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ABSTRACT

This paper reports on the vital role that reflective thinking plays in solving problems involving the mathematics of change and variation particularly multivariable calculus. We report on how reflective thinking is one kind of self-regulatory (in the sense of metacognitive) thought mechanism. We present an outline of the results of a larger study, which investigated the metacognitive behavior of undergraduates solving problems in the calculus. (Author)

THE NATURE OF REFLECTIVE THINKING IN MULTIVARIABLE CALCULUS

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This paper reports on the vital role that reflective thinking plays in solving problems involving the mathematics of change and variation particularly multivariable Calculus. We report on how reflective thinking is one kind of self-regulatory (in the sense of metacognitive) thought mechanism. We present an outline of the results of a larger study, which investigated the metacognitive behavior of undergraduates solving problems in the Calculus.

Theoretial Perspective

> Metacognition refers to one's knowledge concerning one's own cognitive processes or anything related to them, e.g., the learning-relevant properties of information or data ... Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of those processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete [problem-solving] goal or objective. (Flavell, Friedrichs, & Hoyt, 1970, p. 323)

This introductory quote by Flavell et al. set the scene for the next 30 years of inquiry into a complex cognitive mechanism which has received much attention as well as skepticism from various research traditions – from Neuroscience to Mathematics Education.

In Mathematics Education, Schoenfeld (1985, 1987) introduces metacognition through the Artificial Intelligence notion of *Control* as a cognitive strategy for resource allocation. Similarly, Garofalo & Lester (1985) introduce Control as related to metacognition:

Metacognition has two separate but related aspects:

(a) knowledge and beliefs about cognitive phenomena, and

(b) the regulation and control of cognitive actions. (p. 163)

Later Schoenfeld (1985) added a social dimension to the inquiry by observing a Vygotskian perspective on regulation-of cognitive strategies as a process by which we observe and internalize other people's similar thought mechanisms. Through operating in the *Zone of Proximal Development*, learners observe and replicate overt examples of how other people regulate their own problem-solving behavior. Through this practice, we proximally appropriate the behavior as part of own cognition in solving mathematical problems and specifically as part of our own revised regulatory behavior. Brown (1987) also observes this kind of "other-regulation" as being one of four historical

roots for metacognition. So, in attending to studying the metacognitive behavior of students learning mathematics it soon becomes apparent that the relevant literature in Mathematics Education and Cognitive Science (and the overlap) create a multifarious if not multi-headed beast that can lead to confusing lines of inquiry. While it is evident that metacognition is still studied in the fields of Neuroscience (Shimamura, 1994) the word is more so used in mathematics education rather than studied. It is evident in the National Council of Teachers of Mathematics (NCTM) *Standards* (2000) that instructional programs from pre-kindergarten through grade 12 should enable all students to *monitor* and *reflect* on the process of mathematical problem-solving:

> Good problem solvers become aware of what they are doing and frequently monitor, or self-assess, their progress or adjust their strategies as they encounter and solve problems. Such reflective skills (called metacognition) are much more likely to develop in a classroom environment that supports them. (NCTM, p. 54)

It is not evident firstly, what this process of mathematical problem-solving is, hence enabling teachers to observe this behavior in their own students; nor secondly, what kind of classroom environments do actually support these skills as the Standards report. In our work, we have regarded the term metacognition as an umbrella term to embrace multiple perspectives such as Control (from AI), socially-constructed regulation (Vygotskian perspective) and self-regulation (Piagetian perspective).

Regulation-of thinking has played a significant role in Piaget's work (cf. 1978) and that of his contemporaries (Inhelder, Sinclair, & Bover, 1974; Karmiloff-Smith, 1979) where they refer to it as self-regulation and conceptual reorganization. Karmiloff-Smith & Inhelder's ideas explain how these mechanisms are characterized by the internal pressures to systematize, consolidate and generalize knowledge. Following our initial pilot work and study, our analysis highlighted the necessity to concentrate on this one strand of metacognition which offered us a suitable theoretical perspective given the nature of the learners we were studying and our research design.

Garofalo & Lester (1985) cite some evidence of *knowledge-of* thinking. They claim that older children realize that memory ability differs between pupils and that some realize that they can recall better than other individuals. Our pilot work showed similar evidence of this type of thinking and that self-regulatory cognitive strategies were far more apparent.

The main piece of empirical work (Hegedus 1998) further enhanced the meaning of self-regulation in advanced mathematical thinking by observing and verifying four particular strands of this type of metacognition: *Reflection, Organization, Monitoring* and *Extraction* of mathematical resources. In Hegedus (2001), we reported on the role of organizational thinking in the integral Calculus. This refers to the planning behaviors that a problem-solver engages in, in both the exploratory phases and the execution phases of a problem. We now outline the method of data collection and analysis and

then report on another strand of self-regulatory thinking outlined above; the nature of *reflective thinking* particularly when dealing with problems in the Calculus of many variables. As stressed in our previous report, our main theoretical perspective here is grounded in the nature of the mathematics that our students are learning. In detailing the nature of the students' problem-solving behavior in terms of their self-regulation we are examining how particular aspects of the mathematics of multivariable Calculus calls for particular mathematical cognition. Hence our study interprets psychological behavior in terms of mathematical procedure, conceptualization and execution of method.

Methods of Data Collection

Building on the work of Schoenfeld (1985), a pilot study was completed using a think-aloud protocol analysis to observe the role of *knowledge-of* and *self-regulatory* thought processes by undergraduate mathematicians. Our initial research aims focused on highlighting what kind of metacognitive activities the undergraduate students exhibited and how effective these were in their work. The Schoenfeld methodology proved most effective in attending to our primary research aims. We used a method of protocol analysis to focus on the decision-making processes at the executive level or control level. Schoenfeld highlights, in chapter 9 of his book, how this method provides ways of 'identifying three classes of potentially important decision making points in a solution':

1. Decisions at the control level are those that affect the allocation or utilization of a substantial amount of problem-solving resources (including time). It is thus appropriate to look for executive decision-making at points in problem solutions where there are major shifts in resource allocation.
2. The second type takes place when either new information or the possibility of taking a different approach comes to the attention of the problem solver(s).
3. The third category of decision-making points is far subtler. These are times in a solution where nothing has gone catastrophically wrong, but when a string of minor difficulties indicates that it is probably time to consider something else.

We adhere to Schoenfeld's method of data collection and analysis by obtaining verbal data of think-aloud accounts by our respondents, which we then record and transcribe. Each item of dialogue is then numbered. We then parse the protocol into six episodes and using the numbering system make the data more referable. An illustration of the parsing of the solution process (see http://merg.umassd.edu/heg/thesis) was derived by a consensus opinion of a team of three undergraduates working with Schoenfeld. The team had been trained in parsing the data in this way and the results were found to be very reliable. These are then plotted on a time-line diagram to offer a representational image of how the solver changed problem-solving states with respect

to time (see figure 1) completed in Excel. The six states are *Reading, Analyzing, Exploring, Planning, Implementing, and Verifying*. Triangles are used on top of the bars to show times of overt management activity – we did not attend to this final detail, as the respective activities were too numerous to illustrate.

Schoenfeld (1985, p. 297-301) outlines a description of the six episodes and associated questions, which are relevant to our analysis. These episodes refer to particular behavioral states, under which Schoenfeld believed that most problem-solving activity fell.

In his analysis Schoenfeld includes local assessments and introduction of new information with other overt managerial behaviors. He describes managerial behaviors to include:

> [S]electing perspectives and frameworks for working a problem; deciding at branch points which direction a solution should take; deciding in the light of new information whether a path already embarked upon should be abandoned; deciding what (if anything) should be salvaged from attempts that are abandoned, or adopted from approaches that were considered but not taken; monitoring and assessing implementation "on-line" and looking for signs that executive intervention might be appropriate; and much, much more. (Schoenfeld, 1985, p. 152)

We use the time-line diagrams to give an overview for the solution attempts of our students across a variety of problems as well as comparison with expert's solutions we similarly categorize and analyze. We then utilize the questions set out by Schoenfeld for analyzing the times of transition between episodes and the times where new information and local assessments occur. These include:

New-Information and Local Assessments

N1: Does the problem-solver assess the current state of his knowledge? (Is it appropriate to do so?)

N2: Does the problem-solver assess the relevancy or utility of the new information? (Is it appropriate?)

N3: What are the consequences for the solution of these assessments or the absence of them?

Transition Points

T1: Is there an assessment of the current solution state? Since a solution path is being abandoned, is there an attempt to salvage or store things that might have been valuable in it?

T2: What are the local and global effects on the solution of the presence or absence of assessment as previous work is abandoned? Is the action (or lack of action) taken by the problem-solver appropriate or necessary?

T3: Is there an assessment of the short- and/or long-term effects on the solution of taking the new direction, or does the subject simply jump into the new approach?

T4: What are the local and global effects on the solution of the presence or absence of assessment as a new path is embarked upon? Is the action there appropriate or necessary?

This method of analysis highlighted further categorization of what self-regulation means in the problem-solving behavior of Calculus students and its effect on mathematical efficacy.

Data Sample

Primarily, two classes of Calculus and Advanced Calculus were observed for two semesters at a leading British University. Working closely with the students in problem-solving classes certain students were chosen for their ability to think-aloud well and their problem-solving behavior. This said, the students chosen were still of varying ability and gender. We then completed a pilot study with three groups of 2 to 3 students per group to explore the existence of various metacognitive behaviors. Over the course of the following year, these three groups met periodically through two stages of the main data collection. Both sessions involved solving problems from the integral calculus. They had covered the relevant material that same semester yet they had not seen the problems previously. Examples of the problems include:

Problem 1

$$\textit{Evaluate} \iint_R x^2 \sin(x^4 + 2x^2y^2 + y^4)\,d(x,y)$$

$$\textit{where } R \textit{ is the region satisfying } x^2 + y^2 \le 1 \textit{ and } y \ge 0.$$

Problem 2

$$\int_0^1 dx \int_x^{x^{1/3}} \sqrt{1 - y^4}\,dy$$

These questions call for a degree of spatial awareness and flexibility, manipulation of co-ordinate systems and the geometric consequences, and retrieval of algorithmic and algebraic techniques from other areas of the Calculus.

Stage 1 of the empirical work comprised of two exploratory studies. These studies illuminated certain characteristics of metacognitive thinking which affected their problem-solving skills and led to the development and refinement of the four main

items of self-regulatory thinking - one of which we report on now. Stage 2 aimed at verifying and further developing the four main areas of self-regulatory thinking evident in stage 1. The observer intervened in stage 1 to assist in thinking-aloud which was in direct opposition to Schoenfeld's methodology. In stage II, a strict interventionist agenda was set up to direct the line of inquiry. In both stages an interventionist analysis was conducted to observe any bias in contriving the reported metacognitive behavior. Limitations of this report prevent us from detailing this method but it is incorporated in our analysis.

Analysis of Reflective Thinking

Throughout the study it was evident that the students' metacognitive behavior in the form of self-regulation was intimately bound up with the mathematics.

Three types of reflection describe this behavior:

1. Forward-reflection,
2. Backward-reflection,
3. A-temporal reflection

The first refers to looking forward into the solution and how it might develop; predicting outcomes or expecting changes based upon the solution so far or through experience. The second refers to checking and verifying the algebraic and geometric structure in the solution so far; why these were chosen and how they have an affect on how the solution might develop. The third is line-by line and on-the-fly checking often performed without close regard but nevertheless contributing to the coherence or development of the solution. This latter one is harder to distinguish yet more understood as part of problem-solving. The former two are what we particularly concentrate on in our analysis.

It is evident that each type of reflection is largely guided by the form and shape of the various mathematical signs and symbols in the students' problem-solving domain.

In general, the students involved in the study often only examine the variables of integration when they are substituted for another rectangular variable or converted into polars. Reflection on such a change of variable is often in the form of backward reflection. The students reflect upon the suitability of the change of variable with respect to simplification of the algebra. The change is not a function of the theoretical suitability or effectiveness of the new variable, which is a form of effective forward reflection. This has repercussions on the limits of integration that were formulated through the region of integration.

It is evident in the students' behavior that changing the order of integration could minimize the levels of difficulty in the algebra (i.e. forward reflection) but the theoretical implications of such actions again are often not acknowledged.

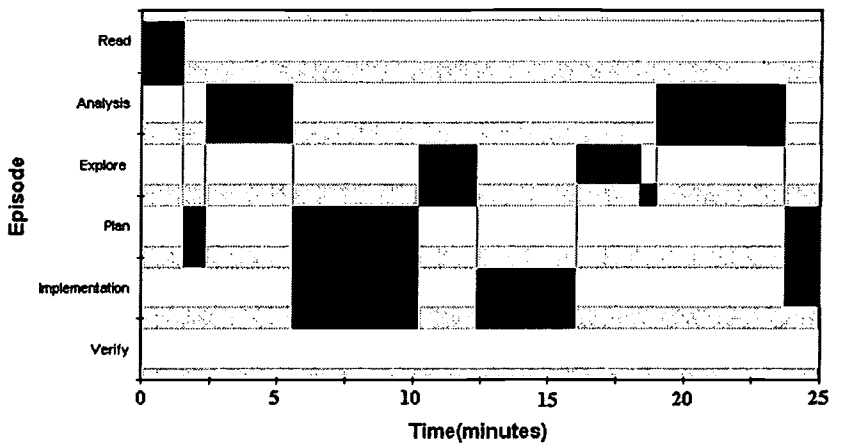

Figure 1

In general, it is reflection on the shape of a region and the form of the algebra that leads to a change in the variables used. We offer extracts of various pieces of work where two average students are working on the first and second problem above.

Following a Schoenfeld protocol analysis we constructed time-line diagrams for each of our sessions. One of the groups completing problem 1 is given in figure 1. It is evident that the students' work followed many periods of transition (grey block areas) and circulated through many plans. We performed the same analysis for expert problem-solvers (Mathematics Professors at the same university), and what is comparative with their solution and the student's work at the meta-level is the extended use of analysis time.

Our protocol analysis has highlighted particular aspects of their work (albeit ineffective) which relate to their reflective thinking as a self-regulatory thought mechanism through analysis of the verbalizations of the student with their written work. This was intimately bound up with the mathematics they were engaged with and their comprehension of the various concepts, algebraic and geometric techniques that they struggled with.

We offer one example of the verbal data which highlights the students' reflective behavior in attempting example 1. Backward reflection on the circularity of the region R: $x^2 + y^2 < 1$, already drawn in their work (see figure 2), leads one of the students to change the rectangular variables to polars (i.e. cylindrical):

G: Erm I think we might have to change the er ... the variables to polars because er ...

S: Why do you say that?

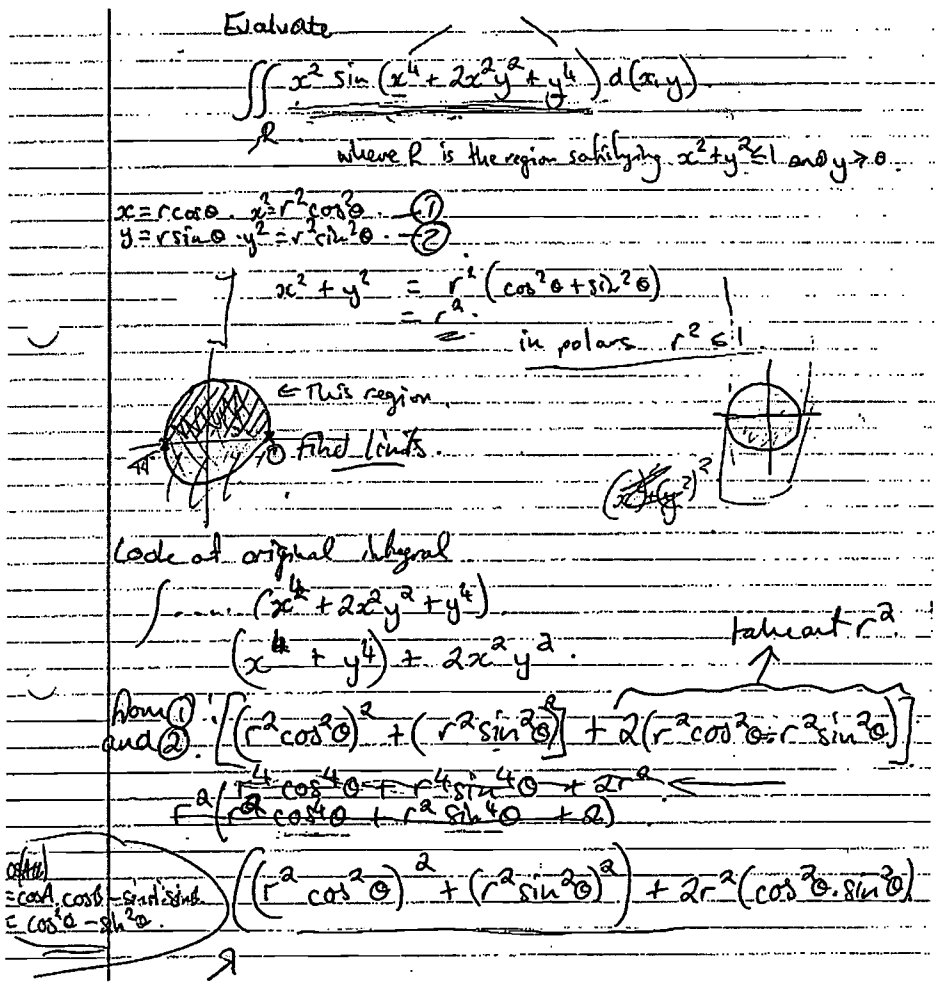

Figure 2.

G: Well because we're doing a double integral we going to end up with a volume over that the base .. which is $x^2 + y^2$.. and because it's less that or equal to 1 it's the disc not a circle so .. we've got a circle and everything inside of it so you've got like this base .. horizontal and you're going to be evaluating up in the air and you need to find what the limits are .. and polars are a bit easier to work with.

S: Why would they be easier? What would they convey? What would that expression be in polars? The $x^2 + y^2$...?

G: Erm .. I'll write it down. Because it's $x = ...$ r cos$q$, I think .... y's r sin. [Pause]

G: Right so that's r^2 ... should have known that ... but it's in polars so you've got r^2 less that equal to 1.

S: So that region has led you to say polars?

G: Yeah. [G - Student S - Researcher]

Having completed this stage, the students then thought about the implications of their substitution on the solution.

The region of integration plays an important role in determining the limits of integration and suitable methods of integration. It is evident in many of the students' work that the provision of a diagram for the region of integration is useful to these effects, but it is only effective if a theoretical understanding of the region is evident.

Reflection on the region often concerns the method of 'slicing', either by using washers or discs, in single integration. In double integration, it affects the construction of the limits of integration. It is not evident in their reflective behavior, though, by what means the region could effectively establish a suitable procedure of integration that would decrease difficulty in algebraic manipulation. Whilst reflection on the integrand in geometrical terms is not evident (i.e. as a surface $w = f(x, y)$, say, in double integration, or a hypersurface $w = f(x, y, z)$ in 4 space for triple integration) this behavior does not have any direct effects on their problem-solving processes. Reflection on the form and shape of the integrand and its algebraic construction is evident and this affects the choice of method, and the algebraic manipulation of it. In the example below, G & D (students) are attempting to solve the integral in problem 2:

S: Why can't you integrate it?

D: Well ... its in terms of y's?

S: Yeh.

D: Because .. I can't remember why but the function is er ... erm ...

[Pause]

G: Is it just because it's improper because it's y to the 4. I know you can do it if it's like something squared .. minus something squared square rooted.

S: Yes you saw that last term .. last semester, didn't you?

D: Yeh.

G: You've got to make a substitution with the y to the 4 .. you've got to substitute like er $z = y^2$. So like you then you got $1 - z^2$.

D: Which is $1 - z^2$ which equals er .. a standard integral, which is the back of the book. which I can't remember what it is?

The students then debate whether to change the order of integration and thus integrate the integrand with respect to x first rather than y. This, of course, would be the most suitable method, as long as suitable changes in limits are made.

In other examples the acknowledgement of oddness/evenness or symmetry in the integrand lead students to effectively reduce the levels of algebra in the integration procedures.

Reflection on the limits of integration seems only evident in the substitution of them. Even then, reflection is minimal. Furthermore, line-by-line reflection is not evident which results in algebraic difficulties and mistakes.

The shape and form of the algebra is often the driving force behind the organization and retrieval of procedural tools, but the functionality and efficiency of these tools is poorly reflected upon. These tools are both algebraic and algorithmic, and include for example, methods of anti-differentiation, trigonometric substitution, and the substitution of trigonometric identities.

Conclusion: Links to Practice

In general, reflection on the form and shape of the algebra lead to a variety of conceptual and procedural thinking. In conclusion, backward reflection caused the manipulation of algebra and the interpretation and reconstruction of geometric objects, which were part of the Calculus problem. Such behavior was conducted to varying degrees of efficiency. Progress occurred when reflection was also in a forward sense that looked critically at the choices made and understanding the theoretical implications (in terms of the Calculus) of their implementation. In addition, reflection is supported by the implementation and integration of the three other main self-regulatory behaviors (organization, monitoring and extraction) highlighted earlier.

This work has shaped a theoretical perspective to observe how self-regulatory thinking is a function of the mathematics being investigated, particularly the Calculus, and our present line of inquiry is exploring how these insights help inform pedagogy and how they might be implemented.

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