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ABSTRACT

This paper offers an argument for the return to a consideration of the basic issues in mathematics education research in order to better understand the mechanisms of mathematics learning. A return to these basic issues can also shed light on the difficulties students encounter in learning mathematics. In organizing the argument, three kinds of difficulties are distinguished that students encounter in mathematics learning: temporary difficulties, recurrent difficulties, and standing or insuperable difficulties. (Contains 22 references.) (DDR)

Basic Issues for Research in Mathematics Education

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Mathematics education covers a very broad range of topics, from primary school to university. It can be analysed from different points of view, epistemology, psychology... But, whatever topic and point of view may be, research in mathematics education entails theoretical and methodological choices on some core problems about the nature of mathematical knowledge with regard to all other kinds of knowledge. Does it depend on the same thought processes as the other kinds of knowledge or does it require the development of some specific ways of cognitive working ? Can be mathematics learning mainly described as a concept acquisition ? Which kind of representation is relevant in mathematical understanding ? Which field of phenomena can show the conditions of understanding and knowledge acquisition in mathematics ? These problems provide alternative choices. We can assume that thinking works in mathematics like in the other areas or that it works in a very specific way. We can focus either on objects and concepts particular to one mathematical area or on constant features of the mathematical activity. We can also focus either on mental and individual representations or on semiotic systems of representation. We can focus either on class room activity or on individual acquisitions over several years.

These basic issues are not purely theoretical. The choices lead to different ways of specifying relevant variables for mathematics learning, and they do not yield equal possibilities to explain the variety of difficulties that students come against up throughout their studies. From primary school to higher secondary level we can notice a strong contrast beween very spontaneous simple mathematics for every child and a little more advanced mathematics, for example when new concepts are introduced or when algebra is brought into use, when theorem proving is required or when graphs are used in analysis as an obvious tool of visualization... And we can see an increasing gap for learning : more and more students seem to reach a breaking point in their understanding of mathematics. Are we faced with the same kind of phenomena ? More precisely is there something similar in the process of mathematics learning at the first levels and at upper levels ? In fact, because of teaching requirements which are peculiar to each level of study and, also, because of internal evidence of mathematics, for teachers and mathematicians, some choices appear essential and obvious.

However, we must pay more attention to these basic issues, at least if we want to understand deep mechanisms of mathematics learning and difficulties most students encounter throughout their curriculum. Our purpose in this paper is to come back to these basic issues and to explain why our research has progresively led us to choices which are diverging from those considered as essential and obvious. In other words, the main question about mathematics learning is : does mathematics understanding require specific ways of cognitive working in comparison with the other fields of knowledge ? Or, from a phenomenological point of view, do visualization, language and conceptualisation work in mathematics in the same way as in other situations ? If it is not the case, what kind of cognitive working is required in order to understand mathematical objects and processes, in order to become equally able to apply them, and how can any student master it ?

I. Analysis of knowledge acquisition in mathematics education research

(1) What do analysis of mathematics knowledge focus on : concepts understanding or underlying thought processes ?

Mathematics are divided into various areas : arithmetic, geometry, algebra, calculus, statistics... And for each area we have, on the one hand, a set of concepts relative to objects such as numbers, functions, vectors, etc. and, on the other hand, specific algorithms, procedures, methods of problem solving which are close connected to concepts/objects. From preprimary schools to senior secondary schools, students must discover or learn some basic concepts and algorithms with their applications within these various areas... Thus we are faced with a large scope of teaching goals And each one leads to focus on the concepts/objects according to specific problems that their teaching can involve : what kind of situations to introduce them or to justify their introduction, what kind of mistakes can occur, what kinds of progression...? In these conditions, it seems difficult to avoid a certain compartmentalization in research. But above all, what concerns the common deep processes which underlie mathematics understanding are put off investigation. Learning processes are assimilated to the construction of such-and-such a concept.

Whatever the concept/object you choose, mathematics knowledge requires thought processes which are multidisciplinary and typical of what it is to understand in mathematics. That appears through validation, through proving, through using symbols and various visualization forms (cartesian graphs, geometrical figures..). For example, it is usual to observe a gap between the use of words and the use of symbols, between «the use of mathematical expressions and the way they are understood» (Sierpinska 1997 p.10), or between the spontaneaous ways of seeing geometrical figures and the mathematical ones. Learning mathematics is not only to gain a practice of particular concept/objects and to apply algorithms, it is also to take over the thought processes which enable a student to understand concepts and their applications. And these thought processes cannot be assimilated to construction of such-and-such a concept.

In the case of proof learning, that alternative between mathematical concepts/objects side and involved thought processes side appears clearly with proof, one of the most difficult topics in mathematics education. Because the ways to show why a proposition is true are not the same for theorems in mathematics as for statements about phenomena of the external world. How to help students gain insight into these very specific mathematical ways ? And why teaching does not succeed in finding such help with most students ? One can emphasize the need to provide not one but several proof methods or the importance to be confronted with rich epistemological context such as a physics problem ... That requires exploration of a particular set of data and activities for each theorem. But what matters is not only to gain insight why such proposition can be true, but to understand how proving in mathematics works and to gain the thought processes involved in proving. That changes the perspective within the educational problem of proof appears. Why, for example, cannot students really understand mathematical ways or reasoning, whenever natural language is used and whatever the proposition they have to prove ?

(2) From what kind of phenomena can the specific problems raised by mathematics learning be examined ?

In order to study the complexity of mathematics learning, we must take into account the students and not only the epistemological complexity of the taught concepts. But there are many ways to refer to what the students do, to their explanations, to their achievements, etc. We can try to observe live behaviours and productions over the learning time or, on the contrary, evolution of mathematical skills within further various situations over a whole curriculum. We can also focus on individuals or on the activities of the class including the teacher and the teaching organisation, or on the whole population of an age group. Thus, we have several possible areas of observation (both scale of time and field of study). Each area requires a specific methodology *because parameters and variations that can be checked are not at all the same.* And when we change the area of observation the problems of learning appear in an other light.

Scale of time →

		ongoing learning of mathematical concepts (over a short-lasting time)	**curriculum** (over several years of teaching) **What transfer ?**
Field	Individuals	one or several sessions - inside class activiities (*some particular productions*) - outside class activitties (*interview, experimental frame..*)	— feed-back of new acquisitions on the previous learnings — skills that can be mobilized in further situations at a higher level (*transversal or longitudinal methodology*)
of	One particular classroom: `the teacher or/and the students ?	one or several sessions (*case study....*)	
study	Population of an age group	at the end (*assessment*)	— feed-back of new acquisitions on the previous learnings — skills that can be mobilized in futther situations at a higher level (*assessment*)

Figure 1. The various areas of observation of mathematics learning.

There are many reasons and social demands which lead to emphasize one area rather than the others or to consider one particular kind of phenomena as the most relevant or the most significant. But the problem is not in this heterogeneous range of possible areas. It is about the depth of acquisition and the possibilities of transfer. Where and how can we gain data about this crucial aspect for any learning ?

For that, we must distinguish three kinds of difficulties that students come up against in mathematics learning :

— **temporary difficulties** in order to succeed the local goals of any learning sequence : they depend on degrees of newness for students, on misleading similarities to what is already known, or on the background of the underlying epistemological complexity

— **recurrent difficulties** whenever context is changed (for example, heuristic using of geometrical figures in problem-solving leads to such changes), or whenever new objects are introduced

— **standing or insuperable difficulties** (for most students) : they underlie local ongoing acquisitions and inhibit further acquisitions. They appear whenever students are faced with a proof task or with some verbal problem in arithmetic or in algebra

Hence the following question : what kind of difficulties do we have to examine, if we want to analyse the thought processes which are required for mathematics understanding and therefore the specific conditions of mathematics learning ? Temporary or standing difficulties ? Obviously all kinds of difficulties must be taken into account in teaching. But over ongoing learning and in the field of class activities, they cannot be truly discriminated. And, in fact temporary, difficulties are necessarily uppermost in the didactic purposes of the teachers. And we cannot avoid the question whether results at local scales can be extrapolated at global scales. Anyway when analysis is turned towards temporary difficulties, phenomena relative to epistemological complexity are favoured and, on the contrary, when it is towards insuperable difficulties, phenomena relative to the cognitive functioning of subject become the most significant.

II. Kinds of representation involved in mathematical thinking

There is no knowedge without representation. But from Descartes until now, through Peirce and Piaget, many changes have taken place in the way to consider the relationship between knowledge and representations, and the nature of representations appears to be more and more complex (Duval article 1998b, Duval *&alii,* 1999). When we talk of "representations" the four following aspects must be taken into account :

— (a) *the system by which representation is produced.* Any representation is produced through a particular system. It can be a physical device such as camera, or an brain organisation as for memory visual images, or even semiotic system such as various languages. And the content of the representation of an object changes according to the productive system of representation which is used. It means content of any representation depends on its productive system and not only of the represented object. The content of a verbal description of a man in order to recognize him is not the same as the content its sketch portrait, or the content of the graph of a function is not the same as the content of its analytic expression. Human thinking require the mobilization of several heterogeneous productive system of representation and their coordination. Do thought processes especially require semiotic systems as the main constituent of the cognitive architecture which enables any individual to understand mathematics ? And, in mathematics education, what is it crucial for learning, (a1) taking into account the global and spontaneous individual state of beliefs about a subject (Peirce, the first Piaget), or (a2) making the students aware of the ways of functioning of the semiotic productive systems which are used in mathematics ?

— (b) *the relation between representation and the represented object*. There are two kinds of productive systems of representation : on the one hand physical devices and neuronic organisations, on the other hand semiotic systems. In the first kind (b1) (physical and mental images), the relation is based on action of an object on the system (causality), and in the second one (b2) (words, symbols, drawings) the relation is only denotation. In mathematics education, when we talk of "mental images" what kind of relation are we referring to ?

— (c) *the possibility of an access to the represented object apart from semiotic representation.* We have representations (c1) which are an evocation of what has already been perceived (Piaget 1926, 1946) or what could be perceived and representations, or (c2) about objects (mathematical objects) which cannot be perceived.

— (d) *the reason why representation using is necessary* : either (d1) mainly communication or (d2) processing (computation or discursive expansion (Frege 1891, 1892), anamorphosis, etc).

According to the way these aspects are taken into account, what is referred as representations change. I will confine here to the relevant issues for mathematics learning.

(3) Which brings about the most misunderstanding : subjective representations of students or manifold semiotic representations used in mathematics ?

Many studies have examined students mistakes over the learning of concepts for each level and some failures remain whatever teaching method is adopted. In order to explain these structural misunderstandings, subjective representations (a1, c1) are emphasized as being the root of obstacles encountered over learning. Thus, in the triadic conceptualisation of Peirce (2.228) {Object, "representamen" (sign), "interpretant"}, interpretant is emphasized in such a way that representations are mainly mental phenomena and individual beliefs.

Progress in mathematics has involved development of several semiotic systems from the primitive duality of cognitive modes, image and language, which are linked with the more informational sensory receptors : seeing and hearing. For example, symbolic notations stemmed from written language and have led to algebraic writing. For visualization, the construction of plane figures with tools, then that of figures in perspective, then that of graphs in order to "translate" curves into equations. Each new semiotic system provides new means of representation and processing for mathematical thinking. So that for any mathematical object we can have different representations produced by different semiotic systems (a2, c2). Thus we must change the triadic conceptualisation of Peirce in the following way : {Object, one of the various semiotic systems, composition of signs). But that necessary variety of semiotic systems raises important problems of coordination.

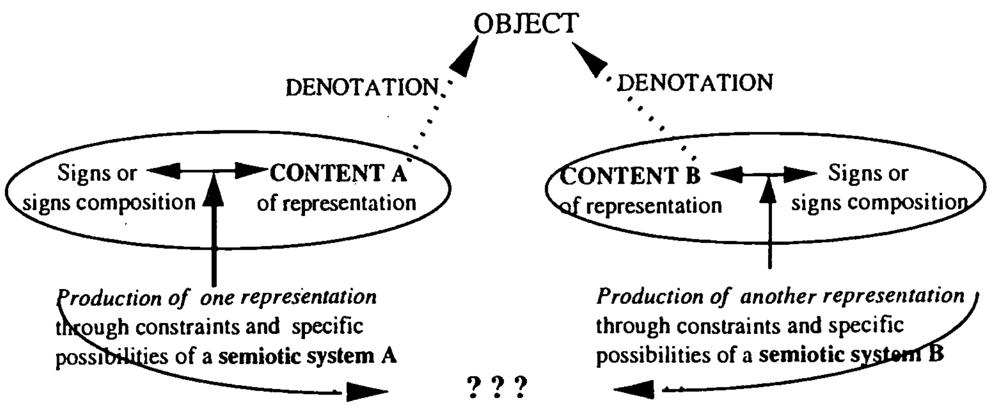


Figure 2. Representation and understanding for mathematical knowledge

In that perspective, deeper causes of misunderstanding appear. Whenever a semiotic system is changed, the content of representation changes, while the denoted object remains the same. But as mathematical objects cannot be identified with any of their representations, many students cannot discriminate the content of representation and the represented object : objects change when representation is changed!

Here the issue is to know what kinds of representation is crucial for mathematics learning. Emphasizing individual beliefs, as for physical phenomena (Piaget), leads to assume a purely mental cognitive model in order to analyse acquisition of mathematics knowledge. And semiotic representations are considered as external to thought processes. Is such an assumption obvious and, above all, relevant ?

(4) Is the distinction between mental and material representations relevant for the use of semiotic representations in mathematics knoledge ?

This distinction is based on three considerations. First, the dualistic opposition, for any sign, between signifier and meaning, between what must can be perceived and what is evoked in the mind (c1). Then understanding is about objects and goes beyond the content of any semiotic representation. Lastly, semiotic representations are needed for communication (d1). Hence the opposition between purely mental representations which would be enable anybody to understand and semiotic representations which would be mainly for communication and social interactions. And it is often argued that semiotic representations used by somebody else are sometimes difficult to understand.

However, in mathematics, semiotic representations does not fulfill first a communication function but a processing function (d2). It is only through semiotic representation that mathematical numbers can be reached and used. Progress in the human numbers knowledge has been closely connected with progress in numeral systems. In fact, the opposition between mental and semiotic systems is deceptive because it is the outcome of the confusion between two heterogeneous aspects in representation production : the phenomenological mode and the used system. Moreover, in external phenomenological mode, we must distinguish oral and visual (writing, drawing) modes. Semiotic representations are neither mental as images of memory (b1) nor material as pieces which can be physically handled.

		Phenomenological MODE of production		
		Mental	Material	
			oral	visual (writing, drawing)
Kind of SYSTEM	SEMIOTIC (intentional production)	*objectivation and processing functions*	*communication functions*	*processing, communication objectivation functions*
of production	NATURAL (automatic production)	*objectivation function (mental imagery)*		

Figure 3. Production of representation and relationships between thinking, semiotic system and the main cognitive functions.

Semiotic systems of representation play an essential part in all the main cognitive functions : not only communication but also processing, that is the transformation from one representation into another one inside the same system and without resort to further external data. And semiotic reprentations too are necessary to enable anyone to become aware of something new. Objectivation is an expression or representation for himself, which can be either mentally or materially produced. But its constrainsts are quite different from these one in social interactions.

The way we take into account semiotic representations involves an implicit model of cognitive working of human thought and entails choices concerning mathematics learning.

Thus one can really wonder why pure mental models, that is off-semiotic representation models, are always postulated to explain mathematics understanding. What seems simple or purely mental in the inner evidence of understanding, especially when you have become an expert, is the outcome of a very long process of internalization of semiotic representations.

One can also ask whether the variations of apprehension between the oral mode of production and the visual mode of production do not lead to introduce a distinction between two kinds of mathematics : spontaneous mathematics which can be discovered, or done, by everyone, child or adult, at school or ouside school, and hardly more advanced mathematics which require, on the contrary, skills in extensive processing of semiotic representations. So that the jump in learning would be between mainly oral practice of mathematics and necessarily writing practice of mathematics. The passage from additive to multiplicative operations, or this one from natural numbers to decimal numbers seem to require such a change of practice. But also proving in the discovery of which writing can be a necessary stage for purpose of objectivation and not only of communication (Duval 1999).

III. What kind of model is relevant to explain the mathematics learning process ?

Somehow, any model must refer to the organisation of a field of phenomena and describe its way of working. With regard for the mathematics learning, we can distinguish two great kinds of models : the developmental models and the cognitive models.

The developmental models focus on the increase in knowledge. Initially they referred first to two fields of phenomena. On the hand, the historical ways whose mathematical concepts/objects were discovered and on the other hand the ways in which young children become aware of natural numbers, geometrical shapes, schematic representation of environnemtal space... And a relative parallelism was postulated between these two fields of phenomena in order to explain acquisition of mathematics knowledge. Thus there a link was established between epistemology and developmental psychology. On this basis, constructivist model of development appeared as the sketch description of every acquisition of mathematics knowledge. And therefore learning processes over the curriculum would have to follow constructivist "laws" of knowledge acquisition

Developmental models have led to enhancing a third field of phenomena, the interactions between students inside classroom, especially while they are solving problems. These interactions present three advantages : they correspond to a main factor in the constructivist model of acquisition, they can be managed by the teacher, they enable researchers to observe live learning processes (see above I. 2).

In a developmental model, explanation of learning process is refered to common schemes which would underlie any increase in knowledge at historical scale of discoveries, at genetic scale of child's intelligence growth (outside of any teaching from the Piaget's view point) and also at the local scale of group work. The cognitive complexity which underlies mathematics understanding is not taken into account except for subjective representations when they seem to hinder learning (see above II.3).

The cognitive models focus on the cognitive complexity of the working of human thought. At first sight, they seem far from mathematics learning. And classical models developed in psychology laboratories cannot be used as they are (Fischbein, 1999). By the simple reason that the learning of mathematics raises specific and fundamental questions about reasoning modes, about the treatment of figures, about the understanding of mathematical concepts —and infinity is a very important instance— which are not envisaged by psychologists. Nevertheless, there is a core question which cannot really be raised in the framework of the developmental model :what are the ***internal cognitive conditions required*** in order that ***any student can understand mathematics at any level*** *of primary or secondary school ?* Note that we are talking now of "understanding" and not only of "learning". These internal cognitive conditions refer to what was called the archictecture cognitive , that is an organisation of several systems (Kant, p.619) : in such an archictecture several semiotic systems must be included or more precisely incorporated into natural systems.

We have already evoked two important facts. Whatever the phenomenological mode of production of representations, working of human thought involves using one or several semiotic systems : the first of all is the native language. But acquisition of mathematics requires other semiotic systems such as the decimal numeral system, algebraic writing or formal languages..; which are suited to mathematical operations. Unlike oral native language, the semiotic system used in mathematics as well as written language, are not natural. In the context of the core question, research on learning processes, must take into account how such semiotic systems can be internalized by students and under what conditions they can become operative for each student on.

The alternative bewteen developmental models and cognitive models concerns directly the way the problem of mathematics learning is raised and analysed : either an increase of knowledge according to common and general processes or a minded-opening to quite specific thought processes.

IV. The paradoxical character of mathematical knowledge

Concerning the cognitive mode of access to objects, there is an important gap bewteen mathematical knowledge and knowledge in other sciences such as astronomy, physics, biology, or botany. We do not have any perceptive or instrumental access to mathematical objects, even the most elementary, as for any object or phenomenon of the external world. We cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations. *This conflicting requirement* makes the specific core of mathematical knowledge. And it begins early with numbers which do not have to be identified with digits and the used numeral systems (binary, decimal).

Obviously, it is not a significant characteristic for mathematicians and epistemology does not take it really into account. From an intrinsic mathematical point of view the semiotic side, which is the only directly accessible, seems to be transparent or subsequent to non-semiotic actions. But from a comparative epistemological point of view, the conflicting requirement cannot be erased. On the contrary it appears as the crucial problem of mathematics learning. In the other fields of knowledge, semiotic representations are images or descriptions about some phenomena to which we can gain a perceptive or instrumental access, ouside any semiotic representations. In mathematics it is not the case. In these conditions, how can a student *learn to distinguish* a mathematical object from any particular semiotic representation ? And therefore, how can a student *learn to recognize* a mathematical object through its possible different representations ? At every level, among many students, inability to convert a representation from one semiotic system into a representation of the same objet from another system can be observed as if both representations refer to two different objets. This inability underlies the difficulties of transfer of knowledge and also the difficulties to translate verbal statements of any problem into relevant numerical or symbolic data for mathematical solving.

This conflicting requirement, which is typical of mathematical knowledge, can be approached otherwise. It is very often assumed that mathematics resort to the most common thought processes : reasoning and visualization. And this assumption is particularly strong in the teaching of plane and solid geometry. But there teaching comes up against difficulties which indicate *an imperceptible but deep gap between common thought processes and mathematical processes*. Considering always persistent understanding blocks about theorems proving and heuristic using of geometrical figures in problem solving is enough to ask questions about the specific cognitive working that mathematical knowledge requires.

The recurrent confusions between hypotheses and conclusion, between a statement and its reciprocal, and other dysfunctions are only the expression of the natural discursive practice in the ordinary way of reasoning. In fact, under similar pratices of speech, there is a discrepancy between the kind of organisation of propositions within a valid reasoning and the one in any common argumentation or explanation (Duval to be published). In order to make students become aware of this discrepancy a cognitive detour is required (Duval 1991). Understanding what is being proved in mathematics is not at first a matter of learning methods, facing different proofs for the same theorem or even convincing other students...

Nothing seems more obvious than a geometrical figure. It seems providing directly to see, even if every figure is always a particular configuration. In fact when the goals of teaching go beyond recognizing or constructing elementary cultural shapes, the gap between figures perception and mathematical way of seeing figures is widening. Mathematical visualization, in the case of geometrical figures, leads away from any iconic representation of physical shapes. Unlike iconic representations, figures are not sufficient to know what are the denoted objects (Duval 1998a). Besides, for the same object, we can have quite different possible figures : thus, for example, there are two typical figures for a parallelogram and only one is iconically a paralelogram shape. But when it is a matter of solving a geometrical problem, the complexity of using geometrical visualization increases fast for most students, even at upper levels. And there we are faced with a field of phenomena which cannot be explained only by the epistemological complexity of such-and-such a concept !

We can focus on the paradoxical character of mathematical knowledge or put it on the fringe. That means to emphasize what is specific to cognitive working in mathematics understanding or to confine cognitive structures that would be common to any kind of knowledge. That means also either to take a comparative viewpoint with other fields of knowledge or to take one only within mathematics. In order to study mathematics learning, we must take into account mainly the insuperable difficulties. And these difficulties, which are the most inhibiting for students, must be analysed in relation to the conflicting requirement and to the gap between common thought processes and mathematical processes. Which raises the following question : what is the cognitive working that underlies understanding in mathematics ? And that leads to highlighting the importance of representations not in the ordinary sense (a1, c1, d1) but in the alternative one (a2, b2, c2, d2).

V. The cognitive working that underlies understanding in mathematics

We cannot talk about representation without relating it to its system of production. But to take into account semiotic systems means focusing on **the transformations** of representations. Thus we must distinguish two kinds of transformations : "processing" and conversion.

Some semiotic systems provide specific possibilities of intrinsic transformations of representation. Any transformation produced in one system can be changed in another representation of the same system. Thus, paraphrase, reformulation, computation, anamorphosis, reconfiguration, etc. are transformations of semiotic representations which can be achieved only in one specific register. We referred to this kind of transformation as "processing" and we referred to semiotic systems which provide such possibilities as registers of representation.

For any representation of an object which is produced within a system, we can also produce another representation of this object into another system. We referred to this kind of transformation as conversion. Thus constructing a graph from a given equation or writing an equation from a graph, translating a verbal statement into a litteral expression or into a equation... Geometry is a field where conversion is very much in demand, as well implicitly as explicitly. But numbers required also changes of representation which are more similar to conversion than to processing, even with the simple change from decimal expressions to fractionary expressions, apart from a few frequent associations such as 0,5 into 1/2.

Researchers do not pay very close attention to **the gap between these two kinds of cognitive operations.** In mathematics processes and in analysis of mathematical tasks, they are not really separated, whenever they are implicitly or explicitly needed. They are looked upon as a whole. For example mathematical activity, in problem solving situations, requires the ability to change register, either because another presentation of data fits better an already known model, or because two registers must be brought into play, like figures and native language. From a cognitive view point the real problem is to know whether these two kinds of transformations can be considered as depending on the same deep thought processes. All observations show that is not the case.

a. The irreductible cognitive complexity of conversion

Conversion is the transformation of representation of an object by changing register. Two main facts can be observed at any level.

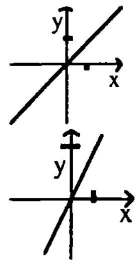
In some cases conversion is obvious and immediate as if the representation of the starting register is transparent to the representation of the target register. In other words, conversion can be seen like an easy translation unit to unit. Conversion is congruent :

set of points whose ordinate is greater than abscissa $\longrightarrow$ $y > x$

In other cases it is just the opposite. Conversion is non-congruent :

set of points whose ordinate **and** abscissa are **with the same sign** $\longrightarrow$ $x \, (\times) \, y > 0$

Non-congruence is the crucial phenomenon for any task of conversion. Difficulties and mental blocks stem often from the inability to achieve a conversion, or to recognize it when it has been made. But what is the most surprising with this crucial phenomenon for mathematics understanding is its unidirectional character. A conversion can be congruent in one way and non-congruent in the opposite way. Congruence or non-congruence are closely connected to the direction of conversion. That leads to striking, typical and particularly persistent contrasts of performances, such as in the following figures.



→ recognition → $y = x$ — **60%** students 15-16 years old (after teaching on linear functions)

→ $y = 2x$ — **25%**

Figure 4. (Duval 1988, 1995b)

Obviously, in the opposite direction, conversion is very easy and there is no more difference between equations (Duval 1996b) . And at higher level we find the same analogous results.

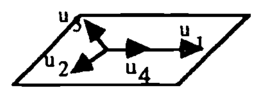
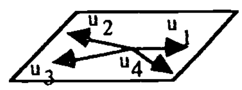
	Starting register	**Target Register**	144 students
(2D Rep.) T → G	$\begin{array}{|c|c|c|c|}\hline 1 & 0 & k & p \\ \hline 0 & 1 & m & 0 \\ \hline \end{array}$		**.83**
G → T		$\begin{array}{|c|c|c|c|}\hline 1 & 0 & a & c \\ \hline 0 & 1 & b & d \\ \hline \end{array}$	**.34**

FIGURE 5. Elementary task of conversion (Pavlopoulou 1993, p. 84)

By bringing into play systematic variations, contrast of successes and failures for the same mathematical objects appear in similar situations! Very accurate analyses of the congruent or non-congruent character of the conversion of a representation into another one can be systematically done. And they explain in a very accurate way many errors, failures misunderstandings or mental blocks (Duval 1995b, pp. 45-59; 1996a, pp. 366-367).

For every couple of registers, typical facts such as these can be sytematicaly observed. What do they mean ? We can see that two representations of an object do not have the same content from a register into another. And when conversion from one into the other is non-concgruent, the two contents are understood as two quite different objects. Students don't recognize it anymore. And there are good reasons for that. The apparent lack of correspondence between two contents of representations of the same object stems from the fact that content of representation does not depend first on the represented object but on the activated system of production. That means not only each register provides some specific possibilities of processing, but also does not explicit the same properties of objets as the other registers.

Now we are coming up against the consequences of the paradoxical character of mathematical knowledge. Since there is not direct access to objects apart from their representations, how can a student learn to recognize a mathematical object through its various possible representations when their contents are so different ? Explaining that as a lack of conceptual understanding is not a right explanation because we have reversals of successes and failures when changing the direction of conversion. In fact the explanation must be searched at a deeper level. Failures or even mental blocks when conversion is non-congruent reveals **a lack of co-ordination between the registers** that have to bring into play together. And if we come back to the schema (figure) we see that conceptual understanding is possible when such a coordination is achieved. Because of this, the condition for mathematical objects are not confused with content of representation. We can complete the above schema (figure 2) in the following way :

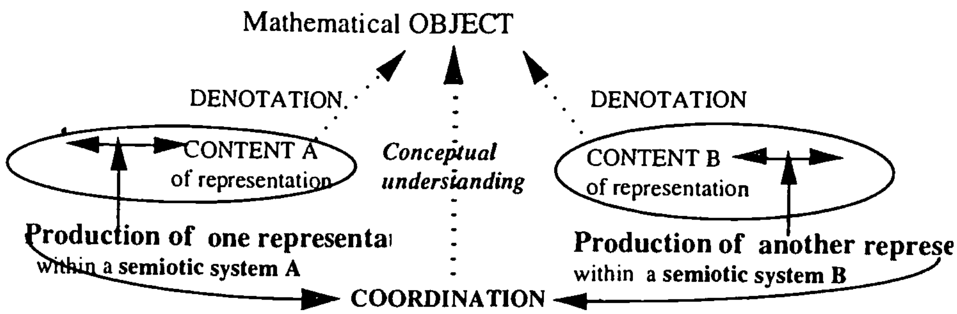


Figure 6. Cognitive conditions of mathematics understanding (see above Figure 2)

But for most students, understanding in mathematics is "confined to some processes within strongly "compartmentalised" registers. Learning mathematics consists in developing progressive coordinations between various semiotic systems of representation.

b. The cognitive ambiguity of some kinds of processing

We must remind that processing is a transformation of representations within one particular register.That means : ways of working of processing do not depend only on the mathematical properties of objects but also on the possibilities of used register. For example we have not the same process of computation with decimal and fractionary notations :

$$0,25 + 0,25 = 0,5 \qquad 1/4 + 1/4 = 1/2$$
$$0,5 : 0,25 = 2 \qquad 1/2 : 1/4 = 4/2$$

And we must also distinguish **multifunctional registers** from **monofunctional registers.** Multifunctional registers are those used in all fields of culture. They are used as well for communication goals as for processing goals. And, above all, they provide a large range of various processings. Thus natural language is necessarily used in mathematics but not with the same way of working as in everyday life (Duval 1995b, cap.II). Within these multifunctional registers, processings cannot be performed or changed in a algorithmic way. On the contrary, monofunctional registers have been developed for one specific kind of processing, in order to have more powerful and less expensive perfomances than those within multifunctional registers. Here processing becomes technical and using signs or expressions depends first on their form. Technical processing are formal processing. That's why processing can be expanded as algorithms.

	DISCURSIVE REPRESENTATION	NON-DISCURSIVE REPRESENTATION
MULTIFUNCTIONAL REGISTERS : **non-algorithmisable processings**	natural language *verbal (conceptual) associations,* *reasoning (argumentation from observatins or beliefs, valid deduction from definitions or theorems...)*	geometrical figures as shape configurations, plane or in perspective *operative apprehension and not only perceptual apprehension* *construction with tools*
MONOFONCTIONAL REGISTERS: **processings are mainly algorithms**	numeral systems symbolic or algebraic notations, formal languages *computation*	cartesian graphs *change of coordinates system, interpolation, extrapolation*

Figure 7. Classification of the four kinds of register used in mathematics processes

Mathematical processes involve at least two of these four kinds of processing as we can see it in any problem solving or in some fields like geometry. Mathematics understanding require the coordination between at least two registers of which one is multifunctional and the other monofunctional. Classic *problématiques* of relations between mathematics and language can be put in an accurate and relevant way only within such a framework of cognitive working. Now if we consider the most advanced level of teaching, the predominance of discursive monofunctional registers seems to increase. Besides it is with this kind of register that both performances and loss of meaning is very often observed. Why? It is wrongly believed that application to daily life or to extramathematical situations can be a source of meaning and therefore of understanding. No! The main problem is first with the multifunctional registers. They are implicitly and explicitly needed for mathematics understanding, but the way they are working in mathematics thought processes is quite different from the one they are working in the other fields of knowledge and, a fortiori, in everyday life. Therefore resorting to natural language as within ordinary speech and referring to geometrical figures as if they were as obvious as other visual images does not help but increases the confusion in understanding and learning. Here a wide field of reseach is opening. If we want to understand the complex mechanisms of mathematics learning we must analyse the specific ways of working of the multifunctional registers, especially for what matters reasoning in proof and visualization in solving geometry problems. We can have already very specific and decisive cognitive variables (Duval 1995a, 1995b, 1996a)

c The cognitive architecture that underlies understanding in mathematics

That quick overview of the complexity of all kinds of semiotic transformations involved in mathematical processes sends us back to the above question : what are the internal cognitive conditions required in order to any subject can understand mathematics ? Now, psychological models of information processing have highlighted that conscious understanding depends on the automatic (unconscious) working of the organisation of various and heterogeneous systems. This organisation makes up the cognitive architecture of the epistemic subject. But mathematics understanding requires a more complex organisation, including semiotic systems, because it depends on the mobilisation of several registers. In these conditions learning mathematics means : integrate into its own cognitive architecture all needed registers as new systems of representation.

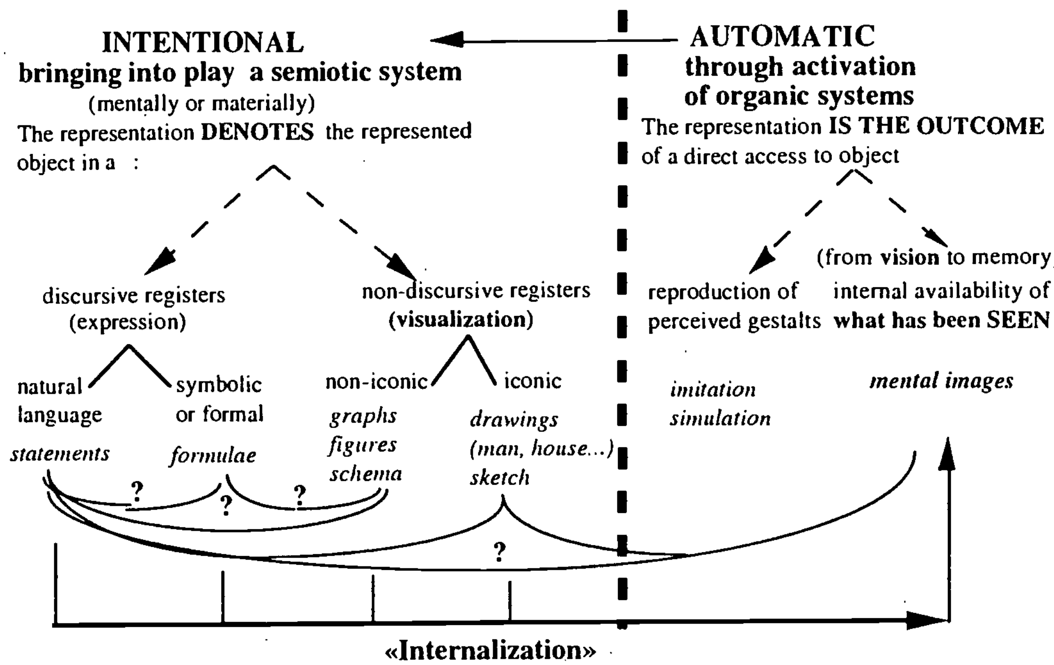


Figure 8. Various coordinations between productive systems required for mathematics understanding

That diagram gives a very simplified presentation of cognitive architecture. For example, we should have to distinguish, for the native language, between common working in social interactions and theoretical working in knowledge areas which are ruled by proof requirement. But it shows the various cognitive coordinations that mathematics understanding requires. Learning mathematics involves both incorporation of monofunctional registers and differentiation of the possible different ways of working within multifunctional registers. But it is not enough. Learning mathematics involves their coordination, or their decompartmentalization. Otherwise, conversion between non-congruent representations will be inhibited. And that is not a side-problem, because registers are non-isomorphous and because showing together two different representations of the same object, in order to create associations, does not work. Learning mathematics is learning to discriminate and to coordinate semiotic systems of representation in order to become able of any transforming of representation.

That can summed up in one sentence. Mathematics learning does not consist first in a construction of concepts by students but in the construction of the cognitive architecture of the epistemic subject. What is at stake in mathematics education through particular content acquisition is the construction of this architecture, because it creates future abilities of students for further learning and for more comprehensive understanding. But this deep aspect is misunderstood because student's individual consciousness, with its beliefs, evidences and interests, is often mistaken for the working of thought processes.

Conclusion

Research in mathematics education is extremely complex, because it must be lead through strained relationships between two heterogeneous kinds of organisation and requirements for knowledge, the mathematical one and the cognitive one. And when we are going from preelementary levels to secondary levels, the predominance from one to the other seems progressively to be reversed. In these conditions, what does research about mathematics learning processes mean ? Are we not confronted with quite different topics, each demanding a particular model ? And would the only common process which could be extracted not be useful mainly in order to organise activity sequences the in classroom ?

In an overview of some basic issues, we have emphasized what in mathematics knowledge is deeply different from other areas of knowledge, rather than what is common. This choice can amaze. Since Piaget's developmental models and also because mathematics are considered as an intellectual subject and are needed in all fields of science and technology, we are inclined to assume common roots between mathematical processes and common thought processes. That is both right and false. It is right because these common thought processes depend on the working of the semiotic system of representation. It is false because the taught mathematics require a more systematic and more differentiated use of semiotic systems than the one needed for anyone who remains at an only oral culture stage, or than the one needed in other fields of culture which do not all resort to mathematics. And thus by highlighting the intrinsic role of productive semiotic systems in mathematics understanding, we emphasize at the same time the gap between natural representations (visual memories, mental images...) and semiotic representations. As we have already said (Duval (Fischbein)) the psychological approach to these fundamental questions requires specific models, which by their turn could contribute to develop the field of cognitive psychology.

In that perspective, conversion of representations and all manifold aspects of non-congruence appear as the typical and basic characteristic of mathematical thought processes. Through conversion we are coming to the core of mathematics learning problems. Furthermore conversion provides a powerful tool of analysis of what is relevant in the content of any representation, because representation is not only considered in itself, but in relation to another register. Thus we can bring out cognitive variables, and not only structural semiotic variations, which determine each register working. It is mainly needed with the multifunctional registers. And by taking into account the two kinds of representation transformations, processing and conversion, a cognitive analysis of problems and exploration of their variations become possible. Unlike mathematical analyses which are downstream analyses, back from various ways of their mathematical solving, cognitive analysis are upstream analyses, from variations of initial conversions forward which can be required in order to start up processing.

All that can seem very far from teaching and especially from questions a teacher can ask in his/her classroom. This deliberate distance reflects the difference between subjective repesentations of individuals and the deep cognitive architecture to construct in order to understand mathematics concepts. The theoretical choices we have made and the model of thought processes that we are developing can lead to many experiments, to other theoretical frameworks, and even in classrooms! But a quite different learning environment than the one of the classroom is becoming more and more important. It requires however the conception of dynamic software which provides very open interactions with learners in order not to be only an assistance for some algorithms learning. The model based on registers of thought cognitive working can be helpful for such a conception and mainly for very complex learning : proving (Luengo 1997) and decimal numbers (Adjiage 1999). In mathematics education issues relative to learning cannot be subordinated to those relative to teaching, because they depend first on the complex cognitive working involved in mathematics understanding. A wide field of reseach is opening ahead of us.

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