's ideas, MU introduces place value instead with measurement units. Students may see proportional measures in a table format that motivate the notion that when you create a quantity of units, at some designated size, the unit is counted differently. For example, in a table format like the following, students combine quantities to create a new quantity. The units (*K* and *N*) are defined by columns, the quantities (*B*, *C*, and *T*) by rows. Next, students explore the exchanging of units, given that a larger intermediate unit is made up of a specified number of smaller units (see table below).In the case of the second table, it takes 4 of the *K* units to make *A*, so an exchange can be made.

K	*N*	
3	2	*B*
2	2	*C*
		T

A	*K*	
1	5 **4** **1**	***L***
2	1	***L***

This idea is then extended to place value, beginning with bases smaller than 10. Students are given a scenario such as there exists an extraterrestrial group of Ternarians that can only use the digits 0, 1, 2, and 3. How would they count? By exploring the smaller bases first, students create means to exchange units to generate other successively larger ones. For example, if using area-unit *E*, in base three, they realize that it takes three of the area-units *E* to create a Place II area. Following the generalization they have made, Place III requires three of the Place II areas; Place IV requires three of the Place III areas, and so on. The exchanging of the smaller units for a larger unit helps student develop algorithms for addition and subtraction that involve regrouping, but it also directly links to counting.

DESCRIPTION OF THE STUDY

The present study is part of the MU research and development project and focuses on second-grade students' ability to count in multiple bases and their understanding of the underpinnings for a place-value counting system. The ten students (six boys and four girls) at the Education Laboratory School (ELS) are representative of the larger student populations in the state. At ELS, for example, student achievement levels range from the 5th to 99th percentile, with students from low to high socio-economic status and ethnicities including, but not limited to, Native Hawaiian, Pacific Islanders, African-American, Asian, Hispanic, and Caucasian. Students at ELS are chosen through a stratified random sampling approach based on achievement, ethnicity, and SES.

The project team has engaged in design research since the fall of 2001. Design research (see *Educational Research, Volume 32, No.1*) in the domain of MU is constituted by a focus on developing a theory about children's learning in elementary mathematics rather than on 'testing' lessons in a write-test-revise cycle. This study used a one-on-one (teacher-experimenter and student) design so that we could study the students' learning in depth and detail (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003). Specifically, we interviewed the students using a method similar to that used by others for curriculum research and development (Rachlin, Matsumoto, & Wada, 1987). Listening to the student's explanation of his/her thinking allows us to analyze student understandings and helps the project team determine approximations of sophistication and complexity levels of the mathematics that students can handle.

Description of Questions

Of particular concern in this study was the students' ability to count in multiple bases and how that ability is connected to an understanding of the structure of a place-value number system. Students had engaged in a series of tasks designed to give them experience with place value using multiple bases in the context of measurement. In lessons previous to the study they had created concrete representations of units using length, area, volume, and mass, and had had much experience combining, grouping and exchanging units. In their class work students demonstrated skill in numbering in multiple bases on number lines and on blank paper. This study was conducted at that point in the lessons where we needed to know more about how the students were integrating the tasks that emphasized the conceptual development with those focused more on building skill and fluidity in counting orally and writing numbers.

The following questions were asked individually of each of the ten students:

1) When you are writing the numbers in any base, how do you know when you need to go to the next place?
2) Explain why you have to go to the next place.

Because our interest is in the conceptual underpinnings of an overt skill, we deliberately made the questions general and open-ended to avoid influencing students' own thinking. [Need more help here]

Analysis of Response

The responses were analyzed along two dimensions: level of generality and basis of explanation. Within each dimension the responses fell into two categories.

Level of Generality

Uses a general rule	Uses specific examples
Generalizes the method to any base. "You can't use the base number." (Dusty) "You can write whatever number before the base; then you change it." (Anthony) "It depends on what base you're in. You can't go to the base number. You can go up to one less." (Justin)	Responds only using particular bases. "You can't go up to six in base 6. When you count six times it means you go to the next place." (Bryson) "Because one, base 3; two, base 3; one-zero, base 3. You can't say 'three' in the [units] place." (Brooke)

Basis of Explanation

Uses procedural explanations	Uses conceptual explanations
Explains what you do. Explains when you change places. Uses ease of communication as a rationale. "You have to do it [move to the next place] to count higher." (Anthony) "You might get mixed up. You don't know how much it [referring to the value of the number] is." (Justin) "You can't go past five because it's in base 5." (Laylie)	Refers to the unit of measure. Refers to the structure of the number system. Relates the base to the supplementary measure E_{II} or place II unit. Refers to grouping the main measures to form the supplementary measure. "The base tells you how many times to use the unit." (Michael) "The base tells you how many units to use." (Alicia)

Table I: Grade 2 Response Categories with Examples

Responses in which students described a general "rule" for numbering in any base were considered more advanced than those in which students only referred to numbering in specific bases. Similarly, when explaining the basis for representing the numbers, responses that included conceptual reasons were considered more advanced than those that referred only to procedural ones. See Table I above.

RESULTS

Table II below summarizes how the students' responses in both dimensions fell into the categories. In each domain, responses that contained elements of both categories were counted with the higher category. For example, with level of generality a response that used specific bases and gave a generalized rule to explain how one knows when to move to place II was counted as conceptual.

	Level of Generality: Specific	Level of Generality: General
Basis of Explanation: Procedural	3	3
Basis of Explanation: Conceptual	1	3

Table II: Second Graders' Responses by Category

In responding to Question 1, all of the second-grade students knew when they had to go to the next place when counting and writing numbers in a base. Three students had to be prompted after the initial question with a more specific context such as, "Let's say, you were writing the numbers in base 5." The group of responses indicates a progression from responding by counting in a specific base to generalizing a rule for any base with no specific reference to a particular one. Laylie answered question 1 by merely counting in base five, "I had to do one, base 5; two, base 5; three, base 5; four, base 5; one-zero, base 5." Logan, used specific examples, but his response indicates the beginnings of generalization. "If I was in base 2, I do 'one' and then I can't go 'two' because I'm in base 2. But if I'm in base 4, I do…" At this point he asked for a pencil and some paper and correctly wrote the numbers starting from 1_4 to 11_4. Justin started by giving a specific example, "In base 3 you can't go up to three, you can only go up to two." He then went on to generalize, "It depends on what base you're in. You can't go to the base number. You can go up to one less." Dusty started his response with a general rule and then gave a specific example, "You can't use the base number. Like in base 5 you can't say 'five-five, base 5." Anthony's response indicates that he is able to make a generalization about counting and numbering, "You can write whatever number before the base; then you change it."

Students' responses to question 2 were procedural, conceptual or a combination of both. Six of the ten students answered question 2 using only a procedural explanation. Procedural justifications either indicated an awareness of some patterning of when to start numbering in place II or referred to ease of communication. Laylie said, "You can't go past five because it's in base 5," and

Dusty said, “If you’re in base 6 you go to the five; then you go to the next.” Justin’s response represents a procedural explanation based on ease of communication: “You might get mixed up.” Students who used conceptual reasoning referred to one or more of the following: the main unit, how many units were needed to create the supplementary unit as determined by the base, or the structure of measures that comprise a place-value number system. Erin started her reasoning by saying, “You can’t know which is the number and which is the base,” but added, “ it [going to the next place] means you made another amount of four.” As with the their ability to generalize writing the numbers in a base, students’ explanations of why one goes to place II showed progressive development toward conceptual justifications.

IMPLICATIONS

The study focused on our interest in students’ counting in multiple bases and their sense making of the ideas underlying counting. Students’ responses were analyzed for degree of generality and the nature of explanation. While all ten students demonstrated skillfulness in their ability to generate and represent counting numbers in multiple bases, they varied in how they described their methods for writing the numbers and in their reasoning of how a place value system works. Students ranged from operating specifically and procedurally to operating generally and conceptually.

The sophistication of the second-grade students’ responses was further verified by asking the same set of questions to fourth-, fifth-, and seventh-grade students at ELS who have not been part of the MU project. Of the 7 fourth and fifth graders and 6 seventh graders only one, a seventh grader, referred to the relationship among the place values. Many fourth and fifth graders responded to the question about how they know when to go to the next place by asking, “What’s place value?” Three seventh graders said they just memorized how to count and didn’t think any more about it.

We note that fewer students answered question 2 using the higher category of response than in question 1. Although the sample is small, we believe these results have several important implications for our work. First, they challenge us to continue to find ways to interweave the concept-development experiences students have with the skill-building ones. Where we once believed that when conceptual understanding was carefully established, students could engage in the type of exercises that promoted automaticity without losing these underpinnings, we now appreciate how much integration of concept and skill development it takes to maintain both. Additionally, this study reminds us that students’ apparent competence may not always be supported by robust conceptual understanding. Implications: the need to interweave the skill but not lose the conceptual understanding.

References:

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in education research. *Educational Researcher, (32)*1, pp. 9-13.

Davydov, V.V. (1975). Logical and psychological problems of elementary mathematics as an academic subject. In L. P. Steffe, (Ed.), *Children's capacity for learning mathematics. Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. VII* (pp. 55–107). Chicago: University of Chicago.

Minskaya, G. I. (1975). Developing the concept of number by means of the relationship of quantities. In L. P. Steffe (Ed.) *Children's capacity for learning mathematics. Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. VII* (pp. 207–261). Chicago: University of Chicago.

Piaget. J. (1973). Comments on mathematical education. In A. G. Howson, (Ed.), *Developments in Mathematical Education: Proceedings of the Second International Congress on Mathematical Education.* New York: Cambridge University Press.

Rachlin, S., Matsumoto, A., & Wada, L. (1987). The teaching experiment: A model for the marriage of teachers and researchers. *Focus on Learning Problems in Mathematics*, (*9*)3, pp. 21–29.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Harvard Press: Cambridge.