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ABSTRACT

This study investigates the development of 5th grade children's understanding of quotient and the classroom norms and practices that constrain or enable that understanding. It reports not only how the children's understandings develop, but also why and under what conditions they develop. The results of this study indicate that children progressed from treating fractions as exclusively part-whole to having at least two parallel conceptions: part-whole and fair share while at the same time beginning to conceive of the division operation as generalizable to any pair of whole numbers. Some children began to see the operation as generalizable to division by fraction. The general learning trajectory of the class is discussed. (KHR)

THE EMERGENCE OF QUOTIENT UNDERSTANDINGS IN A FIFTH-GRADE CLASSROOM: A CLASSROOM TEACHING EXPERIMENT

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Theoretical Orientation

Research on understanding children's rational number concepts has been of keen interest to mathematics educators in the last two decades. Children's difficulties with rational numbers can be attributed to the fact that rational numbers take different meanings across different contexts (Kieren, 1976; Behr, 1983). One of these meanings is the quotient interpretation (e.g., Kieran, 1993). Building phenomenologically from the contexts of fair sharing, under the quotient interpretation, rational numbers become quotients of whole numbers. More specifically, a rational number can be defined as the yield of a division situation. Considering this meaning, the relationship between division as an operation and fractions as quotients becomes a critical consideration for instruction.

Children first learn the concept of division partitioning whole numbers—e.g., fair sharing—and they express quotients as whole number partitions. Division situations with remainders are found to be difficult because instruction fails to engage students in the interpretation of resulting fractional quotients appropriately within the given problem context, instead putting off fractional quotients for later study. However, in research on division, there is a lack of study addressing the relationship between rational numbers and division of whole numbers and whether this traditional separation of whole number quotients-with-remainder and fractional quotients is pedagogically sound practice. Moreover, developmentally, there is little research that documents plausible trajectories by which children come to see the two as isomorphic conceptually (see Behr, Harel, Post, & Lesh, 1992, p. 308 which illustrates the limits of our current understanding in rational number research).

At any rate, current curricular offerings in the United States present whole number division and fractions as separate and distinct topics. Division is treated as an operation one performs on Whole numbers, and fractions are taught almost exclusively as Part-Whole concepts. Only in the late middle grades, when an understanding of quotient becomes necessary for dealing with algebraic entities such as rational expressions, are children expected to connect these two heretofore distinct topics. At no point in the preceding 7 or 8 years is explicit attention focused on teaching fractions as division, nor on teaching division as expressing a rational number: A quotient.

Toluk (1999; Toluk & Middleton, 2000), in a series of individual teaching experiments, studied how children can develop conceptual schemes for making sense of Quotient Situations by explicitly connecting their prior knowledge of whole number division with remainders and their knowledge of fair sharing situations resulting in fractions. Specifically, by pairing problems that generated fractional reasoning, with isomorphic problems that yielded reasoning about division, this work led children to the understanding that $a+b = a/b$ and $a/b = a+b$ for all (a,b), $b \neq 0$. As children progressed through the teaching episodes, they developed the following schemes in order: *Whole Number Quotient Scheme, Fractional Quotient Schemes, Division as Fraction Scheme* and *Fraction as Division Scheme* (see Figure 1).

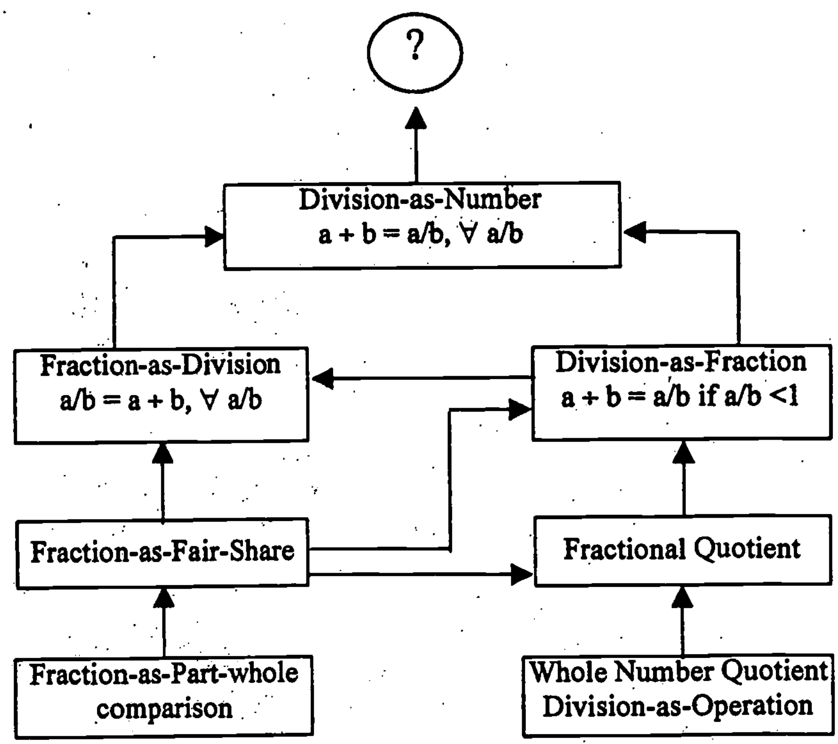


Figure 1. Schematic illustrating children's developing connections among fractions and division schemes (Toluk, 1999).

In most prior work, including that of Toluk, inferences about plausible developmental of rational number concepts and skills have been made from an exclusively individual perspective. There are severe limitations to the kind of pragmatic conclusions that can validly be drawn from exclusively individual research. This study remedies these shortcomings by providing a test of recently proposed depictions of children's quotient understanding (Toluk, 1999, Toluk & Middleton, 2000), and by providing a framework that explains their development and interaction in a classroom teaching experiment from an emergent perspective (e.g., Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1996) that treats *psychological* (i.e., individual) development as reflexive with the development of *social* classroom practices and norms. From this theoretical frame, individual development is constrained by the norms and practices established in the collective, while the development of the collective is made possible by and does not occur apart from, the psychological activity of individual students. The current project targets our work at understanding both the development of individual children's understanding of the quotient and the classroom norms and practices that constrain or enable that understanding. To do this, we triangulate between the microgenetic study of a group of 4 children, with the sociological study of classroom practices within which the group is situated over the same instructional sequence. By understanding the nature of classroom discursive practices aimed at developing mathematical understanding, and the concomitant actions and understandings of children as they engage in those practices, we will be able to report not only how children's understandings develop, but why and under what conditions they develop. Moreover, we hope to be able to project ways in which consistent and coherent understandings of quotients can be fostered in the future.

Method

The case for this classroom teaching experiment centered around a fifth-grade classroom (20 students) in an urban school district in the Southwest United States. The teacher, Ms. Mitchell, just finished her Masters Degree and has been a teacher leader and actively involved in reflecting about mathematics teaching practices within the district. In particular, Ms. Mitchell was involved in a project designed to develop an understanding of children's mathematical thinking in teachers in the areas of Algebra, Geometry, and Statistics. Rational number, as a fundamental conceptual underpinning in each of these areas, was a key focus of the staff development.

The research team (which included Ms. Mitchell) developed a 5-week instructional sequence based on Toluk's (1999) developmental model. A pretest of students' knowledge of fractions and division was administered one week prior to instruction (the students had just completed a TERC unit on Fractions, Percents and Decimals), and again as a post test, one week following instruction. Four children (2 boys, 2 girls), who comprised a cooperative group, were selected as target subjects for the analysis of individual knowledge within the collective. These students were interviewed using the pretest as a protocol the week prior to instruction, and again following instruction. Two clinical interviews of Ms. Mitchell were conducted prior to instruction, as she reflected on the purpose of the sequence, particular strategies she would employ, and her assessment of how she expected students to perform on various tasks. During instruction, Ms. Mitchell was formally interviewed once each week, and informally interviewed following each lesson. Lessons lasted approximately 45 minutes per day. All interviews and lessons were captured on digital video. During instruction, one video camera was focused on Ms. Mitchell and the whole-class interaction, while a second camera was focused on the selected group of 4 students. Additional documentation includes copies of all children's individual work, group models drawn on large sheets of chart paper, the teacher's notes, and field notes by members of the research team. The general structure of the data collection reflects the cyclic process of design experiments and developmental research in particular (e.g., Cobb, 2000).

Results

The results of this study indicated that the practices of the classroom moved *more or less* along the hypothetical learning trajectory abstracted from the model proffered by Toluk (1999; Toluk & Middleton, 2000). That is, they progressed from treating fractions as exclusively Part-Whole to having at least two parallel conceptions: Part-Whole *and* Fair Share (Quotient), while at the same time began to conceive of the division operation as generalizable to any pair of whole numbers, and some children began to see the operation as generalizable to division by fractions Table 1 illustrates the general learning trajectory of the class in three more-or-less distinct instructional Phases (see Figure 2).

In Phase I, key norms were established in the classroom that included the explication of forms of representation in terms of their ability to be partitioned, the acceptance of nested sets of units (such as 2 packets of gum with 5 pieces of gum in each pack) and the need to select an appropriate unit with which to coordinate partitioning. This need was established when the teacher would introduce units at different levels in a nested structure to force students to communicate precisely how they conceived of the quantities under consideration. For Ms. Mitchell, a huge jump in conceptualizing the difficulties in coordinating both partitioning and unitizing occurred early on when students were unable to develop strategies other than repeated halving for "odd" denominators (e.g., Pothier & Sawada, 1990). For example, when Ms. Mitchell asked children to describe "other names" for the (Fair-sharing) fraction 2/3, they were readily able to provide 4/6, 8/12, and 16/24, but stated that 6/9 did not follow the pattern, and thus, was not a legitimate name for 2/3. Students were able to make sense of 2/3 = 6/9 in a Part-Whole context, but were not in one involving Fair-Sharing.

In Phase II, as children began to symbolize fractional quotient situations as answers to "division problems," they encountered a new kind of quotient: One that

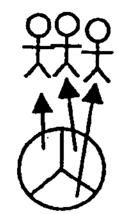
Phase		Focus of Classroom Activity	Forms of Reasoning	Key Transitional Strategies
I	**Representing Quotient as the Answer to a Fair Sharing Problem**	Partitioning: Transforming a whole into countable units. Understanding equivalency of shares. How many pieces make up a share. Representing answer as a fraction. Conceptualizing magnitude of fractional answer.	Primarily Pictorial: Partitioning and assigning. eg: 	Directing students' attention to the magnitude of the partitions, and the composition of shares—eg: away from "How Many?" to "How Much?" Presentation of problem contexts that could be represented by a "mixed number"—a quotient with a whole number portion and a fractional portion.
II	**Symbolizing Fair Sharing Problem Using Standard Divisor and Dividend Notation**	Determining the meanings ascribed to the divisor and dividend in a÷b notation. Mapping the meanings of partitions and whole from Phase I onto a÷b. Symbolically representing fair sharing as division, with a fractional answer.	"Stuff to split" became defined as the dividend (unit) The number of groups to distribute shares across became defined as the divisor (partitions) The bar in the fractional notation came to represent division e.g., "You divide the stuff to split by the number of groups."	Confront conception that "bigger number must divided by smaller" through examination of problem context. Consistently pairing a÷b notation with a/b notation, especially with the introduction of number sentences: $a \div b = a/b$ Using multiplicative relationship as a conceptual support e.g.: $a \times b = c \quad c \div b = a$ where a, b, or c could be < 1.

Figure 2, Part A. Description of classroom learning trajectory as played out in three instructional phases.

Phase		Focus of Classroom Activity	Forms of Reasoning	Key Transitional Strategies
III	**Flexible use of Fraction and Division Notation with Conceptual Understanding**	Examining the relative magnitude of divisor and dividend to anticipate quotient. Predicting the answer to a fair sharing problem without computation. e.g: $6 \div 9 = 6/9$ and $6 \div 9 < 1$ Switch to symbolizing division relationship first, then computing and symbolizing answer. Proving correctness of answers by using inverse operation	Anticipating the magnitude of the result of fair sharing as being either greater than or less than 1. Developing a rule for determining if quotient is greater than or less than one. Connecting the previously separate symbolizations for division and fractions conceptually—e.g., recognizing that they are the same. Prior to this Phase, students focused on figuring out how to symbolize the division relationship and fractional answer separately.	Now pictorial representations were used to justify reasoning following an attempt at solving fair sharing problems, as opposed to being used as tools to come up with solutions.

Figure 2, Part B. Description of classroom learning trajectory as played out in three instructional phases.

had a familiar whole number part, and an extra piece that could further subdivided. In particular, the provision of a continuous partitionable unit in the problem context (e.g., Brownies) enabled children to distribute fractional pieces of objects across groups and begin to conceive of answers to division problems as containing both a whole number part and a fraction part. Later, these fractional parts became the objects of discussion, and "division" problems were introduced that involved only fractional parts.

A watershed moment arose in Phase III when students realized that the notation a + b yielded a fraction a/b for every a>b. Previously, children tended to write given any improper fraction as a mixed number, and this tendency this hindered their ability to recognize the relationship. By providing successive problems that moved towards a numerator less than a denominator, the class was able to come to a taken-as-shared understanding that a + b yields a fraction a/b for every a<b. Not all children reached this understanding individually, however. While in the context of classroom discussion, for example, children would work together to develop a communal solution to a problem utilizing this "Division as Number" scheme, but on the post-test, a number of children did not utilize this understanding for improper fractions.

Discussion

Previous research has shown that young children deal with quotient situations—fair sharing—easily (Lamon, 1996; Kieren, 1988). In a fair sharing situation, a typical behavior is to partition quantities and write the resulting fractions as the quotient of a given situation. From their partitioning behavior, it is concluded that children conceptualize those fractions as quotients. The results of the present teaching experiment, however, suggest that ability to partition quantities into its parts didn't necessarily indicate that children actually conceptualized the resulting fractions as quotients. This was evidenced by the general reluctance in symbolizing quotients less than one as division number sentences.

Children were reluctant to symbolize the situations in this manner for a variety of reasons. First, for them, "division" always yielded a whole number quotient with or without a remainder. It was not that they didn't recognize the partitionability of the remainder, it was the fraction, the result of the partitioning that they didn't conceive of as a part of the quotient. Typically, in their minds, "fractions" resulted from fair sharing situations which involved cutting or splitting of whole quantities into parts less than one, whereas "division" resulted from partitioning quantities into groups with number greater than one. Second, students' resistance to consider a fraction as a quotient was due to the the fact that children had such a strong understanding of the Part-Whole subconstruct which, from prior experience, was the only way they were used to thinking about a fraction.

Moreover the existence of an understanding of fractions representing fair shares and part-whole relationships and division as a whole number operation as separate entities does not necessarily imply that children will connect these understandings into a coherent quotient scheme without intervention. Rather, the results of the teaching experiment showed that some explicit connections have to be made between these concepts. By making the commonalities in context and notation between whole number division and fraction situations problematic, children need to be encouraged to reflect on the equivalency of fair sharing situations and whole number division. Using a common symbolization; children's conceptions of division can be challenged. In this teaching experiment, division number sentences were used to show that the result of a division operation is a fraction represented by the common divisor/dividend notation). These results suggest that, while the mathematics of quantity that connects rational number subconstructs to each other in an epistemic framework is well thought out, without mediation, children's proportional reasoning does not recapitulate this epistemic frame.

The findings of this study suggest that a fragmentary approach to teaching quotients, which arranges instruction into two distinct and separate conceptual dimensions, i.e., division as an operation and fractions (as primarily part-whole quantities, but also applying to simple fair sharing), may lead children to develop a fragmentary understanding of the quotient. Instead, the present cases suggest that providing common problems in both fractional and division forms, and confronting children with the basic equivalencies of these forms may be fruitful in developing a more powerful understanding of the quotient subconstruct.

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