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ABSTRACT

This document contains the proceedings of the 22nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education held October 7-10, 2000, in Tucson, Arizona. Selected papers include: (1) "Dealing with Complexity: New Paradigms for Research in Mathematics Education" (Richard Lesh); (2) "Algebra Reasoning in Elementary Mathematics--Theory and Practice" (Eug'..io Filloy and Teresa Rojano); (3) "Portrait of the Construction of Mathematical Knowledge: The Case of a Deaf and Blind University Student" (Christine L. Ebert); (4) "Preparing to Teach in the New Millennium: Algebra through the Eyes of Preservice Elementary and Middle School Teachers" (Joyce Bishop and Sheryl Stump); (5) "Teachers' Perspectives on Performance Assessment" (Richard S. Kitchen); (6) "Mathematical Explanations: In-Action and as Re-Presentation" (L. Gordon Calvert); (7) "Growing Mathematical Understanding: Layered Observations" (Jo Towers, Lyndon Martin, and Susan Pirie); (8) "Modeling the Development of the Concept of Function" (Mindy Kalchman and Robbie Case); (9) "Student Conjectures in Geometry" (Anderson Norton); (10) "Young Children's Statistical Thinking: A Teaching Experiment" (Arsalan Wares, Graham A. Jones, Cynthia W. Langrall, and Carol A. Thornton); (11) "Symbols and Meanings in Teacher-Student Interaction During Mathematical Problem Solving" (Olive Chapman); (12) "Prospective Teachers' Part-Whole Division Problem Solution Strategies" (Walter E. Stone, Jr.); (13) "Epistemological Analyses of Mathematical Ideas: A Research Methodology" (Patrick W. Thompson and Luis A. Saldanha); (14) "Parents as Learners of Mathematics: A Different Look at Parental Involvement" (Marta Civil, Rosi Andrade, and Cynthia Anhalt); (15)

"Teacher Change in the Context of Cognitively Guided Instruction: Cases of Two Novice Teachers" (Nancy Nesbitt Vacc, Anita H. Bowman, and George W. Bright); (16) "Thought and Action in Context: An Emerging Perspective of Teacher Preparation" (Sarah B. Berenson and Laurie O. Cavey); (17) "Models of Curriculum Use in the Context of Mathematics Education Reform" (Miriam Gamoran Sherin and Corey Drake); (18) "Students Exploring Geometrical Concepts Using Teachers' Web Pages" (Armando L. Bezies); and (19) "Integers versus Fractions: A Study with Eighth Grade Students" (Aurora Gallardo and Rogelio Novoa). (ASK)

Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

Tucson, Arizona, October 7-10, 2000

Volumes 1 - 2

Maria L. Fernandez, Editor

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Proceedings of the
Twenty-Second Annual Meeting

North American Chapter of the International Group for the

Psychology of Mathematics Education

Volume 1

Plenary, Working Groups, Research Reports,
Short Oral Reports, Poster Presentations

Editor:
Maria L. Fernández

PME-NA XXII
October 7-10, 2000
Tucson, Arizona U.S.A.

ERIC *Clearinghouse for Science, Mathematics, and Environmental Education*

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Twenty-Second Annual Meeting

North American Chapter of the International Group for the

Psychology of Mathematics Education

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HISTORY AND AND AIMS OF THE PME GROUP

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and the North American Chapter are:

(1) To promote international contacts and the exchange of scientific information in the psychology of mathematics education;

(2) To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators;

(3) To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implementation thereof.

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Preface

Hosting PME-NA XXII in Tucson, Arizona came about through the collaborative efforts of mathematics educators at the University of Arizona, Arizona State University, Arizona State University-West, and Northern Arizona University. As a group, we felt bringing this conference to Arizona would benefit our mathematics education community, providing an opportunity for local researchers and mathematics educators to engage ideas of and interact with members of the broader mathematics education research community including internationally recognized researchers.

As we entered a new millennium with many present and future challenges, the Arizona Organization Committee felt an appropriate conference theme would be Adapting to Changes of the New Millennium. This theme captures the essence of the present work of mathematics education researchers throughout the world. Noted scholars were invited to give plenary addresses in areas of interest within this theme. Richard Lesh of Purdue University was invited to address the issue of research paradigms, an area of present and future growth for the mathematics education community. Alan Bishop was invited to address social and cultural issues in mathematics education research, an area of increasing interest given the ever-evolving social and cultural diversity of the world communities.

Continuing the community building proposed by the 1998 and 1999 PME-NA organizers, working group leaders were invited to continue their work within PME-NA. The working groups for PME-NA XXII include (1) *Advanced Mathematical Thinking: Implications of Various Perspectives on Advanced Mathematical Thinking for Mathematics Education Reform*, organizers M. Kathleen Heid, Guershon Harel, Joan Ferrini-Mundy, and Karen; (2) *Algebra and Reasoning in Elementary Mathematics*, organizers Euginio Filloy and Teresa Rojano, Graham; *Gender and Mathematics: Emergent Themes*, organizers Diana Erchick, Linda Condron, and Peter Appelbaum; *Models and Modeling: Representational Fluency*, organizers Richard Lesh, Guadalupe Carmona, and

Thomas Post; *Geometry and Technology*, organizer Douglas E. McDougall; *The Complexity of Learning to Reason Probabilistically*, organizer Robert Spieser; and *Representations and Mathematics Visualization*, organizer Fernando Hitt.

In addition to the plenary sessions and working groups, the conference program consisted of 75 research reports, 6 discussion groups, 53 short oral reports, and 48 posters. For the first time in the history of PME-NA, a majority of the submissions and reviews were conducted electronically via the internet. There were 248 submissions overall; approximately 80% of the submissions were electronic. The acceptance rate for research report submissions was 59%. Each submission was reviewed by at least three reviewers with expertise in that area. Additionally, under the direction of the PME-NA Steering Committee, criteria was developed and used to evaluate theoretical proposals distinctly from empirical proposals. Patrick Thompson's leadership in developing that criteria was greatly appreciated.

Submissions for the Proceedings were made electronically, primarily as attached documents within e-mail messages. The format of the documents was adjusted to make them uniform and some editorial corrections were made; however, substantive editing was not undertaken.

The success of the program is based on many factors. Particular thanks are extended to all those who submitted proposals, the reviewers, the 2000 Program Committee, and the 1999-2000 PME-NA Steering Committee for making the program an excellent contribution to the growing body of research and discussions in the psychology of mathematics education. Special thanks are expressed to the mathematics education faculty at the University of Arizona, Arizona State University, Arizona State University-West and Northern Arizona University for their support and generous contributions to the conference.

Maria L. Fernández, Chair
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DEALING WITH COMPLEXITY: NEW PARADIGMS FOR RESEARCH IN MATHEMATICS EDUCATION

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During the closing years of the twentieth century, a number of books and articles have been published describing the status of research in mathematics education and discussing possible ways to increase its power and usefulness (Sierpinska & Kilpatrick, 1998; Grows, 1992; Steen, 1999). Most of these publications have focused on summaries of past research, or on descriptions of the authors' views about which problems or theoretical perspectives they believe should be treated as priorities for future research. — Should teachers' decision-making issues be treated as higher priorities than those that confront policy makers or others who influence what goes on in classroom instruction? Should issues of equity be given priority over those involving content quality, or innovative uses of advanced technologies? Should theoretical perspectives be favored that are grounded in brain research, or artificial intelligence models, or constructivist philosophies? Should quantitative research procedures be emphasized more than qualitative procedures? — In this paper, such questions are not my central concerns. Instead, I'll ask: *"What kind of research designs have proven to be especially useful in mathematics education, and what principles exist for improving (and assessing) the quality of these research designs?"*

In general, the preceding publications suggest that: (a) mathematics education research has made far less progress than is needed, and (b) little attention has been given to many of the most important issues that are priorities for teachers or other practitioners. — I don't dispute these claims. But, in general, I'm far more impressed with the achievements than with the shortcomings of mathematics education research. For example, during the two decades that have passed since the first meeting of PME/NA was held at Northwestern University in 1978, it would be striking to anyone who attended that meeting that the mathematics education research community has made enormous progress to shift beyond theory *borrowing* toward theory *building*. At the 1978 meeting, most of us were doing Piagetian Research, or Vygotskian research — or research based on psychometric models, or information processing models, or artificial intelligence models — where both our theoretical models and our research methodologies were borrowed from these other fields. But, today, examples abound where mathematics educators have developed their own distinctive theoretical models, conceptions of critical problems, research literature, research tools and procedures — and, most importantly, communities of inquiry — in topic areas ranging from early number concepts, to rational number concepts, to early algebraic reasoning.

As a result of the preceding progress, rapid increases have occurred in the volume and sophistication of mathematics education research; and, these increases have ushered in a series of paradigm shifts that involve new ways of thinking about the nature of students' developing mathematical knowledge and abilities, as well as new ways of thinking about the nature of mathematics, problem solving, learning, and teaching — and new systemic ways of thinking about program development, dissemination, and implementation. For example, the paper that describes the PME/NA 2000 *Models & Modeling Working Group* gives a concise description of some of the most significant shifts in thinking that apply to research being conducted by members of this group. Such new ways of thinking have provided primary driving forces behind many of the most successful recent attempts at standards-based curriculum reforms; and, they've also created the need for new research methodologies that are based on new assumptions, and that focus on new problems and capitalize on new opportunities.

Unfortunately, researchers are similar to other busy people who know that *when you're up to your elbows in alligators there's not much time to consider innovative ways to drain the swamp.* Thus, the development of widely recognized standards for research has not kept pace with the development of new problems, perspectives, and research procedures in our field; and, as a consequence of this fact, because there is a lack of clarity and consensus about appropriate principles for optimizing (or assessing) the quality of innovative research designs, three kinds of undesirable results are likely to occur when proposals are reviewed for projects, publications, or presentations. First, appropriate methodologies may be marred by avoidable methodological flaws. Second, studies employing appropriate methodologies may be rejected because they involve unfamiliar research designs, or because inadequate space is available for explanation, or because inappropriate or obsolete standards are used to evaluate them. Third, inappropriate methodologies may be accepted because they employ traditional research designs - even though the assumptions that they presuppose may be antithetical to perspectives the researcher wants to adopt about the nature of mathematics, teaching, learning, or problem solving.

To develop productive ways of dealing with the preceding difficulties, I recently served as the co-editor of an NSF-supported *Handbook of Research Design in Mathematics & Science Education* (Kelly & Lesh, 2000). This handbook includes chapters written by more than forty leading researchers in mathematics and science education. It's aim was to emphasize research designs that: (a) have been pioneered by mathematics and science educators, (b) have distinctive characteristics when used in mathematics or science education, or (c) have proven to be especially productive in mathematics or science education.

Examples of such research designs include several different types of teaching experiments, and distinctive types of clinical interviews, videotape analyses, naturalistic observations, and action research paradigms in which participant-observers may include not only researchers-acting-as-teachers or classroom teachers-acting-as-researchers but also curriculum designers, software designers, and teacher educators whose aims include both optimizing and understanding mathematics teaching, learning, or problem solving. In general, these new research designs draw on multiple types of quantitative and qualitative information; the knowledge-development products they produce often are not reducible to tested hypotheses or answered questions; and, they often involve cyclic and iterative techniques in which participant-researchers include a variety of interacting students, teachers, and other mathematics educators. Finally, and most importantly, they often involved new ways of thinking about the nature of students' developing mathematical knowledge and abilities, and new ways of thinking about the nature of effective teaching, learning, problem solving.

The purpose of our *Handbook of Research Design* was to clarify the nature of some of the most important experience-tested ways to improve (or assess) the usefulness, power, share-ability, and cumulativeness of the results that are produced when the preceding kinds of research designs are included in proposals for research projects, publications, or presentations at professional meetings. Of course, from the beginning of our efforts, participants were mindful of the fact that, if obsolete or otherwise inappropriate standards are adopted, then the results could hinder rather than help. But, as long as decisions must be made about funding, publications, and presentations, it is not possible to avoid issues related to quality assessments. Decisions WILL be made. Therefore, our goal was to try to increase the chances that appropriate issues will be considered and that productive decisions will be made.

For details about factors that contribute to the quality of specific kinds of teaching experiments, or other research designs emphasized in the handbook, readers are referred to the handbook itself. This chapter will restrict attention to some of the most important factors that appear to have strongly influenced the development of virtually all of the research designs that have developed in distinctive ways in mathematics education.

Two Factors Influenced Research Designs That are Distinctive in Mathematics Education

In the development of the *Handbook*, two factors emerged as having especially strong influences on the kind of research designs that have been pioneered by mathematics educators. First, most are intended to radically increase the relevance of research to practice - often by involving practitioners in the identification and formulation of problems to be addressed, in the interpretation of results, or in other key roles in the research process. Second, there is a growing recognition that students, teachers, classrooms, courses, instructional programs, curriculum materials, learning tools and minds are all complex systems, taken singly, let alone in combination. Therefore, regardless of whether we focus on the developing capabilities of students, or groups of students, or teachers, or schools, or other relevant learning communities, the continually evolving ways of thinking of each of these "problem solvers" involve complex conceptual systems that are dynamic, living, self-regulating and continually adapting – and that have competencies that generally cannot be reduced to simple checklists of condition-action rules. They don't simply lie dormant until being stimulated. They initiate action; and, when they are acted on, they act back. So, interactions often involve feedback loops that lead to second-order (or higher-order) effects that overwhelm first-order effects. Furthermore, among the most important systems that mathematics educators need to investigate and understand: (a) many do not occur naturally (as "givens" in nature) but instead are products of human construction, (b) many cannot be isolated because their entire nature may change if they are separated from the complex holistic systems in which they are embedded, (c) many may not be observable directly but may be knowable only by their effects, or (d) when they're observed, changes often are induced that make investigators integral parts of the systems being investigated. So, there is no such thing as an immaculate perception; and, the behaviors of these systems often cannot be described adequately using simple algebraic, statistical, or logical formulas.

Because of the complex, constructed, and systemic nature of most of the "subjects" and "constructs" that mathematics educators need to investigate and understand, it's become commonplace to hear mathematics education researchers talk about rejecting traditions of "doing science" as they imagine it is done in the physical sciences (where, it is imagined, researchers treat "reality" as if it were objectively given). But, when educators speak about rejecting notions of objective reality, or about rejecting the notion of detached objectivity on the part of the researcher, such statements tend to be based on antiquated notions about the nature of modern research in the physical sciences. For example, in mature sciences such as astronomy, biology, chemistry, geology, or physics, when entities such as subatomic particles are described using fanciful terms such as color, charm, wisdom, truth, and beauty, it is clear that the relevant scientists are quite comfortable with the notion that reality is a construct; and, when these scientists speak of principles such as the *Heisenberg Indeterminancy Principle*, it also is clear that they are familiarity with the notions that: (i) the relevant systems act back when they are acted upon, (ii) the observations researchers make often induce significant changes in the systems they observe, and (iii) researchers often are integral parts of the systems they are hoping to understand and explain. Yet, such realities do not prevent these researchers from developing a variety of levels and types of productive operational definitions to deal with constructs such as black holes, neutrinos, strange quarks and other entities whose existence is related to systems whose behaviors are characterized by mathematical discontinuities, chaos, and complexity.

Consider the case of the neutrino where huge vats of heavy water are surrounded by photo-multipliers in order to create situations in which the effects of neutrinos are likely to be observable and measurable; and, notice that, even under these conditions, neutrinos cannot be observed directly. They can be known only through their effects; and, between the beholder and the beheld, elaborate systems of theory and assumptions are needed to distinguish signal from noise and to shape interpretations of the phenomena under investigation. Also, small changes in initial conditions often lead to large effects that are essentially unpredictable; observations that are made induce significant changes in the systems being observed; and, both researchers and their instruments are integral parts of the systems that scientists are hoping to understand and explain.— So, mathematics educators are not alone in their need to deal with systems that have the preceding characteristics.

As the diagram in Figure 1 suggests, in mathematics education, just as in more mature modern sciences, it has become necessary to move beyond machine-based metaphors and factory-based models to account for patterns and regularities in the behaviors of complex systems; and, it also has become necessary to move beyond the assumption that the behaviors of these systems can be explained using simple linear combinations of unidirectional cause-and-effect mechanisms that can be accurately characterized using models from elementary algebra, statistics, or logic.

From an Industrial Age using analogies based on *hardware*		**Beyond an Age of Electronic Technologies** using analogies based on computer *software*		**Toward an Age of Biotechnologies** using analogies based on *wetware*
where systems are considered to be no more than the sum of their parts, and where the interactions that are emphasized involve no more than simple one-way cause-and-effect relationships.	⇒	where silicone-based electronic circuits may involve layers of recursive interactions which often lead to emergent phenomena at higher levels which are not derived from characteristics of phenomena at lower levels	⇒	where neurochemical interactions may involve "logics" that are fuzzy, partly redundant, partly inconsistent, and unstable – as well as living systems that are complex, dynamic, and continually adapting.

Figure 1. Recent Transitions in Models for Making (or Making Sense of) Complex Systems

In the *Handbook of Research Design*, the chapter on "operational definitions" describes how scientists in more mature sciences specify ways to cause, recognize, and measure the occurrence of complex systems (or "elements" whose existences depend on complex systems) without reducing these "operational definitions" to checklists of condition-action rules.[1] For example, when devices such as cloud chambers or cyclotrons are used to observe, record and measure illusive constructs whose existences depend on complex systems, it is clear that: (i) the relevant construct does not *reside in* the device,[2] (ii) being able to measure a construct does not guarantee that a corresponding dictionary-style definition will be apparent, and (iii) even when a dictionary-style definition can be given (for a construct such as a black hole in astronomy), this doesn't guarantee that procedures will be available for observing or measuring the construct. Nonetheless, useful operational definitions usually involve three parts that are similar, in some respects, to the following three parts of traditional types of behavioral objectives (of the type that have been emphasized in past research in mathematics education).

Behavioral Objectives have Three Parts

> GIVEN {specified conditions} THE STUDENT WILL EXHIBIT {specified behaviors} WITH IDENTIFIABLE QUALITY (perhaps specified as percents correct on relevant samples of tasks, or perhaps specified in terms of a correspondence with certain criteria for excellence).

Whereas behavioral objectives collapse three different kinds of statements into a single condition-action rule - and thus treat all knowledge as if it consisted of nothing more than condition-action rules - more general types of operational definitions keep these components separate.[3] For example, when researchers in fields such as physics deal with complex phenomena involving such things as photons or neutrinos, a minimum requirement for a useful operational definition is that explicit procedures must be specified for three things:

1. situations that optimize the chances that the targeted construct will occur in observable forms.[4]
2. *observation tools* that enable observers to sort out signal from noise in the results that occur.
3. assessment criteria that allow observations to be classified or quantified.

Although it's beyond the scope of this chapter to give details about principles that mathematics educators can use to deal with the preceding three components of productive operational definitions, examples can be found in some of the best standards-based "performance assessment instruments" that have been developed during recent years (Lesh & Lamon, 1993). For instance, in the research design book's chapter on *Principles for Developing Thought-Revealing Activities*, examples are given of performance assessment instruments that involve: (i) *thought-revealing activities* that require students, teachers, or other relevant "subjects" to express their ways of thinking in forms that are visible to both researchers and to the subjects themselves, (ii) *response analysis tools* that teachers (or researchers) can use to classify alternative responses, and to identify strength and weaknesses, as well as directions for improvement, and (iii) *response assessment tools* that can be used to evaluate the relative usefulness of alternative responses.

What are Some Relevant Assumptions about Student Development, Teacher Development, and Program Development?

In general, research involves getting clearer about the nature of the subjects (e.g., students, teachers, curricula, etc.) or constructs (e.g., students' conceptual systems) that are under investigation; and, current assumptions about the nature of these subjects or constructs have strong influences on decisions about *whom* to observe, *what* to observe, *when* to observe, and *which* aspects of the situation to observe – as well as decisions about what kind of tools to use to filter, denote, organize, analyze, and interpret the information that is gathered or generated. For example, even when data collection involves tools such as video recordings, which sometimes give the illusion of capturing "raw data", researchers' prejudices about the nature of relevant subjects strongly influence decisions to focus on one situation rather than another, to zoom in on the behaviors of one participant rather than another, or to emphasize one "window" of observation (or one aspect of the situation) rather than another.

The impact of prejudices about what is important and what is not are especially important to consider when tests or questionnaires are used that give the appearance of being prejudice free – because the assumptions on which they are based often are buried under many layers of psychometric fog. In the research design book, most of the chapters in the section on "assessment design" deal explicitly with these sorts of issues. They emphasize that some of the most important factors influencing the quality of research involve the degree of alignment between: (i) intended assumptions about the nature of the subjects (or the constructs) being investigated, and (ii) assumptions presupposed (implicitly or explicitly) by the procedures that are used for selecting, analyzing, and interpreting information. – Consider the following instances of "subjects" commonly investigated by mathematics educators:

(a) *Common Assumptions about Students*:

When mathematics education research investigates the nature of the evolving conceptual schemes that students use to interpret their learning or problem solving experiences, one common assumption is that "thinking mathematically" involves more than simply computation with written symbols. For example, it also involves mathematizing experiences - by quantifying, dimensionalizing, coordinatizing, or in other ways making sense of them using mathematical constructs (systems). Consequently, to investigate the nature of students' constructs and sense-making abilities, researchers generally need to focus on problem solving situations in which interpretation is not trivial; and, in non-trivial situations, most modern theories of teaching and learning believe that the way students' interpret learning and problem solving experiences is based on interactions between (internal) conceptual systems and by (external) systems that are encountered or constructed. Therefore, when interpretation is not trivial, different students are expected to interpret the situation in fundamentally different ways.[5] Furthermore, when the goal of a task involves developing an interpretation, the description or explanation that's produced often involves significant forms of learning. Therefore, when the products involve learning, students who engage in a sequence of such tasks that are "structurally similar" would not be expected to perform in the same way across all tasks.

The preceding observations raise the following kinds of questions in the design of research. When a given student is not expected to perform in the same way across a series of similar tasks, what does it mean to speak about "reliability" in which repeated measurements are assumed to vary around the student's "true" (invariant) understandings and abilities across all tasks? When different problem solvers are expected to interpret a single problem solving situation in fundamentally different ways, what does it mean to speak about "standardized" questions? What does it mean to speak about "the same treatment" being given to two different participants or groups? – By raising such questions, I do not mean to suggest that constructs such as "reliability," "validity" or "replicability" are not relevant to modern research in mathematics education. Indeed, closely related criteria (such as usefulness, meaningfulness, power, and share-ability) always must be part of productive knowledge development in applied fields such as mathematics education. But, the meanings these constructs must be conceived in ways that are consistent with our best assumptions about the nature of systems and constructs being investigated; and, this means that off-the-shelf definitions borrowed from obsolete theories may no longer be appropriate – especially if they are grounded in machine-based models of teaching, learning, and problem solving.

In the *Handbook of Research Design*, many of the chapters identify common mismatches that occur between the assumptions underlying procedures and theories commonly employed by mathematics education researchers. They also suggest procedures in which these assumptions are better aligned. For example, in a chapter about Multi-Tier Teaching Experiments, several interacting levels and types of "subjects" (students, teachers, curriculum designers, researchers) each are engaged in sequences of *thought-revealing activities* in which the repeatedly express their current ways of thinking in forms that go through multiple testing-and-revising cycles. A byproduct of this process is that the trail of documentation that is produced yields a trace (that's analogous, in some ways, to the trace produced by an electron in a cloud chamber in a physics laboratory) that reveals the nature of the constructs that the student produces.

If one asks *"Where is the mathematics?"* in the preceding kinds of thought-revealing activities, the answer is that it resides in the responses that students generate. It does not reside, as many naïve task analyses have assumed in the past, in the tasks themselves.

(b) Common Assumptions about Teachers:

For teachers just as for other types of problem solvers and decision makers (including students), expertise is reflected not only in what they *do* but also in what they *see*. Alternatively, we could say that what teachers *do* is strongly influenced by what they *see* in given teaching and learning situations. For example, as teachers develop, they tend to notice new things about their students, about their instructional materials, and about the ideas and abilities that they are trying to help students learn; and, these new observations often create new needs and opportunities that, in turn, require teachers to develop further. Also, the situations that teachers encounter are not given in nature; they are, in large part, created by the teachers themselves – based on their current conceptions of teaching, learning, and problem solving. Thus, there exists no fixed and final state of excellence in teaching; and, in fact, continual adaptation (development) is a hallmark of teachers who are successful over long periods of time. Furthermore, no teacher is equally effective for all grade levels (kindergarten through college), for all topic areas (algebra through statistics and geometry or calculus), for all types of students (handicapped through gifted), and for all types of settings (inner-city through rural). No teacher can be expected to be constantly "good" in "bad" situations; not everything that experts do is effective, nor is everything that novices do ineffective; and, characteristics that lead to success in one situation (or for one person) often turn out to be counterproductive in other situations (or for another person). Furthermore, even though gains in students' achievements should be primary factors to consider when documenting the accomplishments of teachers (or programs), it is foolish to assume that the best teachers always produce the largest gains in student learning. One reason this is true is because some great teachers choose to deal primarily with difficult students or difficult circumstances?

The preceding observations suggest that expertise (for teachers, for students, or for other problem solvers and decision makers) tends to be plural, multidimensional, non-uniform, conditional, and continually evolving. In general, there is no single "best" type of teacher; every teacher has a complex profile of strengths and weaknesses;

teachers who are effective in some ways and under some conditions are not necessarily effective in others; and, teachers at every level of expertise must continue to adapt and develop. – So, what does it mean to classify teachers into categories such as "experts" or "non-experts" as if it was legitimate to collapse complex profiles of capabilities into fixed positions on a single-dimension "good–bad" scale?[6]

In the *Handbook on Research Design*, one approach for dealing with the preceding issues involves *Evolving Expert Studies* in which thought-revealing activities for *students* often provide the basis for equally thought-revealing activities for *teachers* (and others). For example, using *Multi-Tier Teaching Experiments*, teachers (like students) are put into a series of situations where their views must be expressed in forms that are tested and revised or refined repeatedly and iteratively. Formative feedback and consensus building provide mechanisms to encourage development in directions that participants themselves are able to judge to be increasingly "better" – without basing judgments on some preconceived notions of "best."

(c) Common Assumptions about Programs, Materials, or Classroom Learning Environments:

Because classroom learning environments, schools, and programs are not given in nature, constructs and principles that can be used to construct, describe, explain, manipulate, or control such systems often appear to be less like "laws of nature" than they are like "laws of the land" that govern a country's legal system. Also, researchers who are involved in investigating such systems often are not simply disinterested observers, and, they may be more interested in "what's possible" than in "what's real or typical." Therefore, issues about the truth or falsity of given principles or perspectives may be less pertinent than issues about consistency, meaningfulness, power, and the desirability of outcomes.

Legislated programs, defined curricula, and planned classroom learning environments often are quite different than implemented programs, curricula, and classroom activities; and, complex programs, materials, and activities seldom function like simple functions in which a small number of input variables completely determine a small number of output variables. For example, second-order effects (and other higher-order effects) often have impacts that are highly significant; and, emergent phenomena resulting from interactions among variables often lead to results that are at least as significant as attributes associated with the variables themselves. In particular, tests often go beyond being objective indicators of development to exert powerful forces on the programs, curricula, or activities that they are intended to assess. Consequently, if naïve pretest-posttest designs reflect narrow, shallow, or naïve conceptions of outcomes and interactions, then they often have strong negative impacts on outcomes – so that researchers frequently need to abandon assumptions about their own detached objectivity.

The *Handbook of Research Design* includes a number of chapters that describe alternatives or supplements to pretest-posttest designs. Many of these approaches to research could be called *design studies* – because the products that the research produces consists of (or include) models, conceptual tools, and other artifacts that must be designed to accomplish specified tasks - or to satisfy specific design principles).

Throughout this paper, I use the term "research design" rather than "research methodology." This is because, in mathematics education, the design of research generally involves trade-offs often similar to those that occur when other types of complex products (such as automobiles) need to be designed to meet conflicting goals (such as optimizing speed, safety, and economy). So, whereas the term "research methodology" tends to be associated with statistics-oriented college courses in which the emphasis is on how to carry out "canned" computational procedures for analyzing data, the kind of situations and issues that are most important for math educators to investigate seldom lend themselves to the selection and execution of off-the-shelf data analysis techniques. Combinations of qualitative and quantitative approaches tend to be needed; and, in addition to the stages of research that deal with data analysis, other equally important issues typically arise that involve: (i) developing productive conceptions of problems that need to be solved, products that need to be produced, or opportunities that need to be investigated, (ii) devising ways to generate or gather relevant information to develop, test, refine, revise, or extend relevant ways of thinking, (iii) developing appropriate ways to sort out the signal from the noise in information that is available - and to organize, code, and interpret raw data in ways that highlight patterns and regularities, or (iv) analyzing underlying assumptions and formulating appropriate implications.

It Often Is Said That Good Research Requires Clearly Stated Hypotheses Or Clearly Stated Research Questions

Dealing with complex systems in a disciplined way is the essence of research design in mathematics education; and, it is the central theme of this paper. Relevant perspectives involve cognitive science, social science, mathematical sciences, and a wide range of other points of view. No single means of understanding is sufficient; no single style of inquiry is likely to take us very far; and, relevant research can never be reduced to a formula-based process. Far from being a process of using "accepted" techniques in ways that are "correct", it's a "no holds barred" process of developing shared knowledge about important issues. Doing it well involves *developing a chain of reasoning* that is *meaningful*, *coherent*, *sharable*, *powerful*, *cumulative*, *auditable*, and *persuasive* to a well-intentioned skeptic about issues that are *priorities* to address.

When we emphasize that research is about the development of knowledge, it should be clear that what we know consists of a great deal more than *tested hypotheses* (stated in the form of "if ... then ...." rules) and *answered questions* (using standardized tests, questionnaires, or other techniques leading to quantitative measures of relevant variables). For example, some of the most important products of research also include:

- *Descriptions* and *explanations* (e.g., models and conceptual systems) for constructing and making sense of complex systems. So, truth and falsity may not be at issue as much as fidelity, internal consistency, and other characteristics similar to those that apply to quality assessments for painted portraits or verbal descriptions.

- *Demonstrating possibilities* that may involve existence proofs (with small numbers of "subjects") and that may need to be expressed in forms that are acompanied by (or embedded in) exemplary software, informative assessment instruments, or illustrative instructional activities, programs, or prototypes to be used in schools. So, the quality of results depends on the extent to which these products are meaningful, sharable, powerful, and useful for a variety of purposes and in a variety of situations.

- *Developing tools* such as those that are intended to be used to increase (or document or assess) the understandings, abilities, and achievements of students, teachers, programs, or relevant learning communities. So, again, the quality of such tools depends on the extent to which they are sharable, powerful, and useful for a variety of purposes and in a variety of situations. (Note: These tools may or may not involve measurement or quantification.)

Similar products of research are familiar in the natural sciences. For example, in fields such as physics, chemistry, or biology, some of the most important products of research involve the development of *tools* or *explanatory models* that involve references to phenomena such as waves, fields, and black holes, that provide different ways of describing, explaining, constructing, manipulating, and predicting the behaviors of complex systems. Yet, these tools often generate information that goes beyond characterizing complex systems using a single number; and, they often go beyond comparisons that collapse all relevant attributes onto a single-dimension number line. Similarly, the models often are iconic and analog in nature, being built up from more primitive and familiar notions - so that the visualizable model is a major locus of meaning for relevant scientific theories. They are not simply condensed summaries of empirical observations.

It Often is Said that Math Education Research has not Answered Teacher's Questions

If the point of the preceding statement is to emphasize that projects emphasizing the development of knowledge should make a difference in mathematics teaching and learning, then I concur. But, the view that *"teachers should ask questions and* *researchers should answer them"* is naïve and counterproductive to the point of being a large part of the disease for which it purports to be the cure. Consider the following observations.

- In mathematics education, no clear line can be drawn between researchers and practitioners. There are many levels and types of both researchers and practitioners; and, people who are known as "researchers" often have equally strong reputations as teachers, teacher educators, curriculum developers, or software developers. Similarly, many people who are best known in these latter areas also are highly capable researchers. Also, practitioners whose voices should be heard include not only teachers but parents, policy makers, administrators, school board members, curriculum specialists, textbook writers, test developers, teacher educators, and others whose decisions strongly influence what goes on in schools. So, the process of knowledge development is far more cyclic and interactive than is suggested by one-way transmissions in which teachers ask questions and researchers answer them (see figure 2).

- In mathematics education, productive knowledge development projects often involve some form of curriculum development, program development, or teacher development; and, productive curriculum development, program development, or teacher development projects also should involve knowledge development. Such endeavors shouldn't be artificially separated. In any of the preceding areas of development (including the development of presentations at professional conferences such as NCTM, AERA, or PME), it's obvious that, if progress has been made, it is precisely because *we know more*. Similarly, where less progress has been made, the knowledge base has tended to be weak.

- What people ask for isn't necessarily a wise statement of what they need; and, useful tools and conceptual systems usually need to be developed iteratively and recursively. – For many of the practitioners mentioned on the preceding page, the "problems" that they pose often focus on "symptoms" rather than on underlying "diseases"; or, they sound more like "ouches" (expressions of difficulty or discomfort) than they do like well formulated problems. Consider the politician who says: *"Show me what works?"* Need I say more? Whereas small innovations seldom lead to large results, large innovations seldom get implemented completely. Yet, nearly every educational innovation works some of the time, in some situations, for some purposes, in some ways, and for some students. So, unless it's known which parts work when, where, why, how, with whom, and in what ways, the pseudo-information that *"This program (or policy) works!"* is likely to be misleading to educational decision

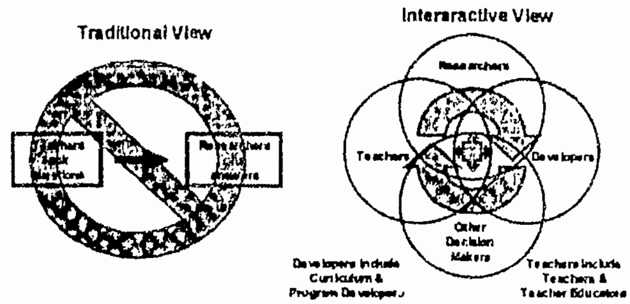

Figure 2

makers. Implementations of sophisticated programs and curriculum materials generally involve complex interactions, sophisticated adaptation cycles, iterative developments, and intricate feedback loops in which breakdowns occur in traditional distinctions between researchers and teachers, assessment experts and curriculum developers, observers and observed.

- Among the challenges and opportunities that mathematics educators confront, most are sufficiently complex that they are not likely to be addressed effectively using results from a single research study. Therefore, rather than thinking in terms of a one-to-one match between research studies and solutions to problems, it would be more productive to insist that results from research studies should contribute to the development of a theory (or a model) – and that this theory should have powerful implications over a reasonable period of time (see figure 3). This is why community building is important; and, it's why, in addition to factors such as usefulness, and share-ability, cumulativeness is another factor that should be considered when assessing the significance of research results.

Mathematics Education is an Exceedingly Young Field of Scientific Inquiry

Many of the most influential leaders who attended the first meeting of PME/NA are no longer among us. Merlyn Behr, Bob Davis, Jack Easley, Nick Herscovics, and Claude Janvier, in particular, were influential in shaping the spirit of PME/NA. Others have retired, or will be retiring soon, who provided leadership through early significant stages in the development of the mathematics education research community. So, with this turn-over, it's not surprising that emerging new leaders sometimes find it difficult to appreciate how brief our history has been.

Many indicators exist to suggest that mathematics education is in its infancy as a field of scientific inquiry. In fact, some cynics might even claim that we need to move beyond Piaget's "unconscious play stage" – where only primitive processes have evolved for planning, monitoring, and assessing our own activities. Instances supporting such claims include (and are partly caused by) the following kinds of commonplace events.

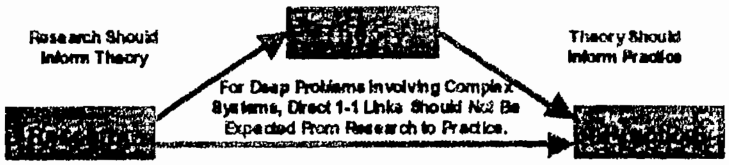

Figure 3

- At professional meetings, it's not difficult to find "research" sessions that consist of little more than stories told about a videotaped episode of teaching, learning, or problem solving.
- In proposal reviews for projects, publications, or presentations, it's not uncommon to hear "quantitative methods" being treated as if they were automatically "scientific" – even when: (i) the "control groups" don't control anything significant, (ii) the pretests and posttests ignore the most important characteristics of the constructs (or subjects) being investigated, and (iii) the analysis procedures presuppose psychometric models whose assumptions are not consistent with modern conceptions of mathematics, problem solving, learning, or effective instruction.[7]
- In promotion reviews for universities with no active researchers on their faculty, reviewers frequently find themselves answering questions like: (i) Could a chapter in a book possibly be as significant as an article published in a journal?[8] (ii) Are all journals equally significant? (iii) Is collaboration or co-authorship a sign of weakness? (iii) Should investigations employing qualitative methods be discounted?[9]

David Johnson didn't edit the first issue of the *Journal for Research in Mathematics Education* until 1970. The first *International Congress of Mathematics Education* (ICME) wasn't held until 1968; and; the *International Group for the Psychology of Mathematics Education* (PME) emerged from Efriam Fischbein's working group on the *Psychology of Mathematics Education* at ICME-II in Exeter, England, in 1972.

One of the main factors leading to the formation of PME/NA was, of course, the formation of PME-International. But, two other significant forces included: (i) activities associated with the *Georgia Center for Research in Mathematics Education* (Steffe, 1974), and (ii) a series of small annual mathematics education research conferences, sponsored by Northwestern University's *Center for the Teaching Professions* (Lesh, 1973), beginning in 1972.

Distinctive characteristics of all three of the preceding initiatives was that they focused on: (i) *stimulating* research (to focus on priority problems), (ii) *facilitating* research (by sharing tools, resources, and perspectives), (iii) *coordinating* research (to form a community of researchers whose work would be more cumulative and mutually supportive), and (iv) in other ways *amplifying* the power and utility of research that already is going on – not on *buying* isolated research projects (through subsidies for salaries, equipment, or supplies), and not on simply providing a venue for people to *talk about results from past projects.*

One reason why these initiatives were so powerful, and why their second-order and higher-order effects have continued to be forceful even today, is that the people who initiated them were active researchers who were intimately in touch with the strengths and needs of the emerging community. In particular, it was clear that a large share of the research that goes on in mathematics education is conducted by:

- doctoral students as part of their degree requirements,
- "researchers" who often have even stronger identities as curriculum developers, program developers, or teacher developers – or as excellent teachers.
- people who do not have large funded projects but who view research as an important part of their professional lives in order to get jobs, promotions, or tenure, and most of all to continue to learn and develop by thinking critically about their teaching and other educational endeavors.
- researchers from fields outside of mathematics education (such as developmental psychology) where mathematics education has a long history of serving as an unusually productive site to explore and test emerging theories.

To harness these resources, PME/NA was intended to be a place that's dedicated to the development of a community of researchers - where participants could: (i) plan future activities as well as report results from past activities, (ii) hear divergent views from colleagues in other fields, and (iii) report research that isn't several years old due to cumbersome review procedures.

I believe that we're currently at a very exciting time in the development of the mathematics education research community. On the one hand, if we compare what is known today with what was known during the "new math" movements of the 1960s, it's clear that a great deal has been learned about children's ways of thinking about elementary-but-deep constructs in topic areas ranging from early number concepts, to rational numbers and proportional reasoning, to geometry and measurement, to early algebra or statistics. On the other hand, as we move from one of these research topics to another, or from one cluster of researchers to another within a given topic, it's equally clear that the accounts of conceptual development often are based on radically different assumptions about the nature of mathematics, problem solving, learning, and teaching. So, mismatches need to be reconciled; and, significant new paradigm shifts can be expected. For example, during the next decade, due to results from fields such as brain research, due to the explosion of ways that mathematics is used beyond schools in a technology-based *age of information*, due to the emergence of new ways to document these abilities, and due to the availability of many new types of modeling tools that are especially well suited to describing the kind of complex, dynamic, interacting, and continually adapting systems that characterize so many of the subjects and constructs that we need to understand in mathematics education, I expect that mathematics educators will need to rethink many fundamental assumptions about the nature of mathematical ability (Lesh & Lamon, 1993; Doerr & Lesh, in press).

Finally, new communication technologies are making it possible for close collaboration to occur among researchers representing multiple perspectives at remote sites. As an example where these new kinds of communication are being used to facilitate the development of a research community, Purdue University, Indiana University, Purdue-Calumet, IU/PUI (Indianapolis) have created a new *Distributed Doctoral Program* (DDP) [10] in which many of the key courses are co-taught by faculty members representing any of the four campuses, and students also can participate from any of the four campuses. Similarly, a loosely knit federation of leading research institutions also have participated in several of the shared courses that are part of the DDP. For example, during the Spring of the 1999-2000 academic year, a course was taught on *Research Design in Mathematics and Science Education.*[11] It included participants from all four Indiana campuses and also Arizona State University, SUNY-Buffalo, Queensland University of Technology in Australia, and the University of Quebec at Montreal in Canada. Similarly, during the Fall of the 2000-2001 academic year, a course is being taught on *Models & Modeling in Mathematics & Science Education,*[12] and, again, participating campuses will include all of the preceding institutions plus Syracuse University. Finally, early in the Spring Semester of the 2000-2001 academic year, the PU/IU Distributed Doctoral Program will play host to the first annual *Distributed Doctoral Research Conference in Mathematics Education.* Like courses in the DDP, this conference will use internet-based videoconferencing and other communication tools to enable participants to interact from remote sites.[13]

One reason why the preceding kinds of collaboratively taught courses are important is because doctoral students in mathematics education are perfect examples of students with highly specialized needs, that do not occur in sufficient numbers on most campuses so that their needs can be addressed effectively. Another reason is that multiple-campus collaboratively taught courses can provide ideal ways to promote the kind of community building that will be needed to address many of the most important issues that mathematics educators need to understand.

Notes

1. Even in everyday situations, thermometers measure temperature; yet, it's obvious that simply causing the mercury to rise doesn't do anything significant to change the weather. Clocks and wrist watches measure time without leading us to believe that they tell what time really is. Symptoms may enable doctors to diagnose a disease;

yet, it's clear that eliminating the symptoms is different than curing the disease.
2. Whereas behavioral objectives treat mathematical ideas if they *resided in* specific problems or tasks, modern mathematics education researchers have turned their attention beyond analyses of "task variables" to focus on analyses of "response variables" – because mathematical thinking resides in students' interpretations and responses, not in the situations that elicited these mathematical ways of thinking.
3. In many respects, the development and assessment of complex conceptual systems is similar to the development and assessment of complex and dynamic systems that occur in other fields - such as sports, arts, or business - where coordinated and smoothly functioning systems usually involve more than the simple sums of their parts. For instance, it may be true that a great artist (or athlete, or team) should be able to perform well on certain basic drills and exercises; but, a program of instruction (or assessment) that focuses on nothing more than checklists of these basic facts and skills is not likely to promote high achievement. If we taught (and tested) cooks in this way, we'd never allow them to try cooking a meal until they memorized the names and skills associated with every tool at stores like Crate & Barrel or Williams Sonoma; or, if we taught (and tested) carpenters using such approaches, we'd never allow them to try building a house until they memorized the names and skills associated with every tool at stores like Ace Hardware and Sears. But, in education, it's common to treat low level *indicators* of achievements as if they *embodied* or *defined* the understandings we want students to develop.
4. Even if it's impossible to reduce Granny's cooking expertise to a checklist of rules for others could follow to duplicate here abilities, it may be quite easy to identify situations where her distinctive achievements are required – and where many of the most important components of here abilities will be apparent.
5. This is because a variety of levels and types of interpretations are possible, a variety of different representations may be useful (each of which emphasize and de-emphasize somewhat different characteristics of the situations they are intended to describe), and different analyses may involve different "grain sizes", perspectives, or trade-offs between factors such as simplicity and precision.
6. Is a Ford Taurus better or worse than a Jeep Cherokee? Clearly, answers depend on purpose, context, and other factors that apply to assessments of any complex system that is intended to function in complex situations.
7. The Handbook of Research Design gives a number of detailed examples of each phenomena.
8. In mathematics education, a great deal of the best work of the most productive researchers has never fit the constraints of the *Journal for Research in Mathematics Education.* Consequently, at least since the early 1970's, there has been a healthy "black market" of research publications – such as those associated with early years of the "Georgia Center" (Steffe, 1974).

9. My experiences coincide with others who write large numbers of review letters for people being considered for jobs, promotions, or tenure. That is, the frequency of such questions appears to be inversely related to the quality of the institution asking them.
10. For information about the Distributed Doctoral Program, contact Terry Wood (twood@purdue.edu), Frank Lester (lester@indiana.edu), Beatrice D'Ambrosio (bdambro@topaz.iupui.edu), Erna Yackel (yackeleb@calumet.purdue.edu).
11. For information about this course, contact Jim Middleton (jmiddleton@)asu.edu) or Marilyn Carlson (carlson@math.la.asu.edu), Doug Clements (clements@acsu.buffalo.edu), Lyn English (l.english@qut.edu.au), or Carolyn Kieren (kieran.carolyn@uqam.ca).
12. For information, contact Helen Doerr (hmdoerr@sued.syr.edu) or Richard Lesh (rlesh@purdue.edu).
13. For information about how to participate in this DDRCME conference, contact Richard Lesh (rlesh@purdue.edu), Judi Zawojewski (judiz@purdue.edu), or Kay McClain (Kay.McClain@vanderbilt.edu).

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CRITICAL CHALLENGES IN RESEARCHING CULTURAL ISSUES IN MATHEMATICS LEARNING

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Introduction

I believe that the major developments in thinking about mathematics education in the last decade have come about through research initiatives, and also that research is where we must look for new ideas and developments (Bishop et al., 1996). The school curriculum in many countries is now so controlled by central authorities, local politicians, or commercial interests, that the opportunities for significant developments coming from teachers themselves are increasingly unlikely. This does not mean that teachers have nothing to contribute - far from it. Nor does it mean that researchers are always free to investigate whatever they wish. What it does mean is that we need to develop more collaborative research between practitioners and researchers in the future if we are to gain the full benefits of what each group can offer.

The focus of my paper will be on cultural aspects of research in mathematics learning, and I will situate my discussion within the research context of mathematics education generally. With the shift to a more socio-cultural emphasis in research that we have seen in the last 20 years has also come an awareness that the dominant research methods used earlier may not now be the most appropriate. I shall address later in the paper one of the ramifications of this point. However, to start with I wish to set the curriculum context.

The Challenge of Culturally-Based Mathematical Knowledge

One of the most significant areas of research development in the last two decades has been in ethnomathematics. It has not only generated a great deal of interesting evidence, but it has fundamentally changed many of our ideas and constructs. The most significant influences have been in relation to:

- human interactions. Ethnomathematics concerns mathematical activities and practices in society, which take place outside school, and it thereby draws attention to the roles which people other than teachers and learners play in mathematics education.
- values and beliefs. Ethnomathematics makes us realize that any mathematical activity involves values, beliefs and personal choices.
- interactions between mathematics and languages. Languages act as the

principal carriers of mathematical ideas and values in different cultures.

- cultural roots. Ethnomathematics is making us more aware of the cultural starting points and histories of mathematical development.

In general, these points have forced us into giving more consideration of the overall structure of the mathematics curriculum and to how it responds, or more usually how it does not respond, to the challenge of culturally based knowledge. In general the mathematics curricula which exist in the countries of the world are not culturally responsive (see Bishop et al., 1993) but are remarkably similar. Whether these similarities exist by choice or are a result of various waves of cultural imperialism is not clear (see Bishop, 1990), but they certainly do not appear to reflect differences in cultural context.

The curriculum structures we generally see have evolved to suit the preparation of an elite minority of students who will study mathematics at university. However when we consider the majority of school pupils who either never go on to study more mathematics or who don't even go to university, this elitist mathematics education is highly inappropriate, and contributes significantly to the widespread problems of alienation felt by many students towards mathematics in particular and also towards schooling in general. Research therefore needs to explore how the mathematics curriculum can be made more culturally responsive, in order to encourage more participation at the higher levels particularly amongst cultural minority groups.

The Challenge of 'Hidden Values' in Teaching

Moving to another critical contextual aspect of learning, let us briefly consider teaching from the cultural perspective. Having already explored several aspects in other writings, (for example, Bishop, 1991) I would like here to concentrate on one often ignored aspect, which is that of values in mathematics teaching. In keeping with a common idea that many people still seem to have, that mathematics education is universal and culture-free, it is also perceived by them to be value-free. This does not mean that they think mathematics has no value, but rather that they do not think it has any values over and above those values a particular society is promoting.

I believe that it is significant that in curricular developments such as Science and Technology in Society the area of values is taken as serious curricular content. In mathematics curricula that is certainly not the case. Beliefs and values in mathematics education are not taken as 'knowledge' with a strong cognitive component, they are instead treated as affective aspects (see McLeod, 1992). What should be of greater concern to mathematics educators is that values teaching and learning does occur in mathematics classrooms, and because most of it appears from our preliminary studies to be done implicitly, there is only a limited understanding at present of what and how values are being transmitted. Given the often-quoted negative views expressed by adults about their bad mathematics learning experiences, one could speculate that the values transmitted to them were not the ones that most educators or educational policy makers would think of as desirable, but that they were transmitted rather effectively!

Rarely does one find explicit values teaching going on in mathematics classrooms, and from our research, few mathematics teachers admit to explicit values teaching. It is however clear from Seah's (1999) research that textbooks do portray certain values, and in our research we are about to document what values teachers do portray. Thompson (1992) summarized the research on teacher beliefs, this time in relation to teachers' actions in the classroom. She points to a repeated finding that mathematics teachers' actions frequently bore no relation to their professed beliefs about mathematics and mathematics teaching. The research by Sosniak et al. (1991) also found striking inconsistencies between different belief statements given by the same teachers. Hence my use of the 'hidden values' words in the title of this section. Values in mathematics education appear to have the role of cultural 'hidden persuaders' (Bishop, 1990). I would contend that this discrepancy between beliefs and values is precisely why it is necessary to focus research on values rather than beliefs, in order to determine the deeper affective qualities that underpin teachers' preferred decisions and actions and that ultimately affect the learners' beliefs and values. My research colleague Phil Clarkson and I have coined a phrase to help us distinguish beliefs and hidden values: "Values are beliefs in action." What we are trying to capture with that phrase is the idea that teachers appear to hold several beliefs, which may or may not be consistent, but that the important transition from a belief into a value occurs in the context of the teacher's actions (see the Values and Mathematics Project web-site <http://www.education.monash.edu.au/projects/vamp>)

In summary then, the challenges for researching values in mathematics teaching include:

- To what extent does explicit values teaching occur in mathematics classrooms?
- Are teachers aware of the values they are transmitting, modelling or portraying?
- Is implicit values teaching more or less effective than explicit values teaching?
- How do teachers facilitate the transitions between implicit and explicit values learning?

Researching mathematics learning: meeting the challenge of culturally situated learning

The importance of the socio-cultural approach to research on learning is due to the fact that the cognitive psychological program, with its focus on individual cognition and intra-individual characterisations and explanations has tended to ignore the crucial socio-cultural context of mathematics learning. However, socio-cultural research in mathematics education has tended to focus on learning within certain cultural practices and communities, and thereby has failed to take into account two crucial aspects. Firstly, the focus on inter-practice differences between cultural groups has obscured important inter-individual differences within those cultural groups. Secondly, the research has tended to ignore the transition aspects of learning between those cultural practices.

Particularly in diverse multicultural societies, we can see that the culture experienced by learners in their homes is rarely the same as that represented by the school curriculum. This kind of disjunction can easily lead to what I have called 'cultural conflicts' (Bishop, 1994). The construct of 'cultural conflict' grew out of educational research in the anthropological tradition. We can find it, for example, as a central idea in McDermott's (1974) classic chapter about 'pariah groups' whose children fail to succeed in mainstream schools. He builds on Barth's (1969) definition of pariah groups, who are those who are "actively rejected by the host population". According to McDermott, "Students and teachers in a pariah-host population mix usually produce communicative breakdowns by simply performing routine and practical everyday activities in ways their sub-cultures define as normal and appropriate....The problem is neither 'dumb kids' nor 'racist teachers', but cultural conflict" (p.173).

Thus for many children around the world the educative experience in schools is not culturally consonant with their home experience. Their situation is one of cultural dissonance and the educational process is one of acculturation, rather than enculturation. The social groupings in which learners exist and learn inside and outside school have their own cultures, customs, languages and values. This is the basis for the development of the research on 'situated cognition' (Lave and Wenger, 1991; Kirshner and Whitson, 1997). The study of the 'failures' of bilingual learners in a monolingual classroom, or of farmers' children studying in totally urban-centred curriculum, or of handicapped learners, all help to shed light on other explanations of failure and success besides the attributes of the learners themselves. However we must not fail to recognize the variation within these groups, and the fact that certain of the learners' attributes will be significant in enabling them, or not, to succeed in the culture of the classroom. Research needs to address those attributes in the socio-cultural context.

Equally more research needs to focus on the transitions in learning experienced by learners in cultural conflict situations (Abreu, Bishop and Presmeg, in press). The learners are clearly faced with negotiating transitions in knowledge, and knowing, but they must also make transitions in values, language customs and behaviours. What is it about learners who succeed with knowledge transitions, or what is it about their learning experiences? What effects do the teacher and other 'significant others' in the social context have on their successful transitions, or otherwise? Here Bronfrenbrenner's (1979) perspective on the ecology of human development is worth revisiting.

These perspectives enable us to see that learners are not just learning the cultural knowledge that they are being taught (as well as other knowledge that they are not taught, of course). They are in fact co-constructing that knowledge. (Note that they are not re-constructing knowledge, since it can never be re-constructed to the same form.) This is to my mind the most important point about constructivism – not that it is the individual who is constructing her/his own personal knowledge. Of course from a psychological point of view that is important, but it is also rather obvious. What is much more important is what is the quality of the *social* situation that enables the learners to socially co-construct their new cultural knowledge. Knowledge changes with every generation, and it is mediated in that change by teachers and by learners of all cultural persuasions.

Thus the challenges for improving mathematics learning through research include:

- How best to represent the 'social situation' in situated cognition research in mathematics education?
- What distinguishes learners, and their contexts, who succeed in making mathematical knowledge transitions between contexts from those who do not?
- What distinguishes cultural constructivism from social constructivism?
- What implications for teaching does cultural construction have as a metaphor for education?
- How to research cultural transitions in mathematics learning?

What About our Research Approaches?

The essential goal of research in mathematics education is to help us understand phenomena in richer ways so that we can improve the teaching and learning situation for as many students as possible. But as we embrace fully the implications of a cultural perspective on mathematics learning, are our research methods and procedures themselves adequate for the task? There are several researchers who argue 'no', and that we need to change how research is carried out and conceptualised if we are to address these socio-cultural aspects in the thorough way that they need to be addressed. As an example of this, at the PME conference in 1998, Valero and Vithal (1998) criticised the mathematics education research community for its imposition of research methods from the relatively developed 'north' onto researchers and students from the relatively underdeveloped 'south' part of the world. They argue that methods developed in one cultural context are not necessarily appropriate or helpful in another cultural context, in terms of what is considered 'normal'.

To develop our field further, we clearly need to take on board the procedures and practices of anthropological and social psychological research, but we also need to recognise that we are working in the field of education, and more particularly in mathematics education. In general I believe that our research approaches must move to a more collaborative style, involving not only practitioners and researchers, but also to include the learners and their peers as partners in the research process.

Just as we have found it necessary and beneficial to do research 'with' rather than 'on' teachers, so I believe we will need to develop ways of researching 'with' rather than 'on', learners, and their peers. Already qualitative methodologies have moved us closer to that goal, and if we are really serious about trying to improve our understanding of how learners deal with the conflicts and transitions in the cultural learning of mathematics then we have little choice but to engage fully with them in the inquiry process. This means as well as taking into account their cultural situation, we must also take into account ours. Just as we recognise the influences that their cultural contexts have on their learning, so we need to recognise the influences that our cultural contexts have on our learning, through our research.

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Working Groups

ADVANCED MATHEMATICAL THINKING: IMPLICATIONS OF VARIOUS PERSPECTIVES ON ADVANCED MATHEMATICAL THINKING FOR MATHEMATICS EDUCATION REFORM

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The Advanced Mathematical Thinking (AMT) Working Group has taken on the task of constructing definitions that capture what appeared to be three different perspectives on the nature of "advanced mathematical thinking." Over the past year, three groups, led by Barbara Edwards, Chris Rasmussen, and Guershon Harel, have developed definitions of advanced mathematical thinking that capture the characteristics each group deemed salient to the issue. Those papers will be made available to participants in the AMT Working Group at PME XXII via the Advanced Mathematical Thinking Working Group list-serve (contact HSUPAO@MAINE.MAINE.EDU for information on how to access the list serve), they will be referenced in this paper, and they (and one other paper) will be a central focus of discussion at the meetings of the AMT Working Group at PME-NA XXII in Tucson.

Three Perspectives on "Advanced Mathematical Thinking"

One can think about advanced mathematical thinking as characterizing the thinking that occurs primarily in the study of advanced mathematics at the collegiate or graduate level. The Edwards team took the perspective that advanced mathematical thinking requires two simultaneous conditions: 1) advanced mathematical thinking requires precise reasoning about mathematical ideas, and 2) these mathematical ideas are not entirely accessible to the five senses (Edwards et al., 2000). The authors point out several examples of ways in which it is insufficient evidence of advanced mathematical thinking that one and not both of these conditions are fulfilled. For example, the authors point out that although "limits" is a mathematical idea that is not entirely accessible to the five senses, "evaluating limits" is probably not advanced mathematical thinking since it may involve only the implementation of an automated routine and not precise reasoning about a mathematical idea. The authors develop several other examples that fit their definition of advanced mathematical thinking.

A second perspective on advanced mathematical thinking offered by Rasmussen and his colleagues focuses on "advanced mathematical activity" since the authors, as supported by Sfard (1998), conceive of mathematical learning as participating in doing mathematics (Rasmussen et al., 2000). These authors specifically do not limit advanced mathematical thinking to undergraduate and graduate mathematics, although the primary examples they develop in their paper are drawn from courses in differential equations and college geometry. They center their conversation about advanced mathematical thinking on the phenomena of horizontal and vertical mathematizing (Treffers, 1987), whose definitions they expand to allow for horizontal mathematizing in pure mathematics settings. The Rasmussen team characterizes horizontal mathematizing as transforming a mathematical or real world problem setting in such a way that it lends itself to further mathematical analysis. The group conceives of vertical mathematization as activities that are grounded in or build on horizontal mathematizing. The authors clarify their stance on advanced mathematical thinking by illustrating horizontal and vertical mathematizing through the activities of symbolizing, algorithmatizing, and defining. For example, the authors describe horizontal mathematizing as using symbols to record and communicate mathematical thinking and vertical mathematizing as using the symbolizations so developed as inputs for further mathematical reasoning.

The third perspective on advanced mathematical thinking, developed by Harel, characterizes mathematical thinking as advanced if mathematics education research can substantiate that "its development necessarily involves epistemological obstacles." Harel holds that advanced mathematical thinking develops over long periods of intensive effort. He gives examples of "ways of mathematical thinking (a) that are essential to the learning of advanced mathematical content and (b) whose development must start in an early age when elementary mathematical content s are taught." (Harel, 2000).

Questions Raised by these Three Perspectives on "Advanced Mathematical Thinking"

Each of the three perspectives on advanced mathematical thinking generates its own list of questions that could be investigated through further refinement of the theories or through empirical research. The Edwards team raises the issue of mathematics that is not entirely accessible to the five senses. To what extent are accounts of instances of mathematical thinking classifiable as "not entirely accessible to the five senses"? To what extent does this characterization capture the mathematical thinking in which research mathematicians engage? Is the viability of the definition largely a function of the type of mathematics being considered? If, as the Edwards team posits, this definition of advanced mathematical thinking lies at one end of a "mathematical thinking" spectrum, what characterizes the role of "accessibility to the five senses" in the intermediate stages between advanced mathematical thinking and elementary mathematical thinking (the thinking at the other end of the spectrum). If one of the goals of secondary mathematics is to prepare students for later advanced mathematical thinking, what will prepare students to conduct mathematical thinking about mathematical ideas that are less accessible to the senses?

The Rasmussen team centers its discussion of advanced mathematical thinking on horizontal and vertical mathematization. Further development of this definition will lead to additional refinement of ways to characterize and identify vertical mathematization. Of interest would be an investigation into the relationship between vertical mathematization and problem solving. Is one a subset of the other? Is vertical mathematization a necessary component of successful problem solving. If so, what are the other components? The authors claim that advanced mathematical thinking is not confined to collegiate mathematics. Are examples of vertical mathematization at the secondary level fundamentally different from those that typify the collegiate level?

The definition of advanced mathematical thinking offered by Harel (2000) is intimately connected to research questions. To qualify as "advanced mathematical thinking," mathematics education research needs to substantiate that "its development necessarily involves epistemological obstacles." Methodological questions arise about how one might investigate whether the development of particular ways of thinking involve epistemological obstacles. To what extent are these epistemological obstacles individual? To what extent are they generalizable? What characterizes growth in mathematic al thinking that evidences the successful maneuvering of epistemological obstacles?

Implications for Teaching and Learning in the Context of Reform-Oriented Teaching

The past two decades in mathematics education might be characterized as decades in which mathematics education reform was conceptualized. The past two decades have witnessed a major effort to reform the teaching of calculus and several major thrusts to reform mathematics teaching at the school level. The most recent document to characterize the nature of that reform is NCTM's *Principles and Standards for School Mathematics* (2000). Principles and standards takes a strident stand on behalf of mathematical thinking. While half of the document's ten standards concern the content of school mathematics, the other half speak to the processes of school mathematics. As such, these later standards help to characterize mathematical thinking at the school level.

Discussions of the nature of advanced mathematical thinking can help to illustrate and illuminate the standards in *Principles and Standards*. Instead of providing those illustrations in this paper, we will simply raise a few questions about the impact of the three perspectives on how we might interpret the standards. The definition of advanced mathematical thinking offered by the Edwards team emphasizes the need for precise reasoning about mathematical ideas. The Reasoning and Proof Standard provides some illustrations of how students at the school level might reason about mathematics. The goal of reasoning about mathematical objects that are not entirely accessible to the five senses, advanced by the Edwards team, suggests that teachers of school mathematics must learn how students develop their capacity to reason in the absence of concrete examples.

The definition of advanced mathematical thinking offered by the Rasmussen team speaks to an expanded notion of horizontal mathematization that includes the communication of purely mathematical relationships. In the context of this definition, implementation of the Communication Standard may require that special attention be paid to students' capacities to symbolize their mathematical ideas. How can students in school mathematics learn not just to symbolize their ideas but also to reason from those symbolizations? What combination of emphasis on reasoning and mathematical connections is needed for students to develop their capacity for vertical mathematization.

Finally, one possible, and very interesting, exercise would be to analyze specific expectations in the *Principles and Standards* from each of these three perspectives; namely, which categories of goals stated in *Principles and Standards* constitute advanced mathematical thinking or are seeds that will form a foundation for advanced mathematical thinking. For example, which of the following mathematical activities can be characterized as advanced mathematical thinking or as a seed for advanced mathematical thinking according to the criteria of: precise reasoning about mathematical ideas not entirely accessible through the five senses; epistemological obstacles; or vertical mathematization?

1. Use representations to model and interpret physical, social, and mathematical phenomena;
2. Make and investigate mathematical conjectures;
3. Organize and consolidate their mathematical thinking through communication; and
4. Understand how mathematical ideas connect and build on one another to produce a coherent whole.

Conclusion

In this paper, we have summarized the perspectives on advanced mathematical thinking offered by three teams of mathematics educators. We have suggested theoretical and empirical research that might be conducted to further understand each of these theories, and we have raised issues about implications for teaching and learning in the context of reform-oriented teaching.

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ALGEBRA REASONING IN ELEMENTARY MATHEMATICS – THEORY AND PRACTICE

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The powerful resources nowadays available from hand held calculators and computers offer new ways for the teaching of mathematics. Numerical-based strategies and visual approaches provided by them currently challenge symbolic algebra as a means of obtaining the competencies desired. These numerical and visual resources allow us to design teaching activities so that students work out mathematical challenges without previously having any formal approach to the mathematics involved. Such an approach implies many theoretical and practical issues.

Recently, much emphasis has been set in the use of computer environments, but less light (from a theoretical point of view) has been cast on how this may result in an innovative organization of the classroom. This is a beneficiary fact without a doubt. Thus it seems relevant to center our discussion in the innovative organization of the classroom environment.

A reflection on the relationship between theory and practice in the context of innovative approaches to the teaching and learning of algebra needs to take into account both the actual classroom practices and curriculum content and the research results from new proposals. In the present working group sessions, we will focus our attention on the changes that have been produced in the classroom practices with the introduction of innovative techniques based or **not** in the use of information technology. We will predominately study the changes in the role of the teacher, the role of the students and the role of the media in which the educational act is produced. In particular, we will discuss the following issues:

- New forms of generalization and formalization;
- The difficulties in the translation from natural language to the algebraic one;
- New approaches to analyze problem solving processes;
- Building-up algebraic syntax with graphic calculators and computers; and
- Innovative organization of the classroom environment.

In what follows, we briefly introduce three possible themes of discussion.

Algebra and Technology: New Semiotic Continuities and Referential Connectivity

This discussion will deliberately take a somewhat different orientation to algebra than is usually the case, investigating how technology affects basic semiotic assumptions and habits, with a special focus on the algebra of functions – the definition, manipulation, and use as models.

Our historical applications of technology to help with both the learning and the doing of algebra have passed through several stages. The earliest involved facilitating manipulations of character strings, as was the case in the late 1960's with MACSYMA being used for complicated symbol manipulations required in General Relativity. In the 1970's the public increasingly used computer technology to plot coordinate graphs of, and generate numerical data from, algebraic functions of one or more variables. In the 1980's these notations were increasingly linked to one another so that by the end of the decade one could make changes in one notation and these changes would be almost simultaneously reflected in any of the others.

Two features were common to all the development up to this point. One was the central role played by character-string notations in both the definition and manipulation of the functions-whether they were closed-form or recursively defined functions. The second was the traditional relationships between the algebraically defined mathematical objects as models and the phenomena or situations that they were used to model or represent. Both of these features reflect a deep, but largely tacit view of formalisms as separate and distinct from informal notations or utterances and from the phenomena that they are taken to represent. In particular, algebraic statements are part of the universe of mathematical notations, with separate rules of reference, with syntax distinct from natural languages, and abstract independence from the media in which they happen to be instantiated. This view in turn is intimately integrated with a Platonist and representationalist philosophical orientation that takes:

- Mathematical objects as pre-existing, to act as pre-given reference objects for mathematical notations;
- Language as an inert representational instrument which does not help create mathematics but only enables us (if we are sufficiently skilled in its use) to see and do mathematics; and
- Mathematics as separate from the material and social worlds we inhabit.

All three of these positions are eroding in the face of an increasing flow or technology-enabled semiotic systems that offer:

1. Increased *semiotic continuity* between mathematical notations and our extra-mathematical methods of manipulating objects in our world, and
2. Increased *referential connectivity* between inscriptions taken to refer as models

to phenomena or situations.

After offering a characterization of the kinds of 21st century mathematical activity that stimulate the need for new representational forms, the remainder of the discussion will be devoted to illustrating and explicating these two assertions, developed in three sections.

1. Parametrization of Mathematical Objects and Relations, and the Increased Steepness of the 21st Century Learning Curve.
2. Semiotic Continuity with Ordinary Physical Actions.
3. Referential Connectivity Between Inscriptions and the Phenomena or Situations They Are Taken to Model.

Verbal Arithmetic-Algebraic Problem Solving

We will address the subject of transference of the algebraic operativity, that has been recently learned, to some other contexts, as would be the case of arithmetic-algebraic verbal problem statements. Among the transfer processes of a given algebra operativity to problem contexts, where it could be used for its solution, are those that can identify the procedures for the solution, in which actually such operativity could be applied. These processes of simple recognition are only part of the complex transfer process (which includes, among others, the analytical reading processes of the statement, the production of a strategy and a representation system, as well). When reasoning through a complex problem, it is more than enough to have some kind of distraction for a child to focus on a certain context in which the recognition of what has already been learned and mastered at an operational level could not be applied. The likelihood of experiencing these types of centering phenomena during the development of the procedure for solving the problem is not overlooked and if this is the case, all the procedure could be upset or still, the possibility of solving the problem could be hindered. The solution to these types of obstacles is a level of transfer of the operativity, in which the already, mastered syntax elements could be drawn from the semantics of the context from which the problem is addressed (or solved).

Progress Towards Semantics

In spite of the confidence that is reflected by some students in being able to solve the new equations operationally when these appear in other contexts; to be able to speak of a true transfer of that operational capacity to the solution of problems, still to be considered are the processes that lead to understanding the statement and writing an equation. Among these processes are those of *representation* of the elements of the problem, and this presupposes reading and analyzing the statement that distinguishes between what is given and what must be found; and that allows the relevant information to be recovered while leaving aside whatever is not essential. This might also precede (or sometimes follow) *the representation*, the production of a strategy to attack the problem. The consolidation of the first elements of algebraic syntax is based on their link with a non-algebraic semantics, in this case that of how problems are stated.

When the performance of the students has put into operation the new elements of syntax, there is evident progress in algebraic semantics (as far as its problem-solving use is concerned) that also implies progress in the use of syntax. The opposite is also true: progress in syntax implies progress in the semantics of algebra; this last-mentioned appears to be a fairly generalized opinion, since, in effect a certain level of syntax is always considered to be a factor in helping to solve problems.

Translating from Natural Language to the Algebraic Mathematical Sign System And Viceversa

This section deals with the translation, in both directions, of natural language (NL) into the Mathematical Sign System (MSS_1) generated by previous learning during the arithmetical and pre-algebraic training of the pupils in primary school and the first grades of high school. This translation between NL and MSS_1 is one of the central features of classical teaching strategies for the solution of word problems by means of the Algebraic Mathematical Sign System (MSS_2).

Children at three different levels of performance (high, medium, low) in mathematics were selected for interviews to work with a basic sequence of four blocks of items:

Block 1. The reading of equalities corresponding to geometric formulae, expressed in algebraic symbols, like $A = \pi r^2$, $A = l^2$, etc.

Block 2. The reading of open algebraic expressions like $(a+b)/2,\quad ab, 3ab, a^2$.

Block 3. The reading of algebraic equivalencies (tautologies) like $(a+b)^2 = a^2 + 2ab + b^2$.

Block 4. The interpretation of sentences expressed in natural language and their translation to mathematical symbols. For example, "the double of **a**," "**a** increased from two." "**a** decreased from two." Only some children with high and medium performance worked with a fifth Block consisting of systems of simultaneous equations of the type $x = a,\quad x + b = c,\quad and\quad y = bx + c,\quad ax + by = d$ with **a, b, c** and **d** particular whole numbers.

Some Results

I. In Block 1, three levels of the interpretation of the formulae were observed:

A) Textual Reading in NL of the expression without reference to any context.

B) Reading as in A), accompanied also by a verbal reference of the elements of the expression to dimensions of a geometric figure, without specification of the latter by the subject.

C) Reading as in B), accompanied also by the association of a specific geometric figure (circle, square) and of the corresponding attribute (area, perimeter); this was not always done in a correct way. These three interpretative levels appeared both in a partial and in a total manner, depending on the level of pre-algebraic performance of the subject.

II. A) With respect to Block 2, the textual reading in NL of expressions like $(a + b)/2$ was accompanied by i) a reference to the dimensions of "ideal" geometric figures (heights, bases); ii) the need to assign specific values to the letters in order to obtain a *result* and "close" the expression *thought up by the subjects themselves*; iii) the elaboration of an equation or equality starting from the expression and the numerical substitution for some of the literals.

B) In some case in Block 2, in the numerical substitution, the election of the values by the subject appeared to be arbitrary; however, in expressions such as **a-b**, identical values for **a** and **b** are not immediately accepted, since the association of different values with different letters and viceversa is present. In children with a low pre-algebraic performance, a resistance to assigning a higher numerical value to **b** than to **a** was observed, given the imminence of a negative result.

C) Furthermore, within the same Block 2, a tendency to give meanings to the open expressions in the context of word problems was observed. This was found very clearly in the case mid-level case in the following way:

Open Expression ——▶ Posing a Problem ——▶ Formulating an "Equation" (the expression is closed) Obtaining a Result.

III. The interpretation of "composite" expressions like $(a + b)^2$ and of algebraic tautologies like the development of the squared binomial (Blocks 2 and 3) presented a high level of difficulty and the majority of the subjects did not get beyond the most primitive level of reading in NL. The reading in NL of $(a + b)^2$ gave the typical error of $a^2 + b^2$.

THE COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

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Over the last several years, we have given serious attention to how students learn to reason probabilistically; that is, how learners build mathematical models, and how these models interrelate with each other and with data. A special focus has been on how students build and work with information. Some of work along these lines has been reported and discussed at Singapore (ICOTS-5, June 21-26, 1998), at PME-NA 20 (North Carolina State University, Raleigh, North Carolina, October 31–November 3, 1998), at the third Robert B. Davis (RBD) Working Conference (Snowbird, Utah, May 22-26, 1999) and at PME-NA 21 (Cuernavaca, Mexico, October 23-26, 1999). Continuing discussion, investigation and collaboration draw on work at sites around the world.

Issues

At PME–NA 20 (Raleigh, 1998), the Working Group began to formulate a joint agenda for research, discussion and investigation. At Cuernavaca, the Working Group at PME-NA 21 (Cuernavaca) developed this agenda further, side by side with research presentations. Central issues that the group discussed include:

(1) Study of how learners work with data, through analyses of learners' images, data representations, models, arguments and generalizations. Attention to learners' success as well as learners' difficulties, in the unifying context of research on the development of learners' understanding.

(2) Attention to how models, reasoning and thinking function in communities of learners, teachers and researchers. Examination of the roles of given tasks, of classroom environments, of student-teacher interactions, and of how learners, in a range of settings, share ideas, reasoning, and information.

(3) Emphasis upon development of mathematical ideas through time, with learners of different cultures, ages, social backgrounds, and with different prior mathematical and scientific experience. Analysis of learners' and researchers' changing views of underlying mathematical and scientific issues.

To help focus and develop this agenda, the Cuernavaca discussion took as starting points the interplay of combinatorial and probabilistic reasoning for constructing images and models in the course of task investigations.

Theoretical Framework

Recent research emphasizes the complexity and subtlety of probabilistic reasoning, even in very basic situations. Models can extend distortions, even as they help support the growth of understanding. Indeed, the variety of representations which learners find useful, and the complex relationships among the models learners build and data which they seek to explicate provide rich opportunities for research investigations focused centrally on sense and meaning. Given its complexity, the development of probabilistic thinking entails building over time, in which earlier inquiries are revisited, reconsidered, extended and reformulated. The tools available, the ways the tools are used, the ways in which ideas and information move among the learners, the teacher's questions, ideas and interventions, all contribute (or fail to contribute) in important ways. In our view, both research and teaching need to take the long-term building and the complexity into account.

Background

Related cross-cultural research on particular dice games, by researchers in several countries, using different methods of analysis across a range of settings and learner populations, was reported in joint sessions at the International Conference on the Teaching of Statistics (ICOTS-5, Singapore, June 21-26, 1998). The Singapore reports (Amit, 1998; Fainguelernt & Frant, 1998; Maher, 1998; Speiser & Walter, 1998; Vidakovic, Berenson & Brandsma, 1998) helped motivate the work at Raleigh. Further discussions at the third RBD Working Conference (Snowbird, Utah, June 1999) addressed important aspects of the Working Group's agenda in the context of the growth of understanding.

The present Working Group, first at Raleigh, then at Cuernavaca, built upon this shared research, enlisted new collaborators, and helped initiate further discussion. An incomplete but perhaps somewhat representative list of active members of the Working Group would include Sylvia Alatorre and Araceli Limon Segovia, both from Mexico; and Alice Alston, Sally Berenson, George Bright, Susan Friel, Regina Kiczek, Clifford Konold, Carolyn A. Maher, Robert Speiser, Draga Vidakovic and Charles Walter from the United States. Further colleagues, in several countries, are engaged in work related to the Group's agenda and concerns.

Plan for Involvement of Participants

At Raleigh, the Working Group considered data drawn from sixth-graders' work on two dice games (Maher, Speiser, Friel & Konold, 1998) which led to an extremely rich discussion. Based on this experience, a list evolved now including four tasks that we invite participants at different sites to explore with diverse learner populations. Here are current versions of these tasks.

A game for two players. Roll one die. If the die lands on 1, 2, 3 or 4, Player A gets one point (and Player B gets 0). If the die lands on 5 or 6, Player B gets one point (and Player A gets 0). Continue rolling the die. The first player to get 10 points is the winner. Is this game fair? Why or why not?

Another game for two players. Roll two dice. If the sum of the two is 2, 3, 4, 11 or 12, Player A gets one point (and Player B gets 0). If the sum is 5, 6, 7, 8 or 9, Player B gets one point (and Player A gets 0). Continue rolling the dice. The first player to get 10 points is the winner. Is this game fair? Why or why not?

The World Series Problem. In a "world series" two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the "world series." Assuming that both teams are equally matched, what is the probability that a "world series" will be won: (a) in four games? (b) in five games? (c) in six games? (d) in seven games?

The problem of points. Pascal and Fermat, in correspondence, discuss a simple game. They toss a coin. If the coin comes up heads, Fermat receives a point. If tails, Pascal receives a point. The first player to receive four points wins the game. Each player stakes fifty francs, so that the winner stands to gain one hundred francs, and then they play. Suppose, however, that the players need to terminate the game before a winner is determined. Further, suppose this happens at a moment when Fermat is ahead, two points to one. In correspondence, Pascal and Fermat discuss the question: How should the 100 francs be divided?

These tasks were developed by Carolyn A. Maher and her collaborators in the Rutgers-Kenilworth longitudinal study. The first two tasks were developed for sixth-graders. The last two tasks were developed later, initially for eleventh-graders. Related research includes (Kiczek & Maher, 1998; Maher & Martino, 1997; Maher & Martino, 1996; Maher, Davis, & Alston, 1991; Maher & Speiser, 1997; Martino, 1992; Martino & Maher, 1999; Muter, 1999; Muter & Maher, 1998).

Parallel research on several of these tasks has taken place at several sites around the world. Work in Brazil (Fainguelernt & Frant, 1998), in Israel (Amit, 1998), and in at least four places in the United States (Berenson, 1999), (Kiczek & Maher, 1998), (Maher, 1998), (Speiser & Walter, 1998), (Vidakovic, Berenson & Brandsma, 1998) has already been reported. Closely related findings, including (Alatorre, 1999) and (Berenson, 1999), were discussed in detail by the Working Group at Cuernavaca.

At the Tucson sessions of the Working Group, additional research will be presented and discussed, including resent work by G. Bright, S. Friel and F. Curcio, and suggestions for continued investigation, discusiion and collaboration will be invited.

Anticipated Follow-Up Activities

Collaborative work, based on case studies drawing on a focused set of tasks, and upon related research, from a variety of points of view, in different sites in several countries, has already helped to focus and extend discussion and collaboration. Based on discussions to take place at Tucson, further work with learners, in a range of settings, as well as further sharing and collaboration, will be initiated. We cordially invite further participants to join a growing and productive enterprise.

Connections to the Goals of PME

This Working Group has emphasized research into the nature and development of probabilistic and statistical understanding, based on collaboration between researchers in several countries, focused by a shared, continually developing research agenda. Recent work by members of the group, which draws on a rich background of psychological, pedagogical and mathematical ideas, has opened opportunities for further study and collaboration. In all these ways, the Working Group supports the aims of PME.

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PME-NA XXII GEOMETRY AND TECHNOLOGY WORKING GROUP

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Introduction

The working group on Geometry and Technology will continue the discussion started in Raleigh, North Carolina in 1998 and enhanced in Cuernavaca, Mexico in 1999. The focus of working groups was on the integration of geometry and technology from the student and teacher perspective in 1998 and on teacher preservice education in 1999. The objectives for our working group were to:

- explore teacher education,
- investigate research questions,
- coordinate future research,
- identify questions to be investigated, and
- identify commonalities and conflicts in research findings.

We began by looking at preservice projects in technology and mathematics education. We identified issues in preservice mathematics education and then investigated how the research was addressing the issues. Some of the questions and issues that were raised included:

- identification of the goals of instruction: technology use vs. geometric content
- how to provide preservice students with experiences to explore and reflect on their learning. *
- in which topics does technology improve learning? Which topics does technology impede learning or is it less useful? *
- how do we fill in gaps in content knowledge while developing pedagogical knowledge?
- how to make technology less procedural and use it as a problem solving tool? *
- how is the problem solving process different with and without technology? *
- how to design activities that integrate hypertext with geometric software and Derive.

The ones with the asterisk were then discussed by small groups in four areas: how the research was addressing the issue, the challenges to the research, the research questions, and how to conduct research in collaboration with others.

Clements and Battista (1994) summarized a number of studies that suggest that geometric computer environments can help develop students' thinking in geometry. According to these studies, students can make conjectures, evaluate visual manifestations of those conjectures, and reformulate their thought (p. 188). In Tucson, we will explore student geometric thinking and reasoning. We will have small presentations on how to increase geometric thinking and reasoning (including proof) through the use of geometric technology tools. Participants are invited to bring journal articles and reports that will further our understanding in this area.

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WORKING GROUP ON GENDER AND MATHEMATICS: EMERGENT THEMES

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This work started at PME XX in 1998, with sessions centered around "Gender and Mathematics: Integrating Research Strands." We explored why we do the work we do and what we know from this work; what the compelling topics are for future study; and how we might further this work. The participants presented short papers and discussed issues introduced through our work. The organizers conducted analyses of discussions and created a preliminary model to relate each of the discussion strands to each other. Two main strands emerged: the "sex-gender system" and the "doing of mathematics." The organizers saw that, in the discussions within the working group, the two strands did not overlap. Thus, the organizers identified the lack of intersection as an absence and an indication of a direction for future study.

Initially, we saw gender and mathematics as two different axes of a matrix, and began placing research topics in the various cells of the matrix. However, our work and our discussions led us to recognize that we needed more dimensions than could be represented with our simple matrix, because the work of feminists, woman mathematicians, and educators regarding issues of gender and mathematics were far more varied and complex to be neatly mapped onto a two-dimensional grid. We turned to webbing as a technique for visualizing the connections among the topics before us. We identified clusters and themes, as well as many linkages between and among the clusters. This illuminated for us the multidimensionalities of our inquiry efforts. We concluded that not only did the web represent research strands in gender and mathematics, but it also represented the new ways in which the group was thinking about the research strands. Figure 1 (courtesy of S.K. Damarin) is a computer rendition of what we produced that year.

Between the 1998 and 1999 sessions, we maintained electronic contact with members and worked to create a website for the Gender and Mathematics Working Group. On-line, we initiated discussion of topics relevant to our work, such as the equity component of the NCTM 2000 Standards, and collaborated on the proposal and paper presentations for the 1999 working group.

In 1999, we returned to PME XXI following the same general guidelines as in 1998. Sessions were devoted to discussion, with the organizers responsible for synthesizing and analyzing each day's work and, based on this between-session work. determining the starting point and framework for the next session. The first session was framed by the summary of work at and since PME XX, and by the work contributed by Dawn Leigh Anderson, Peter Appelbaum, Susanne K. Damarin, and Diana B. Erchick. In the second session, where goals were set by the group, the group made two main decisions: keep our work visible in the mathematics education community and work toward supporting ways to integrate our research findings into mathematics education and mathematics teacher education. Toward that end, we generated suggestions for how we might accomplish our goals.

Toward the goal of maintaining a visible presence in the mathematics education community, we intend to pursue and create outlets for our work. This intention will be realized partially through our original and continuing working group goal of producing a monograph on Gender and Mathematics for the *Journal for Research in Mathematics Education.* Another suggestion was to maintain a strong voice in the formation and development of PME programming.

Toward the goal of finding ways to integrate our research into mathematics education we developed options as viable directions for our inquiry. We hope to support the development of research to help understand the ways in which mathematics educators and mathematics teacher educators can integrate gender research findings into the mathematics classroom – into the teaching, the learning and the representations of the content. We need to conduct and supervise research exploring the teaching practice in our own mathematics and mathematics education classrooms, as well as in the classrooms of other teacher or mathematics educators. As already recommended by Fennema and Hart (1994) in the *Journal for Research in Mathematics Education*, mathematics education needs more research conducted from a qualitative and feminist perspective. We believe that agenda will serve us well, not only in the effort to find ways in which we can and do integrate our research agendas into our teaching, but also in meeting the working group's goals of finding and addressing absences in the research around gender and mathematics.

In the third 1999 session, we then took our agenda into the planning stages. The organizers synthesized discussions from the previous two sessions, identifying emergent themes. We identified those themes as possible topics to present to the group at large (i.e. those who were a part of the group the previous year but who were not able to be in attendance at the 1999 session). Those themes, we expect, will become categories for the call for papers for the proposed monograph for *JRME.* Those topics include, but are not limited to, the following: a) The development of epistemological voice; b) The integration of gender research into the mathematics classroom, K-16; c) The integration of gender research into the mathematics education classroom; d) The role of the content in addressing gender issues in mathematics education research; e) Mathematical success in fast-track and other programs for girls in mathematics; and f) Mathematical success for women in mathematics and math-using fields.

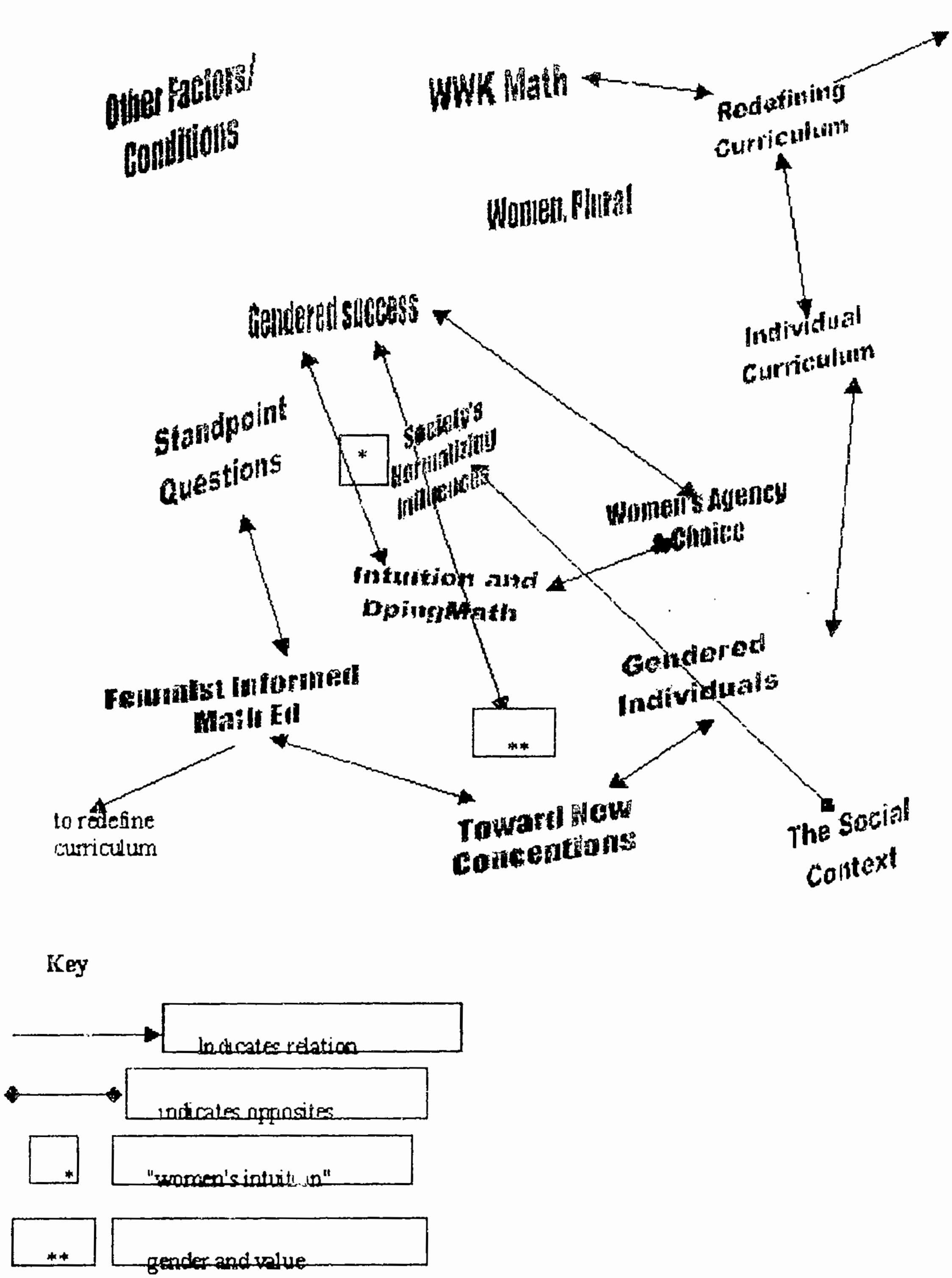
Other Factors/
Conditions
WWK Math
Redefining
Curriculum
Women, Plural
Gendered success
Individual
Curriculum
Standpoint
Questions
*
Society's
Women's Agency
& Choice
Intuition and
Feminist Informed
Math Ed
**
Gendered
Individuals
to redefine
curriculum
Toward New
Conceptions
The Social
Context
Key
Indicates relation
indicates opposites
*
**
gender and value

Figure 1

Since the 1999 working group sessions, we have continued our work by a) working on–line to revise and complete the statements we drafted in Mexico; b) continuing development of the website (we intend to have all papers from the 1998 and 1999 Gender Working Group available on the site this winter); c) generating more topics for themes within the *JRME* monograph proposal; and d) beginning to draft a call for papers to solicit participation from the Gender Working Group members for the proposed monograph.

Our work in the Gender Working Group has been successful in several regards. Most importantly, the group has "broken new ground" in the study of gender and mathematics in that we have developed constellations of issues and questions grounded in mathematics that are pertinent to the problems under investigation. Over the past decade, gender researchers in mathematics education have expressed the need to stop focusing strictly on girls and women and their attributes and failings. Work towards a new focus will be represented in our development and organization of a gender and mathematics monograph.

In that psychology is one of the major areas of research on gender and mathematics, the projects of this group repeatedly intersect it. The research presented through our working group, and ultimately the monograph, is interdisciplinary. It reflects ideas and considerations, and is advanced not only from psychology and mathematics education, but also from sociology, anthropology, philosophy, history of mathematics, feminist theory, and other fields. The study of gender and mathematics is of international interest, and our work reflects influences from outside the US. To be continued at PME 2000, the synthesis of our research, however, is based on work done in the United States. This work parallels synthetic work done in Europe a few years ago.

The three Gender Working Group sessions for PME 2000 are organized as follows:

Session 1: Progress and starting points. Assure all participants are current on the previous work, especially those new to the group or not in attendance during either of the previous years.

- Review the progress made in PME99 and PME98, and review the papers on the website.
- Discuss website contents and its potential use, especially toward monograph goal.
- Set working goals for the three days of the conference.

Session 2: Organization of the monograph. The second session will be devoted to the development of an organization for the monograph based on the papers we have, the themes that have already emerged, absences identified over the previous two years of our working group, and the suggestions of researchers outside our group.

Session 3: Identifying a working plan. Our third working group session will require the group to develop a call for papers for the monograph. This will necessitate agreement on the selection of topics and themes from session two. We will develop a working plan for producing the monograph. We will identify tasks, timelines, and participation in writing and administration for creation of the monograph proposal and the subsequent monograph itself.

We have identified absences in the research and compared that to the research agendas of the participants of the Gender Working Group. Currently and beyond PME 2000, we use that alignment to identify accomplishments, needs and goals toward developing Gender and Mathematics as a more comprehensive field of study than we find it to be now. Making the research on gender and mathematics a more integrated component of mathematics education, drawing together our work into a monograph on Gender and Mathematics, and continuing to develop research agendas to seek answers to as yet unanswered questions about gender and mathematics continue to be a part of our agenda.

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MODELS AND MODELING: REPRESENTATIONAL FLUENCY

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Introduction

The essence of an age of information is that, as soon as humans develop a conceptual tool for making sense of their experiences, they use this tool to create new realities and experiences. Consequently, the world is increasingly filled with comp'-x systems –communication systems, information storage and retrieval systems, economic and finance systems, and systems for planning and monitoring behaviors of other complex systems. Thus, the kind of mathematical understandings and abilities that are needed involve dynamic, iterative, and graphic ways of thinking that are quite different than those that have been emphasized in traditional schooling. For example,

- Beyond computing with numbers, thinking mathematically often involves describing situations mathematically. Furthermore, relevant descriptions often must go beyond simple closed-form algebraic equations; also, relevant mathematizing processes may range from quantifying qualitative information, to dimensionalizing space, to coordinatizing locations.
- Systems that need to be described often beyond those involving simple counts and measures to also include locations, dimensions, shapes, rules, qualities that must be quantified, quantities that are continuously changing and accumulating, quantities that have both size and directions, or quantities that can't be seen (and that must be measured indirectly).
- Rather than operating on pairs of numbers, students often need to work with whole lists (or sequences, or series, or arrays) or quantities; and, rather than simply adding, subtracting, multiplying, or dividing, they often need to investigate patterns, trends, and other regularities.
- Products often go beyond simple numeric answers (e.g., 12 apples) to include: descriptions (e.g., using texts, tables, or graphs), explanations (e.g., using culturally and socially embedded discourse to describe why something that appears to be true is not true), justifications (e.g., using persuasive discourse for recommending one procedure over another), or constructions (e.g., shapes that satisfy certain specifications, or procedures for accomplishing specified goals). Also, requested descriptions, explanations, and constructions usually need to be sharable, transportable, or reusable. So, generalizations and higher-order thinking required as problem solvers must go beyond thinking with the products to also

think about them.

- The problem solvers often include more than individual students working in isolation. They may include teams of specialists who are engaging in socio-cultural relevant discourse while using a variety of powerful conceptual tools. Furthermore, realistically complex problems often need to be broken into manageable pieces –so that results of subprojects need to be monitored and communicated in forms so that a variety of people can work collaboratively.

These observations suggest that thinking mathematically is about constructing, describing, and explaining at least as much as it is about computing; it is about quantities (and other mathematical "objects") at least as much as it is about "naked" numbers, and, it is about making (and making sense of) patterns and regularities in complex systems at least as much as it is about pieces of data.

A Perspective on Models and Modeling

Through this working group, we wish to bring a *models and modeling perspective* to understand how students learn and reason about real life situations encountered in a mathematics and science classroom. We have adopted the language of models and modeling not only because of its well-documented power in so many scientific endeavors, but also because it tends to sound simple, meaningful, and useful to people whose understanding and support we most want to enlist – such as teachers, parents, and leaders in future-oriented science departments and professional schools.

We will discuss the idea of a model as a conceptual system that is expressed by using external notation systems, and that is used to construct, describe, or explain the behaviors of other systems – perhaps so that the other system can be manipulated or predicted intelligently. We will talk about the types of models that children develop (explicitly, not just implicitly), to construct, describe, or explain mathematically significant systems that they encounter in their everyday experiences. In particular, the types of mathematical models that focus on structural characteristics of the relevant systems.

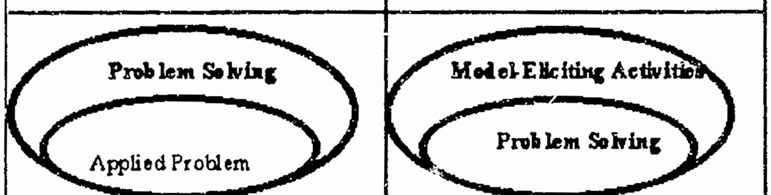
This *models and modeling perspective* has provided a rich context for research and curriculum development. Model-eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000) are kid-versions of adult level case studies. So called because the products that students produce go beyond short answers to narrowly specified questions to involve shareable, manipulatable, modifiable, and reusable models for constructing, describing, explaining, manipulating, predicting, or controlling mathematically significant systems. Thus, these descriptions, explanations, and constructions are not simply processes that students use on the way to producing "the answer"; and they are not simply postscripts that students give after "the answer" has been produced. They ARE the most important components of the responses that are needed. So the process IS the product!

When students solve traditional textbook word problems, it is often the case that the main thing that is problematic for them is to make meaning of symbolically described situations. Model-eliciting activities are different than traditional textbook word problems by going beyond the computational skills that most textbooks intend to emphasize. In traditional word problems, students make meaning of symbolically described situations; where as in model-eliciting activities, students make mathematical descriptions of meaningful situations. According to traditional views of learning and problem solving, learning to solve "real life" problems is assumed to be more difficult than solving their anesthetized counterparts in textbooks and tests. But, according to a models and modeling perspective, almost exactly the opposite assumptions are made.

When we observe students working in problem situations given in the model-eliciting activities, and when we take note of the understandings and abilities that contribute to success, what emerges is often quite different than the understandings and abilities emphasized in traditional textbooks and tests –or even in curriculum standards that attempt to modernize traditional instruction in incremental ways (see figure 1). Although the same mathematical ideas are important (such as those involving rational numbers, proportional reasoning, and measurement) attention shifts beyond asking, *What computations can students do?* Toward asking *What kind of*

Applied Problem Solving ≠ Model-Eliciting Activities

Traditional Views	Modeling Perspectives
Applied problem solving is treated as a special case of traditional problem solving.	**Traditional problem solving is treated as a special case of model-eliciting activities.**



Learning to solve "real life" problems is assumed to involve three steps: 1. First, learn the prerequisite ideas and skills in decontextualized situations 2. Learn general content independent problem solving processes & heuristics. 3. Finally (if time permits), learn to use the preceding ideas, skills, & heuristics in messy "real life" situations where additional information also is required.	Solving meaningful problems is assumed to be easier than solving those where meaningful interpretation (by paraphrasing, drawing diagrams, and so on) must occur before sensible solution steps can be considered. Understanding is not thought of as being an all or nothing situation. Ideas develop; and, for the constructs, processes, and abilities that are needed to solve "real life" problems most are at intermediate stages of development.

Figure 1

situations can students describe (in forms so that computational tools can be used)?

The Working Group at PME-NA XXII

In this working group our sessions will consist of a series of brief 5-minute "elevator speeches" (so called because they're similar to what a researcher might be able to say about "the essence of his or her work" if asked about it on an elevator) followed by discussions related to the following issues.

- Students' models, from a *models and modeling perspective.* It's not possible for researchers or teachers to be certain about many aspects of the conceptual systems that underlie the metaphors, diagrams, symbols, and other media that students use to make sense of their experiences. But, other aspects may be revealed explicitly –such as: (i) what kind of quantities the students are thinking about, (ii) what kind of relationships they believe are important, and (iii) what kind of rules that they believe govern operations on these quantities and quantitative relationships. Clearly, these visible components are part of the students' models and conceptual systems; they are not part of researchers' models.
- Teachers' models, from a *models and modeling perspective.* When teachers use model-eliciting activities in their classrooms, they provide rich opportunities for them to represent and reflect not only upon their students' knowledge, but also in their own understanding and development of mathematical constructs and teaching practice. As students solve the model-eliciting activities, their answers present relevant information of how students are thinking about mathematical constructs. When these are observed and analyzed by teachers, they become a powerful source for teachers to reflect upon students' knowledge achievements, and thus, on their own development.
- A models and modeling perspective in mathematics education curriculum development. Model-eliciting activities are not the only type of curriculum sources that we've developed. In fact, they are only one piece of how we envision an integrated unit we call model-development sequence. Model-development sequences include: (i) model-eliciting activities, which encourage students to repeatedly express, test, and refine or revise their own ways of thinking; (ii) model-exploration activities, that allow different embodiments of mathematical constructs, and (iii) model-adaptation activities, which are more difficult versions of model-eliciting activities, and may focus on problem posing, problem solving, information gathering, and/or information processing. In addition, other supporting activities are envisioned, like *teacher-led activities* that involve the whole class focusing on structural similarities (and differences) among the constructs and conceptual systems emphasized in the embodiments; *students' presentations and discussions* on the results they produced while solving any of the activities in the model-

development sequence; *reflection and debriefing activities* in which students think back upon their experiences in solving these activities; *follow-up activities* that teachers generate to help students recognize connections between their experiences during model-development sequences and more traditional kinds of activities; and an *on-line "how to" toolkit*, where students can find brief explanations of facts and skills that are most frequently needed in designated math topic areas.

Some of the people that will participate with their elevator speeches will be some authors of the book *Models and Modeling* (Lesh & Doer, in press), as well as graduate students from different countries that will be participating in a collaborative seminar on Models and Modeling organized by Purdue University. All of these people will briefly describe their research experiences with students and teachers learning, and development of significant mathematical models, from a models and modeling perspective.

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REPRESENTATIONS AND MATHEMATICS VISUALIZATION WORKING GROUP PME-NA XXII, 2000, TUCSON, ARIZONA, USA

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The Group on Representations and Mathematics Visualization was constituted at the PME-NA XX meeting in the North Carolina State University (Hitt, 1998). In the last meeting in Mexico, the academic agenda of the WG included four presentations followed by corresponding discussions that addressed very important themes. Here, we intend to summarize ideas discussed in each presentation (a full version will be available at each meeting of the WG and at www.Cinvestav.mx/mat_edu/PMENAXXI.html).

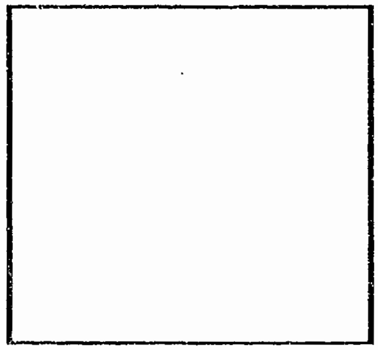
Patrick W. Thompson's (Vanderbilt University): Some Remarks on Conventions and Representations

The work presented by Thompson raised something important dealing with group production versus the memories about the group discussion from the individual perspective. Thompson said: "When we claim that agreement has been reached on a relatively complex idea because disagreement hasn't been expressed, we must consider the possibility that students haven't analyzed their own or others expressions sufficiently to detect severe inconsistencies."

Thompson's study did not stop where usually other research studies stop. That is, usually the research undertaken concerned with groups discussions of a mathematical problem stops when the teacher or the researcher thinks that a consensus among students has been reached. In Thompson's case, he addressed the importance of being cautious about concluding what individuals understand by only considering evidence based on group agreement.

The mathematical problem he studied was to have students "interpret a formula that defined an 'average rate of change' function in regard to the variation in a square's area expressed as a function of the square's side length." Then, given a segment s as an independent variable, he constructed $A(s) = s^2$, and $r(s) = \dfrac{A(s + 0.25) - A(s)}{0.25}$

A(s) = 1.633 square inches
s= 1.278 inches



s	A(s)
1.000	1.000
1.111	1.235
1.278	1.633

Figure 1. Side length and area varies.

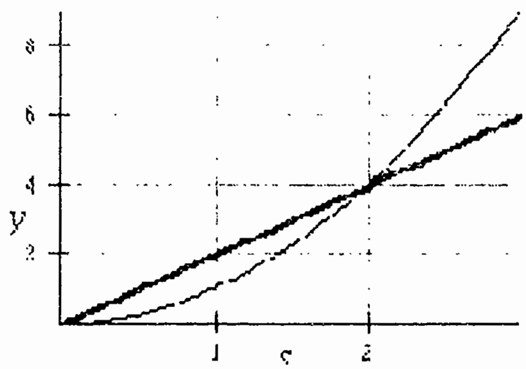


Figure 2. Graphs of A(s) and r(s).

and the Figures 1 and 2. The question to students was: "What is represented by the point on r's graph having coordinates (0.7667,1.635)?" Nineteen students participated in whole-class and small and small-group discussions of the question for 90 minutes in order to arrive at a consensus, both among the whole class and within small groups, as to for what (0.767,1.635) stood. At the end, all stated satisfaction that they all agreed on the meaning. Thompson then asked them to each, individually, write the interpretation with which they all agreed.

Thompson reported: "eight of 19 couldn't remember what their group had said before, that they couldn't reconstruct it, or they couldn't come up with an interpretation". With respect to the responses from the other eleven students he said: "First, none of the responses is internally consistent. Five are relatively close. Six responses are conceptually incoherent, entailing internally conflicting meanings. Second, no interpretation even remotely resembles those that they spent 50 minutes developing and to which they each expressed satisfaction that they had said what they intended".

He finally stated that the two aspects together, lack of internal coherence in students' interpretation and lack of agreement between private and public stated interpretations, points to a matter worth considering.

Adalira Sáenz-Ludlow (University of North Carolina at Charlotte): Interpretation, Representation, and Signification: A Peircean Perspective

The work presented by Sáenz-Ludlow is related to representation and semiotics under a Peircean point of view. Her primary questions were:

1. Is a representation a thing or a process?
2. Is a representation a dynamic process?
3. Is a representation a sign?

Her approach to discuss these questions was from a Peircean perspective. She quotes: Charles Sanders Peirce (1839-1914) considers that "*Semiotics* is the doctrine of the essential nature and fundamental varieties of possible semiosis. That is, strictly speaking, semiosis and not the sign is, for him, the proper of semiotic studies. For Peirce " a sign, or *representamen*, is something that stands to somebody

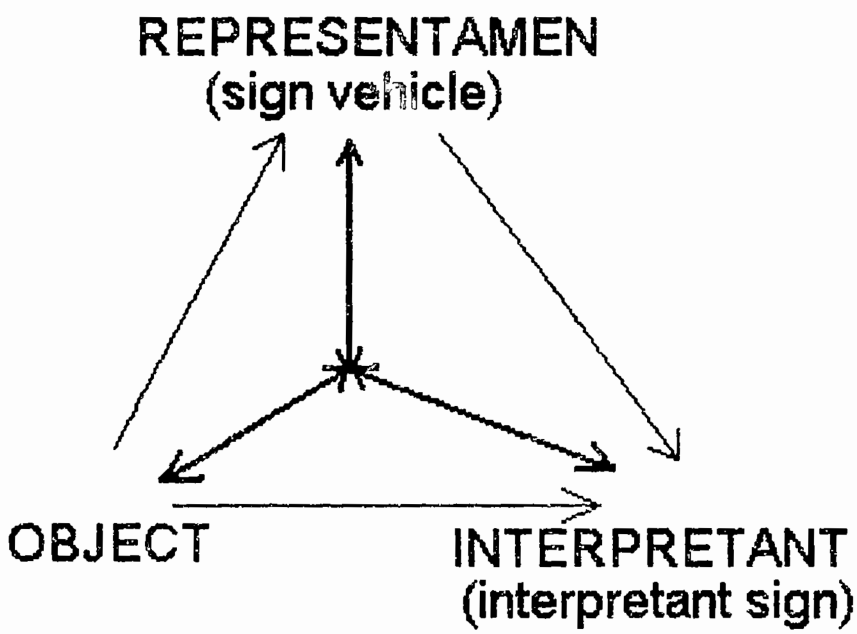


Figure 3. Triadic Sign Relations

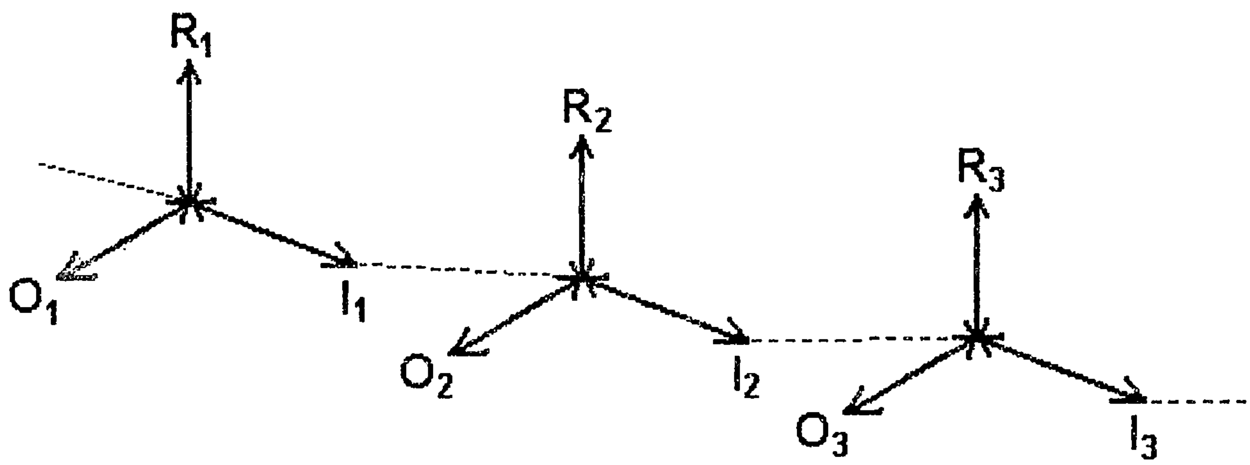
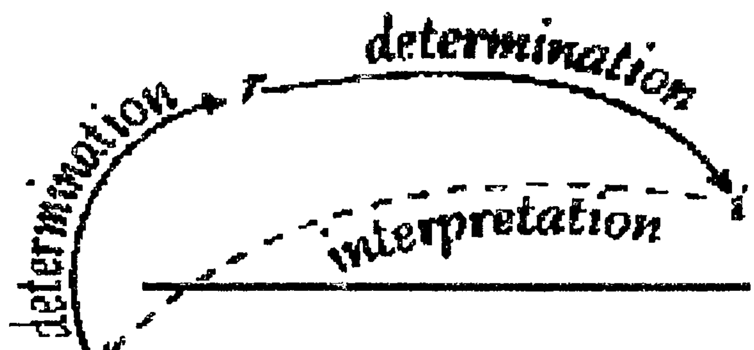


Figure 4

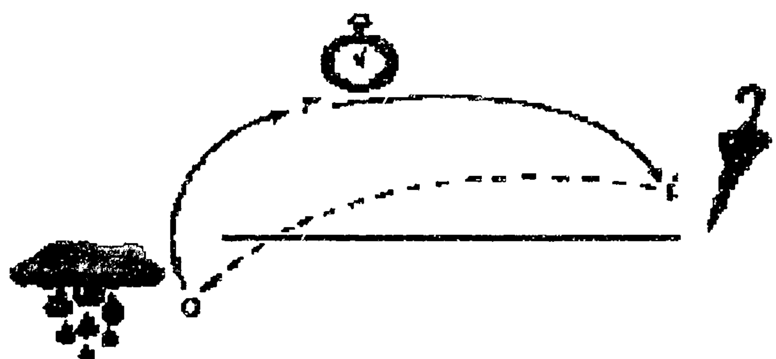
for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. It stands for that object, not in all respects, but in reference to a sort of idea" and "nothing is a sign unless it is interpreted as a sign. "

Figures 3 shows the Triadic Sign Relations presented by Sáenz-Ludlow from the Peircean perspective. Figures 4 shows from the same point of view chains of signification.

Figure 5. Example given by Whitson (1997, p.



Triadic sign relations (Peirce)



Triadic signification: Barometer example

101-102): *Suppose I took a barometer, say "Let's go," pick up my umbrella, and start for the door. You pick up your umbrella and follow. The barometer reading is being interpreted as a sign of rain (the object represented). It is functioning as a sign when it produces as its interpretant the event (me picking up my umbrella) in which the reading is interpreted as a sign of rain. That interpretant can, in turn, function as a sign of rain producing a subsequent interpretant (for example, you taking your umbrella). The two of us both leaving with umbrellas can function as a sign producing (as an interpretant) a co-worker's decision not to go out for lunch.*

Sáens-Ludlow continued with the Peircean perspective and quotes: Peirce defines semiosis as "the triadic action" of the sign in which a sign has a cognitive effect on its interpreter. Semiosis, for him, involves the cooperation of the three elements in the triad object-representamen-interpretant. Such "tri-relative" influence is not resolved by the isolated action between any of the pairs in the triad, but only by the synergistic action of all elements in the three way relationship."

Sáens-Ludlow finished her presentation giving a reflection about these questions:

1. Is it useful for us in mathematics education to think of a representation as a process that presents something other than itself?
2. Is it useful for us in mathematics education to think of a representation as a dynamic triadic process or as a dynamic dyadic process?
3. Is it useful for us in mathematics education to think of a representation as a sign?

Raymund Duval (Université du Lottoral Côte-d'Opale): Figures' Representational Funtion in Geometry and Figure's Multiple and Parallel Entries

The two sessions led by Duval (plenary and working group sessions) were based on theoretical perspective related to semiotic representations and mathematics visualization and specifically in the second one about figures and the learning of geometry. Five questions were at the core of his presentations:

1. Why should semiotic representations be taken into account in order to analyze the learning of mathematics?
2. How do semiotic representations work in mathematics?
   - Are some semiotic representations characteristic of mathematics?
   - In the case where some representations are similar to those within other areas, are they used in the same way?
3. Does the students' use of semiotic representations involve a difficulty that they need to overcome?
   - How can a student learn to recognize a mathematical object through its various representations?
   - How can a student learn to distinguish a mathematical object from any

particular semiotic representation?

4 Does visualization in mathematics work like ordinary iconic representations?

5 What kind of variables, connected with semiotic representations, must the study of learning in mathematics take into account?

The ideas discussed by Duval seem to be framed through a Sausserian approach rather than a Peircean perspective. His idea of register of representation is connected with the analysis of the productions of the students in a restricted system of representations. Duval (1993, p. 40) quotes: A semiotic system could be a representation register, when it permit three cognitive activities related to the semiosis:

1) The presence of an identifiable representation...

2) The treatment of a representation which is the transformation of the representation *within the same register where it has been formed ...*

3) The conversion of a representation which is the *transformation* of the representation into another representation of another register in which it conserves the totality or part of the meaning of the initial representation...

From a cognitive point of view, conversion (and not treatment) is the central process of mathematical thinking for three reasons:

- Related to Question 3 above, the distinction between the representation and the represented object is possible only when students become able to convert a representation of a mathematical object in another representation. It is the cognitive condition for transfer or for decontextualization.
- Some conversions are congruent: The representation in the starting register is transparent (see-through) to the new representation in the target register. But in most cases, conversion is not congruent. That leads most of the students to an obstacle or to compartmentalized understanding.
- They are factors which explain congruency or non-congruency of conversion. They depend on the opposite starting and target registers (Visual or not, Language or Symbolic) thus we can study experimentally these complex phenomena and we can define cognitive variables for the different areas in mathematics education.

On the construction of concepts Duval (Ibidem, p. 46) states that: "Every representation is partially cognitive with respect to what represents" and then: "The understanding (integral) of a conceptual contents based on the coordination of at least two registers of representation, and this coordination is revealed by the rapid use and spontaneity of the cognitive conversion." The last paragraph, as Duval quotes, needs another description of the structure of the semiotic representation and his performance (see Figure 6 [Duval, 1993, 1995] and 7 [Duval, 1999]).

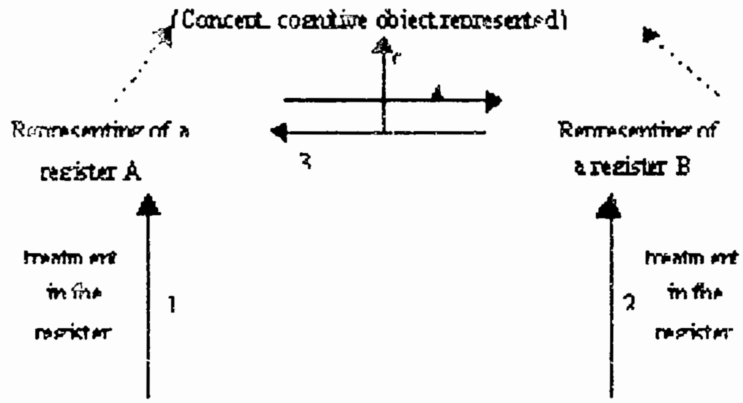


Figure 6. Duval 1993, 1995

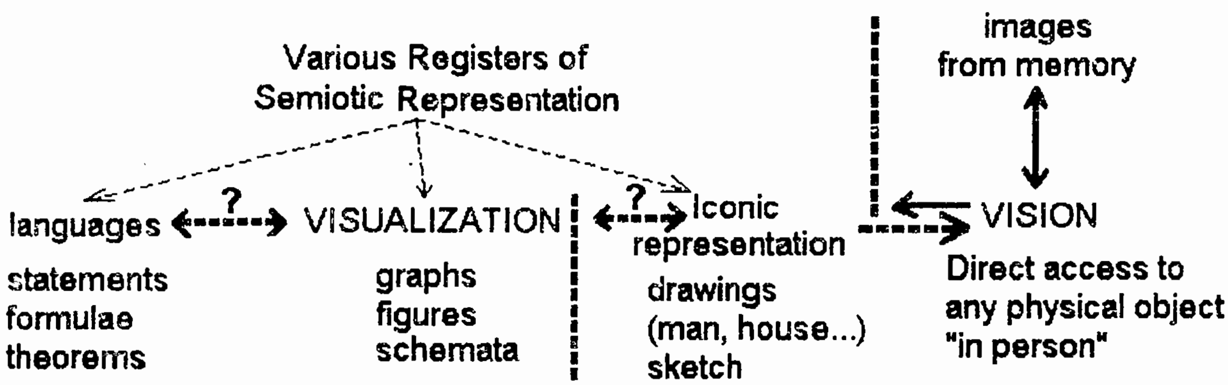


Figure 7. Duval 1999

On the construction of concepts Duval (Ibidem, p. 46) states that: "every representation is partially cognitive with respect to what it represents" and then: "The understanding (integral) of a conceptual content is based on the coordination of at least two registers of representation, and this coordination is revealed by the rapid use and spontaneity of the cognitive conversion."

The last paragraph, as Duval quotes, needs another description of the structure of the semiotic representation and his performance (see Figure 6 and 7).

Abraham Arcavi - Nurit Hadas (Weizmann Institute of Science): Computer Mediated Learning: An example of an approach

Arcavi's presentation addressed directly a theme of our working group: The influence of technology-based multiple linked representation in the students' construction of mathematical concepts and also in regard to the role of visual thinking (see Hitt, 1998).

In his presentation Arcavi showed clear advantages in the use of dynamic computerized environments. He illustrated, via an example, important components of mathematics activities.

- *Visualization*. Arcavi and Hadas cited Hershkowitz (1998, p. 75) to characterize visualization: "Visualization generally refers to the ability to represent, transform, generate, communicate, document, and reflect on visual information." And they added, visualization not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution.
- *Experimentation*. Playing with dynamic environments allows students to learn to experiment, and to appreciate the ease of getting many examples…, to look for extreme cases, negative examples and non stereotypic evidence… (Yerushalmy, 1993, p. 82).
- *Surprise*. The challenge is to find situations in which the outcome of the activity is unexpected or counter intuitive, such that the surprise (or puzzlement) generated creates a clear disparity with explicitly stated predictions.
- *Feedback*. Surprises of the kind described above arise from a disparity between an explicit expectation from a certain action and the outcome of that action. The feedback is provided by the environment itself, which re-acted as it was requested to do.
- *Need for a proof and proving*. Following a surprise, many students may require a proof, maybe not explicitly, but by demanding from others or from themselves an answer to their 'why' (or 'why not').

The Problem Situation Arcavi and Hadas Analyzed was in Two Phases

First phase. Two segments of length 5 with a common end point. Joining the two other end points, produces an isosceles triangle (see Figure 8). The dragging of, for example, the vertex C, yields many possible isosceles triangles whose equals sides are 5.

Question: Predict the shape of the graph of the area of that triangle as the function of its base.

In order to check the prediction, they asked students to make use of the software to draw the graph.

The problem situation -Second phase. Following again the "what if?" approach, the question became: "So far, we explored isosceles triangles where the equal sides have a fixed given value. What would happen if we make a 'small' change, so that the

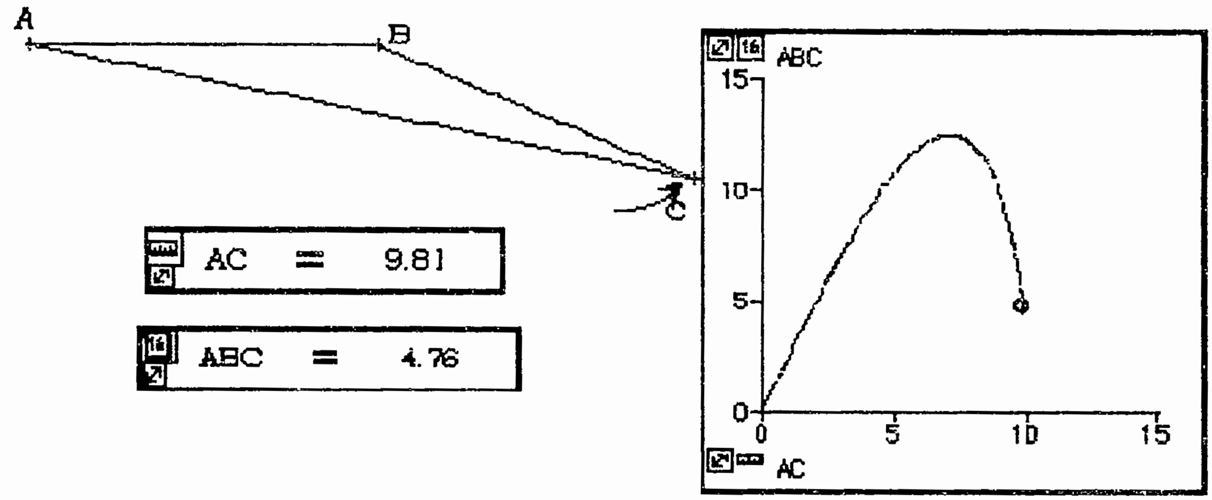


Figure 8

triangle is not isosceles, but "close to being one?"

On the basis of the above tasks and the experience with them, Arcavi and Hadas made a reflection about (see large version of the Working Group on Representations):

1. The role of computerized tools
2. Mathematics and mathematical activity
3. A new way of thinking.

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WORKING GROUP: USING SOCIO-CULTURAL THEORIES IN MATHEMATICS EDUCATION RESEARCH

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Focus and Aims of the Working Group

This Working Group began as a discussion group at PME-NA 1997 and first met as a Working Group during PME-NA 1998. Socio-cultural theoretical perspectives have been used to frame research on learning and teaching mathematics (for some examples see Educational Studies in Mathematics, September 1995 Special Issue). There are multiple interpretations of what socio-cultural perspectives say about learning and teaching and how these phenomena can be studied. The aim of this Working Group is to present and discuss different interpretations of socio-cultural perspectives and different applications of these perspectives to research questions in mathematics education.

The central goal of the Working Group is to develop a shared sense of the contributions that socio-cultural theories can make to research in mathematics education by

a) discussing research conducted using these theoretical perspectives and

b) by analyzing sample data using concepts from these perspectives.

During the three sessions the participants will discuss research conducted using socio-cultural theoretical perspectives, analyze sample data using concepts from these perspectives, and discuss a selected reading. These activities are intended to support participants in a) clarifying which specific versions, aspects, or concepts of socio-cultural theories are being invoked in different research, b) questioning key analytical concepts, and c) exploring which aspects and concepts can be useful for framing further research on learning and teaching mathematics.

The activities and discussion will address several ways to apply these perspectives to research design, data analysis, curriculum development, and teacher professional development. The anticipated follow-up activities for this Working Group include planning for a continuation of the Working Group at PME-NA 2001 and organizing collaborative writing projects on this topic.

Session 1

1) Introduction and overview of the Working Group

2) Two brief (5-10 minutes each) presentations by panel members providing

examples of how they have used socio-cultural theories in their research. The purpose for these short presentations is to provide examples of how socio-cultural theories have been applied and show several different perspectives in a structured way.

3) Participants will discuss a segment of videotape data using a variety of socio-cultural perspectives, sharing their own experiences in data analysis as part of the discussion.

Session 2

1) Two brief (5-10 minutes each) presentations by panel members providing examples of how they have used socio-cultural theories in their research.
2) Discussion in small groups of a selected reading. The reading will be available on the first day of the Working Group.

Session 3

1) Two brief (5-10 minutes each) presentations by panel members providing examples of how they have used socio-cultural theories in their research.
2) Discussion in small groups focusing on the following questions: a) What aspects of socio-cultural theories have participants used in their own mathematics education research? b) What are the different socio-cultural perspectives? and c) Are there common characteristics in studies from socio-cultural perspectives?

Questions for Presenters:

1) How have socio-cultural theories informed your research project(s)?
2) What specific aspects (concepts, methods, etc.) from socio-cultural theories have you used in your research?
3) In what areas of your research were socio-cultural perspectives most useful (research design, data analysis, curriculum development, teacher professional development, etc.)?
4) How has your work extended or expanded socio-cultural concepts?
5) Which concept from socio-cultural theories do you find most puzzling? Most useful? Most misunderstood?

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Advanced Mathematical Thinking

PORTRAIT OF THE CONSTRUCTION OF MATHEMATICAL KNOWLEDGE: THE CASE OF A DEAF AND BLIND UNIVERSITY STUDENT

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Abstract: This study describes the construction of mathematical knowledge of a university student who is blind and deaf. Of particular significance is the issue of mathematical representation both in terms of representing what the teacher was talking about and providing a medium for the student to represent her knowledge. An equally important issue concerned understanding symbol systems and the degree to which Braille text facilitated that understanding. For the teacher, the issue of defining and assessing mathematical understanding in each of the courses was particularly salient. By examining these issues in the case of this unique individual, we may gain new insights into the construction and organization of mathematical knowledge for all learners.

Theoretical Framework

This study, which is set within the context of university-level pre-calculus and calculus courses, is part of a larger study that spans approximately two years in which one teacher worked with the student in the following courses: 1) pre-calculus; 2) calculus (scientific); 3) discrete mathematics; and 4) statistics. Thus, the mathematics discussed represents a transition to advanced mathematical thinking in which "the mind has simultaneous concept images based on earlier experiences that interact with new ideas based on definitions and deductions" (Tall, 1992, p.496). In particular, the mathematical discussions described in the study reflect an approach that builds on concepts that are familiar to the student and provide the basis for later mathematical development. Thinking about and communicating mathematical ideas requires some means to represent them. Communication between student and teacher certainly requires that the representations be external, taking the form of spoken language, written symbols, pictures, or physical objects. Given the perspective that "the nature of the internal representation is influenced and constrained by the external situation being represented" (Kosslyn & Hatfield, 1984), it seems reasonable to assume that "the form of an external representation (physical materials, pictures, symbols, etc.) with which a student interacts makes a difference in the way the student represents the quantity or relationship internally" (Hiebert & Carpenter, 1992, p.66). "Conversely, the way in which a student deals with or generates an external representation reveals something of how the student has represented the information internally" (Hiebert &

Carpenter, 1992, p.66). However, in the case of a deaf/blind student, the choices of external representations are limited. This study describes the various manipulatives and physical objects that were created and utilized to facilitate mathematical communication in which spoken language, written symbols, and pictures were not available. In particular, with respect to the issue of representation, this study provides a careful examination and analysis of how the teaching and learning process interact

Data Source

The data source for this study consists of a 24-year old university student, Beth, who is deaf and blind. She was born deaf and became blind when she was quite young, at approximately three years of age. She has some memories of colors and objects and well-developed iconic knowledge of many common objects.

At the university, the ADA (Americans with Disabilities Act) office coordinated all of the support staff and efforts on behalf of Beth. Communication was possible through a team of tactile interpreters, who communicated with Beth through a tactile form of American Sign Language. Whatever was said or written, including diagrams and graphs, was communicated as accurately as possible to Beth. During class, when she volunteered an answer to a question, she signed her response to the interpreter and the interpreter vocalized her response.

Beth was assigned to my pre-calculus class in the fall of 1998. At that point in time, I had no specific training in teaching students with any type of special needs. For all of the subsequent courses, I served as her tutor, meeting 3 hours per week in a one-to-one setting with the interpreters. The rationale behind Beth's class attendance was that she should receive the information in "real time" just like the other students. However, since she had no way of taking notes and creating her own record of what she deemed important, her attendance was similar to a hearing and sighted person attending a general interest lecture. Certainly, she remembered generally what was discussed in the class but usually could not provide the details.

For the first course, pre-calculus, my notes were transcribed and sent to Beth via email with descriptions of graphs provided. For the two subsequent courses, both the calculus and discrete texts were translated into Braille. Beth's equipment, both at home and at the university (in the library), allows her to read an 80-character line of text in Braille. The equipment does not support equations, tables, pictures, diagrams, or graphs.

For this reason, even the class notes had to be re-written in a more descriptive and less symbolic mode. Thus, faced with these constraints, it was essential to provide concrete representations for the equations and graphs so that Beth did not have to retain everything in memory.

Methods of Inquiry

From the beginning of the experience, I kept detailed notes of all the mathematical discussions that took place during the individual meetings. These notes were edited and transcribed. Several of the sessions were video taped and these were also transcribed. In addition, several of the other people involved were interviewed. These include six different interpreters, the ADA coordinator, Beth, and her mother. These interviews were conducted to provide background information and to augment the existing data.

Throughout this work, many manipulatives and concrete objects were created to facilitate the mathematical communication. These have been documented, photographed, and used in some of the videotapes, to illustrate how they were used. In addition, after the fact, I wrote commentaries describing how the manipulatives were chosen and created. For the most part, these manipulatives were created, as they were needed, to support the discussion of a particular concept or to facilitate a useful representation. I used magnetic letters and numbers and a magnetic board to represent equations. Given that the intended use for these sets is certainly not typical pre-calculus or calculus expressions and equations, I supplemented these with pipe-cleaner symbols. For graphs, we began with a small geo-board and rubber bands but that soon proved to be too restrictive. I created the x-y plane with push-pins on a large cork board (approximately 24 by 36 inches) and used rubber bands and additional push-pins to create the graphs. This device for representing and analyzing the graphs of functions served for pre-calculus, calculus, and statistics (regression).

As a participant-observer, the inquiry was shaped by a *constructivist interpretation* methodology (Noddings, 1990) with each stage of the inquiry iteratively informing the next. Although the notes and commentaries initially provided a simple record of the instruction, it was clear that they also documented information about issues of representation and the mathematical connections between representations. From these notes and commentaries, the videotapes, and the interviews, it is possible to describe the construction and organization of mathematical knowledge in the context of these courses for this unique individual.

Results and Discussion

Writing a linear equation given two points or one point and the slope is a standard task in pre-calculus and calculus. From my teaching experience, I have always believed that the point-slope form is generally easier for students to understand and manage with the fewest opportunities for careless calculation errors. However, during discussions about writing linear equations given two points, I found that Beth could not seem to keep track of all the pieces of the point-slope form of the equation but experienced success with the slope-intercept form. Regardless of the numbers, she always used the slope-intercept form and easily performed whatever calculations were necessary to correctly determine the y-intercept. Beth's preference for the slope-intercept form reflects a far more interesting issue of knowledge organization than simply a preference based on previous experiences. For a sighted student, writing out the point-slope form, replacing the slope and point with specific values and simplifying the result to the slope-intercept form is based on following the pattern in the external representation of the equation. The representation serves both as a record of the ongoing work and a cue for the subsequent steps. However, for a non-sighted student, the point-slope representation requires internally keeping track of several numerical and variable expressions as well as correctly combining the appropriate expressions. It is much easier to remember to multiply the value of the slope by the value of the x-coordinate of the point and then subtract this result from the value of the y-coordinate. Once this result is obtained, it is a fairly simple task to assemble the requisite pieces into the equation of the particular line.

Being able to determine equations of linear functions does not, of course, imply that one understands the nature of linear functions and in particular their graphs. Pre-calculus courses today include a significant emphasis on understanding functions and their graphs and frequently utilize graphing utilities to explore the properties of graphs. Given this perspective, it was important that Beth be able to recognize, create, analyze, and understand the properties of the graphs of the various classes of functions discussed in pre-calculus. I would create the graph on the corkboard and Beth would explore it by touch. However, since graphical understanding is one of the major objectives of pre-calculus, frequently Beth would simply explore the rubber-band graph and we would discuss it.

In discussions about the intervals over which a polynomial graph was increasing or decreasing, Beth exhibited a typical error frequently made by sighted students. Suppose that a function is decreasing over the interval from 5 to infinity. Because the graph is decreasing, students report the interval as 5 to minus infinity. Beth made the same mistake. For a sighted student, one might conjecture that the visual cue of decreasing is so salient that the brain has difficulty focusing on the interval over which these phenomena occur. However, with a non-sighted student, there is no visual cue, only a tactile one. This similarity suggests that a decreasing function over an interval from some value "b" to infinity may create sufficient cognitive dissonance to cause confusion, which leads to the error.

Conclusion

These examples provide only a brief glimpse of how Beth's mathematical knowledge was constructed and organized. They do illustrate, however, some of the representational issues that influenced the knowledge organization. In the case of determining the linear equation, Beth's ability to execute a well-known procedure did not rely upon the representation that requires the least substitution. Rather, the salient representation for her was the one that required the fewest calculations. This choice also indicates an efficiency that is characteristic of knowledge that is part of a network of representations. In the case of the decreasing polynomial function, it seems that both a visual and a tactile representation lead to the same misconception. In this case, the means of accessing the representation are independent of the conclusions that are generated from the representation. It would also seem reasonable that the cognitive dissonance is created by the graphical representation. Thus, for Beth as well as for sighted and hearing students, there is evidence that the internal representation is very definitely influenced by the external representation. These findings provide insights into the interaction between the teaching and learning process and how representations contribute to our mathematical understanding.

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METHOD OF COGNITIVE SYMMETRY: MECHANISM OF INTERRELATIONSHIP BETWEEN EXTERNAL AND INTERNAL REPRESENTATIONS

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Abstract: We consider *representational thinking* as the learner's ability to interpret, construct, and operate (communicate) effectively with both forms of multiple representations, external and internal, individually and within social situations. The key research problem of this paper is how can we develop students' representational thinking more effectively and what would be a sequence of instructional steps that we can follow to improve students' mathematical understanding using multiple representations? Our theoretical and practical findings show, that this problem might be resolved using the *method of cognitive symmetry*, a mechanism of designing a holistic process of developing students' representational thinking, that involves a symmetrical interaction of internalization and externalization of multiple representations.

It is now well accepted that use of particular modes of representation (for example, visual/spatial) leads to improvement of students' mathematical abilities and development of their advanced problem solving and reasoning skills (Krutetskii, 1976; Presmeg, 1997; Tchoshanov, 1997; Yakimanskaya, 1991; Zimmerman, Cunningham, 1990). At the same time, in the fields of psychology, pedagogy, and mathematics education there is an ongoing debate about *how the mind operates with representations*. On one hand, some researchers (Arnheim, 1969; McKim, 1972) believe that the problem of development of students' thinking is directly connected to the problem of operating with mental images (e.g., seeing, imagining, and idea-sketching). On the other hand, the advocates of a "picture" theory of representation (Mitchell, 1994; Wileman, 1980) argue that there is no difference between external and internal (mental) representations: a mental representation is equivalent to what it represents. Furthermore, based on their critique of the "picture" assumption, constructivists reject the "representational view of mind" (Cobb, Yackel, & Wood, 1992) claiming that, translated into instructional practices, a representational view begins with experts' conceptions and attempts to reproduce these ideas by students rather than allowing them socially co-construct meanings of mathematical representations.

All this debate does not contribute to the clarification of the relationship between external (e.g., pictures, diagrams, graphs, charts, 3-D models, computer graphics, etc.) and internal (e.g., mental images, mind maps, webs and hierarchies, schemata, semantic nets, etc.) representations and their role in the development of students' representational thinking. In this paper we advocate the position that the development of students' representational thinking is a two-sided process, an interaction of internalization of external representations and externalization of mental images. We also assume that there is a mutual influence between the two forms of representations: the nature of an external representation influences the nature of the internal one and vice versa. Finally, we argue that representation is not an end in itself but a vehicle for understanding; and the construction of an understanding of representations is an individual as well as a social activity. Therefore, we consider *representational thinking* as the learner's ability to interpret, construct, and operate (communicate) effectively with both forms of representations, external and internal, individually and within social situations.

The educational significance of this conceptualization is in presenting an alternative holistic approach to representational thinking development through construction of students' understanding (internalization) and improvement of their creativity (externalization). Unlike previous studies (Herscovics, 1996; Hiebert, & Carpenter, 1992), that paid attention primarily to the internalization stage, the role of representations in the development of students' understanding, this approach is characterized by its completeness and orientation towards creativity through understanding. We firmly believe that, in the development of students' representational thinking, internalization without externalization is non-holistic and incomplete. The process of interrelationship of internalization and externalization we call *cognitive representation*. The important point here is that despite a tacitly accepted one-sided view of internal representation as a cognitive one (Seeger, 1998), we consider cognitive representation as a zone of interaction of external and internal representations. From our perspective, cognitive representation reflects both the process (internalization) and the products (externalization) of representational thinking.

This conceptualization is further based on recent findings in theory of cognition and brain investigation (Caine, & Caine, 1994; Chabris, & Kosslyn, 1998; et al.). According to these studies, the brain works more effectively while making representational patterns for encoding (internalizing) and decoding (externalizing) information. Unfortunately, as opposed to the varied and complex patterns generated in the human brain, most mathematical content offered to students is basically presented in linear abstract/symbolic forms. However, the patterning capacity of the human brain more closely resembles non-linear representational network forms. It seems reasonable that the language of the brain is a multiple representational language. Therefore, the improvement of students' brainpower requires a development of their representational thinking.

This approach is also built on a number of previous studies done in the field of using multiple representations in teaching and learning of mathematics (Greeno, & Hall, 1997; Lesh, Post, & Behr, 1987; Presmeg, 1997; Wheatley, 1997; et al.).

However, there are still a lot of open questions in the theory and practice of development of students' representational thinking. Some of them are:

- How may different modes of representations, sequences and translations among/ within them support the development of students' mathematical understanding?
- What relationships and connections exist within/between external and internal representations?
- Why are particular modes/types of representation sometimes ineffective in developing students' mathematical understanding?

These open questions led us to formulate and solve the key problem: how can we develop students' representational thinking more effectively and what would be a sequence of instructional steps that we can follow to improve students' mathematical understanding using multiple representations? Our theoretical and practical findings show that this problem might be resolved using the method of cognitive symmetry. In the context of this problem, the *method of cognitive symmetry* is a way of designing a holistic process of developing students' representational thinking. The holistic process involves a symmetrical interaction of internalization and externalization of representations, we call *a cognitive cycle.*

In understanding the nature of the internalization process, we adhere to the Vygotskian conception of mediation. L. Vygotsky and his supporters argue that determination of individual cognition might be presented by the following scheme: collective (social) activity – culture signs/symbols – individual activity – individual cognition (Vygotsky, 1996; Leont'iev, 1978, et al.). In the framework of developing learners' representational thinking we consider the Vygotskian scheme as a basis for designing a cognitive cycle. The importance of the method of cognitive symmetry is that it gives a clue for designing the structure of the externalization process based on the scheme of the internalization one. So, the holistic cycle of students' representational thinking development includes the sequence of the following stages: first, a potential representation transfers to a standard/conventional representation through communication and interpretation; then students construct their internal representation and express it externally as a non-standard/idiosyncratic representation; and finally, they create/generalize a meta-representation through reflection and abstraction (see figure 1). We must make it clear that if internalization is primarily a guided zone, externalization is basically an independent domain of students' activity. Furthermore, if internalization aims at understanding (e.g., seeing, comprehension, interpretation, etc.), externalization tends toward creativity (e.g., construction, generalization, abstraction, etc.). Therefore the key proposition of our vision is that effective methodology for developing students' mathematical representational thinking might be designed using the method of cognitive symmetry that reflects an interaction of the internalization and externalization processes of representations.

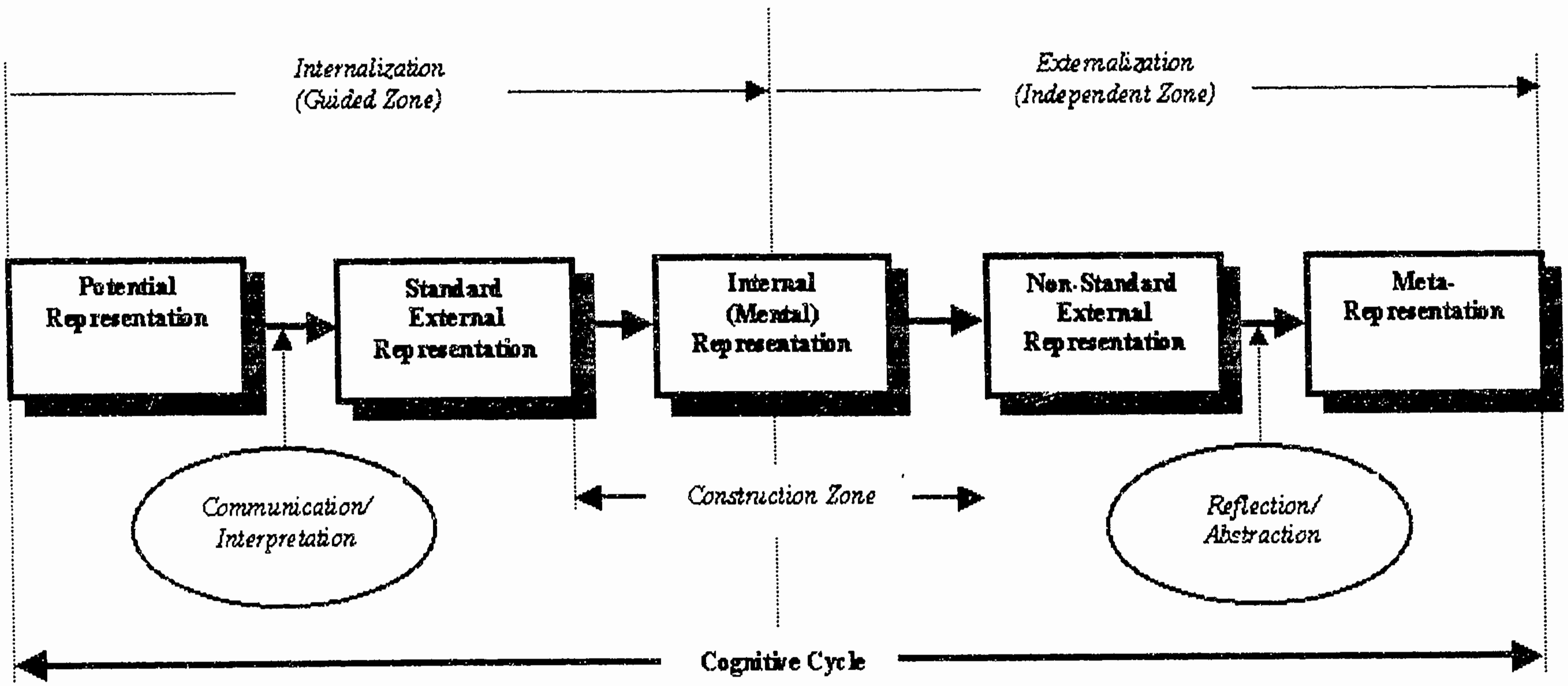

Figure 1. Method of Cognitive Symmetry in Developing Students' Representational Thinking

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Below we briefly discuss some outcomes of the pilot experiment we conducted in 1995-96 with Russian high school students on trigonometric problem solving and proof using method of cognitive symmetry along with multiple representations (Figure 1). The first comparison group of students ("pure-analytic") was taught by a traditional analytic (algebraic) way of trigonometric problem solving and proof. The second comparison group ("pure-visual") was taught by a visual (geometric) way using enactive (manipulative aids) and iconic (pictorial) representations. The third experimental group ("representational") was taught by a combination of analytic and visual ways using translations among different representational modes. The representational group scored 26% higher than the visual and 43% higher than the analytic groups.

This experiment also showed that students in the "pure" (analytic and/or visual) groups "stick" to one particular mode of representation, they were reluctant to use different representations. For instance, students in the pure-visual group (even though there was a random distribution of students with abstract/analytic and visual/geometric preferences of reasoning) tried to avoid any analytic solutions: they were "comfortable" if and only if they could use visual (geometric) problem solving and proof techniques. It means that students are getting used to the particular mode of representation that they were previously taught. Students in the representational group were much more flexible "switching" from one mode of representation to another in search of better understanding of mathematical concepts. We realized that any intensive use of only one particular mode of representation does not improve students' conceptual understanding and representational thinking. Another important outcome of this experiment is that students in the representational group gained more conceptual understanding while using analytic representation before visual one, as well as using more generalized representation before concrete one. The final key issue of the experiment is that the development of students' representational thinking cannot be effective enough if it is not supported by using generalization processes (meta-representations) in mathematical problem solving and reasoning.

The main focus on conducting this experiment using the method of cognitive symmetry was improvement of students' representational thinking in the following contexts:

- students' exploration of alternative ways of mathematical problem solving, inquiry and reasoning;
- students' construction and co-construction (social aspect) of non-standard multiple representations of problem solving and proof techniques;
- involvement of students in a variety of hands-on and minds-on activities (e.g., modeling, drawing, imagining, mapping, etc.) in the process of generalization of mathematical concepts and ideas;

- students' understanding of harmonic relationship between different forms of multiple representation of mathematical knowledge.

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STUDENTS' CONCEPTIONS OF SOLUTION CURVES AND EQUILIBRIUM IN SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract: University courses on differential equations are being reconsidered and reformed, building on the previous reform efforts in calculus. In this paper we present a detailed analysis of semi-structured interviews where 18 students faced problems related to the solution of systems of ordinary differential equations, presented in different settings. We classify students' strategies and show many instances where students' understanding of parametric functions and variation conflict and become an obstacle to their understanding of the meaning of phase space representation and the notions of solution and equilibrium. We also highlight some cognitive and curricular issues that may be taken into account when dealing with these types of problems.

Introduction and Theoretical Framework

The teaching of differential equations at the collegiate level has been questioned in the last years (Artigue,1992; Ramussen, 1998; Trigueros, 1993, 1994; Zandieh and McDonald, 1999). The use of technological support, the development of Dynamical Systems as a growing branch of mathematics and the movement of reform in the teaching of Calculus have raised new concerns about the topics that students should learn nowadays. The decisions taken in this direction would benefit a lot from research on students' conceptions about differential equations, about the strategies they use when facing problems related to differential equations and about the way students deal with complex problems like those appearing in the contexts of systems of differential equations.

The problem of description of equilibrium and solution curves in systems of differential equations is a complex one. It involves the integration of knowledge from different parts of mathematics and the use of different representation tools. This complexity makes the analysis of students' conceptions and strategies a challenging task. Taking into account previous work on differential equations (Artigue,1992; Ramussen, 1998; Trigueros, 1993, 1994; Zandieh and McDonald, 1999), on functions and derivative (Breindenback, 1992; Monk, 1992, Trigueros y Cantoral, 1992, Zandieh, 1997) and on the integration of information when students deal with complex problems. (Baker, Cooley and Trigueros, In press), this study uses the notion of schema from APOS theory (Asiala et al. 1996) in order to understand students' conceptions of solution curves and equilibrium solutions in the context of systems of differential equations.

For that purpose the idea of a double-triad of schema development, as proposed by Baker et al. (in press), was introduced in the context of this problem. Experience from teaching and from previous students' work in different courses on Differential Equations suggested that two main schema might be used in understanding the way students deal with problems related to the systems of differential equations: a schema for the representation of functions in parametric form and a schema for systems of differential equations. A double-triad containing these two schema was thus developed and used in the design and analysis of students' interviews with the aim of understanding students' conceptions and strategies. It is important to remember that the nature of the stages involved in the development of a schema is functional and not structural.

The development of the parametric representation of functions can be described by means of the following stages or levels. At the Intra-parametric level, the student interprets the parametric representation of functions as two isolated functions; the possibility of elimination of the parameter and the representation of a function in a two dimensional plane, where the parameter is not explicitly shown, causes confusion. At the Intra-parametric level, some relationships between the components of the function are found, but the difficulty of using and interpreting graphs for this function persist. At the Trans-parametric level, students are able to describe the function and its different representations in terms of the parameter involved. Coherence of the schema is demonstrated by the student's ability to describe which parametric representations are possible for a given function and how they relate to different graphical representations.

The development of the schema for the solutions of systems of differential equations can be described by the following stages: At the Intra-solution level the student is able to solve a system but is unable to interpret it and to coordinate the system and its solution with its graphical representations. At the Inter-solution level, the student begins to relate some systems with their graphic representation and to interpret the meaning of the solution to some systems, but it is not clear to him or her when a particular representation is convenient or even possible for many systems of equations. At the Trans-solution level, students are able to solve, interpret and describe graphically different systems of differential equations. Coherence of the schema is demonstrated by the student's ability o describe which systems can be solved using analytic methods and which graphical representations are possible depending on the system.

Methodology

Data were collected from two Differential Equations classes at a small private university. One of the courses was taught to 34 Applied Mathematics students and the other to 37 Economics students. Both groups were taught by the author. During the semester, three individual task centered semi-structured interviews were conducted with nine students of each group. Interviews covered only topics covered in class. The interviewer was another researcher not involved in this particular project. The interviews were audio-taped and transcribed, and all the work done by the students was collected. The analysis of the data was discussed with another mathematics educator who also taught courses on Differential Equations.

For this particular project, we selected the following particular tasks to be analyzed: 1) Given a mathematical model that describes the growth of two populations by means of a predator-prey model, and given a a plot that shows two solution curves in a phase plane for different initial conditions, draw the plots that show how each of the populations grow in time and interpret the solution; 2) Given a different mathematical model given by two linear and autonomous equations, solve the system of differential equations, draw the phase plane representation and interpret the solutions found in terms of their representation in the phase plane; 3) Given the x vs. t and y vs. t plots of the behavior of the solution to a model represented by a system of differential equations with different initial conditions, represent the solution curves in a phase plane and interpret the meaning of different points on those curves in terms of the given model.

The focus of the analysis was students' conceptions of solution curves and equilibrium solutions on the one hand and their strategies on the other.

Some Results

The analysis of the interviews showed that the interaction between the schema mentioned earlier can help to understand most of the difficulties of the students. It was found that students have problems interpreting the meaning of equilibrium solution, some of them thought of it as a point in phase space but forgot about variation with time. Some students in both groups could not see the dependence of time in the phase space representation and even considered each differential equation to be a separate equation and the phase space representation as a means to compare the solution of both. Other students were able to recognize the system as such and even to solve it, but when faced with the interpretation of the solution curves in phase space and asked to draw the plots of the dependent variables versus the independent variable, showed confusion because the independent variable did not appear explicitly in the phase plane equations (intra-parametric, intra-solution level). Many students could solve the linear system and interpret correctly the curves in a phase plane, but had trouble justifying their explanations and explaining why they chose a particular method or strategy (inter-parametric, inter- solution level). Most of these students had trouble when the system of equations was not autonomous and/or when it was non linear, but their difficulties could be due to a lack of understanding of the geometric representation of parametric equations (intra-parametric, inter-solution level) or a difficulty in discerning when a particular representation was possible for the problem (trans-parametric, inter-solution level). The results regarding the meaning of equilibrium solutions were consistent with those of Ramussen (1998) and Zandieh and McDonald (1999) who worked with the meaning of a solution to a differential equation.

The meaning of a point in phase space proved to be a problem to most of the students in both groups. They were not able to see it as a representation of the state of an autonomous system at a particular time even when they were able to solve the system. However, their difficulties could also be differentiated as related to their conceptions of parametric functions or to their conceptions of a solution to the system of equations. The construction of phase space, the interpretation of solution curves and the relationship between the graph of the solution in a phase plane and in a configuration plane proved to be a very difficult task for many students, even when they could handle tangent fields for ordinary differential equations.

Students showed some specific tendencies to use particular strategies to make sense of the proposed tasks. For example, many students would start by reading values from a curve represented in a phase plane to make a table of values, but most of them were not then able to draw the other plots or to interpret the solution because they could not see the dependence with the independent variable on the phase plane plot. Other students would first try to solve the equations and then use the solution to construct the graphical representation and to interpret the solutions; within this group of students some tried to use this strategy even if the system was not linear. A very frequent strategy for the task where the solutions in a phase plane were given was to start from the equations and reconstruct the given phase plane from interpretation of the signs of the derivatives in the system and the use of nullclines; but even when they were able to do it, the students showed difficulties in interpreting the solutions and using a different graphical representation for the task. In the construction of a phase plane, a very common difficulty was to relate a zero derivative for one of the variables with a vertical arrow on the phase plane. Two students used the idea of a car traveling along the curve in the phase plane or walking on the curve and centered the attention to the change of the variable shown in the horizontal axis as the moving object goes along the curve; this strategy helped them make sense of the independent variable for one of the dependent variables, but they were unable to use the same procedure for the variable shown in the vertical axis of the phase plane.

Conclusions

As a result of this research project it was found that the integration of different concepts and techniques is difficult for many students. The generalization of concepts and strategies used to deal with ordinary differential equations to systems is not direct and may need to be made explicit in class.

We were able to find different strategies that students use to make sense of solution curves. These strategies can be helpful to design new ways to work in class with students to help them develop a richer schema for the process of solution and interpretation of solutions of systems of differential equations.

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VISUALIZING RATES OF CHANGE FOR WATER TRAJECTORY: A CASE STUDY WITH PRE-CALCULUS AND UNDER-GRADUATE STUDENTS*

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This is an interpretative study of two current issues in the psychology of mathematics education. A first issue was to obtain further understanding on visualization that undergraduate students manifest when asked to imagine and draw the water trajectory that constantly flows from the end of a hose pipe. The second issue concerned how students' conceptions of vertical and horizontal changes in addition to the rate of both changes were influenced by their visualizations of water trajectory. Both issues rose from previous research (Balderas, 1992). This research guided instructional designs for beginning and advanced calculus courses (Balderas, 2000; Balderas & Schäfer, in preparation). The study was based on visualization, problem solving and rate-of-change learning theory (Presmeg, 1997; Zimmerman and Cunningham, 1991; Schoenfeld, 1992; Speiser and Walter, 1994; Thompson, 1994 and 1999; between others). Students' visualizations of the water trajectory as a global and completed phenomenon were centered on numerical data provided by the problem statement. Their answers of rate of change were restricted to those visualizations.

*Note. This study was developed as part of Balderas, P. Visualization in Rate of Change Problem Solving. Research project supported by CONACYT, Mexico, scholarship #145500, for a sabbatical leave at Florida State University, 1999/2000.

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COGNITIVE LINKS IN UNDERSTANDING DERIVATIVE: THE CASE OF HELEN

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In this session, I will present a theoretical framework for analyzing aspects of the cognitive linkages between conceptual knowledge and procedural knowledge (hereinafter referred to as CK and PK). This framework builds on previous research regarding CK and PK of topics in arithmetic (e.g., Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986; Silver, 1986) and extends these studies by examining relationships between one calculus student's CK and PK of derivative.

My theoretical model consists of 3 components: *Associativity*, *Stability*, and *Directionality*. The first two components address the number and strength of the linkages between CK and PK, respectively. The third component addresses Silver's (1986) view that in some settings CK is constructed from a foundation of PK, but in other settings, PK rests on a base of CK.

I will present a method of diagram analysis for interpreting data. I will also explain how my model is a useful analytic and descriptive tool for examining three attributes of the cognitive connections between conceptual and procedural understanding of derivative.

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SOME BELIEFS ABOUT PROOF IN COLLEGIATE CALCULUS

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In this study I compare some of the beliefs that students and their teachers hold about proof in the context of collegiate calculus. Proof, interpreted broadly, refers to any sort of justification that an individual finds convincing. The subjects were five first semester calculus students and five teachers at a large research university. The subjects were asked to prove that the derivative of an even function is odd. Then, in an experimental approach similar to that used to study beliefs about proof in secondary school (Hoyles & Healy, 1999), subjects were asked to evaluate five different responses to this question. These responses included an empirical approach (looking at functions of the type $y=x^n$ for n from 1 to 6), a graphical approach (the slopes of the tangent lines at x and $-x$ are opposite), and a formal proof using the definition of derivative. For each response subjects were asked questions such as: Is it convincing? Why or why not? How many points would this receive on an exam? While students and teachers had similar evaluations of empirical and formal approaches, they differed on their view of the graphical approach. This difference, I claim, stems from students' fragmented understandings of derivative and implicit messages in the curriculum about what constitutes a good mathematical explanation.

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MIXED METAPHORS: UNDERGRADUATES DO CALCULUS OUT LOUD

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We describe our findings from an exploratory study on the use of metaphors in learning calculus among college freshmen. By metaphor, we mean any set of cognitive tools used as a referent system to assist in understanding and employing a mathematical concept. What are the core metaphors that students use to comprehend and work with essential calculus concepts? How are these metaphors used and how do they develop?

During the spring semester of 2000, we conducted task-based interviews with 15 students enrolled in a second semester calculus course at a major southwestern public university. Eight interviews, lasting two to two and a half hours each, were conducted with pairs of students. The students worked on a problem dealing with position, velocity and acceleration and reflected on their work and what difficulties other students of their level would have. The interviews were audio-taped, and transcripts were analyzed for emergent themes.

In our initial analysis, we identified the six most common uses of base metaphors labeling them as "slope," "rate of change," "automobile," "motion detector," "motion on the graph," and "vertical motion," and observed the following: first, most students alternately employed two or three of these metaphors but rarely attempted to do so at the same time; second, each metaphor came with a corresponding set of language use, signs, and images; third, their mode of analysis changed consistently with the metaphor they employed. Often, this would result in the students either unknowingly obtaining contradicting results or creating unnecessary confusion due to cross-talk between similar signs of different metaphors.

This was a planning study for a large-scale study in fall 2000 with over a hundred students in a first-year calculus course. We will track the development of their use of mathematical metaphors in weekly written work, tests, in-class "snapshots," and interviews.

MATHEMATICS COMPETENCY AS A PREDICTOR FOR SUCCESS IN AN INTRODUCTORY ECONOMICS COURSE

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Student success as measured by final grade average in introductory economics courses can be influenced by a multiplicity of factors. In an attempt to establish a profile of the "successful" student and the characteristics which may support this success, research has focused on one of three themes: (a) characteristics of the economics courses, (b) academic abilities of the students, or (c) the social and economic backgrounds of the students. This study focused on the academic abilities of the students. Though economics frequently deals with numerical data and data relationships, not all material is explicitly mathematical in its original formulation. However, the emphasis on mathematics may be a deterrent to students' success in an introductory economics course. Knowledge of mathematical concepts in the areas of functions, basic algebraic reasoning, discrete, statistics, and fundamental calculus seem to support student success in introductory economics courses. An important conclusion from previous research supports the premise of the current study: A measure of basic mathematics skills thought to be aligned with introductory and advanced algebra skills may be a better regressor than knowledge of calculus on final grades in an introductory economics course.

The subjects of this study were 34 students enrolled in two sections of a course in Introduction to Microeconomics at a small university in the southwest. Each student was tested on the first day of class. The test covered topics of algebra included interpretation of graphs, solving linear equations, computing descriptive statistics (mean, median, mode), and graphing linear equations. Course grades were based on exam scores (10% for each mid-term and 17% for the final exam), homework assignments, exercises, and laboratory performance. Student performance on the multiple choice portion of the tests, both individually and collectively, and the final course grade were considered as measures of student success.

Student performance on all instruments (pretest and exams) were significantly positively correlated with each other. Pearson Correlation Coefficients were computed. The pretest and Test 1 were significant at 0.0042, with Test 2 at 0.0068, with Test 3 at 0.0006, and with the final grade at 0.0025. Thus the Mathematics pretest was a significant predictor of success as measured by the final grade. Passing the Mathematics Pretest also contributed more to the students' success than did characteristics of previous economics courses or the number and types of mathematics courses taken either in high school or college.

Results of this study indicate that the Mathematics Pretest has an important role to play in determining which students will be successful in an introductory economics course. With that as a focus, future research should provide avenues for students to enhance their mathematics skills. With this in mind, the following recommendations are made: (a) tutorials would be mandatory for students whose scores are below a pre-established level, (b) review sessions should be held periodically throughout the term to review important mathematics concepts directly related to the economics principles being presented, and (c) review sessions covering the mathematics content would be held prior to each exam. Having students become aware of the mathematics skills that are required in an economics course could have positive effects on students' success and providing continuous support in mathematics could promote positive attitudes not only about economics but the relevance and importance about mathematics as well.

UNDERGRADUATES' REPRESENTATION SCHEMES IN MODULAR ARITHMETIC

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In this exploratory study, we classify four types of representation schemes useful in solving problems involving linear congruences, division-remainder, divisibility, geometric, and equivalence. The schemes were presented both formally (direct instruction) and informally (outside of class) to students in an undergraduate number theory course. After a midterm exam covering linear congruences, students completed a questionnaire, on which they first described all of their representation schemes for ("ways of thinking about") the statement $a \equiv b \pmod{n}$, and then solved two problems, noting which representation scheme they had used. These two items were specifically chosen to be awkward to solve using the students' observed favored representation schemes, division-remainder and divisibility.

In spite of this awkwardness, most students chose to use these favored schemes to solve the problems. On both items, this resulted in several students solving incorrectly or making inaccurate statements in their explanations. In addition, the representation schemes chosen caused most students to use tedious and unnecessarily complex methods to solve the problems. The students' approaches contrast strongly with the flexible use of representation schemes by experts (graduate students and professional mathematicians) on the same items.

This poster reports on the results of this exploratory study and presents examples of activities which may enable students to develop the flexibility and understanding of this topic required for further study in algebra and number theory.

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Algebraic Thinking

PREPARING TO TEACH IN THE NEW MILLENNIUM: ALGEBRA THROUGH THE EYES OF PRESERVICE ELEMENTARY AND MIDDLE SCHOOL TEACHERS

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Abstract: This study examined preservice elementary and middle school teachers' conceptions of algebra using a framework with categories of procedural and conceptual perspectives. Throughout the semester course, the majority of preservice teachers provided either a non-algebraic or a procedural definition of algebra. At the beginning of the semester, more preservice teachers with a conceptual perspective took a problem-solving position, but at the end of the semester, more held a generalization view. The preservice teachers in this study had a limited understanding and appreciation of generalization.

One theme of mathematics education reform is that algebraic reasoning should be integrated throughout elementary and middle school mathematics (National Council of Teachers of Mathematics [NCTM], 1998, 2000). To implement this objective, teachers in the new millennium will need to identify concepts of algebra that are appropriate for elementary and middle school grades and choose appropriate pedagogy for promoting algebraic reasoning. Effective teacher education must insure that preservice teachers are prepared to address the questions: "What is algebra?" and "How can it be taught effectively?" This study begins to explore these issues in the context of an algebra course for preservice K-8 teachers.

Objectives

The purpose of this investigation is to gather insights into preservice elementary and middle school teachers' conceptions of algebra. The research addresses the following questions: 1) What are preservice elementary and middle school teachers' conceptions of algebra? 2) How do their conceptions change while taking an algebra course for preservice elementary and middle school teachers? and 3) What is their understanding and appreciation of generalization?

Theoretical Framework

The nature and role of algebra in school mathematics has been the focus of extended discussion in recent years. Edwards (1990) recommended that we guarantee all students the opportunity to study algebra, and Silver (1997) emphasized that we should guarantee access to algebraic ideas, not just algebraic courses. A consensus has emerged that to ensure such broad access, algebra should be incorporated into elementary and middle school mathematics (NCTM, 1989, 1993, 1998). But what kind of algebra should we incorporate? Moses (1997) observed that the content of algebra is being transformed from a discipline involving the manipulation of symbols to a way of seeing and expressing relationships, "a way of generalizing the kinds of patterns that are part of everyday activities" (p.246). As this view is interpreted in elementary and middle school, the crucial issue appears to be the development of algebraic reasoning, not just the introduction of algebraic concepts (Yackel, 1997; Driscoll, 1999). Mason (1996, p. 65) describes school algebra as the "language for expression and manipulation of generalities," and states that the essence of teaching mathematics is the "awakening of pupil sensitivity to the nature of mathematical generalization, and dually, to specialization." To appreciate the potential of mathematical generalization is to grasp the power of mathematics. As teacher educators, our goal is to help preservice teachers develop an appreciation for these subtle notions.

Discussions of mathematical understanding often distinguish between procedural and conceptual knowledge where procedural knowledge encompasses the formal language and the algorithms used to complete mathematical tasks, and conceptual understanding is characterized as a network of rich relationships (Hiebert & Lefevre, 1986). Particularly where conceptual knowledge is concerned, we have little knowledge about the specifics of teachers' understanding (Ma, 1999). It seems likely, however, that a close relationship exists between the richness of a teacher's mathematical understanding and the quality of mathematical thinking that is promoted in his or her classroom (Ma, 1999; Mason, 1996). A teacher with a rich understanding of the connections between mathematical ideas is more likely to reveal and represent them (Ma, 1999) at the same time that a teacher who lacks this awareness is unlikely to promote deep insight in his or her students (Mason, 1996). We cannot assume that preservice teachers bring appropriate connected knowledge with them.

Although the *Professional Standards for Teaching Mathematics* (NCTM, 1991) calls for the mathematical preparation of all preservice elementary and middle school teachers to include the study of algebra, experiences focusing on algebra as symbol manipulation may do very little to help preservice teachers develop profound understanding of algebraic concepts. To address this goal, a framework is needed for organizing algebra instruction for preservice teachers that promotes understanding of fundamental concepts of algebra beyond the traditional rules and procedures. Bednarz, Kieran, and Lee (1996) present four alternative approaches to the development of algebraic ideas: generalization, problem solving, modeling, and functions. These perspectives suggest a framework for examining preservice teachers' conceptions of algebra, and for providing beneficial experiences for preservice teachers.

Research Methods and Data Sources

The participants in this investigation were students in one of two different algebra classes for preservice elementary and middle school teachers. Two of the common goals for the two courses were 1) to develop preservice teachers' understanding of algebraic concepts, and 2) to explore ideas for teaching algebra to elementary and middle school students. Both classes addressed the first goal by having preservice teachers engage in college-level algebraic experiences involving generalization, problem solving, modeling and functions. They addressed the second goal by exploring algebraic activities for children involving variables, functions, and pattern generalization.

On the first day of class, each preservice teacher responded to two questions designed to probe their definitions of algebra and their beliefs about teaching algebra. Question 1: "How would you explain what algebra is to someone who has never heard of it?" Question 2: "Imagine that you are observing a sixth-grade class where the teacher, Ms. Jones, is trying to promote algebraic thinking. Describe what is happening in the classroom that supports the achievement of this goal." The preservice teachers responded to Question 1 and Question 2 again on the last day of the class.

During the course, the preservice teachers worked in groups to solve two problems selected from Driscoll (1999). The Sums of Consecutive Numbers problem required preservice teachers to identify and extend number patterns and focused on the notion of generalization. The Painted Cubes problem asked preservice teachers to use variables to describe patterns in the number of cubes painted on three faces, two faces, one face, and zero faces of different sized cubes. For each of the two problems, the preservice teachers responded to a set of reflection questions, including the following: "Describe how you used algebra in this problem."

Toward the end of the course, each preservice teacher wrote a lesson plan for students in grade four, five, or six. Choosing from one of three topics (variable, generalizing patterns, or functions), they were to find or create an activity to promote the development of algebraic reasoning. This assignment provided preservice teachers an opportunity to demonstrate pedagogical content knowledge as framed by their conception of algebra.

Data analysis employed axial coding during which data were organized into categories and subcategories and examined to identify relationships among categories (Strauss & Corbin, 1990). Tables were then prepared to organize information and illustrate relationships (Miles & Huberman, 1994).

The researchers worked together to code each piece of data. In order to examine preservice teachers' conceptions of algebra, they analyzed both sets of responses to Question 1 and Question 2, reflections for the Sums of Consecutive Numbers problem and the Painted Cubes problem, and the Lesson Plan. Table 1 explains the categories used to code data addressing the preservice teachers' conceptions of

Table 1. Conceptions of Algebra

Algebraic	Describes algebraic notions such as variables, formulas, unknowns, relationships, patterns, functions, or generalizations.
Non-algebraic	Describes arithmetic or general mathematics without reference to algebraic notions.
Algebraic Subcategories:	
Procedural	Describes rules, procedures, or symbolic manipulation, without relevance to underlying concepts.
Conceptual (one of the following):	
Generalization	Describes the construction of formulas that account for a general procedure or relationship between quantities. The goal is to find an expression or a formula.
Problem Solving	Describes the forming and solving of equations, using letters as unknowns. The goal is to find an answer, to find a numerical value.
Functions	Describes dependence relationships among real-world quantities. The focus is on how a change in one variable produces a variation in the value of the function.

algebra. The responses to various questions were first coded as Algebraic or Non-algebraic. Algebraic responses were then coded as Procedural or as one of three Conceptual subcategories: Generalization, Problem Solving, or Functions. Although Bednarz et al. (1996) included a modeling approach to algebra, that perspective did not appear in this study.

To examine preservice teachers' understanding and appreciation of generalization, the researchers looked for patterns in the categories of preservice teachers' conceptions of algebra across the course. In addition, they analyzed preservice teachers' responses to the following reflection question: "Consider the discoveries you described above (on the Sums of Consecutive Numbers problem). Would you describe them as specific facts or generalizations?"

Results and Implications

At the beginning and the end of the semester, the majority of preservice teachers in the two classes provided either a Non-algebraic or a Procedural definition of algebra. Similarly, when asked to describe a classroom in which the teacher is promoting algebraic reasoning, the scenes they described often contained no algebra or merely procedural aspects of algebra. Table 2 shows the number of preservice teachers in each category of each component of this study.

The Sums of Consecutive Numbers problem and the Painted Cubes problem were chosen as class activities because they provided preservice teachers with opportunities to use algebra as a tool for pattern generalization. However, when asked to describe how they used algebra in the Sums of Consecutive Numbers problem, the majority of preservice teachers focused on procedural aspects. When reflecting on the Painted Cubes problem, a greater number of preservice teachers described using algebra to generalize patterns, but an even greater number focused on the problem-solving aspects of the problem, describing how they used algebra to write and solve equations. Despite having had these experiences with pattern generalization (and problem solving), more than half of the preservice teachers in the two classes later described algebra in procedural terms.

The preservice teachers in this investigation had a limited understanding and appreciation of generalization. In their reflections, only nine students demonstrated on the Sums of Consecutive Numbers problem that they understood the meaning of the term "generalization." Sixteen of the students, 50%, indicated that they thought that a specific fact was more powerful than a generalization about the relationships. Of 15 people who chose to write a lesson plan about generalizing patterns, six worked with patterns but did not actually incorporate generalization. At the beginning of the semester, the majority of preservice teachers with a conceptual view of algebra expressed a Problem Solving conception. On the other hand, at the end of the semester, the distribution of conceptual views shifted toward a Generalization perspective.

Although it is desirable for preservice teachers to recognize a variety of perspectives to algebra and prepare to teach in a manner that incorporates these varied perspectives, it appears that many do not understand what distinguishes arithmetic from algebra, and of those who do, a majority perceive algebra mainly from a procedural perspective. This continues to be true after they participate in either of two courses which emphasize conceptual approaches to algebra.

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Table 2. Preservice Teachers' Conceptions of Algebra Throughout the Course ($N = 32$)

	Non-algebraic	Algebraic: Procedural	Algebraic: Conceptual		
			Problem Solving	Generalization	Functions
Question 1 (first day)	8	16	7	1	
Question 2 (first day)	17	9	5	1	
Consecutive Numbers Problem	6	15	2	6	1
Painted Cubes Problem	1	4	14	12	
Lesson Plan	12	3	2	9	6
Question 1 (last day)	3	19	2	5	3
Question 2 (last day)	18	6	1	6	1

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GENERALIZING AND PROGRESSIVELY FORMALIZING IN A THIRD-GRADE MATHEMATICS CLASSROOM: CONVERSATIONS ABOUT EVEN AND ODD NUMBERS

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Abstract: This study explored third-grade students' capacity for generalizing and formalizing their mathematical thinking in classroom conversations about even and odd numbers. Data were collected during a 90-minute mathematics class in an urban school district multiple times per week over the course of an academic year. Analysis showed that these students have a capacity for making robust generalizations in progressively formal ways and with intuitive supporting arguments. In particular, their activity of generalizing about even and odd numbers was based on diverse forms of reasoning, involved the algebraic treatment of the terms 'even' and 'odd' as placeholders, or variables, in generalized expressions, and extended across mathematical operations. Moreover, the process of argumentation and justification was a factor in how and what students came to know about even and odd numbers.

Perspective and Purpose of the Study

There is a growing recognition that algebraic reasoning takes several forms and can simultaneously emerge from and enhance elementary school mathematics (NCTM, 1998). Kaput (1998) has characterized these forms of reasoning as including (a) the use of arithmetic as a domain for expressing and formalizing generalizations; (b) generalizing numerical patterns to describe functional relationships; (c) modeling as a domain for expressing and formalizing generalizations; and (d) generalizing about mathematical systems abstracted from computations and relations. From this characterization, it is apparent that the processes of generalizing and progressively formalizing a generality are at the heart of the development of algebraic reasoning (see also Mason, 1996). As such, these processes represent the kind of mathematical thinking necessary to extend children's elementary school experiences beyond arithmetic proficiency to include habits of mind that can support the more complex and abstract mathematics that will become increasingly important in the new century (Kaput, 1999; Romberg & Kaput, 1998).

This perspective on the algebraic potential of elementary school mathematics raises important questions about students' understanding of and capacity for generalizing and progressively formalizing mathematical thought in the social context of argumentation and justification. The purpose of this study was to examine these processes within a third-grade classroom by mapping the evolution of students' thinking in conversations about even and odd numbers. In particular, this study explored

(a) Third grade students' capacity for building generalizations and the idiosyncratic ways in which those generalizations were expressed;

(b) the extent to which students were able to formalize their thinking; and

(c) the role of purposeful argumentation and justification in these processes.

Methods/Data Source

The study took place in a third grade mathematics classroom in an urban, under-achieving school district, with eighteen students representing diverse socioeconomic and ethnic backgrounds participating in the research. The classroom teacher (Jan-pseudonym) was concurrently a 2nd year participant in a district–based project, led by the authors, designed to develop elementary teachers' ability to identify and strategically build upon students' attempts to generalize and formalize their thinking and to engineer viable classroom instructional activities that would support this (Kaput & Blanton, 1999). Jan's classroom was subsequently selected as the research site for this study because of her commitment to the development of students' algebraic reasoning.

Data collection occurred during Jan's 90-minute mathematics class approximately three days per week for one academic year. The data consisted of audio recordings, Jan's reflections and field notes that focused on students' verbal discourse. While various other mathematical themes reflecting children's activities of generalizing and formalizing have emerged from these data (e.g., the evolution of children's symbol sense), classroom conversations about even and odd numbers were selected here as the unit of analysis, with the class as the grain-size for analysis. (Ultimately, we will look across other themes in order to get a more complete picture of students' capacity for algebraic thinking.) Viewing learning as a nonlinear series of transformations, we expected that students would be at different points in a continuum of development. As such, the goal of our analysis was not to quantify students' activities of generalizing and formalizing, but instead to use qualitative methods to explore the nature and progression of students' generalizations and how these came to be established within the classroom.

Results and Conclusions

Analysis of the data showed a progression in the complexity of generalizations and supporting justifications that students were able to construct about even and odd numbers. At the beginning of the academic year, students could only identify the parity of a number (Jan had a chart in the classroom that categorized a set of numbers as even or odd). They were unable to argue why a particular number was odd or even and thus were unable to build generalizations based on these properties. During the academic year, we observed both planned instruction (see Jan's reflections) as well as impromptu discussions about even and odd numbers. Through this, students were eventually able to build generalizations and construct spontaneous arguments, even when making such arguments were an abstraction from the task at hand. Classroom protocols and Jan's reflections (italicized) are included here to illustrate this progression.

Jan. *I asked the class what would happen if I added 2 even numbers together. Most of them said that I would get an even number. When I asked what would happen if I added 2 odd numbers together, most of them said that I would get an odd number. When asked about odd and even together, the answers were mixed. In the past I would have told them the answers by giving them some examples (e.g., 5+5=10). But...I wanted them to see how it really works, so that they could see that it would [generalize to all cases]. We did [an] activity combining (square) grid-paper shapes to model adding even and odd numbers. I asked the same questions again. This time they answered with more certainty.*

One student later explained the generalization that 'the sum of any two odd numbers is even' using the idea of adding square shapes: "If you have two odd numbers it makes it even because if you have leftovers the two leftovers go together."

Jan. *The only confusion came when [Sarah] said that odd + even was odd and even + odd was even. [Stephen] responded that that couldn't be. He used numbers in place of odd and even and said that it (using 'odd' and 'even') was the same as [using letters instead of numbers].*

Sarah explained to the class, "I thought that all the time when odd is the first one it was supposed to be odd and when even was first it was going to be even. [But then I figured that that wasn't correct] because once you start turning them around, then it's the same thing. It doesn't make a difference."

This vignette suggests that students initially used the type (e.g., all odd) or, in Sarah's case, the position (e.g., even number first) of the terms 'even' and 'odd', not their intrinsic mathematical properties, to make a generalization about their sums. At this point, Sarah was unable to see the term 'even' or 'odd' in the algebraic way that Stephen did, as a placeholder or variable. Sarah was eventually able to construct a commutativity argument that disproved her initial generalization (based on the position of an odd or even number in a sum), and it was through peer argumentation with Stephen that her generalization was challenged and refined. In this sense, the social context of argumentation and justification ultimately provided the impetus for building a valid generalization and thus became a critical factor in how and what students came to know about even and odd numbers.

Jan extended this activity with students. She wrote, "I asked them, 'If we added odd + odd + odd, what would the sum be?' They figured out that the sum would have to be odd because 2 odds make an even and when you add odd + even, you get odd." In this case, students reasoned that they could associate two numbers at a time and thereby iteratively reduce the task of adding 3 odd numbers. Additionally, they were able to achieve a level of abstraction in which they could reason *with a generalization* to produce a generalization. That is, they were able to reason with the general referent of 'odd' in the expression 'odd+odd+odd' (vs. specific sums of odd numbers, such as 1+5+7) to produce the generalization 'the sum of any 3 odd numbers is odd'.

Several months after the events described above, students extended their generalizations from sums to products of even and odd numbers during a spontaneous whole-class discussion that grew out of a separate mathematical task. After working through different numerical combinations, students conjectured that 'even times even = even', 'odd times odd = odd' and 'even times odd = even'. One student explained that multiplying any number by an even number would always result in an even number because it requires adding pairs of numbers. This episode further speaks to these students' emerging capacity for generalizing. For example, it grew out of *students'* observations about the task at hand (in particular, a multiplication sentence kept producing even number results), and it illustrates students' potential to construct justifications using sophisticated arguments (e.g., the idea of multiplication as repeated addition).

The analysis suggests that these "average" students have a capacity far beyond arithmetic proficiency for making robust generalizations with intuitive supporting arguments. We found that, in their activity of generalizing, students

(a) used multiple forms of reasoning, including representational reasoning (e.g., using graphical or figural objects to model even/odd numbers), numerical reasoning (breaking numbers apart to identify their properties) and pattern-based reasoning ("zero is even because it is in the pattern that includes 2, 4, 6, 8,…");

(b) used the terms 'even' and 'odd' algebraically, that is, as placeholders or variables ;

(c) were able to reason with a generalized referent (e.g., 'odd') to produce a generalization;

(d) were able to extend their generalizations about even and odd numbers across mathematical operations in a spontaneous way and over a sustained period of time;

(e) mediated the validity of their generalizations through peer argumentation; and

(f) were able to construct sophisticated justifications for their conjectures.

Although students were (expectedly) not at a symbolic level of formalism in their generalizations, there was evidence that some students saw even numbers as multiples of 2 (a precursor to the formalism 2n, where n is an integer) and, more significantly, could construct arguments based on this idea.

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THE INTERPLAY BETWEEN INSTRUCTION AND THE DEVELOPMENT OF MIDDLE SCHOOL STUDENTS' ALGEBRAIC THINKING

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Abstract: This study examined the interplay between instruction and the development of middle school students' algebraic thinking. Interviews were conducted with six students over a two-year period during their 7th and 8th grade years. In grade 7, all students received algebra instruction using units from the Connected Mathematics Project (CMP) materials. In grade 8, three of the students took a traditional algebra 1 course while the other three students remained in the regular mathematics class following a typical pre-algebra curriculum supplemented with two CMP units. Our findings indicate that although the algebra 1 students obtained a level of proficiency in algebraic symbol manipulation, their reasoning and sense-making abilities were less pronounced than in grade 7 and generally not as robust as those of the students who received the regular grade 8 mathematics instruction.

For more than a decade, the National Council of Teachers of Mathematics (NCTM) has recommended a broad-based, integrated curriculum for the middle grades that includes topics in probability, statistics, measurement, geometry, and algebra (NCTM, 1989). In support of this recommendation, recently published *Standards*-based curriculum materials and even basal textbooks have included these topics as part of the middle grades curriculum (Reys, Robinson, Sconiers, & Mark, 1999). More recently, the NCTM (2000) has called for "significant amounts of algebra and geometry throughout grades 6, 7, and 8" (p. 212) and has recommended that algebra and geometry instruction emphasize the interconnections between these topics and among other mathematics topics. Nevertheless, a trend adopted by many school districts is to shift mathematics content from the high school to the middle school in the form of a traditional algebra 1 course in grade 8. According to Silver (1995) "mandated algebra instruction in grade 8 can undermine those reform efforts directed precisely at broadening and integrating the curriculum of middle grades" (p. 32). There is also the concern that the content and teaching methods used in traditional algebra courses do not promote conceptual understanding or the development of students' algebraic reasoning (Kaput, 1999).

The purpose of this study was to investigate the interplay between instruction and the development of middle school students' algebraic thinking. The first phase of this two-year study sought to describe the algebraic thinking of students whose mathematics instruction in grade 7 included the study of algebra using materials from the Connected Mathematics Project (CMP) curriculum. In a departure from traditional algebra instruction that focuses on manipulating expressions and solving symbolic equations, the *Standards*-based CMP curriculum uses contextual problem situations to develop students' concepts of variable and their understandings of the relationships among variables (Phillips, 1998). The second phase of the study followed the development of the students' algebraic thinking through grade 8. However, in grade 8, the students split into two groups with one group receiving instruction that continued to incorporate some of the CMP materials and the other group receiving instruction based solely on a traditional Algebra 1 textbook. The development of students' thinking across grade levels and between the regular 8thgrade mathematics and traditional algebra 1 students was examined.

Theoretical Considerations

The theoretical perspective guiding our work assumes that mathematics should be taught for understanding. We refer to the definition of Hiebert et al. (1997) who indicated that understanding something involves seeing how it is related or connected to other things that are known. Understanding is crucial to learning mathematics "because things learned with understanding can be used flexibly, adapted to new situations, and used to learn new things" (Hiebert et al., p. 10). According to Romberg and Kaput (1999), learning for understanding "cannot be viewed as a mechanical performance or an activity that individuals engage in solely by following predetermined rules" (p. 6). Yet it is this mechanistic view of teaching and learning mathematics that characterizes traditional school mathematics (Romberg & Kaput). Within this theoretical perspective, we examined the development of students' understandings of algebra from two curricular orientations: one traditional and the other more *Standards*-based.

Method

Participants

Grade 7 students from a Midwestern school formed the population for this study. Based on the results of a researcher-generated algebra problem-solving assessment, six students were purposefully selected from this population to represent contrasting levels of performance. These six students participated in the study during their 7th and 8th grade years of schooling.

The classroom teacher in this study taught all mathematics classes for grades 7 and 8. He had 15 years of experience at the middle school level. His teaching style was traditional in the sense that instruction was typically teacher directed during whole-class activities but he frequently engaged students in small-group interactions that allowed them to communicate and justify their thinking.

Year One Instruction

In grade 7, approximately 9 weeks of instruction focused on the study of algebra. During this time, we observed the mathematics class on at least two consecutive days each week to document the nature of algebra instruction. We also noted the classroom interactions of the students, asked them to explain their reasoning as they worked on in-class assignments, and obtained copies of their written work when it was collected by the teacher. The CMP texts *Variables and Patterns* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998d) and *Moving Straight Ahead* (Lappan et al., 1998b) were the basis for algebra instruction.

Year Two Instruction

In grade 8, three of the students were selected for participation in a traditional Algebra 1 class based on their overall 7th grade mathematics performance and their performance on a standardized algebra readiness test. The textbook used in this class was *Algebra 1: An Integrated Approach* (Benson et al., 1991). The other three students remained in the "regular" mathematics class. The curriculum for this class incorporated the text *Pre-Algebra: An Accelerated Course* (Dolciani, Sorgegnfrey, & Graham 1985), the CMP texts *Looking for Pythagoras* (Lappan et al., 1998a) and *Thinking With Mathematical Models* (Lappan et al., 1998c), and was supplemented with instructional units in geometry and statistics. Classroom observations similar to those in Year 1 were conducted in Year 2.

Procedure

In Year 1, students were individually interviewed on seven occasions to investigate the interplay between students' algebraic thinking and instruction. Using a semi-structured interview script, we posed problems similar to the ones students were working on in class. We also asked probing questions to assess levels of students' conceptual understanding, and attempted to identify any cognitive obstacles the students were encountering. Interview topics included linear and nonlinear relationships, tabular and graphic representations, concept of slope, and solving linear equations. In Year 2, five interviews were conducted. All of the students were given the same interview items that represented a balanced distribution of topics from the regular mathematics and algebra 1 curricula. As in Year 1, we asked probing questions to assess students' conceptual understanding and attempted to identify students' misconceptions. Interview topics included linear and non-linear relationships, interpretation of multiple representations, concepts of slope and variable, generating expressions, and writing equations. All interviews were audio taped and later transcribed.

Data Sources and Analysis

Data sources consisted of the transcribed interviews, students' written work, researcher field notes, and data displays and summaries generated during the analysis. These data were analyzed using a double coding strategy and data reduction approach (Miles & Huberman, 1994) to discern key thinking patterns for each student. Transcripts were analyzed to characterize each student's reasoning or solution strategy for each problem presented in the interviews. The coding rules used in this process were modified and refined throughout the analysis. They pertained to students' understandings of concepts (e.g., slope, variable), as well as their abilities to describe patterns or functional relationships, generalize relationships either verbally or with symbols, and use and interpret tabular, graphical, and symbolic representations. Codes were reconciled and organized in a Task Coding Matrix that allowed us to examine within-case and across-case trends. Narrative summaries were constructed for each student describing his or her reasoning, solution strategies, and cognitive difficulties across all interview problems. Also, cross-case summaries were developed for each problem.

Results

In *Year 1*, students examined the relationships among graphic, tabular, and symbolic representations of linear situations. In an attempt to uphold the philosophy of the CMP materials, the teacher engaged the students in small-group problem-solving activities and encouraged students' explanation and justification of solution strategies. However, the teacher occasionally supplemented the CMP lessons with worksheets designed to introduce and practice procedures for symbolically solving linear equations. By the end of grade 7, three trends in students' thinking were identified: (1) all students could represent linear situations in symbolic, graphic, and tabular forms; (2) students demonstrated an emerging understanding of the concept of variable in the sense that they could use a variable when it was defined for them but sometimes had difficulty identifying what a variable meant in a problem context; and (3) half of the students (two who later participated in regular 8th grade instruction and one who took the traditional algebra class) consistently referred to the problem context to make sense of a problem situation while the other three students generally ignored the problem context and focused on superficial patterns.

In *Year 2*, the traditional algebra 1 class focused on manipulating symbols in the context of finding unknowns, using properties of generalized arithmetic, and solving problems by representing situations symbolically. The typical lesson followed a mechanistic approach (Romberg & Kaput, 1999) whereby the teacher demonstrated a procedure, provided a few examples for students to work individually, and assigned practice exercises. The algebra instruction in the regular mathematics class continued to build on the algebraic understandings introduced in grade 7. Students informally investigated the graphic, tabular, and symbolic representations involving both linear and nonlinear relationships. It should be noted, however, that only one of the four designated grade 8 algebra units from the CMP materials was used.

By the end of grade 8, differences in the students' understanding and reasoning emerged between the two instructional groups. The algebra students were more proficient at solving equations and simplifying expressions. Although all students could represent situations in various forms when asked to do so, the students in the regular mathematics class accessed a variety of representations to solve problems, while the algebra 1 students focused exclusively on symbolic representations. The regular mathematics students referenced alternative representations to verify their work, whereas the algebra 1 students rarely sought other representations to confirm the validity of their work. Although all students struggled to understand the different uses of variables, students from the algebra group often referred to, and used, variables as unknowns while ignoring the underlying meaning of their symbolic form. In contrast, the regular mathematics students maintained a dynamic view of variables and frequently questioned what variables represented. Justifying one's thinking was another difference that emerged between the two groups. The algebra 1 students often stated "I just know this" rather than offering any justification of their reasoning. For them, providing an explanation did not seem to be an integral part of doing mathematics. The regular mathematics students justified their reasoning when asked or provided an explanation as they talked their way through solving the problem.

Conclusions

Our results indicate that although students in the traditional algebra course acquired some level of proficiency in algebraic symbol manipulation, their reasoning and sense-making abilities were generally lacking. In many respects, the reasoning and sense-making of the algebra 1 students was less pronounced than it had been in grade 7 and was not as robust as that of the students who remained in the "regular" grade 8 mathematics class. The findings of this study underscore the concern that a traditional algebra 1 course in grade 8 may promote a mechanistic view of algebra and fail to develop concepts and understandings that would allow students to reason algebraically in new situations.

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HOW DO STUDENTS ADJUST TO FUNDAMENTAL CHANGES IN MATHEMATICS CURRICULA?

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Abstract: The "Navigating Mathematical Transitions" project is a three-year effort to examine how high school and college students cope with deep changes in mathematics curriculum and teaching. It examines and analyzes students' experiences as they move between "traditional" mathematics curricula and those inspired by the NCTM Standards (1989). This paper presents an overview of our conceptualization of mathematical transitions, our methods for studying them, and our initial analyses of Year 1 data, including our attempt to characterize classroom teaching, some important differences that students report, and a framework for analyzing those differences. Our conference presentation will present a deeper and more elaborate analysis.

Project Objectives

Ten years after the publication of the *Curriculum and Evaluation Standards for School Mathematics [Standards]* (National Council of Teachers of Mathematics [NCTM], 1989), the design of curricula based on these standards is complete; implementation is well underway; and evaluation and assessment studies are appearing (e. g., Schoen, Hirsch, & Ziebarth, 1998; Hoover, Zawojewski, & Ridgway, 1997). But very little attention has been paid to the experience of students who move between "traditional" and "reform" curricula and pedagogy that appear to differ dramatically in their conceptions of thinking, knowing, and doing mathematics.

This project was designed to study these mathematical transitions in one locale (the state of Michigan) where many students cross such boundaries. Two Standards-based curricula, the Connected Mathematics Project [CMP] (Lappan, Fey, Friel, Fitzgerald, & Phillips, 1995) middle school materials and the Core-Plus Project [CPMP] (Hirsch, Coxford, Fey, & Schoen, 1996) high school materials were written in Michigan and have been adopted widely throughout the state. Likewise, the University of Michigan teaches calculus and pre-calculus to freshmen with the Harvard Consortium materials (Hughs-Hallett, et. al., 1994) that are quite consistent with many of the central principles in the NCTM *Standards*.

Theoretical Notions

We take a broadly cognitive perspective in examining the impact of curricular shifts. We understand students carry forward knowledge, beliefs, dispositions, attitudes, and goals from prior experiences, in and out of school. These elements of their mathematical experience are reshaped in the context of new demands and expectations. This shaping and reshaping process is influenced by various social factors, including work and relationships with peers and parents and the norms and activities in their mathematics classrooms.

We distinguish three different conceptual components in clarifying the phenomenon of mathematical transitions. *General issues of transition* refer to developmental processes that are hastened by the move into high school or college that are not directly mathematical in nature, e.g., coping with increased freedom. *Mathematical discontinuities* refer to the differences that students report between current and prior expectations for thinking and acting mathematically. These may involve issues of curriculum, teaching, or the structure of the coursework (see below for details). *Mathematical transitions* are students' active responses and adjustments to such discontinuities (especially at points of struggle) and how they evaluate those actions.

The premise that the content of curriculum matters is central to the project (though students' transitions and our analysis of them involves more than written curricula). Our analysis began with the claim that the three Standards-based curricula named above differ from more traditional curricula in 6 important ways. (1) The *objects of study* are functions and functional relationships in multiple representations (in contrast to equations and symbolic expressions). (2) *Typical problems* require the analysis of situations where quantities grow and change (in contrast to requests to solve, factor, etc.). Likewise, (3) *typical solutions* are more likely to involve tabular or graphical analysis and significantly more verbal explanation (in contrast to numerical and symbolic manipulations). (4) The *role of practice* changes as problems become longer, involve many parts, and do not belong to recognizable types. (5) *Technology* for representing and calculating, i.e., graphing calculators, is an integral component. (6) Deviations from *the typical lesson pattern* of review homework, present new content, and provide time for problem solving are common.

Methods

Our project is tracking 20 to 25 students at each of four sites (two high schools and two universities in Michigan) as they move from a Standards-based program to a traditional one (or the reverse). The site-specific curricular shifts are summarized in Table 1. We recruit students when they "land" in their new mathematics program and follow them for 2.5 years of coursework. At all four sites, the Standards-based curricula listed above have been in place for more than 3 years. Our principal methods of data collection are classroom observation, individual interviews, students'

Table 1. Curricular Shifts of Project Students

Site A (high school) Connected Mathematics (CMP)‡ Traditional high school program	Site B (high school) Traditional junior high program ‡ Core-Plus Mathematics
Site C (MSU) Core-Plus Mathematics ‡ Traditional pre-calculus & calculus	Site D (UM) Traditional high school programs ‡ Harvard Consortium calculus

journal writing, and survey questionnaires. We collect data in six domains of students' mathematical experience: (1) achievement (grades), (2) content learning (key concepts), (3) daily experience, (4) career and educational goals, (5) beliefs about mathematics and themselves as learners, and (6) strategies of adjustment.

Results

At present, we are analyzing Year 1 data. Though our analyses are ongoing and our results preliminary and incomplete, sufficient support exists for some interesting findings.

The Practices of Teachers Using "Reform" Curricula Have Been More Traditional than We Expected

At Sites B and D, the use of "reform" curricula (Core-Plus and Harvard Consortium materials respectively) has generally been coupled with quite traditional teaching practices. This has not, however, negated the impact of new curricula, as indicated by our students' characterization of changes (see below). It is powerful evidence that our efforts to understand students' experience must be informed by analyses of the *enacted* curriculum, not only the written curriculum.

Based on extensive classroom observations, we have found it useful to characterize the teaching our participants have experienced in relation to a theoretical "ideal type." This *Standard Model* (Table 2) is not a representation of any particular teacher's (or teachers') typical daily practice. Rather it is a common base-line against which we describe and compare the practices of all teachers we observe.

Table 2. Standard Model of Teachers' Practice

Teacher Actions	Teacher Expectations of Students
Collect or check students' homework	Listen to teachers' presentations of problems and solutions
Solicit students' questions from homework and produce solutions on the board	Take notes on the content identified by teachers' as important
Present the new content and provide examples on the board	Answer "small" (i.e., "what's the next step") questions from the teacher
Provide time for students to practice solving similar problems	Solve assigned problems (in class or as homework; in groups or individually)
	Study the notes and remember them for tests and quizzes

From our observations we (1) evaluate how well each teacher's practice regularly conforms to these elements, and (2) document the important elements of his/her teaching that are *not* listed in the *Model.* These analyses provide a rough but principled way of characterizing the enacted curriculum, as background for our main efforts to track, understand, and interpret students' experiences.

Students' Report Differences Between Their Prior and Current Experiences but Value Changes in Quite Different Ways

Students at all four sites have experienced elements of Standards-based curricula, either currently (Sites B & D) or in the immediate past (Sites A & C). The differences they report are various, and we are currently analyzing and tabulating them. Some are independent of curricular shifts. For example, some students at all four sites cite a faster pace through the content and less intimate relationships with their teachers than they experienced previously. But many other reported differences are closely related to curricular elements. We present three common ones and illustrate how students take different stances toward them.

Typical problems. Many students have reported the greater frequency of "word" or "story" problems in the Standards-based curricula, in contrast to the prevalence of "book," "number," and "equation" problems in traditional curricula (students' descriptions of "non-story" problems vary widely). Those who value "story problems" cite greater personal meaning via the connections to their everyday world and their ability to make sense of such problems and their approaches to them. Those who prefer more traditional problems tend to complain that "story" problems are harder because they contain more elements to track and understand and because textbooks do not provide model solutions.

Explaining your thinking. In different ways, student have said that more time, value, and attention is or was given to expressing their mathematical thinking in Standards-based programs, both orally (e.g., in whole class discussions) and in writing (e.g., in explaining their answers to problems). Students who value this orientation have cited the value of testing their ideas in discussion, developing their own methods of solution, and testing their understanding. Students who have critiqued this orientation felt that they wanted or needed the teacher's solution methods and that extensive written explanations often felt like "busy work."

Group work. When it is new and emphasized by teachers, solving problems in small groups is reported as a difference. This has been especially true at Site D, where the Harvard calculus courses include mandatory group work outside of class. But three other findings significantly complicate the role of group work in transitions. First, small group problem solving has not been limited to Standards-based programs. Second, "small groups" can mean pairs or groups of 3 or 4 students, and different arrangements affect what happens in small groups and students' experience in them. Third, students have often found ways to ignore or subvert small group processes and work more or less individually.

Curriculum, Teachers/Teaching, and Structure

The term, "enacted curriculum," expresses the idea that curricula and teachers jointly create the content that students experience in classrooms. We have found it useful to add a third component, "Structure," to this scheme (see figure 1).

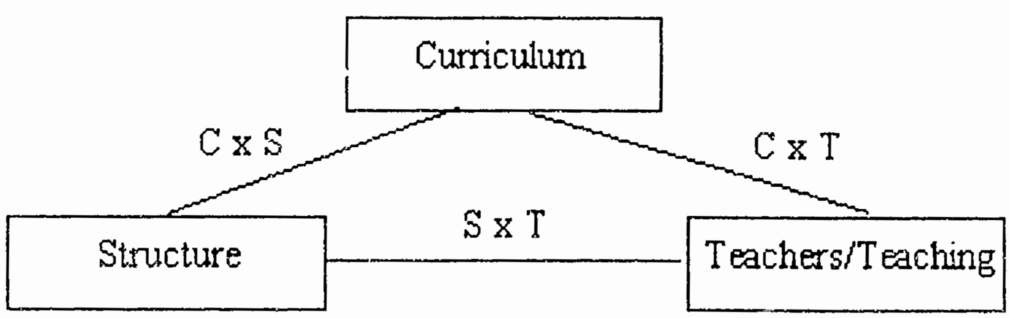

Figure 1. Scheme for Enacted Curriculum

Under "structure" we identify aspects of mathematics courses that are attributable to broader departmental decisions. Issues of "structure" have proven most important at the university sites where many sections of the same course are taught and shared practices are mandated across the sections. For example, at Site D it was the Department, not the teachers or the curriculum, that specified challenging group homework assignments. We have found it useful to locate the differences reported by students at the most appropriate position in this conceptual scheme, either under one specific factor or on the links representing interactions between two factors. This analysis helps us to understand the origins of students' experience and make sense of our results across sites.

Significance

Research that investigates students' experience of Standards-based curricula in relation to more "traditional" curricula will not provide unequivocal evidence for or against either position. Different students, whether by nature or prior influence, will find different programs desirable for different reasons. Such research can, however, provide insights into and analyses of students' experience as input for local, state, and national discussions of broad goals for mathematics education and programs of curriculum and teaching likely to meet those goals. Students' experience is one important factor, though not the only factor relevant to mathematics education policy.

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SECOND GRADER'S DISCOVERIES OF ALGEBRAIC GENERALIZATIONS

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This paper will, necessarily briefly, indicate one pathway by which children can begin to build algebraic generalizations across additive and multiplicative domains. The approach is Vygotskian, and, more particularly, influenced by the work of Davydov and his colleges (Davydov 1975). A central issue for a Vygotskian approach is to indicate precisely how the teacher scaffolds children's entry into what Davydov terms 'theoretical generalization' (Davydov 1990). I will focus here on how the teacher's questions do so.

I will first discuss the earliest algebraic teaching experiment I attempted, with second graders in a low-income inner-city public school. After children had some experience modeling and solving a range of word problems, I just asked them, "What can we find out about *any* situation in which we put quantities together or take them apart? I wanted to find out if discussing a general kind of situation made sense to them at all. There was a very strong response. Children immediately invented a wide range of ways to model general groupings of quantities and name them. Figure 1 shows some of these ways. It also shows the model the class chose for collective use and discussion. Notice its similarity to a Euler diagram, though it was invented from the ground up. Children were able to use their models to make discoveries about what statements are true or false of any Put-Together or Take-Apart situation, as you can see in Figure 2. Notice that this process of discovery builds up a set of statements rather like rudimentary theorems. The models function like rudimentary proofs: Children used them to demonstrate why statements were true or false. They were also able to orally explain why the statements they generated were true or false. Figure 3 gives some examples of their explanations.

Let's consider the role of the question in this process. First, general natural language understandings (e.g., children's understanding of the 'any situation in which we put quantities together or take them apart' question, and their namings of them: 'everything', 'piece', 'piece' etc.) together with prior experience in constructing groupings are important constituents of the process. We are not seeing generalization built up de novo here. Natural language and cultural discourse are suffused with generalization. The question does not so much impart generalization as signal that generalization is being asked for. We are exploring how to continue, focus, redirect and aim children's existing language and experienced based generalizing orientations towards relationships between quantities in general. In particular, what the

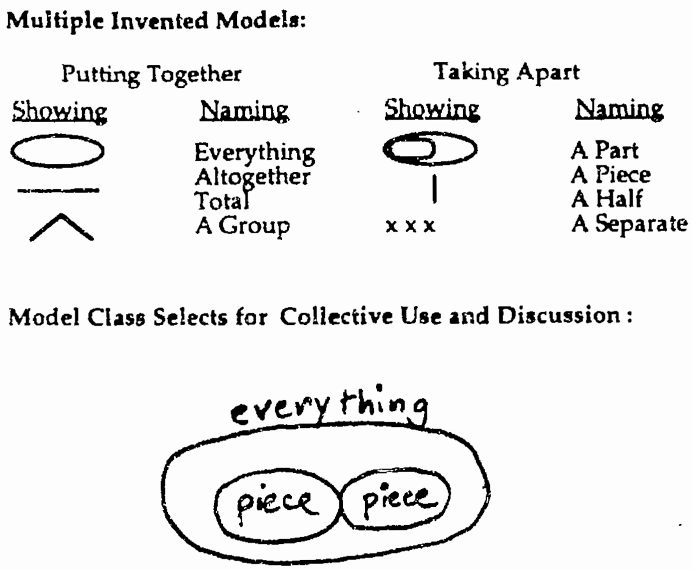

Figure 1. Class invents multiple models for "*any* situation in which we put quantities together or take them apart" and chooses one for collective use.

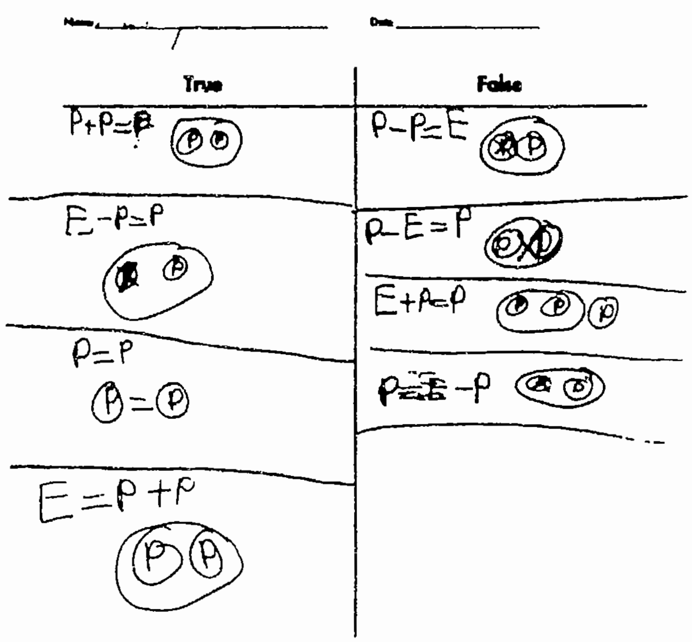

Figure 2. Children individually discover all the true or false statements they can about any put-together or take-apart situation

question focuses children upon is powerful grouping experience they already have ('putting together', 'taking apart'), and apparently can *intentionally* explore, discuss, and model in general terms (in response to the 'any situation in which we put quantities together or take them apart' question) if asked to do so. Once they do so they are able to construct models supporting the discovery of true or false statements about any put-together or take-apart situation, and demonstrate why those statements are true or false of any additive situation, via the model and verbally as well (Figures 2 and 3). But while an existing cultural base of generalization is being drawn upon, mathematical generalization (indeed, any theoretical generalization) requires selection and special constructions of language and experience to direct discovery along lines that are likely to be productive. The role of the question is to help select *from* and direct *towards*.

Which brings us to the next point. If the question is only aiming, rather than imparting, a generalizing orientation, what is it aiming children at? Essentially, 'What can we find out about *any* situation in which we put quantities together or take them apart?' explores aiming children towards the kind of research questions mathematicians undertake, the kind that constructs the field of mathematics itself. Notice that it is *not* focused at a problem solving level. Problem solving, even of the sort focused on discovering alternative solution methods, by definition focuses on the solution of a particular problem. The longer term aims of a sequence of problem solving episodes may remain tacit, unarticulated. Researchers, curricula and teachers of course have such aims, but they are not necessarily discussed with children. Yet mathematics, indeed, any intellectual field, is primarily about the construction of questions that overtly address such long term aims. We see above that children can in fact respond to such questions and begin to intentionally take on such aims. A trajectory of generalization across kinds of quantities and varying kinds of meaningful operations on them is not only under construction, but discussed with children and intentionally aimed for as such. Notice that in figures 4 through 7 length quantities and comparing operations are inquired about in the roughly same way (discovering what statements are true of lengths, where A = B (figure 4) and where B > S (figures 5-7). Figure 7 shows a collective representation of a range of true statements discovered, selected, posted, and discussed by groups of children, not unlike the way an intellectual field might collect, publicize and discuss knowledge.

P + P = E:

"Because you are adding one amount, which is this piece (points to the P in the model and statement) and the other amount (points to other P in model and statement) and altogether its Everything"

"Because if there's a piece and another piece it goes a whole Everything. Like 50¢ + 50¢ equals $1."

"Because there are 2 pieces and 1 piece is half of Everything and the other piece makes a whole Everything."

"Because 2 pieces mean Everything"

E − P = P:

"Because you're like separating out a piece and then its like there's another piece and there's like a part of an Everything. Its like a small part of Everything, cause you're taking out a small piece."

"Everything separate out a piece is a piece. True. Because if you take away a piece out of Everything, there'd be a piece left."

"Everything subtract piece equals piece, cause there's only 1 piece left. Because this is the whole thing. Say there's 6. Take away 3. And there's 3 more left."

"Cause if you put 2 ..., If you get a piece then you get another piece, then you put them together, you put both of them together, is Everything. Now, if you know you can do that, you can take 'em apart. And then it will be, like this is Everything (takes a dime out of his pocket). This is a ten. ... 2 nickles. I take 1 nickle out of it, and there's one nickle left."

Figure 3. Why is this statement true? Examples of oral explanations.

In sum, the role of the question is not merely to extend or 'transfer' *tacitly* general grouping dispositions from situation to situation, but to overtly direct attention towards those very wide ranges of possibility, and the means of capturing them, at which intellectual fields aim, in this case, any put-together or take-apart situation. Prior, general, grouping orientations are overtly focused by the question on the discovery of models and statements about them, publicly displayed, and used to

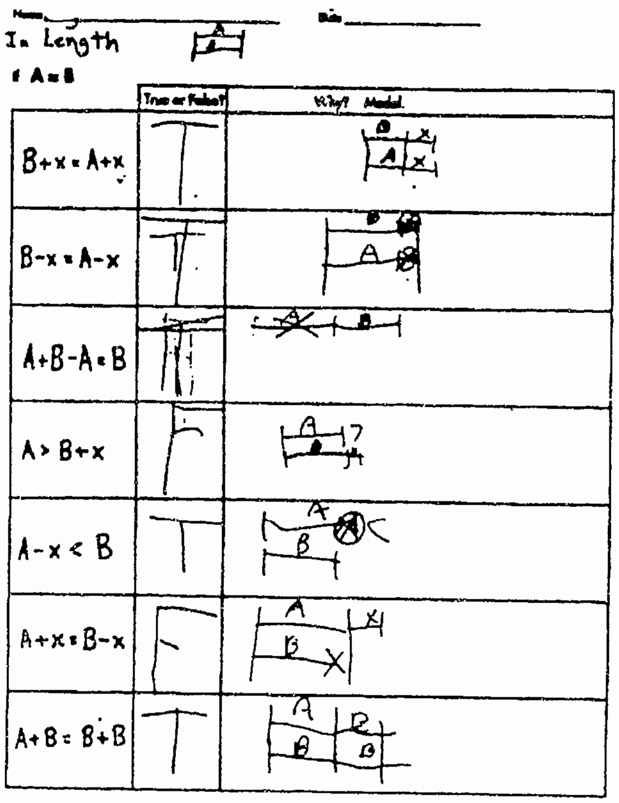

Figure 4. Discovering which statements about length situations are true or false, if length A = length B.

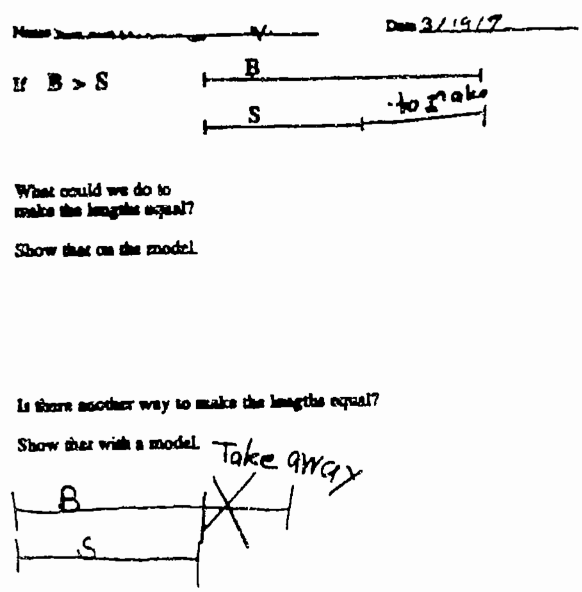

Figure 5. Initial construction of compare situations. Constructing the difference quantity (named the 'extra' by kids)

demonstrate true or false statements about 'any'. The existence, means, and value of intellectual aims are thereby signaled. And a wide range of potential use (any put-together or take-apart situation) is pointed out.

To the extent that the results touched upon above continue to hold, this 'from cultural generalization to mathematical generalization' perspective suggests that something more like a flow than a barrier, more like rapid and universal, rather than difficult and exceptional appropriation of mathematical generality, should be expected, unsurprising. It may be, in some domains at least, that the 'transfer dilemma' of learning psychology, and the 'discontinuities' of developmental psychology are artifacts of misdirecting children's attention. It is not just exclusive foci on 'skills' and 'the answer' that miscommunicates mathematical subject matter, but exclusive foci on problem solving and domain specific contexts as well. We need to communicate the aims and means of intellectual construction if we expect children to have some clue of the value possibilities of mathematics, so that they can orient their attention, aims, and constructive actions accordingly. The argument above is that when they do, in the situation above at least, mathematical generalization is not only enabled, but rapid.

Let's consider an expansion of that argument: mathematical generalization not only in the sense of some range that statements and models about 'any' may cover, but the effective leverage such statements and models promote in situated use. Efficiency is often claimed by basic skills approaches, effectiveness by problem solving approaches. I will discuss two further teaching experiments, because there the algebraic development occurs before certain 'skills' have been developed and problems encountered (2nd graders already have considerable prior additive experience). Hence they bear on the question of efficient skill and problem solving development

First, one cluster of teaching experiments aims kindergartners and older children towards the solution of broad ranges of word problems by *not* focusing them on solving the problem initially. Instead, I ask 'how can you show me which' questions. That is, given any additive word problem context, simple ones at first, such as Marta and Lucille sharing cookies, ask "How can you show me which are: All the cookies? Marta's cookies? Lucille's cookies?" This elicits a rich variety of grouping models and discourse, very like the algebraic models depicted in Figures 1 and 2 above, except that they do not refer to 'any' group or subgroup. By this means not only the simplest word problems (combine: total unknown, change: total unknown) but problems of intermediate difficulty (combine: part unknown, change: change unknown) become immediately accessible to kindergartners and first graders. At a computational level, once kindergartners can model groupings in a situation, they can use the simplest, most widely known computational method, counting-all, to solve more difficult problems (by counting-all within groups or subgroups they have constructed). The 'how can you show me which' question draws from prior general grouping understandings ('which'), and via grouping models kindergartners invent, directs them towards an expanded range of problem solving and computational skill, quite rapidly.

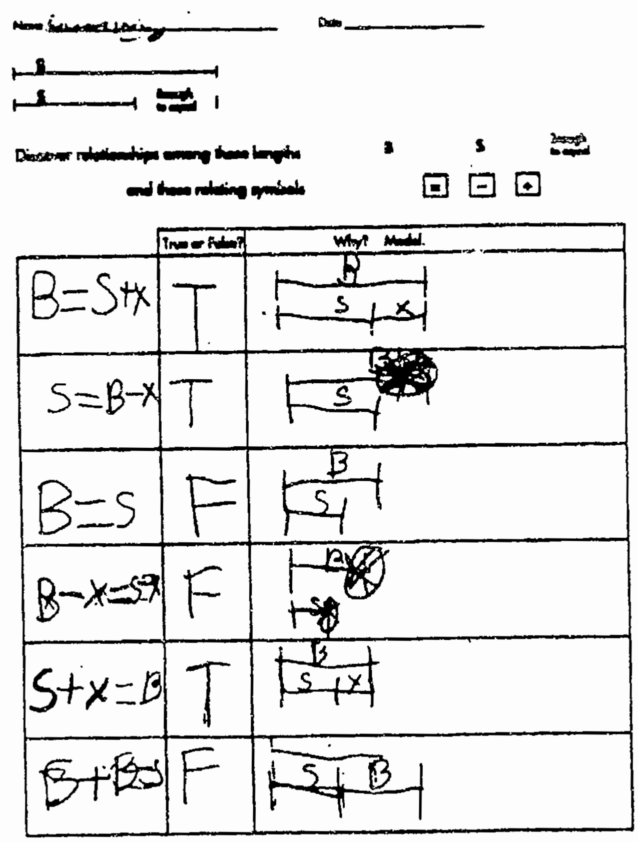

Figure 6. Children individually discover all the true and false statements they can about relationships among Big, Small, and 'Extra' (difference) quantities.

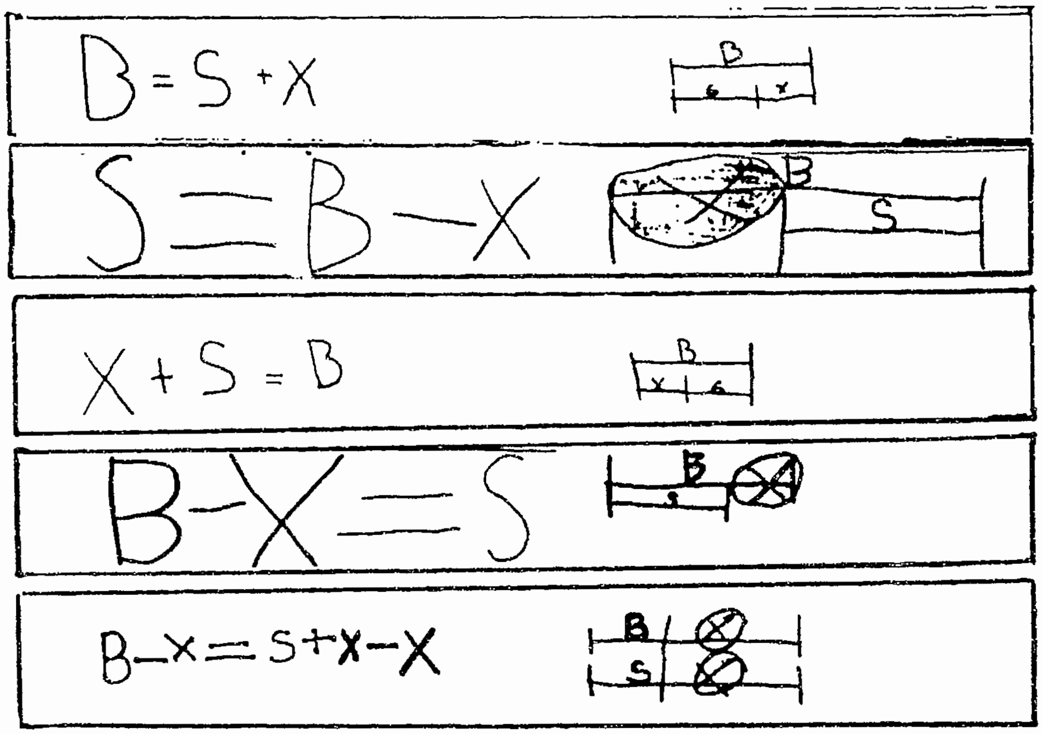

Figure 7. Small groups of children examine the range of statements and models of compare situations discovered individually, select all the different true statements discovered, make a large scale representation of those statements, and post them for the entire class to discuss.

Now let's go to the other end of the spectrum, 4th graders discovery of algebraic relationships in the multiplicative domain, and briefly touch on how this builds proportional problem solving and computational skill. Table 1 displays a progression of mathematical questions posed by the teacher (top row), and summarizes the algebraic discoveries children made in response to them. Again, children compare quantities and discover a broad range of relational statements about them, construct models and verbal explanations to demonstrate why these statements are true, and build inferential connections between alternative statements of the same relationship (If L = 3 S, then S = 1/3 of L, and L/3 = S, and L/S = 3, etc.), so that inferential shifts between multiplicative, divisive, fractional, and ratio perspectives on the same relationship can be made. But let's focus narrowly on 'skill' and problem solving objectives only. On the third day of this teaching experiment, shown at the bottom of the 5th column of Table 1, children begin to solve proportion problems, an important skill (and problem solving) objective. They are tested on proportional problem solving for unknowns in all four possible positions in the proportion the next day. 17/21 get every problem right (six problems total), 3/21 make one mistake, 1/21 makes two mistakes. Even if we ignore all else children learned in those three days, three days for nearly all of the class to learn to solve all four types of proportion problems at that level of accuracy constitutes very rapid and efficient learning (and strictly speaking, they learned proportional problem solving in one day). A look at another aspect of the test suggests why. Children are asked to also describe each proportional relationship (as well as find the unknown) for all the problems (e.g., 9/3 = ?/5, ?/2 = 16/8, 15/? = 25/5, etc.). 17/21 give at least two descriptions for each problem, 11/21 give 3 (e.g., 15/? = 25/5 is described as "5/1 ratio and L/5 = S and S x 5 = L). The ability to describe the relationship from both multiplicative and divisive perspectives, and shift from one to the other, enables solving for any unknown in a proportion problem. The algebraic focus on discovering multiple relational statements and inferential connections between them enables flexible proportional problem solving and speeds up the development of computational skill.

To sum up: We've considered the role questions may have in scaffolding children's entry into mathematical generalization. Questions can select *from* common (cultural) generalizing orientations and experience (about grouping, comparing, any), and direct *towards* models, statements, and rudimentary proof-like demonstrations about 'any' grouping or comparing relationships. This jointly communicates the existence of an expanding range of mathematical value possibilities and means to capture them. It may also scaffold effective, situation-specific problem solving and a rapid and efficient development of computational skill.

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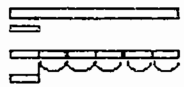
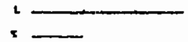
Table 1.
Progression of questions posed by the teacher and student discoveries.

Table 1 **Relational Description ⟶ Algebraic Generalization in the Multiplicative Domain**

How can we figure out how many Shorts are in a Long?	*How can we talk about the relationship of how many Shorts are in a Long?*	*Discover as many true or false statements of the relationship as you can. Why are these statements true (or false)?*		*Suppose L & S keep growing in the same relationship.*	*In what other situations might quantities grow together? Do they grow in the same relationship?*
Children invent multiplicative comparing methods. 	**Children discover many relational descriptions. Multiplicative, divisive, & fractional perspectives on a relationship build up.** 6 S = L 6 x S = L 2 S = 2/3 L L/6 = S, S = 1/6 L L = 6 S 2 S x 3 = L S x 6 = L 2 S = 2/6 L 3 S = 1/2 L 4 S = 2/3 L 3 S + 3 S = L L/3 = 2 S.	**Children construct models and verbal explanations to demonstrate why statements are true.**  3 x S = L S = L/3 2 S = 2/3 L 1/2 S x 6 = L L/6 = 1/2 S etc "L divided by 6 equals 1/2 S .. because, um, if you break L into 6 equal parts ... if 6 times half a small equals L, then L divided by 6 will equal half a small." (L/2 = S is false) "because L divided by 2 equals 1 1/2 S, not 1 S .. L/3 is smaller than L/2"	**Teacher aims children towards extending systematic inferential connections between statements.** *Do these statements mean the same thing? Why?* **If L is 3 S, then ....** **S is 1/3 of L** **and** **L/3 equals S** **and** **L/S equals 3** **etc.**	**Children discover relational statements as defining conditions for generating an infinite sequence of pairs growing in the same relationship and characterizing the relationship in general.** First, a child shifts a statement into a defining condition: (If you double L) "You have to double the S also, because S also has to always be 1/2 the length of L" Children use it to find unknowns and generate a series of pairs in the same relationship, e.g., $\frac{L}{S} = \frac{3\text{ cm}}{1\text{ cm}} = \frac{6\text{ cm}}{\_\_\text{ cm}} = \frac{\_\_\text{ cm}}{4\text{ cm}} = \frac{15\text{ cm}}{\_\_\text{ cm}}$ etc. $= \frac{\_\_\text{ cm}}{\_\_\text{ cm}} = \frac{\_\_\text{ cm}}{\_\_\text{ cm}} = \frac{\_\_\text{ cm}}{\_\_\text{ cm}} = \frac{\_\_\text{ cm}}{\_\_\text{ cm}}$ $\frac{24000}{8000} = \frac{90000}{30000} = \frac{18{,}000{,}000}{6{,}000{,}000} = \frac{45000}{15000}$ etc. Then characterize the relationship in general.: "The small into the large stays 3 because, um, the, um, small goes into it 1/3 each time, and the large is still a length 3 times the small." *Suppose I was to pluck out one pair from that series. Could you describe the relationship? If we know one pair, can we predict an unknown length in another pair?* **Children discover that from a single numeric pair they can extract a relational description and predict unknowns in the same relationship (i.e., solve proportion problems).** $\frac{15\text{ r}}{\text{r}} = \frac{25\text{ r}}{5\text{ r}}$ "5/1 ratio L/5 = S S x 5 = L"	**Teacher introduces relational research questions** (such as exploring the relationship between the height of a dropped ball and the height of the bounce) Children imagine and search for other 'as A increases, B increases' situations on their own Children hypothesize relationships, plan and carry out data collection to test their relational hypotheses. Then formulate their data in interlinked table, graph, and multiple equation forms. And discuss implications of the relational form of the data. **Children begin to initiate research questions on their own** For example: J's initiation of flashlight experiment (the farther the flashlight from the wall, the larger the circle of light) K's discovery of 1/n x A x n of S = L regularity (e.g., where L = 8 x S 1/2 S x 16 = L 1/4 S x 32 = L 1/8 S x 64 = L., etc. K and E.'s discovery of cross products and attempts to explain why they works

THE VARIABLE IN LINEAR INEQUALITY: COLLEGE STUDENTS' UNDERSTANDINGS

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Abstract: This doctoral study examined six undergraduate students' conceptions of variable in linear inequality. Subjects completed a written algebra test and participated in unstructured interviews in which they solved and interpreted multiple representations of six linear inequality problems. A strong understanding of the concept of inequality was exhibited by subjects with an advanced conception of the variable as a *varying quantity* and also by the subject with an elementary conception of the variable as an *evaluated* unknown. Results of the analyses also indicated that an advanced conception of variable as a *varying quantity* appears to be necessary in order to algebraically model, solve, and interpret solutions to linear inequality problems.

There is currently a call for all levels of algebra instruction to emphasize the development of algebraic thinking. Understanding the many and sometimes complex uses of the algebraic variable and understanding the fundamental mathematical relations of functions, equations, and inequalities are necessary conditions for this important goal of algebra instruction. Although much research has investigated students' understandings of variables in equations and functions, little has been done with inequalities. In addition to being important mathematical relations, inequalities form a rich context in which to study the concept of the variable as it is understood by undergraduate students, who have had years of experience with school algebra. Variables in inequalities are manipulated as when solving equations; however, variables in inequalities usually represent sets of numbers, which makes them similar to variables in functions.

The purpose of this doctoral study was to investigate college students' conceptions of the variable in linear inequality. Sfard and Linchevski's (1994) operational-structural theory of reification formed the fundamental theoretical framework for describing subjects' understandings of the mathematical concepts of variable and inequality. Küchemann's (1978; 1981) six categories of uses of the variable (*evaluated, ignored*, used as an *object*, a *specific unknown*, a *generalized number*, or a *varying quantity*) provided the framework to classify subjects' levels of understanding of the concept of variable.

Methodology

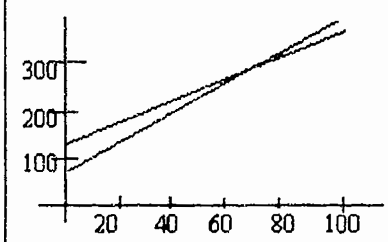
The sample for this study consisted of six college students enrolled in a variety of mathematics courses at a four-year New England college. All six subjects had successfully completed high school mathematics through precalculus; five of the six subjects had successfully completed at least one calculus course. Data were gathered for this study using a quantitative and a qualitative instrument. The *Chelsea Diagnostic Algebra Test* (Brown, Hart & Küchemann, 1985) a valid and reliable paper and pencil test, was used to quantitatively classify subjects' levels of understanding of the variable in expressions, equations, and functions into four hierarchical levels, Level 1 being the most elementary to Level 4, the most advanced.

The qualitative instrument was an Inequality Test designed by the researcher, consisting of six linear inequality problems that were presented in prose, symbolic, and graphical forms. The prose versions of the Inequality Test problems were modeled after the inequality word problems used by Goodson-Espy (1994) in her investigation of college students' abilities to solve linear inequality word problems. The characteristics of the Inequality Test problems were graduated in situational context, mathematical structure, domain, and type of solution. With respect to mathematical structure, Problems 1 and 2 were of the general form, $ax + b > c$; Problems 3 and 4 were modeled by the general form, $ax + b > cx + d$; Problems 5 and 6 were similar to Problems 3 and 4, but their mathematical models required the use of parentheses. With respect to the problem solutions, Problems 1 through 4 resulted in solution sets, but Problems 5 and 6 resulted in contradictions and had no solution.

The Inequality Test was administered to each subject in three sections during an unstructured, videotaped interview. In Section I, the subject was asked to talk aloud while solving six prose problems. During Section II, the subject was asked to solve and interpret each of the symbolic representations of the prose problem relationships from Section I. Finally, in Section III, the subject was asked to use given Cartesian coordinate graphs and the corresponding Section I prose problems to again solve and interpret each problem and its solution. Table 1 illustrates the three presentation forms of one of the Inequality Test problems.

Inequality Test transcripts were coded with Küchemann's uses of the variable, methods of solving the problems, and indices of strength of understanding of the concept of inequality. In order to detect trends, tables were prepared to represent each subject's uses of variable, methods of solving and interpreting the problems, and strength of understanding of inequality, for each problem across all presentation forms and for each presentation form across all problems. Narratives were prepared describing each subject's methods of solving the Inequality Test problems, describing relationships between subjects' performances on the Algebra Test with their methods of solving the Inequality Test problems, and describing subjects' uses of variables for each of the three presentation forms for each pair of similar problems (Problems 1 and 2, 3 and 4, 5 and 6). Finally, Sfard and Linchevski's theory of reification as it was applied to algebraic thinking provided a framework for interpreting the results of the study. Their theory specifies three phases of algebraic thinking: operational, fixed-

Table 1. Three Presentation Forms of Linear Inequality Problem #3

Section I: Prose	
Two fitness centers in one town are competing for business. The Fitness Factory is offering new 6-month memberships for $125 plus a fee of $3.25 per visit. Slimnastics, Inc. is offering new 6-mont memberships for $75 plus a fee of $4 per visit. Both centers have similar equipment and trained personnel. How many visits would a new member have to make during the 6-month period for Slimnastics to be the better choice?	
Section II: Symbolic	**Section III: Graphical**
$75 + 4x < 125 + 3.25x$	

value algebra, and functional algebra, which correspond, respectively, to the stages of interiorization, condensation, and reification of the concept of variable. Sfard and Linchevski's (1994) indicators of these phases of algebraic thinking were used to synthesize and frame the results of the study with respect to subjects' use of variables and their understanding of inequality. A case study for each subject was produced.

Results and Conclusions

Table 2 illustrates subjects' levels of use of the variable as revealed by the Diagnostic Algebra Test; their methods, arithmetic (did not use a variable) or algebraic (used a variable), of solving the prose and symbolic representations of the Inequality Test problems; their overall level of understanding of inequality; their overall use of the variable in the context of linear inequality problems; and their phase of algebraic thinking (operational, fixed-value algebra, or functional algebra) as determined by analyses of the Inequality Test interviews.

The six undergraduate subjects in this study exhibited a wide range of conceptions of variable. Eric, the one subject who tested at Level 1 understanding of the variable, was at an operational phase of algebraic thinking. He used arithmetic guess-check-and-revise procedures to solve the prose *and* the first two symbolic problems (of the form $ax + b > c$). Notably, when he reached the third and subsequent symbolic problems (of the form $ax + b > cx + d$), he said that he could not solve them. He

Table 2. Summary of Results of Analyses of Algebra Test and Inequality Test

ubject	Level of Use of the Variable	Method of Solving Prose Problems	Method of Solving Symbolic Problems	Level of Understand-ing of Inequality	Use of Variable in Linear Inequality	Phase of Algebraic Thinking
c	1	Arithmetic	Arithmetic (Guess & check)	Moderately strong	Evaluated	Operational
rl	3	Arithmetic	Algebraic	Weak	Used as object	Operational *Pseudostructural*
ıgie	3	Arithmetic	Algebraic	Moderate	Specific Unknown	Operational
b	4	Arithmetic	Algebraic	Moderately strong	Specific Unknown	Fixed-value Algebra
nise	4	Algebraic	Algebraic	Strong	Varying Quantity	Fixed-value Algebra
ınk	4	Algebraic	Algebraic	Strong	Varying Quantity	Functional Algebra

appeared to be mired in what Sfard and Linchevski (1994) describe as the 'process-product dilemma.' He saw the left side of a problem, such as $215 + 3x > 374$, as a prescriptive process and, without hesitation, used a guess-and-check procedure to *evaluate* the variable and compare his guesses with the number, 374. His inability to use these same arithmetic methods on Symbolic Problem 3 (illustrated in Table 1) indicated his inability to see that the results of his calculations could then be compared with each other. Filloy and Rojano (1989) describe a cognitive gap between students' abilities to solve equations of the general form $ax + b = c$ and equations of the form $ax + b = cx + d$. They call this gap the 'didactic cut.' Eric did not use algebraic methods to solve the former type of inequality. Yet his ability to solve Symbolic Problems 1 and 2 arithmetically, combined with his revelation that he could go no further when he met inequalities of the latter form, indicate that a 'didactic cut' appears to also exist for inequalities.

Despite his inability to work with variables unless he *evaluated* them, Eric displayed a rather strong understanding of the concept of inequality. In a manner consistent with Kieran's (1988) findings, Eric appeared to be interiorizing the concept of variable and strengthening the relational concept of inequality through his use of guess-check-and-revise procedures.

Two subjects, Carl and Angie, were classified at Level 3 by their performance on the Algebra Test, and analyses of the interview data revealed that they were at Sfard and Linchevski's operational phase of algebraic thinking. However, their work on the Inequality Test problems revealed different understandings of variables. Carl appeared to use variables as specific unknowns only in that manner described by Sfard and Linchevski (1994) as "pseudostructural." He efficiently and correctly solved Symbolic Problems 1 through 4, but was unable to interpret the meanings of either the variables or the critical endpoints in his solutions and could not interpret the meaning of the loss of variables in Symbolic Problems 5 and 6, which had no solutions. It appeared that Carl had never interiorized the concept of variable beyond that of an *object* to manipulate.

On the other hand, Angie's Level 3 performance on the Algebra Test was consistent with her understanding of variables as *specific unknowns* on the Inequality Test. However, Angie appeared to be stuck in an operational mode because she could not use a variable to represent the unknown quantity in the prose problems and, indeed, attempted to use two different input values to determine output values for some of the problems. In Prose Problem 3 (illustrated in Table 1), she stated that she would like to make the two fitness centers' fees equal, but that she could not do that. This appeared to be because she did not see that the processes producing the two fees could also be considered quantities. She was bothered by the 'process-product dilemma' (Sfard and Linchevski, 1994) in her attempts to solve the prose problems; this indicated that she had not yet made the transition to fixed-value algebraic thinking.

Among the three subjects who were classified at Level 4 understanding of the variable, there were also differences in their understanding of variables in linear inequalities. Bob exhibited a strong conception of variable as a *specific unknown*. His inability to interpret his solution steps or the resulting contradictions in the symbolic versions of Problems 5 and 6 of the Inequality Test suggested that he was at the fixed-value phase of algebraic thinking and had not made the transition to functional algebra. Bob was a junior Computer Science major and had developed considerable facility with symbolic algebra. Yet, like Eric, Carl and Angie, he did not generate algebraic models of the prose problems on the Inequality Test, but struggled to solve them arithmetically.

Denise and Frank, both classified at Level 4 understanding of the variable, were the only subjects who used algebraic methods in order to solve the prose versions of the Inequality Test problems. Both used variables as *generalized numbers* and as *varying quantities*. However, Denise had some difficulty interpreting her answers to Symbolic Problems 5 and 6 (the contradictions), and she relied on a guess-and-check process in several of the prose and symbolic problems in order to determine the correct direction of the solution set and to solve Prose Problem 6. This showed a strong understanding of inequality, but it was an indication that she had not yet reified the variable at its highest level and was still at the fixed-value algebra phase of algebraic thinking. Only Frank appeared to have made the transition to the functional algebra phase of algebraic thinking, as defined by Sfard and Linchevski (1994).

It is important to restate the fact that all of these subjects, except Eric, had successfully completed a traditional calculus course. Yet, only Frank, a senior mathematics major, consistently made connections among the representations of the inequality problems and displayed a deep and robust understanding of the variable. Undergraduate students are assumed to have attained a minimal level of understanding with respect to functions and their applications in order to succeed in many, if not all, college mathematics courses. The results of this study indicate that, despite a displayed proficiency with symbol manipulation, many college students do not demonstrate that level of algebraic thinking which Sfard and Linchevski call functional algebra, which is characterized, in this study, by an ability to use variables flexibly, to generate algebraic models of word problems, and to clearly interpret the resulting solutions.

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SUPPORTING STUDENTS' CONCEPTUALIZATION OF ALGEBRAIC EXPRESSIONS AND OPERATIONS USING COMPOSITE UNITS

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Abstract: In this paper we attempt to share understandings that we have developed by investigating how students' notions of algebraic expressions and operations might be fostered in a conceptually sound way. To that end, we describe the generalized candy factory instructional scenario that was used in a classroom investigation in a university developmental-level mathematics class. The analysis emphasizes the utility of the notion of composite units as a guiding construct in understanding and further developing students' conceptualization of operations with algebraic expressions.

Purpose

The developmental mathematics program at our university is designed to help students whose background in algebra is deemed inadequate. Students enrolled in these courses typically have had more than one year of algebra during high school. However, their understanding of algebra consists of an isolated repertoire of proceduralized skills, that they often perform inaccurately. Over the past six years, rather than simply addressing the symbol manipulation difficulties of these students, we have worked toward identifying and developing certain concepts and notions that are foundational to algebraic thinking and reasoning, as well as creating materials and methods to help students develop a conceptual understanding of basic algebra. In this paper we attempt to share understandings that we have developed by investigating how students' notions of binomial multiplication might be fostered in a conceptually sound way.

Methods of Inquiry and Data Sources

The data for this study was collected in a zero-credit, developmental-level mathematics class for entering university students. The class was comprised of 14 female and seven male students. One of the researchers conducted the class sessions during each classroom investigation. Data for the study consist of a graduate assistant's field notes, videotape data, and a documentation of the development and revisions of the instructional sequence.

Theoretical Framework

The overarching theoretical framework used to guide the study is a version of social constructivism, called the emergent perspective (Cobb & Bauersfeld, 1995). According to this perspective, interactionism and psychological constructivism are coordinated to account for learning and teaching. Interactionism is the social perspective that is taken from communal or collective processes, while psychological constructivism is the perspective that is taken from an individual's activity as he or she participates in and contributes to the development of communal processes (Cobb & Yackel, 1996). This theory is compatible with the Realistic Mathematics Education (RME) instructional design approach (Freudenthal, 1991; Treffers, 1987). Using this approach, students are supported in the "guided reinvention" of mathematical concepts through a process of mathematizing activity in problem situations that are experientially-real to the students. Reasoning with conventional mathematical symbols is an end-point of this process.

Composite Units

Our past research has indicated that young children develop increasingly sophisticated concepts of a unit as they participate in classroom situations in which the instruction has been designed to foster this development. We have also noted the difficulties that students experience in constructing composite units when the nature of the instruction is centered on teaching students how to operate procedurally on numbers. It is within this frame that we began to view students' conceptual difficulties with algebraic expressions and operations on algebraic expressions as struggles with constructing and operating on composite units in an unknown base.

Steffe's notion of a composite unit (1992 and personal communication) is useful for making sense of an individual's activity from a psychological point of view. According to Steffe, when a child has developed a notion of composite unit, she can coordinate units of different rank. For example, she can treat a number, such as 23, as a single unit comprised of 23 ones, or as a group of 23 individual units and can move back and forth between these conceptions and coordinate them in flexible ways. She might think of a group of 23 combined with a group of 25 more as two groups of 20 and 8 more, or possibly think of it as 2 fewer than two groups of 25. This is the type of flexibility needed to have an understanding of place value numeration.

In an analogous way, conceptualizing an unknown as a composite makes it possible to think of X+X as 2X where the 2X is the result of counting X units, followed by counting X units again. That is, 1, 2, 3,. . .,X, X+1, X+2, . . .X+X, where X+X is now seen as 2X. In the same way, an algebraic expression such as 3X-2 can, depending on one's current needs, be viewed as a single composite unit, as 3X-2 individual units, as 2 fewer than 3 units of (size) X, or as 2 units of (size) X and 2 fewer than another unit of (size) X. In this way, this conceptualization of algebraic expressions is analogous to conceptions of place value numeration. With this in mind, our teaching-learning goal was to develop an instructional sequence following the RME instructional design theory that would support students' notion of an algebraic expression as a composite unit and that would facilitate their operations on these units.

Instructional Sequence

The instructional sequence that we employ is adapted from a sequence developed for children to facilitate young children's development of place value numeration. For children, we use the scenario of a candy shop, in which pieces of candy are packaged in rolls of ten, ten rolls of ten pieces are packaged in a box, and ten boxes of ten rolls are packaged into a case. In our work, the scenario of the shop is also used, but now the candies are packaged into rolls of some specific, but unknown, quantity. Similarly, the same (specific but unknown) quantity of rolls are packaged into a box, the same (specific but unknown) quantity of boxes are packaged into a case. The purpose of using the scenario is that students' mathematical activity is grounded in real-world imagery. For example, to figure out the result of (3X+5) + (2X-1), students might imagine the activity of combining 3 rolls and 5 pieces of candy with 1 roll and a roll missing 1 piece of candy. (An earlier investigation of students' development of an algebraic expression as a composite unit within additive settings can be found in Underwood and Yackel, 1998)

To promote a notion of arrays, the following scenario is used. When candies are produced in the candy factory they come out of the machine onto a conveyor belt. The number of candies on each row on the conveyor belt is exactly the number needed to make a roll, that is X candies are in each row. Further, the conveyor belt stops after making X rows so that the candies can be packaged. Each row is packaged into a roll and the rolls are packaged into a box. An alternative description is that the machine makes an X by X array of candies before the conveyor belt stops so that the pieces may be packaged. The critical feature in this scenario is that the quantity X now takes on two different roles. On the one hand, X represents the quantity in a row (the number of candies in a roll), and, on the other hand, X represents the number of rows (the number of rolls in a box). Using the language of composite units, we would say that in thinking of the X by X array, the student would need to think of iterating a composite unit (consisting of X ones) X times. The conveyor scenario was further extended to situations in which the machine breaks down. The difficulties that are encountered may be of two types. First, the machine may make too many or too few candies in each row. Second, the machine may make too many or too few rows before it stops so the candies can be packaged. In some cases, the machine may make both types of errors at once. For example, the machine may make two extra candies per row and may make three fewer rows than it should before stopping. From our perspective, the situation just described can be thought of as an X-3 by X+ 2 array.

An Example

Below (Figure 1), we describe the small group activity of one group of four students during their work on the broken machine tasks. Although the students in this group were among the least conceptually advanced students in the class, they were among the most active participants in small-group and whole-class discussions. The analysis of their activity gives us an opportunity to consider the nature of the difficulties that students encountered in this instructional sequence. As can be seen below, the group is attempting to figure out the number of rows, the number of pieces per row, and the total number of pieces which were made by the broken machine that produced X+2 pieces per row and X+1 rows. Just prior to this episode, a research assistant had guided the group through a similar task. Therefore, this particular episode illustrates that the students' difficulties were not due to a lack of understanding of the conventions of the task. The analysis suggests that the students were unable to think about X simultaneously as the number of pieces in a roll of candy and the number of rows and, therefore, move flexibly between these two ideas. Prior to the episode, the group had concluded that the number of pieces in the array in figure 1 was $X^2 + 2X + 1X$. In the vignette below, the group is now attempting to figure out the number of rows and the number of pieces per row in this array.

Marla: For the number of rows, you got $X + 1$. No, $X^2 + 1$.

Christine: No. $X^2 + X$ *(the other group members agree with Christine).* That's the number of rows. Now we got to figure out the number

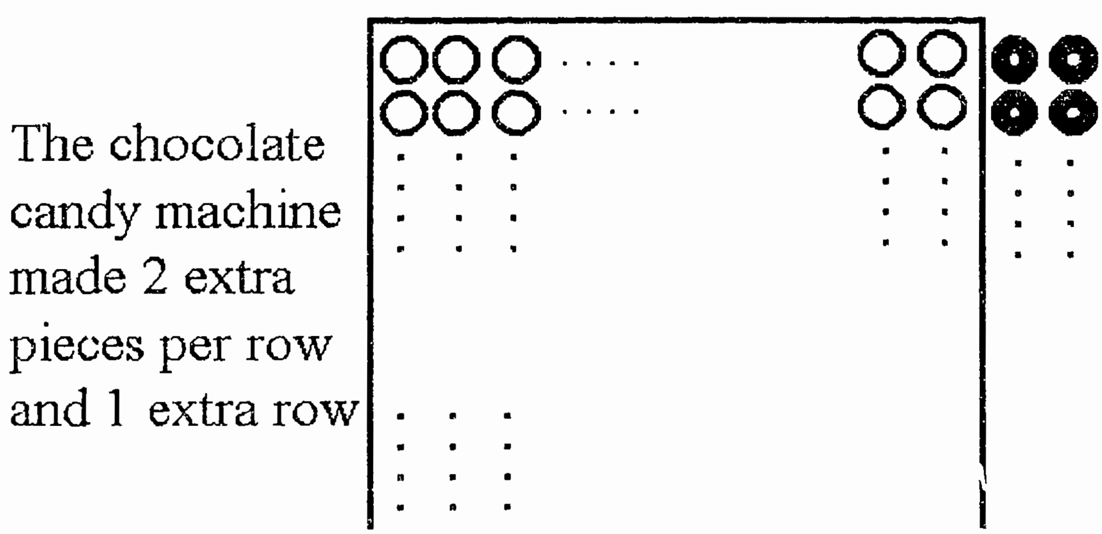

Figure 1

	of pieces per row, which is ,
Jesse:	The number of pieces per row? X.
Marla:	X. It's -- I mean, plus X.
Christine:	No. These are saying the same thing. How many pieces per row? X^2+2X? Because we got—this (pointing to the two columns of X) is our 2X from our X pieces. *(Several minutes later.)* Okay X^2+X is the number of rows. X^2+2 is the number of pieces per row, then X^2+X is the number of rows.

This vignette illustrates the group's inability to keep track of the units that they were counting. For instance, at the beginning of the episode, Marla correctly labeled the number of rows as X+1 but immediately changed it to X^2+1. While there are X^2 pieces in the square array, there is 1 extra row below this array. She was mixing the unit *pieces* with the unit of *rows*. Christine and Jesse had the same problem later in the episode. Christine stated that there were X^2+2 pieces per row and X^2+X rows.

While in subsequent tasks that involved the candy factory sequence there was evidence illustrating that this group had made significant progress in the coordination of units, the group's initial inability to keep track of the units that they were counting is similar to the difficulty that young children have in coordinating units of different rank in base 10 situations. For example, when asked to figure out the number of tens in 230, many second- and third-graders will count 10, 20, 30, … up to 200, while keeping track that they have counted 20 tens. Then, realizing that they have 30 remaining, they add 20 and 30 to get 50 and report that there are 50 tens. The children's activity indicates that they are not able to coordinate units of different rank. That is, they could not keep track of which unit they were operating in as they solved the task, in the same way that the group in the vignette could not coordinate the units that they were counting.

Analyses of the developmental-level students' activity in our study indicate that to distinguish between X as the quantity in a row (the number of candies in a roll) and X as the number of rows is an essential first step in conceptualizing the quantity of candies as an array. The activity of making this distinction is what Steffe (1992) refers to when he says, "For a situation to be established as multiplicative, it is always necessary at least to coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit."

Final Thoughts

The significance of our research extends beyond the analysis of this classroom data. Using the construct of composite units has provided us with a lens for interpreting students' mathematical activity and a framework for developing conceptually sound instruction We believe that other algebraic constructs may also be investigated by considering students' understanding of related arithmetical operations.

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PROBLEMS CONCERNING VERBAL ENUNCIATION IN GRAMMAR SCHOOL TEXTBOOKS IN MEXICO

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Abstract: The present article reports results of the analysis of verbal problems in grammar school textbooks in Mexico. The study aims at gathering relevant information leading to a better understanding of the kind of problems proposed to basic students before they are formally introduced to algebra. Our results reveal that *rate* verbal problems in their various modalities are found in most textbooks. In basic levels, a large amount of problem information is derived from charts and designs. These problems are notably found in fourth grade and decrease from there on. Finally, the inclusion of verbal problems in grammar school textbooks is sloppy.

Introduction

Our work is based on Bednarz and Janvier's research study (1996). We are, as they, interested in analyzing the nature of verbal problems presented to students, and their relative difficulty. This aims at providing new elements for a better understanding of students' transition to algebra. We are particularly concerned with grammar school education as a direct background, where arithmetic provides the students with a variety of procedures they may put to action when faced with "algebraic" problems in high school.

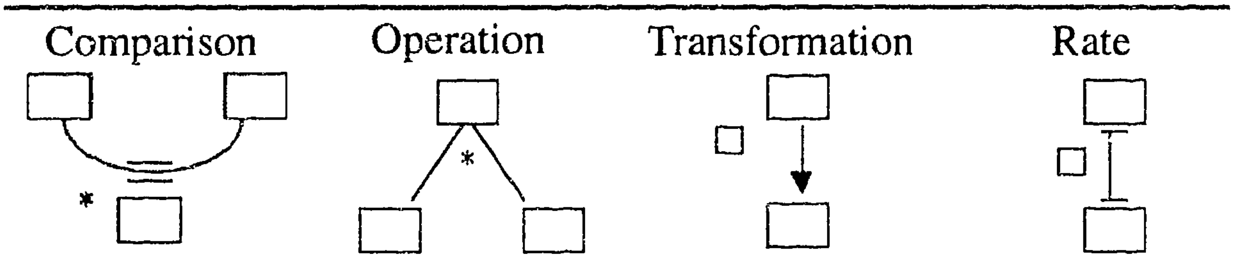
We intend to find out whether there is a spiral phenomenon in the design and structure of verbal problems in grammar school mathematics textbooks; if there are general patterns maintained in the problems presented in these books. We also intent to become acquainted with the elements that are incorporated or omitted in problems which makes them ever more complex, as well as to identify advantages and disadvantages of problems presentation array. We may hereafter make a proposal to facilitate the teaching of such verbal problems.

Theoretical framework

Among the diverse approaches supporting meaningful learning of algebra is Bednarz and Janvier's perspective on problem-solving (1994) that studies the transition from Arithmetic to Algebra in high school students. Bednarz and Janvier consider that the surfacing of algebraic thinking in a problem-solving context brings about reflection on the nature of the problems presented to the students in both the algebraic and the arithmetical domains and on the solving procedures that come to hand. In this respect, their research focuses on the setting up of a theoretical instrument allowing for a classification of problems used in the teaching of arithmetic and algebra. The inherent purpose of this theory is to understand students' difficulties when solving verbal enunciation problems and to understand as much as possible their reasoning in solving them. The theory makes prediction of problem complexity possible, whether arithmetical or algebraic.

Bednarz and Janvier (1994) have created a symbol system to support their theory. Their purpose is to separate the mathematical-relational structure from the context elements. In other words, identifying the nature of data (known and unknown), the relations among them and the structure of such relations. For example, addition problems, that have to do with transforming. The symbolism used by Bednarz and Janvier is the following: known quantities are symbolized by black squares; unknown quantities are symbolized by white squares. A question mark is placed over the white squares representing the problem's unknowns to be unfolded. There are four types of relationships among quantities: *comparison, operation, transformation* and *rate.* The symbolism representing them is shown in Table 1

Table 1. Relationships among quantities.



In Table 1, the sign "*" indicates the position of one of the operations: addition, subtraction, division and multiplication. The symbol = means that there is a comparative relation between quantities.On the basis of this symbolism, Bednarz and Janvier analyzed the verbal enunciation problems -arithmetical and algebraic- in different Mathematics textbooks at various educational levels. The three kinds of problems they identified were: *rate, transformation* and *unequal distribution*. The *rate* problems are those involving comparison between non homogeneous magnitudes; *transformation* problems involve one or several amounts or magnitude transformations, and in *unequal distribution* they deal with the whole-and-parts problem.

Methodology

The theoretical construct supporting this study was developed by Bednarz and Janvier (1996). All of the verbal problems in six grammar school textbooks were classified; problems were analyzed in terms of the known and unknown quantities, the relation between them and the kind of relationship involved, and they were classified according to the three problem categories that were identified: *rate, transformation* and *unequal distribution*. Finally, the problem presentation array is explained in terms of their structure. We may therefrom implement and experiment a didactical proposal focused on the improvement of the presentation array of diverse verbal problems, in order to give further didactical support to teachers.

Advances

The research process is currently dealing with the identification and analysis of verbal enunciations in grammar school textbooks. We have identified 234 verbal problems altogether which are distributed the following way: 7 % in first grade (1°), 22 % in second grade (2°), 16 % in third grade (3°), 26 % in fourth grade (4°), 17 % in fifth grade (5°) and 12 % in sixth grade (6°). Condensed information appears in the graph in Figure 1:

We have found that a high percentage of verbal problems (97% approximately) is arithmetical or *connected*. That is, the student may "build bridges" among the prob-

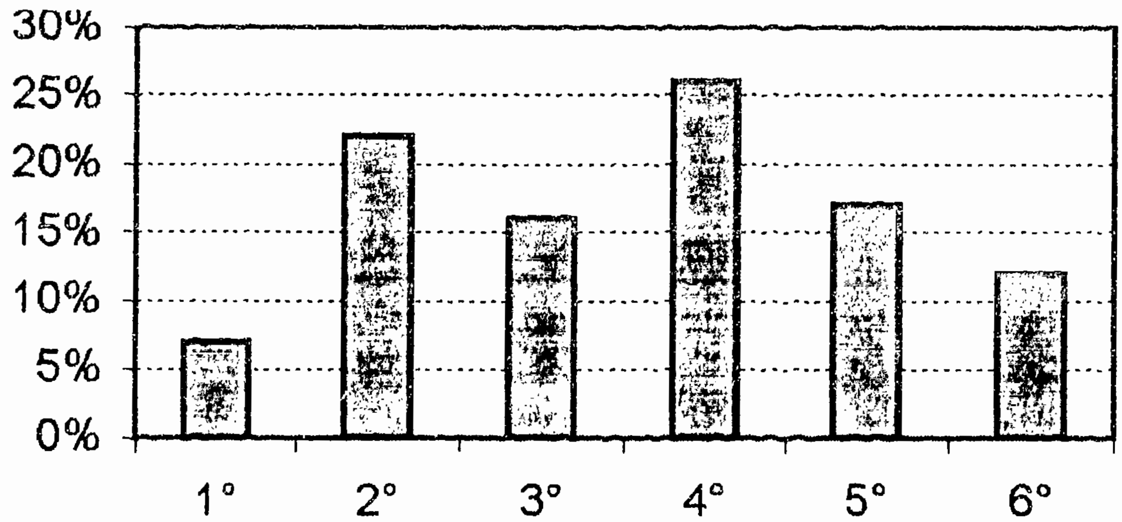


Figure 1

lem's data in order to unfold the unknowns and, then, solve the problem arithmetically. There is no need to work on more than one state simultaneously. When one is obtained, the rest can be obtained. For example: *If Ruben took 8 minutes to get from his house to the football field on his bike and the game started right after, how long was the game if you know that he burnt 208 calories altogether?* (The problem explains that six calories are burnt every minute when riding a bike and 8 calories per minute are burnt playing football [5th grade problem, page 83] [see Figure 2].

The scheme in Figure 2 shows the kind of relations between known and unknown quantities and the problem conditions. On the left you can see the known quantities;

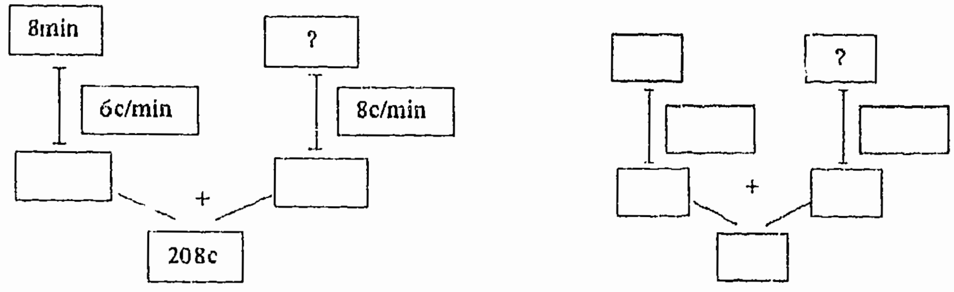


Figure 2

Bednarz and Janvier's symbolism (1996) is on the right. "Stressed" rectangles show the problem's data.

The other 3% are the so called *disconnected* problems, which do not allow for "bridges" among the known quantities. Various states must be operated at the same time, that is, the utilization of an equation becomes necessary. As a consequence, the student ends up exerting algebraic type of thinking. (See Figure 5 and its related problem further ahead.)

As to the general structure of the identified problems, we have classified them as *transformation rate* problems and *unequal distribution* problems. 95% of all of the problems are the *rate* kind. For example: *How many times can a 10 m long rope fit in a 100 m2 football field?* (5th grade problem, page 45) (See Figure 3.).

Transformation problems appear in a low percentage, 4% approximately. They are found in the last three grades of grammar school. For example,
The mountain gorilla walks with flexed legs measuring 1.75m. It is 0.25m taller when stretching its legs. The giraffe is 8.5m tall. What is the difference between the giraffe's size and the gorilla's with stretched legs? (4th grade problem, page 160) (See Figure 4).

Finally, there is only one problem involving unequal distribution:
"I'm glad you mention it because the bus has 18 double seats only and Mrs Dominguez and Mrs. Rocha and myself are going too to see to the group. We are 48 people altogether..." How many people are there in Chela's group? If there are five more girls than boys, how many girls and how many boys are there? (5th grade problem, page

Figure 3

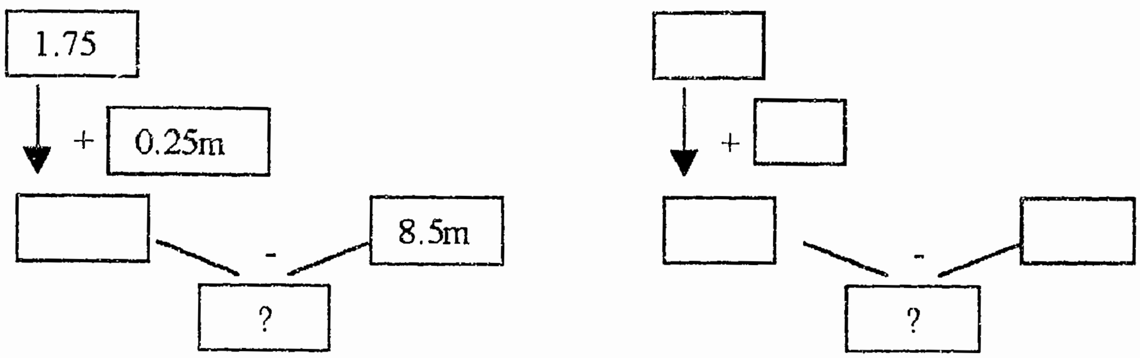


Figure 4

61) (See Figure 5).

Given that rate problems appear the most in grammar school textbooks, they were analyzed in terms of the unknown's placement. The rate is known in 197 out of

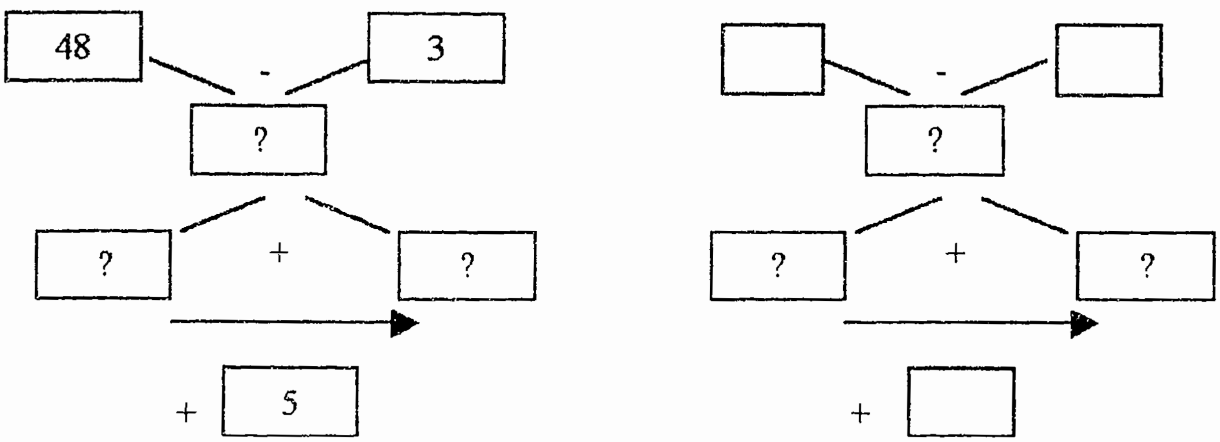


Figure 5

223 rate problems, whereas in the 26 remaining problems (12% approximately), the unknown is the rate. 43% of the 197 rate problems have the scheme in figure 6. Here is a problem example: How many legs must Carlos cut out to make 5 bears out of them? (2nd grade problem, page 56). 81% of the 26 rate problems having the rate as the unknown have the scheme in Figure 7. An example of this kind of problem follows: "I live in a 30 house little village. After the class, I walk to the parcel which is 2 km away from the village to take food to my father. Once I am back, I play with my friends." If an average of 180 people live in the village, how many people average live in a house? (6th grade problem, page 81).

Scheme distribution and their differences are the following: Most of the first grade problems are of the "*adding rates*" type. That is, 71% of the schemes are of the Figure

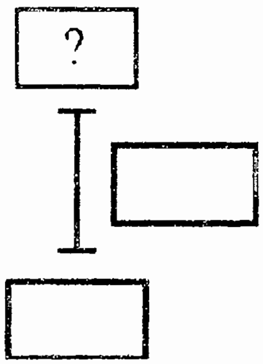


Figure 6

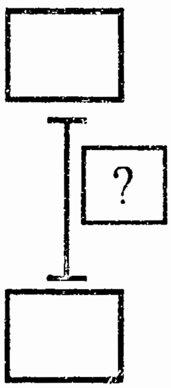


Figure 7

2 type. In second grade, the scheme type varies very little. 32% are of the Figure 2 type, but nere the operation is not only adding but subtracting; and 32% are of the Figure 6 type. In third grade, problems are again focused on the two previous schemes. 41% are of the Figure 6 type, 15% are of the Figure 2 type and a 10% are represented in Figure 7. In fourth grade, 41% of the problems focused again on figure 6, 11% on Figure 7 and 13% on Figure 2. In 5th grade, 33% of the schemes are of the Figure 6 type. Finally, in 6th grade 55% of the problems follow the Figure 6 scheme. The schemes of the remaining *rate* problems are different from one another in general and do not necessarily reappear by grade.

Conclusions

As a result of this advance we may conclude that, from a theoretical perspective, the presentation of verbal enunciation problems in textbooks has not been taken care of, and this happens in the first place because their occurrence is not frequent; most problems are found in fourth grade and the number decreases from there on. On the other hand, problems are concentrated mainly in three types of schemes: those represented by Figure 2, Figure 6 and Figure 7. What we have observed in the verbal enunciation problems we analyzed is the growing complexity of the quantities dealt with and the contexts in which they appear. In the first grades, students may resort to illustrations for support and in the last grades they must derive most information from the texts. On the other hand, the fact that we are in an arithmetical context does not necessarily imply that the problems presented to students must be reduced only to the so called "connected" problems. Introducing oneself into the solution of "disconnected" problems is also important.

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EXPLORING ALGEBRAIC THINKING IN FIFTH GRADE STUDENTS

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Davis (1985) recommended that students in elementary grades be given activities that promote algebraic thinking to serve as a foundation for the later, more formal study of algebra. This researcher examines early algebraic thinking and use of algebraic notation by eight fifth grade students who were brought together for a group interview after their class had been videotaped six times over a two-month period, working on tasks involving fractions. These students were accustomed to sharing ideas and justifying their solutions. The inquiry was motivated by a study of data collected during a three and a half year longitudinal research project conducted by mathematics educators from Rutgers University. Data came from videotapes of the classroom activities and the group interview, student work and researcher notes.

The problem discussed in this talk, required the students to divide a 12 meter length of ribbon into bows that were fractional parts of a meter long. The students discovered that dividing the ribbon into lengths of *1/D* resulted in "the denominator times twelve" number of bows. They first expressed this rule verbally using a specific example and then tried to write their rule using words and letters. Finally they refined this rule to a general case with a numerator other than one, using letters to represent words. This research gives evidence that children's generalizations are originally expressed in ordinary language and, given sufficient time and opportunity, students can develop symbolic notation to express their ideas.

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EARLY INTRODUCTION TO ALGEBRAIC THINKING: THE ROLE OF LOGO AS A LINK BETWEEN ALGEBRA AND GEOMETRY

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Transition from arithmetic to algebra has been studied by diverse authors from the perspective of general arithmetic, break evolution, reification, the sense of operations, symbol interpretation, and methods. The present article reports on a study about transition to algebra incorporating a teaching model that takes into account geometric background and proportional reasoning of the appearance of algebraic language, as well as generalization processes that allow access to more abstract algebraic thinking. In the history of the development of algebraic ideas, geometric thinking appears as a relevant background regarding the meanings underlying the symbols of early algebraic expressions (Witmer, 1983) as well as ideas of geometric proportionality (Radford, 1996).

In light of these considerations, a path to algebra that has been conceived incorporates meaning sources such as proportional reasoning (numerical and geometric), aspects of proportional variation and generalization processes (Mason, 1985) toward the construction of a teaching model (Filloy, 1999) in which students may build meaning sources beyond numbers and its operations. The teaching model thus proposed (Filloy,1999) intends to provide students with early mathematical experience as meaningful initiation to the learning of algebra. In this direction, the Logo environment is incorporated for the relationship among proportional numerical thinking, proportional geometric thinking and generalization processes to become explicit. The study includes 10-11 year olds and it involves a pre-questionnaire to analyze notions of geometric proportionality and the arithmetic handled by students at the beginning of the study; experimental work involving pairs of students working in a Logo environment; and a post questionnaire to study influences of the three components (numerical, geometric and generalization) in the construction of early algebraic notions. Another aspect of the study focuses on student-student, student-interviewer, student-environment interactions, through the study of meaning negotiation while discourse takes place. Structural analysis of the text by Clarke (1998) is included where social interaction levels proposed by Kieran and Dreyfus (1999) are identified.

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THE SCALE AS A FUNDAMENTAL FACTOR TO CONSTRUCT THE ALGEBRAIC EXPRESSION

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This paper presents a study of the algebraic expression based upon its graphic and the consequences that the scale has upon it. This means to identify the visual variables from this representation (Duval, 1994). These values are linked to the categorical values (Duval, 1994), that integrate the algebraic representation. However, the identification of visual values is not enough to construct its algebraic expression. It is necessary to have another treatment that allows the analysis of the visual variables from the numerical perspective. This treatment is named quantitative and is defined as the identification of the numerical values from the visual values. The identification of numerical values agrees on the use of at least two representations to analyse the contents of the graphical representation. Its final goal is to construct the algebraic expression using the scale factor. The modification of the scale factor affects the global apprehension of these visual variables. Therefore, the visual and numerical values change, although, the algebraic expression may or may not be affected. This generates two situations explored in Figures 1 and 2. These graphics present several modifications due to the scales, which are of figural order. In order to identify the modification of the numerical values due to the influence of the scales it is important to analyse the graphic in a global fashion. This treatment can be applied to the graphical and the numerical representation. For the experimentation, a group of 55 students from high school (ages from 15 to 17 years old) resolved different activities. The sequences of activities explore the graphic representation based upon the global interpretation based upon the straight line.

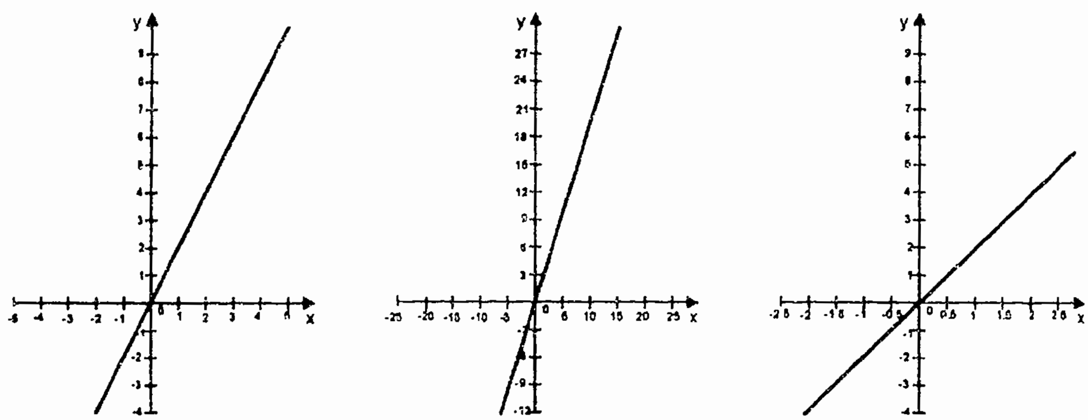

Figure 1. Different Scale, Unaltered Straight

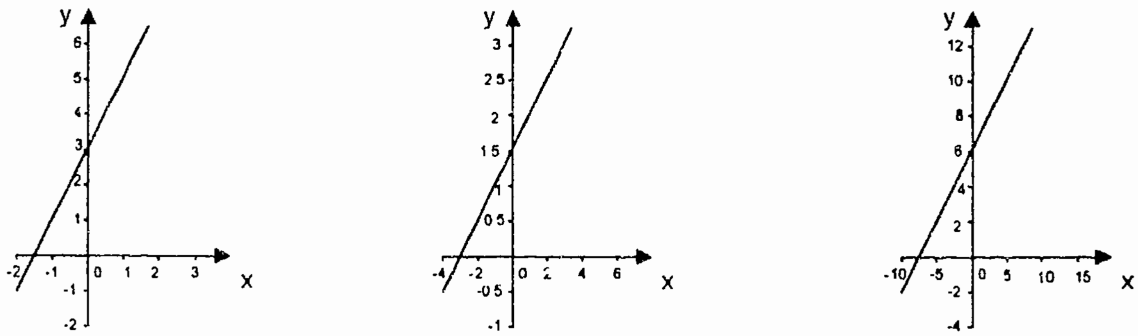

Figure 2. Different Scales, Different Straight Lines but Similar Figure Forms.

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GUESS AND CHECK AS A TRANSITION TO ALGEBRAIC THINKING

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This study examines whether the reasoning pre-algebra students use when employing a systematic guess and check strategy to solve algebra word problems helps promote algebraic thinking. First, algebraic thinking and arithmetic thinking are characterized. Using these characterizations, the reasoning of pre-algebra students' written solutions to algebra word problems are examined. The students' solutions take three forms: arithmetic, writing and solving an algebraic equation, and systematic guess and check. It is proposed that the reasoning used in the systematic guess and check solutions has a close resemblance to algebraic thinking and that developing this form of reasoning has the potential to help students in their transition from arithmetic to algebraic reasoning.

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Assessment

TEACHERS' PERSPECTIVES ON PERFORMANCE ASSESSMENT

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Abstract: In this research report, six mathematics teachers describe the process of writing and implementing performance assessment tasks for use across all mathematics classes at their middle school. Writing the tasks helped the teachers to articulate the entire mathematics curriculum. This process of identifying the important mathematics that they wanted their students to learn also unified the faculty and made supporting the learning of low-achieving students a priority at their school. The teachers also discuss how the project is one aspect of the mathematics reform effort that began at their school a decade ago.

Introduction

According to reform documents, student results on mathematics performance assessment tasks provide teachers with immediate feedback regarding students' mathematical strengths and weaknesses (NCTM 1995). Little research has been done to demonstrate just how teachers take advantage of the knowledge they acquire from the use of performance assessment tasks. Less is known about how teachers benefit by collaboratively writing, revising, implementing and scoring performance assessment tasks.

In this research paper, six mathematics teachers at Borel Middle School in San Mateo/Foster City School District in the San Francisco bay area describe how they were impacted by an assessment project that they undertook. Funded by the Coalition of Essential Schools, the teachers began writing performance assessment tasks in the summer of 1997 that aligned with their mathematics curriculum. The teachers decided to create performance assessment tasks for each unit of their mathematics curriculum to define the important mathematical ideas that needed to be taught in each unit. The goal of the project was to create performance tasks that would be cognitively demanding and assess students' understanding of important mathematical concepts.

A premise of the project was that reforms in assessment would promote higher order thinking in the teachers' classrooms (Kulm, 1991). The teachers also wanted to create tasks that did not simply assess facts and students' skills in isolation, but that would require students to apply their knowledge in real-life contexts. This approach is supported by researchers who advocate revising assessment practices to support changes in instruction that are based on how children learn (see O'Day & Smith, 1993). In addition, tasks were designed to be as authentic as possible, to require students to communicate their mathematical thinking, and to elicit a range of potential responses (Wiggins, 1993). I served as a facilitator for the project, giving teachers feedback about the tasks and the alignment of the tasks to their curriculum.

Research Participants and Site

April, Joanne, Judith, Maria, George and Mike were the six participants in the study. All six have extensive experience as mathematics teachers. Five of the teachers have more than 10 years of experience teaching at Borel Middle School. Borel Middle School is a school with approximately 850 students, with a student population that is 45% minority, representing about 35 nationalities. The six participating teachers had witnessed the demographics of the school change in the past decade from a predominately white, upper-middle class school to the school with the most socio-economic diversity in the district.

Methodology

The approach to voice scholarship chosen in this study was narrative inquiry. This research methodology allowed the research participants to define the important issues in their own terms (Riessman, 1993). In the spirit of a narrative inquiry approach to qualitative research, the research participants were asked several "broad questions" to motivate discussion, supplemented by "probe questions" to illuminate various aspects of the respondents' responses. Several major themes emerged from the interview that are illuminated below: 1) how the process of writing, revising, using and scoring mathematics performance assessment tasks impacted the teachers; and 2) how the project was an aspect of larger reforms that the teachers had initiated at Borel.

Research Findings

The Process of Writing, Revising, Implementing and Scoring the Tasks

Mike described how writing and revising the tasks impacted the mathematics teachers at Borel: "It's made us become more focused on certain concepts that we're trying to teach... Really has developed into a team and everyone has a significant role. Has helped with the pacing without compromising our personal style... Keep hearing mile wide, inch deep and I took it real seriously." By meeting regularly to discuss and write assessment tasks, the teachers were able to articulate the entire 6-8 mathematics curriculum at Borel. Maria discussed how the process of articulating the curriculum had additional benefits: "We have a common goal, doesn't matter if it's not your grade level. We're each interested in what's going on at various grade-levels and how students progress in math (from 6th grade on). This project was bigger than writing tasks.

It gave us a thread that keeps us very connected with each other's students."

Becoming more connected with each other's students over the course of the three years since the assessment project began has translated into changes in the teachers' conversations and practices. Maria alluded to these changes: "We're all very invested in the lower-end kids. Our discussion has changed over the past couple of years. Assessment, teaching, lead up to giving assessments have changed." It has become a priority at Borel for the teachers to work together to help all of their students succeed in mathematics. The project has clearly helped to further unify the faculty. According to Joanne: "We have something we're working on together consistently. If somebody's running something off, we run it off for everyone. So, the consistency from one class to the next has really become a practice here. We're all going in the same direction. Because we talk, we meet a lot, we're unified and we like each other."

During implementation of the tasks, the teachers noticed that most students viewed the assessments positively because their teachers had written the tasks. The students also understood the five-point rubric and how their work would be scored. According to Judith: "(We) gave them the rubric up front. Everyone has the possibility of getting a five. Everyone has the possibility to shine!" Data from the third year of task administration demonstrated that students are consistently performing better on each task at each grade-level from year to year. In addition, from examining students' work, the teachers have discovered that many students attempt to do the extra work necessary to achieve a score of five. This finding demonstrated that the tasks stimulated the students to do their best possible work, an important characteristic of quality assessment tasks (Cooney et al., 1993).

Lastly, the teachers have discovered that analyzing students' work on the tasks has improved their teaching. Maria found that: "By doing this, we picked up students' misconceptions. They obviously picked up the misconceptions somewhere, probably from us... I think of better questions to ask kids, help kids follow thinking of assessment. It definitely carries over (to instruction)." Maria also discussed how assessment influenced her view of the curriculum: "I go back to the units now to be sure that we're not spending too much time on periphery material."

Project as Part of Larger Reforms

This project was one of many reforms undertaken at the school during the past decade. The initial stages of reform at Borel Middle School involved detracking mathematics classes. George discussed de-tracking: "(We) did tracking for years, that's how everybody did it. When went to heterogeneous grouping, we weren't very good at it at the beginning because we had never done it before." April continued: "We also didn't have the curriculum for it, we had tracked curriculum." The faculty adopted a new reform mathematics series in the early 1990's that made it possible to move away from tracked classes at Borel.

Not surprisingly, as the mathematics teachers began to de-track their classes with the support of the administration, angry reaction from parents ensued. According to George: "We took it on the chin. Parents were lined up for conferences. We had math nights to explain (the reforms) to parents. First part of using (our reform curriculum) was brutal. Most problems now are with 6th grade parents." After almost a decade of de-tracking the mathematics curriculum at Borel, the only vestige of the traditional curriculum is several Algebra sections. The teachers were adamant in their belief that all students, not just their lower achieving students, benefited from heterogeneously organized mathematics courses.

There is also reason to believe that parents are satisfied with the changes in mathematics education at Borel. In a district-wide survey, parents were asked to respond to the statement: "I am satisfied with my child's instruction in mathematics." Seventy-nine percent of Borel parents taking the survey responded strongly agree or agree. This compares to the other three middle schools with percentages of parents who responded similarly at 66%, 62% and 56%. Interestingly, students at one of these schools consistently score higher on SAT 9 tests than students at Borel and the other middle schools in the district.

The teachers began working with elementary and high school teachers in the district to articulate the mathematics curriculum in the 1990's. April discussed how parents viewed the articulation efforts positively: "Another thing that helped a lot with parents is talking with the high schools, the math articulation team. When we can tell parents we know what they expect, that we're preparing them for high school, that's a big parental concern,." All the elementary schools in the district began using a NSF funded curriculum in grades K-5 several years ago. George stated: "The whole district is using (an NSF curriculum) at the elementary school. Now they have a calendar that they are asked to follow." Judith talked about the 6th graders at Borel: "Last year to this year it became much more obvious that the kids are getting it. We got a much more uniform group." The implementation of a challenging mathematics curriculum in the elementary schools in the district is helping to better prepare students for a more rigorous mathematics education at Borel Middle School.

Conclusion

This project proved to be an important piece of the reform puzzle for the mathematics teachers at Borel Middle School. The teachers indicated that the project provided them the time and resources to define their mathematics curriculum. Meeting regularly, the teachers at Borel articulated the entire 6-8 mathematics curriculum at the school and understood what their colleagues were teaching. The teachers' analyses of students' work on the assessment tasks informed their instruction by giving them insight about which topics needed further instruction and about student misconceptions. The students also strove to achieve high scores on the tasks, demonstrating that the tasks motivated the students to do their best work. Finally, the project supported

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MATHEMATICS INTERVENTION: THE IDENTIFICATION OF YEAR 1 STUDENTS MATHEMATICALLY "AT RISK"

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Abstract: In this paper I discuss *Mathematics Intervention,* a program established for students "at risk" of not succeeding with Year 1 mathematics. The program is based on current research that shows that students become numerate by progressing through five counting stages. The importance for classroom teachers to be able to identify each student's strategies and thus their counting stage will be stressed as a starting point for numeracy teaching in the early years. The presentation will highlight those strategies used in the intervention program that can be modified for classroom teachers to incorporate into their mathematics program.

Objectives

Mathematics Intervention is an ongoing research project involving the Principal and staff of a state elementary school in the eastern suburbs of Melbourne, and a mathematics educator from a nearby university (Pearn & Merrifield, 1996). Developed in 1993 *Mathematics Intervention* was designed to identify, then assist, Year 1 students "at risk" of not coping with the mathematics curriculum as documented in the National Statement on Mathematics for Australian Schools (Australian Education Council, 1991). *Mathematics Intervention* features elements of both Reading Recovery (Clay, 1987) and Mathematics Recovery (Wright, 1991) and offers students the chance to experience success in mathematics by developing the basic concepts of number upon which they build their understanding of mathematics. Students are withdrawn from their classes and work in small groups with a trained specialist teacher. This paper focuses on the results of clinical interviews conducted to identify Year 1 students considered "at risk" and needing to participate in the *Mathematics Intervention* program.

Theoretical Framework

The theoretical framework underpinning *Mathematics Intervention* is based on recent research about children's early arithmetical learning (Steffe, von Glasersfeld, Richards & Cobb, 1983; 1988; Wright, 1991) and about the types of strategies used by students to demonstrate their mathematical knowledge (Gray & Tall, 1994). In particular, the *Mathematics Intervention* program documents and promotes students' progression through the counting stages (Steffe et al., 1983, 1988) which are summarized below:

1. *Perceptual.* Students are limited to counting those items they can perceive.

2. *Figurative.* Students count from one when solving addition problems with screened collections. They appear to visualize the items and all movements are important. (Often typified by the hand waving over hidden objects.) If required to add two collections of six and three the student must first count the six items to understand the meaning of "six", then count the three items, then count the whole collection of six and three.

3. *Initial number sequence.* Students can now count on to solve addition and missing addend problems with screened collections. They no longer count from one but begin from the appropriate number. If adding two collections of six and three, students commence the count at six and then count on: six, seven, eight, nine.

4. *Implicitly nested number sequence.* Students are able to focus on the collection of unit items as one thing, as well as the abstract unit items. They can 'count-on' and 'count-down', choosing the most appropriate counting strategy to solve problems. They generally 'count down' to solve subtraction.

5. *Explicitly nested number sequence.* Students are simultaneously aware of two number sequences and can disembed smaller composite units from the composite unit that contains it, and then compare them. They understand that addition and subtraction are inverse operations.

Gray and Tall (1994) have shown that young students who were successful with mathematics use different types of strategies from those who were struggling with mathematics. Students struggling with mathematics, were usually procedural thinkers dependent on the procedure of counting and limited to strategies such as "count-all." Gray and Tall (1994) defined procedural thinking as when

> ... the numbers are used only as concrete entities to be manipulated through a counting process. The emphasis on the procedure reduces the focus on the relationship between input and output, often leading to idiosyncratic extensions of the counting procedure that may not generalize (p. 132).

For example, when asked to give the number before a given number, students were heard to count up to each number before responding. While some students were dependent on rules and procedures other students gave instantaneous answers. When asked "How did you do that?" they usually gave several different strategies they could have used and checked that their solutions were correct. According to Gray and Tall (1994), this use of known facts and procedures to solve problems, along with a combination of both conceptual and procedural thinking, indicated that these students were proceptual thinkers. Gray and Tall (1994) defined proceptual thinking as

> ... the flexible facility to ... enable(s) a symbol to be maintained in short-term memory in a compact form for mental manipulation or to trigger a sequence

of actions in time to carry out a mental process. It includes both concepts to know and processes to do (pp. 124-125).

Methods

The initial assessment for the Year 1 *Mathematics Intervention* program required teachers to assess the extent of the student's mathematical knowledge by observing and interpreting the student's actions as he/she worked on a set task. Researchers have advocated encouraging students to talk about their mathematical strategies as the superior method of obtaining information on students' own mathematical constructs and knowledge (see for example, Peck, Jenks & Connell, 1989).

A clinical interview protocol was developed, administered and consequently modified by three teachers. This is called the Initial Clinical Assessment Procedure-Mathematics [ICAPM] -Level AA (Pearn, Merrifield, Mihalic, & Hunting, 1994). By carefully observing the student's solution methods, interviewers ensured that they were aware of the strategies being used and if required the following prompts were given: "How did you work that out?" or "How did you do that?"

Data Sources

Since 1993, 357 Year 1 students have been clinically interviewed at the beginning of their second year at school. Teacher-clinicians used the ICAPM -Level AA protocol. Each clinical interview took approximately ten minutes and included tasks that ascertained students' verbal counting skills, their knowledge of the number word sequence and tasks that helped ascertain their counting stage level. For example, the verbal counting tasks included

"Can you count out loud for me, beginning at one, until I tell you to stop?"

"What number comes *after* 4?"

"What number comes *before* 15?"

In a counting stage task, six counters were displayed and three other counters were hidden under paper:

"There are six counters on the table. Can you count them?"

"Under this paper there are three counters." (Lift paper briefly)

"How many counters do I have altogether?"

The results from these clinical interviews were recorded by the interviewers and have been collated and analysed by the researcher.

Results

The clinical interview results shown in Table 1 indicate that most Year 1 students were successful counting forwards by ones to 20 and backwards by ones from ten,

Table 1. Year 1 clinical interview results (in percentages).

Year	ones	Back 20-1	Back 10-1	twos	fives	tens	6 -12
1993-1999	97	60	93	98	41	64	62

Year	before/ after	14 beads	pattern	numeral	6+3-	10- +2
1993-1999	77	86	83	41	76	62

counted patterns of dots and counted out exactly 14 beads. They were less successful identifying the numbers between the numbers six and twelve or determining numbers "before" or "after" a given number.

Year 1 students considered mathematically "at risk" and in need of additional assistance in the *Mathematics Intervention* program generally exhibited the following characteristics:

1. Students had difficulties in elaborating the number sequence. For example, they
   - used the right words but in the wrong order: one, two, three, four, six, five.
   - omitted a number: one, two, three, five, six.
   - confused two number sequences when counting by ones: 10, 11, 12, 30, 40, 50, 60, 70, 80, 90, 20.
   - experienced difficulty in counting backwards from 20. This was hardly surprising as in most cases students' forward counting sequence was not accurate.
2. Students exhibited little or no one-to-one correspondence. Their verbal number sequence was not consciously co-ordinated with the actual counting of objects.
3. Students exhibited confusion with place-value concepts. For example, 13 and 31 were both considered 'thirteen' as they had "a three and a one."
4. Students were also receiving additional support for reading, that is, Reading Recovery. However, not all students receiving Reading Recovery needed assistance for mathematics.

. Students generally had poor language skills. When asked to explain their response to a task their explanations would include expressions such as 'I knowed.' and 'It was in my brain.'

6. If unsure of a response students guessed with no check as to whether the response was sensible or logical.

The interviews highlighted the differences in students' mathematical knowledge and the types of whole number strategies they used when solving tasks in different contexts. Successful Year 1 students counted fluently by ones, twos, fives and tens from a given number, and demonstrated their ability to choose and use an efficient and appropriate strategy. These students appeared to exhibit proceptual thought (Gray & Tall, 1994) and were articulate in their responses. Year 1 students requiring *Mathematics Intervention* experienced difficulties with the verbal counting sequence and were at either Counting Stage 0, or 1. When unsure of an answer these students guessed with no attempt to confirm their answer. All students' answers used minimum words and highlighted poor language skills.

Conclusion

The importance of providing additional assistance as early as possible to "mathematically at risk" students cannot be over-emphasised. There will always be a need for a program such as *Mathematics Intervention* that is specifically designed to cater for students "at risk" in the early years of schooling. By being aware of each student's mathematical knowledge, types of strategies usually used, and language ability, the teacher should be able to design appropriate activities to extend each student's mathematical understanding and language.

This paper reports findings from an Australian research study designed to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics of young students in their early years of schooling. The research builds on previous international research in psychology of mathematics education and implications for teaching and learning. The findings that show some students required assistance in both mathematics and reading highlights a need to promote and stimulate interdisciplinary research in the area of young students mathematically "at risk" with the cooperation of psychologists, mathematicians and mathematics educators.

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CRITERION REFERENCED TESTING: AN OPPORTUNITY FOR RESEARCH AND REFORM

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It was predicted in the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989, p. 189) that "A common response to the challenge of the *Standards* [would be], 'Yes, but who will change the tests?'" The tests have indeed changed. The national scene for evaluation and assessment in the schools has evolved from standardized tests and tests of basic skills to more comprehensive assessment that also includes criterion referenced tests. States also began to develop curriculum frameworks based on the *Standards*. Criterion referenced testing aligned with these frameworks was then developed. Of course these new tests have had their critics: "Isn't this the tail wagging the dog?" "Aren't we just teaching to the test?" "This is just a fad that will go away."

However, these tests are not going to go away. They reflect what is valued in mathematical learning and strongly influence what is taught. They can provide valuable information for mathematics educators in the areas of reform and research. While test results have caused backlashes against *Standards*-based curricula and the use of calculators in the classroom, we do know that we get a different understanding from administrators and teachers who have truly participated in curriculum/testing alignment activities. They see that teaching skill in isolation and using traditional curricula will not prepare students for problem solving items or the writing required on open-response items of these tests. They are more willing to decide to use innovative materials and technology and to adopt *Standards* based programs, and test results are beginning to support their decisions.

The mathematics educators' interpretation and analysis of test results is crucial in the support of these teachers and administrators who need a data- informed position on *Standards*-based curricula compared to traditional curricula. Item analysis enables educators to draw conclusions about instructional needs, progress toward the goals of a curriculum, and evaluate the effectiveness of a program. Furthermore, this data can be a tool to study professional development and teacher knowledge.

The discussion will focus on three main issues: 1) What are the testing programs in different states and countries and what is the impact on the schools and teaching in the region? 2) What do the scores show about statewide systemic initiatives and about individual districts? 3) What is the impact on professional development at the preservice and inservice levels?

We will begin with an overview of state scenarios in which we have had experience and with additional states' information from our recent inquiries and from discussion participants. Next, we will present examples of presentations and interpretations of state data and ask for input from the audience. We also will show how we are incorporating data presentations into curriculum alignment and other professional development activities. These activities have formed a basis for a research project in professional development.

Participants should bring information about their state or country assessment plans and projects that use the assessment data in research or professional development. We will provide a handout that gives an overview of large-scale assessment and a general outline of alignment activities.

LESSON QUALITY IN MATHEMATICS CLASSROOMS AND THE INTRODUCTION OF A STATE TESTING PROGRAM

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In this report, analyses are presented of the mathematical content, investigatory practices, and lesson quality observed in mathematics classrooms whose teachers were involved in over 100 hours of professional development over a four-year period. In 1995, the National Science Foundation initiated the Local Systemic Change through Teacher Enhancement program with a goal of providing professional development to teachers for the implementation of high-quality mathematics curriculum. Cohen and Hill (1998) reported that educational policy can play a favorable role in influencing teachers' practice. However, during the data collection period, a state-wide standardized testing program was introduced along with other educational policy changes that were a source of concern for the teachers. The study was undertaken to ascertain what, if any, impact the testing program had on the quality of the mathematics lessons that were observed.

In each year of the four-year study, ten classrooms were randomly selected for observation from a large school district in California that has adopted *MathLand* (Creative Publications, 1995), designated a promising program by the U.S. Department of Education expert panel. Pre- and post-observation interviews were conducted. Lessons were assessed according to the current standards for mathematics education (NCTM, 1989).

The results indicate that, in general, the lesson quality was higher when the curriculum was followed more closely. Higher quality lessons had less time devoted to computational practice than lessons rated lower on the evaluation scale. Generally, there was an increase in rigorous mathematical content and investigatory practices over the first three years of the project, however, a downward shift in the quality and duration of learning opportunities in the classrooms occurred in the fourth year after the institution of the state testing program and the reporting of the state-wide school rankings based upon standardized test results.

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Discourse

TALKING MATHEMATICS WITH THE TEACHER WHILE WORKING IN SMALL GROUPS IN AN ALGEBRA I CLASS: SOME DIFFERENCES AND SIMILARITIES IN DISCOURSE FOR STRONGER AND WEAKER STUDENTS

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Abstract: The purpose of this study was to better understand what and how students learn through their talk by examining the discourse between teacher and student as students worked on problems in small groups in a diverse Algebra 1 class. This summary report is an overview of group results based on data gathered in relation to the questions: How do students differ from each other in their talk about their mathematical work? and How does the teacher respond to differences in students' talk about their mathematical work? The full report includes further analyses of individual transcripts and provides more detailed discussion of differences and similarities among students.

The national reform effort in mathematics education calls for a constructivist approach to teaching mathematics. Recently developed curricula, including the program used in this study, are based on the theory that students construct their own understanding of mathematics and that teachers and materials can be prepared to better serve students in helping them to develop this understanding. These materials generally include problems designed for small group work so students can talk about their work as they are doing it.

Articles written in the mid-nineties urged researchers to study classrooms in which teachers were attempting to help students develop their own understanding through the social negotiation of meaning as they worked together and talked with each other and the teacher (Cobb, 1996), and there is still much to be learned in relation to language use in mathematics classrooms particularly high school. Few researchers in mathematics education work as regular teachers in classrooms and report on their attempts to work with diverse groups of children in that setting. Lampert (1985), Ball and Wilson (1996), Parker (1993) and Romagno (1994) have been exceptions. Lampert at the middle school and Ball in a third grade classroom have focused their work on generating thoughtful whole group discussions. Parker at the fifth grade level and Romagno in a ninth grade basic mathematics class, team taught with the regular teacher and focused on developing alternative curricula and teaching methods to engage students in thinking more deeply about mathematics.

There are not long term studies of high school classrooms in which the researcher is the teacher, nor are there year-long studies focused on the discourse that occurs between students and teacher as the teacher circulates to work with the students as they work on problems in their small groups.

Methods

Two questions guided this study. How do students differ from each other in their talk about their mathematical work? How does the teacher respond to differences in students' talk about their mathematical work? Because these were two of seven questions about what happens in a typical classroom when students work in small groups, I used an ethnographic approach. I arranged to teach an Algebra 1 class from October to June, in an inner-city school with a mixed population of Asian, African-American, Latino, and White students. The class used Mathematics 1 materials, a replacement course for Algebra 1 designed to enhance students' problem solving, reasoning, and communication skills and to use the other areas of mathematics, geometry, graphing and functions, probability and statistics, as a basis for understanding algebra.

As the teacher, fully responsible for all aspects of the class including attendance, grading, and talking with parents, I could be an insider, but because I was only teaching one class I would still be an outsider in the some important respects. While I did not live through a regular teacher's long and demanding five-period teaching day, my other full-time responsibilities did not allow me any more preparation time for teaching than a regular teacher wo ' have for such a class, so it was easy to stay with my plan of using the materials as recommended and not supplementing or changing them significantly. Using the materials as they were written was important to my goal of working in a classroom situation that might be considered close to a normal class where I would experience many of the same pressures and dilemmas as a regular teacher (Ball and Lampert).

Data

I gathered data in the four categories described by Eisenhart (1988): participant observation, interviews, collection of artifacts, and reflections including "emergent interpretations, insights, feelings, and the reactive effects that occur as the work proceeds." (p.106). To supplement my observations as a participant, I used an audio tape recorder. To record teacher-student exchanges, I wore a small tape recorder and an external microphone as I moved around the classroom from group to group. Artifacts gathered included written work and records: all the assignments the students did throughout the year, my grade book records, their mathematics grades in previous and following courses, and information from the counseling office.

The class was composed of 33 students who were in grades 9-12. The larger study focused on the 21 ninth and tenth graders. The analysis in relation to change in discourse is based on a chronological sequence of transcripts for each of the eleven ninth and tenth grade students that were in the class from November through May and for whom I had both early and late recordings on the 28 classroom tapes that were transcribed. Analysis was based on on-task talk, which meant the exchange was related to the mathematics we were working on. On-task turns were categorized as *About, In, With,* or *Beyond* mathematics based on an idea from Brenner's (1995) work.

Another part of the analysis included identifying topically related sets, TRS's (Cazden, 1988) in the on-task talk and looking for discourse patterns in those sets. In considering differences among students, I examined the initial questions they used to open the dialogue and categorized opening turns as requests for help, requests for verification, or requests for clarification. I also categorized degrees and forms of closure.

At the end of the school year, before transcribing all of the tapes, I ranked the 21 students from strongest to weakest (numbered 1-21) based on my holistic assessment of their level of understanding of algebra, problem solving, their facility with algebraic operations, and the likelihood of success in Geometry and Algebra II. As I considered the use of language *In, With,* or *Beyond* mathematics, I realized that much of the language *In* mathematics used by the weaker students appeared in one to four word phrases. I then re-examined a sample of the transcripts of all the students to classify on-task turns as questions, phrases, or statements.

As I compared TRS's three forms of teacher-response emerged. Level 1 involved questions or prompts to get started, but then leaving the work to the student. Level 2 included explanation and direct instruction and usually meant longer turns for the teacher. At Level 3 the teacher became involved in working through the problem with the student.

Results

As might be expected stronger students required less closure and posed a much higher percentage of initial questions that asked for verification, while weaker students were more likely to ask for help. Figure 1 shows the relationship between class rank (1-16) and percentage of initial questions that were for verification ($r = -0.57$). Only 16 students are included because the other five almost never initiated a TRS.

On the scatter plot Hoang and Sam are not in line with the others, and Hoang is clearly an outlier. Without Hoang the correlation is $r = -0.76$ and without Hoang and Sam $r = -0.89$. As an able problem solver, Sam never hesitated to ask for help, unlike most of the other good problem solvers who were not so sure of themselves. Hoang, on the other hand, was not as strong a problem solver as his class rank might imply. He was an extremely hard-working and thorough student who asked about everything he was unsure of.

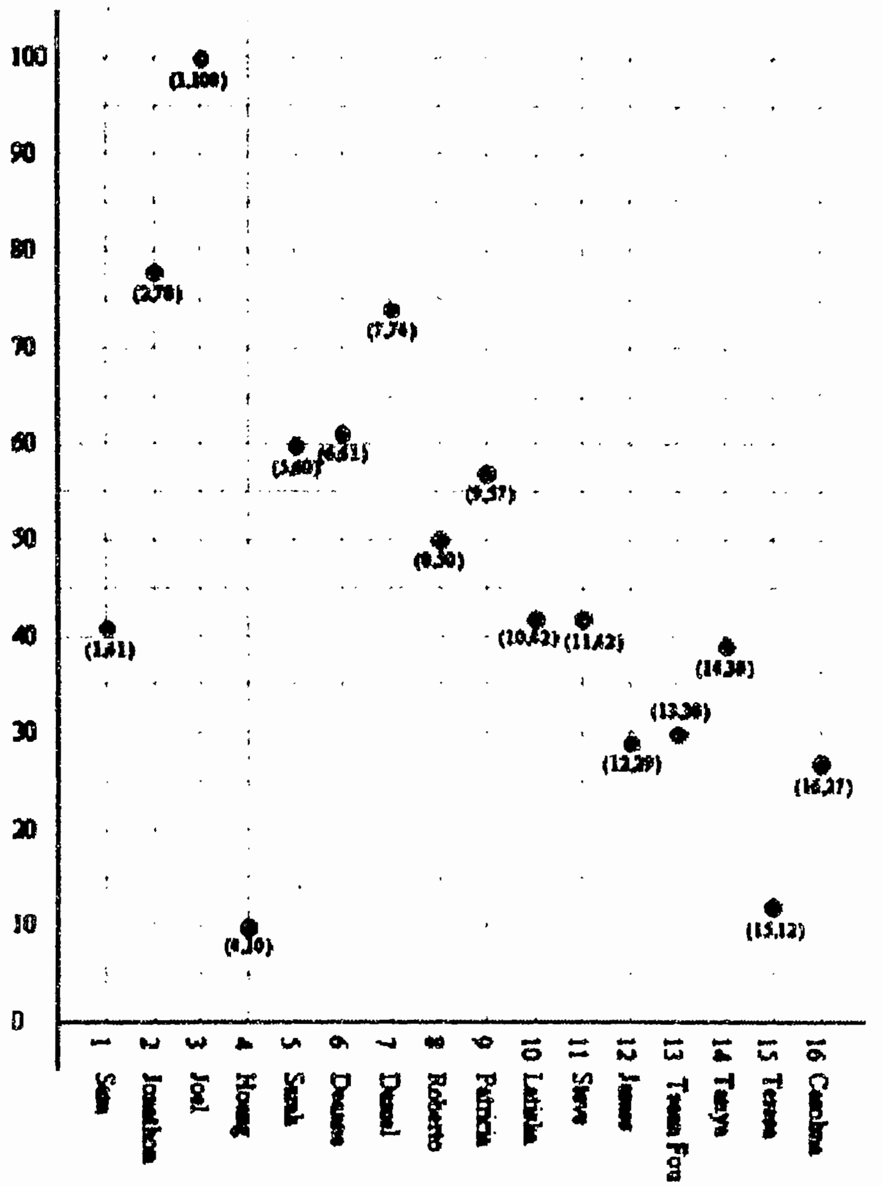

Figure 1. Class rank x% of Initial Question to VERIFY

The results for closure showed a strong correlation between the need for more closure and class rank (r = 0.70). Again Hoang was an outlier, and without Hoang the correlation increased to 0.84.

More interesting, I think, is the correlation between class rank and percentage of responses that were statements. Stronger students more often completed their sentences while weaker students responded with one or two words, or phrases. I discovered this pattern when comparing November and May transcripts for the highest and lowest ranked students. Table 1 shows the number and percent of turns classified as statements, responses, or questions for two early TRS's for Sam and Jonathan and for

Carolina and Tsaan Fou. Sam, a confident problem solver, consistently made statements, without being overly concerned as to whether they were right or wrong. He gained information about the solution of a problem either way. Jonathon, also a strong student and an independent thinker made statements. Tsaan Fou, one of the weaker students, like Carolina gave short responses that were not sentences. While one reason for Carolina's short responses might have been that she was still learning English and was more comfortable speaking Spanish, Tsaan Fou was bilingual and spoke only English in school. Both lacked confidence in their ability to solve most of the problems they asked about.

Table 1. Student Responses

	Number			Percent		
	Statement	Response	Question	Statement	Response	Question
Sam	7	1	3	64	9	27
Jonathon	4	1	2	57	14	29
Carolina	9	36	11	16	64	20
Tsaan Fou	0	7	3	0	70	30

When I compared transcripts from early in the year with those recorded in May, I found an increase in percentage of statements and length of responses for both Carolina and Tsaan Fou. There was no change for Sam or Jonathon. The fact that stronger students used a greater percentage of complete sentences means that they took more opportunities to articulate mathematical relationships, a practice that could be very useful in terms of both understanding and remembering those relationships. A goal for teachers might be to focus on tailoring their questions to elicit full sentence responses from all students and to avoid settling for phrases.

Surprisingly there was no discernible difference in percentages of language use *In* or *With* mathematics between stronger and weaker students. Figure 2 is a scatter plot comparing class rank and percentage of all turns (not just initial questions) *In*, *With*, and *Beyond* mathematics (r= –0.097).

Arranging students in order from highest to lowest percentage of use of language *In*, *With* and *Beyond* mathematics led to some further observations about gender differences and English language learners. The list shows percentages largest to smallest with the girls percentages underlined. The Spanish speaker's percentages are in **bold** and Chinese, Mien, and Vietnamese speaker's percentages are in *italics*. Percentages for those who spoke only English are in regular type.

70, 68, 60, **57**, 55, 55, **53**, 53, **52**, *52*, *52*, **48**, 47, **45**, *42*, 41, *39*, 32, *26*

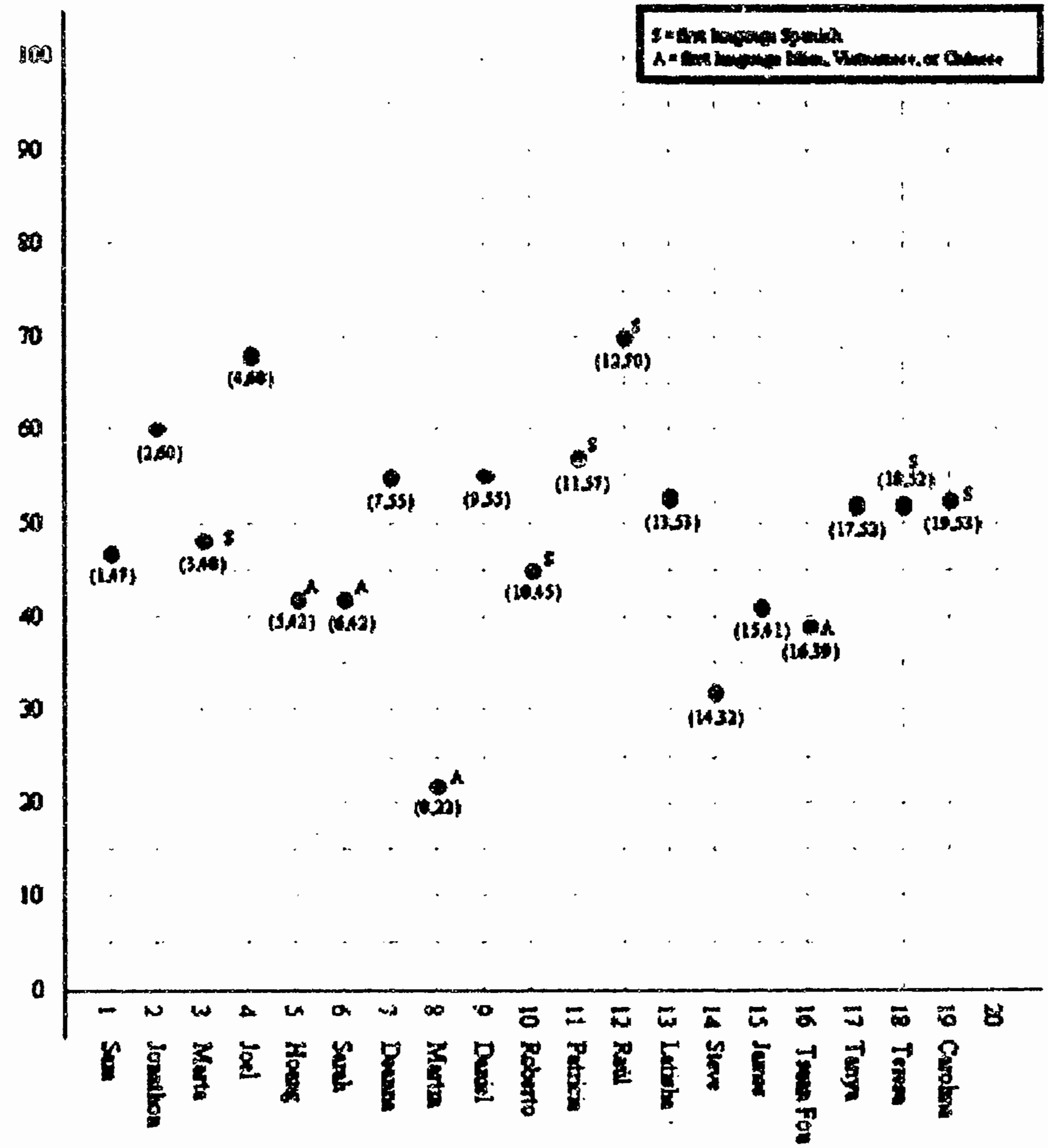

Figure 2. Class Rank x In, With, Beyond % Turns

The boys are at one extreme or the other, while the girls cluster around the median. The Spanish speakers show no clear pattern while the Asian language speakers tended to use a lower percentage of mathematical language. While the transcripts for each individual provided reasons for his or her position in this hierarchy of language use, generalizations were not possible.

For Level 1 and Level 3 teacher responses, comparisons of TRS's for strong and weak students showed that the type of response seemed to depend on the type of problem or question posed more than on who posed it. While weaker students elicited more Level 3 responses, stronger students elicited their share, and comparisons of TRS's line-by-line, for strong and weak students showed similar discourse patterns.

Level 2 responses, on the other hand, were regularly elicited by some students and not by others based on the teacher's assumptions about the student's expectations. Students most likely to receive Level 2 responses were three students who were reluctant to respond and two who were in their "silent phase" of learning English where they could understand but lacked confidence in their ability to speak. My response to their reluctance to respond, in either case, was often a too lengthy explanation, and this seemed to become my habit in talking with these students throughout the year.

The results of this study, many of which are in the details of examining each student's talk, have implications for teachers' expectations in relation to working with students in small groups particularly in how we pose our questions and our expectations for their responses. Clearly further examination of student discourse is needed. This study is just a small part of beginning.

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MATHEMATICAL EXPLANATIONS: IN-ACTION AND AS RE-PRESENTATION

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The purpose of this ongoing study is to explore the nature of mathematical explanations and make comparisons between explanations that arise in-action and those provided as re-presentations or summaries of formative efforts either to the teacher or to the whole class. Some of the contrasts that arose during the study were in relation to how explanations were posed or offered to others; the purpose or need the explanation appeared to fulfill; the criteria used to accept or reject an explanation; how that acceptance or rejection was signaled; and what was hidden, lost or ignored as explanations were re-presented.

In brief, explanations expressed in-action were offered to both oneself and to others in the group in an effort to broaden understanding in and for that moment, often with the assumption that they could return to the ideas later if necessary. The explanations were hesitant, incomplete and viewed as plausible, viable and consistent with previous experiences with that task and with previous tasks. The incompleteness allowed other participants to add on and revise in the course of interaction. Accepted explanations were significant points in the path of activity as they allowed participants to move on. The explanations were then incorporated into subsequent actions and explanations. In contrast, explanations offered as re-presentations were often posed to outsiders as complete arguments in an effort to convince others of the correctness of the explanation. As such, they were generally not used to initiate further investigation but as an endpoint to activity. Acceptance was signaled when no errors, inconsistencies or disagreements were stated with the explanation presented.

An awareness of how explanations are altered in content and intention has implications for understanding the generative features of mathematical explanations in the relation to human perceptions of the nature of mathematics.

Epistemology

VIEWS OF THE NATURE AND JUSTIFICATION OF THREE DOMAINS OF KNOWLEDGE: MATHEMATICS, MATHEMATICS TEACHING, AND MATHEMATICS LEARNING

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Abstract: This study examined 2 preservice secondary mathematics teachers' views of sources, evidence, and certainty of knowledge in three domains. Data were collected throughout a year-long program using multiple open-ended data sources. Sources of and evidence for *mathematical knowledge* were predominantly external, based on the words of an authority. In the context of open-ended mathematical activities, they acknowledged that their peers' approaches were unique, yet mathematically legitimate. This raised concerns about *mathematics learning*; however, there was little indication that they searched for knowledge in this domain. For *knowledge of mathematics teaching*, Brenda exhibited an internal orientation and relied on experience, herself, and intuition whereas Liz continued to rely on perceived authorities for this knowledge. Corresponding implications for teacher education are discussed.

An individual's epistemology, or philosophy about the nature and justification of knowledge, influences learning in that how an individual perceives a learning experience is contingent on who or what the individual views as sources of and evidence for knowledge, and to what extent the individual perceives that knowledge as certain. Mathematics teacher education programs are environments for learning about domain-specific knowledge such as mathematical knowledge, knowledge of mathematics teaching, and knowledge of mathematics learning. This study examined two preservice secondary mathematics teachers' views of the nature and justification of knowledge in these three domains in the context of a reform-oriented secondary mathematics teacher education program.

There is no research that concurrently investigates views of knowledge in all three domains. The studies of Arvold (1996), Etchberger and Shaw (1992), and Eggleton (1995) were specific to views of the nature and justification of mathematical knowledge, whereas Mewborn (1999) identified changes in sources of knowledge of mathematics teaching and learning. Arvold and Albright (1995), Cooney and Wilson (1995), and Cooney, Shealy, and Arvold (1998) addressed orientations toward authority (sources and evidence) primarily in the domains of mathematical knowledge and mathematics teaching. This study asked the following: For each participant, who or what constitutes sources of and evidence for mathematical knowledge, knowledge of mathematics teaching, and knowledge of mathematics learning, and to what extent is that knowledge certain? An understanding of views of knowledge in these three domains has implications for mathematics teacher education.

Similarities across theoretical perspectives about epistemological development (Baxter Magolda, 1992; Belenky, Clinchy, Goldberger, & Tarule, 1986; Perry, 1970) provided a framework for characterizing sources, evidence, and certainty of knowledge (see Figure 1). For example, all three perspectives have positions of development in which the individual views sources of knowledge as external, internal, or a combination of both. All three perspectives have positions that characterize evidence to be based on what an authority says, personal opinions and experiences, or reasoned judgment. Finally, all three perspectives highlight positions in which the extent of certainty of knowledge ranges from absolute certainty, free of context and human values to relative certainty, based on context, and subject to the values of those involved. A detailed description of this framework can be found in Chauvot (2000).

Two preservice secondary mathematics teachers (Liz and Brenda) from a larger research project, Research and Development Initiatives Applied to Teacher Education (RADIATE[1]), were informants for this study. Data collected through RADIATE served as the data for this study. The goals of the RADIATE instructional program were consistent with a reform vision of mathematics teaching and learning (NCTM, 2000). The RADIATE research questions addressed the nature and structure of the preservice teachers' knowledge and beliefs about mathematics, mathematics teaching, and mathematics learning, and the extent to which the preservice teachers were able to be reflective about their teacher education experiences. Data collection occurred throughout a four-quarter sequence of two content/pedagogy courses, a student-teaching quarter, and a post-student-teaching seminar. Data sources included 34 journal entries, 6-13 semi-structured interviews of at least 45 minutes in length, open-ended surveys, course artifacts, and fieldnotes. Journal prompts, interview protocols, surveys, course artifacts, and detailed descriptions of selected mathematical activities of RADIATE can be found in Chauvot (2000).

The data and perspectives provided in the work of Baxter-Magolda (1992), Belenky et al.(1996), and Perry (1970/1999) provided insight for coding data for each knowledge domain. For example, the authors used information about their participants' expectations of the learner, peers, and instructors in the learning process as a means to discuss participants' epistemological views. Therefore, for the domain of mathematical knowledge, I identified indications of each participant's expectations from herself, peers, and instructors while involved in mathematics courses and any mathematics-related activities of the RADIATE program. For the domains of knowledge of mathematics teaching and mathematics learning, I identified indications of each participant's expectations of herself, peers, instructors, and cooperating

EXTERNAL SOURCE	INTERNAL SOURCE
KNOWLEDGE IS CERTAIN	
• Dualism & Multiplistic pre-legitimate (Perry), Silence & Received knowing (Belenky et al.), Absolute knowing (Baxter Magolda) • Multiple perspectives are not epistemologically legitimate • EVIDENCE: One right answer based on words of the authority	
KNOWLEDGE IS EVENTUALLY CERTAIN	
• Multiplistic subordinate (Perry) • Multiple perspectives are epistemologically legitimate • EVIDENCE: One right answer based on words of the authority when authorities figure it out	
KNOWLEDGE IS PARTIALLY CERTAIN	
• Relativism subordinate (Perry), Impersonal Transitional knowing (Baxter Magolda) • EVIDENCE: Multiple right answers based on logical reasoning processes that use evidence based in context. • Reliance on authorities for when to use reasoning processes	• Multiplistic correlate (Perry), Interpersonal Transitional knowing (Baxter Magolda), Subjective knowing (Belenky et al.) • EVIDENCE: Multiple right answers based on personal experiences • Reliance on intuition
EXTERNAL/INTERNAL SOURCES	INTERNAL/EXTERNAL SOURCES
KNOWLEDGE AS UNCERTAIN	
• Relativism (Perry), Separate Procedural knowing (Belenky et al.), Individual Independent knowing (Baxter Magolda) • EVIDENCE: Multiple right answers based on logical reasoning processes that use evidence based in context. • Recognition of self's abilities to apply reasoning processes in new contexts.	• Connected Procedural knowing (Belenky et al.), Interindividual Independent knowing (Baxter Magolda) • EVIDENCE Multiple right answers based on reasoning processes that strive to understand the personal experiences of others. • Authorities are sources and partners.
INTERNAL/EXTERNAL SOURCES: KNOWLEDGE AS RELATIVE	
Commitment within relativism (Perry), Constructed knowing (Belenky et al.), Contextual knowing (Baxter Magolda) EVIDENCE : Knowledge is relative, based in context, and subject to the values of those involved	

Figure 1. Characterizations of Source, Evidence, and Certainty of Knowledge

teachers in the program experiences. Inductive analysis strategies (Patton, 1990) were used to identify patterns in the data. Figure 2 summarizes the findings in terms of the framework. In the discussion that follows, the findings are organized by source that then includes explanations related to corresponding assumptions about evidence and certainty.

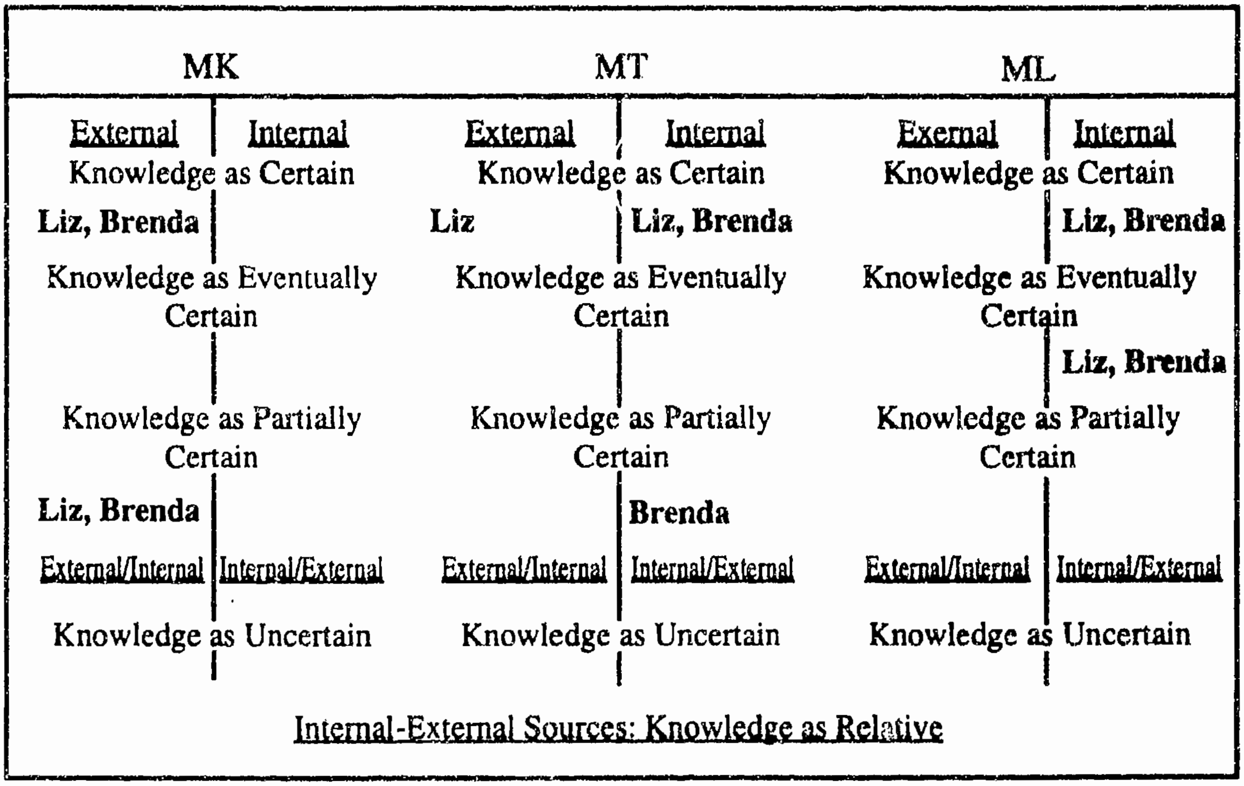

Figure 2. Views of the Nature and Justification of Knowledge in All Three Domains

For the domain of mathematical knowledge, sources were predominantly external; mathematical knowledge was certain with correctness based on the words of the teacher or textbook. This was evident in Liz and Brenda's recollections of past learning experiences and characterizations of good mathematics teaching. Mathematics teachers were expected to explain material clearly, be willing to answer students' questions, and be willing to re-teach the material when necessary. The student was expected to pay attention in class, study regularly, and get help from the teacher when needed. Also, both participants spoke of learning by doing, learning by seeing, and learning through discovery. They expected students to practice problems, use manipulatives in small groups, and use technology. In practice, students were actively involved but student contributions to the mathematical discourse was limited. Sources and evidence for knowledge remained with Liz or Brenda or the textbook. The mathematical knowledge existed with certainty, free of context and human values.

The RADIATE mathematical activities sometimes provided a context that indicated an awareness of partial certainty of mathematical knowledge. Both participants were surprised to realize that their peers approached mathematical problems in unique, yet mathematically legitimate ways. The uncertainty of mathematical knowledge was due to different mathematical thinking by peers. Evidence for mathematical knowledge was not necessarily the words of an authority. For example, in response to the Function Card Sort Activity (see Chauvot, 2000), Liz "found it interesting to see how students with similar backgrounds in math could see totally different answers (backed with perfect logic) to one problem" (4/7/94).

For the domain of knowledge of mathematics teaching, one source was personal experience as learners. Both participants admitted that their own learning of mathematics in the context of the RADIATE mathematical activities served as evidence for knowledge of mathematics teaching. Liz was quite adamant about this. Brenda was more reserved. The evidence for knowledge focused on the perception of an increased understanding of mathematics. Also, these activities fulfilled the participants' desires to have their students more actively engaged in the learning process. Consequently, beliefs about learning mathematics (e.g., students learn by doing) also served as evidence for knowledge of mathematics teaching.

A second identified source of knowledge of mathematics teaching was practical teaching experience. This was especially important to Brenda:

> The field experience I think is really important [for my professional development] because experience is something you can learn from. And no matter how much you talk about teaching geometry, you're not going to know what it's really like until you get up there and actually do it. I think that [experience] is probably the most important thing there is. (Interview 4, 10/94)

A third source of knowledge of mathematics teaching was best expressed as a combination of opinions of self, RADIATE instructors and peers, and cooperating teachers. What became significant in this analysis was the weighting system each participant used to decide the veracity of the knowledge she heard from the different sources. Liz's focus was external and authority-oriented. She placed more value on the opinions and experiences of the RADIATE instructors and her cooperating teachers as compared to the views of her peers.

> I get so sick of everyone. We'll spend like three days in class doing something, and then the first comment is "Well you can't do it in a classroom." And I'm like "Give it a chance." I mean we're not here for no reason. I just get kinda upset with people just immediately saying they can't do it in a classroom. (Interview 4, 10/94)

In terms of self, she frequently assumed she was wrong. "I might be totally wrong cause I've found out that I'm totally wrong about a lot of things recently" (Interview 2, 5/94).

Brenda focused on herself, experience, and intuition. The opinions of her cooperating teachers and RADIATE instructors and peers were valuable, but secondary. She did not expect the RADIATE instructors or her cooperating teachers to transmit knowledge of mathematics teaching to her. Instead, she intended to rely on experience and herself as the primary source.

> I realize that many of her [the cooperating teacher] teaching skills come from experience. I hope to be able to determine what amount of class discussion, lecture, and guided practice is appropriate for each of my classes as she seems to have done. (Paper, 5/26/94)

Also specific to Brenda was an unwillingness to pass judgment on the opinions of her peers, her cooperating teachers, and the RADIATE instructors.

> I think [my cooperating teacher] did a real good job with the class that she had. I don't know how I would have done anything different. ... Because of the students that were in there. I mean, different students in the same class may have responded differently and students in a different class may have responded the same way. I don't know. (Interview 3, 6/94)

Brenda seemed to feel that everyone had a right to his or her own opinion about mathematics teaching. Opinions were based on practical teaching experience. Brenda was aware that individual experiences varied; consequently, she seemed reluctant to pass judgment.

There was little indication that the participants searched for knowledge of mathematics learning. Consistently, each participant assumed that students learn mathematics in the same manner that she had learned mathematics. However, the participants were concerned about using "it worked for me" as evidence for this knowledge. They were particularly concerned about how lower-level students learn mathematics. Both concluded that such students simply needed more time and patience: Learning occurred the same way, but at a slower pace. In addition, the awareness that their RADIATE peers, who were assumed to have "similar backgrounds in math," approached mathematical problems in different, yet legitimate, ways created concern. Consequently, they turned toward practical teaching experience as a source of knowledge of mathematics learning. Both hoped that the field experiences and beginning years of working with students would inform them about student thinking. This conclusion seemed to discontinue the search for any other sources of knowledge of mathematics learning. Interestingly, there was no indication that they turned toward the RADIATE instructors or their cooperating teachers for this knowledge.

Findings from this examination of teachers' views of knowledge in the three domains have implications for teacher education. For example, interactions with peers in the context of open-ended mathematical activities stimulated new thinking about the certainty of mathematical knowledge (see Borasi, Fonzi, Smith, & Rose (1999) for similar results). This in turn led to concern about mathematics learning. These are promising results, particularly because there was little evidence to indicate that the participants were even searching for knowledge of mathematics learning. This limited indication in itself is a significant finding for teacher educators. Perhaps the program was not as explicit in this domain of knowledge. Or perhaps the participants were too naïve at this point in their development to reflect on student thinking. Finally, regarding the domain of knowledge of mathematics teaching, teacher educators must be cautious of teachers (e.g., Liz) who seem to eagerly and readily accept new ideas about mathematics teaching without careful deliberation. At the same time, almost total reliance on experience and self (e.g., Brenda), also without careful deliberation, has limitations as well. Teacher educators need to consider program experiences that both honor teachers' views as well as promote further professional growth.

Note

1. RADIATE was directed by Dr. Thomas J. Cooney and Dr. Patricia S. Wilson and funded by the National Science Foundation (#DUE 9254475) and the Georgia Research Alliance. Any opinions or conclusions expressed by this report are those of the author and do not necessarily reflect the views of the funding agencies.

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ON BEING EMBODIED IN THE BODY OF MATHEMATICS: MATHEMATICS KNOWING IN ACTION

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Abstract: Mathematics knowing is explored in this interpretive study of a parent and child's actions and interactions as they are occasioned by a variable-entry prompt, each other and the artifacts of their own thinking. We explain their knowing in interaction as embodied in the body of mathematics in so far as it is observed to fit within and contribute to the local, contemporary and historical mathematics communities.

Introduction

There is considerable interest in mathematics education in how a person's mathematics knowing arises in interaction with others and with elements of his or her environment (e.g., Cobb and Bauersfeld, 1995; Davis, 1996). Over the past several years, we have contributed to this conversation by exploring the consequences of thinking of mathematical knowing in action not as the matching of pre-given models or answers to mathematical queries (although a knower might provide answers as part of their actions), nor even as problem solving (although some of the data discussed here could be viewed in those terms), but as a fully embodied phenomenon which brings forth of a world of significance (including mathematics) with others in a sphere of behavioural possibilities (Maturana and Varela, 1992; Varela, Thompson and Rosch, 1991). Observing mathematical knowing in the actions and interactions through which such bringing forth occurs, involves observing the personal/structural, social/interactional and cultural/mathematical dynamics of the situation all at once. In previous papers, we have focused on the interplay between personal constructive activity and social interaction (Simmt, Kieren, 1999; Kieren, Simmt, Mgombelo, 1997).

Research Focus

In this paper, we turn our attention to the ways in which a person's embodiment in the body of mathematics co-emerges with his or her mathematical actions and interactions and the potential that arises out of mathematics knowing in action. In other words, we interpret a person's mathematics (actions, utterances and forms of reasoning, in part) and demonstrate how it can be viewed as occasioned by the mathematical practices and artifacts of his or her local community as well as by the practices and artifacts of the broader contemporary and historical mathematics communities. This perspective allows us to ask, *"What occasions such embodied* *actions and what are the possibilities for embodiment in the body of mathematics that arise when, let's say, adults and children engage in a particular mathematical setting differently?"*

Research Setting and Methods

In our work we have been considering such questions by discussing and interpreting the work of parent-child pairs (the children are between the ages of 8 and 13) as they respond to variable-entry prompts offered in an extra-curricular mathematics program (Simmt, 1997). It is important for us to note that we understand the "starters" used in the program as prompts rather than problems in that we observe it is the participants, in their various actions, who specify the problems with which they engage rather than the starter itself. Such a characterization of the prompts also points to the feature that a person can enter into, or engage with these prompts through a variety of appropriate mathematical actions. Thus the parent's or child's mathematical actions (mathematics) should not be thought of as caused by the prompt in the sense that these are simply an effect of it; nor should one think of the parent's and child's actions and mathematics independent of the prompt. We find it useful to say that the mathematical actions (the mathematics) of the parent and the child are *occasioned* (see Kieren, Simmt, Mgombelo, 1997) by the prompt and co-emerge with it and their environment.

Our method of study is interpretational in nature; we invoked the second-order cybernetic method of observing observers (von Foerster, 1981). It involves the gathering of a variety of data (e.g. audio or video tapes; mathematical artifacts created by the participants; field notes) and the creation of other "data" (such as transcripts of the tapes; mathematical activity traces; sequences of still pictures from video with related annotations). All data are studied recursively to build up the interpretations which we use in our paper and are studied by teams of researchers, each of whom brings their own interests to the study. This overall research philosophy is consistent with our view that the statements we make in our interpretations are radically contingent ones; they are co-dependent on the structures of the participants, the conditions and interactions in the settings, the nature of the observations, and the lived histories and structures of the observers.

In our paper, we elaborate on our conception of being embodied in the body of mathematics. We show that while the inter-action between the parent and child affected the mathematical actions of each, both their lived histories and their embodiment in the body of mathematics (the ways in which they engaged in mathematical activity) were unique; hence different spheres of possibilities were created for each person. While we focus on only one parent-child pair, we believe their case is illustrative of the many pairs we have observed working with variable-entry prompts in the parent-child mathematics program.

Interpreting the Mathematics of a Father-Daughter Pair

Imagine posing the following prompt to a father (Jake) and his 9 year-old daughter (Cathy) (see Figure 1).

Mark off a rectangle on your graph paper. Now draw in a diagonal. The object of th activity is to determine for any rectangle how many unit squares the diagonal passe: through. For example, in this 3 by 5 (3x5) rrectangle, the diagonal passes through 7 unit squares.

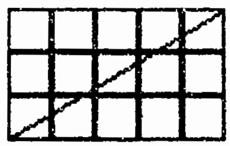

Figure 1. Variable-entry prompt offered to Cathy and Jake

Assuming this is a variable-entry prompt (extensive data on this support our assumption), we anticipated that it would be approached in a variety of appropriate ways by the participants in our study and that different individuals and parent-child pairs would be occasioned to consider very different problematics. This certainly proved to be true for Jake and Cathy. They entered into mathematical activity by specializing; that is, they considered the particular case of *n*x*n* squares. Cathy drew a sequence of three squares beginning with a 2 x 2 square and then superimposed a 3 x 3 square on the 2 x 2 square and finally a 4 x 4 square on the 3 x 3 square. (I have drawn what her image looked like over time in figure 2.) Although the diagonals drawn were not accurate, both the father and his daughter, in their own ways, were able to imagine how the diagonals 'should' look and they quickly saw a generalization for squares. Jake explained, a "four by four is four 'cause you just have to go to the middle one—cross diagonally each one."

Cathy then turned her attention to (non-square) rectangles. "Can I just show you something?" she said to her father as she drew a 3 x 5 rectangle. Jake responded by commenting, "Let's do this step-by-step." As Cathy started to draw a 4 x 6 rectangle in a different part of the paper Jake stopped her and said, "Don't go all over the place. What we want is to go over the top of the three by five ... and just keep expanding ... otherwise we will not have enough room."

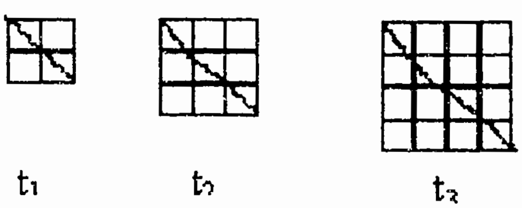

Figure 2. Replication of the image Cathy drew with its changes over time

At one level, we might observe the exchange above as a feature of the interaction between a parent and a child— in this case an interaction whereby the parent exerts control over his child.

Viewed in another and not necessary disjoint way we might note how this interaction demonstrates Jake's desire for or value of order and neatness. In action, he "knows" the value of step-by-step activity, or the value of ordering his activity in particular ways. Jake's statements and consequent actions (his ordering) could be observed as an embodiment in the body of mathematics in that the actions of ordering and progressing in a systematic manner are common mathematical actions and ones which are promoted in school mathematics.

Although Cathy followed Jake's instructions not to go all over the place, she found her father's strategy of drawing one rectangle over another one too confusing. She whispered to herself about how "messy" and "complicating" she found it to work in this way. On one hand, it may be that Cathy's structure and her lived history prevented her from seeing the value in her father's strategy of systematically generating cases. On the other hand, from previous work, we know that Cathy can envision geometric patterns and is oriented to representing and exploring her thinking geometrically, with diagrams. Thus, even though we might conclude that her father is directing her in a mathematically appropriate direction, Cathy starts to work divergently and soon she and her father are working quite independently of each other, each keeping their own records (figure 3).

It is not surprising that the nature of Cathy's mathematical practice is different from her father's but it is notable for a number of reasons. Most importantly (for this paper), we are given the opportunity to observe and consider the mathematical potential in what Cathy and Jake are doing given that their utterances, forms and reasoning differ from each other's. While their records do not look much alike, it is clear that

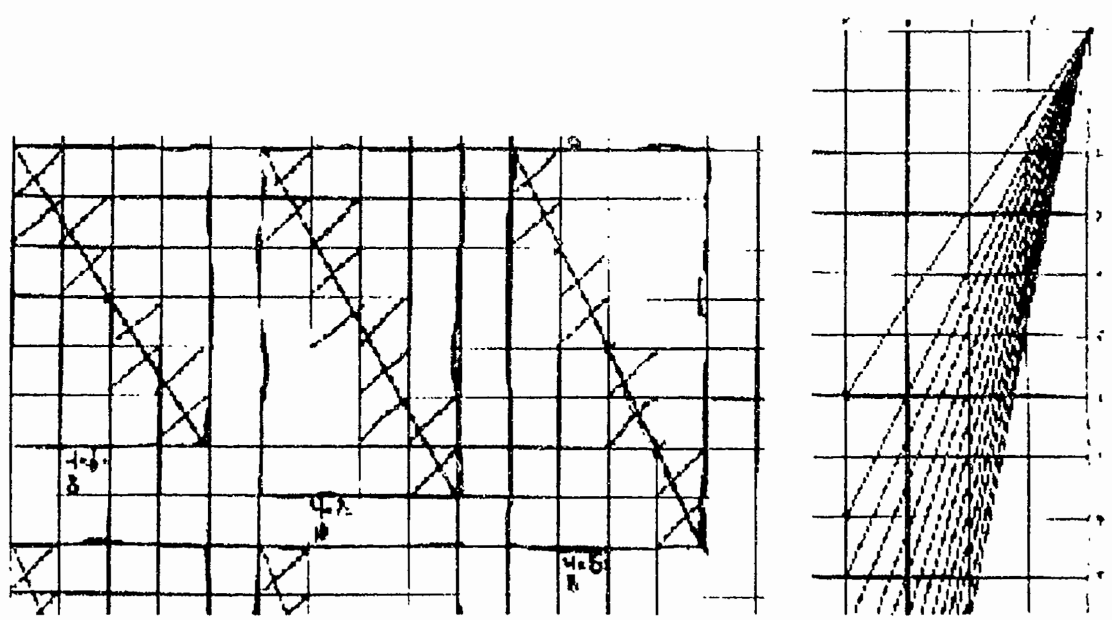

Figure 3. Extract from Cathy's (left) and Jake's (right) working papers

both Jake and Cathy, in their own way, are searching for patterns. Notice that such a search can be observed as part in parcel of Jake's mathematics and his engagement in the body of mathematics, certainly at the level of the local practice of the parent-child mathematics program. In fact, this practice appears to lead him to powerful mathematical patterning involving the dimensions of the rectangles in terms of factors and relatively prime factors. From his work with the rectangles, Jake created a series of tables in which he noted the dimensions of the rectangles and the number of unit squares the diagonal passed through. Eventually he stated a relationship between the number of unit squares and the dimensions of the rectangle, if the dimensions were relatively prime—length plus width minus one ($m + n - 1$).

In contrast, Cathy's forms are quite distinct from Jake's and her "work" lends itself to different mathematics. Cathy's approach allows her to organize her actions and keep her records in her own fashion. However, she was influenced by the tables her father kept and the utterances he made about the relationship he noted for relatively prime rectangles. After hearing her father explain his reasoning to the researcher, Cathy made a table of values based on her drawings and tried to find a relationship among the numbers in her table. But more than that, a careful study of her work shows that her mathematical actions opened her to mathematical ideas not available to her father in his work. In her drawings one can find small marks she made on the 4 x 6 and the 4 x 8 rectangles where she has distinguished (visually) patterns along the diagonal and hence the simplest ratio between the two sides. We might observe that by marking the "prime" rectangles in each of the "composite" rectangles her sphere of behavioural possibilities holds different potential from her father's. From her mathematics one could begin to develop notions of equivalent ratios, for example—something that is not easily developed from her father's diagrams. Further, an observant teacher might notice the potential in this prompt for promoting a geometric basis for thinking about equivalent and non-equivalent ratios.

Consequences of Embodiment in the Body of Mathematics

It is clear that Jake's and Cathy's mathematics and actions were determined by their own lived histories (personal, structural and dynamic) and hence the worlds they brought forth were at once different but co-emergent with the prompt. At the same time, their mathematics was also occasioned by their interaction. Jake, occasioned by Cathy's method of presentation for squares, developed his sequential overlapping method which, in turn, occasioned him to contrast various cases and make generalizations based on the relationship of the factors. While Cathy did not follow her father's work, she was occasioned by it to engage in pattern making and generalizing. Although Cathy and Jake engaged the prompt in distinct ways, their approaches were mathematically appropriate; further, both Cathy and Jake were, at once, participating in the practices of contemporary and historical mathematics and contributing to that culture.

So what might these observations of a very unique mathematical activity session offer to the mathematics education community? In our paper we explored the divergences between Cathy's and Jake's actions and interpreted them within a framework of mathematics as fully embodied. In particular, we explored the cultural dimensions of their embodiment by considering the mathematical potential that was triggered from Cathy's and Jake's different forms, utterances, and reasoning. We assert that engaging differently in mathematical practices or being embodied differently in the body of mathematics by one's actions, opens one to different mathematical forms and practices within the local, contemporary and historical mathematics communities. Finally, we believe that thinking about personal mathematical knowing in terms of its embodiment in the body of mathematics provides both researchers and teachers with new insights into pedagogical mathematics knowing and into ways in which mathematics knowing in action affects the curriculum in action. It suggests to us that by offering even a slightly rich prompt, the teacher is providing herself multiple opportunities to look not simply for expected behaviour or results but to be tuned into the actions, utterances and reasoning which reveal various embodiments in the body of mathematics. Although the teacher may offer certain practices in a particular setting, she would be wise to take up the mathematics and mathematical practices of her students as an opportunity to observe just how students belong to and are part of the body of mathematics through their actions.

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ARTICULATING THEORETICAL CONSTRUCTS FOR MATHEMATICS TEACHING

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Abstract: We articulate and explicate a theoretical framework for mathematics teaching in which we specify mechanisms of students' conceptual learning and the role of teachers in promoting that learning. We consider the notion of the learning paradox as defining the problem that our framework addresses. We explicate how conceptualizing learning in terms of reflection on activity-effect relationships can address the learning paradox and provide a basis for specifying an approach to mathematics pedagogy that can foster conceptual advance in students.

The current mathematics education reform has promoted a large-scale movement away from direct instruction, leaving the field of mathematics education without well-articulated theories of teaching. In this paper, we elaborate and examine constructs that can contribute to re-conceptualizing mathematics teaching. We approach the need for re-conceptualizing mathematics teaching not only from our role as mathematics educators (teachers), but also from our role as mathematics teacher educators and researchers of mathematics teacher development. It is only with clearly articulated conceptions of mathematics teaching that the goals of mathematics teacher development can be defined and the approaches to and results of mathematics teacher education effectively analyzed and evaluated (Simon, 1997).

We work from the notion that theories of teaching must build on and be integrated with theories of learning, but that theories of learning do not, in themselves, prescribe approaches to teaching. We have asked ourselves: what aspects of our understanding of learning might provide a solid basis for re-conceptualizing teaching? We identified ways of understanding mathematics learning that exist in the literature, synthesized and further elaborated these understandings in order to articulate a framework for teaching based on those understandings of learning. Examples of teaching that are consistent with these ideas can be observed occasionally in practice. However, it is our experience that the theoretical basis for such teaching is generally unformulated. The consequence of the lack of a well-formulated theory of teaching is that effective mathematics pedagogy is not produced consistently and that the pedagogical ideas involved are not accessible subjects of discourse.

Overarching Conceptual Framework

In our theoretical and empirical work, we employ a social constructivist perspective in which we coordinate social and cognitive (constructivist) perspectives. In this article, we focus on the cognitive; that is we articulate theoretical constructs with respect to teaching that are built on radical constructivist interpretations of learning. We classify cognitive perspectives based on constructivism as "conception-based." We use this term to emphasize that the researcher postulates conceptions as a way to characterize learners' current organizations of their experiential realities. Conception-based perspectives are based on the following principles:

1. Mathematics is created through human activity. Humans have no access to a mathematics that is independent of their ways of experiencing/knowing.
2. What individuals currently know (i.e., current conceptions) affords and constrains what they can assimilate—perceive, understand, and learn.
3. Learning mathematics is a process of transforming one's ways of knowing (conceptions) and acting. This has two implications. New understandings are the result of change in current understandings, and current understandings afford and constrain what understandings can be developed at any point in time.

The idea of learning as transformation of current understandings is in contrast with perspectives that assume that learners can acquire new concepts by perceiving relationships that exist in the world around them. In the latter case, the notion of how to foster conceptual learning is relatively straightforward. However, from a conception-based perspective, promoting the transformation of current understandings toward the development of more advanced understandings is problematic. In particular, the most important and challenging aspect of fostering cognitive growth is promoting the development of new cognitive entities (e.g., number, ratio, function).

Identifying the Pedagogical Challenge: The Learning Paradox

Bereiter (1985) made a distinction that is at the core of the framework that we are presenting: "The distinction is between kinds of learning that can be accounted for on the basis of knowledge schemas that the learner already possesses and learning that involves new cognitive structure to which already existing schemas are subordinated" (p. 217). Fostering the latter type of learning is the theoretical challenge that we are attempting to meet.

In grounding this exploration in constructivism, we continue to eschew the notion that more powerful concepts can be infused into learners and embrace the idea of promoting an internal process of construction. This quest however, puts us face-to-face with what has been called "the learning paradox" (Pascual-Leone, 1976), the need to explain how learners "get from a conceptually impoverished to a conceptually richer system by anything like a process of learning" (Fodor, 1980, p. 149 cited in Bereiter, 1985). This is conceived of as a paradox for the following reason. Piaget's (1970) idea of assimilation, a core idea of constructivism, suggests that one needs to have concept X in order to make sense of one's experience in terms of concept X. Thus, it seems impossible for one's experience to lead to a more advanced concept, because that concept would need to be already available for assimilation. We stress that this paradox is a function of adopting a constructivist understanding of knowing. It does not exist for those who view learning as taking in relationships from the outside world (See Simon, Tzur, Heinz, Kinzel, & Smith, in press). Our challenge, therefore, is to find a theoretical explanation for how humans construct more powerful concepts out of less powerful ones that can serve as a basis for articulating a role for pedagogy in promoting such learning processes.

Addressing the Learning Paradox: The Activity-Reflection Cycle

In this section, we introduce a way of conceptualizing learning that addresses the learning paradox and provides a powerful basis for re-conceptualizing teaching. We introduce these ideas with a non-mathematical example (mathematical examples would have required more space than we had available) and then consider the theoretical issues in greater detail.

Developing Strategy for the Game of Checkers

This is a thought experiment involving children's learning of a game strategy. As such it is not an example of mathematics learning, and therefore should not be considered as evidence to support our framework for conceptualizing mathematics learning and teaching. Rather, we use it as an easily understandable context for presenting a set of ideas.

Consider two children who have learned the rules of checkers. They understand the goal of the game and can play a game according to the rules and determine a winner. However, at this point, they have developed no strategy, that is, they have no basis for choosing one move over another. If these two children were to play checkers regularly over several weeks, without any coaching or contact with more advanced players, would they develop some strategy? We imagine that you responded as we do, "Yes."

How is this example related to the challenge that we have articulated? The children's progression from having no strategy to having strategy can be viewed as an example of developing more powerful conceptions from subordinate ones, the issue raised by the learning paradox. Our explanation is structured as follows:

1. The children had a goal (to win the game) and an activity sequence that they could use in service of the goal (moving pieces in accordance with the rules). The activity sequence that led to the development of the new conception was one that the children could carry out without and before the cognitive advance.
2. From the outset, the children were able to distinguish effects of their activity that advanced their goal from those that did not (e.g., taking an opponent's piece versus losing a piece).

3. As they played more games, they began to reflect (we are not claiming conscious thought) on the relationship between their activity (particular moves) and the effects of that activity (taking or losing a piece).
4. Their reflection resulted in *abstracting an activity-effect relationship* (between certain moves and changes in the number of pieces), the rudimentary form of a new conception (Tzur, 1996).

Re-conceptualizing Teaching

Based on the description of conceptual development as a process of reflection on activity-effect relationships, we now articulate the teacher's role in promoting the development of a new cognitive structure. The teacher's role is to specify and engage students in an activity sequence that the students are capable of carrying out, independent of the teacher, that can lead to the students' identification of regularities in activity-effect relationships contributing to an intended cognitive advance. Let us look at some of the entailments of such a formulation of teaching. Note that although we are focusing on the teacher's role and the activity of teaching, much of what we are describing could be carried out by curriculum developers; the same principles would apply.

Instructional Planning

Specifying students' current knowledge. Central to the conception-based perspective that is the basis for our framework on mathematics teaching is Piaget's (1985) notion of *assimilation.* That is, what individuals know affords and constrains what they can learn. In our formulation of teaching, the teacher endeavors to understand the students' conceptions in order to anticipate interpretations students can make of proposed tasks, goals that they can set, and activity sequences in which they can engage to work towards their goals.

Specifying the pedagogical goal. In addition to understanding the students' knowledge, the teacher must be able to specify the conceptual advance intended. This is a difficult undertaking. Not only is it insufficient to specify what the student will be able to do (the traditionally employed behavioral objective), it is insufficient to specify the mathematics to be learned (e.g., "The student will understand the distributive property."). It is essential to specify the understanding and how it differs from the identified prior state. Thus, in the example of the distributive property, the teacher would likely describe and contrast two stages of understanding multiplication.

Identifying an activity. Once the teacher has a useful specification of the conceptual advance that she wants to promote and a useful understanding of the students' relevant knowledge, she is ready to consider the key activity involved (e.g., counting as the basis of developing number). The teacher conceives of the new understanding as an abstracted activity-effect relationship. Once the key activity has been identified, the teacher considers whether the students are currently able to generate an activity sequence embodying that activity. In some cases, the students may not have the requisite activity and the goal for instruction will be modified. In identifying an appropriate activity sequence, the teacher must hypothesize about the types of distinctions students could make among the effects of their activity and how these distinctions could lead to the new conception.

Selecting a task. Once the activity sequence is identified, a task can be selected. The task is one that, based on the teacher's conjectures, will result in the students setting a particular goal and engaging in a particular activity sequence.

We see steps 2-4 as further elaboration of Simon's (1995) hypothetical learning trajectory.

Instructional Phase

During the instructional phase, the teacher may need to negotiate with the students a shared interpretation of the task (including the goal). Once this is in place, the teacher is engaged in monitoring each aspect of her conjecture (students' selection of an activity sequence, sorting of effects, and reflection on activity-effect relationships). As a result of monitoring students' activity, the teacher may revise her understanding of the students' conceptions, the task employed, or both. Task modification can range from asking the students a question related to the task to changing the entire task (cf. Tzur and Simon, 1999, for specification of tasks).

Discussion

The theoretical work that we describe in this paper is part of an ongoing effort to understand and explain mathematics teaching and learning in powerful ways. Theoretical advances in this area can contribute to mathematics teaching, curriculum design, teacher education, and research on mathematics teaching and teacher development. Our efforts have been guided by two assumptions: that the mechanisms of learning and teaching underlying successful lessons can be understood, and that this understanding can lead to a more methodical approach to teaching resulting in more consistent generation of successful lessons and informed modification of unsuccessful ones.

Our position is *not* that this framework is indicated for every instructional situation involving conceptual learning. Indeed, students learn some things spontaneously and other things through relatively unstructured inquiry lessons. Rather, we endeavor to understand basic mechanisms of teaching and learning for approaching the more intractable pedagogical problems. Towards this end, we have described organizing instruction to foster reflection on particular activity-effect relationships. As Bereiter (1985) pointed out, "certain kinds of learning really are problematic . . . the learning paradox helps us see into the heart of the problem" (p. 221). We have described an approach to teaching that addresses the learning paradox.

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RE-"CONCEPTUALIZING" PROCEDURAL KNOWLEDGE IN MATHEMATICS

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Abstract: I propose that many mathematics educators have lost sight of the critical importance of the mathematical "understanding" which underlies procedural competence, in part because we do not have a language to refer to this kind of understanding. The modal way of categorizing mathematical knowledge -- conceptual and procedural knowledge --is limited in that: (a) it is almost exclusively focused on the mathematics of elementary school, and (b) in studies using this framework, knowledge of procedures and concepts are assessed in very different ways. Three students' solutions to a linear equation are presented and compared. I suggest that the differences between what these three students know about linear equation solving can be framed in terms of students' planning knowledge of the procedures -- in other words, "conceptual "knowledge about procedures.

By many accounts, the U.S. is beginning to experience a backlash against current mathematics education standards-based reform efforts, with many advocating a renewed emphasis on "basic skills." Many mathematics educators recognize the similarities between this current push toward basic skills and prior backlashes against mathematics reforms in years past. Why does this oft-commented-on pendulum between "skill-based" and "reform-based" curricula appear to be swinging yet again? One reason for the emergence of the current back-to-basics movement is the failure of mathematics education research to adequately study and thus appropriate the topic of "skill." Current reform efforts and research are primarily focused on the type of understanding commonly referred to as "conceptual understanding." In doing so, we have lost sight of the critical importance of another kind of mathematical understanding -- that which underlies procedural competence. Current work lacks an emphasis on *doing* mathematics -- in other words, using and understanding the mathematical procedures and skills that are an essential part of our discipline. In this theoretical paper, I suggest a way in which mathematics education research can return to this issue of procedural competence -- not in place of a focus on "conceptual understanding" but in addition to it.

Conceptual and Procedural Knowledge

Particularly since the publication of Hiebert's book (1986) on the topic, the terms "conceptual knowledge" and "procedural knowledge" have served as a widely-used framework for thinking and analyzing mathematical knowledge. Hiebert and Lefevre

(1986) define conceptual knowledge as "knowledge that is rich in relationships" (p. 3). They note the following example of conceptual knowledge: the construction of a relationship between the algorithm for multi-digit subtraction and knowledge of the positional values of digits (place value). Procedural knowledge is defined as "rules or procedures for solving mathematical problems" (Hiebert & Lefevre, 1986, p. 7). Hiebert and Lefevre write that the primary relationship in procedural knowledge is 'after,' in that procedures are step-by-step, sequentially ordered, deterministic instructions for how to solve a task. They note the following examples of a procedure: the adding of two fractions of unlike denominators.

These two "types" of knowledge are assumed to be distinct yet related. Much effort has been spent examining the relationship between them, particularly determining which optimally comes first. On this issue, Rittle-Johnson and Siegler (1998) conclude that there is no fixed order in the acquisition of mathematical skills versus concepts. In some cases, skills are acquired first; in other situations, the order is reversed.

Two main criticisms of Hiebert's (1986) conceptual/procedural knowledge framework emerge from a more thorough review of this literature (see Star, 1999). First, almost all of the studies underlying this framework (both in the 1986 book and in subsequent work) are from the topic areas of counting, single-digit addition, multi-digit addition, and fractions -- all areas of study in elementary school. Notably absent are studies of the development of procedural and conceptual knowledge in algebra, geometry, and calculus. Second, knowledge of concepts and knowledge of procedures are typically assessed in very different ways. Knowledge of concepts is often assessed verbally and through a variety of tasks. This would suggest that conceptual knowledge is complex and multi-faceted. By contrast, procedural knowledge is assessed uni-dimensionally and non-verbally by observing the execution of a procedure. A student either knows how to do a procedure (and therefore can execute it successfully and automatically) or does not know how to do the procedure.

Elsewhere I have examined the conceptual/procedural framework in more depth, as well as other terminological distinctions among knowledge types (Star, 1999). Here I present an example of an alternative conception of how a procedure can be known, as a way of arguing that the current framework's treatment of procedural knowledge is inadequate.

A "Procedural" Example

Consider the following equation, which was given to middle school students in a unit on solving linear equations: $4(x+1)+2(x+1)=3(x+4)$. What are some different ways that students successfully solved such an equation?

One student, Joanne, performed this sequence of steps: (1.) Use the distributive property on the left and right sides of the equation to "clear" the parentheses.

(2.) Add (-4) to both sides. (3.) Add (-2) to both sides. (4.) Add ($-3x$) to both sides. (5.) Combine like variable terms on the left side and constants on the right. (6.) Divide both sides by 3, yielding $x=2$. Upon completing her solution, Joanne was asked if she could solve the same equation using a different order of steps; she said that she could not.

A second student, Kyle, used this sequence of steps: (1.) Use the distributive property on the left and right sides. (2.) Combine like variable and constant terms on the left side. (3.) Add ($-3x$) to both sides. (4.) Add (-6) to both sides. (5.) Combine like variable and constant terms on both sides. (6.) Divide both sides by 3, yielding $x=2$. Upon completing the problem, Kyle was asked if he could solve the equation using a different order of steps; he said yes and generated a solution that was identical to the one used by Joanne. Kyle was then asked if he could solve the equation in yet another order of steps. He said that steps 3 and 4 of his initial solution could be done in the opposite order, but he knew of no other alternative sequence of solution steps.

A high school sophomore, Leah, initially solved the problem using Joanne's sequence of steps. Upon being prompted for a different solution, she used Kyle's sequence. Upon further prompting, she generated this sequence: (1.) Combine the two terms on the left side, yielding $6(x + 1)$. (2.) Divide both sides by 3, yielding $2(x+1)=x+4$. (3.) Use the distributive property on the left side. (4.) Add (-2) to both sides. (5.) Add ($-x$) to both sides. (6.) Combine like variables and constant terms, yielding $x=2$. When prompted for more, alternative orders, she said (in a somewhat exasperated tone) that there were lots and lots of different orders, but that they were all the same.

It is difficult to draw broad conclusions about what these students know about equation solving from these limited anecdotes. However, it is possible to make some preliminary inferences about the differences in what these students know about the operators of the domain of equation solving. Joanne was only able to solve this equation using one ordering of steps. Her knowledge of the operators of this domain is relatively inflexible. She knows that applying operators in a particular order is very likely to lead to a solution; she perhaps has not considered the question of whether or not any variation in her ordering will also be successful. Kyle knows and is able to use the same series of steps as Joanne. However, in addition, Kyle realizes that one can add constants or variable terms to both sides in any order without affecting the successful solution of the equation. He also knows that one can choose to simplify each side individually before any "moving" of terms. Leah realizes that there are a very large number of ordered arrangements of steps that can lead to a successful olution. She is able to look at the specific details of a particular problem and choose an ordering of steps that works. For example, if a term of the left hand side of the original problem were changed from $4(x+1)$ to $4(x+2)$, Leah would realize that her first two solution orderings (identical to Joanne's and Kyle's) would work, but that her third ordering would not.

It is possible to frame these differences between Joanne, Kyle, and Leah in "conceptual" terms, perhaps referring to each students' relative conceptual understanding of the commutative or distributive properties. However, it is equally plausible that one could characterize the differences between these students' knowledge in terms of what they know about the procedures and operators of this domain. The procedure for solving linear equations has a limited number of possible operators which, when correctly applied in particular combinations, lead to the solution. Kyle knows more than Joanne (and Leah knows more than Kyle) about how these operators fit together to achieve particular goals (e.g., getting the numbers to one side and the variables to the other side, transforming the equation to the form ax=b), what each operator does, and under what conditions operators and chains of operators can be used and to what end.

This kind of knowledge about a procedure has been referred to elsewhere as *teleological semantics* (VanLehn & Brown, 1980). The teleological semantics of a procedure is "knowledge about [the] purposes of each of its parts and how they fit together. ... Teleological semantics is the meaning possessed by one who knows not only the surface structure of a procedure but also the details of its design" (p. 95). VanLehn and Brown (1980) note that a procedure can be cognitively represented on a very superficial level (as a chronological list of actions or steps) or on a more abstract level (incorporating planning knowledge in its representation). *Planning knowledge* includes not only the surface structure (the sequential series of steps) but also "the reasoning that was used to transform the goals and constraints that define the *intent* of the procedure into its actual surface structure" (p. 107). In other words, planning knowledge of a procedure takes into account the order of steps, the goals and subgoals of steps, the environment or type of situation in which the procedure is used, constraints imposed upon the procedure by the environment or situation, and any heuristics or common sense knowledge that are inherent in the environment or situation.

Within Hiebert's framework, this type of knowledge does not fall nicely into either conceptual knowledge or procedural knowledge. In essence, teleological semantics is conceptual knowledge *about* a procedure -- it is both procedural and conceptual knowledge. It is knowledge that is rich in relationships, but where the relationships in question are abstract and are among procedural steps.

Conclusions

Brown and colleagues (Brown, Moran, & Williams, 1982; as cited in Nesher, 1986) argue that procedures should not be assumed to be rote but rather as objects with several different sources of meaning. The example described above illustrates a case where students may possess rather sophisticated knowledge about a procedure and its operators. Neither of the terms "procedural knowledge" or "conceptual knowledge" seems appropriate to describe the understandings that students such as Leah have and can use.

Especially given the renewed emphasis among anti-reformers on "basic skills", it behooves us as a community to be more explicit about what we mean by understanding. As Ohlsson and Rees (1991) note, a complete theory of understanding includes both an understanding of concepts and an understanding of procedures. At present, the terminological framework that is most widely used in the field makes it difficult to investigate the understanding of procedures.

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GROWING MATHEMATICAL UNDERSTANDING: LAYERED OBSERVATIONS

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Abstract: This paper discusses the use of the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding. In particular it illustrates and draws upon two recently extended elements of the theory, those of 'folding back' and 'teacher interventions'. From an initial analysis drawing upon the Theory, significant instances of 'teacher interventions' and 'folding back' are identified and focused upon in greater detail. We suggest that by analysing these points in the pathway of growth a fuller, thicker, and more complete account can be developed of the ways in which the understanding of learners grows. This analysis illustrates the power of the Pirie-Kieren Theory as a theory for the growth of mathematical understanding, and illuminates the way in which different elements of the theory can be combined to provide a rich, multi-layered account of what is observed.

The Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding

The Pirie-Kieren theory has been fully presented and discussed in a number of previous PME meetings (see, for example, Pirie & Kieren, 1992; Martin, Pirie & Kieren, 1996; Martin & Pirie, 1998) and many of its features have been set out and elaborated there and elsewhere. It will therefore only be briefly reviewed here. This theory provides a way of considering understanding that recognises and emphasises the interdependence of all the participants in an environment. It shares the enactivist view of learning and understanding as an interactive process (Davis, 1996; Varela, Thompson & Rosch, 1991) and is based in a belief that although understanding is still the creation of the learner, the classroom, curriculum, other students, and actions of the teacher occasion such understanding, which can be seen as co-emergent with the space in which it was created (Varela, Thompson & Rosch, 1991). The theory, developed to offer a language for, and way of observing, the dynamical growth of mathematical understanding, contains eight potential levels for understanding. These are named Primitive Knowing, Image Making, Image Having, Property Noticing, Formalizing, Observing, Structuring and Inventing.

A diagrammatic representation or model is provided by eight nested circles (see Figure 1). This nesting and the associated traces of students' understanding illustrate the fact that growth in understanding need be neither linear nor mono-directional. Growth occurs through a continual movement back and forth through the levels of knowing, as the individual reflects on and reconstructs his or her current and previous knowledge. When faced with a problem at any level that is not immediately solvable, an individual will need to return to an inner layer of understanding. This shift to working at an inner layer of understanding actions is termed folding back and enables the learner to make use of current outer layer knowing to inform inner understanding acts, which in turn enable further outer layer understanding. The theory suggests that for understanding to grow and develop, folding back is an intrinsic and necessary part of the process. This element of the theory was explored and expanded by Martin (1999) who developed a framework for describing in detail an act of folding back. Included in this framework is the development of the notion of folding back to collect (Martin, Pirie & Kieren, 1996). An act of collecting involves the retrieval of an earlier understanding without the modification of the earlier understanding. Instead, there is the action of reviewing it in the context of the present problem.

The second recently extended element of the Pirie-Kieren theory on which we will focus is the notion of teacher interventions. Understanding what occasions the growth of mathematical understanding requires an understanding of the coemergence of teachers' and students' actions and understandings in classrooms. In studying these interactions and co-evolving understandings, Towers (1998a, 1998b) has developed a number of intervention themes that describe teachers' actions-in-the-moment. The intervention themes identified (showing and telling, leading, shepherding, checking, reinforcing, inviting, clue-giving, managing, enculturating, blocking, modelling, praising, rug-pulling, retreating, and anticipating) acknowledge that the growth of students' mathematical understanding is dependent on, but not determined by, teaching (Davis, 1996).

Layering a Story

The Pirie-Kieren theory provides a way to analyse the mathematical actions of a learner or a group of learners and can describe and account for the way that their mathematical understanding grows and develops. Through the use of a technique known as "mapping", a pathway of growth can be traced out on the model diagram, indicating the ways in which the understanding actions of the learners shift within the layers of the theory (see Figure 1).

We will focus particularly in this paper on two learners who were Grade 9 students in a middle-ability grouping taught by Towers. The session was concerned with finding the area of any segment of any circle. The prompt offered to the students was in the form of a diagram of a circle with a sector and segment drawn as shown

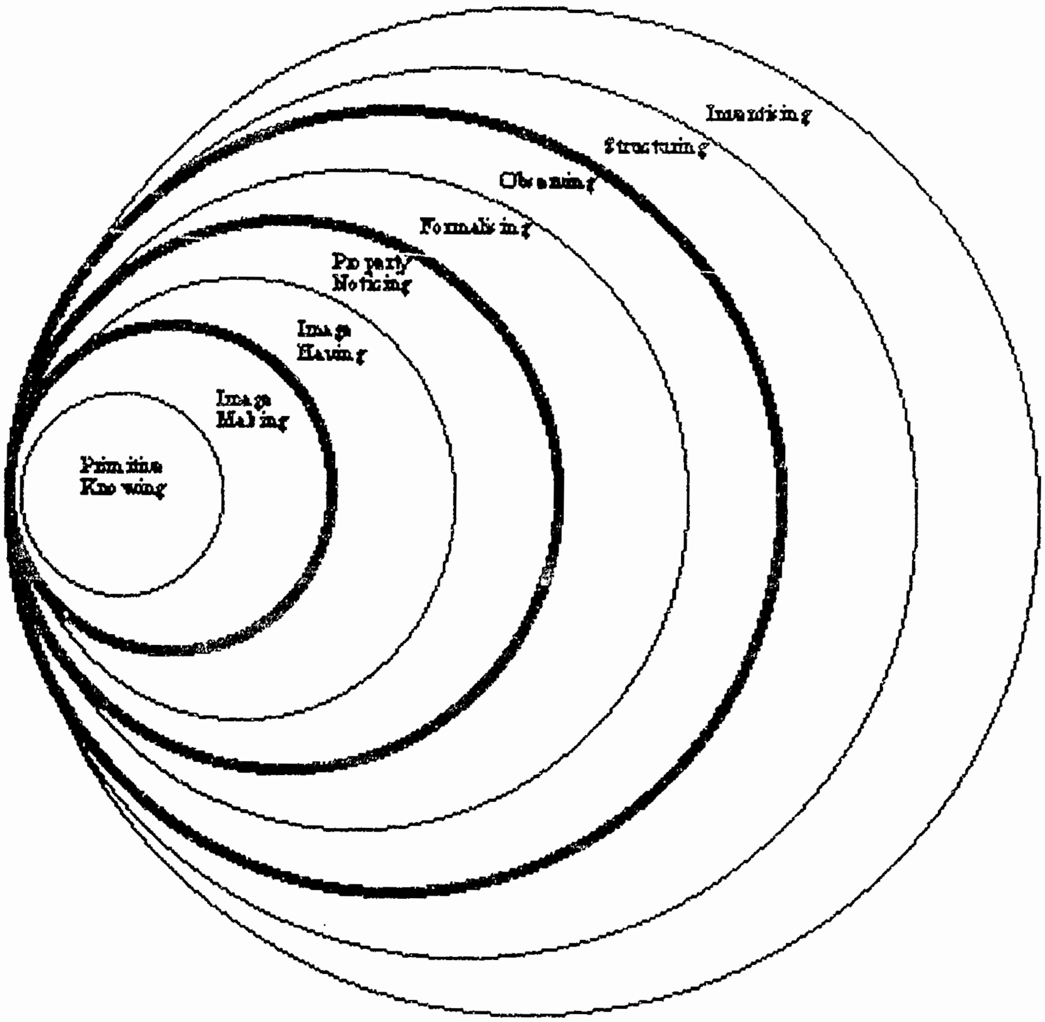

Figure 1. The Pirie-Kieren model for the dynamical growth of mathematical understanding

in Figure 2, but with no dimensions offered. The students in question established the need to find the area of the sector of the circle first, and then proposed to subtract the area of the enclosed triangle. This is the point (Image Having) at which the mapping seen in Figure 1 starts

The predominant impression of Towers' teaching during this lesson is of a teacher who continually wants to encourage the pupils to move from working with the specific to thinking in more general ways. In doing this, she intervenes to help the pupils realise some of the limitations of their current understandings, and to consider how they need to broaden their existing images for the concept. This nudging towards a deeper understanding is an intervention style that Towers (1998a, 1998b) has named "shepherding". A close analysis of the video of this lesson reveals that the teacher invites the students to explain their thinking in order to have a place from which she hopes to occasion their growth of understanding (for example she uses phrases such

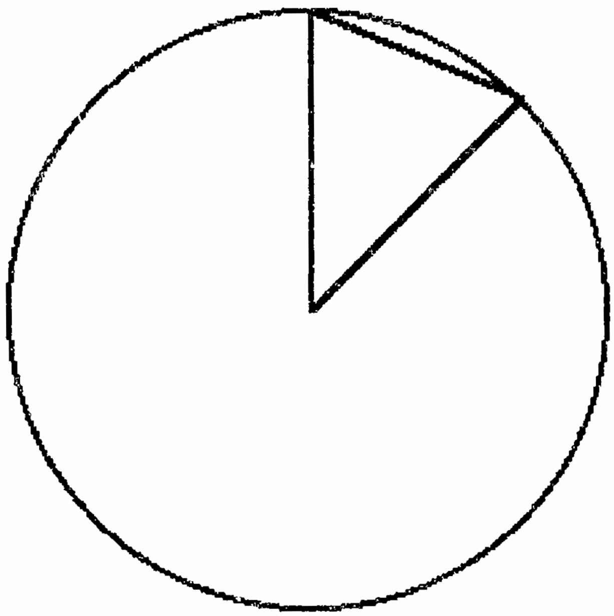

Figure 2. Finding the area of a segment of a circle

as "Tell me what you think you are doing..."). This intervention, occurring at a point at which the students are Image Having (Figure 1) but with an incomplete and partially incorrect image of the problem, enables the teacher to engage in an interaction which prompts the students to fold back to Image Making to build a thicker understanding that recognises the need for trigonometry. She challenges the students' method (as they initially assume that the angle of intersection of the radius of the circle and the chord is a right angle), and offers a new perspective (prompting them to consider the chord as the 'base' of the enclosed triangle) when the students had been struggling for some time working with one of the radii of the circle as the base of their triangle. Rather than halt the flow of the moment with an explanation of why it would be better to use the chord of the circle as a base for the triangle, though, she simply smiles when one of the students recognises the value of the suggestion (evident in the student's sudden recognition that she might "chop the triangle in half" thereby creating a right-angle), and instead prompts the students to access their own Primitive Knowing about trigonometry to help them continue (reminding them that "we've done something like that very recently, haven't we?") Such teaching requires a teacher to be willing to refrain from telling (at least immediately), and be aware that to move forward, these students need to fold back.

When we tried to consider in detail the folding back actions of the learners at this point, analysing the video suggests that this is not a simple automatic process, and that the learners are not able to instantly apply trigonometry to solve the new problem. They know what piece of their Primitive Knowing is needed and why they need it. They are able to fold back and begin to collect from their Primitive Knowing the formalisings and images for trigonometry that they need to use in the current problem. They know that they know what is needed, and yet their understanding is not sufficiently structured for the automatic recall of the appropriate piece of usable knowledge. It is this lack of ability to simply instantly recall that we call collecting. There is however, no act of modification of the earlier understanding, instead there is the action of finding and reviewing it in the context of the present problem. It is this reviewing that gives the 'thickening' effect of folding back. The students are able to re-collect the role of angle in trigonometry and especially the way in which dropping a perpendicular to the base from the apex of the triangle creates a right angle and halves the angle at the apex. They are then able to state what they think is the necessary trigonometric formula. However, they are not totally confident with this as an answer, thus confirming that the formula was not immediately accessible to them for automatic use. They deliberately choose to use an exercise book as an aide-memoir to confirm their collecting and quickly find a similar example that supports the formula they have stated. Throughout this episode they have to work on collecting small fragments of their understanding. There is a sense of the learners knowing and being aware that they have the necessary understandings but that they are just not immediately accessible.

Observations and Conclusions

The micro-analyses presented above of one small classroom incident have been offered as an illustration of a way of getting at the complexity of growing mathematical understanding. Although the mere noting that folding back and teacher interventions play a part in this episode begins to tell us something about the way that the understanding of these two students is growing, the subsequent more detailed study of these aspects of the theory provides a greater insight into the process through which this growth occurs. The power of the Pirie-Kieren theory is that it is a theory *for*, not a theory *of*, the growth of mathematical understanding, and as such it is validated by its usefulness to someone seeking to make sense of the growing mathematical understanding of learners. This paper illustrates how the theory might be useful both from the perspectives of teaching and of learning, through offering to an observer two different yet complementary lenses within the Pirie-Kieren theory through which to view particular elements of the process of coming to understand.

The choice of which lens to use in a particular situation is of course determined by the needs of the observer. The extended aspects of the Pirie-Kieren theory, folding back and teacher interventions, provide a way of layering an account of the developing understanding of learners and an observer may choose to make use of both of these elements, only one, or indeed neither – depending on what they want to observe and to account for. This paper has aimed to briefly illustrate the multi-layered, flexible and dynamic character of the Pirie-Kieren theory and, more specifically, the ways in which it can be used to look closely at growing mathematical understanding as it is seen to emerge.

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ONTOLOGY AND PHENOMENOLOGY IN MATHEMATICS EDUCATION

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Abstract: To its credit, the discipline of mathematics education has a tradition of concern for theory, both in attempts to theorize about learning and teaching mathematics, and in making theoretical assumptions behind research explicit. We acknowledge as critical the role that the philosophical background plays in research. In discussing an approach that has both phenomenological and ontological validity, based on Heidegger's careful study of *being*, we introduce this tradition, along with some of its terminology, and briefly address how we have made use of it as we have studied the teaching and learning of mathematics.

Introduction

As Kilpatrick (1992) has pointed out, mathematics education as a discipline has traditionally turned to psychology to provide theoretical grounding for its inquiry into how mathematics is learned. More recently, it has also turned to sociology and cognitive anthropology to explain the phenomenon of teaching and learning. It is entirely proper that mathematics educators concern themselves with theory. Such a concern distinguishes mathematics education as a scholarly discipline and separates it from the technology of teaching.

In its short history as a separate discipline, mathematics education has embraced several theoretical positions–running the gamut from behaviorism, through cognitivism, genetic epistemology, constructivism, and situated cognition, to structuralist views such as semiotics and interactionism. Most recently, attempts have been made to ground theories of learning and teaching mathematics in post-structuralist or post-modern thought (Gallagher, 1992, Davis, 1996, Brown 1997). In many cases, these philosophical marriages have been based on practical considerations–providing lenses through which scholars viewed mathematical activities and behaviors. In a few cases, most notably within the constructivist tradition, an equally important consideration has been metaphysical, involving ontological or epistemological stances. We argue, along with Slife and Williams (1995), that the theory that lies "behind the research" is important, perhaps even critical, and needs to be taken seriously.

We suggest that, in order to work, a good theory must succeed at both an *ontological* and at a *phenomenological* level. That is, it must function both as a good description of how things seem to human subjects and as a consistent and reasonable description of *how things are.* For example, constuctivism functions at one level in solving an ontological problem but, we argue, does not function as a phenomenology except at the *trivial* level. Within constructivism, humans feel that they actively construct knowledge but also feel that there is an external reality about which they can gain more or less reliable knowledge. Other theories used to ground studies of mathematics teaching and learning seem to function well as phenomenologies but leave their ontological positions unexplored or largely implicit. As an example, we can consider recent works in situated learning or cognition, in which knowledge is viewed as participation in a certain community. Here, humans' embeddedness in a context plays a role, but it is not always clear whether a Cartesian dualism (between abstract, subjective cognition and the external, contextual situation) is being maintained or not.

We do share, with the constructivist tradition, a view of Cartesian dualism as problematic and, with scholars who study situated cognition, a concern for the importance of context and *situatedness* in the world. Along with many other colleagues in mathematics education, we share a concern for the importance of the social world in the acquisition/construction of knowledge. It is our hope in this paper to briefly sketch a philosophical position that addresses ontological problems head-on and, at the same time, rings true at both phenomenological and methodological levels. We argue for the appropriateness of Heidegger's ontological project as an appropriate grounding for studies of teaching and learning. We go back to Heidegger, rather than drawing on more contemporary postmodern thinkers, because we agree with Krell (1977) that Heidegger's *Being and Time* (1962) "brooks no comparison in terms of influence on Continental science and letters or genuine philosophical achievement" (p.17). Heidegger's careful explication of being, particularly *human* being, remains unparalleled in philosophical thought.

In this paper we briefly explore the implications of three aspects of Heidegger's thinking: 1) the view of human being as *being in the world*, radically contextualized by nature, and *thrown* into an historical world that is already there; 2) a careful elucidation of the structures, including the temporal structures, of being that define the ways humans understand and engage with the world; and 3) his view of *the mathematical* and how it relates to our task as mathematics educators.

Heidegger's Fundamental Ontology

Heidegger confronts Cartesian dualism head-on by a careful study of *being*. He suggests that the particular kind of being, for whom its own being is an issue, is uniquely human and calls this type of being *Dasein*. *Dasein* is not to be understood as a subjective ego, but rather as a kind of being that is always already in a world, with projects, equipment, and concerns. Similarly, *world* is not to be understood as a separate, external reality, but as radically linked to *Dasein*. Thus *world* in general is understood much the way we understand *the world of the theater* or *the world of work.*

The world presents itself to us as an equipmental whole, already bound up with our concerns. This is not to say that humans cannot consider themselves as separate from the world, but that all understanding and meaning making is based on the fundamental fact of finding ourselves already in a world, already having concerns and projects, and unable to step entirely out of it. Everything we understand, then, is contextual in this radical way. Our being is essentially interpretive. Moreover, for Heidegger, being-in-the-world is always also being-with-others; our worlds are inherently social, and our projects always have meaning socially.

Heidegger distinguishes three modes of being, or three ways in which the world can be disclosed to us: as *ready-to-hand*, as *unready-to-hand*, or as *present-at-hand*. The most fundamental level is the ready-to-hand or what we will call *available* (following Dreyfus 1991). Things in the world present themselves as available when they are being used to further our projects, in such a manner that we can be largely unaware of them. Chalk is usually available in this sense as we use it during a lecture. We are hardly aware of it or its properties, and it functions within the larger project of our teaching. If the chalk should break, or if we suddenly wish we had a second color of chalk, it begins to be *unavailable* to us–it becomes unready-to-hand. At this point, it requires our attention and it stands out somewhat within the context of our teaching. At this point, we can easily fix the problem (pick up another piece of chalk, perhaps) and return to an unconcerned coping. Finally, we can strip the chalk of its context to consider its properties–size, shape, cost, dustiness. In this case, the chalk becomes *occurrent* or present-at-hand. In this mode, we might do *science* with the chalk, study what makes chalk behave as it does, and so forth.

Heidegger's treatment of *understanding* parallels his development of these three modes of being. For Heidegger, the primary way we understand the world is through a concernful coping with the available. This kind of knowing about the world, what Dreyfus (1991) calls *primary understanding*, is most like know-how, for example, being able to drive a car or strike a tennis ball without having to deliberately focus on the activity. At times, of course, we have to act in a more deliberate way, as when we see something unexpected on the road before us, but we still act to solve the problem within the context of our current project, i.e., driving somewhere. In this case, breakdowns in the background flow cause us to have to *interpret* the events within the context of our driving. Finally, we can understand the world through what is occurrent–we can study driving a car as a decontexualized act, focusing on response time or studying driver's eye movements–we can *theorize*. These modes of understanding all have value, but, for Heidegger, the point is that they are all based primordially on the everyday, concerned coping with available equipment.

In his later writings, Heidegger discusses what he calls the *mathematical*. He distinguishes the mathematical from mathematics proper, arguing that what we think of as mathematics is a special case of the mathematical, or "the fundamental position we take toward things by which we take up things as already given to us, and as they must and should be given" (Krell, 1977, p. 254). It is, basically, the desire to reason from first principles, or axioms, or the will to "ground itself in the sense of its own inner requirements. It expressly intends to explicate itself as the standard of *all* thought and to establish the rules which arise thereby" (p. 275). Thus, the mathematical, and by implication mathematics, is seen by Heidegger as the ultimate in *theorizing*–seeking to ground understanding in such a way that only that kind of understanding is recognized as understanding at all. For Heidegger, this typifies modern scientific and technical thought.

Some Implications

Heidegger's ontological project has implications for how we view what mathematics is, how we view the understanding of mathematics and, hence, its learning and teaching, and how we view classroom life. For example, if the being of our students is being-in-the-world, we cannot separate their "taking up" mathematics as either a tool or as a way of theorizing, without dealing with the world in which they find themselves. We must do so on the background of the world as it is disclosed for students–against the background of their projects, social relationships, and prior understandings. This means more than trying to take into account what we might call social variables or social context. Rather, it means we must approach our venture as Heidegger approached his–grounded in the average everydayness of both students' worlds and mathematics as a tool for their projects.

As an example, we have in the past reported on ways in which students' social projects can radically alter teachers' intentions and the mathematics that is learned (Williams & Ivey, 1994; Ivey, 1997). The social projects of the students can essentially remake mathematical tasks so that social, rather than mathematical ends are met. This can happen when members of groups create a miniature classroom, complete with a teacher, or when students become marginalized despite their overall competence with the mathematical tasks. It can also happen when students' goals, desires, and world views do not have a place for the mathematics they meet in schools (Walen, 1993; Ivey, 1994). This takes the problem of making mathematics "meaningful" from the domain of psychology, into the realm of fundamental ontology.

Thus, when we address the learning of mathematics, it becomes clear that we must deal with mathematics at both the level of the available and at the level of the occurrent. Certainly part of the goal of school mathematics is to make mathematics available (in Heidegger's sense) as a tool to be used in our everyday projects. This is complicated by the assertion that mathematics is the tool of the theoretical–the most radically occurrent kind of being. These dilemmas play out every day in the classroom lives of mathematics students, as they grapple with the role of mathematics in their world, accept or reject mathematics as occurrent, and determine the availability of mathematics. One example given by Ivey (1994) is of a class discussion of divisibility by four, in which the class as a whole was engaging with the concept at the level of the available, simply as a tool to be used in problems. One student, because of her desire to theorize–to address mathematics as occurant–was able to influence the classroom discussion by altering the mathematics discussed and presented to be learned for the class as a whole. The study of the subtle interplays between the occurant and the available yield deeper insights into the nature of the mathematics classroom, and into how mathematics itself is taken up by students.

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MODEL-DEVELOPMENT SEQUENCE PART I: DESIGN PROCESS FOR MODEL-ELICITING ACTIVITIES

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This poster presentation is the first of a four-part sequence that describes Model-Eliciting and Exploration Activities. This poster presents the process that activity designers have typically engaged in to develop the Model-Eliciting Activities. The activity designers work individually and in teams to craft the activity relative to the needs of the person or groups requesting the task. The process is both multi-step and iterative.

A Model-Eliciting Activity asks students to develop a mathematical model that describes, explains, manipulates, or predicts the behavior of a real world mathematical or scientific system. Students are given a problem statement that represents a real-world dilemma that needs to be solved for an identified client. The students' solutions are models in that they represent students' reasoning about the different elements of the complex situation in which the problem is cast. Typically, the problem statement asks the students to describe how the client can use the students' solutions in the future to solve similar problems. A Model-Exploration Activity is a follow-up activity for the Model-Eliciting Activity.

The design process typically starts with a rough idea of a problem and a context that is presented to the design group. The design team identifies a context in which the problem is realistically situated and crafts an appropriate focus problem statement. Several design principles have been identified to ensure that the activities elicit models of student thinking. In addition, several mathematical concepts have been identified that resist Model-Eliciting Activity development.

A THEORETICAL FRAMEWORK FOR THE DEVELOPMENT OF MATHEMATICAL SOPHISTICATION

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I have developed a framework that describes students' habits in learning mathematics (*How should I study it?*), and practices in checking mathematical truth (*How do I verify it?*). My work stems from theories of college students' intellectual development (e.g., Perry, 1970), which state, in brief, that students begin with narrow, black-and-white, uncomplicated views of the world but slowly develop more complex, contextual, shades-of-gray views. I took these theories on "ways of knowing" and adapted them into "ways of knowing mathematics" to help frame students' intellectual understandings *about mathematics.* The results of my research is an evolving framework that describes stages of development of mathematical sophistication in undergraduates (much in the spirit of Piaget's description of the stages of learning of schoolchildren).

My poster displays the theoretical framework. My presentation is highly interactive – as people walk by my poster, I buttonhole them, ask them to bring to mind a student they know well, and we collaboratively discuss the framework, which describes five stages of sophistication in two separate categories. I therefore have the chance to give each participant some hands-on experience for the framework, and I provide them a summary sheet and contact information to enable further exploration.

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Functions and Graphs

MODELING THE DEVELOPMENT OF THE CONCEPT OF FUNCTION[1]

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Abstract: As children move from a process-based to an object-based understanding of mathematical functions, we suggest they pass through three phases of processing: (1) procedural; (2) interval; and (3) object-based. Analysis of a microgenetic instructional study showed that each of these processes is developed through a particular stage in an experimental curriculum. We present here the (i) psychological processes that characterize each phase of processing, (ii) the component of the curriculum that supports each phase, and (iii) examples of students' reasoning within each phase of processing. We suggest that standard curricular approaches to the teaching of functions do not adequately address the many psychological connections children must make in understanding functions.

Much has been written about the importance of the *process to object* conceptual shift for developing a deep understanding of mathematical functions (e.g. Sfard, 1992). However, analyses that characterize students' psychological p cesses when going from a process- to object-based understanding are lacking. In this paper, we present the results of such an analysis.

We consider a process understanding to be one where students use a procedural or computational (Sfard, 1992) approach for deriving pairs of values, which are then used to represent a function in a tabular and graphical way; and we regard an object-based understanding as one where a function is conceptualized as something to which processes may be applied (Harel & Dubinsky, 1992). Three phases of psychological processing have emerged from our analyses of children going from a process to object understanding of function: (1) procedural; (2) interval; and (3) object-based. Because we see a circular relationship between structural modeling and curricular design (Kalchman, Moss & Case, in press), we derive our cognitive models from the analysis of instructional studies (Kalchman & Case, 1998, 1999).

The Study

Three sixth-grade female students attending a university's laboratory school participated in the study. They spent nine 45-minute sessions engaged in an experimental instructional unit on functions. The first five days were spent in a classroom setting; the next four were spent using spreadsheet technology; and the final day was spent meeting on a one-to-one basis. On days 1, 2, 5, and 9, students were asked the same set of items either at the beginning or the end of the session. This repeated questioning was to identify at which points in the curriculum students made conceptual gains. Students worked at their own pace on the computer for days 6, 7, and 8.

Phases of Processing: Curriculum, Phase Attributes, and Students' Thinking

Table 1 summarizes the central curricular features that support each phase of processing, the cognitive attributes that characterize children's thinking in each phase, and examples of how children conceptualized a particular item while working within each level of understanding. The item analyzed was one where students were shown a graph of the function $y = x + 7$, seen only in the upper right quadrant of the Cartesian grid. Both axes were labeled from 0 to 10, with the points (0, 7), (1, 8), and (2, 9) plotted within the line. Students were asked to give an equation for a function that would pass through the linear function seen, and then to explain their reasoning.

The Procedural Phase

In the first phase of processing we identified, children go from no understanding of what a function is to a procedural understanding. They become able to generate a table of numbers and a graph for any function of the form $y = f(x)$ from $x = 0$ to $x = n$ (where n is a positive integer).

We support the development of this understanding by beginning instruction with the context of a walkathon, where a specified amount of money earned per kilometer walked is symbolized, calculated, and then graphed. For example, if the sponsorship arrangement is earning $4.00 per kilometer walked, students first construct an algebraic symbolic representation for the rule, e.g., $\$ = km * 4$. They then calculate the amount of money earned at each kilometer walked by multiplying each kilometer by 4, e.g., 0 km = $0.00, 1 km = $4.00, 2 km = $8.00, etc. They then represent the results of these calculations in a tabular fashion. When the table has been completed for 10 km walked, students place markers at each coordinate point on a Cartesian grid. The markers are then joined to form a line. Several different rules of sponsorship are explored in this way, including those that produce curved lines (e.g., the amount of money earned is equal to the number of kilometers walked times itself [$\$ = km^2$]), and those that involve initial "starting amounts," which correspond to the y intercept (e.g., one is given a starting bonus of $5.00 and still earns $4.00 per kilometer [$\$ = 5 + 4 * km$]).

The psychological attributes of this phase are summarized in the middle column of the first row of Table 1. In this phase students work from a procedural standpoint. That is, they perform calculations on the independent variable in order to obtain values for the dependent variable and then organize their results in first a tabular and then graphic way. General properties of these representations are not abstracted, but the

Table 1. Phases of processing, their attributes, and students' thinking

Phase of Processing	Central Curricular Features	Psychological Attributes	Example of Students' Thinking
Procedural	Walkathon -- calculation of dollars earned per kilometer walked; creating a table of values; and plotting discrete points on a grid	Mastery of calculating numerical values for the `ependent variable given a set of values for the independent variable and arranging them in a tabular form Mastery of plotting pairs of x and y coordinates that correspond to the pairs of values found in the above table Recognition that each pair of numerical values is associated with a unique point on the graph and vice-versa Mastery of heuristics for moving from a set of points found in a tabular form or graph to the rule that generated them	"Let's say someone paid you $9.00 for every kilometer you walked. So, I was thinking the equation could be x times 9 equals money. At 0 kilometers you have $0 and then [when] x is 1 … 1 times 9 is 9 so y = $9, which has already crossed."
Interval	Calculation of numeric intervals between values of the dependent variable in both tabular and graphic representations	Recognizing particular properties of intervals between successive y values as a constant increasing value for functions such as y = 4x (with 4 being that constant value) Recognizing that constant intervals between y values on a graph always result in straight lines Connecting a concrete mathematical rule, the pattern in the sequence of numbers it generates (e.g.., a constant interval), and the pattern seen in the graph (e.g., a straight-line) Repeating above for properties such as y-intercept, non-linearity, and the negative or positive slope or "direction" of any function	"I started off with 3, … and each time I'm going to move up 6. So, it's going to be a straight line because it's going up by the same amount and it's going to be steep because it has a slope of 6 and it does pass through (the existing line) The equation is y = x times 6 + 3."
Object-Based	Computer spreadsheet activities where students operate on y = x or y = x2	Use of the graphic representation of y = x and the y = x2 as mental referents, or concrete objects, for abstracting properties and features of a function and judging the "steepness" of a line, degree of a curve, "directionality", and y-intercept	"y = x times 5. It goes up by the same amount and it's a straight line and it's steep enough because it goes up by 5 each time. You could also do other steep ones but I just did this one."

link between the coordinate pairs found on the graph and the pairs of values recorded in the table is recognized and understood to have been generated from a specific rule such as multiplying each x by 4.

When asked to provide an equation for a function that would pass through the one described above, prior to instruction students made a series of dots at the coordinates (0,9), (1,8), (2,7), and (3,6) but did not express the meaning of these dots as any sort of functional relationship. Then, on day 2, CP explained: "*Let's say someone paid you $9.00 for every kilometer you walked. So, ... the equation could be x times 9 equals money. At 0 kilometers you have $0 and then [when] x is 1...1 times 9 is 9 so y equals $9, which has already crossed.*" With each calculation for the dependent variable, CP plotted the coordinates on the grid.

The Interval Phase

The second phase of processing is one where students develop an *interval* understanding of a function by noticing the second-order features found in the tables and graphs. These second order features include the value of the numeric and coordinate *jumps* found between successive y values. The second row of Table 1 describes the particulars of this phase.

We promote an interval understanding of function by having students investigate what generally determines the linearity, degree of "steepness", and y – intercept of a function. For example, in the tabular expression of the function *$ = km* * 4, students add a column to the right of the *$* column and write in the difference between successive values, which in this case is consistently 4. Students also graph functions by sketching the graph in intervals by moving their pencils "up by" for example 4 as they move over one unit, rather than as discrete points. Intervals are also calculated for non-linear functions, and students realize that the degree of the "steepness" of a curve is related to the span of the intervals. Students eventually generalize that straight-line functions always have constant intervals between y values, and curved lines do not.

By noting second-order properties found in the tables and graphs of particular functions (i.e., numeric or graphic *jumps* or intervals), students abstract general features of functions such as (a) linearity, (b) specific slopes, (c) the degree of an exponent, and (c) the y-intercept.

In response to the item, on day 2, HJ explained that "*if your starter offer was 1 and your next point was 6, so you go up by 5...then your next one is 10 so each time you go up by 5.*" Her interval understanding appears incomplete at this point, however, because she did not include the $1.00 starting value in the equation or mental computation. This omission may have been due to a still emerging understanding of how to connect the mathematical rule, the pattern in the sequence of numbers it generates, and the pattern seen in the graph. On day 5, however, HJ seemed to make these connections: *"I started off with 3...and each time I'm going to move up 6. So, it's going to be a straight line because it's going up by the same amount and it's going to be steep because it has a slope of 6 and it does pass through. The equation is y = x times 6 + 3."*

Object-Based Phase

In this phase, students construct and abstract third-order properties such as slope and y-intercept from the second-order properties already noted. The curricular feature that supports this phase is students' use of spreadsheet technology. Students work with a pre-configured computer screen, which displays the graph of $y = x$ (or $y = x^2$) on the left-hand side, and a spreadsheet with the corresponding table of values (found in columns **X** and **Y**) on the right. A console displays the equation of the function. On the spreadsheet, students are asked to change individual parameters of the function (slope or y-intercept) in order to move $y = x$ through randomly placed pre-plotted points. All actions carried out are reflected instantly and automatically in the graph and in the numeric pattern found in the **Y** column. Then, using columns to the right of **Y**, students program their own functions with given specifications such as a slope greater than that of the original line ($y = x$) and a y-intercept < 0. Students also carry out these sorts of activities for curved lines.

Using $y = x$ and $y = x^2$ as concrete mental *objects*, students are able to operate on the functions as entities in order to produce new functions that have, for example, constant intervals greater than 1, and thus, are steep relative to $y = x$. Others with constant fractional intervals are relatively flat. Students may operate on $y = x^2$ to produce functions whose curves are steeper than $y = x^2$ by making the exponent larger than 2, the coefficient greater than 1, or some combination of the two. Likewise, for functions that curve down, the degree of "steepness" is still described by the exponent and the coefficient, and the "down by" is defined by the x^n being multiplied by a negative coefficient. With respect to the y – intercept, students see the addition or subtraction of a constant value as a means to qualitatively and quantitatively translate the base function.

On the final day, students seemed to be using the given line as a mental referent for determining a suitable equation for a function that would pass through the one given. HJ drew a straight line starting from the origin and passing through (1,5) and (2, 10). She explained: *"y = x times 5. It goes up by the same amount and it's a straight line and it's steep enough because it goes up by 5 each time. You could also do other steep ones but I just did this one."*

Conclusions

We attribute students' success with this difficult topic to their engagement with the innovative curriculum, which allowed them to work within the walkathon context to master the necessary sorts of procedures. First, they calculated values for the dependent variable from values for the independent variable. Then they calculated the intervals between successive values for the dependent variable – a calculation that essentially defines the general shape (i.e., linear or curved) of a function. As students moved among tables, graphs, and symbolic and verbal rules, they recognized and generalized certain important features of a function such as *y*-intercept, degree of "steepness" and linearity. We think that experts in mathematics underestimate the time it takes to make the many psychological connections proposed here and neglect to provide an adequate body of examples necessary for abstracting the entailments of particular functions.

Notes

1. This work was made possible through the generous support of the James. S. McDonnell Foundation and the Social Sciences and Humanities Research Council of Canada. Many thanks to Karen Fuson for her support and comments.

2. Robbie Case passed away while this work was in progress.

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USING THE FUNCTION MACHINE AS A COGNITIVE ROOT FOR BUILDING A RICH CONCEPT IMAGE OF THE FUNCTION CONCEPT

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Abstract: We examine the question of whether the introduction and use of the function machine representation as a scaffolding device helps undergraduates enrolled in a developmental algebra course to form a rich, foundational concept of function. We describe students' developing understanding of function as an input/output process and as an object, tracing the internalization of the function machine concept as it relates to various representations of functions.

Introduction

Students' use of expressions, tables, and graphs in understanding functions has been studied extensively over the past several decades. Much of the literature on students' concepts of function examines what they do not understand and their misconceptions, offering explanations as to why this might be so (Goldenberg, 1988; Janvier, 1987; Kaput, 1989; Tall & Bakar, 1992; Thompson, 1994). Two recent studies suggest that the introduction of the function machine as an input/output box enables students to have a mental image of a box that can be used to describe and name various processes, often without the necessity of having an explicit process defined. Other forms of representation may be seen as mechanisms that allow an assignment to be made (by a table, by reading a graph, by using a formula, or by some other assignment procedure). The evidence of our research suggests that the function machine provides a powerful foundation and is a cognitive root for developing understanding of the concept of function. A *cognitive root* (Tall, McGowen, and DeMarois, 2000) is a concept met at the beginning of a curriculum sequence that:

(i) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence.

(ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction.

(iii) contains the possibility of long-term meaning in later theoretical development of the mathematical concept.

(iv) is robust enough to remain useful as more significant understanding develops.

Students' internalisation of the function machine concept were examined against these criteria, addressing the question of whether use of the function machine representation leads to a rich, foundational understanding of function.

Data from two previous studies (DeMarois, 1998, McGowen, 1998) are examined for evidence of the function box as a cognitive root. The subjects of these studies were undergraduate students enrolled in developmental Introductory or Intermediate Algebra courses that do not carry general education credit. Many students had encountered the content before, so these studies used a restructured curriculum centred on the concept of function using function machines. The two studies include (a) quantitative methods of data collection used to indicate global patterns generalizable across populations to document changes in students' understanding and to measure improvements in their mathematical competencies, and (b) qualitative methods that add depth and detail to the quantitative studies that allowed the researchers to focus on individual students within the context of the quantitative studies.

All students were given pre- and post-course surveys to establish what they knew about functions initially and after sixteen weeks. Several students from each course participated in interviews subsequent to the course. Data routinely collected in the Intermediate Algebra study also included student work and concept maps collected at five-week intervals, as well as mid-term and end-of-course student interviews. Growth in students' understanding and improved flexibility of thought was documented in descriptions/explanations of their work in terms of an input/output process and in their improved ability to (i) interpret and use ambiguous function notation, (ii) translate between and among various function representations, and (iii) view a function as an object in its own right. Various types of triangulation were used including data triangulation, method triangulation and theoretical triangulation (Bannister, et. al., 1996).

Examination of Data

Two questions asked on the pre- and post-course Introductory Algebra survey are shown in Figure 1. The summary of responses in Table 1 indicates that two-thirds of Introductory Algebra students were able to interpret a function machine diagram flexibly at the beginning of the course, an indication that the function machine representation is an accessible starting point for many students.

Students in both studies were asked on pre- and post-course surveys to find output(s) given a graph and input. They were also asked to find input(s) given a graph and an output. The questions on the two surveys differed in some respects. The Introductory Algebra question displayed a window indicating the scale and graph of a parabola (see Figure 2). A correct response included recognition that there are two

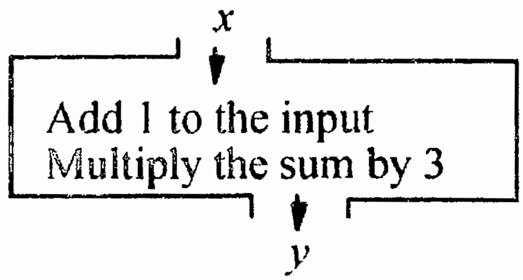

Figure 1. Consider the diagram above. (a) What are the output(s) if the input is 7? (b) What are the input(s) if the output is 18?

Table 1. Funtion Machine Input and Output (Introductory Algebra Student Responses)

Question	Pre-course (n = 92) number (%) correct	Post-course (n = 92) number (%) correct
a) Function machine: input given	62 (67%)	79 (86%)
b) Function machine: output given	44 (48%)	64 (70%)
Function machine: both parts correct	43 (47%)	61 (66%)

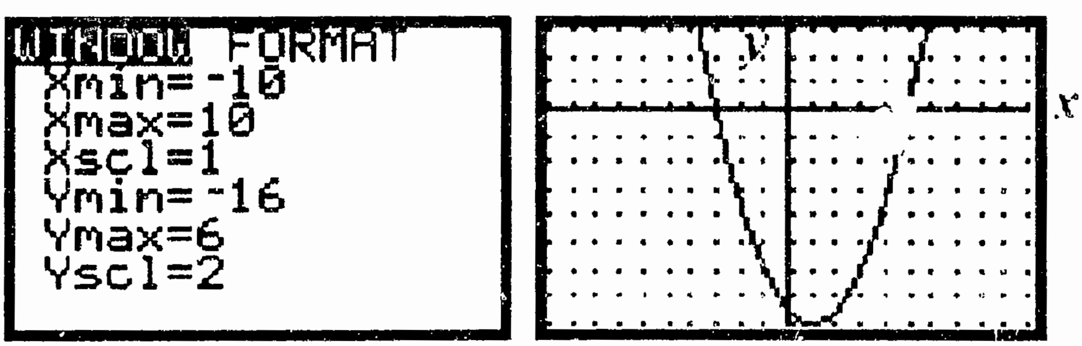

Figure 2. Consider the viewing window and graph copied from a TI–82 graphics calculator. (a) What are the output(s) if the input is 3? (b) What are the input(s) if the output is 0?

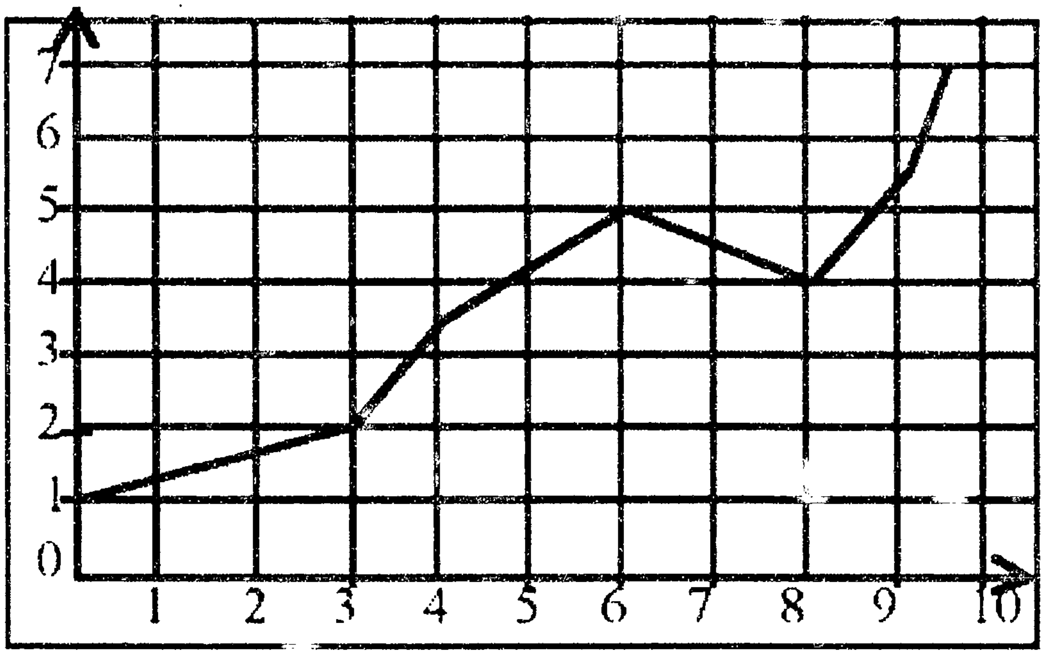

Figure 3. Given the graphabove. (a) Indicate what y(8)=_______. What first comes to mind? (b) If $y(x) = 2$, what is x? _________ What first comes to mind?

answers to part (b). Given the form of the question, students were not required to interpret function notation in order to solve the problem though they were required to switch their train of thought to answer part (b). Students' responses to both parts were considered a measure of their improved ability to think flexibly.

The Intermediate Algebra survey question asked students to determine output(s) given the graph of a piece-wise function and an input, then to determine the input, given an output, using the same graph (see figure 3). The form of the question required students to interpret function notation as well as to change the direction of their train of thought to answer part (b).

Table 2 displays the results of student responses to the survey questions. The results indicate that only 41% of Introductory Algebra students were able to find output given input and only 22% were able to reverse the process at the end of the semester. This suggests that, even when understanding of functions based on a function machine representation was demonstrated, these students demonstrated little connection between function machines and graphs (DeMarois, 1998). By the end of the semester, 71% of Intermediate Algebra students were able to find output(s) given an input and 46% were able to reverse the process. Only 21% of Introductory Algebra students and less than half (40%) of the Intermediate Algebra students were able to do both processes by the end of the semester. However, when one considers students' initial responses and their improved flexibility of thought over the sixteen weeks, the

Table 2. Graph Input and Output Student Responses

Survey Question	Introductory Pre-course % correct (n = 92)	Introductory Post-course % correct (n = 92)	Intermediate Pre-course % correct (n = 52)	Intermediate Post-course % correct (n = 52)
Graph: input given	1% (1/92)	41% (38/92)	38% (20/52)	71% (37/52)
Graph: output given	0% (0/92)	22% (20/92)	17% (9/52)	46% (24/52)
Graph: pair correct	0% (0/92)	21% (19/92)	6% (3/52)	40% (21/52)

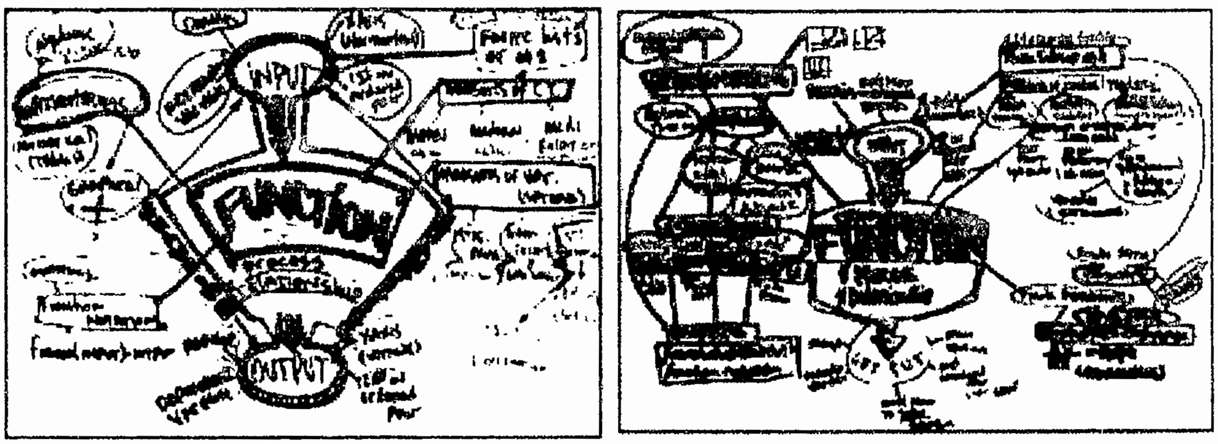

Figure 4. Concept Maps of Function (Week 4 and Week 9): Cognitive Expansion

results are encouraging. The average change in correct responses for the Intermediate Algebra students was statistically significant (two-tailed paired t-test, $p < 0.001$).

Concept maps done throughout the semester document how the function machine idea allowed initial development through cognitive expansion of Intermediate Algebra students' developing understanding of function. Figure 4 illustrates how a student's concept image of function developed and was impacted by use of the function machine representation.

A closer examination of these maps in Figure 5 documents the student's growing understanding of *representations* that occurred over time. In an interview at mid-term, the student described his growing ability to make sense of and interpret functional notation in terms of input and output:

> I feel that I have really made sense of input and output when dealing with function notation. Problems on the Unit II individual test used to look so

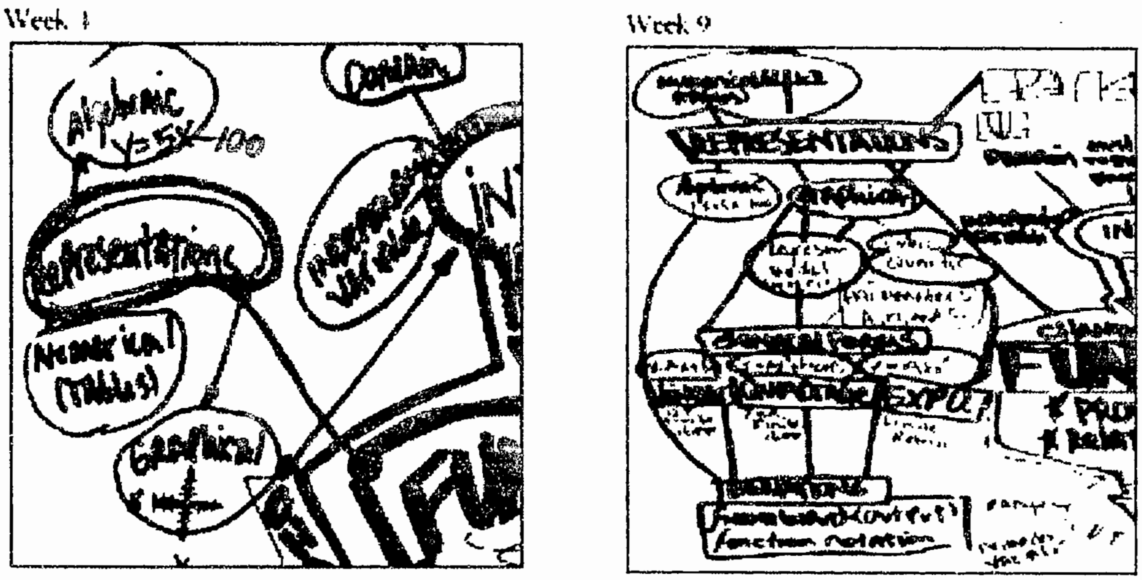

Figure 5. Concept Map Close-ups (Week 4 and Week 9): Representations

unfamiliar to me, but now make perfectly good sense.... I'm learning how these algebraic models are set up and what the variables that they contain represent. I'm no longer just blindly solving for x, but rather understanding where x (input) came from and how it was found from the data given. Through this kind of learning I have developed an understanding for the use of function notation [$f(x)$ = output] and how it replaces the dependent variable, y.

By Week 15 the student internalised the function-as-process concept. Evidence of the input/output cognitive root was still present in his final map, which was colour-coded to indicate concepts connected with input or output. By the end of the semester, the student was able to translate flexibly and consistently among various representational forms (tables, graphs, traditional symbolic forms and functional forms). He expressed confidence in the correctness of his answers. In his final interview of the semester, the student spoke of his understanding of function notation:

I think the most memorable information from this class would be the use and understanding of function notation. A lot of emphasis was put on input and output which really helped me comprehend some algebraic processes such as solving for x.

Conclusion

The evidence presented suggests that the function machine is a cognitive root for the function concept for the subject population and that function machines provide a foundation on which to further develop the function concept. Function machines impacted students' thinking and learning as evidenced in their work and in their written and oral statements. They were able to interpret the instructions in a function machine diagram flexibly at the beginning of the courses, an indication that the function machine representation is an accessible starting point for many at the beginning of a learning sequence that made sense as representative of the function input/output process.

Further analysis of the data documents the profound divergence that occurred over time between the most successful and least successful students. References to input and output occurred in the work and interviews throughout the semester of students who were successful. They used the function machine notion to organise their thinking as they worked problems and interpreted notation. Axes on graphs were labelled in terms of input and output, as were questions using symbolic notation. The function machine representation provided students with access to the function concept and became a meaningful unit of core knowledge upon which they built subsequent understanding about functions. Their concept maps document the cognitive expansion that occurred over time and provide evidence that the function machine as cognitive root is robust enough to remain useful as more significant understanding develops.

Strikingly, the least successful students generally did not make use of the function machine notion except in limited instances. In contrast to the more successful students, the least successful students made very few references to function machines in their work or in the vocabulary they used. The least successful students demonstrated little or no improvement in their ability to think flexibly. Such rigidity of thought extended to arithmetic computational processes. Their ability to reverse a train of thought appeared frozen, regardless of which representation was used. On the other hand, the most successful students demonstrated flexibility of thinking in their ability to use various representations. They were able to translate among representations, intelligently choosing among alternative procedures.

The usefulness of function machine as a cognitive root continues to be examined as students attempt to deal with the function concept at the College Algebra level. We are investigating the question of how their development of the function concept compares with that of developmental algebra students. The search for possible cognitive roots for other mathematical concepts is also on going.

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THE FUNCTION MACHINE AS A COGNITIVE ROOT FOR THE FUNCTION CONCEPT

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Abstract: The concept of function is considered as foundational in mathematics. Yet it proves to be elusive and subtle for students. In this paper we suggest that a generic image that can act as a cognitive root for the concept is the *function box*. We see this not as a simple pattern-spotting device, but as a concept that embodies the salient features of the idea of function, including process (input-output) and object, with the various representations seen as methods of controlling input-output.

The notion of "function" has often been used as an organizing principle in the teaching of mathematics (Yerushalmy and Schwarz, 1993). However, the subtlety of the function concept with its various representations and process-object duality proves to be highly complex, leading not only to a concept with wide ranging powers, but also with widespread misunderstandings (see for example, Dubinsky and Harel, 1992; Sfard, 1992; Cuoco, 1994; Thompson, 1994). We consider how the function concept may be introduced in a manner which is potentially more meaningful across a wide spectrum of students with differing abilities and needs. In doing so, we develop general principles that relate to other theories of cognitive development in mathematics education. This will therefore have wider implications at a theoretical level, particularly at this point in time as technology affords us entirely new ways of interacting with and constructing conceptual ideas.

The APOS theory of Dubinsky and his colleagues, for example, sees cognitive development in the light of Piaget's theory of reflective abstraction. APOS theory suggests that the individual first performs *actions* (on already existent objects) that are then interiorized into *processes*, later to be encapsulated into *objects* to be built into a wider cognitive *schema*. The embodied theory of Lakoff and Johnson, (1999) on the other hand suggests that *all* thought is built upon embodied perceptions and actions. A vast proportion of the brain is dedicated to vision, for the perception and analysis of objects. It is therefore natural for the brain to construct cognitive concepts not only through encapsulation of processes, but also by focusing on objects and their properties. We contend further that even the encapsulation of a process to a mental object does not occur *only* by a shift in which a process becomes conceived as an object. We suggest that other embodied mental connections are involved. For instance, a *symbol* may act as a pivot between process and concept (Gray and Tall, 1994).

More generally, encapsulation will involve a much wider range of mental structure, including visual images, properties, relationships, perceptions, actions, emotions, and so on, which are already present in the mind. These will be modified and integrated as part of a conceptual object-schema that links and retains both process-driven and object-focused aspects.

Thompson (1994) suggests that an appropriate initial focus builds not from the various representations, but from a meaningful context that embodies the function concept:

> I agree with Kaput that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function. We should instead focus on them as representations of *something* that, from the students' perspective, is representable, such as aspects of a specific situation. (Thompson, 1994, p.39) (our italics))

We suggest that the "something", rather than being a variety of different contexts from which the student is expected to abstract the function aspect, could usefully be a generic embodied image that exhibits as many of the important aspects of the function concept as possible. We also intend that such an initial image should be appropriate for a wide spectrum of students.

Tall (1992, p.497) defined a *cognitive root* to be "an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built." An example of a cognitive root, is the notion of "local straightness" in calculus.

To help us formulate our theory, another relevant construct is the notion of *cognitive unit*—"a piece of cognitive structure that can be held in the focus of attention all at one time" together with its immediately available cognitive connections (Barnard and Tall, 1997, p. 41). Its power "lies in it being a whole which is both smaller and greater than the sum of its parts — smaller in the sense of being able to fit into the short term focus of attention, and greater in the sense of having holistic characteristics which are able to guide its manipulation." (Barnard, 1999, p. 4).

This allows us to propose a refined definition of the notion of cognitive root:

Definition: A *cognitive root* is a concept that:

(i) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence,

(ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,

(iii) contains the possibility of long-term meaning in later developments,

(iv) is robust enough to remain useful as more sophisticated understanding develops.

A cognitive root certainly does not guarantee that every student will progress to later theoretical developments, but it has the advantage of embodying ideas which are potentially meaningful at the time (in the sense of Ausubel, Novak and Hanesian, 1968) and lay groundwork for possible later theories. As the theory develops, the cognitive root will become more sophisticated with a richer interior structure and more appropriate links to other related concepts. Some reconstruction will undoubtedly be necessary as old ideas are seen in a new light. At such times these changes may be threatening to some learners. What is important is that the curriculum designer is aware of reconstructions and their related difficulties and takes account of them in the learning sequence. It is hoped that a firmly based cognitive root will allow the learning sequence to build from meaningful foundations that may be enriched and adjusted whilst maintaining the strength of the entire structure.

Given the complexity of the function concept, we seek a cognitive root that embodies both its process-object duality and also its multiple representations. A highly likely candidate is the *function machine* as an *input-output* box (see figure 1). This already has iconic, visual aspects, embodying both an *object*-like status and also the *process* aspect from input to output. The usual *representations* of function (table, graph, formula, procedure, verbal formulation, etc) may also be seen as ways of representing or calculating the inner input-output relationship as shown in figure 2.

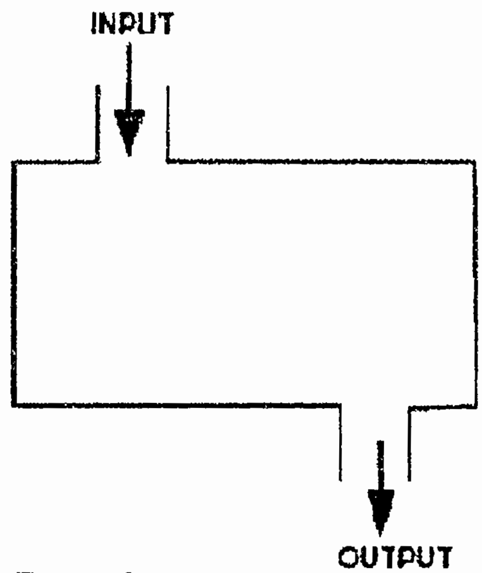

Figure 1

We note that the function box is often used in the early stages of the curriculum. However, this is usually as a "guess my rule" problem, to guess the internal formula expressing the rule. This activity gives rise to the epistemological obstacle that *all functions are given by a formula*. The function box may be used in a different way to retain greater generality through everyday

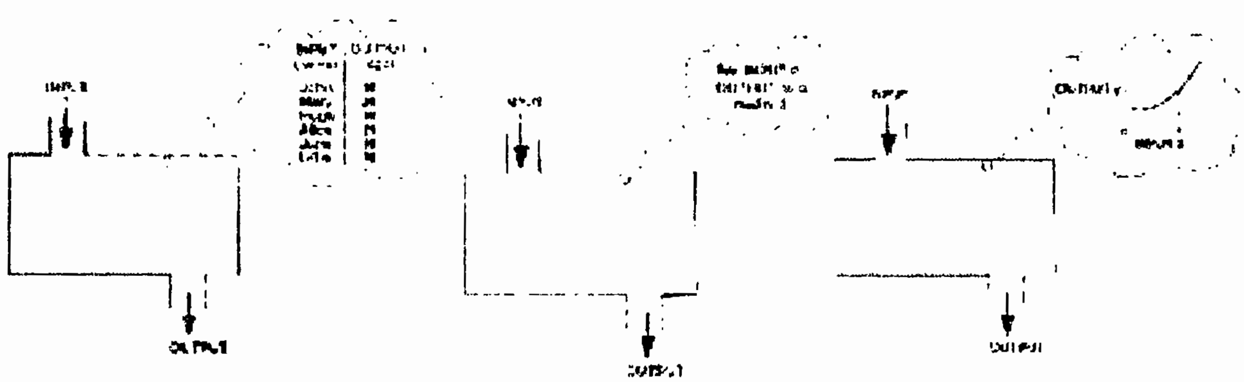

Figure 2. The funtion box as a table, a formula and a graph.

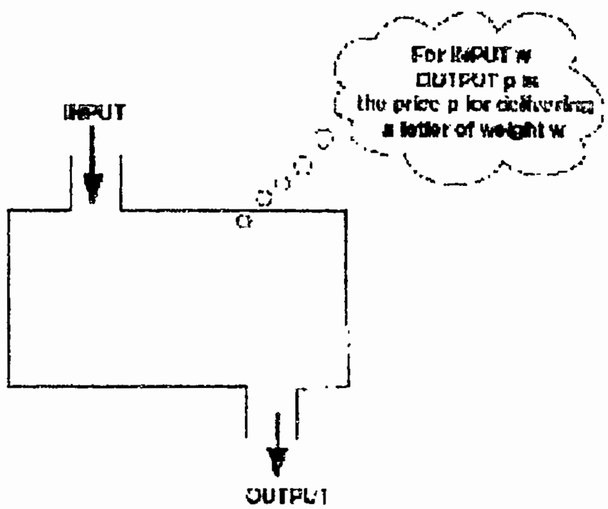

Figure 3

examples with functions given by a procedure rather than a simple formula, for instance, the cost of delivering a letter of given weight.

We hypothesise that the function box in this wider context is an *embodied* version of the more general function concept. It can be imagined and represented in various ways that link directly to human perception and sensation. It allows simple interpretations of profound ideas, for instance that two function boxes are "the same" if they have the same output for each input in the domain, regardless of the particular inner workings of the box. We interpret this perception of two function boxes being the same as occurring at the *process* level in the sense of Dubinsky.

Empirical data to test the use of the function box is given in detail in a parallel presentation in these proceedings (McGowen, DeMarois and Tall, 2000). The data arose, not from a curriculum with a specific focus on the function box as a cognitive root, but from the performance of students on a college course based on various representations of linear relations using function boxes, linear equations, graphs and tables. This showed that 49% of the students began the course operating at a process level for the function box, on a par with their use of a numerical table, but considerably higher than their process use of algebra (20%) or of a graph (1%). (DeMarois, 1998, p. 147). An in-depth study of several individuals revealed successful students expanding their concept maps of function building on the function-box, whilst less successful individuals worked superficially on the current topic of study, making few long-term links (McGowen, 1998, p. 174–179). A parallel study of the growing sophistication of the meaning of a function box revealed a similar spectrum of student performance. A successful student operated at an object-level, a mid-range student acted at a process level, and an unsuccessful student was only able to use the function box in a step-by-step manner, without attending to the possibility of different procedures giving the same input-output process (DeMarois, 1998, pp. 173–175).

Cognitive obstacles with the function box

The function box, as with any other initial starting point, gives rise to a range of cognitive obstacles requiring cognitive reconstruction in later developments. A major weakness is that it does not have an explicit range or domain. The domain can be introduced in a "natural" way as "the set of possible inputs", and in contexts such as real functions, there is a "natural" range, namely the real numbers. This may later embody a belief that a function will always have a "natural" domain and range, rather than the domain and range being *specifiable* in the definition. It would therefore be an advantage at an early stage to embody the function box as an input-output arrow taking the elements from a specific domain A into a range B to attempt to move closer to the formal definition.

In developing our theory, we note that the function concept itself is rarely a concept of study. Instead, the term "function" usually applies to a special kind of function—linear, quadratic, trigonometric, given by a formula, differentiable etc. We refer to such concepts as "function plus," where the "plus" refers to the relevant additional properties which significantly change the nature of a function. (For instance, a linear function only requires two pairs of input-output values to determine it uniquely). Sometimes the "plus" is extremely subtle; the graph of a real function incorporates the *order* of the real numbers on the two axes. Attempting to represent it only as sets without order would be foolish indeed! For such reasons, we see an important role for the function box as a cognitive root *before* considering specific types of function. In this new age of technology, we also consider the importance of the study of a wider range of functions that is now available in spreadsheets, symbolic manipulators and graphic calculators.

Comparison with other theories

The approach advocated has much in common with other theories, however, it reveals a significant underlying difference: the cognitive foundation of mathematical concepts is here based on meaningful scaffolding involving thought experiments with generic objects—in this case a "function box". The difference with the theories of Thompson and Kaput is a matter of emphasis. Our starting point builds out from the function-box metaphor, while their viewpoints focus either on a specific problem or on the links between several related representations.

We take a different position from the development sequence suggested by an (over-simplistic) interpretation of APOS theory. The first (Action) stage, is described as "... a reaction to stimuli which the subject perceives as external," (Czarnocha, Dubinsky, Prabhu, and Vidakovic 1999). The theory seems to intimate an initial stage in which the student does not, and cannot, have a view of the broad future development of the theory. The full schematic (S) part of the theory is, essentially, impossible to envisage until the student has reached the later stage (O) of encapsulation of objects. More recent interpretations of APOS (e.g. Czarnocha et al., 1999) suggest a broader dialectic in which "the development of each level influences both developments at higher and lower levels," but even this manifestly ignores the richer embodied activity of the brain (Lakoff & Johnson, 1999). Gray, Pitta, Pinto and Tall, (1999) show that a focus on objects in arithmetic leads to the less successful remaining with images and procedures, whilst the more successful develop a reflective hierarchy from primitive imagery to the powerful use of more refined mathematical ideas. At a later stage successful individuals often focus far more on the powerful higher levels with little emphasis on more primitive detail, however, this does not mean that such a level does not require a more primitive scaffolding at an early stage. It is our belief that the use of an embodied image can provide a foundation for the widest range of students, giving a good insight for some and laying a firm foundation for more subtle, highly compressed modes of thought that form the basis for more sophisticated mathematical thinking.

Note

The authors wish to thank the Reverend Gary Davis for his insightful comments in the preparation of this article.

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DIFFICULTIES IN THE INTERPRETATION OF GRAPHS OF QUADRATIC FUNCTIONS AND THE USE OF THE TI-92 GRAPHIC CALCULATOR

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Our study investigated students' interpretation of graphs of quadratic functions as mediated by the use of the TI-92 graphic calculator. Emphasis has been placed on the importance of developing the ability to change from one register of representation (such as the graphical one) to another (e.g. the algebraic) (e.g., Duval, 1993). However, difficulties occur when attempting to relate the information given in a graphical register to that in the corresponding algebraic register: Students have difficulties when asked to gather information from the graphic representation of a function in order to deduce the corresponding algebraic representation (Duval, 1993). In an extensive study of 800 subjects, Zaslavsky (1997) found and described 5 types of obstacles in the learning of quadratic functions. Using these obstacles as a basis for our research, we aimed to take advantage of the representational elements provided by the graphic calculator in order to facilitate the construction of relationships between the graphic and algebraic registers; specifically, between the visual variables and symbolic units (see Duval, 1993), since modification of one affects the other.

For the equations of quadratic functions, we used the form $y = ax^2 + bx + c$ and defined the following visual variables: (i) the concavity of the parabola (given by the *sign* of a); (ii) the width of the parabola (given by the value a); and (iii) the position of the curve with respect to the vertical axis, in relation to the origin of the graph (given by the value of c). Ten first-year college-level engineering students participated in 4 sessions of activities with the TI-92 calculator, designed to explore the relationships between the visual and symbolic variables of quadratic functions. Through direct observation, questionnaires and clinical interviews, we observed positive results in that many students who were previously unable to do so, were able at the end of the study to identify the different visual and symbolic variables and construct relationships between them.

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STUDENT INTUITIONS CONCERNING DESIGN PRINCIPLES FOR REPRESENTATIONS OF MOTION

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The NCTM *Principles and Standards for School Mathematics* (2000) emphasize the need for students to be able to construct and interpret data in various forms occurring over time. While an extensive number of studies have examined students' difficulties in creating or interpreting an expert's orthodox graphical representation, few research agendas have considered the design principles that guide and constrain the construction of representations. The process of examining representations as an object of thought has been referred to as metarepresentation. This paper examines the intuitive notions of metarepresentation that appear in students prior to instruction.

The researcher individually interviewed all 25 students from a combined 4th, 5th, 6th grade classroom, following a semi-structured protocol. The students (a) constructed 3 representations of motions enacted by the researcher, (b) described the motion they thought was depicted by 3 representations provided by the researcher, (c) critiqued all 6 representations, and (d) constructed a representation for a new motion.

Analysis focused on design principles reflected in the students' construction, preference, and critique of representations. Indicators of design principles manifested by the students were identified and grouped into two categories: (a) normative principles (those used by an expert mathematician in creating a representation of motion), and (b) non-normative principles.

Few students adopted normative design principles, even after viewing and critiquing representations constructed with these design principles (such as appropriate abstraction, homogeneity, etc.). However, many of the subjects' violations of normative design principles, as well as evidence of non-normative design principles from the interviews, point out an instructional challenge. In some contexts, the principles the subjects applied would make sense. It is only when viewed within the particular intellectual enterprise of the mathematical representation of motion, that they do not work. The challenge here is helping students gain knowledge of the goal structure that permeates the expert's field of mathematics.

SOME DIFFERENT WAYS HIGH SCHOOL STUDENTS SEE THE EQUALITY OF STRAIGHT LINES ON THE PLANE

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When we teach linear equations, we state analytic conditions for straight lines (i.e., they are equal when the slope and the y-intercept are the same). When we tested three samples of high school students (17-18 years old) about the straight line equality, one question arose that caught our attention. The task was to recognize equal straight lines of one set of twelve graphs. This activity is a part of one investigation related to the visual variables of graphs from a semiotic point of view[1].

The questions were: (a) One straight line is equal to another straight line when they have the same slope and the same ordinate to the y axis. In the set of graphs below, how many different types do you recognize?_____ ; (b) Draw one of each type below.

The frequencies for (a) in the recognition of six different types were Sample A 12.2%, Sample B 21.7% and Sample C 17.1%. The following figure shows the frequencies of the recognition of each type.

Sample A	65.9%	63.2%	41.5%	58.5%	48.8%	29.3%
Sample B	56.5%	47.8%	30.4%	39.1%	26%	47.8%
Sample C	34.3%	28.6%	25.7%	22.9%	22.9%	14.3%
They appear in the task	twice	twice	twice	twice	three times	once

In the students' answers to the questions, we found other ways students look at the equality. The most representative features were:

1. The students recognized only the classical Euclidean position; there the difference between straight lines are justified by their relative position. A straight line is a straight line regardless of the position. There were no framelike axes, there was no gestalt relation, and the focus was on the form outside of the plane.
2. There is an important difference when they speak about slant and vertical lines and parallel and perpendicular lines. In the first case they describe the picture using perceptual apprehension, but in the second case they make an unexpected

operation, combining two of the pictures and ignoring the premise of *equal*, working only with *different type*.

3. The axes remain in the picture but the students do not use them. There is no gestalt relation; thus t o horizontal straight lines are equal although the y-intercepts are different.

The incorrect beliefs found in this investigation support three important facts: one is relative to the way the students are working with definitions missing premises; the other is that they miss the gestalt relation and they don't have enough contact with "different types of straight lines".

Note

1. R. Duval, 1995, Sémiosis et Pensée Humaine. Registres Sémiotiques et Apprentissages Intellectuales, Peter Lang Editions Scentifiques Européennes

ON UNDERSTANDING OF TRANSFORMATIONS, DOMAIN, AND RANGE OF FUNCTIONS

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There has been considerable research in students' understanding of functions (Dubinsky & Harel, 1992). As a result, there have been many suggestions of working with multiple representations of functions and transformations of basic functions such as quadratic, rational, exponential, etc. to help students increase their understanding of functions and their properties.

This research project focuses precisely on this question by examining students' interview responses to questions about domain, range, and transformations. Twenty four students were interviewed at the end of a college algebra course where the emphasis was on studying families of functions, their graphs, and other properties. The research questions addressed in this study examined the effects of student understanding of graphical transformations on students' construction of function concepts.

This project uses APOS (Action, Process, Object, Schema) theory (Asiala, et al., 1996) to analyze student responses. The theory is used to measure the level of understanding a student exhibits when identifying domain and range for functions represented either algebraically or graphically. Secondly, the student is asked to sketch a graph of a quadratic on which some transformations have occurred. Finally, there is a comparison question concerning a different function having the same transformations. The results are interesting. While it seemed that the use of multiple representations was accessible to the students during the course, it did not appear to help them as much as expected in constructing a rich function concept but did show that most students expressed a clear preference for the graphical context.

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MODEL-DEVELOPMENT SEQUENCE PART III: DELIVERY ROUTES AND THE MATHWORLDS™ EXPLORATION

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This poster presentation is the third of a four-part sequence that describes Model-Eliciting and Exploration Activities. This poster presents a Model-Eliciting Activity that requires students to develop a model that describes relationships among distance, rate, and time to create reasonable delivery routes. It also presents a follow-up Model Exploration Activity that has students explore position versus time and velocity versus time graphs.

A Model-Eliciting Activity asks students to develop a mathematical model that describes, explains, manipulates, or predicts the behavior of a real world system. Students are given a problem statement that presents the students with a real-world problem that needs to be solved for an identified client. Typically, the client asks the students to describe how the client can use the students' solutions in the future to solve similar problems. A Model-Exploration Activity is a follow-up activity for the Model-Eliciting Activity. The Exploration Activity allows the students to explore patterns, regularities, traditional mathematical notations, and different representational systems that further develop the models that they constructed in the Model-Eliciting Activity.

The Rush Popcorn Delivery Model-Eliciting Activity asks the students to prepare delivery routes according to the constraints of means of transportation, location, and time for orders of holiday popcorn. The Model-Exploration Activity uses *SimCalc MathWorlds*™ to simulate the path of a jeep along a delivery route. By exploring this simulation students examine the relationships among the several variables to extend the models they first constructed.

CONSTRUCTING MEANING FOR POLYNOMIALS: EXPLORING DUVAL'S REPRESENTATIONS-CONVERSIONS WITH CABRI GÉOMÈTRE II

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Recent studies in mathematics education stress the importance of visual representations of mathematical concepts. Related investigations inquire into the effects on students' learning of working with and converting different semiotic representations of the same mathematical object. According to Duval, the reading of graphical representations pre-supposes the discrimination of the pertinent visual variables and the perception of the corresponding variations in the algebraic writing.

In this poster (aided with a lap-top computer) we present a design for Cabri-Géomètre II for the graphic construction of a segment whose magnitude corresponds to $p(x) = a_n x_0^n + a_{n-1} x_0^{n-1} + \cdots + a_2 x_0^2 + a_1 x_0 + a_0$, when a segment of magnitude x_0, and $n+1$ segments with magnitudes $|a_n|$, $|a_{n-1}|$, ... $|a_2|$, $|a_1|$, $|a_0|$, are given.

The procedure is based on work described by Thurnbull (1994) as "Lill's method" and permits not only the construction of polynomials but the graphic display of the dynamic behavior of $p(x)$ as the different constant coefficients of the polynomials are modified at will. The handling of coefficients allows the visualization of the simultaneous variation of algebraic writing and its corresponding graphical representation, helping students' articulation of two registers of semiotic representation.

The design is now also being used in exploratory studies in the classroom to investigate the effects on students' learning of representations-conversion, and to collect empirical evidence of the plausible pertinence of Duval's representations-conversion ideas, in the treatment of related (workable with the same tool) topics such as Relations between $p(x)$ and its derivative, Polynomials and its roots, and Newton's method to approximate roots of polynomials.

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Geometric Thinking

SPATIAL STRUCTURING AS A COORDINATION OF MENTAL MODELS AND SCHEMES FOR MEASURING PERIMETER AND PATH LENGTH: SECOND-GRADE CHILDREN'S ACCOUNTS OF LINEAR QUANTITY

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Abstract: Children's ways of measuring length and perimeter for polygons are often thought to depend on their experiences with the appropriate tools, and their ability to estimate length. My interest is to identify obstacles and supportive structures from a psycho-cognitive viewpoint. This report emphasizes the need for coordination of collections of unitary line segments beyond the 1D world of linearity, emphasizing the 2D aspect of most perimeter tasks. Part of a study of 4 second-grade children's understanding and practices for measuring length and perimeter, this report examines the role of spatial structuring as Reba worked to coordinate units of length along sides of polygons, and around their perimeter.

The cognitive constructions students make in measuring perimeter for polygons, or path length in a plane have been described by researchers in terms of one-dimensional, one-directional constructs (Cannon, 1992; Clements, 1997; Hiebert, 1981). In contrast, other researchers have attended to spatial structuring as a theoretical construct allowing analysis of students' work in more than one dimension, emphasizing the importance of coordination of units without strict boundaries on the properties of dimensionality (Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998). In this report, I employ a theory of spatial structuring as a way of describing children's evolving strategies for measuring paths and finding perimeter for polygons like triangles and rectangles during task-based interviews. This report focuses on children's ways of coordinating length units to relate 1D quantity along a linear object to its 2D context as a polygon in a plane. By accounting for children's strategy development, this report examines the psychological aspects of learning to measure length, informing task development (Tzur, 1999).

The Principles and Standards for School Mathematics (NCTM, 2000) calls for increased attention to measurement concepts and practices by including measurement as one of the five content standards in the K-12 curriculum. If teachers can learn what kinds of trajectories, and what kind of cognitive stumbling blocks to expect as children learn to measure length and perimeter, then they may develop instructional practices that promote increasingly abstract levels of understanding for length measurement. Much of the current research on length identifies the problem of *point counting* as an imagistic stumbling block, but fails to explain the cognitive aspects of this struggle.

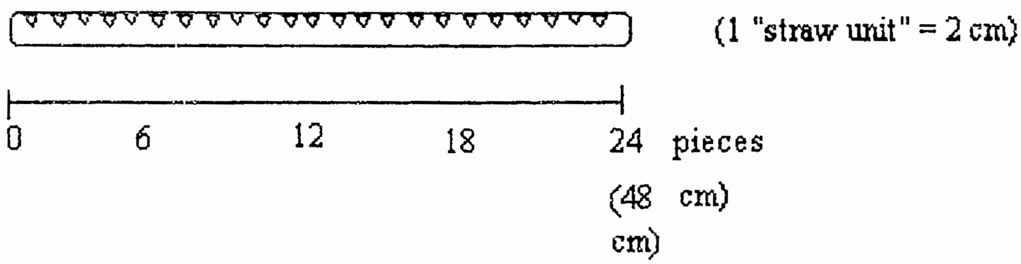
Point counters report the number of subdivision marks rather than the number of intervals.

Specifically, this study will describe children's ways of organizing collections of line segments as iterative sequences of length units for enumerating perimeter as a way of explaining the phenomenon of point counting for children measuring perimeter. The most critical mental process for organizing collections of unitary segments along a path is *spatial structuring*. Constructivist research on children's learning of geometry emphasizes spatial structuring as a fundamental mental process whereby children come to connect and organize collections of objects taken to be related in a whole; by synthesizing collections of related parts, one may establish an ordered sequence to be used for storing and retrieving images of that whole. For example, Battista and Clements (1996) explained children's ways of understanding and enumerating 3-D cube arrays by describing children's use of terms like 'layers' or 'rows', or 'stacks' to identify relevant spatial structures. They emphasized the connections children used as they composed parts of cube arrays into a whole, attending to their capacity for coordinating various sub-collections, or their failures to integrate several viewpoints as they were counting up the unitary cubes. These researchers argued that such structuring implies a complex interaction between numerical and spatial structuring (1996, p. 291). The present report extends this perspective to examine student's ways of relating units of length through iterative sequences in one and two dimensions.

This study examines patterns and reasoning habits exhibited by second-grade students as they attempted to find several cases of triangles or rectangles having a perimeter of exactly 24. I examined the generality of the four-part account of length knowledge and strategies supported by earlier research with fourth-graders (Barrett, 1998). I examined whether this would provide adequate discrimination among strategies exhibited by second-grade level students. There are four observed length strategies:

- Level 1-Partially structured efforts to assign number for length,
- Level 2a-Enumerating single rows by making hash marks,
- Level 2b-Enumerating sequences of rows by making hash marks,
- Level 3-Establishing quantity by applying imagistic nested units of units to coordinate enumeration for collections of parts of paths (Barrett, 1998; Clements, 1997)

I interviewed ten children individually in videotaped, transcribed sessions of approximately 40 minutes duration. I followed Goldin's (1999) method for task-based structured interviewing, using a task from previous studies with modifications to emphasize point counting as a potential cognitive obstacle for enumerating perimeter (Cannon, 1992).



A flexible plastic straw, segmented by triangular notches cut at regular intervals. The overall length of the straw is 24 "straw pieces" or 48 cm.

Figure 1. Side View of a Plastic Straw, as Used in Interviews.

As an example of the data suggesting the importance of spatial structuring for measuring perimeter, consider Reba, a student who was moving from a strategy described as Level 1: *Partially-Structured Collections of Segments* into a more abstracted strategy described as Level 2a: *Enumerating Single Rows of Countable Items Along Segments.* Reba described length along a decreasing sequence of straw sections (I asked her to show me first a length of three, then two, one, and finally zero units) by explaining that the countable items along the straw must begin and end exactly, "in the middle" of the notches. Her way of structuring the observations of the straw eventually lead her to describe a segment of length one as having a length of one and a half. Reba tried to be consistent in identifying countable items by calling a unit the distance between one middle and the next:

I: Why don't you show me 3 pieces on your straw?

REBA: [bends the straw at the 3rd notch]

I: Okay, now show me 2 pieces.

REBA: [bends the straw at the 2nd notch]

I: Okay, that's 2 pieces, now show me one.

REBA: [bends the straw in between the 1st and 2nd notch]

I: Now show me none.

REBA: [looks up and puts the straw down, indicating none.]

I: So how about if you were going to show me 2 [again]?

REBA: I would probably...fold it in the middle, [takes the straw and folds it in between the second and third notch] like this.

Reba did not impose a consistent part-whole structuring to partition the straw, even when a consistent partition was visible with the notches. Unlike students who simply counted notches (common point counting), Reba counted the center points between each pair of notches. Nonetheless, Reba tried to enumerate length by counting collections of specific points along the object (center points between notches), and thereby failed to isolate unit objects with length. Point counting appears to be a mismatch of dimensions, a counting of points that do not have dimensionality to measure a linear path with dimension one. It is interesting that Reba struggled to enumerate perimeter around this straw based on straw units, but she was later able to impose inches as interiorized units. This suggests children's strategies for imposing units may be interiorized for a specialized context, especially those based on standard units, yet still lack generality.

Like Reba, other students exhibiting strategy Level 2a often failed to make use of rows of segments as a composite unit. Many of the second-grade children in this study did not represent parts of collections of length units in relation to the other parts at the internal, representational level of figural-imagistic processing (Steffe & Cobb, 1988). They were not yet able to visually iterate a single unit, or disembed the unitary structure that is implicit in the notched intervals along the straw. For example, for Reba the zero point was not at the position along the straw that she expected. Thus, she struggled to find a way of identifying exactly one unit. This corroborates the claim that without the spatial structuring process of iterating a unitary piece along the straw, one is left to guess at the meaning of the sub-dividing hash marks, the notches along the straw. Students ways of structuring sequences of unit-length parts to coordinate along various sides of polygons affects their ability to enumerate length along the entire perimeter (cf: Battista et al., 1998). Further research is needed to explain students' ways of appropriating verbal narratives, their own drawings, and diagrams in measures of perimeter and path length.

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AN INVESTIGATION OF THE PROBLEM-SOLVING KNOWLEDGE OF A YOUNG CHILD DURING BLOCK CONSTRUCTION

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Abstract: This is an investigation of the problem-solving knowledge demonstrated by two five-year-olds during block construction. Behavioral and verbal indicators for declarative, procedural, schematic, and metacognitive knowledge were used to identify the child's problem-solving knowledge as they solved construction problems. Children were videotaped during month-long free play sessions using specifically-designed Arcobaleno blocks. Results indicated that although both children were successful solving the construction problems, each child demonstrated the use of different knowledge in their play. Most noteworthy, their use of metacognitive knowledge was especially obvious as they naturally shared their knowledge with their peers.

Early childhood educators have long emphasized the importance of block construction in early childhood classrooms. With special emphasis on the cognitive and social significance, theorists and educators alike have strongly advocated the importance of block play and its aid in developing problem-solving skills (Cuffaro, 1991; Gura, 1992; Vygotsky, 1976). Mathematics educators have agreed, acknowledging that "the ability to look at situations geometrically, spatially, and analytically enhances understanding and problem-solving success" (Lappan, 1999). Because the foundations of geometric thought begin at a very early age and because of our dismal performance on recent international and national tests (TIMMS and NAEP), geometry, spatial thinking, and problem solving have become a focus of mathematical teaching even at the pre-kindergarten level (Clements, 1999; National Council Teachers of Mathematics, 1999). Although problem solving is often mentioned as a natural outcome of block construction, it is not clear what type of problem-solving knowledge young children naturally demonstrate when constructing with blocks.

Objective

This study was designed to investigate the problem-solving knowledge of two young children during block construction. To accomplish this purpose, behavioral and verbal indicators of metacognitive, declarative, procedural, and schematic knowledge were used to identify children's specific knowledge in month-long free play sessions.

Theoretical Framework

Cognitive development research suggests that young children's cognitive performance is "profoundly variable and that performance variability is a reflection of important properties of their knowledge." (Sophian, 1999, p. 19). Thus, observing performance in context is necessary over a period of time to make accurate assessments of children's understanding. In addition, informational processing proponents view the effective use of specific processes and operations as necessary for successful problem solving. Hamilton and Ghatala (1994) explain four types of knowledge that contribute to the effective solution of problems. Declarative knowledge is the domain-specific, verbalizable knowledge that increases the number of cues for accessing relevant knowledge. Procedural knowledge involves effective strategies for solving problems. Schematic knowledge is the larger chunks of related thematic knowledge within the content area. Metacogntive knowledge is the planning and monitoring of problem-solving attempts. While declarative, procedure, and schematic knowledge alone cannot ensure effective problem solving, there must be an awareness of knowledge about one's own knowledge and the active monitoring and regulation of learning (Baroody, 1993).

Method or Mode of Inquiry

This is an observational case study of two five-year-olds, one female and one male, during block play. Both Wesley and Aimee[1] were members of a summer preschool program in an urban southwest city in the United States. Both children demonstrated success on the reproduction and visualization problems after free play. Their block play was videotaped at three different times: (a) during the completion of visualization and reproduction problems before free-play sessions, (b) during month-long, free time block-building sessions with peers, and (c) during the completion of visualization and reproduction problems after free play sessions.

Arcobaleno blocks (Learning Materials Workshop Blocks designed by Karen Hewitt) were used for this study because of their special spatial relationships. The beveled, curved blocks are twelve half circles with six different radii in six different colors; their design increased the number of aesthetic variations and possible geometric patterns. Six construction models specific to Arcobaleno blocks and rated from easiest to hardest were used for the visualization and reproduction problems. Each problem or model had a name that was easily identifiable to young children: A "House", B "Caterpillar", C "Hut", D "Snake", E "Bowl" and F "Snail". To create each model, children had to use the correct blocks, place them in the correct order, and balance them in unique ways. The same problems were used before and after the free play sessions. Visual models of the problems were then posted near the free play construction sites for reference.

Data Sources and Evidence Collected

The visualization and reproduction problems required children to identify specific drawings of block structures that they believed could be made with the Arcobaleno blocks and then to reproduce them using a completed model as a reference. When a model became too difficult, the next model was attempted, and when that model proved to be frustrating, the task was discontinued. The number of correct solutions (0 to 6 models) as well as the time required for building each model were recorded.

During the free play sessions and the visualization and reproduction tasks, all experiences with Arcobaleno blocks were videotaped. Tapes were transcribed specifically noting children's play behaviors and verbalizations. Written and validated during a pilot study of ten children during block construction (Oto, 1997), specific behavioral and verbal indicators for declarative, procedural, schematic, and metacognitive knowledge were used to identify children's problem-solving knowledge as they constructed with the blocks. Each videotape was viewed independently by at least two researchers and behaviors or verbalizations labeled as "demonstrating declarative, procedural, schematic, and/or metacognitive knowledge" with an initial inter-rater reliability of more than eighty-five percent.

Results

The cognitive performances of both Wesley and Aimee were variable as expected; however, during the final reproduction problems, both children completed most reproduction problems with little error. During the initial and final reproduction problems, Wesley built all models correctly with only a few variations. However, the initial building construction took three times longer than did the completion of the final models. Aimee, on the other hand, experienced no success on any of the models during the initial task and yet completed five of the six models with little error during the final reproduction. In contrast to Wesley, Aimee's final building time was twice as long as her initial building time.

During the free play sessions, Wesley spent half of the time building with the Arcobaleno blocks. Nearly all of that time, Wesley worked directly with one or more of his peers asking them to "make it like this " or "look" as he demonstrated the procedure. Aimee spent one fourth of the free play time building with the blocks. Of that time, half of the time was spent socializing with peers about the blocks, with the remaining time spent in observation of other's work and independent building. Both children's verbalizations and behaviors indicated that they had some declarative, procedural, and metacognitive knowledge. Indicators for schematic knowledge were few in number.

Declarative Knowledge. Both children talked about the blocks using a variety of descriptive terms. Aimee's language was quite specific during her self-talk and her discussions with peers. Her verbalizations indicated that she had some declarative knowledge about the blocks and their positions. Wesley, on the other hand, easily recognized when one of his friends had the blocks turned incorrectly or when the edges had the wrong orientation. Although he did not verbalize the information, his demonstrations along with "not this.... this" terminology indicated that he used his declarative knowledge about the position of the blocks in space and balancing principles.

Procedural Knowledge. Trial and error was easily the most frequent procedure used by both children. Wesley was very systematic in his approach, while Aimee was more random. In one particular session, Wesley spent more than an hour enlisting peer's help in making a symmetrical wave creation. In another session, he worked on one of the reproduction tasks that were difficult to balance. He systematically tried it three times one way; it collapsed. Three times another way; it collapsed. He then moved one block, tried it three times, it stood! Aimee's strategies were very interesting. She seemed to place the block in a boundary pattern rather than systematically placing the blocks in order. Her procedures were place the outside block, place the block opposite in position, and then place the block in the center. She continued that type of "pattern" building throughout the sessions.

Metacognitive Knowledge. Most noteworthy, children's use of metacognitive knowledge was indicated during both the free play sessions and the reproduction tasks. Wesley's plans were systematic and flexible enough to adjust when he found problems. During his free time, he continually monitored his building by frequently checking the posted visual models. When he worked with peers, Wesley always took the initiative and suggested the division of labor. He also spent most of his time demonstrating to others how to make the pictured reproductions, even saying, "just do it like me.... watch." Evidence of his planning was obvious by his behaviors. Aimee, on the other hand, planned "out loud." She constantly verbalized her plan of reproduction such as "I will make this (pointing to the picture of models)" and "I'll do this once more" or "I won't do that one. It is just too hard!"

Interpretations

Several interpretations can be made from this data. First of all, the two young children in this study demonstrated a wide variety of problem-solving behaviors or verbalizations. Although each child verbalized or behaved using different problem-solving strategies or connections, each one demonstrated informal problem-solving knowledge that far exceeded our expectations. Second, both Aimee's self-talk as she monitored her constructions and Wesley's step-by-step demonstrations while he directed his peers indicated that these two five-year-olds had an amazing, albeit incomplete metacogntive knowledge. Both Wesley and Aimee reflected on their constructions comparing their constructions and commenting on their relationship to the pictures. Third, both subjects initially used trial and error procedures as they built. As they became more proficient, however, Aimee systematically placed blocks by first building boundaries while Wesley seemed to identify "up-down" and "curvy-straight" patterns as he placed them sequentially. Finally, although children demonstrated the use of problem-solving knowledge, their skills were frequently a random occurrence and often inconsistently applied. Because early mathematical cognition is full of variability, teachers need to be aware of a young child's informal knowledge and build on that knowledge as they teach a more formal approach to mathematics. Sophian

(1999) describes the young child as "fertile ground for instruction," a statement that is supported by this study.

Note

1. Names of children have been changed to ensure privacy.

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QUILT DESIGN AS INCUBATOR FOR GEOMETRIC IDEAS AND MATHEMATICAL HABITS OF MIND

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Abstract: We investigated student design activity as an incubator of important mathematical ideas and as a forum for developing mathematical habits of mind. We report a case study of one second grade classroom where students used transformations in the plane to design quilts (Watt and Shanahan, 1994). The case study was conducted with an eye toward characterizing the relationship between collective and individual forms of activity, because such a relationship is at the heart of design. Our investigation was guided also by examination of how design activity and teacher assistance jointly crafted mathematical learning.

In earlier research, we found that second-grade students engaged in quilt design learned important habits of mind, like the limits of case-based generalization (Lehrer et al., 1998), and important ideas, like composition of transformations (Jacobson & Lehrer, 2000). Here we employ a case study approach (Stake, 1994), to track the experiences of two students. Data sources included videotape of the classroom activities, observation notes prepared by both participant and non-participant observers, teacher interviews, artifacts of student work including portfolios and journals, and pencil and paper assessments.

Theoretical Framework

Scientific and mathematical inquiry through design enables exploration of the space of potential solutions, engages students in problem solving through successive approximation, and highlights the functionality and the acceptability of solutions (Lehrer, Schauble, Carpenter, & Penner, 2000). Design, as a process of invention, requires representational capacities that enable feedback loops to promote reflection and revision (Goodman, 1976). The forms of assistance provided to accompany design activity play a major role in the development of mathematical reasoning and clarify that design is a form of mediated activity (Tharp & Gallimore, 1988; Wertsch, 1998). The development of student ideas and the acquisition of habits of mind are continuous during activity, as students are enculturated in a community of designers (Brown, Collins, & Duguid, 1989). The nature of the tasks and the nature of the teacher's interventions jointly shape the form and the sophistication of the mathematical practices that the students undertake during this apprenticeship (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996).

The Quilting Curriculum

The quilting curriculum employed in this classroom employs transformations using different base designs (e.g., a core square design or a rectangular strip design) to create quilts (Watt & Shanahan, 1994). The teacher promotes the investigation of mathematical concepts by varying the pallet of shapes and colors available to the students, thereby changing the complexity of the design space. Consistent with design as a craft activity, the students continually revise their quilt designs during the course of instruction. The process of design and revision provides a meaningful venue for the class to engage in mathematical discourse about transformations in the plane. The linguistic and representational resources developed provide students with the means to raise conjectures and to consider arguments pertaining to the mathematical properties of figures in the plane, including symmetry and congruence.

Results

Our interpretative analysis of classroom activity suggested several themes. First, much of the structure of everyday activity resulted from the dialectic between public display and discussion on the one hand and pursuit of particular individual (or small group) goals on the other. Public forums provided much of the space of reflection, where the teacher orchestrated student discussions about the mathematical consequences and implications of the explorations of particular students or groups of students. For example, during an early conversation the teacher encouraged the students to consider the functional role of flips and turns in the act of quilt design. After demonstrating a quilt design produced using only rotations of a core square, the teacher asked the students to consider using flips in producing quilt designs.

> Teacher: If we proved we can do it [make a design] with only turning, why would we need to flip? What do you think Susan?
>
> Susan: If you turn it or flip it then you can get different 2x2's... Just by turning you can't get...you can't get all the different 2x2's.
>
> [5 second cut]
>
> Susan: Right, but you can only turn it from the outside corners or something like that. But, if you flip it, you will get the same exact pattern only... the same exact core square, only it will give you a different place to put it.
>
> Teacher: okay, and just refresh my memory, why would it be important to get that core square in different places (December 1, 1994).

During this exchange, Susan's remarks portray her burgeoning understanding of the functional role of flips and turns in creating designs and expanding the space of potential designs. However, the italicized statements suggest that she regarded flips as a physical relocation rather than a reflection of a core square. The teacher did not attempt to change this conception at this point in time but simply concluded by recruiting students to the original goal of examining the functions of reflection in this design space.

Discussions like these were often accompanied by public display of various quilting products. Both the physical display of student designs and the patterns of discourse in the classroom highlighted the students' learning histories. The students appreciated this norm and followed it by recalling their own designs to provide warrants for the conjectures that they offered during class discussions:

> Teacher: could there be a core square that looked the same after a sideways flip as it did after an up-down flip?
>
> Students: yeah, yes [several students]
>
> Eugene: yeah [with his hand raised]
>
> Teacher: are you thinking of one Eugene?
>
> Eugene: I remember the time that I made... this one... that had... that it could only... that every way you turned it it would only look the same. And, I bet if you flipped it up and flipped it sideways I bet it would still look the same.
>
> Teacher: that was a pinwheel design wasn't it, four triangles of yellow and four triangles of blue.
>
> Eugene: yeah (December 1, 1994).

The teacher extended this particular interaction by providing Eugene with Polydron™ manipulatives and asking him to produce copies of his original pinwheel design (figure 1) so that the class could test his conjecture about the invariance of his design under both flips and turns.

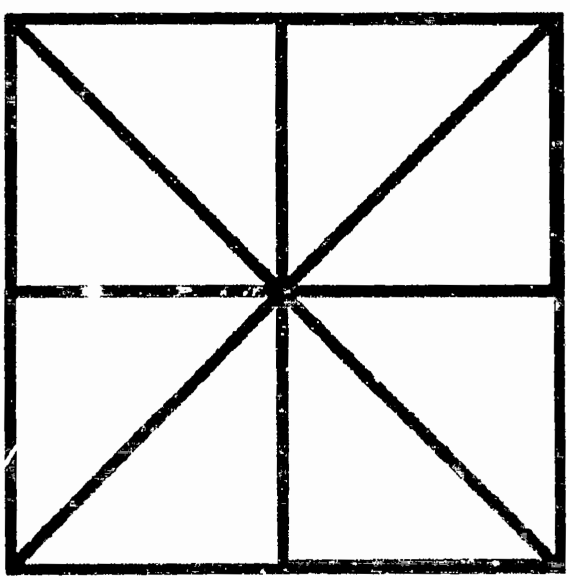

Figure 1. Pinwheel

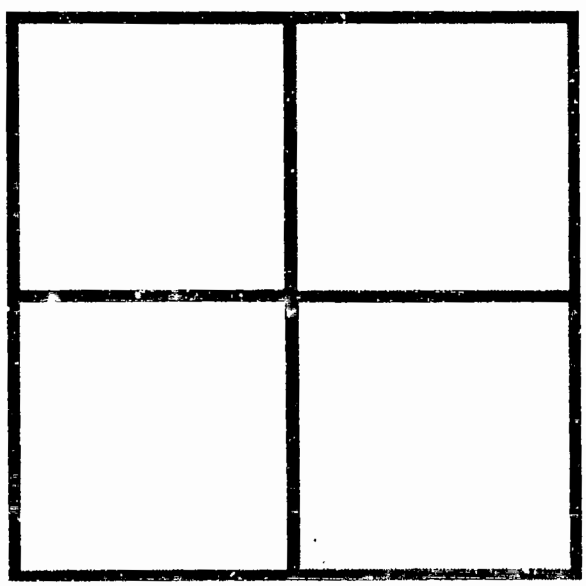

Figure 2. Checkerboard

After the class tested both the pinwheel design and a checkerboard design as cases that fit Eugene's conjecture about invariance under vertical and horizontal flips (reflections), Susan observed that Eugene's conjecture could be extend to diagonal flips for the pinwheel design, but not for the checkerboard design (figu):

> Teacher: [after Susan demonstrates a diagonal flip] That one [checkerboard] didn't fit our rule, did it? I wonder why that is? I mean, this one [pinwheel] all of the ways, diagonal flip, sideways flip, up-down flip ended up looking exactly the same.
>
> Susan: I knew it was not going to be the same because [points to two shaded squares].
>
> Teacher: These two were, you mean you knew the diagonal flip wasn't going to be the same?
>
> Susan:...Right, because these are going to be right here [points to shaded squares on core flipped diagonally], these are going to be here [points to two unshaded squares on core flipped up], and these are going to be right here [points to unshaded squares on core flipped sideways]. Diagonal flip, this is how we started out [demonstrates original position], but a diagonal flip is going to be exactly like that because the red [points out that red is opposite to red] (December 1, 1994).

Susan's generalization to other cases suggests an increasing orientation toward understanding reflection as a transformation of an image about an axis. We find further evidence of growth in her knowledge five days later when she describes how a core square that she has designed (figure 3) is invariant for diagonal flips, but changes under sideways or up-down flips:

> Susan: after an up-down flip this corner [shaded triangle in top left] would be where this corner is [unshaded triangle in bottom left]...ta dah! But a diagonal flip, it [the core square] would just be exactly the same. Either a sideways flips or an up-down flip it would be...different (December 6, 1994).

Collective activity structures, such as the class investigation of Eugene's pinwheel conjecture, were complemented by individual and small-group investigations, where the teacher modified the nature of activity in accordance with her understanding of how individuals or small groups were thinking. Often, the teacher constrained the design space and asked children to consider the mathematical implications of these constraints. For example, early in the cycle of designs, she asked students to consider the effects on design of limiting their choice of color to only two. Later, she asked students to explore the consequences of using three colors rather than two for quilt designs. Changes in the variables constraining the design space often resulted in the production and investigation of new conjectures. The teacher's explicit promotion

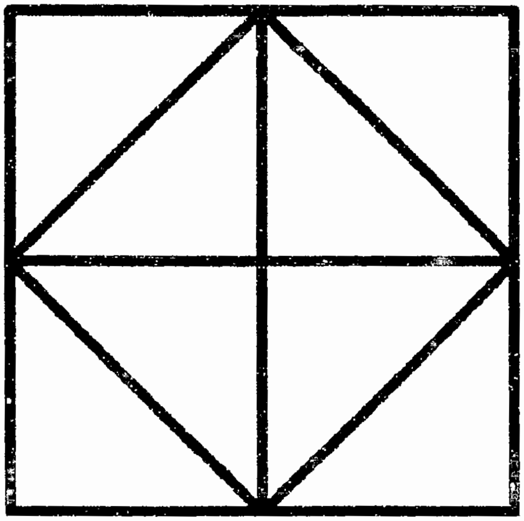

Figure 3. Susan's Core Square

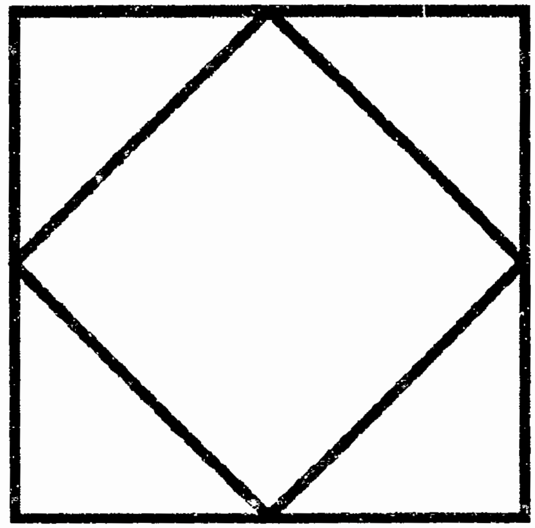

Figure 4. Eugene's Invariant Design

of students' attention to their own histories of design (as evidenced by the above interaction between the teacher and Eugene in the pinwheel segment), enabled student exploration to be governed not just by the moment but also in light of past activities and products.

Student histories provided a nice window to the interplay between the social and individual planes of thought (Vygotsky & Cole, 1978) because conversations about particular histories first elicited publicly were later prominent features of the conversations of individual children (recall the development of Susan's ideas about flips). As they talked about their designs with the participant observers, children spontaneously included "lessons learned" as a platform for current activity. For example, using computer software designed to accompany the curriculum, Eugene demonstrated for a researcher that a core square (figure 4) with four lines of symmetry and two colors could only be used to produce a single quilt design, noting that "It's because… it matters on the outside of it…the blue keeps on going in the blue places. So it never gets in a place…you could have …a yellow diamond [in the middle] and then blue, blue, blue, and then red [triangles bordering the diamond]. That would be good, but, you can't …just have like all blues on the outside of it" (December 13, 1994).

Eugene's investigation of the relationship between symmetries and the space of designs led him to conclude that base designs with four lines of symmetry close the design space and resulted in "boring" quilt designs. This privately arrived at conjecture was later discussed in a group design activity as a potential "design standard" (Erickson & Lehrer, 1998) governing "interesting quilts":

Researcher: Eugene, [can you explain] why you like this design?
Eugene: Because you can turn it [an asymmetrical 2x2 design] and it looks different.
Carol: The part that Eugene hates about mine [a different 2x2 design that has four lines of symmetry] is that no matter how you turn it or flip it, it will always look the same (December 15, 1994).

Although some students conjectured that multiple forms of reflection symmetry might limit the space of potential designs, other students, like Carol, argued that symmetrical designs have attractive qualities. Carol investigated how different transformations or compositions of transformation might result in different degrees of (reflection) symmetry, given base designs of particular patterns. Consideration of competing design standards like these were grounded in mathematical exploration and justification.

In summary, design in this second grade classroom had a dual aspect. The collective design space was populated by reflection about the mathematical consequences of design activity and by public products that encapsulated the history of design in this particular community. We found that articulation of design standards fostered reflection and prompted further exploration. The semi-private design spaces of individual and small-group activity were forums for generation and test of conjecture, and for pursuit of design guided by personal interests and proclivities. Learning was mediated in this design context by the affordance of the tasks posed (or those generated by children), the material means (paper, glue, and electronic design tools) used to generate the various designs, the inscriptions and notations children invented or appropriated, and by cycles of conjecture and refutation. Teacher orchestration of each of these resources made the dialectic between collective and private design spaces mathematically fruitful and enabled children to participate meaningfully in mathematical activity.

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STUDENT CONJECTURES IN GEOMETRY

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Abstract: A better understanding of plausible reasoning and conjecture and their role in learning is vital in describing the development of students' mathematics. The works of Lakatos (1976), Peirce (Fann, 1970) and Polya (1954) already offer descriptions and examples of plausible reasoning and conjecture but are not supported by actual episodes and are not fitted in a comprehensive model of mathematical learning, such as von Glasersfeld's scheme theory (1998). The present study presents two possible models developed from teaching experiments with three tenth-grade geometry students. These models, which are based upon von Glasersfeld's "pattern of action" (1998, p. 8), describe how abduction (Fann, 1970) and repeated assimilation often serve as conjecture for geometry students.

It has been called "guess" and "hypothesis," and has been attributed to notions as vague as intuition. However allusive the concept may be, the crucial role conjectures play in mathematics is recognized by logicians (Fann, 1970), mathematicians (Lakatos, 1976; Polya, 1954), mathematics educators (Steffe & Thompson, 2000), and psychologists (von Glasersfeld, 1998) alike. In order to handle conjectures in the present study and their nature in geometrical reasoning, it is important to begin with an explicit working definition. At least for now, conjectures are ideas formed by a person (the learner) in experience which satisfy the following properties: the idea is conscious (though not necessarily explicitly stated), uncertain and the conjecturer is concerned about its validity. Though this exhaustive list of descriptions may render the working definition broader in comparison to some or more restrictive than others, it is useful in the present study to adopt these and only these restrictions. Further directions for study, that are discussed below, may require alternative definitions.

Once a working definition of conjecture is adopted, plausible reasoning may be considered as that reasoning we use to support our conjectures. Even with these working definitions in mind, this study does not snare the allusive concept of conjecture, let alone the results reported here. Rather, these results offer a couple of possibilities and suggestions for further study in addressing the following questions: what is the nature of plausible reasoning and conjecture in geometry? and how might the role of conjecture be described in a larger theory of learning? The research hypothesis of this study follows from the work of Peirce on abduction and von Glasersfeld's (1998) use of this work in providing a solution to the learning paradox. In short, conjectures from geometric experience may be described as abductions that serve in accommodating the learner's cognitive structure to the new experience.

Theoretical Framework

As hinted above, this research was conducted from a radical constructivist perspective using von Glasersfeld's (1998) scheme theory to build models for students' reasoning in their mathematical environments. His pattern of action describes the implementation of schemes in assimilation. When perceived results do not fit expected ones, the learner experiences a perturbation. Von Glasersfeld asserted that one reasonable response to such a perturbation is the re-examination of situational aspects that were, in assimilation, previously ignored. The present study was intended, in part, to affirm, refute or revise von Glasersfeld's claim through teaching experiments with three geometry students. Indeed, an extension of his scheme theory model, including re-examination, can often be used to describe the larger concern of the study: the nature of students' conjectures and plausible reasoning in geometry.

For Peirce, conjecture is synonymous with abduction in which "we find some surprising fact which would be explained by supposing that it was a case of a certain rule, and thereupon adopt that supposition" (Fann, 1970, p. 21). If the "surprising fact" can be considered a perturbation for the observer, Peirce's definition fits nicely with the stated working definition of conjecture and with von Glasersfeld's scheme theory. In fact, in describing how the re-examination of a situation might lead to accommodation, von Glasersfeld (1998) claims that "if the accommodation were done consciously, it would be an abduction, because, at the moment the changes are made, they are hypothetical in the sense that their usefulness has not yet been tested in further experience" (p. 9). The results reported in this study reveal the depth of the connection between the theories of Peirce and von Glasersfeld and their relevance for geometric reasoning.

Methodology and Analysis

Three students were selected from a tenth-grade geometry class in a southern United States public high school. Students of low, average and high ability were selected based on the classroom teacher's evaluation of their classroom performance. An initial van Hiele interview was conducted to corroborate the diversity in levels of reasoning of the students selected.

Data collection consisted of five 45-minute teaching episodes conducted with each student individually. Teaching episodes were designed for the researcher to build and revise models of students' reasoning where interaction with the student in a learning environment is crucial. Specifically, students were investigating geometric constructions and relations using the Geometer's Sketchpad (Jackiw, 1991) as a medium.

During the episodes, the researcher's role was to interpret the students' statements and actions in order to form a working hypothesis for each student's reasoning. Thus, as the students were forming and testing conjectures about their geometry, the researcher was forming and testing hypotheses about the nature of those mathematical conjectures (note: "hypothesis" is used here to describe the research-related nature of conjectures concerning models of learning, so as not to be confused with the "conjectures" students were forming). In order to test the hypotheses, the researcher posed questions to students, not so much to guide the students, but to compare student responses to predicted responses based on the experimental hypothesis.

The researcher wrote journal entries for each teaching episode and videotaped the last three teaching episodes with each student. In order to describe a general nature of plausible reasoning and conjecture in geometry, these materials were thoroughly reviewed. Instances of various activities were categorized into groups, such as "student examined key properties of a geometric construction." These categories were used to identify students' tendencies in plausible reasoning and forming conjectures. Then, two or three specific episodes were highlighted for each student in which the student exemplified such tendencies; that is to say, these episodes were transcribed and a model for the student's reasoning was offered to explain her or his actions. Further analysis of such episodes yielded descriptions of the student's geometric reasoning in forming plausible conjectures about her or his experiences.

Results

In terms of von Glasersfeld's scheme theory, the researcher assumes that the realm of conjecture is in that interval following perturbation and preceding a re-established equilibrium. Students begin plausible reasoning when the use of an existing scheme yields an unexpected result – a perturbation. The results reported here describe models for students' reasoning following perturbation and leading up to conjecture; this reasoning is referred to as pre-conjectural plausible reasoning. As von Glasersfeld (1998) suggested, it was found in this study that pre-conjectural plausible reasoning often begins with the re-examination of situational aspects that were ignored in the initial assimilation. At such time, two courses in conjecture were found to be common amongst the three students of this study.

Upon re-examining a situation following perturbation, students in this study often devoted attention to aspects of the problematic situation which appeared uncommon or out of place. Students' conjectures were often centered around these "key properties." In such cases, they commonly related, in a causal manner, the key property to the surprising result of the initial assimilation. Recall that this is what Peirce referred to as abduction (Fann, 1970). This study not only corroborates Peirce's model for abduction, but also suggests that students attend to unusual aspects of the situation in developing the rule or conjecture. For instance, one student in this study, after observing the surprising result that the sum of the top and bottom angle measures of a diamond (rhombus situated like a compass) was equal to the measure of each side angle, determined that this was true for diamonds in general because of their symmetry with respect to each diagonal. For him, the unusual property of the quadrilateral in question was this symmetry. When further perturbed by the measure of a second diamond not satisfying the ascribed property, again the student identified a key property of the first figure: that it was a diamond formed by two equilateral triangles. This constituted a second abduction for the diamond, from associating the surprising result of measurement with the unusual property of containing equilateral triangles.

Another course of action observed in the research was caused by the recognition of previously ignored properties. Attention to these properties sometimes triggered a second scheme that called for a specific action. Since these successive assimilations of the situation occur in a conscious attempt to eliminate a perturbation, they also serve as conjecture. This iterative process often allows the student to eliminate the perturbation and alter the problem. This is exemplified by a second episode in which a student was trying to justify that a given angle in a construction was a right angle. The student independently used three consecutive schemes before she arrived at an explanation with which she was satisfied. Each time that she realized one scheme was unsuccessful, she re-examined the situation until she was able to assimilate it into another existing pattern of action. Though this study did not attend to changes in the schemes of assimilation after the conjecture, it seems reasonable to assume that the result of this experience altered the schemes used, at least in terms of the trigger for the schemes. Under this premise, it can be said that accommodation, or learning, took place in such episodes as a result of the assimilation conjectures.

Discussion

The first case reported in the results demonstrates the usefulness and compatibility of the theory of abduction and scheme theory in describing students' plausible reasoning. The second case offers a possible alternative to abduction in resolving perturbations through conjecture. Of course, arguments for each of these would be, by necessity, relative to one's definition of conjecture. For Peirce, it is clear that abductions were conscious conjectures. In describing learning in a less formal manner, it may be useful to include unconscious conjectures as well. Conscious or otherwise, conjectures play a crucial role in accommodating schemes. Thus, for a deeper understanding of conjecture, researchers need to examine students' scheme structures and their modifications resulting from conjecture.

In using a definition of conjecture that is compatible with abduction, the goal in analysis is to explain how we decide on which supposition to adopt. The results of this study describe some of the tendencies of such pre-conjectural plausible reasoning, in the context of students' geometric experiences. Peirce (1998) himself offered a few guidelines for the formation of abductions, though he warned against the extensive use of logic in describing it. In addressing an extreme case of abductive inference (perceptual judgment), he described how a researcher might be led down Achilles' path of analysis in chasing the inferential tortoise. Thus, the psychological implications for studying plausible reasoning may run deeper than the consciousness of the learner, whereas the logical implications may run no deeper than the study of students' schemes. As the reader is left to ponder, Peirce's (1998) statement may apply equally well to all abduction (conjecture) formation: "because it is subconscious and not so amenable to logical criticism, [the process of forming perceptual judgement] does not have to make separate acts of inference but performs its act in one continuous process" (p. 227).

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TOOLS FOR INVESTIGATING HIGH SCHOOL STUDENTS' UNDERSTANDING OF GEOMETRIC PROOF

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Abstract: Although proof and reasoning are seen as fundamental components of learning mathematics, research shows that many students continue to struggle with understanding geometric proofs. In a preliminary study to a larger research project, we investigated two components of students' understanding of proof – their beliefs about what constitutes a proof and their ability to construct proofs. This preliminary study focused on the development of two instruments to gather information regarding these two components of students' understanding of geometric proofs. Results from the study have aided us in revising the instruments for the larger study, and have given us some information that has been useful for focusing our research questions in the current larger research project.

Introduction

Proof is fundamental to the discipline of mathematics because it is the convention that mathematicians use to establish the validity of mathematical statements within a given axiomatic system. In addition, the teaching of proof as a sense-making activity is important for developing student understanding in geometry and other areas of mathematics. Despite the fact that student difficulty with proof has been well established in the literature, existing empirical research on pedagogical methods associated with the teaching and learning of geometric proof is insufficient (Chazan, 1993; Hart, 1994; Martin & Harel, 1989). Our work in this area has begun to address the need for research into the pedagogy of geometric proof instruction. In a preliminary study we developed and piloted research instruments for measuring students' understanding of geometric proof. We are currently undertaking research funded by the National Science Foundation (NSF) in which revised versions of these instruments are being used to help us develop an empirically grounded theoretical model that relates pedagogy to student understanding of proof.

The objectives of the preliminary study were (1) to construct, administer, and refine two instruments for measuring students' understanding of geometric proof; and (2) to conduct preliminary analyses of the data collected from students' responses to the instruments in order to inform the current NSF project. In developing the research instruments, we focused on two components of student understanding of proof, namely, students' beliefs about what constitutes a proof and students' proof-construction ability.

Perspectives

In trying to make sense of students' difficulties with geometric proof, Dreyfus and Hadas (1987) articulate six principles that form a basis for understanding geometric proof. These principles address many of the student misunderstandings of proof cited in the literature (Chazan, 1993; Hart, 1994; Martin & Harel, 1989; Senk, 1985). A revised version of the six principles guided the development of the research questionnaire to assess students' beliefs about what constitutes a proof. Other perspectives on students' reasoning abilities that we considered when developing our instruments can be found in Harel and Sowder (1998), Hoyles (1997), and Simon and Blume (1996).

Methods

In order to measure beliefs about what constitutes a proof, we constructed a questionnaire that assessed students' agreement with a revised set of six principles. It was necessary to add more detail to Dreyfus and Hadas' (1987) principles in order to develop items that could be reliably identified with particular principles. Questionnaire items consist of items modified from instruments used by Chazan (1993), Healy and Hoyles (1998), and Williams (1979), as well as some original items. Part I of the questionnaire, requires students to indicate whether they agree, disagree, or are unsure about statements that correspond to the revised six principles. Part II of the questionnaire includes open-response items related to the same principles.

The instrument designed to assess proof construction ability includes items in which students must construct partial or entire proofs, as well as generate conditional statements and local deductions. In addition to some original items, the instrument includes items modified from Healy and Hoyles (1998), Senk (1985), and the Third International Mathematics and Science Study [TIMSS] (1995).

Data Sources

In the pilot study, the instruments were given to first and second semester geometry students in a local high school summer program. The same version of the questionnaire was given to all students early in the semester. There were four versions of the performance assessment: before instruction and after instruction versions for the first semester students as well as before instruction and after instruction versions for the second semester students. Students received about two weeks of instruction on proof between the two administrations of the performance assessments.

In the process of piloting the instruments, we used a number of techniques to determine the quality of the instruments. For example, four experts examined all questionnaire items in order to assess the content validity. In order to determine the construct validity of the questionnaire, we compared responses to open-ended items with responses on parallel multiple choice items and assessed the consistency of those responses. Consistency issues are discussed in the results section. In order to estimate the questionnaire's reliability, Part I of the questionnaire was divided into two 14-item half tests, matched on content. The split-half reliability estimate, with a Spearman-Brown correction, was 0.67. To assist us in interpreting these measures, we also collected student feedback regarding the clarity of the questionnaire items.

The quality of the performance assessment instrument (proof quiz) was also assessed using several measures. Construct and criterion-related validity of the proof quiz were measured by a simple correlation between students' scores on the pilot instruments and their final examination grade. The correlation coefficient of 0.56 is fairly high and indicates that the proof quiz did a good job of differentiating between high- and low-ability students. The experts also confirmed that before and after versions of the proof quiz were roughly equivalent. The coefficients of stability and equivalence for the versions given to first and second year students, respectively, were 0.78 and 0.91, indicating that the two forms were reasonable equivalents and that we could expect student performance on these instruments to be stable over time. Cronbach's a coefficients, which estimate reliability, were 0.66 and 0.86 for the before and after versions of the quiz given to first year students and, likewise, 0.57 and 0.68 for the second year versions. The researchers and the classroom teacher used a common written rubric to score students' responses to the performance assessment items. Rater agreements for the four versions of the performance assessment were 96%, 93%, 89%, and 90%.

Results

There are two types of results for this study. The first type of result is the set of instruments for assessing student understanding about geometric proof. Based on student responses as well as validity and reliability estimates from the pilot study, we have revised the instruments. We are currently using these revised instruments in our ongoing work and will continue to assess their quality and make further revisions as necessary. We recognize that creating appropriate and useful instruments is problematic because "understanding of proof" is difficult to define and more difficult to measure.

A second type of result from the study is evidence of students' beliefs about proof and abilities to construct proof from administration of the instruments. For example, by their responses to related items on the questionnaire -- which assesses students' agreement with the six principles of understanding proof -- most students (89-100%) showed evidence of agreeing with the principle that proof has a dual purpose, to convince and to explain. In contrast, few students (22%) indicated agreement with the principle that proofs have internal logic requirements. Both as a group and individually, many students gave inconsistent responses to different items that addressed the same principle. An example of this relates to the generality requirements of a proof. On one item, students claimed to believe that a proof must be general (78% agreement on the item), but on another item, 78% of students accepted as valid a "proof" based on a few particular examples. In some cases, students who provided consistently correct responses to items related to a particular principle in the multiple choice section of the questionnaire were unable to produce correct, coherent reasons for their conclusions to items corresponding to the same principle in the open response component of the questionnaire. For this reason, we are developing an interview protocol so that we may learn more about inconsistencies in students' beliefs. In our pilot study, many of the inconsistencies were due to wording of the instrument or students' inability to communicate ideas in writing. However, by questioning the students, we hope to be able to identify instances in which their beliefs about a particular principle are incomplete or contradictory.

The performance assessment instruments revealed results that were not surprising since they echoed the findings of Chazan (1993) and Senk (1985). For example, student scores were lowest (ranging from 0% to 33%) on quiz items that required them to construct a proof, even when provided with an outline of the proof. Student scores were a bit higher (ranging from 29% to 44%) on an item that showed a proof with missing statements and reasons, for which students were asked to fill in the missing steps. Student scores were highest (ranging from 29% to 60%) on an item that asked them to rewrite a conjecture in "if-then" form and determine the given information as well as what was to be proved.

In summary, the instruments we have created appear promising for collecting evidence about students' beliefs about proofs and their ability to construct proofs. We realize that results from the initial administration of the instruments may not be readily generalizable due to the fact that summer students may not be representative of the general high school population. However, initial findings identify several specific weaknesses and inconsistencies in proof understanding for further consideration.

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DEFINING AS A MATHEMATICAL ACTIVITY: A REALISTIC MATHEMATICS ANALYSIS

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Abstract: The purpose of this paper is to report on a college level classroom teaching experiment in geometry and to discuss one aspect of students' activities with mathematical definitions. The instructional design theory of Realistic Mathematics Education is used as a framework for understanding students' mathematical activity involving the modification of new definitions out of familiar ones. Our analysis extends the work of Gravemeijer (1999) and suggests that the creation of new mathematical objects and their definitions involves four interrelated types of activities: situational, referential, general, and formal.

Introduction and Theoretical Perspective

The purpose of this paper is to further the notion of definition as a mathematical activity (Mariotti & Fischbein, 1997; Freudenthal, 1973) and to elaborate a theoretical means to make sense of students' defining activities where new mathematical objects and their definitions are created out of familiar ones. From a mathematical point of view, definitions should be useful in that they serve to single out a concept with certainty, they should be minimal and elegant (Vinner, 1991), they should capture or synthesize the mathematical essence of the concept (Borasi, 1992), and they are links in deductive chains of organization (Freudenthal, 1973).

Freudenthal (1983) described mathematical concepts, structures, and ideas as our inventions, created to organize the phenomena of the physical, social, and mental world. In the research reported here, we view mathematical definitions as one aspect of the mathematical structures and ideas that learners' create. Drawing on the work of Freudenthal, we view mathematics itself as essentially a "human activity" in which one engages for the purposes of generality, certainty, brevity, and exactness, where defining is one aspect of this activity (Gravemeijer, 1994). Freudenthal (1973) distinguishes between two different types of defining activities in mathematics, descriptive and constructive. Descriptive defining "outlines a known object by singling out a few characteristic properties," whereas in constructive defining one "models new objects out of familiar ones" (p. 457). Focusing on descriptive defining, DeVilliers (1998) argued that students should be actively engaged in the defining of mathematical concepts and elaborated the notion of descriptive defining, framing his analysis within the cognitive theory of Van Hiele levels. In this report, we focus on what Freudenthal refers to as constructive defining, framing our analysis within the theory of Realistic Mathematics Education (RME).

Previous work within the theory of RME has mainly centered on the learning and teaching of K-12 mathematics (for some exceptions, see Gravemeijer & Doorman, in press; Rasmussen & King, 2000) and has not elaborated the role of definitions in the learning and teaching of mathematics. The theoretical significance of this report is in its elaboration of definitional activities in the theory of RME. In this paper we draw parallels between the types of activities elaborated by Gravemeijer (1994, 1999) in his analysis of the role of emergent models and the role they play in fostering the constitution of formal mathematics. In Gravemeijer's analysis, there is a "global transition in which 'the model' initially emerges as a model of informal mathematical activity and then gradually develops into a model for more formal mathematical reasoning" (p. 155). In a similar way, our analysis suggests that definitions first come to the fore as a definition of students' previous activity and later these definitions serve as tools for further mathematical reasoning.

Methodology

We conducted a classroom teaching experiment (Cobb, 2000) during a 5-week summer session with 25 students at a large Southwestern university in the United States. The course used the textbook, *Experiencing Geometry on Plane and Sphere* (Henderson, 1996) and instruction generally followed an inquiry-oriented approach. The course was taught by one of the research team members. A second researcher attended every class session. Data consisted of videotape recordings of each class session, copies of students' written work, bi-weekly student journal entries to specific questions developed by the research team, researcher and instructor field notes, audio-recorded debriefing sessions between the instructor/researcher and the other researcher, and videotape recordings of individual interviews conducted with 22 of the 25 students at the beginning and end of the course.

Typical class sessions consisted of a brief introduction of the problem by the instructor followed by small group work on the problem and whole class discussion of students' reasoning and interpretations. Students' activities that form the crux of the data for this paper occurred during the second and third weeks of the course. During this time period, students sorted through for themselves the definition of a triangle on the plane, explored the world of triangles on plastic spheres, and investigated several questions about spherical triangles.

Results and Discussion

We posit that students' defining activities in which new mathematical concepts and their definitions are created out of familiar ones can be understood in terms of four interrelated types of activity: situational, referential, general, and formal. Consistent with the manner in which Gravemeijer (1999) described these types of activity in the modeling process, we do not view these levels as a strict hierarchy or as a strict developmental progression, but rather as a general trend in students' ways of reasoning, acting, symbolizing, etc. Students' activities at one level often fold back (Pirie & Kieren, 1994) to another level. Next we clarify what we mean by these four levels of activities.

Situational Activity

As described by Gravemeijer (1994), situational activity involves interpretations and solutions that depend on understanding how to act in the problem setting. The hallmark of situational activity is the fact that students deal with experientially real settings. This may include imagined experiences like packaging candies in a factory or people getting on and off a bus. In such settings, students' interpretations and solutions are very much grounded in the real world context, using their understandings of the experientially real setting serves as a means to solve the problem. From the perspective of RME, however, such experientially real situations need not be limited to real world settings. Depending on the level of the student, experientially real settings can also refer to more abstract mathematical contexts. This latter interpretation is the one taken in this paper. In our example, the problem setting is the plane where straight lines have zero extrinsic curvature and the task posed was the formulation of a definition for a planar triangle. In this case, students have a way to proceed because they have had many prior experiences (both in school and out of school) with straight lines and planar triangles and thus have built up a rich concept image (Vinner, 1991; Edwards, 1999) about triangles.

Referential Activity

Referential activity involves interpretations and solutions that refer to activity at the situational level. For example, as students explored the world of triangles on the sphere there was considerable amazement at how "different" these triangles are from planar triangles. Considerable discussion occurred as to whether they really were triangles (since, for example, the interior angles didn't sum to 180 or a triangle had two obtuse angles). Students' decisions as to whether such objects were actually triangles ultimately referred back to their definition of a triangle (where straight line segments were taken to be great circle segments), rather than to their imagery of planar triangles. It is in this respect that students' activities are referential, both to their imagery of planar triangles and to their definition of triangle. The following journal excerpt, which begins by referring to a figure that a group of students were sharing with the whole class, illustrates what we are calling students' referential activity.

> From the figure [they had] drawn, it didn't seem like it was a figure at all, but in close observation it was a triangle! Yes, a triangle. It was a triangle based on the definition we chose in class. The definition of a triangle matched up with the figure. Though this was true, the figure did not look like a triangle. I did not see the triangle until someone brought up that it was a triangle by definition. Better yet, there were two triangles! Yes, the inside AND the outside were both triangles."

General Activity

Students activities with triangles on the sphere begin to take on a life of their own, independent of the situation specific imagery. That is, independent of planar specific imagery. Evidencing this independence, students began to formulate conjectures (without prompting from the instructor) about the range for the sum of the interior angles. Illustrating this point is the following quote from a student's journal. "Another surprising observation was when Group 1 gathered information about triangles on a sphere and concluded that the maximum number of degrees that a triangle can have with respect to its angles is 1080 degrees. These observations [have] changed my view on spheres. All along I was thinking and limited to a 2-dimensional perspective." Furthermore, this excerpt illustrates that students are working within a new mathematical reality, characterized by new mathematical objects and relationships on the sphere that previously did not exist (for students, that is). From our perspective, the creation of such a space of possibilities is reflexively related to students defining activities.

Formal Activity

At the formal level, triangles on the sphere become an object in their own right that can be modified to suite particular purposes and needs. For example, when investigating whether two spherical triangles for which two sides and the included angle are congruent necessarily mean the triangles are congruent, students created a new class of spherical triangles for which this theorem is true. They defined this new mathematical object as a "small triangle" and these definitions varied from group to group. Furthermore, students considered the equivalence of these various definitions of small triangles and used small triangles as links in chains of deductive reasoning.

In this report we furthered the notion of defining as a mathematical activity from an RME perspective. Our analysis suggests that the RME constructs of four different levels of activity, first elaborated in the context of modeling, are a viable framework for making sense of students' activities where new mathematical objects and their definitions are created out of familiar ones. One direction for future research would be to investigate the extent to which the framework elaborated here might guide the design of instructional sequences where the defining is a prominent aspect of students' activities.

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THE FORMULATION OF CONJECTURES IN GEOMETRICAL ACTIVITIES WITH CABRI-GÉOMÈTRE

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Some advances will be presented here concerning a project whose objective is to explore the relationships existing between the geometrical activities resulting from students' activities in an environment of dynamic geometry and the production of written formulations of conjectures and proofs. The Cabri-Géomètre scenarios favor in the student the knowledge of geometrical facts, the validation of which is based on the "dragging" technique, which is an exploration tool to discover the invariant relationships that are present in the geometrical constructs (Hölzl, 1996). However, the scientific consolidation of these geometrical facts demands that they be constituted in a system of results to be expressed in written representations in a language adapted to the demands of deductive proofs; hence, the importance that conjectures and propositions are formulated in writing. In the case of the student's geometrical ideas, his or her activity with dynamic geometry allows him/her, even before the written phase, to have a representation that is free of ambiguity and imprecision, and also a conviction of the validity of such ideas. This representation should favor his/her written formulation, since the student's linguistic competence is now acting on less unshaped material. A question, however, arises: How does the data processing representation of the geometrical facts favor in the students the possibility of the written representation of such geometrical ideas?

Method. In order to progress towards an answer to the question above, a geometry workshop (eight 4-hour sessions) was organized with 8 students in the second year of high school (16-17 year olds). Training activities were performed in which the students acquired the ability to work with Cabri-Géomètre. Next, activities were proposed for them to discover important geometrical facts concerning the triangle and its medians. The activities included the task of formulating in writing the conjectures they discovered. The students were requested to measure certain quantities, to vary objects, and to observe invariants to arrive at their conjectures. Finally, although not analyzed here, the students were asked to construct a proof for their results.

Results. The students possessed different performance levels in writing; yet, this did not influence the knowledge and the acceptance of the geometrical proposition that was at issue. The Cabri-Géomètre program provides a field for geometrical experimentation that favors the knowledge of geometrical results; however, the external representation that the students can make of such results is a consequence of actions on the computer and its software, whereas it would seem that the competence to formulate the propositions in a geometrical language is not developed only through the activities with such a software.

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INSTRUMENTAL MEDIATION AND THEOREMS IN GEOMETRY

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Exploring with computational tools eventually allows students to generate and articulate relationships that are general to the computational environment in which they are working. That means students can develop an ability to state general propositions in the language of the environment. A *situated proof* is the result of a systematic exploration mediated by a computational environment. It could be used to build a bridge between situated knowledge and some kind of formalization. In our study, whose goal was to explore how students "proved" a mathematical proposition within a computationsl environment, we worked with students, between 15 and 17 years old, trained in dynamic geometry–as it comes in the calculator TI-92. For the development of the activities, teams of two or three students were formed. Each participant was given a notebook to take notes on both individual and team observations and conclusions. One of the activities was organized around the theorem of the central angle inscribed in a circle. The objective was that students constructed the idea of an invariant property. In this, as in other related cases, students became aware of invariants and they could express the relevant ideas *but only within the expressive medium* made feasible by the calculator. During the oral presentation, we will give full examples as well as references.

MODEL-DEVELOPMENT SEQUENCE PART IV: TILING A PLAYGROUND AND GEOMETER'S SKETCHPAD®

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In this poster session, the fourth of a four-part project, we present a Model-Development Sequence: a Model-Eliciting Activity followed by a Model-Exploration Activity, designed for upper elementary through middle school mathematics curriculum. The math concepts elicited on this sequence include: rotation, translation, reflection, and analysis of angles in regular polygons.

Model-Eliciting Activities present a problem, based on a real-life situation, to be solved by students in small groups. The solution calls for a mathematical model to be used by an identified client, or the person who needs to solve the real-life problem. In order for the client to implement the model adequately, the students must be very clear in describing their thinking processes that justify their solution. Thus, they need to describe, explain, manipulate, or predict the behavior of the real world system to support their solution as the best option for the client. Like in real life, there is not a single solution but there are optimal ways to solve the problem.

Model-Exploration Activities are a follow-up for Model-Eliciting Activities. Through the use of computer microworlds, these activities allow students to encounter new representational systems that help them formalize some of the math concepts involved in the solution of the Model-Eliciting Activity.

The Model-Eliciting Activity presented, *Tiling a Playground*, is about a contest to design a creative tiling pattern for a school, and the winning design will decorate the playground. Students are asked to explain the process of creating a tiling pattern using given geometrical shapes. *Tessellations* is the Model-Exploration Activity that follows *Tiling a Playground*. Through *Geometer's Sketchpad*®, students are encouraged to explore and generalize properties of vectors and translations, while developing their own tessellation.

BUILDING THE SPATIAL OPERATIONAL CAPACITY (SOC) OF THE PRIMARY SCHOOL CHILD THROUGH RICH LEARNING EXPERIENCES: A GEOMETRY CURRICULUM FOR GAUTENG PROVINCE IN SOUTH AFRICA

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The changes in South African curriculum design, called curriculum 2005 (Revised version: Curriculum 21) has major implications for mathematics (number and space) teaching and learning in the primary school of South Africa. A new program for spatial development has been designed. This program agrees with Freudenthal namely that geometry education should be about the way in which children (at the lowest level) grasp the space in that they live and move. Geometry, as a logical system, should never be introduced to the child at too early an age. If so, it tends to be a fruitless exercise.

The way in which the young learner investigates this space can be organized by educationists in different ways. This paper is an attempt to give a guideline or theoretical framework through the introduction of the SOC (Spatial Operational Capacity) model, of how space should be organized in the curricula for young learners. This model does not only propose the spatial content that needs to be addressed but also a specific teaching methodology as well as a suitable assessment tool to evaluate the process of spatial development.

The curriculum guidelines that have been proposed in the form of Progress Maps were designed with the SOC model as theoretical framework. Three content areas were chosen on which the model was superimposed namely, Shape, Vision and Location. This view implies that the spatial development of the learner is not strictly limited to the development of their understanding of shape, but also of their positioning in space as well as their ability to look and see things in space. Levels 1-6 as indicated in the progress maps, were designed to show order and growth in the spatial thought process. The design of these levels is based on the Van Hiele thought levels for the development of spatial knowledge of the young child.

An additional aspect of geometry that is addressed by the SOC model is the availability of dynamic computer software namely Geometer's Sketchpad. The utilisation and introduction of computer software that utilises the dynamic aspects of transformations, are crucial components in assisting understanding in the development of spatial knowledge of young children.

Aspects of this model have been tried out, for the past seven years, with multilingual/multicultural children between the ages of 5 and 13 years. The research approach that was followed made use of a Developmental Research Methodology where teaching, learning and assessment are viewed as interrelated components of the total development of spatial knowledge.

The assessment that was and should be used in assessing the spatial knowledge of young children can be called didactical assessment. This means that the purpose of the assessment as well as the content, the methods applied and the instruments used are all of a didactical nature (see Van den Heuvel-Panhuizen).

Probability and Statistics

PROBABILISTIC INTUITIONS IN TRADITIONAL VS. ALTERNATIVE CONTEXTS

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Abstract: Research and assessment of probabilistic reasoning has often presented text-only multiple choice problems which ask students to reason about common *random generating devices*, (e.g., coins, dice). This study presented problems in three contexts: traditional ("COIN/DIE"), pinball diagrams ("GALTON") and database frequencies ("PLANET"). Research questions focused on a common source of students' errors and biases: the distribution of probabilities associated with *ordered* sequences of independent events. Results indicated that general purpose cognitive heuristics described in previous research change or are not as salient in some alternative problem contexts. Students' problem solving frameworks, heuristics, and atmospheric beliefs were interpreted by extending Fischbein's theoretical framework of *primary* (every-day) and *secondary* (instruction-related) intuitions.

In recent years educational organizations have given increased attention to probability curricula and have recommended introduction of probability concepts and activities at earlier grade levels (NCEE, 1995; NCTM, 2000). However, students' difficulty in reasoning *normatively*[1] has posed considerable challenges for curriculum design and implementation (Algren & Garfield, 1991; Jones et al., 1999; Shaughnessy, 1992). These pedagogical challenges are due in part to the fact that, unlike many mathematical topics, naive intuitions (i.e., immediate apprehensions or cognitions[2]) about probability are entrenched in everyday discourse and activity. Cognitive as well as educational researchers have noted that there are large gaps in our understanding of how these intuitions interact with instruction-based knowledge, and how best to deal with intuitions in order to facilitate students' construction of normative concepts and problem solving skills.

Research Questions

Traditional cognitive research investigating probabilistic reasoning has often focused on intuitive errors and biases which indicate that human fallibility is associated with natural or everyday thinking. However, many traditional research instruments utilized only limited response sets (e.g., multiple-choice responses). Furthermore, problem situations presented in traditional studies typically involved only standard random generating devices (e.g., coins, dice, spinners, etc.). For example, consider traditional versions of the three problems presented in this study (correct choices for the three problems are 1-c, 2-e, and 3-b):

A primary motivation for the study was to explore how students' responses differ

1. Referred to below as the "Four Previous Heads" problem:

If a coin is flipped four times and comes up heads every time, which is more likely on the next flip?
a. head
b. tail
c. head and tail equally likely

2. Referred to below as the "Equiprobable Ordered Sequences" problem:

Which sequence of coin flips is most likely-?
a. HHHTT
b. THTTH
c. HHHTH
d. HTHTH
e. All sequences are equally likely.

3. Referred to below as the "Skewed Probability Samples" problem:

A die is painted black on five sides and white on one side. What is the most likely outcome of six rolls?
a. 6 blacks and no whites
b. 5 blacks and one white

Figure 1. Basic multiple choice versions of the problems presented in this study.

in alternative contexts. In particular, do naïve intuitive responses found in *heuristics and biases* research vary across the three contexts? Fischbein (1975) suggested that investigating such differences can help us identify *secondary intuitions* (i.e., acquired in traditional contexts through school curricula) as well as *primary intuitions* (i.e., general purpose heuristics and biases acquired through everyday experience). In other words, school-based instruction—as well as general purpose everyday cognitive processes—can be a significant source of misappropriated intuitions. Of particular interest are the *Equiprobable Ordered Sequences* and *Skewed Probability Samples* problems in which students appear to misappropriate formal probabilistic knowledge. Intuitions about *ordered sequences*, (in which multiplicative-analytical skills facilitate understanding that probability is related to conditional dependence of events in a sequence), can erroneously substitute for, or be substituted by, intuitions about *unordered samples* (in which the probability is directly related to the 'partitive' proportion of independent outcomes in the sample).

Theoretical and Historical Framework

The traditional *normative* theoretical framework provided a basic measure for evaluating students' judgments in this study—a response was considered correct if it was consistent with generally accepted mathematical models of probability. However, even when responses are clearly correct or incorrect in a normative framework, theorists are sometimes divided into opposing camps depending on whether human thinking is characterized as more, or less, rational. On one hand, early cognitive investigations provided evidence which indicated that everyday *heuristics and biases* are irrational obstacles to normative probabilistic reasoning (Kahneman et al., 1982). On the other hand, some researchers contended that a critical analysis of experimental paradigms is crucial to interpreting research that focused on erroneous heuristics and biases:

> The paradigms [are] so limited and inadequate that generalisations from current research on heuristics and biases cannot be justified. In particular, the view of people as 'intellectual cripples', who exhibit severe and systematic biases in making judgements, [is] a value judgement on the part of the investigators. (Phillips, 1983, p. 525)

In general, theories of probabilistic reasoning have differentiated 1) frequentist-distributional vs. local-atmospheric thinking, and 2) intuitive vs. analytical thinking. Interpretation of results in this study was influenced by Fischbein's (1975) theoretical framework which viewed 'intuitional' cognition to be as important for human functioning as formal-operational cognition (cf. the Piagetian focus on formal-operational cognition as *the* end state in the development of higher order thinking).

Research Method

Virtually all 168 participating undergraduate students were drawn from psychology department subject pools in two U.S. universities. Students were given a question sheet packet and worked alone; most finished the problems in about ten to fifteen minutes. Each question sheet packet contained three problems presented in only one context.

Figure 1 shows the basic structure of problems presented in the traditional "COIN/DIE" context and Figure 2 gives a basic idea of how diagrams looked in the alternative contexts problem (note that the diagrams in Figure 2 are small iconic versions of the diagrams presented to students). Problems in the "GALTON" context showed diagrams of a simple pinball machine in which the path of a pinball was traced through a triangular set of pins (a right/left branch of the pinball is analogous to head/tail in a coin flip). Problems in the "PLANET" context described space exploration probes whose data indicated that there was a 50% chance that planets in a solar system were inhabitable (the 50% chance analogous to head/tail in the traditional context).

In addition to reasoning about the diagrams students were asked to estimate frequencies and percentages for every multiple-choice alternative, and to write a short justification of the alternative they selected. These multiple representations and response modes were designed to facilitate correct interpretation and evoke a more informative response set.

Results

Two general outcomes related to problem format and context will be briefly described, followed by results showing how responses in the alternative contexts differed from the *heuristics and biases* model of reasoning in traditional contexts. A first general observation in considering the overall response sets is that problem formats with multiple representations and modes of response seemed to have little positive effect. For example, many students gave frequency and percentage estimations that were incoherent (e.g., estimated percentages weren't consistent with estimated frequencies, or the sum of percentage estimations for each alternative was over 100%, even when alternatives were a subset of all possible outcomes).

A second general observation involved differences across contexts in students' willingness to accept assumptions about the independence of a future event.

Table 1
Percent Correct on Multiple Choice Responses in Each Context and Problem Type

Context	N	Four Previous Heads	Equiprobable Ordered Sequences	Skewed Probability Samples
COIN/DIE	56	93 %	27	38
GALTON	57	63	32	41
PLANET	55	45	2	62

Performance on the traditional version of the *Four Previous Heads* problem (93% correct) indicated that most students are familiar with and willing to accept the idea that a future coin flip is independent of previous events. In contrast, students weren't as likely to accept assumptions about independence of events in nontraditional contexts (63% correct in the GALTON context, and 45% correct in PLANET context). Students tended to hypothesize physical causes in the GALTON context and nonprobabilistic dependencies in the PLANET context:

GALTON student: "The ball has momentum and will continue going in that direction."

PLANET student: "Since one planet is inhabitable the others are more likely to be inhabitable."

The situated nature of students' reasoning in the alternative contexts was interesting because the study was designed in part to explore how computer-based activities in the alternative contexts might help students reason normatively.

Significant differences from traditional results may be related to how events are represented as well as students' tendency to hypothesize nonrandom causes or associations in the alternative contexts. Figure 2 shows which outcomes students thought were most likely in the *Equiprobable Ordered Sequences* problem. First note that results in the traditional COIN/DIE context were consistent with the *heuristic and biases* model. Outcome #2 ("THHTH") tends to be viewed as most likely because it is more 'representative' of a random process: 68% of students in the COIN/DIE context ranked "THHTH" among the two most probable outcomes. The other alternatives tend to be viewed as less representative of a random process because "HHHTT" has runs, "THTTT" has a lopsided proportion of heads, and "HTHTH" alternates too regularly (see rankings in Figure 2).

Students in the GALTON and PLANET contexts more often chose Outcome #1 ("HHHTT" in the traditional context) as most likely. Moreover, the second highest ranking in both alternative contexts was Outcome #3 ("THTTT" in the traditional context) which has a run as well as the least balanced proportion. Thus in contrast to results found in traditional studies, runs of the same outcome in real world scenarios (i.e., a pinball repeatedly branching in one direction, or similar sized planets being inhabitable) may not appear as unnatural.

Compared to previous studies a surprisingly high number (57%) of students in the COIN/DIE context ranked Outcome #4 ("HTHTH") among the two most likely outcomes. Thus, in contrast to responses in the alternative contexts, COIN/DIE students seemed particularly prone to view runs of a particular outcome (e.g., "HHH") as indications of a nonrandom process and alternations between heads and tails as indications of a random process. In sum, differences between contexts indicate that strings of letters representing sequences of coin flips appear to have symbolic characteristics which evoke specific situated intuitions—different from intuitions about the most likely paths of a pinball or the most likely patterns of inhabitable planets.

Reasoning about symbolic representations of random events in the COIN/DIE version of Equiprobable Ordered Sequence problem may have been associated with *secondary* school-based intuitions. Other evidence for *secondary* intuitions was indicated in students' performance on the COIN/DIE version of the *Skewed Probability Samples* problem: Students with stronger science/math backgrounds scored lower than students with weaker backgrounds (42% vs. 45% correct respectively). Recall that

	COIN/DIE	GALTON	PLANET	Context Effect? $\chi^2_{(df=2)}$
Outcome #1	H H H T T			
% of students who ranked it among two most probable	22%	50%	45%	p < .030
Outcome #2	T H H T H			
% of students who ranked it among two most probable.	68%	41%	39%	p < .018
Outcome #3	T H T T T			
% of students who ranked it among two most probable	19%	47%	41%	p < .033
Outcome #4	H T H T H			
% of students who ranked it among two most probable	57%	41%	35%	p = .116

Figure 2. Differences between contexts in students' ranking of most likely outcomes in the *Equiprobable Ordered Sequences* problem.

students were asked to judge the most likely outcome in six rolls of a die that was painted black on five sides and white on one side. COIN/DIE students' justifications for judging six blacks most likely often referred to the high odds for black on each *independent* roll. This erroneous focus on independence appeared to be associated with *secondary* intuitions acquired in (school-based) contexts.

In summary, comparison of reasoning across contexts raised some questions about traditional research that suggests specific universal/innate heuristics and biases in probabilistic reasoning. Some of these cognitive processes may in fact be tied to intuitions acquired in formal contexts. Identifying *secondary* as well as *primary* intuitions would help educators design early learning environments that instill a more rigorous set of normative conceptual schemata.

Notes

1. Normative, in this context, means "in accordance with mathematical principles of probability."
2. Webster's Collegiate Dictionary (10th Edition)

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YOUNG CHILDREN'S STATISTICAL THINKING: A TEACHING EXPERIMENT

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Abstract: This study designed and evaluated a teaching experiment in data exploration with two grade 1 classes, one of which collected the data used in instruction. The teaching experiment was informed by a cognitive framework that described elementary students' statistical thinking. Following the teaching experiment, the children showed significant gains on some but not all of the statistical thinking processes associated with the framework. While the two classes (collection and non-collection groups) changed in different ways, the evidence did not support stronger overall growth for the collection group. Case-study analysis revealed that: experience with the data context reduced children's idiosyncratic descriptions; data values of zero were problematic for these children; children possess intuitive knowledge of center and spread; and making meaningful predictions from data was difficult for these children.

In response to the critical role that data play in our technological society, there have been ongoing calls for reform in statistical education beginning in the primary grades (National Council of Teachers of Mathematics [NCTM], 2000). Notwithstanding these recommendations there has been relatively little research c. primary children's statistical thinking and even less research on the efficacy of instructional programs in data exploration (Shaughnessy, Garfield, & Greer, 1996). Moreover, the studies that have been undertaken have not developed and used the kind of cognitive models that researchers like Fennema et al. (1996) deem necessary to guide the design and implementation of instruction.

This study addressed the above-mentioned void in the research literature by developing and evaluating a teaching experiment on data handling with young children. More specifically, the study sought to (a) use a cognitive framework that describes students' statistical thinking to design and implement a teaching experiment with two grade 1 classes, and (b) to evaluate the effect of the teaching experiment on children's learning.

Theoretical Perspectives

The conceptualization of this study drew on two theoretical perspectives. First, it was grounded in teaching experiment theory (Cobb, 1999). Second, the teaching experiment was informed by a cognitive Framework (Jones et al., in press) (Figure 1) that describes students' statistical thinking. In making the link between these two perspectives the Statistical Thinking Framework served as the research base for designing a hypothetical learning trajectory (Simon, 1995) for students as they engaged in the teaching experiment. We also used the Framework to interpret classroom events and examine changes in children's statistical thinking.

The *Statistical Thinking Framework* incorporates four key data handling processes adapted from Shaughnessy et al. (1996): describing, organizing and reducing, representing, analyzing and interpreting data. Describing data involves the explicit reading of data contained in visual displays. Organizing and reducing data involves ordering, grouping, and summarizing data using measures of center and spread. Representing data incorporates the construction of visual displays and analyzing and interpreting data includes what Curcio and Artzt (1997) refer to as "reading between the data" and "reading beyond the data" (p. 124). The Framework descriptors for each statistical process built on previous research (e.g., Beaton et al., 1996; Bright & Friel, 1998; Zawojewski & Heckman, 1997; Curcio & Artzt, 1997). For each of these processes, four levels of thinking were hypothesized and validated (Jones et al., in press). Level 1 is associated with idiosyncratic thinking, Level 2 is transitional between idiosyncratic and quantitative thinking, Level 3 involves the use of informal quantitative thinking, and Level 4 incorporates analytical and numerical thinking. These levels of thinking are consistent with neo-Piagetian theories that postulate the existence of thinking levels that recycle during developmental stages (e.g., Biggs & Collis, 1991).

Method

Children from two intact grade one classes in a midwest school participated in the teaching experiment on data exploration. One class collected the data on number of missing teeth from its own members (Collection Group, n=20) and the other class also used this data (Non-collection Group, n= 18). We hypothesized that this collection activity might work in favor of the Collection Group by building a better contextual base for data exploration. In addition to the overall analysis involving all children, three children from each class were purposefully sampled as target for more detailed case-study analysis. For each class, one student was chosen from each of the upper and lower quarters and one from the middle 50% on mathematics achievement.

The intervention phase of the teaching experiment for these grade 1 children comprised *four* 40-minute sessions spread over a 3-week period. Sessions opened with a whole-class exploration posed by the third author. Ten teacher education undergraduate mentors facilitated children's solving of data exploration problems that were based on the missing-teeth data and linked to the Framework. To ensure that the inquiry orientation of the intervention phase was met, mentors participated in weekly seminars that explored strategies for fostering children's statistical thinking.

Data were gathered from three sources: (a) researcher-designed interview assessment protocols administered in the weeks preceding and following the intervention, (b) mentor evaluations from each instructional session, and (c) researcher field notes on the six target students. The *assessment protocol* (Jones et al., in press) based on the Framework comprised 37 items (7 on describing, 13 on organizing and reducing, 2 on representing data, and 15 on analyzing and interpreting data) in three contexts: How Many Friends Came to Visit?, Beanie Babies, and The Beanbag Game. A double-coding procedure (Miles & Huberman, 1994) was used to establish pre- and post-intervention statistical thinking levels for *all* students in both classes. In this procedure, the first two authors independently coded, according to the levels of the framework, all questions on each child's protocol. The modal response level for each statistical thinking process was used to determine children's dominant thinking levels. The authors reached agreement on 82% of children's dominant thinking levels. A Wilcoxon Signed Ranks Test (Siegel & Castellan, 1988) was used to compare pre- and post-intervention statistical levels for each of the two classes. Both "within" and "cross-case displays"(Miles & Huberman, 1994) were used to guide the analysis of qualitative data from the six target students. Mentor evaluations and researcher field notes were coded and synthesized to discern learning patterns exhibited by these students during the intervention.

Results and Conclusions

The Effect of the Teaching Experiment: Quantitative Analysis

The Wilcoxon Signed Ranks Test (Siegel & Castellan, 1988) revealed differences between the pre- and post-intervention thinking levels of the grade 1 children that were significant for some statistical thinking processes and not for others. For *describing data*, only the Non-collection Group showed a significant difference (Collection Group, $p < .08$; Non-collection Group, $p < .01$); for *organizing and reducing data*, both Groups showed significant differences (Collection Group, $p < .04$; Non-collection Group, $p < .01$); for *representing data*, neither group showed significant differences (Collection Group, $p < .17$; Non-collection Group, $p < .42$); and for *analyzing and interpreting data* only the Collection Group showed a significant difference (Collection Group, $p < .01$; Non-collection Group, $p < .65$). While the statistical thinking of the children in the two groups changed in slightly different ways, the evidence does not support a stronger overall growth in favor of the Collection Group. When the two classes were combined the differences between children's pre- and post-intervention thinking levels were significant for all statistical processes except representing data.

Process/ Level	Level 1 Idiosyncratic	Level 2 Transitional
Describing Data Displays (D)	•gives a description that is unfocused and includes idiosyncratic/irrelevant information, has no awareness of graphing conventions [e.g., title, axis labels] of the display	•gives a description that is hesitant and incomplete, but demonstrates some awareness of graphing conventions
	•does not recognize when two displays represent the same data OR indicates some recognition but uses idiosyncratic/ irrelevant reasoning	•recognizes when two different displays represent the same data, but uses a justification based purely on conventions
	•considers irrelevant/subjective features when evaluating the effectiveness of two different displays of the same data	•focuses only on one aspect when evaluating the effectiveness of two different displays of the same data
Organizing and Reducing Data (O)	•does not group or order the data or gives an idiosyncratic/ irrelevant grouping	•gives a grouping or ordering that is not consistent OR groups data into classes using criteria they cannot explain
	•does not recognize when information is lost in reduction process	•recognizes when data reduction occurs, but gives a vague/irrelevant explanation
	•is not able to describe data in terms of representativeness or "typicality"	•gives hesitant and incomplete descriptions of data in terms of "typicality"
	•cannot describe data in terms of spread, gives idiosyncratic / irrelevant measures	•invents a measure--usually invalid--in an effort to make sense of spread
Representing Data (R)	•constructs an idiosyncratic or invalid display when asked to complete a partially constructed graph associated with a given data set	•constructs a display that is valid in some aspects when asked to complete a partially constructed graph associated with a given data set
	•produces an idiosyncratic or invalid display that does not represent or reorganize the data set	•produces a display that is partially valid, but does not attempt to reorganize the data
Analyzing and Interpreting Data (A)	•makes no response or an invalid/irrelevant response to the question, "What does the display not say about the data?"	•makes a relevant but incomplete response to the question, "What does the display not say about the data?"
	•makes no response or gives an invalid/incomplete response when asked to "read between the data"	•gives a valid response to some aspects of "reading between the data" but is imprecise when asked to make comparisons
	•makes no response or gives an invalid/ incomplete response when asked to "read beyond the data"	•gives a vague or inconsistent response when asked to "read beyond the data"

Figure 1, Part A. Statistical Thinking Framework

Process/ Level	Level 3 Quantitative	Level 4 Analytical
Describing Data Displays (D)	•gives a confident and complete description and demonstrates awareness of graphing conventions •recognizes when two different displays represent the same data by establishing partial correspondences between data elements in the displays •focuses on more than one aspect when evaluating the effectiveness of two different displays of the same data	 •recognizes when two different displays represent the same data by establishing precise numerical correspondences between data elements in the displays •provides a coherent and comprehensive explanation when evaluating the pros and cons of two different displays of the same data
Organizing and Reducing Data (O)	•groups or orders data into classes and can explain the basis for grouping •recognizes when data reduction occurs and explains reasons for the reduction •gives valid measures of "typicality" that begin to approximate one of the centers (mode, median, mean), reasoning is incomplete •uses an invented measure or description which is valid, but the explanation is incomplete	•groups or orders data into classes in more than one way and can explain the bases for these different groupings •recognizes that data reduction can occur in different ways and gives complete explanations for the different reductions •gives valid measures of typicality that reflect one or more of the centers, reasoning is essentially complete •uses the range or invented measure that has the same meaning as range
Representing Data (R)	•constructs a display that is valid when asked to complete a partially constructed graph associated with a given data set, may have difficulty with ideas like scale or zero categories •produces a valid display that shows some attempt to reorganize the data	•constructs a valid display when asked to complete a partially constructed graph associated with a given data set, works effectively with scale, zero categories. •produces multiple valid displays, some of which reorganize the data
Analyzing and Interpreting Data (A)	•makes multiple relevant responses to the question, "What does the display not say about the data?" •gives multiple valid responses when asked to "read between the data" and can make some global comparisons •tries to use the data and make sense of the situation when asked to "read beyond the data," reasoning is incomplete	•makes a comprehensive contextual response to the question, "What does the display not say about the data?" •gives multiple valid responses when asked to "read between the data" and can make coherent and comprehensive comparisons •gives a response that is valid, complete, and consistent when asked to "read beyond the data"

Figure 1, Part B. Statistical Thinking Framework

For the three significant statistical thinking processes, the most salient feature of the data was that the number of children exhibiting Level 3 increased following the intervention and this was accompanied by a corresponding decrease in the number exhibiting Level 1 thinking.

The Effect of the Teaching Experiment: Case-Study Analysis

A number of learning patterns and trends were discerned by examining the relationship between target students' thinking during instruction and their thinking at the pre- and post-intervention assessments. These patterns are described and interpreted for each of the four statistical thinking processes. With regard to *describing data*, students brought some prior knowledge to the classroom. For example, they recognized how categorical data such as "days of the week" were shown on a scale, how to find values on a line plot using counting, and how to read data in a table. Target students had more difficulty reading bar graphs than line plots and they made only cosmetic comparisons between a line plot and a bar graph of the same data. However, during instruction they gave less idiosyncratic responses when reading visual displays and became more facile in comparing different displays and organizations of the missing teeth data. With regard to *organizing and reducing data*, students' informal knowledge prior to instruction was limited. However, during instruction and in the post-assessment most target students demonstrated informal notions of mode or middle (median) in dealing with center. A smaller number showed some idea of clustering in discussing spread. Although most children were able to organize the familiar Beanie Baby data, they were unable to construct different organizations of the missing teeth data. Nevertheless, when the children were shown two different organizations of the missing teeth data they were able to recognize that they represented the same data. With regard to *representing data*, students were more capable at completing unfinished graphs based on a given set of data than they were at representing a given graph in a different way. Their limited skills in reorganizing data also constrained their representations of data. Interestingly more than 50% of the target students were not able to represent "zero" on a display and zero data values also caused problems when children analyzed and interpreted data. With regard to *analyzing and interpreting data*, our students, like those of Curcio and Artzt (1997), demonstrated more normative thinking in tasks that involved reading between the data than in tasks that involved reading beyond the data. However, even though students read zero values they ignored them when analyzing and interpreting data. The instructional sequence helped target students to gain better facility in dealing with zero values and in reading beyond the data particularly as they became very familiar with the missing teeth data. By the end of the intervention, more students in both classes were able to make and justify meaningful predictions of how many friends would visit Sam in the next month based on the data for the previous week.

Given the prior knowledge and growth that children showed on the four statistical processes, there is evidence that they can accommodate a broader approach to data exploration. However, if instruction is to reach its full potential in the elementary grades, further research is needed to build learning trajectories that link different levels of children's statistical thinking.

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DEVELOPING STATISTICAL PERSPECTIVES IN THE ELEMENTARY GRADES

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An idea that has emerged as one around which we might organize an introductory sequence in data analysis is that of data as an aggregate (Cobb, 1999). Many studies have pointed to the importance in statistical reasoning of this construct (e.g., Hancock, Kaput & Goldsmith, 1992; Mokros & Russell, 1995), but we still know little about how this perspective develops or of student ideas that might serve as precursors to it. We analyze 34 case studies written by teachers about their experiences and reflection in teaching data analysis in grades K-5. Analyses suggest three perspectives students use in their approach to data. Used as (1) pointers, data serve as shorthand records of more complex events. When they have observed the events, very young students use recorded data to help them recall other information about the observed event. The most prevalent idea among elementary students is that of data as (2) classifiers. As classifiers, data provide ways to compare individual data values or types, to easily locate a value with respect to others, and especially to determine who is the most and least. We see a few students in upper elementary grades beginning to focus on data as (3) distribution, attending to emergent features of distributions such as centers and spreads. However, when they first begin focusing on these distributional characteristics, students often disconnect plot features from the situations and questions of interest. These orientations towards data are closely tied to the questions students have when they collect and analyze data. This being the case, instruction should stress reasons for collecting data and for looking at data with those questions in mind.

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TASK VARIABLES IN A QUESTIONNAIRE ON PROBABILITY'S PRODUCT RULE

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The content core as defined by the product rule in probability —problems which can be solved by means of the $P(A \cap B)=P(A) \cdot P(B)$ formula, or by some equivalent procedure— is a particularly difficult matter in its solving procedure by the students. Following the methodology suggested in Goldin's (1984) compiled works, we have identified and defined *task variables* concerning the probabilistic product. The ability to classify and to define task variables will allow both the researcher and the teacher to have a systematic control to determine the effects of such variables on the solving behaviour [Kulm, 1984, p. 1].

We have prepared a questionnaire consisting of six items that are related to the concept of the product rule in probability where such rule can be used to solve them, although a combinatory scheme and the classic probability formula can also be employed. We have regarded three main task variables: 1) *time*, refers to situations that can be of two kinds: a) *synchronic*, if two (or more) events occur simultaneously, and b) *diachronic*, if the events happen in succession; 2) '*distinguishability*', refers to the possibility of distinguishing —or not— objects from one another (it has already pointed out that this 'small' difference propitiates confusions in combinatory tasks [Batanero et al., 1994] and 3) the *amount* (few, many) of objects considered —if they are but a few, the solution is easier than if they become a big bunch. In constructing choices to each question, we considered two wrong *strategies*, which had been previously detected among high school students [Buendía, 1994]: a) *additive* and b) '*pseudoadditive*'.

The problem in each item can be divided into two moments, and for each of them the probability must be calculated —the correct answer is the product of the probabilities. Many students perceive the two moments and they calculate their respective probabilities. However, not all of them perform their product (the multiplicative strategy); some perform an addition (the additive strategy); and there are some others who apparently perform an incorrect addition of fractions, as if they were applying the following rule: (a pseudoadditive strategy). This answer could be produced by a model other than an incorrect addition. The questionnaire was applied to 196 students from different levels and two institutions. We analyzed the results and made observations about performance, order of difficulty in the items and the relationship among them considering the task variables.

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CONCEPTUAL ISSUES IN UNDERSTANDING SAMPLING DISTRIBUTIONS

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This study investigated students' abilities to conceive the ideas of sampling distribution and margin of error. Twenty-seven 11th- and 12th-grade students participated in a teaching experiment addressing ideas of sample, sampling distributions, and margins of error. Our aim was to produce epistemological analyses of these ideas (Glasersfeld, 1995; Steffe, 1996; Thompson, in press) – ways of thinking about them that are schematic, imagistic, and dynamic– and hypotheses about their development in relation to students' classroom engagement.

Better performing students and students exhibiting coherent discourse during class had developed a multi-tiered scheme of conceptual operations centered around the image of repeatedly sampling from a population, recording a statistic, and tracking the accumulation of statistics as they distribute themselves along a range of possibilities. These operations seemed to be grounded in an image of samples as quasi-proportional, mini-versions of the sampled population. Poorer-performing students (1) tended to view samples simply as some of the population, (2) did not extend their sense of variability to ideas of distribution. Instead, variability meant only that if we were to draw more samples and compute statistics from them, those statistics would differ from the ones of previously drawn samples, and (3) had difficulty coordinating the various levels of activity (drawing one sample and calculating a statistic, repeating this process many times, analyzing outcomes from the second-level process, etc.).

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MODEL-DEVELOPMENT SEQUENCES: PUPPY GENETICS AND STARLOGO™ PART II

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This poster presentation is the second of a four-part sequence describing model-eliciting and exploration activities. The Puppy Genetics Model-Eliciting Activity introduces basic genetic inheritence concepts. In the subsequent exploration activity, the students explore the probability concepts underlying the genetic behaviors.

A Model-Eliciting Activity asks students to develop a mathematical model that describes, explains, manipulates, or predicts the behavior of a real-world system. Students are given a problem statement that presents a real-world problem that needs to be solved for an identified client. Typically, the client asks the students to describe how the students' solutions can be used in the future to solve similar problems. A Model-Exploration Activity is a follow-up activity for the Model-Eliciting Activity. The Exploration Activity allows the students to explore patterns, regularities, traditional mathematical notations, and different representational systems that further develop the models that they constructed in the Model-Eliciting Activity.

In the Puppy Genetics Model-Eliciting Activity, a dog owner wants to know why her two black Labrador Retrievers had a litter with a brown puppy. Students are asked to develop a representation system that illustrates the mechanism of the inheritance pattern.

The Exploration Activity to follow Puppy Genetics presents a StarLogo™ environment that shows the result of multiple possible matings. The students explore the probability of different colored puppies occurring in different parent matchings.

Problem Solving

SYMBOLS AND MEANINGS IN TEACHER-STUDENT INTERACTION DURING MATHEMATICAL PROBLEM SOLVING

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Abstract: Teacher-intervention during problem-solving instruction was investigated for six inservice elementary teachers. The study was framed in the perspective of symbolic interaction, i.e., a focus on the teachers' personal meaning and supporting symbolic systems. The findings indicated that the teachers constructed a symbolic system to guide their interactions with students in a way that emphasized an autonomous role for the students during problem solving. Their approach to intervention consisted of a 5-action sequence associated with particular symbols and embodied a process of separation and connection between teacher, student, and/or problem. The findings suggest the importance of helping teachers to understand such patterns in their own teaching as a way of facilitating changes to it.

In the current reform movement in mathematics education, problem solving is being emphasized as a basis of mathematics thinking and a basis of learning mathematics (e.g., NCTM 1989). Implicit in this perspective is that problem solving is an open-ended process that requires flexibility in the thinking and behaviors of both teachers and students. This view of problem solving forms the basis of this paper. The paper reports on a study of teacher-student interaction during mathematical problem-solving instruction from the perspective of symbolic interaction, i.e., a focus on teachers' personal meanings and symbolic systems that formed a basis for how they conceptualized and facilitated students' problem-solving behaviors.

There is a large body of literature on problem solving, but very little deals with problem solving instruction from the teacher's perspective. Historically, a number of differing viewpoints regarding instruction on problem solving have been proposed; the most common of these are based on Polya's four-stage model (Polya, 1957). Charles and Lester (1982) identified two contexts in which teacher behavior during problem solving has been described in the literature. One context dealt with the types of teacher behavior that should be used at each of Polya's stages. The other considered teacher behaviors in terms of ten teaching actions grouped into three distinct time periods that make up a problem-solving session: before, during and after the problem is solved. While these prescribed behaviors form useful guidelines to problem-solving instruction, they ignore the humanistic aspect of classroom interaction, e.g., they involve surface or observable behaviors in isolation of the teachers' thinking. Since it is the teacher who must interpret such guidelines in the context of real classrooms, then her/his perspective becomes a necessary lens through which to understand classroom behaviors during problem solving as opposed to the prescribed behaviors

Symbolic Interaction

Symbolic interaction was used as the basis to interpret teacher-student interaction from the teacher's perspective. The central idea of symbolic interaction (Blumer, 1969) is that human interactions are carried on through the medium of symbols and their meanings. Reality is not disclosed directly, but is experienced through symbols and activities mediated by language and culture. As Blumer (1978) explained: Human beings interpret or "define" each other's actions instead of merely reacting to each other's actions. Their "response" is not made directly to the actions of one another but instead is based on the meaning which they attach to some actions. Thus, human interaction is mediated by the use of symbols, by interpretation, or by ascertaining the meaning of one another's action. This mediation is equivalent to inserting a process of interpretation between stimulus and response in the case of human behavior [p. 97].

In relating symbolic interaction to the classroom, the assumption is that teachers and students rely on symbols, whether consciously or not, both to create and "read" the learning environment. Teachers and students do not typically respond directly to each other's actions as stimuli, but assign meanings to the actions and act on the basis of the meanings. Such meanings are socially derived through interaction with others rather than inherent in the actions themselves or idiosyncratically assigned by the teacher or student. Thus, when viewed through symbolic interaction, the mathematics classroom can be seen as interweaving symbols and signification systems that students and teachers use, whether consciously or not, as texts of mathematics learning and teaching. These symbols emerge from a variety of situations in the classroom. During a lesson, who can talk, when, how, and about what are examples of symbols. Other examples are the way a teacher structures a lesson (e.g., focus on drill); the way the teacher uses time (e.g., time spent on a particular concept) and space (e.g., arrangement of desks); and the location of the teacher (e.g., circulating among students). These symbols convey what should be valued in the mathematics classroom and about mathematics.

Research Process

This study is based on a larger project that investigated the effect of a humanistic approach to teacher development as a basis for facilitating change in teachers' thinking and teaching of problem solving (Chapman, 1999). The approach, the problem solving inservice [PSI] program, was found to be effective in allowing the participants to make significant shifts in their thinking and teaching that were consistent with the reform perspective of teaching mathematics in terms of recognizing the active, social, and constructive nature of the learning process. This study is an analysis of the participants' teaching after the PSI program with a focus on teacher-student interaction during students' problem solving. The participants were six inservice elementary school teachers (Grades 3 to 6) who volunteered for the study. They participated in the inservice program over a 4-week period during their summer break. Prior to the PSI, they had little or no experience solving non-routine problems as learners. The PSI activities involved non-routine problem solving, role-play, and narrative reflection of personal meaning of past, present, and future experiences with mathematical problem solving.

Data included transcripts of open-ended interviews on the teachers' thinking and teaching of problem solving and all oral aspects of the PSI activities (e.g., group discussions and narrative reflections). Copies of all written work during the PSI program (e.g., solutions of problems, journals of individual reflections, summaries of group discussions) were also obtained. The teachers were observed in their classrooms while conducting lessons involving problem solving, and teacher-student verbal interactions during these lessons were audio-taped and transcribed.

An interpretative research approach (Creswell, 1998) was used to determine meanings associated with the teachers' actions. The data were scrutinized to identify recurring themes of how the teachers viewed and practiced intervention. Themes from interviews were triangulated with themes from the teachers' group discussions and their actual classroom discourse to determine the final set of themes. The symbols and descriptors (e.g., separation and connection) to reflect the essence they embodied for the teachers were deduced from the data based on these themes. The analysis built on the findings of the PSI study in terms of the personal meaning the participants constructed during and after the PSI experience as a basis for interpreting the themes identified in relation to their symbol systems.

Teacher Intervention

The outcome is presented here only in terms of when and how the teachers intervened and their bases for intervention. In addition, teacher intervention is considered only in terms of what the teachers did when and after a problem was assigned to students to solve. The teachers' behaviors and thinking reflected one general pattern of intervention consistent with the knowledge they constructed during the PSI program. The general theme of this knowledge was that intervention should be both passive and active. During passive intervention, the teacher should only listen to the students to become aware of their thinking and to give them time to think on their own. Active intervention, however, required that the teacher communicate with the students, not to tell them how to get the answer, but to stimulate their thinking to get beyond obstacles and to make sense of their processes. Based on their PSI experience, the teachers selected a set of terms they felt specified the essence of the problem-solving experience that was relevant to the classroom context. Although this was done individually after the PSI experience, for the most part, the teachers selected the same terms, e.g., obstacle/barrier, stuck, off-track, challenge, make sense, interpretation/meaning, strategy, listening to. These terms became symbols that helped to facilitate mutual interpretation of when and how the teacher should intervene in order to create a learning environment that allowed the students to be more autonomous during problem solving.

During the PSI program, the teachers had constructed for themselves particular meanings for the terms. Of particular importance were stuck, off-track and lost, all of which were considered to be important indicators for active intervention. The teachers considered stuck to be when students tried everything they could by themselves and were about to become frustrated. Here the student should initiate the intervention and the teacher should intervene by asking open-ended questions and/or make an open-ended suggestion of something to try. Off-track was considered to be when students were doing something incorrect based on how *they* interpreted the problem or on the strategy *they* were using to solve the problem. Here the teacher should initiate the intervention and intervene by asking open-ended questions or making an open-ended comment. Lost was considered to be when students were confused and disoriented, lost control of the problem, and could no longer make sense of the problem or any help provided. Here the teacher should take control of the situation in order to re-orient them to a specific solution. This could involve the teacher explaining the problem and a possible solution directly or with the help of students who were able to do it.

Another common theme that emerged from the teachers' behaviors and thinking was that intervention was a process of separation and connection. Separation involved a form of decontextualization in which the teacher consciously removed herself from the student's experiences with the problem or from her experience with the problem. Connection involved a form of contextualization in which the teacher participated in the student's experiences with the problem or relived her own experience with the problem. The teachers conceptualized the problem-solving lesson in three stages for a problem: problem presentation, problem solution, and solution sharing. A summary of the pattern of intervention, connection and separation for each stage follows. Only common patterns are discussed instead of individual situations/differences.

Problem-Presentation Stage. The dominant goal of this stage was to let students own the problem. As Susan explained:

> I am more aware of things like [the student] owning the problem, like is this really a problem for me [the student] or is this just something that I have to do to make it through the next 20 minutes.

In order to own the problem, students were required to interpret if for themselves. For example,

> [My] focus the whole year was on meaning and thinking through things. ... I give them a problem, which is written, they read it themselves, then I ask for their interpretations.

There were variations in how this stage evolved. For example, Mary led her Grade 3's in a large group discussion while Pam had her Grade 3's work, first individually, then in partners to arrive at their own meaning of the problem. Rose told her Grade 5's, "Tell what you think the problem means to you." Most of the teachers required that the students write their interpretations.

In this stage, the teachers became detached from the problem in order to allow students to connect to it. Thus the teachers did not intervene with any predetermined interpretation of the problem. The teachers, for a few minutes, also became detached from the students in order to allow them to connect to the problem. They only reminded students of their task, e.g., write your meaning, discuss your meaning with a partner. Finally, the teachers helped the students to connect with each other and the problem as they shared their interpretations. For example, the teachers would ask for volunteers to respond to any queries raised and remind students to listen to each other's interpretations. Most of the teachers chose passive intervention at this point. They listened to the interpretations but did not try to make corrections.

Problem-Solving Stage. The dominant goal in this stage was for students to become decision-makers in deciding on a strategy and testing it. The teachers continued to be detached from the problem to now allow students to develop and work on a strategy. However, they became connected to the students and the learning environment by circulating and constantly interacting with the students either passively or actively. During passive intervention the teachers tried to see what the students saw, thus trying to connect with the students' perspective. During active intervention, they focused on when students were stuck, off-track, or lost. For stuck and off-track, Pam, for example, first intervened with questions like, "What have you tried?" "Why did you add?" "What part of the problem asked you to do that?" "What else do you think you can try?" "Why don't you try drawing a picture or using a chart?" "Is that what you really want to do?" She gave them time to resolve difficulties on their own. For lost, she provided more direct guidance by telling them what was wrong or how to get started. But in general, she allowed them to arrive at a solution in their own way even if incorrect.

Solution-Sharing Stage. The teachers remained detached from the problem in order to allow students to share and justify their solutions, but only if the teachers' solution was very different from the students' that it got presented as an alternative and not *the* solution. The teachers intervened to encourage reflection and discussion of the solutions in a variety of similar ways. For example, Mary asked questions like: "What do you think of ....?" "Which of the answers do you think is/are correct and why?" "Why does it make sense?" Pam asked questions like: "What do you think about their method?" "Does it make sense?" "What doesn't make sense?" "How can they fix it?" Students were also encouraged to talk about what they thought about the problem, e.g., what they liked or did not like about it. In general, then, intervention during this stage was not simply to check solutions but to connect students, teacher, and problem in meaningful ways.

Discussion

In this study, the symbols the teachers used to frame their interventions (e.g., stuck, off-track, lost) were triggered by particular actions of the students in relation to getting to a solution for a problem. These actions were students' questions, oral and written responses, and physical expressions. They provided cues for the teacher about when students were successful, stuck, off-track, and lost. These cues were dependent on the teacher's judgement or personal meaning. For e.g., what was considered stuck for one student could be considered lost for another based on how the teacher perceived the student's ability to solve problems. Thus the teacher's personal meaning of the cues and not the cues in themselves guided intervention. This interpretation often evolved from communicating with the students. When considered necessary, the teachers were able to suspend their interpretations until after communicating with students to make sense of the context embodying the cues. For e.g., a student was perceived to be off-track only after the teacher got a sense of what the student was trying to do after questioning him or her. The goal of this communicating was to listen *to* the students as opposed to listening *for* specific behaviors.

In general, intervention involved a sequence of related teacher behaviors: awareness of students' actions, identification of cues in the actions, communication with students about context embodying cues, relating cues to symbol, enacting symbol. Intervention also involved a process of separation and connection with students and/or problem. Separation appeared in the teachers' behaviors as a distancing from, a low level of awareness of, or a decontextualizing of an experience, while connection appeared as the opposite. For the most part, during teacher-student interaction, the teacher tried to separate from the problem while the students were encouraged to stay connected to it. The teacher also tried to stay connected to the students and thus to the problem through the students. Each teacher's commitment to the students' interpretation and solution of the problem was the focus of when and how separation and connection with students' experiences occurred. The teacher [T], students [S] and problem [P] existed as a triad of two-way relationships in which the teacher recognized a T-P connection (i.e., T specified P and P specified T), a S-P connection (i.e., S specified P and P specified S) and a T-S connection (i.e., T specified S and S specified T) i.e. Figure 1:

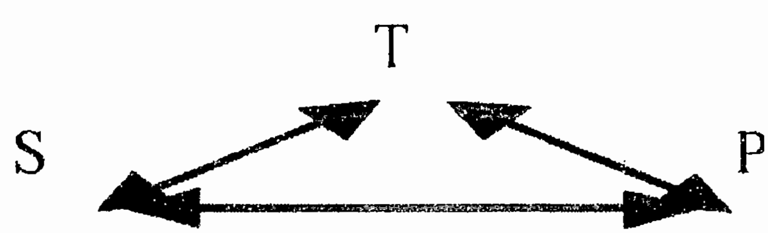

Figure 1. Connections between teacher [T], students [S], and problem [P]

The specifying between two entities refers to how each allows the other to perceive it/him/her (Merleau-Ponty, 1962). For example, in the T-P and S-P situations, the specifying is dependent on the problem providing opportunities for alternative solutions and the teacher and students being able to see one or more of them.

Conclusion

The pattern of intervention discussed in this paper involved a sequence of five related teacher behaviors associated with particular symbols for teacher-student interaction. This pattern also embodied a process of separation and connection between teacher, students and/or problem that characterized their interactions. Helping teachers to understand such patterns in their own teaching could be useful in facilitating changes in their teaching. In particular, teachers would need to understand the symbols and meanings they use that influence intervention and what alternative symbols they could adopt to change their approach to it.

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MENTAL PROJECTIONS IN MATHEMATICAL PROBLEM SOLVING: THE ROLES PLAYED BY ABDUCTIVE INFERENCE AND SCHEMES OF ACTION IN THE EVOLUTION OF MATHEMATICAL KNOWLEDGE

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Abstract: Combining aspects of Piaget's scheme theory and Peirce' theory of abduction, this paper examines the novel problem solving actions of a college student. The analysis documents and explains the important role of abductive inference in the solver's novel solution activity.

Abstraction and abduction describe creative processes in mathematical problem solving. The solver's ability to abstract mathematical relationships from their problem solving actions enables them to create mathematically powerful ideas (Schoenfeld, 1985). The process of abduction, as described by Charles Saunders Peirce, wherein explanatory hypotheses are generated and tested, enables solvers to reflect on and scrutinize their potential solution activity and make conjectures about its usefulness (Anderson, 1995; Cifarelli, 1998; Fann, 1970; Mason 1995).

The purpose of this paper is to demonstrate how a focus on the abductive reasoning activities of solvers enhances and extends contemporary constructivist analyses of problem solving. The first part of the paper provides a brief overview of the theories of Peirce and Piaget, focusing on how each explains the construction of new knowledge as involving acts of problem solving. The second part of the paper focuses on a Piagetian study of problem solving previously conducted by the author. Through re-examination of selected episodes of solution activity, the revised analysis demonstrates the prominent role that abduction plays in problem solving activity and shows how the Peircean analysis enhances and extends the Piagetian analysis.

Using Peircian and Piagetian Perspectives to Study Problem Solving

Peirce and Piaget each had high regard for problem solving activity and its role in the evolution of knowledge. Peirce's focus on the importance of logical reasoning that individuals use to explain unexpected or surprising facts, and Piaget's focus on how a learner's thinking proceeds in the face of cognitive perturbation, suggests they each saw a fundamental connection between problem solving and learning: an individual solving a problem engages in learning activity and has constructed new knowledge.

Both Peirce and Piaget viewed the construction of new knowledge as involving dynamic, creative activity (Table 1). Peirce identified *abduction* (the generation of plausible hypotheses to account for surprising facts) as the process that introduces new ideas into the reasoner's actions.

Table 1. Problem Solving-Based Explanations of New Knowledge

	Peirce	Piaget
Constructs	abductive inference	reflective abstraction
key processes	hypothesis generation and testing: plausible hypotheses are self-generated and tested to explain surprising results	through resolution of perturbations, learners construct and reconstruct their knowledge at increasingly abstract levels
new knowledge	hypotheses evolve through intertwining of induction, deduction, and further abductions	learners develop structure in their problem solving activity (schemes of action)
growth of awareness	explanatory hypotheses	anticipation

In contrast to Peirce's views, Piaget maintained that learners organize their sensori-temporal actions into mental structures, or ***schemes***, that can be evoked to aid the learner's interpretive acts when problems are encountered. Schemes become ***operative*** as they are generalized and extended. Piaget explained the development of new knowledge in terms of reflective abstraction as the primary process that explains the re-organization of action as schemes are revised.

While the differences above involve different emphases regarding what counts as meaningful problem solving, they also indicate that Peirce and Piaget held different ideas about how learners mentally project their ideas through time and space. The following sections analyze the solution activity of a college student named Marie. By examining critical junctures of her solution activity using a Peircian lens, the revised analysis will help clarify these differences.

Marie's Problem Solving

Marie was interviewed as she solved a set of nine similar algebra word problems (see examples in Table 2). These tasks were designed by Yackel (1984) to induce problematic situations across a range of similar mathematical situations.

Table 2. Sample of Algebra Word Problems Used in the Study

Task 1: Solve the Two Lakes Poblem

The surface of Clear Lake is 35 feet above the surface of Blue Lake. Clear Lake is twice as deep as Blue Lake. The bottomof Clear Lake is 12 feet above the bottom of Blue Lake. How deep are the two lakes?

Task 2: Solve a Similar Problem Which Contains Superfluous Information

The northern edge of the city of Brownsburg is 300 miles north of the northern edge of Greenville. The distance between the southern edges is 218 miles. Greenville is three times as long, north to south as Brownsburg. A line drawn due north through the city center of Greenville falls 10 miles east of the city center of Brownsburg. How many miles in length is each city, north to south?

Task 3: Solve a Similar Problem Which Contains Insufficient Information

An oil storage drum is mounted on a stand. A water storage drum is mounted on a stand that is 8 feet taller than the oil drum stand. The water level is 15 feet above the oil level. What is the depth of the oil in the drum? Or the water?

Task 4: Solve a Similar Problem in Which the Question is Omitted

An office building and an adjacent hotel each have a mirrored glass facade on the upper portions.The hotel is 50 feet shorter than the office building. The bottom of the glass facade on the hotel extends 15 feet below the bottom of the facaade on the office building. The height of the facade on the officed building is twice that on the hotel.

The Piagetian analysis of Marie's activity is summarized as follows. She was inferred to have constructed a conceptual structure while solving Task 1. While solving Tasks 2 to 9, Marie's sense of problem "sameness" (Lobato, 1996) evolved to the extent that she could begin to reflect on and *anticipate* results of potential solution activity prior to carrying it out with paper-and-pencil. This development of her solution activity was interpreted as Marie having constructed a conceptual scheme that enabled her to see each successive task as "the same" and act accordingly to solve the problems (Figure 1). Her growing awareness of the efficacy of her solution activity was characterized as increasing levels of abstraction of the scheme (Table 3).

A Revised Analysis of Marie's Solution Activity

Marie's solution activity while solving Tasks 1 & 2 will be summarized. H subsequent solution activity in Tasks 3 & 4 will illustrate and explain the gradu generation of novelty into her evolving solution activity in terms of abductive reasoning.

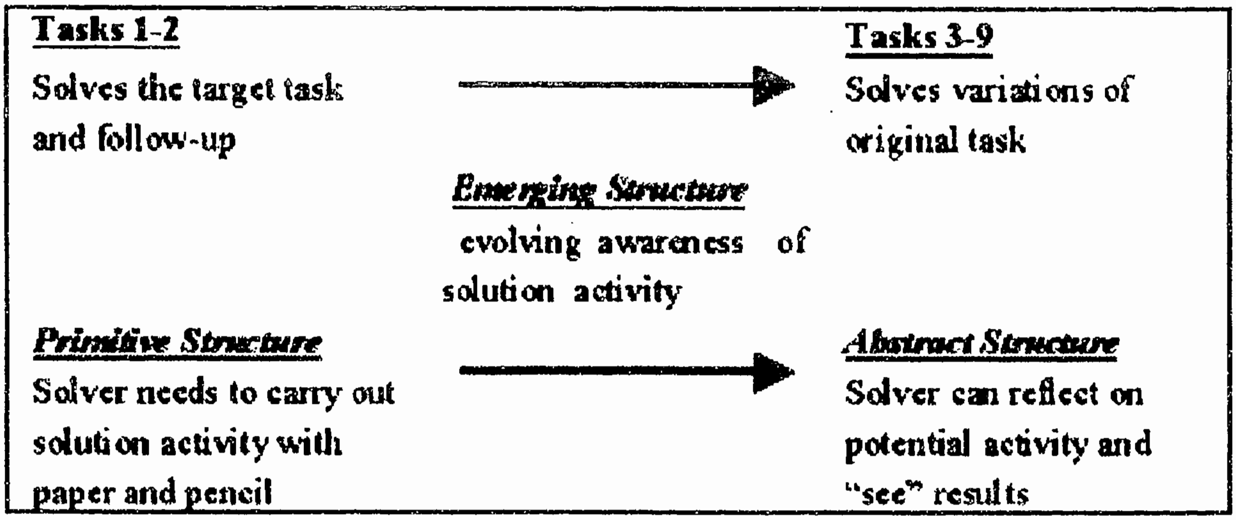


Figure 1. Marie's Evolving Scheme

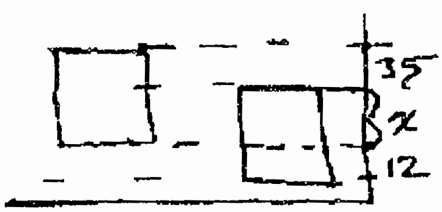
Table 3. Marie's Solution Activity as Levels of Abstraction

Level of Activity	Characterization	Examples
Abstraction	Solver can coordinate potential actions and "run through" potential solution activity in thought and operate on its results.	Solver can "see" or anticipate results of potential activity (and draw inferences) prior to carrying out solution activity with paper-and-pencil.
Re-Presentation	Solver can coordinate prior actions and "run through" prior solution activity in thought.	Solver can "see" or anticipate potential difficulties in new problem situations.
Perceptual Expression	Solver uses diagram to aid reflection.	Solver can reason from diagrams to anticipate potential problems.
Recognition	Solver sees the relevance of using previously constructed solution activity to solve new problems	Solver recognizes usefulness of diagrammatic analysis used to solve Task 1 to solve Tasks 2-9

Marie's solution activity for Task 1 was by no means routine. She initially interpreted the task about the two lakes as an "algebra word problem in two variables" and generated a system of several equations, no two of which were consistent (Figure 2). When she realized that this approach did not lead to a solution, she constructed a side-by-side diagram of the lakes, and translated relevant lengths from the diagram to a vertical axis, which served as a reference aid in constructing relationships. This solution activity eventually led to a correct solution (Figure 3).

$$S_c - S_b = 35$$
$$B_c - B_b = 12$$
$$(S_c - S_b) = 2(B_c - B_b)$$

Figure 2. Marie's initial equations for task 1.



$$35 + x = 2(12 + x)$$
$$35 + x = 24 + 2x$$
$$11 = x$$
$$C = 35 + 11 = 46$$
$$B = 23$$

Figure 3. Marie's solution to task 1.

After solving Task 1, Marie interpreted Task 2 as similar, remarking that "the first thing that strikes me is that this problem is a lot like the first one" and constructing a diagram similar to that she constructed to solve Task 1 (Figure 4). While her anticipation indicated some sense of similarity between the solution of the current task and Task 1, it did not allow her to see and address the potentially problematic information. It was only after she carried out her solution activity that she realized the potentially problematic information:

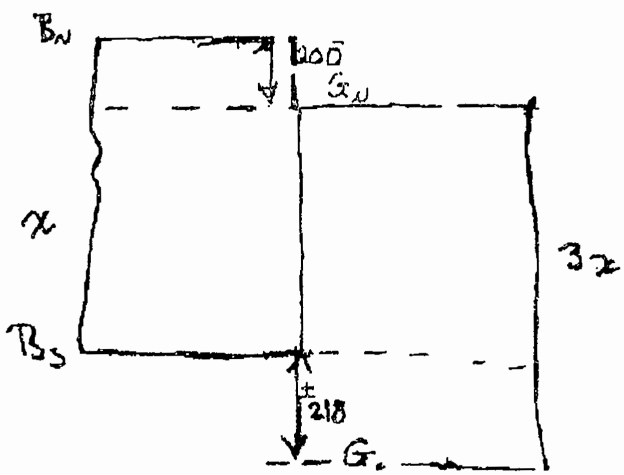


Figure 4. Marie's diagram for Task 2

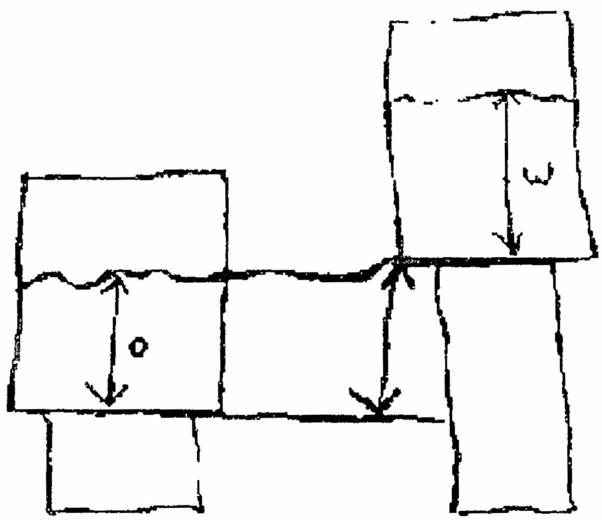


Figure 5. Marie's diagram for Task 3

Marie: This information seems to have nothing to do with the problem. So, I'll just consider all of the other relationships first.

Marie went on to construct a correct solution for Task 2. Marie attempted to solve Task 3 in the same way as she solved Tasks 1 & 2. However, she soon found herself faced with a problematic situation she had not anticipated, which galvanized her with a sense of excitement and wonderment, as she looked to explain the new problem she now faced (Figure 5):

Marie: I am going to draw a picture. Here is my oil stand. And we have a water storage 8 feet taller. And here's level water. And here's the oil level. *(reflection)* So, solve it ... the same way. *(She smiles, then displays a facial expression suggesting sudden puzzlement)* Impossible !! It strikes me suddenly that there might not be enough information to solve this problem. I suspect I'm going to need to know the height of one of these things *(points to containers).* I don't know though, I am going to go over, all the way through.

Marie's anticipation that "the same way" would not work was followed by her *abduction* that the problem might not contain enough information, later refined to the hypothesis that she needed more information about the heights of the unknowns.

While her hypothesis contained uncertainty, it helped to organize and structure her subsequent solution activity as she explored and tested its plausibility as an explanatory device. She spent much time pursuing the elusive information and finally concluded that the problem could not be solved.

Marie's solution activity while solving Task 3 suggested a qualitatively different level of inquiry than she demonstrated while solving Tasks 1 and 2. While she could temporarily *suspend* conditions of the problem to solve Task 2 (by ignoring the extra information), Marie's sudden experience of surprise while solving Task 3 fueled her desire to generate and entertain novel explanations that were radically different from her prior sense-making actions; this indicated a major opening-up of her conceptual boundaries. She ventured to explore her ideas and convictions with a sense of open-endedness, free in the sense that she no longer was constrained by the conventions she previously operated within. The crucial point here is that her abductive actions opened up conceptual boundaries for potential action, and did not merely suspend the constraining conditions that constituted the problem (as was the case in Task 2).

A second indication that the solver had transformed her actions to a new level of inquiry was the shift in her *reflective orientation*, whereby she began to formulate goals for action in terms of drawing from *potential states* of the problem. This change of orientation came out of her need to explain a result in "present time" for which there was no room for explanation given her current understandings. With her abduction she generated plausible explanations within the world of future events and imagined action, thereby forging her deliberation over future events that ultimately served to constrain her current actions. She organized her sense-making actions in terms of future events (specific actions concerning the problem conditions she needed to perform in order to verify that a solution was possible) which then *beckoned* back to her to make them real. This drawing from the future to chart a present course of action helped the solver make-sense of her current problem and paved the way for her to pose new problems.

In what ways did Marie's abductions help to evolve her solution activity while solving later tasks? A partial answer to this question is that she became more cautious in her activity, spending increased time reflecting on her potential activity. However, her reflections on potential solution activity continued to exhibit hypothetical qualities that led to novel conjectures. For example, while solving Task 4, Marie quickly noticed the omission of a question from the problem statement yet was able to hypothesize potential problems for her to solve from the information.

> **Marie:** There's no question! *(Long reflection here)* ... The things they *could* ask for are things like ... *(HYPOTHESIS)* ... the height of one of the buildings but ... *(ANTICIPATION)* ... there's not enough information to get that.... The only thing we have information about is ... *(HYPOTHESIS)* ... Ah, the relative heights of the two facades. So, if I were ... if somebody wanted

me to solve any problem, that's probably what they're asking for

Marie's hypothetical statements about potential problems that could be solved were provisional in the sense that they lent themselves to further scrutiny, and plausible since, based on her current understandings, these were problems that could conceivably be solved. Marie's anticipation following her first hypothesis indicated she had deduced from her hypothesis the necessary conditions of the problem, and had performed a mental "run through" of the imagined action of trying to solve the problem, the result of which she rejected her hypothesis. Similarly, she explored the plausibility of her second hypothesis, concluding that it made more sense to her that the problem of finding the heights of the two facades was a problem that could be solved.

Unlike her solution activity in Tasks 2 and 3, where her solution activity involved making explicit comparison to Task 1, here Marie employed hypothetical states of new and future problems in initiating her solution activity. Her anticipations were now connected to specific hypotheses. The solver demonstrated this highly abstract activity prior to constructing her diagrams. She constructed a solution to the problem, utilizing diagrams to construct relationships, in much the same way as she solved earlier tasks (Figure 6).

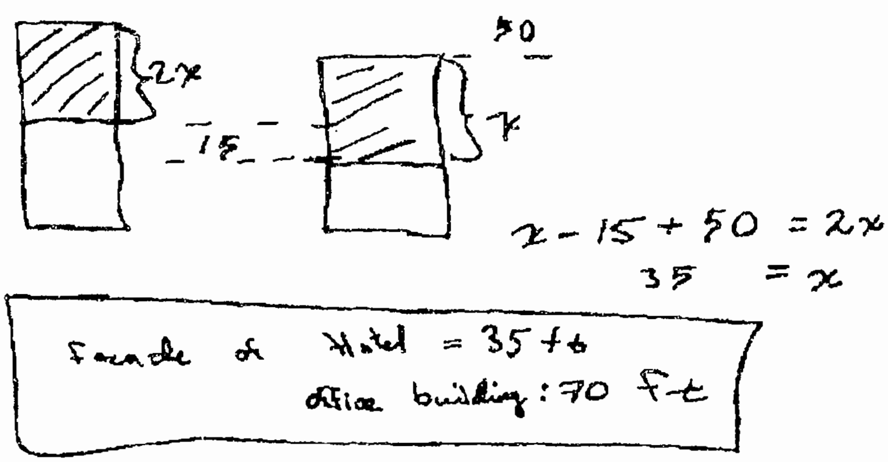


Figure 6. Marie's solution to Task 4

Marie: Okay. Let's see if there is anything here that will at least give me information. Okay, the hotel is 50 feet shorter than the office building. So we have distance here which is 50. The facade of the hotel extends 15 feet below the facade of the office building. That distance would be 15. The height of the facade on the office building is twice that on the hotel. *(Long reflection here)* So I call this distance X, this distance here is 2X. All right!

> And then I can say that X minus ... I'm trying to find a relationship between these two. And I know that ... X minus 15 plus 50 is going to equal 2X. So, 35 equals one X. So that would indicate that the facade on the hotel is 35 feet. On the office building is 70 feet.

Discussion

Abductions as Motivating Orientations for Future Actions

As Marie elaborated and extrapolated her hypotheses, her reflective scope widened in the sense that she could 'see' among many options for action and determine those which aided her progress towards solving her problem. In this way, those future events *beckoned* to Marie for her to *actualize* specific trials, the results of that provided feedback for her evolving hypotheses. This reflective phenomena of formulating explanations in terms of future action is an aspect of abductive reasoning that involves the ability to coordinate images of action with one's evolving goals and purposes. The philosopher Bertrand de Jouvenel explained how images of action are projected and 'stored' into the future:

> *"Our actions seek to validate appealing images and invalidate repugnant images. But where do we store these images? For example, I "see myself" visiting China, yet I know I have never been there .... There is not room for the image in the past or present, but there is room for it in the future. Time future is the domain able to receive as "possibles" those representations which elsewhere would be "false". And from the future in which we now place them, these possibles "beckon" to us to make them real." (de Jouvenel, 1967, p. 27)*

The Role of Anticipations

The original Piagetian analysis explained Marie's growth of awareness in terms of the process of anticipation: "anticipation is nothing other than application of the scheme to a new situation before it actually happens" (Piaget, 1971, p. 195). As Marie solved variations of the original problem, she developed awareness of the structure of her solution activity, enabling her to anticipate results of potential solution activity prior to carrying it out with paper-and-pencil.

By considering aspects of Marie's solution activity in terms of abductive reasoning, her anticipations were seen to be connected to her evolving hypotheses, and hence took on greater impact -- they were constituted within Marie's hypothesis-elaborating and hypothesis-testing activities, and thus helped her confer degrees of clarification and certainty in her on-going reasoning. In this way, problem solving for Marie was less about resolving problematic situations by revising her current scheme, but more about making her hypotheses work for her.

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THE ROLE OF THE METACOGNITIVE ASPECT OF SELF-QUESTIONING IN MATH WORD PROBLEMS

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The findings of the Third International Mathematics and Science Study indicate that students in the United States fared above average on algorithmic math problems, but ranked below average on non-routine, multi-step problems, i.e., those that require higher-order thinking. Math word problems fall into the latter categories, because they require the activation of metacognitive thought processes, especially the self-regulatory aspects of self-questioning. The solutions to these problems require more than simple algorithmic procedures. Students must reflect upon the problem, then analyze, strategize and attempt a solution. Word problems model real-life experiences, and offer an insight into whether students truly understand the math involved. Through modeling and then coaching students to self-question, teachers can encourage the activation of necessary metacognitive processes.

A problem-solving schema in the form of *guided self-questioning* offers a tool for students to use when solving word problems. This self-questioning strategy training helps them unveil the meanings of the math word problems, while simultaneously prompting them to more actively monitor their own comprehension. Students are guided to a solution through a series of questions that help them clarify, depict, predict, solve, and re-check the word problem.

When instruction also incorporates group learning and a reciprocal teaching format, students can become active learners, and they can be scaffolded into solving word problems. An increase in teachers' attention to the metacognitive self-questioning aspect of solving word problems might be one solution to the problems U.S. students face in this area of mathematics.

FIRST THOUGHTS AND SOLVING PROBLEMS: WHAT DO WE LEARN FROM PH.D. MATHEMATICIANS

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Although the nature of expert mathematical problem-solving performance has been documented, several issues need further investigation. As a result, 15 Ph.D. mathematicians from research I universities were asked to think aloud while solving four complex mathematics problems and respond to questions regarding their solutions. Information regarding participants' "first thoughts" about a problem and control behavior exhibited on the problem was organized in a contingency table and a chi-square test was employed to analyze the data. Results indicated that: (1) in the majority of problem-solving protocols, participants were unable to identify the deep structure of the problems prior to solving them, and (2) there was no significant relationship between identifying the deep structure of a problem and exhibiting efficient control behavior. In addition, the 60 problem-solving protocols were coded and a cross-case analysis of the protocols was employed. General themes regarding efficient and inefficient control behavior with respect to domain knowledge, problem-solving skills, and beliefs were identified. Further analysis revealed that the use of appropriate heuristics and efficient control behavior enabled participants to solve the problems in spite of their inability to recognize the deep structure of a problem or identify an approach to the problem prior to solving it. This study was, in part, exploratory and more work needs to be done to understand control behavior exhibited by experts in the midst of solving complex mathematics problems.

AN EXPLORATION OF SUCCESSFUL STUDENTS' PROBLEM SOLVING

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This paper reports the results of a study that explored student solving behavior in the area of linear equation solving. The study seeks to develop better theory, from a procedural perspective, of why some students develop rote knowledge of algorithms while others seem to be able to execute procedures "with understanding."

Ten 7th grade students (50% male) from a large public middle school were recruited to participate. Students (as a group) were given a 15-minute "benchmark" lesson on the basic operators of linear equation solving. Following this initial lesson, each student met individually with a researcher, once a week for four weeks, with each session lasting approximately 30 minutes. Students were given a progression of several types of tasks, including a total of 25 equations to be solved (e.g., two-step problem such as $2x+1=11$, progressing to more complicated problems such as $2(x+1)+6(x+3)+5x=4(x+8)+3x$). During each session, students were first asked to verbally describe or plan all steps needed to complete each problem. After planning was completed, students completed each problem in writing.

In a close analysis of the videotaped problem-solving sessions, I identified three dimensions of differences among the successful solvers: ability to plan to the solution, language used to refer to operators in planning, and ability to solve in multiple orderings of steps. These results offer preliminary support for the hypothesis that solvers with rote knowledge of equation solving procedures are less likely to show advanced ability in these three dimensions than solvers who are competent.

PRESERVICE ELEMENTARY TEACHERS' MODELING STRATEGIES WHEN SOLVING NON-STANDARD ADDITION AND SUBTRACTION APPLICATION PROBLEMS INVOLVING ORDINAL NUMBERS

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The purpose of this study is to examine the modeling strategies that preservice elementary teachers use when solving non-standard subtraction and addition application problems that involve ordinal numbers. In such problems, addition or subtraction of the two given numbers produces either 1 more or 1 less than the correct solution. Thirty-four preservice elementary teachers completed a paper-and-pencil questionaire that contained 9 experimental items and 6 buffer items. Four of the nine experimental items can be solved by straightforward addition or subtraction of the two given numbers. The experimental items of the questionaire were adapted from Verschaffel, De Corte, and Vierstraete's (1999) study. Students performed poorly on the five non-standard experimental items. Out of the 170 answers, 142 (83.5%) were incorrect. The number of correct responses varied from 1 (In November 1994, the twenty-fifth annual school party took place. In what year was the school party held for the first time?) to 17 (There was a summer market in our city every summer up through 1990. Since then the summer market was canceled 7 consecutive times. In what year did the summer market restart?) The main factor accounting for the incorrect solutions was students' tendency to subtract or add the two given numbers without realizing the inappropriateness of these actions. In another paper (Contreras, 2000), I continue to examine prospective elementary teachers' difficulties with modeling non-standard arithmetic word problems.

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Verschaffel, L., & De Corte, E. (1997). Teaching realistic mathematical modeling in the elementary school: A teaching experiment with fifth graders. *Journal for Research in Mathematics Education*, 28(5), 577-601.

PRESERVICE ELEMENTARY TEACHERS' USE OF REALISTIC CONSIDERATIONS WHEN SOLVING PROBLEMATIC ARITHMETIC WORD PROBLEMS

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In this paper I examine the extent to which preservice elementary teachers use real-world knowledge and realistic considerations when solving problems in which the application of straightforward arithmetic operations is problematic, if one takes into consideration the reality of the context. A paper-and-pencil test was administered to 34 preservice elementary teachers. The test consisted of 10 experimental items and 5 buffer items. The experimental items were problematic in the sense that their correct solution is obtained by taking into account realistic considerations. The experimental items were based on Verschaffel and De Corte's (1997) study. Preservice teachers' performance was poor. The number of realistic responses varied from two to 25. One of the two lowest numbers of realistic responses was for the problem: At the end of the second year, 50 elementary school children try to obtain their athletics diploma. To get the athletic diploma they have to succeed in two tests: running 400m in less than 2 minutes and jumping 1.5m high. All the children participated in both tests. 9 children failed the running test and 12 failed the jumping test. How many children did not get their diplomas? One of the two highest numbers of realistic responses was for the problem: 1175 supporters must be used to the soccer stadium. Each bus can hold 40 supporters. How many buses are needed? As a summary, 82 (24%) out of 340 responses were correct or involved a realistic comment. Preservice teachers' tendency to ignore contextual realities of word problems is also examined in Contreras (2000).

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INTERACTIONS BETWEEN MATHEMATICS ANXIETY AND COGNITIVE FACTORS IN THE PREDICTION OF PRESERVICE ELEMENTARY TEACHERS' PROBLEM SOLVING PERFORMANCE

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In efforts to enhance the effectiveness of mathematics problem-solving instruction on the part of elementary teachers, mathematics anxiety is often indicated as a mitigating factor. The present study attempts to explore this relationship a bit further, in that it investigates whether the influence of mathematics anxiety on problem-solving performance is stable across cognitive types and levels of mathematics achievement. The prevailing view is perhaps that low mathematics achievers and those with less analytical cognitive styles are more susceptible to the effects of mathematics anxiety. The present study hypothesized that in actuality, although they possess less mathematics anxiety, it is primarily the more analytically and mathematically oriented types that have their problem-solving performance succumb to the negative effects of such anxiety.

For 120 preservice elementary teachers, interactions were explored between mathematics anxiety (Revised Mathematics Anxiety Rating Scale [R-MARS]) and three cognitive factors, field independence (Group Embedded Figures Test [GEFT]), spatial visualization (Differential Aptitude Test [DAT]), and mathematics content and applications knowledge (ACT mathematics subscale). Results were significant for two of the three hypothesized interactions. Mathematics anxiety was more strongly related to problem solving performance among field independent subjects than among field dependent subjects, and was also more strongly related to problem solving among high ACT scorers than among low ACT scorers. The key implication is that the more analytical cognitive types and those strong in mathematics ability are often even more susceptible to negative effects from their individual level of mathematics anxiety than are the more global cognitive types or the mathematically under prepared.

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Volume 2

Discussion Groups, Research Reports,
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RATIO COMPARISON IN TWO DIFFERENT CONTEXTS. A METHODOLOGY FOR THE STUDY OF INTUITIVE STRATEGIES

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Abstract: A methodology, originally constructed for the study of intuitive strategies used in double urn probability tasks, was applied in an experiment involving two ratio comparison tasks: a probability task and a mixture task. This methodology, previously described in Alatorre (1999), consists of two parallel systems: one for classifying the answers given by subjects (strategies), and one for designing the questions to be asked (situations) in order to observe those strategies. The results are analyzed for comparison between both contexts from three viewpoints: strategies, situations and performance. The concentrate task is easier for the adult subjects considered in this study than the probability task. The methodology can be effectively used in both contexts. It can then be asserted that it is useful for the anticipation and study of strategies used in any mathematical problem involving ratio comparison.

Purposes and Theoretical Framework

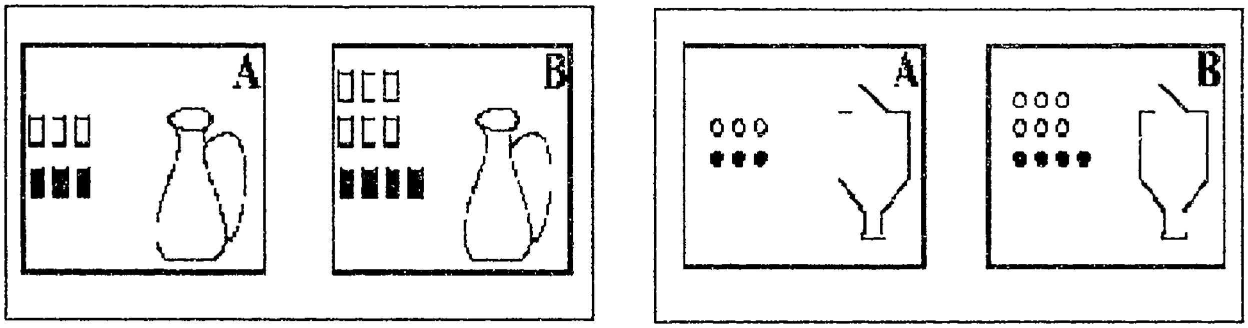
The main aim of this work was to apply a methodology reported earlier (Alatorre, 1999) for the study of strategies used by adults in probability experiments, in another ratio comparison frame, for which Noelting's (1980) classical orange juice experiment was chosen, and to compare the results obtained in both.

As a theoretical framework several studies were considered. Worthy of mention are Piaget's (1951) classical work on the acquisition of the concept of randomness, Fischbein's (1975) concept of intuition, Kahneman and Tversky's (1982) studies on adults' intuitions in probabilistic problem solving, Falk's (1978) work with probability in young children, Noelting's (1980) work on strategies used in proportional reasoning, and Maury's (1986) research with high-school students in a probability task. The contributions of these authors to the framework of the research are more detailed in Alatorre (1999). Also considered for this work is Tourniaire and Pulos' (1985) comment about the lack of research comparing proportional reasoning in different contexts.

Methodology

In this work the same ratio comparison problems were posed to subjects, in two versions (see figure 1). One of them consisted of the probability problem of two open urns with simple extraction, equivalent to the cards task reported in Alatorre

(1999); the other one was adapted to a Mexican context from Noelting's orange juice experiment. The graphical disposition of the figures shown to subjects was similar, and the differences among both versions consisted in the randomness in the first vs. the mixture in the second, and the discreteness of the first vs. the continuousness in the



second. In both versions subjects were asked to justify their choices in written form, and in some cases a clinical interview was conducted afterwards.
Figure 1. In the juice concentrate task glasses containing concentrate and water would be poured inside two jars; it was asked which of the jars had a strongest taste (or if it was the same). In the probability task white and black marbles would be thrown inside two bottles; one bottle would be closed and agitated and one marble would come out of it; it was asked which bottle would the subject choose (or if it was the same) if the desired result was a black marble.

The methodology constructed for the study of strategies used in probability tasks reported in Alatorre (1999) was applied. The following paragraphs summarize it in terms of the application to both tasks considered in the research here reported. It involves two parallel lines: one designed to interpret the answers given by subjects and the other designed to pose the adequate questions in order to observe those answers. For the methodological construction, each problem is defined as an array of two ordered pairs of favorable (f) and unfavorable (u) cases. Also defined are the total cases or glasses n = f+u, the differences d = f–u, and the ratios p = f/n.

The first methodological line contemplates the construction of categories for the interpretation of the answers given by the subject. A strategy was identified in each answer, which could be of a simple or composed form. Simple strategies can be *centrations* or *relations*: a centration is the observation of only the favorable cases, the unfavorable ones or the total ones. In a relation two of those three elements are observed and compared by means of an order relationship, a difference or subtraction, or a proportion. The main centrations and relations are displayed in table 1. Composed strategies include two or more simple strategies joined by a logical operation (of which there are four types); the intervening simple strategies may be dominant or dominated or subtraction, or a proportion. The main centrations and relations are displayed in table 1. Composed strategies include two or more simple strategies joined

Table 1. Strategies

	Coding	Name and description
CENTRATIONS	{CN--}	**negative centration in total cases:** choosing the side where there is a smaller amount of marbles (glasses)
	{CN=}	**equality centration in total cases:** saying "it is the same" because in both sides there is the same amount of marbles (glasses)
	{CF+}	**positive centration in favorable cases:** choosing the side where there is a larger amount of black marbles (concentrate glasses)
	{CF=}	**equality centration in favorable cases:** saying "it is the same" because in both sides there is the same amount of black marbles (concentrate glasses)
	{CU–}	**negative centration in unfavorable cases:** choosing the side where there is a smaller amount of white marbles (water glasses)
	{CU=}	**equality centration in unfavorable cases:** saying "it is the same" because in both sides there is the same amount of white marbles (water glasses)
RELATIONS	{ROlw}	**lose-win order relation:** choosing the side where there are more chances of winning than of losing (more concentrate glasses than water ones), whereas in the other one there are more chances of losing than of winning (more water glasses than concentrate ones)
	{ROld}	**lose-draw order relation:** choosing the side where there are as many chances of winning than of losing (as many concentrate glasses as water ones), whereas in the other one there are more chances of losing than of winning (more water glasses than concentrate ones)
	{ROdw}	**draw-win order relation:** choosing the side where there are more chances of winning than of losing (more concentrate glasses than water ones), whereas in the other one there are as many chances of winning than of losing (as many concentrate glasses as water ones)
	{RO=}	**lose-lose or win-win order relation:** saying "it is the same" because in both sides there are more chances of losing than of winning (more water glasses than concentrate ones), or because in both sides there are more chances of winning than of losing (more concentrate glasses than water ones)
	{RD+}	**largest difference relation:** choosing the side where the difference of black minus white marbles (concentrate minus water glasses) is the largest
	{RD=}	**equal difference relation:** saying "it is the same" because the difference between black and white marbles (concentrate and water glasses) is the same in both sides
	{RP+}	**largest quotient proportionality relations:** choosing the side where the quotient f/n or the quotient f/u is the largest
	{RP=}	**equal quotient proportionality relations:** saying "it is the same" because the quotient f/n or the quotient f/u is the same in both sides

by a logical operation (of which there are four types); the intervening simple strategies may be dominant or dominated.

Strategies are also classified according to their correctness. {RP} strategies are always correct, as are most {RO} strategies, with the exception of {RO=}, which is incorrect. Some centrations may be correct in certain situations, e.g., {CU–} in a possibility-certainty situation (no white marbles or water glasses in one of the two sides). Some composed strategies may also be correct, e.g., {CF+ & CU–} (choosing the side where there is a larger amount of black marbles or concentrate glasses *and* where there is the smaller amount of white marbles or water glasses).

The second methodological line is the construction of the situations: the specific amount of black and white marbles or concentrate and water glasses in each collection for each problem posed. They were built in order to allow the detection of the simple strategies described above. The different possible arrays were grouped in categories determined by several variables. The *combination* is a succession of the possible results of the order relationship between these five elements of sides A and B of the array: n, f, u, d and p. The *location* unites the possible results of both ratios, each in five forms: "surely lose" ($p=0$), "lose" ($0<p<0.5$), "draw" ($p=0.5$), "win" ($0.5<p<1$) and "surely win" ($p=1$). Some locations do not exist in some combinations; Piaget's original 10 categories are broken down in 85 different situations, which allows a finer analysis.

Results and Discussion

The research was done in the National Pedagogical University in Mexico City. 65 first year university students aged 17-28, majoring in Educational Psychology and without any prior instruction in probability participated in the study. Each received two paper and pencil tests with 16 items each, first the concentrate task and then the probability task; in both the arrays were the same and in the same order and graphical disposition (see figure 1). The situations of the 16 arrays were chosen so as to favor the happening of different strategies, as shown in table 2.

The comparison of the results obtained for the probability and the juice concentrate tasks was carried by means of three analysis procedures, centering in strategies, situations and performance.

Strategies

The expected strategies (table 2) did happen in both contexts, but their occurrence (percentage of times a strategy occurs among the situations where it can possibly happen) was different in both tasks. The results are displayed in table 3.

Situations

The percentage of times a given situation was correctly solved in each task (whatever the correct strategy used) was analyzed, and the results compared for both contexts. The main result is that in 10 of the 16 situations considered in the tests,

Table 2. Expected Simple Strategies in the 16 Items

		Strategies Leading to Choosing		
Item	(f1,u1)(f2,u2)	Side A	Side B	It is the same
1	(3,3)(4,6) *	{CN–}, {CU–}, {ROld}, {RD+}, {RP+}	{CF+}	
2	(6,2)(5,5)	{CN–}, {CF+}, {CU–}, {ROdw}, {RD+}, {RP+}		
3	(4,3)(5,4)	{CN–}, {CU–}, {RP+}	{CF+}	{RO=}, {RD=}
4	(2,5)(2,7)	{CN–}, {CU–}, {RD+}, {RP+}		{CF=}, {RO=}
5	(2,2)(3,3)	{CN–}, {CU–}	{CF+}	{RD=}, {RP=}
6	(2,5)(3,7)	{CN–}, {CU–}, {RD+}	{CF+}, {RP+}	{RO=}
7	(2,4)(4,5)	{CN–}, {CU–}	{CF+}, {RD+}, {RP+}	{RO=}
8	(6,4)(7,3)		{CF+}, {CU–}, {RD+}, {RP+}	{CN=}, {RO=}
9	(2,0)(4,2)	{CN–}, {CU–}, {RP+}	{CF+}	{RD=}, {RO=}
10	(6,1)(7,3)	{CN–}, {CU–}, {RD+}, {RP+}	{CF+}	{RO=}
11	(2,3)(4,6)	{CN–}, {CU–}, {RD+}	{CF+}	{RO=}, {RP=}
12	(2,4)(4,4)	{CN–}	{CF+}, {ROld}, {RD+}, {RP+}	{CU=}
13	(1,2)(2,3)	{CN–}, {CU–}	{CF+}, {RP+}	{RO=}, {RD=}
14	(2,1)(6,4)	{CN–}, {CU–}, {RP+}	{CF+}, {RD+}	{RO=}
15	(4,1)(8,2)	{CN–}, {CU–}	{CF+}, {RD+}	{RO=}, {RP=}
16	(2,3)(5,4)	{CN–}, {CU–}	{CF+},{ROlw}, {RD+}, {RP+}	

* Note: Item Nº 1 is the one displayed in figure 1

Table 3. Occurrence of Strategies in Both Tasks

Strategy family	Concentrate task	Probability task
{CN} centrations	3%	12%
{CF} and {CU} centrations	22%	18%
{RO=} (incorrect)	17%	27%
Other order relations (correct)	76%	60%
{RD} relations	15%	12%
{RP} relations	20%	12%

the juice task problems are significantly better solved than the probability ones (items 3,4,7,8,10,11,12,13,14 and 15); only in one situation the reverse is true, and this is the possibility-impossibility situation (item 9).

Subjects' Performance

The consistency showed by subjects between their answers to both tasks was analyzed. The answers given by the same subject to the same array in both contexts were compared: in only 46% of the cases one can speak of a consistency, but these results are similar to the ones obtained in a previous work (Alatorre, 1999) where only a version of the probability task was considered. However, there is significantly less consistency in locations of the "lose-lose" or the "win-win" types.

Concluding Remarks

1) Although the methodology was originally constructed for the study of strategies used in ratio comparison in a discrete probability task, it proved to be useful also in the context of a continuous mixture problem: to select the items' situations, to predict and examine the strategies used by subjects, and to analyze the results. It could be ventured that the same methodology can be applied to any ratio comparison problem, with the possible exception of primitive strategies which could be used by younger subjects and which were not considered in this exposition.

2) The two ratio comparison problems considered in this work are not equivalent: the juice concentrate is an easier context than the probability task. This was apparent in the three analysis procedures. On the first hand, all correct strategies of the {RP} family and the correct order relations occur more in the context of the juice concentrate than in the probability task, whereas all incorrect strategies, including {RO=}, occur more in the probability task. On the second hand, with one exception the situations in which the probability task proved to be more difficult than the concentrate task are

precisely those in which the incorrect relation {RO=} can be applied: locations of the "lose-lose" and the "win-win" types. Finally, it is in those same situations where the subjects show more inconsistency.

3) Although the mixture task was presented as discrete amounts of glasses, there seemed to be no transfer among contexts (few students realized that the questions were equivalent). This could have implications in the design of learning experiences.

4) Some remarks about the {RO} and the {RP} families may be of interest. The incorrect strategy {RO=} is more appealing to subjects in the probability task than in the concentrate task, but this works the other way around for the other order relations, {ROlw}, {ROld} and {ROdw}, which are correct: they are more attractive in the juice concentrate task than in the probability task. As for {RP}, one possible explanation for the larger occurrence of this family in the juice concentrate context could be that liquid is a continuous medium as opposed to the discrete number of marbles in the probability task: it is thus easier to think in terms such as "one and a half glass of water for each glass of concentrate". Further research should then compare the results obtained in discrete mixture problems with the two open urns problem, and continuous mixture problems, such as the juice concentrate, with a continuous probability problem.

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USING TECHNOLOGY TO PROMOTE AND EXAMINE STUDENTS' CONSTRUCTION OF RATIO-AS-MEASURE

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Abstract: This paper examines students' construction of ratio as a measure of speed, in the context of a teaching experiment, and discusses the role of computer software in the ratio-as-measure process.

Introduction and Theoretical Framework

Much of the research on ratios and proportions has focused on numeric strategies, like unit rate and factor of change methods (Cramer, Post, & Currier, 1993). Simon and Blume (1994) argue that the construction of a ratio as the appropriate measure of an attribute (which they call "ratio-as-measure") has been inadequately addressed. In an effort to greater understand the ratio-as-measure process, Lobato and Thanheiser (1999) and Lobato (in preparation) conducted two studies of 17 high school Algebra 1 students. None of the students were able to successfully create a ratio as a way to measure the steepness of a wheelchair ramp, the "proteinness" of nutrition bars, or how fast a mouse travels. The researchers identified two previously unreported sources of difficulty in the ratio-as-measure process: 1) isolating attributes (e.g., steepness was often conflated with attributes like "work required to climb" or "materials required to construct" in the wheelchair ramp situation), and 2) identifying which quantities affect an attribute and in what ways (e.g., whether or not the number of steps taken affects how fast one walks). Furthermore, even those students who understood how changing the height and the length of a ramp affect the steepness of a ramp or how distance and time affect speed, still did not appear to view the relationship between the quantities in these situations as proportional in nature.

Consequently, the teaching experiment described in this paper was prompted by our interest in helping students construct ratios as appropriate measures of attributes. We hypothesized that re-conceiving static situations (like the wheelchair ramp situation) as dynamic, perhaps with the help of computer software, might help students determine when it is sensible to construct a ratio between two quantities. This hypothesis developed, in part, from a disagreement in the literature regarding the distinction between ratio and rate.

For Thompson (1994) and Kaput and West (1994) a rate signifies a whole structure where the two quantities co-vary dynamically in a constant ratio, and a ratio is a static instance of a rate. In contrast, Confrey and Smith (1995) reject ratio as an instance of a relationship between quantities, claiming instead that ratios are constructed "by objectifying and naming that which is the same across proportions," i.e., to construct a ratio, one needs to first identify what is the same across more than one instance (p. 74).

On the one hand, it is clear that one can conceive of a static multiplicative comparison or ratio without the ratio continuing across more than one instance, e.g., a 35-year old father is 5 times as old as his 7-year-old son, but this ratio does not hold as both individuals age. On the other hand, Confrey and Smith's idea of perceiving sameness across multiple instances as instrumental to the construction of ratio may be more applicable to a second way of constructing ratio, namely the creation of a composition of two composite numbers (Lamon, 1995), which we call a "two-number." For example, when buying 3 lbs of candy with $5, one can form a composite "three-five" number (which differs from the multiplicative comparison of 3 as 3/5 of 5). However, it is difficult to assess whether a student has constructed a "two-number" as opposed to thinking of "two numbers" (e.g., conceiving of $5 for *every* 3 lbs of candy, not simply $5 *and* 3 lbs) unless one sees evidence of the student iterating or partitioning the new composite number. Furthermore, we hypothesize that constructing a "two-number" ratio might be linked to conceiving the *feasibility* of the extension of that ratio.

A critical reader might argue that one cannot construct a family of ratios without first forming a single ratio. However, computer environments might allow students to generate a "family of values" with a given attribute (e.g., same speed) by guessing and checking, without mentally constructing a ratio. Subsequent discussion of these values might support the creation of ratios. The use of families of values representing an attribute that is visually part of a dynamic software environment permits a middle ground position between "ratio as static" and "rate as dynamic."

Purpose. This study will examine the ratio-as-measure process while students engage in computer activities that are hypothesized to be propitious for the construction of ratio-as-measure.

Method of Inquiry

A teaching experiment was conducted during the summer in a university computer lab for about 30 hours over two weeks. Nine average-performing students (i.e., those who earned Bs or Cs) were recruited from 8^{th}-10^{th} grade math classes. The authors team-taught the course. All sessions were videotaped. Two "family of related values" tasks were used (see Figure 1). Prior to each of these tasks, students worked on isolating an attribute (i.e., motion or steepness) and identifying quantities that affected the attribute (e.g., time and distance). Due to space limitations, only results from the "same speed" task will be reported. Students used the SimCalc Mathworlds software (see Figure 2) to enter a time and distance for two animated characters and then ran the computer simulation to see if the characters walked at the same speed.

1."Same Speed" Task. The clown travels 10 cm in 4 seconds. Find as many different ways as you can to make the second character, the frog, travel at the same speed as the clown by entering a distance (other then 10 cm) and a time (other than 4 seconds) for the frog.

2. "Same Steepness" Task. Make as many ramps as your can (using Geometer's Sketchpad) that have the same steepness as ramp with a height of 2 cm and a length of 3 cm.

Figure 1. Two "Family of Related Values" Tasks

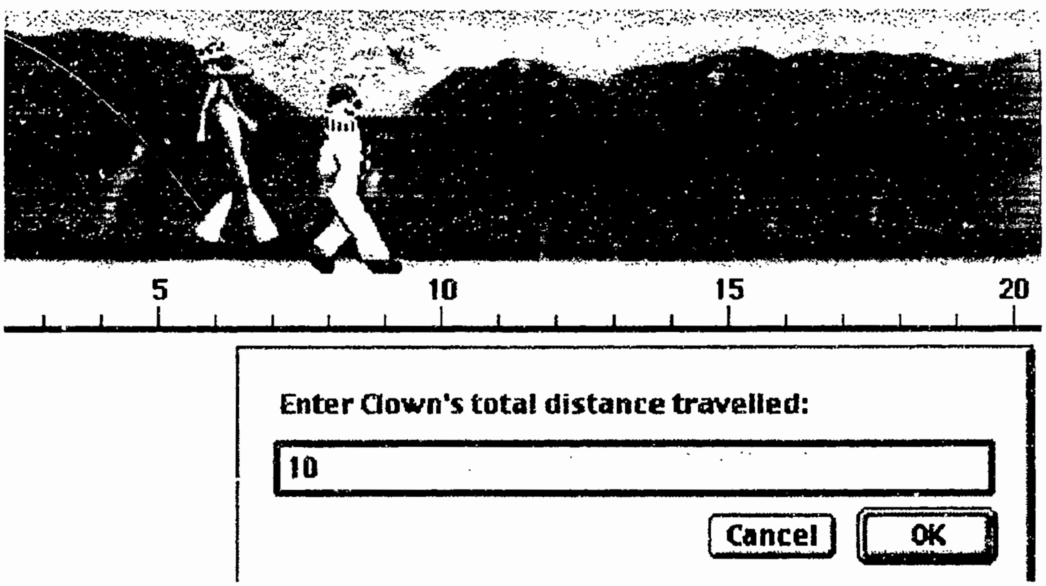

Figure 2. Screen capture from SimCalc Mathworlds

Results and Discussion

In the individual interviews conducted during the first day of the teaching experiment, no subject provided evidence of the creation of a ratio as a measure of the steepness of a wheelchair ramp, and all but two students showed serious proportional reasoning problems. In this section we present an example of the construction of a ratio as the measure of speed during a class discussion of the "same speed" activity. Three findings follow:

1. Students' numeric strategies may project a misleading image of proportional reasoning ability The "same speed" task was difficult for all students, as evidenced, in part, by the incorrect entries that each student recorded in his/her time and distance chart while working individually at a computer. Most students relied on a "guess and check" strategy (e.g., entering 15 cm and 8 sec and then adjusting the time until arriving at 15 cm in 6 sec). Four students found numeric patterns: three used factor of change strategies like doubling the distance and time, and one student used a unit rate strategy. However, three limitations to the numeric strategies were found. First, when the first author walked around to each computer station and questioned students, no one was able to explain *why* their numeric patterns worked.

Second, during the hour-long class discussion that followed the computer work, numeric explanations were not understood by other students. For example, Brad shared his "solution" of 90 cm in 35 seconds (as a "same speed value" as 10 cm in 4 sec). Terry disagreed, arguing that "10 goes into 90 nine times and 4 goes into 35 eight times and a little bit left over." But the other students said that they couldn't follow Terry's explanation.

Third, students' illustrations indicated a lack of connection between the numeric strategies and the quantities involved in the situation. For example, the teacher asked the students to draw a picture to explain why the doubling strategy worked, i.e., why walking 20 cm in 8 sec was the same speed as walking 10 cm in 4 sec. Terry represented the distances of the two characters without attempting to show that the frog's distance was double the clown's distance (see Figure 3). In fact, Terry asked whether he was working on the 20 cm or the 90 cm problem *after* he had represented the frog's distance with a line. He explained that for both characters to have the same speed, they would need to walk 10 cm in 4 sec at the same time. Neither his verbal explanation nor his visual representation included frog's distance and time after the initial 10 cm in 4 seconds. He relied on calculations, stating that "if you want frog's distance to be 20, then you have to multiply 10 x 2 to get 20; since you multiplied 10 by 2, you also need to multiply 4 (the time it took the clown) by 2 to get 8," without explaining why time and distance need to be doubled or how multiplying by two could be represented in his drawing The next student to go to the board, Jim, offered a numeric explanation almost identical to Terry's. The discussion appeared to stall, when suddenly Brad had a new idea that he seemed anxious to share.

2. Ratio-as-measure construction appears to involve an understanding of covariation and relationships between quantities in the situation. A breakthrough occurred when Brad appeared to construct a "two-number" ratio. Brad explained that doubling works "because the clown is walking the same distance; it's just that he's walking the distance twice... he's walking it once, going li, li, li, li, li, li, [Brad retraced the line Terry drew, up to 10 cm and drew a vertical mark], all the

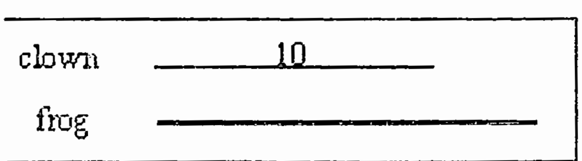

Figure 3. Terry's first diagram

way here. Four seconds. OK. He's going to walk it again. Another four seconds, li, li, li, li, li, li, li, li. Another ten centimeters in four seconds. He's done."

Brad's explanation involved three elements lacking in both Terry's and Jim's work, suggesting a greater understanding of the covariation between distance and time. First, his picture illustrates what happens to the frog character after the initial 10 cm in 4 seconds by noting that the frog walks another 10 cm in 4 seconds. This observation may be closely tied to what permits the construction of ratio in this walking situation, namely an understanding that if one walks at a constant pace for x cm in y sec and then repeats the exact action, then one will not go faster or slower but will walk at the same speed for both journeys, as well as for the combined journey. Second, Terry seems to pick one quantity, namely 20 cm and then produces the other related quantity of 8 sec. In contrast, Brad's work is consistent with a more sophisticated image of distance and time varying simultaneously, or at least 10 cm in 4 sec "chunks." Finally, Brad appears to coordinate the quantities of time and distance by using sounds to represent time while he retraces a line segment to represent distance, an important component of covariation.

3. A "two-number" ratio can be iterated and partitioned to form other ratios that represent the same speed. The construction of the "10 cm in 4 sec" unit was adopted by other students and combined with iterating and partitioning to create additional "same speed" values. For example, Denise added another 10 cm in 4 sec section onto Brad's drawing, concluding that 30 cm in 12 sec also works. Later on, Terry explained why walking 2.5 cm in 1 cm was the same speed as walking 10 cm in 4 sec. He partitioned the "10 cm in 4 sec" unit into four segments, formed a new "2.5 cm in 1 sec" segment (as indicated by the section in Figure 4 that Terry circled), and then iterated the "2.5 cm and 1 sec" unit four times to end up with 10 cm and 4 seconds. He stated that "it would be like he's walking one fourth of the 10 and 4; it's like one fourth of each thing" [meaning 1/4 of the 10 cm and 1/4 of the 4 seconds]. Terry's idea can form the basis for a very powerful generalization that if one travels x cm in y sec at a constant speed, then if one goes a/b of this journey, one will travel (a/b)•x cm in (a/b)•y sec. Although more work remains for students to fully develop an equivalence class of ratios and to see distance and time as flowing quantities, this approach suggests a promising avenue for further research.

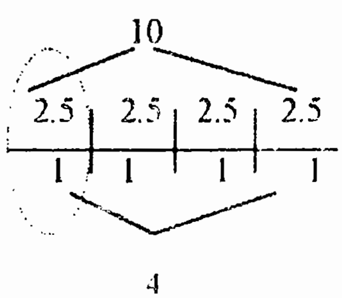

Figure 4. Terry's second diagram

Reflections on the Role of Technology

The computer environment, combined with a whole-class discussion, helped support the construction of ratio as a measure of speed. By allowing students to test same speed values, the software seemed to support an image of other distance and time pairs that would produce the same speed as the initial 10 cm in 4 sec value. By asking students to explain why their values worked, a condition in which ratio-as-measure could be constructed was supported. However, it is unlikely that students would have progressed beyond guess-and-check strategies or numeric patterns without the class discussion.

The software also afforded three unintended actions or conceptions. First, students developed a practice of checking to see whether the characters walked at the same speed by running the simulation and then looking to see if the characters walked "neck-and-neck" for only the duration of the shorter journey. This might explain why students like Jim and Terry did not initially account for the time and distance of the character that kept walking after the first character stopped. Second, throughout the discussion, despite the teacher's efforts to focus on explanations and reasoning as ways to settle mathematical disagreements, the students seemed to consistently view the computer simulation as a highest authority. Finally, the students were unable to visually distinguish which character was going faster if the ratio is close, e.g., a character traveling 11 cm in 4.5 sec *appears* to be going the same speed as a character traveling 10 cm in 4 sec, though no one raised this issue as worrisome; students simply accepted these values as correct.

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THE DEVELOPMENT OF THE PART-WHOLE AND MEASURE SUBCONSTRUCTS OF FOUR SIXTH-GRADERS DURING AN STRUCTIONAL UNIT

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Abstract: Four sixth-grade students were interviewed prior to and following a six-week unit that focused on fraction concepts. The measure subconstruct of the rational number construct was tested using a number line model and an eighth-inch ruler, and the part-whole subconstruct was tested using rectangular regions. The four students' ability to model and identify proper fractions, improper fractions, mixed numerals, and equivalent fractions with these three models are described. The students performed differently with the three models in both interviews.

The measure subconstruct is one of the four subconstructs included in Kieren's (1993) analysis of the rational number construct. Behr, Post, Lesh and Silver (1983) have included a distinct part-whole subconstruct. Kieren (1993) and Behr et al (1983) agree that for students to understand rational numbers, they must understand all of the interpretations of rational numbers and the interrelationships between them. Kouba, Zawojewski and Strutchens (1997) and Larson (1987) report that students are better able to associate fractions with parts of geometric regions than to locate fractions on number lines. Larson (1987) and Bright et al. (1988) have shown that locating fractions on a number line is very difficult for both elementary and middle school students. In past research, rulers and number lines have been considered to be essentially the same model, with the result that few rational number research studies have compared students' identification of fractional parts of linear units to locating fractions on a number line. The purpose of this research is to explore sixth-graders' development of a measure subconstruct of rational numbers as represented by two different exemplars of the measure subconstruct: the number line and the scale on an eighth-inch ruler. Another aim is to contrast their part-whole subconstruct with their measure subconstruct.

Design of the Study

The data presented here is part of a study that focused on one sixth-grade teacher, Anne, teaching a six-week unit on rational numbers at the beginning of the school year. The content addressed in 18 of the class periods was for students to understand proper fractions, mixed numerals, improper fractions, and equivalent fractions using area models (seven classes), set models (seven classes), and the number line (four classes). Students used models or drawings to represent fractions and equivalencies in groups of four students. They also participated in large group discussions led by the teacher. No instruction was given on fractional parts of an inch. The researcher observed and took notes during 15 of the 23 class periods in which the unit was taught. These classes were also audio-taped.

The focus in this paper is on the understanding of four students in Anne's class of the measure and part-whole subconstructs for rational numbers prior to and following instruction. The students were selected by the teacher to represent the various levels of student achievement in her classroom. The students responded to interview tasks prior to and following instruction, all of the interviews were audio-taped and later transcribed. Part of each interview included tasks where the students related proper fractions, improper fractions, mixed numerals and equivalent fractions to area models, number lines, and an eighth-inch ruler. Two types of tasks were used to test the area and number line models: 1) the students were shown a shaded region (or regions) or an indicated point on a number line scaled from 0 to 3 and asked to identify the fraction and 2) the students were given the fraction and asked to generate the model by shading one or more nonpartitioned rectangular regions or showing the location of the fraction on a number line scaled from 0 to 3 on which only the points associated with the whole numbers 0 to 3 were marked. All of the ruler tasks involved the accurate measurement of strips of paper using an eighth-inch ruler.

Results

The results are presented for each sixth-grade student in order to show the development of knowledge in each student for each model.

Laura

The teacher identified Laura as a high achieving student.

Area Tasks. Laura modeled proper fractions with an area model and wrote proper fractions for area models in both interviews. In the pre-interview she was unable to identify 1 1/4 regions and when asked for a fraction she gave 5/8 rather than 5/4. When asked to model 3/2 she divided 3 by 2 and then shaded in 1 1/2. She could not give a second fraction to describe the region that was 6/8 shaded, but after a period of trial and error could shade 2/3 of a rectangle with 6 equivalent parts. In the post-interview she correctly responded to all area tasks similar to the ones on the pre-interview.

Number Line Tasks. In both interviews when given a number line with only whole numbers marked, she was able to correctly mark where proper fractions and mixed numerals would be located. In both interviews when faced with an improper fraction such as 5/4 she immediately divided and then correctly marked 1 1/4. In the pre-interview when asked to indicate the number associated with a point on the number line, Laura counted points rather than segments so that in each case she was

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at focused on one sixth-grade teacher, nbers at the beginning of the school eriods was for students to understand tions, and equivalent fractions using classes), and the number line (four

the post interview she not only correctly line but indicated two or three names for 2/3, 1 1/2, 9/6, a lot of names.

ıra correctly measured all strips and wrote ımerals, but s' ꞓ never ind.cated the unit,) indicate the unit after successfully writing . 1/2 was centimeters. For 2 3/8, in both t that she didn't know what the little ones give equivalent fractions, she did so without

:rage achieving student.

ıhn could successfully do all area model tasks ınd mixed numerals. He could not do the two ıons. Given a rectangle with 6 of 8 equivalent ; shaded but not 3/4. Similarly when given could not shade in 2/3 of the rectangle. In ıll proper fractions, improper fractions, mixed s.

ıterview John was unable to correctly respond post-interview, he correctly completed all tasks erals, and improper fractions.

v, John correctly indicated that the length of a :stimated to the nearest half-inch the other two n the post interview he gave accurate measures tions, e.g., 1 2/8 inches was also 1 1/4 inches.

ı average achieving student.

v Anita could give a proper fraction, and a mixed s. When asked to describe 1 1/4 regions with not indicate that 6/8 of a region could also be shade in regions to show fractions, she could not could not show 2/5 of a rectangle was because she ılent parts. In the post-interview, Anita correctly and 7/4 shaded, and she successfully modeled the :d to have problems in partitioning a sheet of paper ıe needed five equal parts but could not do it. She ectangle was shaded and then reduced to 4/5 when could not relate the fraction 4/5 to the model. Also, she could not shade 3/4 of a rectangle partitioned into 12 equivalent parts.

Number Line Tasks. In the pre-interview, Anita was unable to correctly respond to any of the number line items. In the post-interview, she correctly completed all tasks involving proper fractions, mixed numerals and improper fractions.

Ruler Tasks. In the pre-interview when measuring strips, all lengths were incorrect. When the length was a mixed numeral, she always had the correct whole number but the incorrect fraction. In the post-interview she rounded each measure to the nearest inch, e.g., a strip 2 3/8 inches long was "about 2 inches." In both interviews she correctly labeled all units as inches.

Jim

The teacher identified Jim as a low achieving student.

Area Tasks. Jim was unable to respond correctly to any of the area tasks in the pre-interview. In the post-interview he correctly identified 8/10 of a shaded region but could not give the equivalent fraction 4/5. He correctly identified 1 3/4 regions that were shaded but said a related fraction was 7/8. When asked to model fractions the only one he could do was 3/5.

Number Line Tasks. In the pre-interview, Jim was unable to successfully do any of the number line tasks. In the post-interview, he correctly completed all tasks involving proper fractions, mixed numerals and improper fractions.

Ruler Tasks. In the pre-interview when measuring with an eighth-inch ruler, Jim called all fractional parts of an inch "quarters." He labeled only the whole number part of measures as inches, e.g., the length 2 3/8 inches was called "2 inches and 3 quarters." In the post-interview, he eventually identified the length of two strips in terms of mixed numerals but only labeled the whole number part as inches. So a length of 2 3/8 inches was first called "2 inches and 3 millimeters" and after a discussion called "2 inches and 3/8." When asked what you could call the 3/8, he said he didn't know.

Discussion

All of the four students in the study increased their knowledge and understanding of rational numbers as a result of the instructional unit. Each student's profile of knowledge prior to and following instruction was specific to that student. The four students responded differently to the three models for fractions. The scale on a ruler and the number line were not treated the same by the students. The area in which they made the most progress was in associating fractions with points on the number line. Prior to instruction, three of the students could not associate proper fractions, improper fractions and mixed numerals with points on the number line. The fourth student, Laura, could indicate a correct point given the fraction but could not correctly indicate the fraction for a given point. About a week after the completion of the unit all four students could correctly do all number line tasks.

In the post-interview, Laura and John could identify and model proper fractions, improper fractions, mixed numerals and equivalent fractions using rectangular regions. Anita and Jim were still not proficient with all the modeling and identifying tasks with area models. Jim could not model an improper fraction with an area model yet he successfully modeled an improper fraction on the number line.

Measuring to the nearest eighth-inch with a ruler was not addressed in instruction as the teacher ran out of time and decided that she needed to begin a unit on a new mathematical topic. So a question to examine is: Did the three students, John, Jim, and Anita, who could not correctly identify fractional parts of an inch prior to instruction, improve in this area? Following the unit, Anita simply estimated to the nearest inch, John and Jim read the scale on the ruler and identified the correct mixed numerals. John also indicated the length in terms of the correct number and units, inches. Jim was confused about the unit as it related to the whole number and fractional parts of mixed numerals. Laura also showed this same confusion in both interviews. Laura and Jim indicated that the whole number was inches but at times that the fraction was centimeters (Laura) or millimeters (Jim). After questioning in the post-interview, both students said that they knew the whole number was inches, but didn't know what the "little ones" were. One explanation for this confusion could be that throughout elementary textbook series, inches and centimeters are taught one or two days apart. There is seldom emphasis on units such as, inches and centimeters, being part of different systems of measurement and why. Another aspect of measurement that was apparent from the interviews is the role of estimation in measuring. John and Anita both estimated lengths to a specific unit. In the pre-interview, John systematically measured to the nearest half-inch; in the post-interview, Anita estimated to the nearest inch. The students never made estimates of this type when trying to identify the number to correspond with points on the number line or when associating fractions with an area model.

This research indicates that the use of rational numbers in measurement situations needs to be taught to show the connection between the mathematical topics of measurement and rational numbers. This is an example of the need for mathematical ideas to be interconnected as described in the new NCTM Standards (National Council of Teachers of Mathematics, 2000). Students would benefit from comparing various measurement scales to related number lines and on discussing measurement units when fractional parts are involved.

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PROSPECTIVE TEACHERS' PART-WHOLE DIVISION PROBLEM SOLUTION STRATEGIES

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Abstract: The purpose of this paper is to report results from a piece of a larger research study (Stone, 1999) that investigated the factors that contribute to prospective teachers' abilities to solve part-whole division problems. The factors investigated were fraction type, that is, how prospective teachers' success was related to the type of fraction (unit fraction or common fraction) presented in part-whole division problems and whether or not a visual image of the data presented in the problem contributes to successful solution strategies. Prospective teachers' part-whole division problem solution strategies are the focus of this paper. Prospective teachers use a combination of proportional reasoning, concept of unit, and fraction as operator strategies when solving part-whole division problems.

Introduction and Rationale

Prior research suggests that preservice and inservice elementary through intermediate level teachers misunderstand many fraction concepts and procedures (Ball, 1990; Post, Harel, Behr, & Lesh, 1991). Ball (1990) found that preservice teachers' knowledge of fraction division is particularly weak. Post and his colleagues (1991) found that many practicing teachers had difficulty ordering fractions, finding equivalent fractions, performing fraction operations, and solving proportional reasoning problems.

Prospective teachers' abilities to solve part-whole multiplication (or fraction as operator) problems has been investigated by Behr, Khoury, Harel, Post and Lesh (1997). Behr and his colleagues found that prospective teachers' abilities to solve part-whole multiplication problems were strong. These researchers paid special attention to strategies used by the subjects and found that they were able to effectively use two related operator subconstruct strategies; duplicator-partition reducer, and stretcher-shrinker.

Part-whole division problems are problems that define part of a quantity and ask for the whole of that quantity (Greer, 1992). For example, "Two-thirds of the children order chocolate milk with their lunch. Six children order chocolate milk. How many children order lunch?" is a part-whole division problem. Post and his colleagues (1991) found that intermediate level teachers were not as successful when solving part-whole division problems. These researchers found that a mean percentage of 69% of their subjects could successfully solve part-whole division problems. Approximately 87% of the teachers were successful at solving an item in which a unit fraction was presented. This percentage dropped to 78% when common fractions were presented in the problems. Post's study focused only on achievement. Teachers' understanding of part-whole division relationships nor the strategies they used when solving part-whole division were not presented. The research reported in this paper presents solution strategies and errors prospective teachers make when solving part-whole division story problems.

Methodology

The sample for this study consisted of 66 students enrolled in a graduate elementary mathematics methods course at a private urban college in the Boston metropolitan area. During November of 1998, data were gathered for the study using two forms of a written instrument called "The Part-Whole Test," an instrument consisting of twelve part-whole division problems. One-half of the problems presented employed a combination of prose and illustration. For example, the problem "One-twelfth of the cans of soda in the refrigerator is diet soda. There are three cans of diet soda in the refrigerator. How many cans of soda are in the refrigerator?" was accompanied by an illustration of three cans of diet soda. One-third of the fractions presented in the problems were unit fractions; two-thirds were common fractions.

Data Analysis

Subjects' solutions to problems from the Part-Whole Test were scored for correctness and coded by solution strategy. Solution strategies included concept of unit and the thinking of a fraction as an operator or a ratio (Behr et al., 1997; Vergnaud, 1983). Subjects who used the fraction presented in the problem in an equation or divided the whole number part by the fraction were classified as using an operator strategy. Subjects who set up a proportion were classified as using a ratio strategy. Subjects were classified as using a concept of unit strategy when the number that corresponds to the unit fractional amount was used to find the whole. Consider the following problem:

> Thirty-five gallons of paint is of the paint needed to paint a building. How many gallons of paint are needed to paint the building?

Subjects were classified as using an operator strategy if they set up the equation or performed the operation . Subjects were classified as using a ratio strategy if they set up the proportion .

When employing a concept of unit strategy, subjects used strictly multiplicative or a combination of additive and multiplicative reasoning when solving part-whole division problems. Using multiplicative reasoning, a subject stated that of the paint needed is 7 gallons. Since is one whole, 7 times 7 is 49 gallons; the amount of paint needed to paint the building. Using additive reasoning as well as multiplicative reasoning, a subject stated that since of the paint is 35 gallons, more of the paint is needed since is equivalent to one whole, or the amount needed to paint the whole building. Since of the paint is 7 gallons; of the paint is 7 + 7, or 14 gallons. Thus, the total amount of paint needed is 35 + 14, or 49 gallons.

Subjects' errors were coded as multiplication or addition errors. Multiplication errors occurred when a subject directly multiplied the fraction and the whole number presented in the problem and addition errors occurred when subjects added the fraction and the whole number presented in the problem.

Results

Strategies

The number of subjects and the percentages of subjects using particular combinations of strategies are listed in Table 1.

Table 1. Strategies Subjects Used to Solve Part-Whole Division Problems (Number, Percentage of Subjects)

One Method	Two Methods	Three Methods
Operator (17; 25.7%) Ratio (9; 13.6%) Concept of Unit (16; 24.2%)	Operator/Ratio (6; 9.1%) Operator/Concept of Unit (9; 13.6%) Ratio/Concept of Unit (4; 6.1%)	Operator/Ratio/Concept of Unit (5; 7.6%)

Results from Table 1 show that 36% of the subjects did not consistently use the same strategy to solve part-whole division problems. Strategies used by each subject varied from problem to problem. The concept of unit strategy was used by approximately 50% of the subjects. Thirty-six percent of the subjects treated the part-whole division problems like proportional reasoning-missing value problems.

In addition, many of the solution methods involved the use of visual images. In 36% of the successful solution strategies, subjects drew pictures, charts, or diagrams. Subjects appeared to use images to model the data and the relationships present in the problems. Only 12% of subjects used the drawings provided on the Part-Whole Test to model successful solution strategies.

Errors

Of the errors subjects made on part-whole division problems, 25% were multiplicative errors. Twenty (20) percent of the errors subjects made were additive. Non-completion of solution procedures, non-attempts, or calculation errors while applying correct solution procedures accounted for 55% of errors subjects made when solving part-whole division problems.

Conclusions

As can be seen in Table 1, it may be inferred that prospective teachers that participated in this study have flexible conceptions of fraction (the conception of fraction as a multiplicative operator as well as a sum) as proposed by Kieren (1993). Having a flexible conception of fraction involves understanding fraction as both a multiplicative and an additive entity. Sowder and her colleagues (1998) suggest that teachers need this broad, flexible knowledge of fractions in order to help students solve problems that have a "multiplicative structure," or belong to the multiplicative conceptual field.

It is surprising that additive solution methods such as the concept of unit strategy were used to solve many of the part-whole division problems. The persistence of additive solution methods when solving problems that belong to multiplicative structures has been investigated by many researchers (e.g., Kaput & West, 1994). Resnick and Singer (1993) believe that the preference of additive solution strategies finds its genesis in the early development in school of additive properties of number in contrast to the later development of multiplicative properties. Subjects in this study may have known more about the additive than the multiplicative composition of numbers when relating fractional parts to wholes. Additional research is needed on the reasons why some prospective teachers prefer the use of additive solution strategies.

When making errors to part-whole division problems, subjects may have chosen multiplication to solve part-whole division problems because they had difficulty conceiving the whole, or answer to the problem as a fraction of the part (Vergnaud, 1983). The findings in this study lend credence to Vergnaud's (1983) premise that most of the errors made by subjects making multiplicative errors involve multiplying the whole number by the fraction presented in the problem.

Subjects may have based their choice of solution method on the misconceptions that "multiplication makes bigger" and "division makes smaller" (Fischbein, Deri, Sainati Nello, Sciolis Marino, 1985). These are misconceptions based on the ideas that multiplication is "repeated addition" and division is "repeated subtraction." Additional research is needed on how the misconceptions of "multiplication makes bigger" and "division makes smaller" are related to prospective teachers' abilities to solve part-whole division problems.

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INVESTIGATING MATHEMATICAL LEARNING PROBLEMS FROM THE STANDPOINT OF REPRESENTATION THEORY

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Students who have difficulty in learning mathematics usually attribute their plight to certain situations which suggest that mathematical learning difficulties may, in essence, be problems of *representation*. Teachers are often accused of being unable to represent mathematical concepts and processes in forms which students can easily assimilate, and students sometimes cite their own inability to transform adequate representations into meaningful knowledge. Therefore, to minimize leaning problems and enhance students' opportunity to learn mathematics, it seems prudent that *all* mathematics teachers should acquire rich repertoires of connected representation knowledge that they can use flexibly and effectively, especially when teaching topics like fractions, which many students find hard to learn.

The purpose of this study was to investigate the ability of a group of thirty pre-service elementary school teachers to represent fraction concepts, and translate given representations into different forms. The study was done partly to inform decisions about the kind and amount of representation experience that these prospective teachers would need.

Lesh's (Lesh, Post & Behr, 1987) 5-mode model of representation provided the framework for this study. The participants did two kinds of tasks requiring performance of representational acts that are required to do fraction problems in two current textbook series.

The results of the study indicate that participants were good at representing fraction concepts involving single regions. However, many of them were weak in making translations within and between different modes of representation. The presentation will give full detail of the study.

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LEARNING TO TEACH DIVISION OF FRACTIONS MEANINGFULLY

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We describe how two teachers learned to teach division of fractions for conceptual understanding in terms that would be meaningful for students, and discuss their success and limitations in learning to teach a hard topic on their own in relation to their knowledge of mathematics and pedagogical content knowledge.

Each teacher was teaching a combined fourth-fifth grade. The teachers had a double role, as the subjects of study and also as active participants in the research. Sources of data are a baseline interview, a questionnaire, the notes and materials developed by the teachers, and a final interview. Data for the learning process are the materials written by the teachers.

Both teachers had a solid understanding of the fraction concepts they teach in 4th and 5th grade. They had explicit conceptual understandings of procedures such as finding equivalent fractions, adding fractions, finding common denominator, adding, subtracting and multiplying fractions. They knew how to use concrete representations to do division of fractions in terms of measurement or sharing, without having to use rules. Both were familiar with the "invert the second fraction and multiply" procedure, but did not know why this rule works. The two teachers share some of the characteristics, but not all, of teachers with profound understanding of fundamental mathematics. Their knowledge of elementary mathematics is still not a unified body of knowledge. In the case of division of fractions their repertoire of situations and story problems was limited to the measurement interpretation of division. They are acquainted with basic ideas such as the use of reciprocals, the identity nature of 1, units of different kinds, and the inverse nature of multiplication and division. However, these basic ideas do not seem to guide their mathematical activity.

The connection from previous understanding of division of fractions in terms of measurement or sharing to gaining understanding of the reason of why the "invert and multiply" method works occurred through three different approaches. First, they used common denominators. For fractions with the same denominator, they noticed that the denominator will not appear as part of the answer. Later, they understood how finding common denominators and dividing relates to the original fractions, and they made the connection to the algorithm of multiplying by the reciprocal explicit. Second, they explored the inverse relation of multiplication and division, first with whole numbers and then with fractions. Third they used examples to illustrate that when you divide by a number, it is the same as multiplying by the inverse.

Using their previous understanding of division of fractions with concrete representations and the measurement interpretation of division, they learned to explain the "multiply by the reciprocal" algorithm, via fractions with common denominators. However, they did not make completely explicit the role of reciprocals and their connection to the inverse relation between multiplication and division. Both their knowledge of mathematics and concerns about teaching influenced teachers' approaches and explained their success and limitations. It is possible for teachers like these to learn to teach for conceptual understanding. However, it is striking how long the process was.

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A TEACHING MODEL: A QUALITATIVE-TYPE ANALYSIS IN PROBLEM SOLVING

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In the research being carried out with undergraduate mathematics students at the *Escuela Normal Superior de México,* aspects linked to a qualitative analysis in the solving of non-numerical and school problems related to ratio, proportion, and proportional variation are being examined. The theoretical framework is described in López & Figueras (1999), and a report is also there on the findings of the first two stages. For the third stage of this study, a methodology is employed that combines a clinical-type inquiry, and experimentation through a simulation of didactic sequences. The latter requires the construction of a teaching model, which is put to the test by means of individual interviews. The model's didactic sequences are structured in the form of interview protocols which are organized as a double-entry matrix. Three main axes are considered in its design: a) the findings from the analysis of the teaching models underlying the textbooks used at the aforementioned educational institution; b) the construction of networks of non-numerical and school problems dealing with mixtures, mobile objects, proportional sharing, work/time, water pumps, etc.; and c) the various representations used by students to solve these kinds of problems, such as qualitative and numerical variation tables, drawings and schemes. The direction of the lines in the matrix is based on a series of problems pertaining to the same class; the first of these is a non-numerical one, and the last one is a school problem. The direction of the columns in the matrix defines routes to link various contexts, and also to graduate the complexity of the different classes which integrate the matrix to the didactic sequence.

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BUILDING ON INFORMAL KNOWLEDGE: COMPARING PART-WHOLE TO RATIO AS FOUNDATIONS FOR RATIONAL NUMBER UNDERSTANDING

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This study examined how two groups of fourth grade students constructed formal understandings of rational numbers, from informal knowledge when they received instruction that encouraged them to emphasize either a ratio perspective or a part-whole perspective of the rational number domain in their problem solving. Analyses of the eight lesson curricular intervention explored differences in each group's ability to work in pairs to complete analogous problems using the same rational numbers. Every lesson was organized into three sections which investigated if students could (a) recognize the rational number relation in contextualized word problems and illustrations (b) correctly apply the rational number relation to quantities without context and (c) correctly link formal notations to representations of quantity.

Comparisons of the two groups' performance over time indicated that students who received the ratio curricula displayed a higher level of covariance between the three lesson components indicating greater integration and less compartmentalizaion of their knowledge. A posttest sorting task, which asked students to group index cards showing components of a formal domain analysis of rational numbers into categories that made sense to them, indicated that students were more likely to make pairings that were consistent with the rational number perspective to which they were assigned. Further, ratio students more frequently paired representations of quantity with their corresponding formal notations demonstrating greater transfer. The ratio group also displayed greater transfer on a novel problem solving task that was part of the posttest assessment.

STUDY OF SOLVING STRATEGIES AND PROPOSAL FOR THE TEACHING OF RATIO AND PROPORTION

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This document reports an investigation in process, which approaches the topics of ratio and proportion, whose importance and current interest have been shown in studies done in several countries and during various decades. The research problem consists of identifying the strategies employed by students who are finishing their primary education as they solve problems involving ratio and simple and direct proportions, in order to identify qualitative and quantitative components linked to these topics and their diverse modes of representation. These strategies will be the basis on which a proposal for teaching these topics will be designed and applied. The strategies used by the subjects are important because they permit the recovery of certain passages of thought, and they exhibit a wide diversity of resources as well as different modes of representation, all of which are fundamental for the design and application of the teaching proposal to be carried out.

The theoretical background consists of work by researchers from different countries in different epochs, beginning with the theory of Piaget (1972, 1978) and culminating with the recent tasks Lesh (in press) has designed to teach ratio and proportion. The authors mentioned in the theoretical framework are mostly constructivists like Hart (1988), and the rest have basic coincidences with constructivism. These have attempted to create consistency in the conjunction of didactics with mathematical reflection on ratio and proportion.

The research instruments establish a relationship with the objectives. In terms of the sequence of the application of these instruments, first direct observations have been done in the classroom; following this are indirect observation of the activities carried out by the students in their notebooks. After that, a questionnaire has been given to the students on their form of work when dealing with ratio and proportion problems to look more closely at the students. In order to confront the situations lived by the student in this framework, a series of sessions have been conducted by the researcher and have been observed by people involved in the field of mathematics education, as she gives students ratio and proportion problems to be solved by the students. Once the work sessions were concluded, a second questionnaire was given to the students in order, in this case, to check the progress the students might have made. To look more deeply into this progress and into the strategies that can be identified through the teaching process put into practice by this researcher, "semi-structured" interviews will be used.

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FRACTION INSTRUCTION THAT FOSTERS GENERAL MULTIPLICATIVE REASONING

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Research indicates that students are relatively unsuccessful on tasks that require multiplicative reasoning, including those involving operations on fractions or proportional reasoning. The present study was undertaken to see if an instructional program on fraction operations--one that emphasized multiplicative relations--would transfer to unpracticed multiplicative problems, including proportionality.

The 51 participants in this study consisted of volunteers from three sixth-grade classes. Two control classes received traditional textbook-based fraction instruction, and the experimental class received instruction developed and piloted by the first author. The experimental instruction had the following features: (1) partitioning experiences with objects and extensive use of drawings that represented the partitioning; (2) partitioning experience that dealt with noninteger factors and complex ratios; (3) experience with ratio tables; and (4) instruction for multiplying a mixed number by a whole number based on a distributive model.

Participants were pre- and posttested using four dependent measures. Three measures were group-administered, paper-and-pencil tasks: computation of simple fractions, equivalent fractions with noninteger factors, and missing-factor sentences. The fourth measure was an individually administered oral test of contextualized proportionality problems.

ANCOVA analyses indicated that the experimental class did equally well as the control classes on fraction computation ($p=.412$). There was a significant difference among the other measures (all ps=.000), with the experimental class outscoring the control classes. Analyses of interview protocols indicated that the experimental participants were able to apply multiplicative reasoning to unfamiliar proportion problems.

INVESTIGATING PRESERVICE TEACHERS' SOLUTION STRATEGIES OF EQUIVALENCE ITEMS

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Much of the research in rational number has centered around the idea of the fundamental subconstructs that underlie rational number. Equivalence has been proposed as a concept that unifies these subconstructs. Research has shown that preservice teachers lack fractions content knowledge. However, it has been shown that it is productive to build upon what they do know- teachers have similar intuitive knowledge of fractions concepts and operations as do children. In an effort to develop preservice teachers' knowledge of equivalence, it is productive to assess their current understanding.

This study examines the solution strategies of 270 preservice teachers for solving three equivalence problems. These three problems are variants of Kieren's Fraction Kit items. Not surprisingly, the majority of the solution strategies were procedural. The poster session will share these procedural solutions as well as preservice teachers' solutions that demonstrate conceptual understanding.

MATH, MUSIC AND THE PART-WHOLE CONCEPT: LINKING STEFFE AND COBB'S MODEL OF NUMBER SENSE DEVELOPMENT TO CURRENT MUSIC RESEARCH

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This paper proposes that the "gate" concepts between the three stages of Steffe and Cobb's (1988) 3-Stage model of number understanding can be experienced in music rhythm. The "gate" between Stage 1 and Stage 2, the concept of the equal distance of 1 between neighboring whole numbers, is the same as the equal distance between the steady melody beats of simple children's songs. The "gate" between Stage 2 and Stage 3, the part-whole concept (the coexistence of all the partitions of a quantity and the whole quantity), is the same as the coexistence of the melody beats (the parts) grouped into the unchanging meter or "big beat" (the whole) of a song. Rauscher et al. (1997) recent work appears to show that the "equal distance" concept can be assimilated through rhythmic, sequential finger training at a music keyboard. A classroom-based research project is underway to determine whether non-keyboard music instruction targeted to the part-whole concept will significantly improve 2nd- and 3rd-graders' grasp of that concept.

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PRE-SERVICE TEACHERS' PROPORTIONAL REASONING STRATEGIES

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Proportional reasoning is basic to the middle school curriculum and is an issue in current reform efforts. There have been few research articles dealing with pre-service teacher's proportional reasoning abilities. This study examines solution strategies used by 71 pre-service elementary teachers, 5 male and 66 female enrolled in mathematics methods courses, to solve a proportional reasoning problem. They were asked to solve an application problem involving variation between slow and fast speeds of videotapes.

A variety of strategies were found in their solutions including the traditional proportion algorithm, diagrams, and arithmetic reasoning. An analysis of these showed that approximately 82% of the students used a strategy other than the proportion algorithm. Approximately 58% of the pre-service elementary teachers gave a correct solution and of those giving an incorrect solution the majority displayed incorrect conceptual reasoning, whether they used proportions or basic arithmetic. For example, 8.5% of the students responded with the same incorrect answer, ignoring the relationship between tape speeds. None of the students with incorrect solutions attempted to use a diagram and only 4% of those with correct solutions attempted to use a diagram. The majority if not all students had completed a mathematics course that dealt with proportions and ratios; however, almost half of these students did not respond with a correct solution. The results of this study suggest a change in the curriculum from a mainly procedural presentation to a more conceptual presentation of proportions is needed to enhance pre-service teachers' preparation.

Research Methods

EPISTEMOLOGICAL ANALYSES OF MATHEMATICAL IDEAS: A RESEARCH METHODOLOGY[1]

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Abstract: This paper discusses a methodology for researching the question, "What does it mean to understand x, and how might people develop such an understanding?". We call this methodology *epistemological analysis* (EA), and we view it as appropriate for creating didactic models of mathematical understanding. We give an overview of aspects of EA by outlining its links to a constructivist epistemology and related research tools and analytic methods.

Introduction

A large part of mathematics education research investigates the question, "What does it mean to understand x, and how might people develop such an understanding?" This paper gives an overview of a methodology for researching that question. We call this methodology *epistemological analysis* (EA), and we view it not only as appropriate for creating models of mathematical understanding, but also as supporting the design of instructional strategies intended to support the development of such understanding.

The aim of EA in any setting is to produce an *epistemic subject*, a framework for creating models of individual persons' mathematical thinking and knowing that explain why they behave as they do, both individually and in interaction with others. We borrow this phrase from Piaget, who described the epistemic subject as "the cognitive core that is common to all subjects at the same level" (Piaget, 1970, p. 120, Quoted in Glasersfeld, 1995, p. 72). However, we use this term slightly differently than did Piaget. To Piaget, something like "the conserving child," the child who conserves one-one correspondences even after spatial transformations, would have been an example of an epistemic subject. It is a description of a "common" child at a particular stage of development. We extend the notion of epistemic subject to reflect Piaget's aim to develop theoretical frameworks that describe knowledge in a particular area as being organizations of mental operations any of which could be at one of several levels of development within any particular person. When accomplished, the framework turns out to be useful in describing the actual composition of any one person's knowledge when positing where in a developmental sequence each mental operation is. When the framework is particularized for a particular person or group of persons, you have a model of that person's, or group of persons', ways and means of operating within the contexts being investigated.

An Overview of Epistemological Analysis

Characterizations of an epistemic subject must have very special natures because of the traditions from which EA draws its notions of knowledge and knowing (radical constructivism and thereby operational analysis and genetic epistemology) and because of the tools it employs to research them (cognitive task analysis, constructivist teaching experiment, and conceptual analysis). Developing these links completely is beyond the space limitations of this paper. Instead, we offer a brief overview of EA's intellectual heritage as a way to describe its aspects.

Task analysis (TA) focused on performance capabilities a person must have in order to perform a specific mathematical task (Gagné, 1977; Thorndike, 1922). Where TA focused on behavior, cognitive TA viewed behavior as an expression of goal-directed cognitive operations (Klahr & Siegler, 1978; Newell & Simon, 1972). Cognitive TA became very influential in some areas of mathematics education research (Brown & VanLehn, 1981; Davis, Jockusch, & McKnight, 1978; Greeno, 1978; Resnick, 1975; Schoenfeld, 1989). It provided not only models of cognitive processing, but a method for describing the knowledge that schooling should help students develop (Bransford, Nitsch, & Franks, 1977). One shortcoming of TA, both behavioral and cognitive, was that people using it often confounded correct performance and understanding, thereby ignoring issues of individuals' motives and conceptions of the contexts in which their behavior was observed (Cobb, 1987; Steier, 1991). One positive aspect of cognitive TA was that it produced methods for modeling complex organizations of cognitive processes and it developed criteria for assessing a model's viability — would a model "run" if operationalized and would the model be expressed in behavior roughly consistent with observations?[2]

Piaget's *genetic epistemology* (Piaget, 1971, 1977) was an interdisciplinary approach to understanding human intellectual, moral, and social development. Genetic epistemology made deep connections among biology, philosophy, psychology, and logic, and used both structural and functional approaches to understand what might constitute human knowledge. The ideas that knowing is always a dynamic process, always involving mental operations, and that mental operations are always part of a larger system of operating, were central to Piaget's work. On the other hand, while Piaget described mental structures as being organizations of mental operations, he emphasized the structural aspect of knowledge over the operational aspect of knowing. But he always grounded his notion of knowledge firmly in the idea that knowledge is not a copy of reality, but rather is built from and within a person's total neural activity. Working from a tradition distinctly different from American psychology and independently of the Piagetian school, Silvio Ceccato outlined what he called *tecnica operativa*, or operational technique, in which one must "consider any mental content (percepts, images, concepts, thoughts, words, etc.) as a result of operations" (Cecatto, 1947 as cited in Bettoni, 1998). That is, one must describe *consapevolezza operativa*, or conceptual operations (translated literally as "operating knowledge")[3] in order to answer the question "which mental operations do we perform in order to conceive a situation in the way we conceive it?" (Bettoni, 1998).

Glasersfeld combined aspects of Ceccato's operational analysis and Piaget's genetic epistemology to devise a way to talk about reasoning and communicating as imagistic processes and of knowledge as an emergent aspect of them (Glasersfeld, 1978). This produced an analytic method, that he called *conceptual analysis* (CA), whose aim was to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do. CA resembles TA in its attention to detail and its focus on creating models of thinking and reasoning that ostensibly explain why people behave as they do, both individually and in interaction with others, but it differs from TA in the nature of the operations it posits. Where TA describes reasoning as production systems, systems of "if-then" propositions, CA describes reasoning as grounded in imagery and meaning, where meaning and imagery are described in terms of conceptual ope. ations someone might employ to have them. CA's emphasis on conceptual operations derived from Ceccato's operational analysis. Glasersfeld's key contribution was to tie Ceccato's operational analysis to Piaget's genetic epistemology. The combination produced a way of talking about a person's conceptions of specific situations with an eye toward placing them in a context of larger systems of knowing.

In the same way that Glasersfeld accommodated operational analysis to the constraints of genetic epistemology's conception of knowledge, a number of researchers accommodated the Soviet-style teaching experiment, with its roots in Vygotsky's socioculturalism, to conceptual analysis and genetic epistemology (Steffe, 1991; Steffe & Thompson, 2000b; Thompson, 1979). The constructivist teaching experiment's primary purpose has always been to have second-order models of students' understandings built by observers who are reflectively aware of interactions with them. In a sense, the constructivist teaching experiment produces the interactions between students and knowledgeable persons that are at the root of Vygotsky's notion of cultural transmission (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997) and allows investigations of the emergence of intersubjectivity (Steffe & Thompson, 2000a) among participants in instructional settings.

EA is not a new methodology as much as it is a synthesis of those already described. Its purpose is to create models of knowing that attempt to characterize students' understandings in specific settings. At the same time EA also allows us to place those understandings within large systems of knowing in a way that can be useful across students who differ greatly in their conceptions and capabilities. EA's greatest overlap in analytic method is with Glasersfeld's CA and it owes its greatest intellectual debt thereto, but there are differences between them. First, EA is used to model what might be called systems of ideas, like systems of ideas composing concepts of numeration systems, functions and rate of change, or even larger systems like those expressed in quantitative reasoning. The added complexity of modeling large knowledge systems leads to issues not addressed by CA, issues such as what might constitute principled, coherent, and general conceptions of sophisticated ideas and how immature conceptions of sophisticated ideas might evolve into them. Second, by its use of constructivist teaching experiments, epistemological analysis gives explicit attention to instructional contexts in students' construction of these conceptual systems. These two points support our assertion that models generated by EA have potential didactic value; because they are created in tandem with the design of instructional environments and activities intended to support the development of students' ways of reasoning and thinking, these models capture aspects of how various levels of understanding might evolve into sophisticated, advanced, coherent understanding as a function of students' engagement in instructional activities. They therefore necessarily address the nature of instructional strategies someone might take to foster that development.

Notes

1. Research reported in this paper was supported by National Science Foundation Grant No. REC-9811879. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of NSF.
2. A profound difference between EA and TA that we cannot develop here is that TA is rooted in encodingism while EA is not (Bickhard, 1991a, 1991b).
3. Quotations translated by M. Bettoni. Phrases translated by GO Translations, http://translator.go.com/.

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CROSSING TO A NEW MILLENNIUM BY CROSSING THE BOUNDARIES OF OUR RESEARCH PARADIGMS

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We have crossed to a new era; some say a new millennium. What will the new millennium bring? How will this affect each of us - as researchers and as teachers of mathematics? This discussion group topic is about research boundaries and what crossing those boundaries has meant for our research and teaching.

With this new millennium comes excitement that follows the unknown and the anxiety of possible change due to this unknown. How do we travel to the unknown with respect to our research and teaching? This presentation will initiate conversations about contributions to the field of mathematics education through various research paradigms. By researching through paradigms that may not be as frequently used, our research holds the potential of helping us find answers that have not yet been heard. We could obtain a wealth of knowledge to use in the furtherance of mathematics education and contribution to mathematics teaching.

The session will begin with a brief overview of research from three scholars in the field of mathematics education whose studies are from varying paradigms: positivism, interpretivism, and critical theory. However, this is not simply a presentation of three different research studies. Instead, the focus of this session is on the process of research and how our alliances to a particular paradigm may limit our findings and how this affects our teaching and further research.

Following a brief overview of each of the scholar's work, a facilitator will introduce methods on how to use the process of identifying and developing a paradigm and the impact of that paradigm on a strong research design to appropriately answer research questions. The discussion group will focus on questions that the presenting scholars had before and after their research and how this connects to all participants' research. How do we (researchers) decide to work from within a particular paradigm? What influences our decision to stay within this paradigm? How can our work bridge to other studies within alternate paradigms to enrich all findings? How does our research affect our teaching? The implications of examining questions within mathematics education, new and old, from a variety of paradigms will lead to novel answers that emerge from our research. This is what makes this work useful and most relevant to us as mathematics educators and mathematics teacher educators. By raising the questions of paradigms and the possibility of crossing the boundaries of the paradigms, we will begin to learn from a variety of perspectives to more broadly influence mathematics education research and mathematics teaching.

Our intent is not to research every issue from within each paradigm. Rather, our aim is to raise questions about why we choose the methodologies and paradigms that we do for a particular study and what more we might learn upon investigation grounded in particular paradigms. In addition, how can this decision influence our contributions to the field of mathematics education and mathematics teaching? Our goal is to increase the influence of our work to be of greatest value for the teachers of mathematics. Our premise is that we can do this by investigating research from multiple paradigms. This will then lead to our discovery of alternative methodologies within mathematics education in general, and mathematics classrooms, specifically.

Social and Cultural Factors

CONTEXTUALIZING THE ACTOR AND THE PLAY: A CASE STUDY OF THIRD GRADE STUDENTS DEVELOPING UNDERSTANDINGS OF PLACE VALUE

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Abstract: Recent trends in mathematics education research reflect the view that learning is an inherently social and cultural activity. This shift can be seen in the rise of studies conducted using social constructivist and situated perspectives (cf., Beach, 1995; Boaler, 2000; Cobb, Stephan, McClain, & Gravemeijer, in press; Greeno, 1997; Lave, 1997; Saxe, Gearhart, & Seltzer, 1999). On one hand, this focus enables researchers to situate learning in forms of social co-participation (Lave & Wenger, 1991). On the other hand, it can be interpreted as minimizing the importance of individual students' reasoning (Anderson, Reder, Greeno, & Simon, 2000; Cobb, et al., in press). The objective of this paper is to develop an analysis that is based on the situated perspective, but also seeks to place individual actors' progress within the context of the play.

Theoretical framework

Identifying Practices Using the Situated Perspective

One advantage of the situative perspective is that it can be used to document differences between social practices that are established among groups in different settings. For example, in their seminal work, Lave & Wenger (1991) described different apprentice relationships found in four contrasting communities of practice: midwives, tailors, quartermasters, and alcoholics. Their goal was to compare different practices and participation structures among the groups, but not to document differences occurring *within* any one community. In situative studies focusing on schooling, Boaler (2000) and Beach (1995) each conducted studies that contrasted forms of arithmetical reasoning developed by two groups of students who participated in different learning practices in different school settings. In each of these studies, the researchers took participation in two broad forms of collective activity as the primary unit of analysis.

The approach taken in this analysis builds on the idea of identifying practices within a community, but also attempts to account for individual learning that occurs among members *within* a group. Following the situated perspective, learning is defined in terms of the process by which students actively reorganize their ways of participating in classroom practices. However, the major premise of this report is that the relation between individual students' learning and the evolution of communal practices is viewed as reflexive (Cobb, 1994). Students contribute to the evolution of the classroom practices that constitute the immediate social situation of their mathematical development as they participate in the evolving practices of the culture. The methodological implication of this reflexivity is that research must document the emergence of specific mathematical practices and then identify different ways in which students participate in them.

Psychological Constructs for Assessing Individual Participation

The question of how to assess the ways in which individuals contribute to the classroom practices involves developing psychological constructs to describe students' mathematical understandings. In the current study, the students were learning concepts of place value. The constructs that were developed to assess students' understandings were based on the work of Cobb & Steffe (1983) who identified several ways in which students solve tasks involving collections. For the purposes of this study, the list was narrowed to three main distinctions: (1) creating numerical composites, (2) creating composite units, and (3) using part-whole reasoning (Cobb & Steffe, 1983). These may best be described by considering different ways that students solve the task of figuring out how many groups of ten crayons could be obtained from a bag containing 360 crayons. Students who are creating numerical composites and composite units often begin by counting by tens from 10 to 360 and keeping track using their fingers. Upon reaching 360, students who have created numerical composites state that the answer is 360 rather than 36. When asked to explain, they cannot reconcile how their fingers relate to countable groups of ten. In contrast, students who have created composite units are able to explain that the answer is 36 because they have counted 36 groups of ten items. Students who partition the collection into a group of 300 (containing 30 composite units of ten) and 60 (containing 6 composite units of ten) are reasoning in terms of part-whole relations.

Setting, Data, and Methodology

Setting and Data Collection

The setting for the teaching experiment was a third-grade classroom in a public suburban school in the southern United States. The class consisted of 23 students (14 boys, 9 girls). Seven of the students (30%) were members of ethic minority groups (four African American boys, two African American girls, and one Indian girl). The data corpus for the entire study consists of: videotapes from each lesson, field notes, copies of all students' written work, and protocol sheets from pre- and post- individual interviews conducted with all of the students just before and after the teaching experiment.

Instructional Sequence

The goal of the Candy Factory sequence (developed by Cobb, Yackel, & Wood, 1992) was to support students' construction of increasingly efficient but not necessarily standard algorithms that reflected their developing understanding of place value numeration. To this end, the students first pretended to be workers in a candy factory where ten candies were packed in a roll, and ten rolls in a box. In subsequent instructional activities, the students developed a variety of ways of symbolizing the process of combining and separating quantities of boxes, rolls, and pieces. These included drawing pictures, making tally marks, and writing numerals.

Methodology

The method used to collect and analyze the data involved two steps. First, researchers documented the overall mathematical practices that emerged in the classroom over the course of the 9-week teaching experiment (See Bowers, Cobb, & McClain, 1999 for a full description of the five mathematical practices). The critical issue in identifying a mathematical practice was that it was not defined by conducting a modal trend analysis to see when a majority of students began to change their ways of acting. Instead, practices were distinguished by seeing how contributions made during whole-class discussions served as catalysts for shifting the types of explanations and justifications that eventually became taken-as-shared (cf., Cobb, Yackel, & Wood, 1992).

The second step of the analysis method involved using the psychological constructs described above to analyze how individual students were thinking about place value as they participated in the identified practices. To this end, four target students were chosen by the research team (on suggestion from the teacher) at the outset of the teaching experiment. These four students, who together to represented a variety of different skill levels, were interviewed and carefully observed by the research team over the course of the teaching experiment.

Results

The Mathematical Practices

Table 1 includes a brief description of each identified practice, the accompanying ways of justifying that were established during each practice, and the precipitating event that initiated the practice (i.e., the particularly significant observation or suggestion made by a student that contributed to the shift in ways that students explained and justified their reasoning).

Individual Ways of Participating in the Mathematical Practices

The constructs for gauging students' participation in classroom practices can be arranged in order of increasing levels of sophistication from creating numerical

Table 1. Mathematical practices that emerged over the course of the teaching experiment.

Practice	Methods of Explanation and Justification	Precipitating Event
I. Enumerating a given collection	Students enumerated pre-drawn collections of boxes, rolls, and pieces. Justifications involved counting or iterating units.	(first practice, no precipitating event)
II. Creating different arrangements	Students drew different ways that a given number of candies could be packed. Justification involved counting by hundreds, tens, and ones.	Students described the conjecture that collections of candies that were "all packed up" (in canonical form) were easiest to enumerate.
III. Transforming quantities	Students continued to draw different ways, but changed their ways of justifying from packing to interpreting the results of transformations preserve quantity.	Students noted that it was possible to anticipate new collections without actually drawing them and that the number of pieces would have to be the same.
IV. Adding and subtracting	Students drew pictures to show addition and subtraction activities. Ways of justifying involved counting their drawings, and then recording their answers on an inventory form.	Students used their insights about imagining unpacking to facilitate the process of sending out orders of candy.
V. Partitioning and recombining collections	Students were again enacting addition and subtraction tasks, but anticipated unpacking rather than	Students established different (but not equally efficient) ways of notating transformations.

composites to composite units to reasoning in terms of part-whole relations. Based on this ordering, the chart in Figure 1 indicates that, although they shifted at different times and to differing degrees, each of the four target students did develop increasingly sophisticated ways of thinking about place value through the course of the teaching experiment.

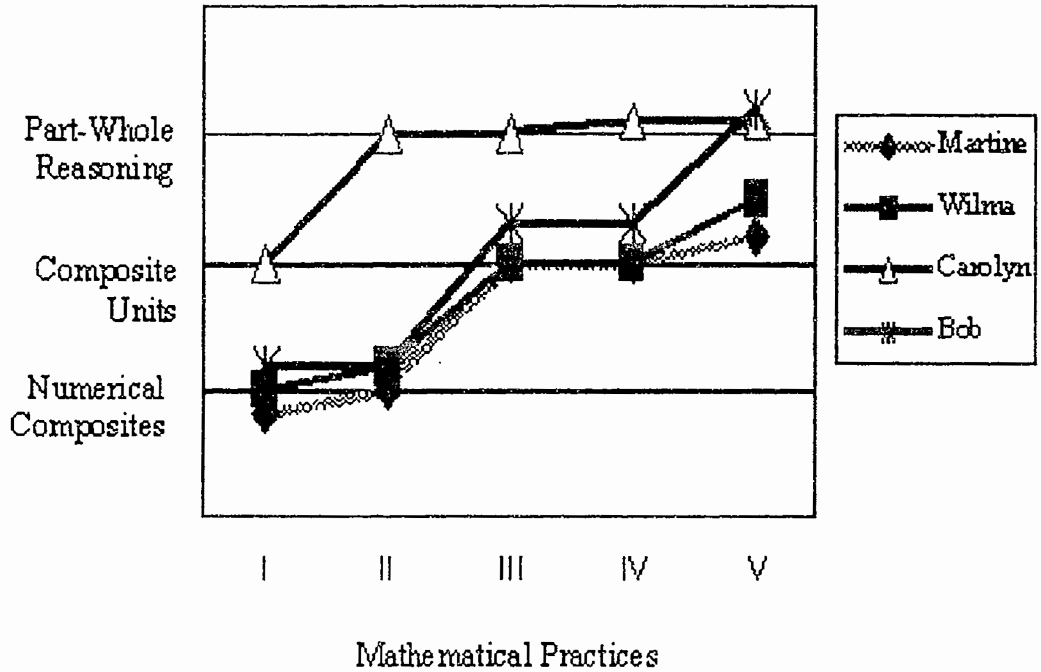

Figure 1. Ways in which target students participated in mathematical practices.

Martine participated in mathematical practices I and II by interpreting the rolls as numerical composites. For example, during one whole-class discussion in which students were asked to draw different ways that 532 pieces could be partially packed, Martine drew 50 rolls on the board but described them to the class as "500 rolls." Although his comment may have meant 500 pieces *arranged* in rolls (as Carolyn, another target student suggested), this response, along with other written work, indicates his construction of numerical composites rather than composite units. As he participated in practice III, he began to interpret the rolls and boxes in terms of composite units. This shift was initiated when he began to reorganize his activity of creating different arrangements of candies by unpacking previous ones rather than starting from scratch each time.

Wilma also made a shift from interpreting the rolls and boxes as numerical composites to composite units as she participated in the third math practice. For example, when drawing different ways to show how a collection could be partially packaged, she initially justified each of her new solutions by counting each item (by hundreds, tens, or ones accordingly) whenever she unpacked a new box or roll. As she participated in practice III, she curtailed her counting when she realized that she did not have to recount because the total quantity of candies did not change.

Carolyn interpreted the boxes and rolls in terms of composite units early on as she participated in practice I. As she participated in the activity of drawing combinations of 532 candies described earlier, her drawings on the blackboard and accompanying justifications indicated that she was reasoning in terms of part-whole relations. That is, unlike Martine and Wilma, Carolyn participated in class discussions by explaining that she imagined a series of unpacked boxes before drawing any of them. This indicates that she interpreted collections as both one whole quantity and as composite units that could be partitioned into constituent parts simultaneously.

Bob was a vocal participant who was actively involved in negotiating two of the shifts in the mathematical practices that occurred over the course of the teaching experiment. As Figure 2 indicates, Bob participated in practices I and II by interpreting the boxes, rolls, and pieces in terms of numerical composites. The precipitating event that contributed to his construction of composite units was the observation he made in class that he could anticipate the results of multiple packings without actually carrying out the transformations. This observation, and a similar one regarding addition and subtraction, led the class to agree on a notation system to keep track of the results of each imagined transformation and served as a catalyst for negotiating the emergence of new practices.

Discussion and Conclusions

The overall objective of this analysis has been to document how individual students contributed to, and were consequently affected by, the emergence of mathematical practices. This approach builds on a situative view of learning as participation in social practices, but also attempts to account for how shifts in the mathematical practices influenced and were influenced by individual students' participation. One advantage to this approach is that it provides teachers and curriculum developers with a clearer picture of what activities and discussions initiated shifts in the ways that students engaged in various tasks. The analyses of the practices indicated that two major transitions emerged. The first occurred during the transition from practices II to III, when the students reorganized their activity from drawing pictures to create different configurations to realizing that packing and unpacking a collection did not change the total quantity of candies. This is not to say that *all* students suddenly shifted their ways of acting at the same time or in the same way. Instead, the precipitating events involved noticing that transformations preserved quantity and that a collection that was "all packed up" could be most easily counted. Taken together, these conversations supported students' efforts to make their work more efficient. This can be seen, for example, in the target students' efforts to shift from justifying their answers by circling ten rolls or pieces and describing packing to implicitly agreeing that each new arrangement was the result of transforming a previous arrangement.

In closing, the underlying assumption of this analysis was based on the view that learning is a social process in which students reorganize the ways that they participate in activities. This report has revealed that the two critical shifts in mathematical practices were each precipitated by conversations that involved collective anticipation such that students began describing what *might* happen rather than what *did* happen and using sophisticated justifications that became taken-as-shared. These observations provide researchers with insights regarding the types of arguments that raise the level at which students justify their answers, and also provide a methodology for placing each actor's contributions within the overall play.

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PARENTS AS LEARNERS OF MATHEMATICS: A DIFFERENT LOOK AT PARENTAL INVOLVEMENT[1]

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Abstract: This paper explores the nature of parental involvement in mathematics education and presents a model that involves parents sharing their beliefs, ideas, and concerns about their children's mathematics education. This paper draws on qualitative data from two research projects on parental involvement in mathematics education in minority and working-class communities. Our model focuses on engaging parents in doing mathematics and talking about mathematics education, thereby, increasing parents' confidence, understanding and information in these areas. These three aspects are likely to play a role towards these parents becoming advocates for quality mathematics education for all children.

This paper focuses on the nature of parental involvement in mathematics within two research projects located in minority and working-class communities. In our work we seek more relevant forms of parental participation and engagement. By these we mean ways that are respectful of parents and communities, yet challenging to them in that these mathematical experiences will authentically promote their roles as informal educators in the home.

The parent component in one of the projects is comprised of seminar-like mathematics workshops in which a small group of mothers explores mathematics to enhance their knowledge base. The parent component of the second project is much larger in scale and it seeks to promote the mathematics leadership of parents within a school district. This project has several components, but a main one is the engagement of parents as learners of mathematics. In this paper we focus on this aspect –engaging parents as learners of mathematics—as a key component of this model of parental involvement. In this proposed model, parents are viewed as intellectual resources (Civil, 1999) who want to contribute as such, both at school and at home. We seek to develop a different view of involvement, one that moves away from what we have seen in our local context with working-class, minority mothers for whom "involvement" in schools has often been limited to activities such as monitoring the cafeteria, sharpening pencils for upcoming standardized tests, or working on bulletin board displays. Our experience in a vast number of classrooms in our area has shown us that parents do enjoy doing mathematics and that they want to gain a better understanding of current issues such as reform in mathematics education or high stakes tests to be better informed to help their children. Becoming better informed, acquiring more confidence as learners of mathematics, and developing a stronger understanding of mathematics are three areas that we are looking into in our research as we seek to redefine parental involvement through a model that sees parents as advocates for a quality mathematics education for all children.

Theoretical Framework

We draw on three bodies of literature that explicitly reject a deficit view of families and parents that portrays them as somehow lacking, as "the problem." One of these bodies is the literature on parental involvement, in particular that which critically examines issues of power and perceptions of parents (especially minority and working-class parents) and that moves away from stereotypes and deficit views by giving parents a voice in the process (Henry, 1996; Vincent, 1996). Our work reflects an awareness that, as Weissglass and Becerra (n.d.) write, "often classes or programs for parents are one-way transmissions of information and materials from school to the parents. Rarely do parents, particularly those from groups underrepresented in mathematics, have an opportunity for their beliefs, ideas, and concerns to be heard." (p. 2).

As Peressini (1998) points out, mathematics educators often stress the importance of involving the parents in the road to reforming school mathematics. However, as he observes "in both the larger arena of general educational reform and the subset of school mathematics reform, these calls for parental and community involvement have been at an abstract level and have not been closely examined" (p. 557).

A second body of literature is that of research on adult education, especially that grounded on critical pedagogy (Benn, 1997; Frankenstein & Powell, 1994; Harris, 1991; Knijnik, 1996). These researchers stress the importance of there being different forms of mathematics, while pushing us to reflect on what counts as mathematical knowledge, and suggesting pedagogical approaches that can be very powerful in working with adults that have often been marginalized. Directly related to these pedagogical approaches is the third body of literature that informs our work, that based on a socio-cultural approach to education (Forman, 1996; Moll, 1992; Rogoff, 1994; van Oers, 1996). This approach takes for granted the dynamic nature of social and cultural experiences, while rejecting those commonly-held notions about minority and working-class families (e.g., "they are not interested in education"; "these parents cannot help their children"). More specifically, while appreciating the difference in experiences that children and their families may share, we believe that these same differences when viewed as positive can propel and enrich educational practice and learning experiences in more productive and meaningful ways.

Method

Our modes of inquiry are qualitatively based involving the use of observations and field notes, researchers' journals, interviews, evaluation protocols, and collaborative writing exercises with participating parents. Many of the mathematics sessions with the parents are being videotaped, thus allowing us to examine their participation in mathematical discussions, their questions, and the overall nature of the discourse.

We are interested in understanding the essence of parents' experiences in the education of their children, with the school and teachers, and in their experience of participating as adult learners of mathematics. To do so, we ascribe to a phenomenological methodology (Van Manen, 1990) which relies heavily on participants' contributions to the experience, then strives to triangulate the data through multiple experiences and sources of data. The lived experience of each parent is considered significant. Developing an appreciation of this lived experience constitutes the basis of our research.

A Glimpse at our Model

In our proposed model for parental involvement in mathematics, we challenge the common approach that looks at superficial aspects such as number of parents at a given event, and focus instead on the process of involvement itself that centers on parents engaging in doing and talking about mathematics. As one mother writes:

> Engaging with our children in the mathematics, allows us to see them differently, that it is not sufficient to attend to all their other needs, but that it is important that we as parents have these types of discussions. We also realize that though we may not have a certificate in hand, we are also teachers. [Collaborative writing]

Our model focuses on three aspects that have become salient in our analysis of data in both projects: a) participants' growing confidence as learners of mathematics; b) participants becoming better informed about mathematics education; c) participants' growing understanding of mathematics. These three aspects go hand in hand as the following brief vignettes show.

The Case of Alicia

Alicia has never missed a single session in the program and has commented on how the homework she gets has become a family affair as she engages all her family members in it. Although hesitant to present her mathematical thinking to the group of parents in the sessions, towards the end of the academic year, she promptly volunteered to assist with a demonstration of the Calculator Based Ranger (CBR) in a fifth grade classroom. Alicia then talked to her daughter's teacher (her daughter is in another fifth grade classroom at that same school) and asked her to allow her daughter to attend these special sessions on the CBR. Alicia was disappointed by the kind of mathematical experiences her daughter was receiving and decided to act on it, even if only through these special sessions. The teacher agreed and sent in an additional ten students to this other classroom so that they could also benefit from the experiences with the graphing calculator and the CBR. Alicia came in ready to help and so did three other mothers from the project and another who volunteers at the school. Alicia's confidence as a mathematics learner and her interest and enjoyment in the subject seem to be helping her become more assertive in advocating for her daughter's mathematics education. She has since been hired as a parent assistant for that school.

The Case of Emilia

Emilia, a mother participating in the other project, writes about her growing awareness with respect to the impact of mathematics on future life choices for her children:

> As the mother that I am of family of four children, on a personal level I worry about their academic development – mathematics is one of the most difficult subjects for them and if one cannot help them at home they are even more difficult.

This paragraph is taken from a newsletter that other women in the group have produced to share their thoughts on mathematics in the home and the school. Emilia's advocacy role as an "informal" educator is confirmed in the following excerpt written by her 15 year-old son in that same newsletter:

> Now that she [Emilia] is attending the [mathematical workshops], she can teach me other ways of learning mathematics.... She shares it with the entire family and we all get involved in a mathematics reunion that is fun. We are all teachers and students at the same time, there is no difference.

Conclusion

In their mathematical autobiographies and through informal initial interviews, many parents shared their fears and apprehension towards learning mathematics. As we listen to parents reflecting on their past and current experiences, we are gaining a better understanding of factors that affect their confidence, which in turn informs our further work with them. As parents visit mathematics classrooms, they are becoming more aware of how and what mathematics is being taught. This added to their experience as learners and as teachers when they share the activities at home with their families, is resulting in some parents taking an active role in wanting to change their children's experiences with mathematics in school. In some cases we are starting to see an awareness of their potential roles as advocates for all children:

> I also want to know not so much for this one particular child I have here [in the middle school], I really want to know if the teacher is focusing and figuring out which students...are staying behind...the students who are quiet and go with the flow, and they might say that they know, but they don't. [Debriefing after classroom observation]

Note

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TEACHERS' USES OF SEMIOTIC CHAINING TO LINK HOME AND MIDDLE SCHOOL MATHEMATICS IN CLASSROOMS

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Abstract: In this case study, two middle grade teachers became increasingly autonomous users of semiotic chains for the purpose of bridging the gap between their students' everyday activities and school mathematics. By using the semiotic chaining model, the teachers were able to address specific curriculum standards by selecting appropriate familiar activities of their students that would, through the creation of signs, link to the predefined mathematical goals. While the teachers exhibited some difficulties with creating appropriate linkages in the early stages, their continued participation generated increased ownership of the process resulting in the reduction of errors. The results of the study seem to indicate that teachers are able and willing to create semiotic chains that ground the teaching of mathematics in their students' realities.

Objectives or Purposes

The purpose of the research described in this paper is to investigate the implementation in two middle grades mathematics classrooms of a pedagogical model that bridges situated activities of students and the mathematical concepts required to be taught in schools. In this model, teachers increasingly take the initiative in building semiotic chains that lead from activities in which their students engage, to selected topics from the school mathematics curriculum.

Theoretical framework

Literature in situated cognition (Lave, 1988) and cultural psychology (Abreu, 1998) suggests that mathematics is perceived by students to be more relevant to their lives if "realistic" situations are involved in its learning. However, "real life" activities undergo transformations when used in the classroom, that may cause them to be seen as less than real by students (Walkerdine, 1988). The most that can be said is that such activities are "realistic," as in the situations used in the "realistic mathematics" of the Freudenthal institute (Treffers, 1993). Researchers such as Civil (1995 and 1998) and Presmeg (1994; 1998a and b) have experimented with ways of introducing activities that are authentic to the home cultures of students into mathematics classrooms. The current project is grounded in this literature, but takes a semiotic conceptual framework a step further than Presmeg (1997 and 1998b) did in asking teachers to enter a process in which they learn to construct semiotic chains from their own students' activities and use these in their mathematics lessons.

We believe that mathematics is "the science of detachable relational insights" (Thomas, 1996) or, resonating with this conception of the nature of mathematics, "the systematization of relationships" (Ada Lovelace, in Noss, 1997). Accordingly, in attempting to use everyday activities of students in the learning of school mathematics, teachers need ways of helping students to accomplish such a systematization of relationships. The conceptual framework of our research involves Lacan's inversion of Saussure's posited semiotic relationship between signified and signifier in a sign. When signifiers are given free play, they may be linked in chains of increasing generality that may provide a symbolic bridge between realistic activities and classroom mathematical concepts (Cobb et al., 1997; Presmeg, 1997 and 1998b; Walkerdine, 1988; Whitson, 1997). In the current project, two teachers were introduced to this semiotic chaining model, and the research investigated their use of the model in increasingly autonomous ways in their own practice. The methodology of the research was as follows.

Methods of inquiry

In order to gain a rich understanding of how teachers might use semiotic chaining for the purpose of teaching academic mathematics from everyday activities, a case study of two middle grade teachers was employed. The development of the case study was designed in a manner so as to familiarize the teachers with the process and purpose of semiotic chaining while fostering their increased independence in creating chains. To this end, the study was designed in three phases. In the first phase, the teachers were introduced to semiotic chaining by means of provided chains. These chains were designed by the first author to develop a mathematical topic that was specified by the teachers from an activity of the students. In order to create such a lesson, a realistic activity that was experienced by the students needed to be found. The chosen activities were the result of multiple researcher interviews and observances of the students during their free-time at school. During this phase, the teachers requested a lesson on converting fractions to decimals. From several interviews, it was clear that many of the students in the classes were interested in baseball and could quote some statistics for many popular players. Among these statistics was the batting average. Thus, baseball appeared to be an appropriate everyday activity that could be linked to the predefined mathematical goal. The semiotic chain in figure 1 was given to the teachers to structure the development of a lesson.

This type of chain came to be known as an intra-cultural chain since the culture of baseball enveloped the whole chain. That is, the semiotic chain did not move away from the culture it was grounded in, but rather moved within the culture toward the specified goal. The type of chain that was created (intra-culture or inter-culture) appeared to depend upon how explicit the target mathematical concept existed within the foundational culture.

To increase the teachers' autonomy in creating semiotic chains for their own purposes, the second phase of the study permitted the researcher to provide the

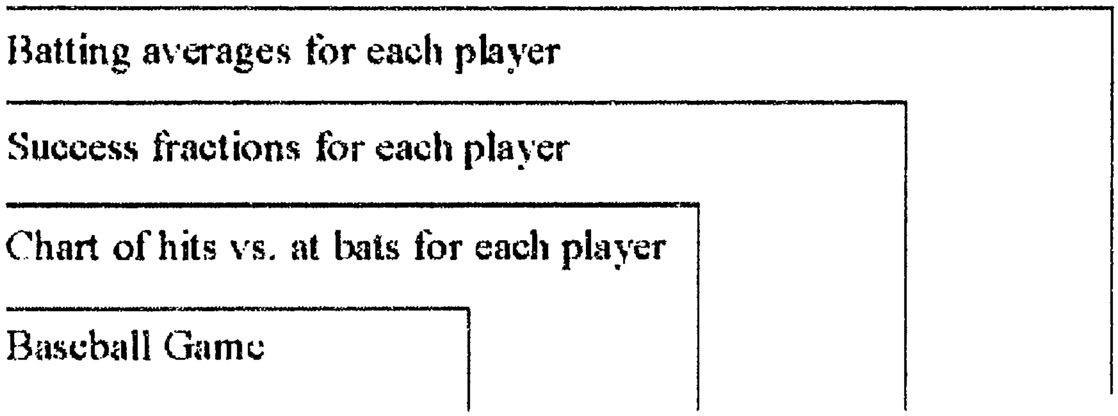

Figure 1. Intra-cultural semiotic chain for lesson development

everyday activity that might lead to a specified mathematical topic, but required the teachers to develop the links in the semiotic chain. This not only allowed the teachers to become more familiar with the intricacies of the chaining process, but also allowed them to take greater ownership of the chain by working their own ideas into the chain itself.

Finally, the third phase required the teachers to develop their own semiotic chains from their observations of the students' everyday activities towards desired mathematical topics and without any direction from the researcher. In order to teach basic algebra, the teachers created a semiotic chain that moved the students from music to algebraic equations (see figure 2).

The teachers began this lesson by listening to and conducting informal discussions about different popular songs. Once they concluded that the differences in the songs had a lot to do with the beat, the teachers reviewed how the students could represent the beat of a song using different notes and time signatures. Next, they used the beat values of different notes to determine which single note was needed to fill a measure under a specified time signature. Finally the teachers moved the students away

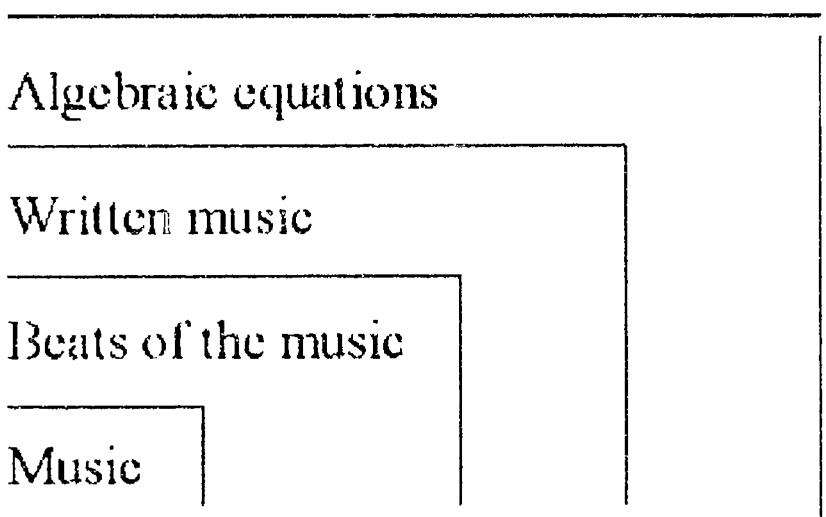

Figure 2. Teacher created semiotic chain

from music toward more formal algebra by using equations to represent that which they were doing. As an example, when the time was four-four (four beats per measure) and a quarter note (one beat) and a half note (two beats) were present in a measure, the following equation was used to determine the missing note:

$$1 + 2 + \_\_\_\_ = 4.$$

Data sources

The data for the project were collected in a university lab school. In order to gain a better understanding of the everyday activities of the students, the research began with interviewing several children in each class. The interviews were videotaped and transcribed for the purpose of ascertaining appropriate beginning points of instruction. Before each semiotic chain was used in the mathematics classrooms, the teachers and the researcher met to discuss the instructional process that was to be undertaken. Field notes were taken during and after these meetings. During the lesson using the semiotic chains, the teachers were videotaped and field notes were also taken. After their presentations, the teachers and the researcher met again to discuss the semiotic chain and suggestions for improvement. Finally, the teachers' links in phase two and semiotic chains in phase three were collected.

Results and conclusions

The data from the study indicate several results that are important for mathematics education. First of all, it seems that following a path of increasingly abstract signs needs to be undertaken carefully. When a semiotic chain is developed as a pedagogical tool from a student's everyday activity to school mathematics, a gentle transition from one link to the next is most appropriate. It was observed that if the abstraction from one sign to another is too great, then students appear to lose the link to reality that is fundamental to this process. Once this was done, it was difficult to undo the lost connection without retreating to the realistic situation and beginning the building process again.

Furthermore, the logical progression in a semiotic chain appears to be important for similar reasons. If a mathematical topic is not developed from the everyday activity through the process outlined by the semiotic chain, then links to previous signs are lost. On several occasions during phase one, the teachers confused the signs in the chain and presented them out of order. This created many problems with children not realizing why or how the current activity related to the previous one. Here again, once the damage was done, the teachers had to return to the everyday activity and progress through the semiotic chain in order to correct the problem.

Finally, it appears that teachers are able to create semiotic chains that link their students' activities to desired mathematical topics. Initially, it was unclear as to the impact that semiotic chaining should have at the classroom level. That is, would it be more of a teacher initiated tool or one usable only by curriculum developers?

However, from the chains that were produced by the teachers, it appears that they were not only able but willing to participate in this type of lesson development. This is most assuring since the everyday activities of children are far from being universal. Thus, it seems most appropriate that teachers who witness their students' activities be the developers of their mathematical curriculum. The issue of ownership is also important here. An increased sense of ownership on the part of teachers and students may enhance students' learning of mathematics (Presmeg, 1998a), and teachers are in a position to judge activities of their own students that have the potential for ownership of related mathematical constructs.

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A COMPARISON OF PROBLEMS IN SELECTED CONTENT SECTIONS IN AMERICAN AND SINGAPORE MATHEMATICS TEXTBOOKS

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Abstract: This paper reports a comparative study on mathematical problems that follow content presentation of graphing linear equations/functions in several U.S. and Singapore textbooks. The results show the differences in problems' three dimensions: mathematics, context, and performance requirements. In particular, the U.S. textbooks tended to include various problems for developing students' problem-solving ability in general, whereas the Singapore textbook tended to include purely mathematical problems that have high mathematics requirement to facilitate students' acquisition of newly taught concepts and procedures. The results provide a basis for understanding students' mathematics performance documented previously and further support the use of problem-analysis approach for understanding curricular expectations of developing students' mathematics competence.

Efforts to identify contributing factors for cross-system differences in students' mathematics achievement have led to the contention that curriculum is one of the key factors (e.g., McKnight et al., 1987; Schmidt, McKnight, & Raizen, 1997). In particular, researchers have analyzed textbooks to understand their potential effect on students' mathematical achievement in the United States and other countries (e.g., Li, 2000; Mayer, Sims, & Tajika, 1995; Schmidt, McKnight, & Raizen, 1997). The results from previous textbook studies have shown cross-system similarities and differences in textbook content presentation (e.g., Mayer et al., 1995; Schmidt, McKnight, & Raizen, 1997) and exercise problems (e.g., Li, 2000). Unlike textbook content analyses that often focus on content exposure to students, an examination of textbook problems can illuminate the cross-system similarities and differences in expectations for developing students' mathematics competence. Li (2000) exemplified the feasibility and importance of textbook problem analysis in the case of China and the United States. To extend the previous study, this study was undertaken to compare the mathematical problems in several American and Singapore textbooks. Due to variations of topic sequence and content, it was necessary to designate a common content topic in the textbooks selected for comparison. Thus, in this study all mathematical problems presented in selected content sections were compared for several American and Singapore textbooks.

Theoretical Framework

Mathematical problems can be analyzed from various perspectives (e.g., Goldin & McClintock, 1984). An examination of mathematical problems can reveal their characteristics in mathematics and context (e.g., Stigler et al., 1986). Moreover, differences in problems' performance requirements can also dramatically influence students' mathematics performance (e.g., Zhang, 1992). Therefore, a three-dimensional framework was developed for examining textbook problems (see Table 1).

Table 1 Coding Framework

<u>Framework for Coding Mathematical Problems in Textbooks*</u>

1 Mathematics[#]		• same content (S)
		• different content (D)
		• mixed content (M)
2 Context		• purely mathematical context (PM)
		• illustrative context (IC)
3 Performance Requirements**		
	(1) Response type	• no explanation or solution process required (NES)
		• explanation or solution process required (ES)
	(2) Cognitive requirement	• conceptual understanding (CU)
		• performing routine procedures (RP)
		• using complex procedures (CP)
		• problem solving (PS)
		• special requirement (SR)

[*] An elaborate description of the framework can be found in Li, 1999.
[#] A problem's requirement in mathematics is specified in terms of whether the mathematics content of the problem is the same as, different from, or mixed other contents with the content that is introduced in the

Method

Materials

Five U.S. textbooks and one Singapore textbook were selected for comparison. These textbooks are the ones that were analyzed in the TIMSS curriculum study (Schmidt, McKnight, Valverde, Houang, & Wiley, 1997). All textbooks were developed and intended for use in the eighth grade. The American textbooks were commonly used across the country in various settings and with diverse populations. The Singapore textbook was commonly used and bore the approval of the Ministry of Education of Singapore. An examination of these textbooks showed that there are striking differences in their content topic inclusions (see Li, 1999). In particular, only one common content topic on algebra, graphing linear equations/functions, can be found across the five U.S. textbooks and the Singapore textbook. Therefore, mathematical problems that were included in sections on graphing linear equations/functions were examined and compared.

Problem Analysis

Mathematical problems selected from the textbooks were those exercises or questions that did not have accompanying solutions and/or answers. In both countries' textbooks mathematical problems appeared under the headings: 'check for understanding', 'exercises', 'problems', 'practice', 'application', or 'problem solving' within or immediately following the selected content sections. Each mathematical problem was coded using the four categories listed in the above framework. A total of 308 problems were examined. The data used in this report were from a larger textbook study (Li, 1999).

A second rater independently coded problems randomly selected from each textbook. The inter-rater agreement of all corresponding codes was 97%.

Results and Discussion

Table 2 shows the percentages of mathematical problems classified in terms of problem requirements in mathematics (same content, different content, or mixed content), context (purely mathematical context or illustrative context), and performance for the American and Singapore textbooks. Problems' performance requirements include two aspects: response type (explanation or solution process required, no explanation or solution process required), and cognitive requirement (conceptual understanding, performing routine procedure, using complex procedures, problem solving, or special requirement).

Table 2 shows a similar distribution pattern for the classifications in each of the three categories: mathematics, context, and response type. It was found that in both systems, the mathematical problems in sections on graphing linear equations/functions overwhelmingly required the same mathematics content as introduced, contained a purely mathematical context, and required no explanation or solution process. Although a similar pattern is generally presented across these two systems' textbooks, some substantial variations exist. Specifically, in the dimension of mathematics, the U.S. textbooks contained a small percentage of exercise problems that required the use of mathematics content which was different from or mixed with the newly introduced content, whereas none of those included in the corresponding section in the Singapore text. The results indicate that the U.S. texts tended to provide students with problems that served the purposes of content review or connections and the Singapore textbook emphasized on students' practice in using newly introduced concepts and procedures. In the dimension of context, the U.S. texts contained a high percentage of exercise

Table 2 Percentages of Textbook Problems Classified According to the Categories of Mathematics, Context, Response Type, and Cognitive Requirement

	Singapore Text	U.S. Texts
Mathematics		
S	100	92
D	0	7
M	0	1
Context		
PM	100	89
IC	0	11
Response Type		
NES	99	93
ES	1	7
Cognitive Requirement		
CU	13	32
RP	72	56
CP	15	8
PS	0	4
SR	0	0

problems given in illustrative contexts but none in the Singapore text. The results suggest that the U.S. texts tended to include fewer purely mathematical problems but more real-world-like problems for students' practice. In the first aspect under the dimension of problems' performance requirements, response type, the results show that the U.S. texts put more emphasis on explanation and solution process in exercise problems than did the Singapore text. There are very limited requirements for explanation in the exercise problems in the selected section from the Singapore text. In contrast, the U.S. texts contained 7% of the problems that required an explanation or solution process.

For cognitive requirement the percentage of problems classified also shows a similar pattern, but to a less degree. 56% of the U.S. textbook problems in the selected sections and 72% of those in the Singapore textbook were found to require performing routine procedures. Such a big difference was also evident between the percentages of problems classified as requiring conceptual understanding for solution in the textbooks from the Singapore (13%) and the United States (32%). However, the Singapore text included a higher percentage of problems that required using complex procedures, whereas the U.S. texts tended to put less emphasis on using complex procedures but more on problem solving.

These results show that the U.S. texts tended to include problems that vary in mathematics content, context, response type, and cognitive requirement. The

Singapore text tended to include purely mathematical problems that had high mathematics requirements. The cross-system differences in textbooks' problem inclusion suggest that the U.S. texts emphasized problems' variations that may have advantages for developing students' problem-solving competence in general, whereas the Singapore text emphasized the use of exercise problems to facilitate students' acquisition of newly taught concepts and procedures. Evidence exists that Singapore students outperformed their U.S. counterparts in solving school mathematics problems (Beaton et al., 1996). This study illustrated that the Singapore text did expect students higher and more on solving mathematically difficult problems. In contrast, the U.S. texts tended to provide students with opportunities of solving various problems that is less mathematically demanding. The cross-system variations in curricular expectations of developing students' mathematics competence, as illustrated from this study, show a pattern of curricular differences that is similar to what were found in a previous comparison of problems in Chinese and American mathematics textbooks (Li, 2000). Taken together, both this study and the previous study (Li, 2000) suggest not only the relevance between curricular expectations and students' mathematics performance, but also the feasibility of using problem-analysis approach for understanding curricular expectations of developing students' mathematics competence.

Note

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CREATING A MINDFUL CLASSROOM: ONE TEACHER'S BEGINNINGS

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Abstract: The research reported here concerns itself with two types of "beginnings." First, there is the literal beginning associated with the start of the school year. Within this context, the research looks at how a mindful learning community is initiated and built through the establishment of learning routines and practices. The second type of beginning is an investigation of the psychological origin or knowledge base from which these teaching practices stem. These two types of beginnings are explored through case-study research of a middle school mathematics teacher whose classroom exemplifies the principles of mindfulness (Langer, 1989, Ritchhart & Perkins, 2000). This particular case comes from a larger ethnographic study having four main components: teacher selection, teacher interviews, classroom observation and videotaping, and a repertory grid methodology.

There has long been a concern in the mathematics-education community that students' instruction go beyond the basic development of skills and knowledge to cultivate an understanding of mathematical concepts and cultivate students' ability to reason and think mathematically (NCTM, 1980; NCTM, 1989). The justification for this position rests on the awareness that without understanding knowledge and skills are often inert and compartmentalized, leading to inappropriate or inexpert application (Gardner, 1991). Likewise, without the ability to think and reason, it is impossible to create new knowledge and engage in effective problem solving (Perkins & Salomon, 1987). However, the ability to reason, think, and understand also have limitations when it comes to performance. Ability in and of itself does not necessarily imply action. There must be the disposition to use those abilities as well as a sensitivity *to* and awareness *of* occasions for that ability's use (Perkins, Jay, & Tishman, 1993). To avoid such an ability-action gap, instruction must extend beyond the cultivation of skills, knowledge, and understanding to the enculturation of a disposition toward thinking and of awareness. In short, classrooms must strive to be mindful places in which students can become wise, not just smart.

To better understand what such environments might look like and how they are established, this study investigated how exemplary middle school teachers develop and nurture students' disposition toward thinking and mindfulness. While this research extended over the course of the school year with six teachers, this particular report focuses on the case of one mathematics teacher and his instruction during the first days of school. The specific questions being addressed are: How do teachers set the stage for student mindfulness during the first days of school? And, how does a teacher's own thinking, values, and beliefs about thinking and the discipline play out in beginning of the year instruction?

Theoretical Framework

Mindfulness. Langer (1989) describes mindfulness as a facilitative state that promotes increased creativity, flexibility, and use of information, as well as memory and retention. It is characterized by an increased recognition of possibilities and formation of new categories, openness to new information, and an awareness of more than one perspective. According to the theory, mindfulness results from drawing novel distinctions, exploring new perspectives, and being sensitive to context while mindlessness is fostered through the premature formation of fixed mindsets, overgeneralizations, automaticity and acting from a single perspective.

While experiments often focus on the promotion of mindfulness as a temporary state, Ritchhart and Perkins (2000) argue that the cultivation of mindfulness as an enduring trait is a worthwhile and achievable goal of education. However, the accomplishment of such a goal requires educators to challenge many of the norms of schooling, including traditional conceptions of what it means to be smart and do well in school. Rather than focusing on developing ability, education for mindfulness is more dispositionally based. This means that in addition to developing students' abilities, such as the ability to consider multiple perspectives, educators must also seek to nurture students' inclination to engage that ability and a sensitivity to occasions for the appropriate deployment of that ability. This model of education is more about enculturation in a set of norms and patterns of thinking than the dispensing of knowledge and training of skills (Perkins et al., 1993).

Mental Models. Just as teachers' beliefs and conceptions about subject matter, pedagogy, and their students' abilities have been shown to affect classroom practice (Thompson, 1992), it was hypothesized that teachers' thinking and beliefs; that is, their mental models (Johnson-Laird, 1983) of thinking, would also influence their instruction and their ability to create thoughtful classroom environments. In particular, teachers' thinking about thinking might influence the types of thinking they choose to develop and their ability to spot and exploit occasions for students' thinking. Just as good mathematics instruction proceeds from a foundation of solid content knowledge (Stodolsky, 1988), good instruction in thinking and mindfulness might likewise be dependent on the teacher's understanding of what it means to think and be mindful.

Methods

This study was designed primarily as an effort to learn from best practice. Therefore teachers who valued the promotion of student thinking and were effective at doing so were sought out through a process of community nomination (Ladson-Billings, 1994). Prospective teachers were then screened using criteria of classroom thoughtfulness developed by Onosko and Newmann (1994). John, an eighth and ninth grade mathematics teacher at an independent school in the West, was identified through this process. John was a mathematics major in college with nineteen years of teaching experience at the time of the study. As a study participant, John was interviewed six times and his classroom observed for a total of four weeks during the 1998-99 school year. Audiotaped interviews served to elicit John's instructional goals and values, explore his thinking on thinking, and gain his perspective on classroom events. John's mental model of thinking was inferred using a repertory-grid methodology (Kelly, 1955) adapted from Munby (1984). Classes were videotaped and fieldnotes taken during each observation in order to develop rich portraits of classroom instruction (see, for example, Ritchhart et al., 2000).

Analysis of John's first-week instructional practice sought to isolate key instructional moves and features through repeated viewing of the videotapes. These instructional moves were examined to see how they might relate to the promotion of student mindfulness and to what extent they coincided or differed with traditionally-advocated instructional practices for the first day of school (Wong & Wong, 1997). John later reviewed and offered comments on this initial analysis. Analysis of the repertory gird, used to infer John's mental model of thinking, employed cluster analysis to reveal an underlying structure in the data. John then reviewed the dendogram produced from this analysis and interpreted its meaning. Based on this interpretation, a Venn diagram of John's mental model of thinking was generated to visually represent the relationships he expressed. This diagram was again reviewed by John.

Findings

John's beginning-of-the-year instruction differs greatly from traditionally advocated approaches (Wong & Wong, 1997). Rather than focus on developing management and behavioral routines, John's initial instruction focuses on the development of learning routines that help to set expectations for how learning will take place in his class. During the first days of the school year, John's instruction contains three principal moves: *looking closely, exploring different perspectives,* and *introducing ambiguity*. These moves clarify John's expectations, establish an atmosphere of thinking, and help to develop students' inclination toward thinking.

Within the first minutes of class on the first day of school, John presents his students with a problem taken from the *Phantom Tollbooth*. In the book, the character Milo claims that mathematics is magical in that it can make things disappear. Milo gives the equation, $4 + 9 - 2 \times 16 + 1 \div 3 \times 6 - 67 + 8 \times 2 - 3 + 26 - 1 \div 34 + 3 \div 7 + 2 - 5 = ?$, as an example. John asks the class to work in pairs to figure out what the equation means, adding that he will also work on the equation. Over the next four days, the class not only solves the problem but also engages in a discussion of order of operation rules, exploring the origin and application of those rules. Below I present brief highlights from these classes to demonstrate the three principles mentioned above.

In devoting time to this problem and its discussion, John provides students with an opportunity to think beyond the surface of the problem, demonstrating the principle of *looking closely*. When his students discover that order of operations rules are the key to solving the problem, John asks, "Who thought of those rules? Why all that instead of doing it from left to right?" When the class responds with silence, John adds that he doesn't know the answer to that for sure, "But the real question is: Could we do it another way? Could we do things in a different order?" Thus, rather than being an exercise in solving an equation by applying previously acquired rules of arithmetic, John uses the problem to explore a much bigger disciplinary issue: Where does knowledge and truth come from in mathematics? While the *Phantom Tollbooth* does not deal with this issue or even that of order of operations, John demonstrates that bigger issues and principles often lurk within problems if one takes the time to look closely.

When one student tentatively answers John's query with, "Well, it depends," the opportunity arises for students to *explore multiple perspectives* on this problem. Students have already been presented with the story characters' perspective. In addition, they have argued to justify their various answers to the problem and in doing so presented the reasons backing up their own positions. Now, they are offered the chance to challenge the standard order of operations rules and present an entirely new perspective. To make this a more real and plausible venture, John has students use their calculators to solve the initial problem. In doing so, students quickly see that even their calculators, which contain different programming rules, represent a perspective. As a homework assignment, John asks students to devise new order of operations rules and to try them out. As the class explores these alternatives, the importance of parentheses and the problematic nature of exponents quickly emerges. As one student states, " I think that parentheses only exist because we have order of operations, you wouldn't need them otherwise."

Throughout his initial instruction, John has taken arithmetic and tried to make it problematic for students in an almost Socratic way. Rather than present mathematics as straightforward, he has *introduced a degree of ambiguity* into situations to make students more mindful of what is going on. An example of this can be seen on the fourth day of school when John asks the class to interpret the different meanings of x^2, $(x)^2$, $-x^2$, and $-(x)^2$ when $x = -2$. Immediately, the class erupts in opinions as students argue about the meaning of x^2 when $x = -2$. When the majority of the class is convinced that the answer is -4, John comments, "Where we're getting bogged down is that we're trying to remember a rule rather than think about what is going on. I need you to think about what is going on here. Let's go back to something that was brought up in the discussion. What does x^2 mean?" The point that a variable has to be treated as an entity just as an expression in parenthesis is treated is then made. Exasperated, a girl in the second row asks, "Why didn't you just put the parenthesis in the problem then?" John turns the question back, "Why didn't I?' With a sigh the girl responds, "To make us think?" John responds and concludes, "Yes, that's the main reason. This isn't something just to memorize. I need you to understand it."

John's mental model of thinking (Figure 1) reveals a rich and nuanced conception of thinking. As part of the repertory-gird process, John generated thirty-two synonyms for thinking, which he placed in six major categories. The importance of playing with ideas, taking risks, and considering different perspectives emerge in the "open thinking" cluster. Thinking associated with looking closely can be found in the "keying in" category. While it is difficult to say what an all-inclusive or complete mental model of thinking might be, John's model does tend to capture a variety of types and facets of thinking useful in problem solving, developing understanding, creating new ideas, and decision making. While one cannot say that John's instruction is dependent on his mental model of thinking, the richness present in both certainly suggests that, in at least John's case, his teaching is relatively compatible and in line with his mental model.

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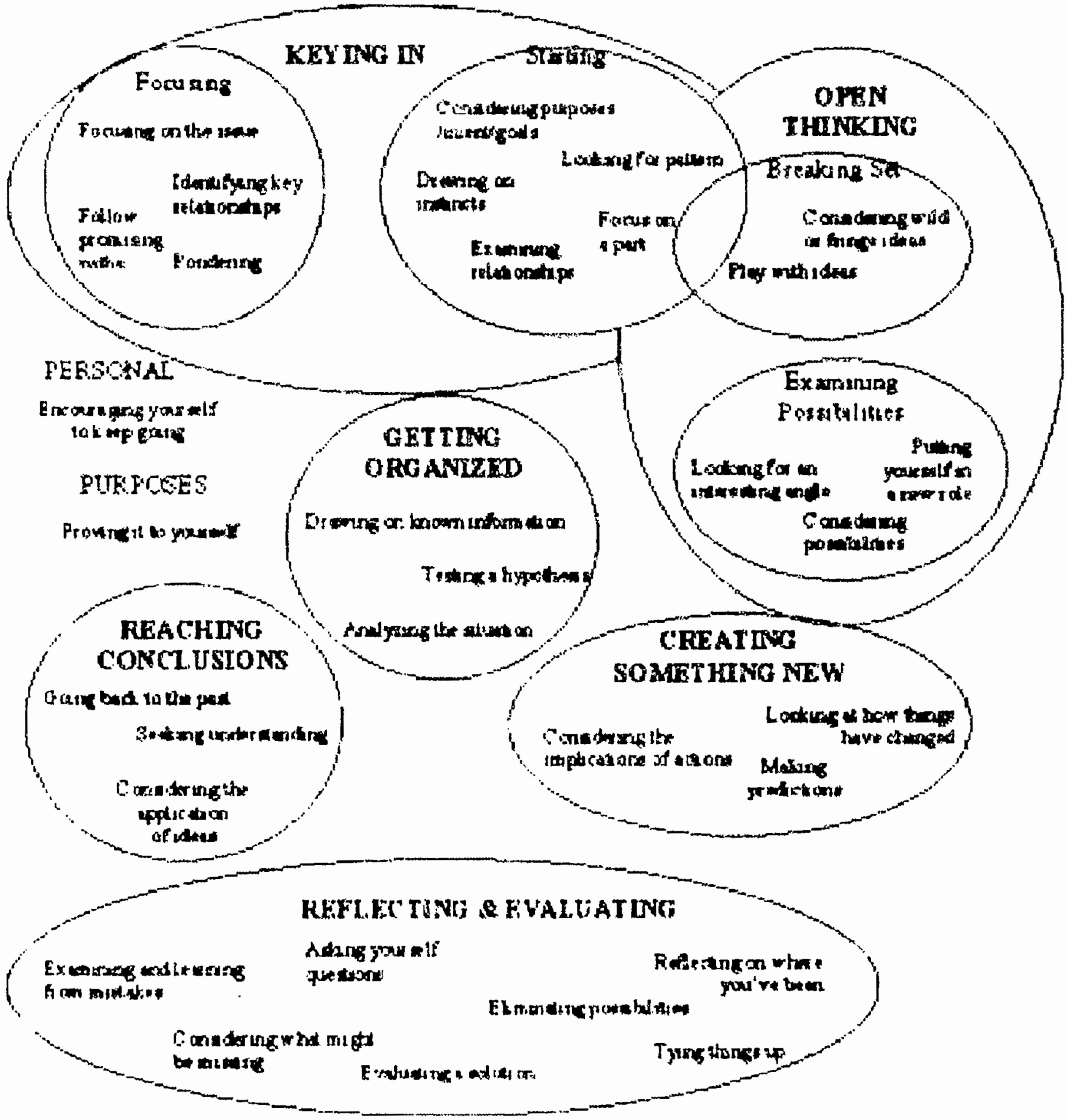

Figure 1. John 's mental model of thinking

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THE STRUGGLE FOR MATHEMATICS STANDARDS: TAKING A CLOSER LOOK

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Abstract: The development of Florida's mathematics standards was researched from 1994-1996 using a symbolic interactionist perspective. In the first year of development, the writing team, consisting entirely of mathematics educators, strategically used the language and structure of the standards to convey their goals for reform in mathematics education. During the second year of development, as the standards underwent revision and non-mathematics educators entered the process, numerous debates erupted. Although debates occurred primarily over issues of language, it was the multiple and symbolic meanings of the language and structure of the standards that formed the basis for these debates.

Introduction

The publication of the National Council of Teachers of Mathematics' [NCTM] *Curriculum and Evaluation Standards for School Mathematics* (1989) set off a flurry of activity in the development of standards around the United States not only in other national subject areas, but in the revision of standards among the states (Ravitch, 1995). Throughout the 1990s, mathematics standards became the basis of a variety of state systemic reform initiatives including assessment and accountability programs. The debates over how to write these standards and who should write them have led to "math wars" in numerous states (see *Phi Delta Kappan*, February 1999). However, despite their importance and controversy, little is known on how mathematics standards are conceptualized, developed, and debated within political contexts.

Research Focus

Florida developed new mathematics standards from 1994 - 1996[1]. The purpose of this research was to understand how participants conceptualized writing standards, why they conceptualized the standards in this way, and to uncover the meanings they ascribed to these experiences. An additional goal of this research was to place the development of the mathematics standards in the social, political and historical contexts within which they were developed. This paper explores the debates that erupted during the second year of development when the initial set of standards underwent revisions.

Methodology

Data collection included interviews, document analysis [e.g., memos / letters, drafts of the standards, reviews, meeting artifacts (i.e., overheads, agenda sheets, handouts, official state documents)], and field observations (e.g., writing, editorial and advisory meetings). The participants included members of the mathematics writing team, Department of Education (DOE) officials, numerous reviewers, and a variety of other Florida educators involved in writing, reviewing, or editing the new standards. Data were analyzed inductively (i.e., patterns, themes came from the data rather than being imposed on them prior to data collection and analysis). Triangulation, prolonged submersion / engagement at the research site, negative case analysis, and extensive member checking were used to help ensure the integrity of the research.

Theoretical Framework

Using a symbolic interactionist framework, the focus of this research was on understanding the intent of the writers as they developed the mathematics standards and to place this within the larger historical and political context. A symbolic interactionist framework focuses attention on how individuals interpret and give meaning to their experiences, to other people, and to "objects" in their lives (in this case, mathematics standards), and endeavors to understand how this process of interpretation leads to particular behaviors. As Jacobs (1987) explains,

> Symbolic interactionists' assume that individuals' experiences are mediated by their own interpretations of experience. These interpretations are created by individuals through interaction with others and used by individual to achieve specific goals. Symbolic interactionists are interested in understanding how these interpretations are developed and used by individuals in specific situations of interaction. (p. 27)

Symbolic interactionism frames mathematics standards as social constructions that result from the interplay of diverse political interests. This perspective encourages the collection of evidence that reveals the differing intents / interests of the participants (Hall, 1997), and focuses on the language and the politics of meaning in standards development (Placier, 1998).

Major Findings of the Study

Historical Background

Florida's systemic reform initiative for education, known as *Blueprint 2000*, was developed in 1991 and called for more local control but greater state accountability. However, what this should look like in practice and how it was to be achieved politically were unclear. As a result, the policy specifics of *Blueprint 2000* were intensely debated between 1991 – 1994. In 1993, as these debates continued, the Florida DOE decided to developed new standards in all subject areas. Because of the uncertainties regarding the interpretation and implementation of *Blueprint 2000*, the new subject area standards were originally conceptualized as voluntary standards that local districts could use to help design their own curricula. It was unknown how or even if these standards would be used for state level policies (e.g., state assessment).

First-Year Effort to Develop Standards: "Managing the Message"

During the first year of development, the writing team consisted only of mathematics educators (K-12 / university teachers and district curriculum specialists) and was given considerable flexibility in how they could write the standards. Underlying the entire development process was the writing teams' belief that any set of standards (even if only to delineate content to be learned) implicitly carries messages regarding curriculum and pedagogy based on how they are written. Therefore, since they believed that standards are naturally interpreted on multiple levels, they tried to "manage" these interpretations to be consistent with their own goals for reform and the role they wanted the standards to play in Florida's systemic reform effort.

The writing team wanted the standards to reform mathematics education by encouraging teachers to reconsider their views of mathematics and how it should be taught. As a result, the writers developed a set of strategies for writing standards based on how they felt teachers might interpret and use them in their classrooms. Some of the strategies used by the writing team to achieve these goals included:

a) using "dynamic" verbs to encourage a more active pedagogy (e.g., verbs such as *explore*, *investigate*, and *analyze* rather than "traditional" verbs such as *solve*, *compute*, or *factor*).

b) integrating process strands such as problem solving and communication into the standards to encourage their integration into the curriculum;

c) integrating a variety of activities and content into single standard statements in order to encourage a more holistic and integrated view of mathematics, for example, a standard in the September 1994 Draft included the following: Recognize, describe, extend, estimate, analyze, generalize, transform, and create a wide variety of mathematical relationships by using models such as tables, graphs (both one- and two-dimensional), matrices, verbal rules, expressions, equations, and inequalities; and

d) writing broad standards (lacking specifics) in order to provide both flexibility in classroom practice and to encourage teachers to engage in curriculum development at the local level (i.e., by requiring teachers and district curriculum specialist to fill in the details).

Revising the Standards

In early 1995, the political context changed with the election of a new education commissioner who called for a strong accountability system based on new content standards. Analyzed from within this new political context, the DOE officials felt it was unlikely that the new standards – written more to reform teaching than delineating what students should know and be able to do – could be used to guide the development of a new state accountability and assessment program.

The Revision Process

In early 1995, McREL Institute was hired to help the DOE facilitate the revisions. Since the standards were to be used for state assessment, the mathematics writing team initially (and reluctantly) agreed to make some changes. These included: (a) adding more specificity, (b) removing non-assessable verbs (e.g., "explore," "construct meaning for," and "investigate"), (c) including language more understandable to the general public, and (d) including additional "basic" content (e.g., computational skills). However, despite these changes, the mathematics standards continued to be criticized throughout the revision process. The standards were often characterized by McREL and DOE officials[2] as (a) too broad and vague; (b) more like classroom activities than statements of knowledge and skills; (c) "overloaded" (i.e., containing too much information in single standard statements); (d) too sophisticated for grades K-2; and (e) lacking clear and consistent distinctions between grade levels.

Mathematics Writing Team's Response

To the criticism that the standards were too broad and vague and lacking clear and consistent distinctions between grade levels, the writing team explained that although the standards were to be used as a basis for state assessment, the standards were also for classroom instruction, and they wanted to preserve a set of standards that provided for local flexibility. With respect to other criticisms, the writing team disagreed that including verbs such as "communicate," "describe," or "analyze" in the standards turned them into classroom activities instead of content standards. The writing team felt the use of these "dynamic" verbs encourages an active view of teaching mathematics (which was one of their original goals). With respect to some standards being "overloaded" with too much information, the mathematics writing team felt that these standards would help convey mathematics as integrated and holistic. To separate them, the writing team felt, would lead to a portrayal of mathematics as a set of isolated facts and topics.

Although compromising at times, the writing team used a variety of strategies to counter the pressure to make changes to the standards and to maintain as much of their original intent for the standards as possible. Strategies included: (1) keeping the number of standards to a minimum forcing the standards to be written at broader levels of generality; (2) agreeing to make more changes to the Number / Measurement standards than those in Geometry, Algebra and Data Analysis; and (3) using performance activities (examples that were included alongside the standards showing how they could be implemented in the classroom) to: (a) counter standards written in language more traditional than they wanted; (b) provide clarity to what the

DOE felt were "vague" standards; (c) show increasing levels of sophistication for standards that were not significantly different across the grade levels; and (d) provide examples of how students could achieve standards deemed too sophisticated by reviewers (particularly for grades K-2). Despite the on-going debates and numerous concerns over the standards, DOE officials eventually acquiesced to the mathematics education community's wishes and the new mathematics standards were officially adopted by Florida's State Board of Education in May 1996.

Summary

In the first year of development, the writing team, consisting entirely of mathematics educators, strategically used the language and structure of the standards to convey their goals for reform in mathematics education. During the second year of development, as the standards underwent revision and non-mathematics educators entered the process, numerous debates erupted. Although debates occurred primarily over issues of language, it was the multiple and symbolic meanings of the language and structure of the standards that formed the basis for these debates.

Notes

1. In 1998, Florida once again revised their mathematics standards by adding sample, grade-level benchmarks; however, that effort was not included in this study.
2. "McREL" and "DOE officials" as used here does not imply unanimity of opinions by the various individual members within the DOE / McREL who participated in the process to revise the standards.

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DISCUSSION GROUP ON GENDER AND MATHEMATICS

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The purpose of this discussion group is to make connections with members of PME-NA at large who are not necessarily able to commit to full participation in the Gender Working Group. Our plan is briefly to share our work, elicit feedback, and invite participation of interested scholars. We believe we are at a critical point in our work. We have had little peer review and feedback from outside the group but we are preparing to embark on a major task of creating a monograph of our work. We feel we would be remiss in our scholarship were we not to hear what the colleagues who would apply our findings have to say about our directions and efforts.

During these past three years, the Gender Working Group has been able to reach a broad spectrum of participants. We differ across interests, experience, and goals within our working group. Our work has been fruitful and seems to be thorough in its reach. However, the work we have done has been conducted *within* our group, either on-line, within the working group sessions at PME, or at outside conferences where we might connect in our scholarly pursuits. What we hope now, even as the working group works toward a publishable monograph of our research, is to seek feedback from interested PME members who are *not* members of the working group. In our experience at PME discussion groups, we know that scholars who might not necessarily participate in our working group and may not have a gender and mathematics research agenda, still have a vested interest in our work. Whether because of their own experiences with mathematics, or their efforts in teaching mathematics or mathematics education, these professionals recognize the value in our work and have important feedback on what we might do to better meet their needs in integrating work on gender and mathematics into their teaching and research. Thus, to gather their feedback on our developing directions and intentions for research and publication, a discussion group will serve us well. We see this both as a kind of peer review, an opportunity to have those outside our group assess our efforts, broaden the discussion in our group, and contribute to our plans and also as an opportunity to incorporate the voices of those who might be able to apply our work into our future directions.

The session will begin with a short 20 minute presentation and explanation of our work to date as a working group. That presentation will include the web of research agendas created by our group at PME-NA XX, the themes and plans generated by the PME-NA XXI group, an introduction to the website of the Gender Working Group, and the current proposals for research and publication we intend to address at PME-NA XXII. We will then conduct an open discussion to elicit from the participants responses to the following questions:

1. In what we have begun to explore, what do you see as most and least helpful to you in your efforts to integrate gender issues into your teaching and research?
2. What suggestions might you make to help us organize or reorganize our efforts?
3. Even though we have found absences in the research, what absences do you find in our analysis?
4. What kinds of studies would you like to see addressed by gender and mathematics researchers?
5. What do you see as obstacles to completing this kind of work?
6. What do you see as supportive issues?

PATHWAYS TO EARLY NUMBER CONCEPTS: USE OF 5- AND 10-STRUCTURED REPRESENTATIONS IN JAPAN, TAIWAN, AND THE UNITED STATES

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This study examines what representational supports for early number concepts appear across curricula in Japan, Taiwan, and the United States and what related supports exist in everyday lives in those cultures. Although there is some difference between Japanese and Taiwanese representational supports for early number concepts, their practice can be characterized as the extensive use of concrete and semi-concrete objects and a focus on the numbers 5 and 10 as units. Textbooks and teachers often show numbers in a vertical and/or horizontal 2x5 ten-pattern array. Linear representations organized by groups of ten help children see other numbers in the relation to 5 and 10. The groups of ten are clearly stated in the number words in these languages (12 is "ten two" and 53 is "five ten three"), but the groups of 5 are not reflected in the language. The use of groups of ten and of five is long-standing; present teachers used such representations as students.

In Japan and Taiwan, many aspects of the cultures emphasize groups of 10. Money is grouped into tens as it is counted. The metric system is used in many places in children's lives; rice, their staple food, is sold in 5kg and 10kg bags, and bottled drinks come as 1 liter and 500 mg. The fives and tens both appear in the abacus that has been used for calculations for centuries in both countries. In both abaci, one bead stands for a group of 5. In the classroom, however, all five entities in a group are shown. Few everyday uses of five are present in Taiwan. In Japan, five and ten are used frequently to package objects for sale, such as eggs, dishes, dried noodles, socks, and postcards. Children have rich exposure to the number 5, as heroes often come in groups of 5s in their stories, and 5 dolls are displayed as a part of traditional girls' day celebration.

In the United States there are few everyday occurrences of groups of five or of ten except for money. Dozen (12) and half dozen (6) rather than 10s or 5s are used pervasively in packaging, and the metric system still makes relatively few appearances. Some textbooks use ten-frames for a few pages, but the 5-structure as a way of seeing the numbers 6, 7, 8, 9, and 10 is infrequent. Therefore, using 5s and 10s in adding and subtracting numbers is less familiar and seldom arises spontaneously.

EL MERCADO IN LATINO PRIMARY MATH CLASSROOMS

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We sought to clarify the potential and the limitations of an ethnomathematical perspective through an analysis of attributes of successful mathematical learning outside of school (e.g., Nunes, 1992; Saxe, 1991) and an analysis of our work in U.S. urban Spanish-speaking Grade 1 and Grade 2 classrooms focused heavily on aspects of buying and selling situations ("El Mercado"). We sought to ascertain what previous experiences children had in this area (e.g., over half of one second-grade Spanish-speaking class had previous experience in Mexico or in the U.S. selling things with their family) and then designed classroom experiences that related to these experiences. Our theoretical perspective combined a constructivist view of learning as individual meaning making by each participant, a Vygotskiian view of teaching as assisting the performance of learners by adapting to the perspective of the learner while helping the learner move toward more culturally adapted conceptions, and an ethnomathematical and "funds of knowledge" (Moll et al., 1992) view of searching for experiences in children's lives to which school content could be related to form coherence and meaning.

We found that both first and second graders possessed robust "mercado scripts" that enabled them to engage in buying and selling pair activities. Playing mercado worked well as a classroom activity structure. Children enthusiastically and creatively role-played buying and selling and embellished with talk, objects, and physical actions the basic situations given to them in various ways to make them socially detailed and personal. The use of money was positively charged for children and also created sustained involvement even in activities outside the buying/selling pairs. There were also substantial limitations in the buying/selling situations as sources of learning. Complexities included difficult features of coins and bills, counting these different quantities in the face of obstructive features, and inadequate understanding of aspects of giving "change". In the real world the nonmathematical aspects of buying/selling are more salient and obvious than are the mathematical aspects. Thus, much work must be done in the mathematics classroom to support children's construction and use of the quantities involved in money.

COGNITIVE RESTRUCTURING FOR CHANGING MATHEMATICS BELIEFS

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One of the foremost problems confronting higher education is the lack of student success in remedial college mathematics courses. Beliefs about learning mathematics inhibit students' ability to focus on being successful in mathematics courses that they see as emotionally draining and cognitively difficult. This paper discusses the theory of cognitive restructuring as used in psychotherapy along with the implications of this framework for the mathematics classroom.

Most mathematics teachers see themselves as cognitive specialists, comfortable with mathematics, but believing that the affective component of their students' learning is out of their domain. This lack of focus on the affective domain leaves students who are unsuccessful with traditional teaching techniques with little hope for success and without an approach to learning. The affective component must be addressed. One of the most promising approaches offered by psychology is cognitive restructuring-a therapeutic approach seeking to modify irrational beliefs. The goal is to increase awareness of negative self-statements and images, resulting in a change of beliefs. The merging of cognitive restructuring and mathematics education gives hope to teachers. Routinely questioning negative self-statements from their students is a first step in the use of cognitive restructuring. Students may then reflect on the validity of these long-held self-statements. With cognitive restructuring the student "can come to understand that his affective experiences and maladaptive behaviors are a result of his particular thinking process—processes that he is capable of changing and correcting by himself (Beck, 1970)."

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A FRAMEWORK FOR COORDINATING WHOLE CLASS AND SMALL GROUP JUSTIFICATION NORMS

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The research presented here is part of a larger effort to coordinate analyses of individual student justifications, small group reasoning, and whole class patterns of justification in an Algebra I classroom. The focus is on how students' ways of justifying differed in small groups when compared to their whole-class discussions and furthermore how the presence of technology affected these ways of justifying. I report particularly on the degree to which students constructed personal ways of judging and changes in their use of imagery in justification.

Researchers and the teacher designed a technology-intensive unit to support students' development of meaning of graphs of linear equations in a classroom teaching experiment (Cobb, 2000). The analysis is based on a framework described by Cobb (2000) that takes students' individual reasoning as a unit of analysis while simultaneously viewing that reasoning as an act of participation in communal practices.

Analysis of small group and whole class justification norms when coupled with articulation of mathematical practices indicated that, at the beginning of the teaching experiment, authority resided with the teacher and computer. As the class developed in ways of reasoning, they began to use the imagery developed through their experience with the software, and hence developed more robust personal ways of judging.

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RETENTION OF MINORITY MATHEMATICS TEACHERS: A SURVIVAL ANALYSIS

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Attrition and retention of teachers is of critical importance within the educational community. Some researchers indicate that high levels of teacher turnover are disruptive to programs and correlate with decreases in student performance (Bempah, 1994; Theobald, 1990).

The number of minorities taking advanced mathematics needs to increase. National organizations have emphasized the importance of role models for minority students. According to researchers there is a rather stark and troubling mismatch between the diversity of the student population and the relative homogeneity of the teaching force (Wilson, 1988). This disparity brings up issues of both recruitment and retention of minority teachers. There are programs in place aimed at recruiting minority teachers, but are the minority teachers being retained?

The purpose of this research is to examine attrition rates of minority mathematics teachers. Using the techniques of survival analysis and data from California Department of Education, this study tracks the "class of 1986". This group of over 3,000 teachers is followed over an eleven-year period in order to examine their attrition rates.

The results indicate that the survival rate is different depending on the ethnicity variable. Over the eleven-year period, the Black and American Indian teachers had the lowest survival rates. The literature supports high hazard rates the first year, but the rates depicted here approach 60 percent. First year rates, especially for some minority teachers, are disturbingly high. Awareness of the significant loss of these teachers is the first step in addressing the question of what can be done to reverse the exodus of minority teachers, especially in their first year.

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THE MERIT WORKSHOP PROGRAM: UNDERREPRESENTED SCHOLARS IN MATHEMATICS

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Many students struggle in calculus especially minorities, women, and students from small high schools. The Merit Workshop Program, developed by Uri Treisman, targets these groups of students, who tend to be underrepresented in the areas of mathematics, science, and engineering. In this presentation I include a comparative study and describe in general the student outcomes, benefits, and frustrations of the Merit Workshop Program..

The Merit Program is not a remedial program (Jackson, 1989). Students are invited into the program based on high academic potential and a commitment to excellence. One of the main goals of the Merit Program is to develop a community of scholars among the Merit students. The students in the program work together to solve difficult course problems, develop friendships based on common academic interests, and inspire each other to maintain a high level of commitment to excellence. Learning mathematics is not a passive activity in which students absorb facts from a teacher as a sponge absorbs water. Merit students are responsible for learning calculus through active participation in both the teaching and learning process. In order to "think like a mathematician" students must learn to solve complex problems by understanding the fundamental concepts of mathematics rather than simply using algorithms to get correct answers.

The results of the Merit Workshop Program support the notion that active learning. The findings suggest Merit Workshop students perform better in their courses than students in traditional settings. The students have continually done better than their counterparts in the traditional sections of calculus. A community of scholars seems to evolve, which often lends support and encourages success in later courses. The Merit Workshop students seem to gain confidence in expressing themselves and their abilities in calculus.

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ON LEARNING AND KNOWING MATHEMATICS: ONE AFRICAN AMERICAN STUDENT'S PERCEPTION

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It may seem surprising, but NAEP data suggests that students experiencing low achievement in mathematics often retain positive views about mathematics. Many African American students even identify mathematics as their favorite subject even though they exhibit very little understanding about the usefulness of mathematics (Martin, 2000). Martin asserts that a difference between successful and unsuccessful African American students is that the unsuccessful students fail to demonstrate "the same kind of attitudinal and behavioral investment made by their more successful peers" (p. 13). Researchers have used various evaluative tools (e.g. standardized test scores and school records) to gain perspectives on success attributions (Bempechat, 1996; Willig et al., 1983), but few have interviewed students to gain the students' perspectives on their mathematics performance.

This presentation shares an up-close look at the perspective of a student in what teachers have designated as a low-tracked class in order to expose her reasons for her perceptions about mathematics and the level of achievement she experienced. This study examined students' perceptions about their knowledge of mathematics, how they learn mathematics, and th barriers that hinder their learning of mathematics. The research was conducted in an Algebra I Extended class. Data collection included field notes and artifacts from several weeks of classroom observations and audio-taped interviews. Emergent themes guided subsequent observations and interviews. The results of the study showed that while the student experienced high grades in the Algebra I Extended class, she was able to identify many gaps in her understanding. She believed that her lack of understanding plagued both her continued performance in mathematics as well as her level of confidence. Moreover, the participant demonstrated limited understanding of the importance of mathematics. Analysis suggests that influences include: a decontextualized curriculum, a desire to avoid failure rather than to achieve success, and a limited understanding of foundational mathematical concepts.

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GIRLS ON TRACK: PURSUING ADVANCED MATH FROM MIDDLE SCHOOL TO COLLEGE

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Girls on Track is an NSF program for middle-school girls that gives participants the opportunity at a two-week summer camp to learn math and technology through activity-based investigations related to urban-planning issues and the opportunity to gain public-speaking experience by presenting the results of their investigations. The program is directed by faculty members at North Carolina State University and Meredith College, both in Raleigh, North Carolina. In 1999, 40 girls participated in the first year of the program, and we expect close to 80 girls at the second year of the camp in July, 2000.

Data sources for the first year include summaries of girls' responses to survey questions about confidence in mathematics, attitude toward mathematics, interest in computers, computer skills, and career interests. The girls also completed a proportional-reasoning test, the scores of which are highly correlated with their scores on the state's standardized algebra test. A longitudinal study will gather information about the girls' choices of math courses in middle school and high school, their college major, and their career goals. We hope to identify factors that are associated with girls' course selections, choice of college major, and career goals through middle grades, high school, and into college. By the end of summer 2000, we will have the initial data for the girls who participate in 2000 and the first set of follow-up data for the girls who participated in 1999. One of the primary research interests is to conduct longitudinal case studies with selected girls to gather more in-depth responses about attitudes and confidence and about factors that may influence their academic and career choices.

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INVISIBLE BARRIERS THAT DISABLE MATHEMATICS EQUITY AMONG NON-MAINSTREAM STUDENTS

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Introduction: Information on Research Funding

Grant funding from the National Science Foundation and from the McDonnald Foundation was received to inform how to provide teachers, parents, and students with math knowledge among grades 1-4. This research has resulted in developing and writing mathematics curriculum in both English and Spanish for grades one through four. We have many teachers piloting our Children's Math World curriculum, which contains a strong family component. We have 23 Arizona teachers and 15 California teachers that are using our curriculum through the Arizona State University West research team.

Findings from the research involving teacher and family interviews and classroom video taping

Teachers do not receive enough training to know how to prepare children that come to schools with many mathematics gaps. They rely on teacher manuals to help guide them with students who are struggling to understand mathematics content only to find that the teacher guide does not give enough information for serving the needs of students whose ability level does not "fit" within the narrow range provided.

Teachers need to know how to tap into the students' knowledge base and make the necessary connections required to expand knowledge. The problem is that many of our students need to learn by having sub-steps that help link their knowledge in the learning process. Textbooks do not provide these sub-steps in their teacher manuals. Sub-steps make it possible for a wider knowledge range to connect to the learning process.

Teachers who have used our Children's Math World curriculum for grades 1-4 report that sub-steps have helped them teach math concepts more effectively and are remarkably surprised that their "lower" ability students are able to learn at a faster pace *once* they understand a process they had been having trouble with.

Sub-steps offer multiple entry levels so that all students find ways of connecting their knowledge base to what is being taught. Incidents like these that expand student learning help to create boosts in *teacher confidence* and help to push them into seeking more ways to connect with all their students. We find teachers more willing to seek different methods that create similar results for all their students who struggle with the learning process.

COOPERATIVE LEARNING AND MATHEMATICS ANXIETY IN THE COLLEGE ALGEBRA CLASSROOM

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Mathematics anxiety and cooperative learning are two highly researched areas in mathematics education today. Research by Johnson and Johnson (1998) and by Norwood (1994) reveals that the use of cooperative learning in mathematics classrooms results in higher student achievement and in reduced levels of mathematics anxiety than the traditional lecture method of instruction. Mathematics anxiety correlates negatively with high student achievement (Ma,1999).

During the fall semester 1999, at North Carolina State University, three college algebra instructors (all graduate students) participated in a cooperative learning study, led by Dr. Karen Norwood. Each instructor taught two sections of college algebra: an experimental section, taught with an emphasis on cooperative learning, and a control section, taught in the traditional lecture style. Students in all sections completed Aiken's Revised Attitude Scale at the beginning and at the end of the course. All sections used graphing calculators.

Student achievement data and the data from the attitude scale are still being analyzed. This poster session will outline the results of the study.

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COPING STRATEGY ANALYSIS: A NEW APPROACH TO STUDYING MATHEMATICS ANXIETY

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The purpose of this study, the second in a series of studies on coping strategies, was to evaluate the relationship between level of mathematics anxiety and the strategies chosen to cope with it. Two-hundred seventy-nine college students enrolled in either a remedial algebra course or a nonremedial precalculus course completed the Composite Math Anxiety Scale in order to provide a mathematics anxiety score. Afterwards, the students rated ten Likert type coping strategies with regard to frequency of use and helpfulness (or value).

A multivariate analysis of variance (MANOVA) was performed on the student data. The independent variables were mathematics anxiety (high or low), and course enrollment (remedial or nonremedial). The dependent variables were the ten coping strategies, each of which was rated for frequency of use and helpfulness. Low mathematics anxiety students utilized and valued the majority of coping strategies more than did high mathematics anxiety students and algebra students did so more than precalculus students. Completing homework assignments on time, letting your instructor know if you don't understand the course material, setting aside extra study time before exams, and asking questions in class received the highest ratings. This cluster of strategies was characterized as approach strategies inasmuch as they all focus upon confronting the stressful situation at hand (i.e. the study of mathematics) in comparison to avoidance strategies in which the student attempts to reduce anxiety by at least temporarily leaving the stressful situation (such as exercising to reduce tension) and returning to it later when he or she feels better.

CONFIDENCE AND ACHIEVEMENT IN FAST TRACK MIDDLE SCHOOL GIRLS

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The American Association of University Women (1992) found that confidence plays a role in the achievement of young girls in mathematics. As a result of declining self-confidence throughout middle school, young women tend to enroll in less challenging mathematics courses as their education continues (Boswell, 1985).

The purpose of this study was to analyze a set of statements made by fast track middle school girls in order to determine a set of criteria for confident and non-confident statements. The study was split into two parts. The 14 subjects for part 1 were randomly selected for a videotaped interview from a group of 40 middle school girls who attended the *Girls on Track* summer camp. The 3 subjects involved in part 2 were selected from among the girls interviewed for a case study. The participants in both parts were interviewed during the summer before they entered Algebra I. Three questions from the videotaped interviews were selected for an analysis of confidence in part 1. Each girl's response to the three questions was coded as either confident or non-confident based on the level of empowerment shown in her response. In part 2, the interview responses were used in an analysis of the effect of confidence on achievement for these three middle school girls. Interview responses gave information about the confidence of each girl.

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URBAN SCHOOLS ADAPT TO CHANGES IN THE NEW MILLENNIUM: DETERMINING IMPLEMENTATION LEVELS OF A REFORM K-5 CURRICULUM

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This poster session reports on an investigation of the factors which led to a successful implementation of an NSF supported K-5 Mathematics Curriculum, *Math Trailblazers*. We were particularly interested in looking at an urban school in Chicago which is 100% black, is located in a neighborhood with the highest crime rate in Chicago and has high levels of poverty and mobility. What were the factors that led to a successful implementation of this rigorous, standards-based curriculum? As we interviewed teachers, attended their grade level meetings, observed classes and read journals of the teachers, administrators and parent aides, it also became important to determine how one determines what the level of implementation is and what are meaningful criteria for success. For most schools, it is most frequently judged by standardized test scores. That is not the case for this school.

As Hiebert (1999) indicated, our lack of knowledge about the level of implementation of such a program always leaves us guessing about the reasons for the success or failure. Romberg and Collins (2000) assert that there is a pressing need to investigate how classrooms that promote understanding in mathematics and science can be created.

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Teacher Beliefs

TEACHER CHANGE IN THE CONTEXT OF COGNITIVELY GUIDED INSTRUCTION: THE CASE OF MRS. R

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Abstract: We documented Mrs. R's change across four years of implementation of the principles of cognitively guided instruction. Data included annual interviews, her written reflections on instructional issues, and our observations of her mathematics instruction. Her beliefs shifted immediately toward a constructivist view and remained stable throughout the project. By the end of the project, Mrs. R (a) saw student-student interaction as critical to development of mathematical thinking, (b) viewed students' struggles with mathematics ideas as desirable, (c) helped students reflect, (d) made explicit decisions about when children would share solutions, and (e) focused questions to help children see mathematical structures.

Cognitively guided instruction, or CGI, (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989) is an approach to teaching mathematics in which knowledge of children's thinking is central to instructional decision making. Teachers use research-based knowledge about children's mathematical thinking to help them learn specifics about students and to adjust instruction to match students' performance. The tenets of CGI fit well within Fosnot's (1996) principles of learning derived from contructivism: (a) learning is not the result of development, learning *is* development; (b) disequilibrium facilitates learning; (c) reflective abstraction is the driving force of learning; (d) dialogue within a community engenders further thinking; and (e) learning proceeds toward the development of structures (pp. 29-30). These principles were our theoretical framework for understanding teacher change.

Method

Project. The project (NSF Grant ESI-9450518), conducted from January 1995 to December 2000, included five teams (originally, two teacher educators and six classroom teachers on each team); participants learned to use CGI as a basis of mathematics instruction. Workshops were held in May 1995 (3 days), July 1995 (10 days), June 1996 (8 days), June 1997 (7 days), June 1998 (4 days), and June 1999 (2 days). Between workshops, teachers implemented the principles of CGI in mathematics instruction, each team met about once a month to continue learning about CGI and to discuss progress, and each teacher was visited about once a month by one of the team's teacher educators. Project staff visited each teacher once each semester to provide general support. In 1997, teams began to deliver CGI professional development for their colleagues.

Instrumentation. Data sources were (a) transcribed annual interviews, (b) written responses on several instruments (described in Bowman, Bright, & Vacc, 1997) administered each year, and (c) three sets of classroom observations (two consecutive days each) in Spring 1998, Fall 1998, and Spring 1999. Field notes were taken during each observation, and after each observation there was a debriefing interview; all interviews were transcribed.

Subject. At the start of the project, Mrs. R was a 3rd-grade teacher with 15 years of experience, pre-K to 3rd grade. She was licensed for K-3 instruction and held a master's degree in early childhood education. She had served as a university-level child development trainer. During the first 3 years of CGI implementation, Mrs. R taught 3rd grade at one school. Mrs. R indicated a general lack of interest in CGI by other teachers in her building. Her principal was verbally supportive of her participation in the project, but the principal did not attend the principal days during the summer workshops and appeared to be most interested in whether the state test scores of Mrs. R's students increased more or less than the scores of other 3rd grade students. During the 4th year, Mrs. R moved to a different school and taught 2nd grade. The principal seemed supportive, but there were no opportunities for that principal to attend CGI inservice. By the project's end, Mrs. R seemed to be at a high level of CGI implementation -- possibly level 4b (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996).

Analysis. Reflections on teaching were grouped into three categories, based on both Fosnot's principles and the principles of CGI: discourse, children's thinking, and instructional planning. The classroom observations served as evidence of the extent to which Mrs. R attended to these notions simultaneously in creating coherent mathematics instruction. Reflections were also grouped into four time periods (i.e., pre-CGI, early CGI, mid-CGI, late CGI) but the limited space here prevents explicit presentation of the evidence from each period.

Results and Discussion

Beliefs. Mrs. R's total Belief Scale score (240 maximum) increased substantially between the first administration (196 at the beginning of the initial workshop) and the second administration (227 at the end of the first summer workshop) and remained essentially stable (227, 224, 220, 228 on four additional administrations) during the remainder of the project.

Discourse. Mrs. R said little directly about discourse, in terms of either the teacher's role or students' roles. She was consistent over the four years in identifying "teacher as facilitator" as a critical component of instruction, and this seemed to be an important part of a teacher's role in discourse. She was not very specific about what she meant by "facilitation" until toward the end of the project, when she began to identify some specific aspects of facilitation (e.g., carefully sequencing the solution strategies that students were asked to share).

Three changes seem to stand out in Mrs. R's views about discourse. First, the roles of students became more prominent in instruction, through both more student sharing of solution strategies and more student-to-student interaction. Second, questioning became more useful for helping reveal students' thinking, partly through use of more high-level questions. Third, classroom instruction became more responsive to children's needs, for example, through explicit decisions about whether unusual solutions were shared with the entire class.

Children's Thinking. Helping students develop confidence was a pervasive theme across all of Mrs. R's reflections. Throughout the project, Mrs. R seemed alert for children's notions of number sense. These orientations are certainly consistent with the principles of CGI, and it seems that her developing knowledge of problem types and children's solution strategies reinforced, rather than changed, her views. The specific evidence that she cited about either children's confidence or understanding of number sense was typically general and generic.

Three changes stand out in Mrs. R's perceptions about children's thinking. First, the process of revealing thinking became a joint effort between Mrs. R and the students, rather than being the responsibility only of Mrs. R. Second, thinking was inferred from a wider variety of situations, and it was tied to more specific mathematical concepts. Third, Mrs. R began to talk more about the thinking of individual children and less about the thinking of groups of children collectively.

Instructional planning. For Mrs. R, it was important that CGI held "value" for children, for example, by helping increase students' knowledge. Specifically, by learning to ask each other questions, children learned mathematics. The notion of creating a risk free environment for children pervaded Mrs. R's reflections about instructional planning, and she increasingly talked about having patience with children to let their thinking develop.

Three changes stand out in Mrs. R's perceptions of instructional planning. First, the knowledge-base for her planning moved from general knowledge to specific knowledge, and the specific knowledge-base expanded across the life of the project. Second, her planning became more explicitly designed to move children's thinking forward. Third, her planning became more responsive to individual students' needs.

Mathematics instruction. Mrs. R was knowledgeable about the needs and mathematical understandings of individual students in her class. She used this knowledge to adapt problems for individuals. She asked questions and altered problems to probe students' understanding.

- Carter shared his solution to this problem: *In the ocean there was a school of 155 fish swimming around the reef. 50 fish swam away. How many fish were still in the school of fish?* He wrote

$$\begin{array}{r} 155 \\ \underline{-\ 50} \\ =105 \end{array}$$

> and then said, "I went 1 take away 0 and I put 1; 5 take away 5 is 0; and 5 take away 0 is 5."... Mrs. R gave Carter a new problem: 150 - 49 and asked him to think about that. Carter wrote
>
> $$\begin{array}{r} 150 \\ \underline{-\ 49} \\ =119 \end{array}$$
>
> Mrs. R asked Carter if he could take 9 away from 0. He said no, but still wanted to write 9. Mrs. R then said that the problem would be his own challenge to think about. (Classroom Observation Field Notes, May 17, 1999)

Mrs. R was concerned about the "comprehension" of her students. She wanted them to understand problem situations as a whole and not focus on isolated words or phrases.

- I'd say 12 to 13 of them have pretty solid understanding of tens.... [and] comprehension of the problem. There's a real problem with this group; it's comprehending what the problem is asking you to do.... When you stop throwing numbers together is when you are willing to look at the problem and try to figure out what you know and what you don't know. (Classroom Observation Debriefing Interview, November 4, 1998)
- I purposefully try not to put in the problems, like that fish problem ... [the] key words. I use different words so they're really not comprehending an isolated word but the whole concept of what I was asking. (Classroom Observation Debriefing Interview, May 17, 1999)

One of the problems was a multiplication problem: *A rare find was discovered in a deep section of the ocean. It was a group of 12 octopus. Knowing that octopus have 8 arms each, how many arms were there in that group of octopus?* In her conversations with individual children, Mrs. R changed the number of octopus to 5 for one child and 10 for another child. As children shared solutions to this problem, they used a variety of skip-counting and counting-on strategies, without any apparent connection to place value (e.g., 10 eights equals 8 tens or 80).

- I know they know eight tens, but they don't see yet that ten eights are the same as eight tens. I really thought that Stan and Evelyn would pick up on that right away, and maybe James. (Classroom Observation Debriefing Interview, May 17, 1999)

Evidence for Constructivist Principles

Learning is not the result of development, learning is development. During the project, Mrs. R began to encourage more student to student interaction. She saw interaction as critical to development of children's mathematical thinking. Illustrative of this is her focus on not avoiding conflicts among students' answers; good discussion, with resulting increase in understanding, arose from addressing those conflicts. Mrs. R's emphasis on comprehension by the students further illustrates her implicit agreement with this construct. Mrs. R also reorganized her own thinking about instruction. Her interpretations of children's work moved from general development concerns to specific mathematics development concerns. She expanded both the range of situations where she looked for mathematical thinking and the frameworks within which she interpreted children's thinking.

Disequilibrium facilitates learning. Mrs. R was conscious of the fact that students need to struggle in their learning. Struggles were not viewed as stumbling blocks or barriers but rather as opportunities to make sense of mathematical ideas. Mrs. R's realization that making sense of students' understanding was not solely her responsibility, but was a shared responsibility between her and each student, is an indication that she could accept her own limitations. She acknowledged that she, too, had to struggle to understand how to plan and implement instruction.

Reflective abstraction is the driving force of learning. Mrs. R came to understand that having students reflect on their own learning is important. She acknowledged that every child has to reflect in order to learn. Mrs. R herself clearly became much better at reflecting on her own teaching, and she came to realize the importance of doing so. She could recognize the mathematics that occurred in lessons even when she had not planned for that mathematics to be addressed. One result of this was her recognition first, that there were changes in her own teaching that she needed to work on and second, that she needed to learn more mathematics.

Dialogue within a community engenders further thinking. Dialogue among students became a focal point for her planning and implementation of instruction. She made conscious decisions about when to involve the entire class in dialogue and when to involve only selected students. This dialogue was intended to further students' understanding and reflection.

Learning proceeds toward the development of structures. The discussion about 8 tens versus 10 eights indicated how important students' understanding of structure was for Mrs. R. She recognized when structures were being developed, and she explicitly focused her questions to help students see those structures. Mrs. R's structural knowledge of teaching evolved from her growing base of frameworks for understanding children's thinking. By the end of the project, she identified quite a list of things that teachers need to know in order to be effective.

Summary. First, the perspectives that a teacher brings to a professional development project are important and need to be identified and acknowledged. Mrs. R's background in early childhood education seemed critical in framing her development. Second, there are identifiable stages of development as teachers work through significant professional development programs. For Mrs. R, the stages included teacher as teller, students as tellers, and students and teachers working together in a community of learners. Third, a teacher's working environment is an important factor influencing a teacher's development. Mrs. R might have developed faster or differently if she had had collegial support in her building or a stronger professional community with which she could have interacted. Fourth, Fosnot's (1996) constructs provide a framework for understanding not only students' mathematical growth but also teachers' instructional growth. Making this framework explicit to teachers might help them in self-reflection and self-analysis.

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PSYCHOLOGY STUDENTS' CONCEPTIONS ON THE TEACHING AND LEARNING OF ARITHMETIC: AN ANALYSIS OF REPORTS ON WORK SESSIONS WITH CHILDREN

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Abstract: In the study we are now reporting, research was performed on the conceptions that psychology students uphold concerning the changes in arithmetical knowledge of first and second grade-school pupils, as well as on the role that such psychology students assign to the didactic activities they carry out with the children. Through a qualitative analysis of reports from the work sessions, that were written by the students, three major tendencies were identified in the conceptions on change regarding arithmetical knowledge. One of them focused on the strategies, a second one on the assessment of answers, and the last one on the explanation of change. In the matter of the role assigned to didactic activities, it was found that, from the students' perspectives, such activities may serve for a consolidation of the acquired knowledge, for the promotion of new knowledge, or for the evaluation of such knowledge.

Buenrostro and Figueras (1999) have suggested the need that educational psychologists be provided with both the theoretical and the practical tools to face the problem of a low school performance in mathematics, and particularly in arithmetic. Obtaining information about the conceptions upheld by psychology students about the various aspects related to the teaching and learning of arithmetic is an important step towards a proper orientation of the teaching that these students receive. One way to approach an understanding of these conceptions is to examine the writings that the students produce as part of their training in the intervention process that they carry out with children in the early grade-school years.

The purpose of the present research was to inquire into the conceptions of a group of psychology students, with respect to:

- the way they conceived changes in arithmetical knowledge of first and second grade pupils which were considered low-performance children, and
- the role they assigned to the didactic activities used to promote such changes.

Theoretical Perspectives

Thompson's (1992) position is adopted here, concerning the notion of conception as "a more general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like" (p. 130); and also, his statement relating to the aspects that are included in a teacher's conception concerning the teaching of mathematics as "desirable goals of the mathematics program, his or her own role in teaching, the students' role, appropriate classroom activities, desirable instructional approaches and emphases, legitimate mathematical procedures, and acceptable outcomes of instruction..." (p. 135). The research being reported in this article must be placed along with the studies analyzing the written reports of teachers, as a means to understand their beliefs (Gellert, 1999; Kaplan, Rosenfeld & Appelbaum, 1999).

On the other hand, a concern is shared with various researchers about whether teachers, and, in our case, educational psychologists, are becoming aware of the various thought processes, and of the actions and justifications that children carry out when facing situations of a mathematical nature. Our starting point is that the decisions made both by teachers and psychologists in their respective realms will be better based if these persons have acquired a greater knowledge of these issues. Through the application of Cognitively Guided Instruction, Carpenter, Fennema, Franke, Levi, and Empson (1999, p. 105) have found that "learning to understand the development of children's mathematical thinking could lead to fundamental changes in teachers' beliefs and practices, and that these changes were reflected in students' learning". Other researchers have highlighted the need for teachers' use of the flexible interview (Ginsburg, Jacobs, & López, 1998) or the clinical interview methods (Hunting, 1997) as tool that permits uncovering children's thinking about mathematics. Doig and Hunting (1995), and Buenrostro and Figueras (1999) emphasize the importance that in programs focused on the understanding of the processes of the children's mathematical thinking, participants thereof carry out a practical and direct work with children.

In order to analyze the students' conceptions, we will lean on several concepts from authors like Watzlawick (1989) and Keeney (1987). For the latter, epistemology on a sociocultural level, "is tantamount to the study of the way in which persons or person systems get to know things, and of the manner in which they think they know things" (Keeney, 1987, p. 27). The punctuations they establish, (i.e., the issues which are relevant to them), "create various realities, in the strict sense of the word" (Watzlawick, 1976, p. 75). Thus, beliefs are placed within a frame with a specific sense. To reframe these beliefs implies a change in their framework in order to give them a different sense.

The Context of the Research

This research is performed within the framework of the application of a teaching model (see Buenrostro & Figueras, 1999) wherein the psychology students from the *Facultad de Estudios Superiores Zaragoza* of the *Universidad Nacional Autónoma de México*, are carrying out work sessions, twice a week, with children from the first and second primary grades that have been reported by their teachers as pupils with a low school performance. In these sessions, a first assessment is made of the pupils' arithmetical knowledge and, based on this, various didactic activities are carried out with the purpose of enriching their knowledge and promoting an improvement of the children's school performance. An average of ten students participate, with ten children and a professor in educational psychology who provides advise to the students concerning the instruction that they provide the children.

Mode of Inquiry and Data Collection

The research has a qualitative and reconstructive nature. Attention is focused on the comprehension of the processes, conceptions, and actions by the participants. Therefore, we concur with Gellert (1999) when he asserts, in reference to the purposes of an investigation on the conceptions of mathematics teachers, "The aim is to reconstruct the intentions and strategies of the actions of the people under study. The focus is on understanding and not on predicting" (p. 28). What we try to do, then, is to understand the psychology students' conceptions through a reconstruction exercise which permits to arrive not to a precise and representative image of such conceptions, but to the identification of certain cognitive and action tendencies. In order to attain this, an analysis was performed of the contents of the reports that the students prepare for each of the work sessions that they hold with the children. In the reports, the purposes to be attained in the session are specified, a description is drawn, and an assessment is made both of the didactic activities and the strategies used by the children; also, suggestions are made for the following session. The analysis is based on the procedure employed by Gellert (1999) to get to know the convictions of future teachers about various aspects of mathematics teaching through the analysis of such teachers' diaries. The analysis of the reports was done in three clearly designed stages: 1) the performance of successive revisions of the reports, with an aim to derive specific categories; 2) a new revision, now guided by the categories which were identified in the former stage; and 3) the specification of tendencies in the students' conceptions.

Results and Discussion

Regarding the way in which the students conceive the changes in the children's arithmetical knowledge, three major tendencies are seen which can be characterized as follows:

Focus on the Strategies

A change is considered to have been promoted when the child makes use of a strategy that differs from the one used before to solve the situation posed. The following statements are representative of this tendency: "a change occurred in the type of strategy, for instead of counting everything, he started with the greatest addend"; "as opposed to former sessions where he counted the cubes in a bar one by one, he now counted each bar while saying ten, twenty, thirty, etc." It is important to note that the assertions corresponding to this tendency contain specific descriptions of the actions performed by the children, and that belong in a student's conceptual framework allowing him/her to identify such actions as significant and indicative of a change in the children's knowledge levels.

Focus on the Evaluation of the Answers

In this tendency, more than describing change, judgement is passed about it: "an improvement was seen in the solution of problems..."; "the child's performance was good". Or else, the change is qualified as right or wrong: "she was asked to solve the last problem, which was correctly solved"; "the child answered the activities correctly". One additional element touches on the help provided by the psychology student: "he solved most of the problems without any help from the instructors". As it can be seen, it is difficult within this tendency to realize in what way the change took place, for what seems to matter is the result in the posed situations, or the final answer.

Focus on the Explanation of the Change

Here, the child's action, which is not described, serves to infer an internal cognitive process that explains the action: "the child understood how the numbers should be read and written"; "she realized that it was easier to count by twos."

These tendencies reflect three different ways of punctuating the children's behavior that would seem to stem from different observational perspectives. In the first of these, there is an intent to describe the child's actions and to frame them within a context of meaningfulness that could allow the student to interpret them as part of the construction processes of the children's arithmetical knowledge, and which, in this case, are indicative of some progress in the children's arithmetical thinking. In the other two tendencies, the children's actions are taken as reference points, so that, starting from such actions, an evaluation can be made of them, or a number of inferences can be drawn with respect to internal processes that account for such actions. In both cases, the description of the actions is left aside, and the evaluation is done in terms of how "correct" the answers are, or on what the improvement has been concerning the actions. In the case of the inferences, it is the internal status that is responsible for the children's actions.

With respect to the way in which the students conceive the didactic activities that are carried out to propitiate a change in the children's arithmetical knowledge, our findings are that these are closely related to the purpose of the activity. Thus, the activities can serve to

- Give continuity to teaching: "the same mechanics for activities will be followed".

- Consolidate the acquired knowledge: "problems on... will be presented in order to reinforce the child's knowledge".
- Increase the level of difficulty: "problems with a higher degree of complexity will be posed to her".
- Facilitate learning: "simple multiplication problems will be posed to him".
- Evaluate the acquired knowledge: "we will try to analyze whether the child is already able to solve any kind of problem".

It is interesting to observe how the purpose of the activities is conceived in different ways, not necessarily excluding each other, and how these different approaches reflect three conceptions as to the purposes of teaching: the consolidation of knowledge acquired; the promotion of new knowledge; and the evaluation of such knowledge. Apparently, in the first two cases above, it is considered that the children's behaviors are appropriate, and that it is therefore necessary to continue to apply a certain set of activities, with an aim to consolidate the knowledge that such activities had promoted. In the third and fourth cases, the intention is that the child progresses, either through proposing situations which, in some way, lead him/her to more complex resolution forms, or through a graduation of activities going from the simple to the complex. And finally, activities are conceived as an instrument to evaluate the children's knowledge.

Conclusions and Implications

The psychology students' conceptions concerning the changes in the children's arithmetical knowledge pertain to different punctuation processes; this leads to conceive the change from various standpoints. If one of the purposes of the programs of the initial formation of psychologists and teachers consists in linking such programs with the children's mathematical thought processes, then it is important that the faculty in charge of these programs are clear about the students' conceptions so that, if the case arises, they can help them understand the children's behavior from different perspectives than those they have used before, and thus open new ways for them to conceive such a behavior. In other words, a reframing of the situation ought to be promoted –i.e., a change in the manner of understanding an assertion or a behavior, to attribute a different sense to these-- which could allow the students to become aware of those aspects in the children's behavior that are relevant in the process of structuring arithmetical knowledge.

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THE NATURE OF TEACHER CHANGE AND ITS LESSONS FOR THE EVAULATION OF INNOVATIVE EFFORTS IN THE TWENTY-FIRST CENTURY

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Abstract: This paper reports the results of a five-year study of teacher change among calculus teachers at both the high school and college level and discusses the implications of these results on future evaluations of professional development efforts. The fact that change occurs in steps and over time implies that the evaluation of change should be longitudinal and employ a variety of assessment instruments.

The purpose of this paper is to report the results of a research project that began in 1993 as an evaluation of the National Science Foundation (NSF) sponsored Calculus Reform Workshops and grew into a longitudinal study of teacher change at the high school and college level. This paper reports the results of that study and the implications that those results and the methodology of the study have for evaluation of educational change in the twenty-first century.

Theoretical Framework and Literature

This study is grounded in certain beliefs of the researcher in the relationship between teacher change and learning and in how change/learning occurs. In terms of learning, the researcher believes that the mathematical understanding an individual creates is dependent upon that individual's point of view and his or her previous knowledge. Thus the creation of understanding has both social and cognitive aspects (Carlson, 1997; Yackel, 1995). Teacher change involves a learning process – a development of each individual's pedagogical understanding – with both social and cognitive aspects. Certain cognitive requisites for change are indicated (Shaw & Jakubowski, 1991); and social aspects exist since negotiating these cognitive steps may be enhanced by outside support (Wasley, Nonmoyer, & Maxwell, 1995). Although all change efforts seem to involve certain recognizable steps, the pedagogical results are not necessarily the same from one individual to the next. What reform looks like in any given classroom is dependent upon several personal factors among which are a teacher's past experiences and beliefs (Cohen, 1990; Drake & Hufferd-Ackles, 1999), and his or her knowledge of mathematics and pedagogy (Lloyd & Wilson, 1998).

Cohen (1990) recounts the innovative efforts of Mrs. Oublier, a second grade teacher who had spent her early teaching career teaching mathematics in a very traditional way of memorized facts and procedures. Then she attended a workshop that, in her view, completely changed her ideas about the teaching of mathematics.

Curricular materials are also believed to facilitate change: "Because many teachers rely on textbooks as a core for their teaching, a textbook is a reasonable candidate for communicating and providing guidance for change" (Ball, 1990, p. 257). The curriculum used by the teacher in this study aspires to this goal. It is one of many reform calculus texts commercially available and is often considered a moderate attempt at reform. For certain teachers, the use of reform orientated curricular materials has promoted change (Edwards, 1995). For others, their lack has been found to impede change (Wasley et al., 1995).

Research Design and Methodology

This is part of a larger study that examined changes in practice and beliefs of a high school mathematics teacher as she implemented HC. At the time of this study, the teacher had taught mathematics for 13 years during 7 of which she had taught one section of a non-Advanced Placement [AP] calculus course. She was the only calculus teacher in her small, rural, Midwestern high school. The teacher was the main influence on the choice of the HC and expressed a desire to change her instruction. Her first year implementing the HC her class consisted of 10 students. The teacher described these students as "atypical" relative to those she had taught in the past; she believed that they complained more and seemed less studious.

The data were collected as follows. In the summer before implementing the HC, baseline interviews with the teacher were conducted. These interviews focused on teacher beliefs, instructional practices, and on reconstructing her lessons taught from LH. The following school year, data collection included observations of the teacher's instruction in HC. A total of 52 lessons were observed and videotaped. Detailed narratives of the lessons were constructed. Teacher interviews were also conducted after all observations, before and after each chapter in the HC, and at the end of both semesters. These interviews were audio recorded and transcribed. Artifact collection included teacher lesson notes, student notebooks, and handouts from both curricula.

Qualitative methods were used to analyze the data. Data were analyzed using grounded theory (Strauss & Corbin, 1990). The particular manner in which the data were used is as follows. Field notes, interviews, and written documents were coded. Coding the data helped the researcher find commonalties. Initially, data were Open Coded for rough categorization. During this process, the focus was on mathematics content, teacher actions and beliefs, assessment, technology use, and representations used in instruction. Axial Coding techniques were then used to relate categories and subcategories discovered during Open Coding. Relationships between the different categories were then examined to determine the presence of more abstract concepts that might link less abstract categories (Strauss & Corbin, 1990). Direct comparison of instruction in both curricula was also made. Multiple data sources were used to validate trends in the data.

Results and Conclusions

When compared to her practice in the LH, the teacher, on the surface, maintained a similar mode of instruction in the HC. Her practice consisted of demonstrating prototype examples. However, the teacher was able to change her practice in important ways: (a) an increased focus on conceptual knowledge, (b) the use of graphing calculators to learn calculus, and (c) changes in evaluation that included testing concepts in addition to procedures.

Three major types of intervening conditions were noted in this teacher's process of change: internal, external, and first-year challenges. Among the internal conditions were teacher beliefs, the teacher's desire to make students feel comfortable, her open attitude toward change, and teacher concerns. External conditions included the HC curriculum, a reform calculus workshop, her students' beliefs, and school factors. The first-year challenges primarily related to the newness of the HC to the teacher and the challenges that this brought to her planning and classroom teaching.

Internal Factors

The teacher's beliefs were found to influence the ways in which she planned her lessons and subsequently her instruction. Among the beliefs that influenced her practice were those regarding how students learn, the importance of reading the text, the role of the teacher and students, and her beliefs about calculus and its teaching and learning.

The teacher tended to hold what many would consider 'traditional' views of how students learn mathematics: "I think that students learn calculus by coming to class prepared every day, by listening in class to the presentation, by going home and trying their homework on their own." These beliefs, in part, influenced her to plan instruction that consisted primarily of presenting ideas found in the reading and showing students how to solve sample problems.

Another influential belief in this teacher's change process was a desire to make her students feel more comfortable. This resulted in practices of rarely assigning students homework problems beyond the sample problems solved in class, giving students 'homework hints', telling students what types of problems would be found on tests, and carefully guiding students to the solutions of problems covered in class.

The teacher's open attitude toward change was another internal factor that influenced her change process. This was characterized by her desire to modify her instruction, her choice of the HC text, and the fact that she liked it both during and after her first year using it. Finally, the teacher's concerns and frustrations were also intervening internal factors. These concerns centered around pacing, students dropping out of the course, and students who were cheating.

External Factors

Among the external factors that influenced this teacher's change process were the HC, a reform calculus workshop, her students, and school factors. The HC materials were very influential in the teacher's change process. Its conceptual and technology foci helped the teacher center on them to a greater degree. The HC's ancillary materials were also a very important aid to her in her planning. They influenced not only the sample problems that she solved in class, but also her homework assignments and the tests she constructed.

A weeklong reform calculus workshop that the teacher attended before her first year using the HC was another external influence on her change process. This workshop had two primary influences: it enhanced her desire to include projects in her instruction and reassured her that her plans for change were "on the right track."

Students' attitudes and motivations also influenced this teacher's change process. Early on in the teacher's implementation of the HC, students resisted the teacher's plans for them to become more independent learners. Over time, this led to a high level of teacher frustration. This finally led her to abandon some of her early plans for change, which included her students reading the text independently and trying some problems before they were addressed in class.

The final external factor that influenced the teacher's process of change involved the school and its structure. The limited amount of lesson planning time during the school day inhibited somewhat the teacher's ability to move beyond those ideas found in the HC text and its ancillary materials. Time constraints were also influential in her inability to implement projects.

First-year Challenges

The final type of intervening condition can be best described as 'first-year challenges'. The most influential of these related to teacher planning. The newness of the HC to this teacher forced her to spend an increased amount of time on planning. She believed that in the first year using a new text, it was more difficult to know what to focus on in instruction and where students would have difficulties. This was significant because it made it difficult for her to determine how long and difficult student assignments would be and how long it would take to teach the content. This appeared to influence her to focus more on day-to-day operations and less on long-term goals for change, such as implementing student projects.

The influences on this teacher's change process were diverse and came from various sources. They impacted it in both positive and negative ways. When confronted with the complex environment that is a mathematics classroom, teachers construct their own ways of creating a learning environment that will result in student learning. For this teacher, the three primary influences on her construction were the internal, external, and 'first year challenges' discussed here. The HC was an especially strong influence, and its role in this teacher's change process reinforces Ball's (1990) belief in the positive role that reform curricula can play in influencing change. The role of this teacher's students in her change process was also strong.

Note

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SOME PRIMARY-SCHOOL TEACHERS' CONCEPTIONS ON THE MATHEMATICAL NOTION OF VOLUME

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Abstract: The concept of volume is one of the topics in the Mexican basic education curriculum. Inservice primary-school teachers have pointed out that this is one of the topics where their students face difficulties. On the other hand, although the volume mental object is quite a broad issue (Freudenthal, 1983), there is evidence that teachers limit themselves in relating it to the notion of capacity and with its calculation by means of formulas. This situation motivated the creation of a research project to investigate teachers' ideas on this mathematical concept and its teaching; a part of the findings are reported herein.

Theoretical Perspectives and Methodology

The methodology chosen to develop the theoretical framework wherein the teachers' observation is inserted was Filloy's (1999), which proposed the initial creation of a local theoretical framework; its construction assumes four components related to each other. The component dealing with models of cognitive processes was prepared after a documental research of the works of Piaget and other reseachers who have studied childrens' cognitive difficulties with the concept of volume, as well as the works of other persons who have shown an interest in the general study of teachers' beliefs and conceptions. In order to structure the component of teaching models, a review was made of the textbooks used in Mexico during the last 100 years (Sáiz, 1998). This review allowed a comparison of the pedagogical trends that have formed the basis of the initial development of Mexican school-teachers with those upheld by international experts; and, in turn, permitted an analysis of the teaching proposal in force in our country, which covers all of Mexico, and is mandatory. To prepare the component of formal competence models, use was made of the phenomenological and didactico-phenomenological analysis that Freudenthal (1983) devised for this concept, albeit enriching it by means of contributions from other researchers, mathematicians, and historians. A summary of the local theoretical framework thus constructed, and the conceptual network that organizes it in a schematic form appears in Sáiz & Figueras (1999). In the present research, a combination of techniques from qualitative research methodologies were used for data collection and obtaining evidence for the analysis. In principle, by using the aforementioned conceptual network as a support, three types of questionnaires briefly described in Table 1 were designed and applied.

Table 1. Questionnaires

Questionnaire	Type of data	Examples of questions posed
1	The participants' professional profile and work experience.	In what grade have you worked the most during your years in charge of a group?
2	The teacher's ideas about the teaching of mathematics in general, and of volume in particular. Among other things, there is an interest in finding out whether the teachers are using the bibliographic material provided by the Ministry of Public Education, and how deeply they have studied it, especially concerning the part related to volume. On the other hand, hypothetical situations are proposed to the teachers with the intent that they evaluate these situations dealing with the learning of volume and other volume-related situations.	1. Teacher Julián asked his students to calculate the volume of a cup. a) Amalia poured water into the cup and then measured the liquid by means of a graduated container. What was it that Amalia measured? b) Ana put the cup inside a graduated vessel containing water, and measured how much the water-level rose. What did Ana measure? c) Which of the two girls measured the volume of the cup? 2. Julio, a sixth-grade student, was asked by his teacher to calculate the volume of a parallelepiped made up of cubes (as shown in a figure that contains a picture of a 2 X 4 X 8 parallelepiped). Julio's answer was: "the volume is 64, because I can undo the parallelepiped and, with the same number of small cubes, I can build a cube measuring 4 on each side. I know that to calculate the volume of a cube you multiply 4 X 4 X 4, and when I solve this operation the result is 64". If Julio were your student and he responded to you the way he answered his teacher, a) what would you say to him? b) Why? c) Would you recommend to him to do things differently? If so, what procedure would you suggest to him? If not, why?
3	The teacher's ideas on the mathematical notion of volume, such as the various meanings attributed to that word, and the properties of bodies in relation to volume.	Considering the following list of properties of bodies, answer "yes" or "no" depending on whether you believe that the property in question is, or is not, related to volume. Finally, comment on why you say "yes" or "no". 1) mass 2) temperature 3) lateral area 4) capacity 5) weight

Taking into account that written answers do not provide complete information about teachers' ideas and knowledge (Thompson, 1992), a workshop was designed where the teachers could spontaneously comment and share their ideas with their peers when working with tasks related to the concept of volume. The workshop was presented as an instrument for data collection, and not with the purpose of assessing the impact it could have on the teachers' knowledge and ideas, even if one is aware that some learning or a new conceptualization might occur. A description of the structure and the topics dealt with in a workshop with 24 teachers, consisting of five

Table 2. Workshop Sessions

Session	Description
1	Tasks on spatial imagination and the estimation of capacities without the use of conventional units. Review of textbooks in order to identify the lessons related to volume.
2	Activities dealing with relationships between volume and capacity, and problems related to the conversion of capacity and volume units.
3	Activities permitting reflection on methods to calculate or to compare the volumes of irregular bodies, and the variations between volume and lateral area, and vice versa.
4	Analyses on the effect on a body of lengthening or shortening the linear magnitudes of such a body.
5	The construction of bodies, and the make-break transformations to calculate and compare volumes, and to deduct formulas for geometrical solids. The analysis of the formulas used in grade-school. A reflection on the reading of some historical passages related to the origin of some formulas.

sessions, each lasting four hours, is presented in Table 2.

The data were subjected to iterated scrutinies. During the first of these scrutinies, evidence was identified and classified on the uses of the teachers' mathematical knowledge on the topic and on their beliefs about mathematical teaching in general, found in their written answers and their verbal comments. In the next stage, the classifications obtained in the first scrutiny were constrasted with the conceptual network, in order to identify the uses of qualitative and quantitative aspects that teachers make regarding the general aspects of the mental object of volume as characterized in such a network. The purpose of the next stage was to identify those elements of the mental object that were tied to the mathematical properties closely linked to the volume concept from the standpoint of the formal component.

Discussion and Findings

From the present stage of the analysis of the extensive data derived from the work with the teachers in the workshop, three findings have been selected that are considered to be significant; these will be presented in the following paragraphs.

1) The belief in the existence of a dependency relationship between the lateral area and the volume. In order to collect information on this issue, a question in Questionnaire 3 shown in Table 1 was designed. Nine of the twelve teachers answering the question stated that the lateral area and the volume are related, and the arguments of four of them agreed in pointing out that: *lateral area is determined by the dimensions* which, in turn, *determine its capacity or volume.* This evidence shows a tendency among the participants to consider a dependency between the lateral area and the volume, a finding that appeared later during the third workshop session (see Table 2), designed precisely to provoke a discussion about this belief. In Table 3, a transcript excerpt illustrates the point that problems such as the one shown, and the ways in which teachers solve them, tend to strengthen such a belief.

It should be mentioned that in order to solve the problem in Table 3 all the teams used a particular case, a rectangle 5 cm x 10 cm, from which they derived a general rule. In three of the teams a calculation was performed using those numbers, and in one, work was done with paper models and the pouring of seeds. In spite of their realization that the volumes of the constructed and calculated cylinders were different and the workshop conductor focused their attention on the equality of the lateral areas,

Table 3. The problem of the water tanks and the subsequent discussion

Problem	Transcription of the discussion
A factory produces water tanks by means of rectangular steel sheets; if it is intended that such tanks have a maximum capacity, how is it convenient to roll the sheets, lengthwise, along their width, or is it indifferent?	Researcher: Did you think initially that one was bigger, or that they were both equally big? Teacher No. 1: Here we even established that they were both equal; because we said to ourselves, look, if the sheet is like this and I fold it (he rolls a sheet of paper along its width) like this, or if I fold it like this (he rolls it lengthwise) **the result is the same, because the area does not vary.**

the teachers argued that the area of the base was missing, and that was the reason for the difference in the volumes. The workshop included other activities where it became clearer that there could be two bodies with the same volume and different lateral areas, work was done on these later on.

2) The association of the change in the level of a liquid with the weight of the body when submerging it in a vessel containing water. In order to find evidence on this belief, question 1 from Questionnaire 2 was posed (see Table 1). On the questionnaire one teacher, from thirteen, associated the weight with the change in the level of water. Then during the third workshop session, when it was suggested that a comparison be made of the volume of irregular bodies in order to evoke the use of the Archimedes' principle, the trend favoring this belief manifested itself more vigourously. The excerpt from a discussion during the team-work, shown in Table 4, where certain phrases have been emphasized by the use of bold-face, comes to prove our former

Table 4. Extract of a discussion around Archimedes' principle

Researcher: [...] Do we all agree that we are measuring volume?

A chorus of the whole group: Yes.

Teacher No. 2 (from one team): It is volume. In the experiments we performed we saw that volume was actually measured; we concluded that volume is the place that a body occupies in space. In order to prove this –considering that there were some doubts among our colleagues—we took one of these cubes (takes a plastic cube which can be opened and closed) and we filled it with modeling clay; **it was heavier, it sank**. The one that was empty, we had to push with a pencil; in other words, weight helps immersion. **Yet, in the level of water with modeling clay and with the empty, closed cube, the same displacement occurred. That means that weight has nothing to do** with displacement.

Teacher No. 3 (from another team): You mean that **with modeling clay inside, or without it, the rise is the same?**

Teacher No. 2 and his team in chorus: Yes.

statement.

The fact that the teachers designed the experiment of filling a plastic cube in order to have two bodies with identical volumes but with different weights, to see if this affected the change in level, shows that more than one of the teachers had this doubt; and this is corroborated when a teacher from yet another team asked, incredulously, whether the level of water had risen the same amount.

3) A qualitative didactic approach, versus the use of formulas. One final aspect that was explored had to do with the teacher's ideas concerning the teaching of volume. In order to collect information on this point question 2 in Questionnaire 2 was designed (see Table 1). Out of nine participants who answered items b) and c) of this question, three of them stated that they would not suggest to Julio a different way of doing things, arguing that children *must use their own procedures.* The written responses are now given of those teachers who answered in the affirmative, including the way they argued their opinions. *Teacher No. 8: Yes, the application of the formula V = the area of the base x the height; but first we would have to perform activities leading to the comprehension of this measurement. Teacher No. 15: Yes, to measure what the base contains (how many cubes will fit in it) and then multiply this by the number of "floors" [i.e., the number of levels] that are repeated.*

Other teachers would try to direct the children towards a more profound relationship between the linear dimensions of the body and its volume, and some of them intended for the child to deduce such a relationship. *Teacher No. 2: Let him/her leave four cubes as a base. Teacher No. 10: Yes, he should try to find some regularity akin to the one he/she found for the cube, in order to obtain the volume of a parallelepiped. Teacher No. 6: Yes, I would ask him/her: How would you calculate the volume without breaking it up. Imagine that I cannot dismount it because the cubes are glued.*

As it can be seen, a major tendency towards a valuation of the quantitative aspect in the study of the notion of volume made itself evident. The use of the make break transformations that Freudenthal and other researchers consider fundamental for the formation of the volume mental object was not valued. This trend also arose during the workshop. To illustrate this episode, a dialogue taken from the transcript pertaining to the first session is included. Among other things, the National textbooks were examined –these being, as it will be remembered, of nation-wide application, the only ones, and mandatory—where it is intended to develop a teaching model for volume through a primarily qualitative approach. *Teacher No. 5: Oh! Here, even a problem on how to measure is shown, [...] how to deliver 12 liters [...] we say that this does pertain to volume. [...later, about a different lesson] yes, the fact is handled that the box is filled up. But not because a box is filled this must pertain to volume; that's the way we feel about it.* These kinds of expressions were fairly abundant in all the workshop sessions, including those dealing with spatial imagination, where the calculation of volumes was not involved.

Concluding Remarks

In this article, three trends about the teachers' beliefs have been discussed: two of them have to do with their knowledge about the concept of volume, and the other falls into the sphere of the teaching of such a notion. One aspect that deserves greater attention, and perhaps further study is the one linked to the difficulty of changing some of the teachers' beliefs, as we have already said. The problems commonly posed to generate a discussion on such beliefs, and the methods the teachers use to solve the problems –such as generalizing particular cases– only succeed in reinforcing the aforesaid beliefs. Another aspect deserving reflection is that a reform was carried out in Mexico, that included changes in the curricula and in the approach to the educational proposal for primary-school education. It also included the preparation of new didactic materials such as textbooks, activity files, and teachers' books that reflect such changes. These materials have been used since 1993, preceded by workshops and courses for the teachers who worked with them. Throughout this process, a vision of mathematics was proposed where many topics are approached with a qualitative focus, as in the case of volume. A fact that has been confirmed by the present investigation, however, is that this effort becomes clouded by the value that many teachers confer to the quantitative aspects. This situation highlights the need for studies leading to strategies and situations that may facilitate a change in the teachers' conceptions about what teaching, learning and doing mathematics might mean.

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STRUCTURING MATHEMATICAL BELIEF STRUCTURES – SOME THEORETICAL CONSIDERATIONS ON BELIEFS, SOME RESEARCH QUESTIONS AND SOME PHENOMENOLOGICAL OBSERVATIONS

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Abstract: This article is to be understood as a theoretical-analytical contribution on the definition of mathematical beliefs and their possible structuring. The understanding of possible clusters of mathematical beliefs is our central concern. Whereas data collection concerning mathematical beliefs has received substantial attention in specialized literature, leading to the identification of a considerable number of beliefs, contributions focused on the definition and on the categorization of beliefs or the description of correlating interdependencies are comparatively rare. Alongside a short overview of various approaches, implication patterns for beliefs in relation to calculus are used as an example here. There is the impression that domain-specific beliefs are induced by certain mathematical global beliefs.

The Starting Point: The Lacking Definition of Mathematical Beliefs

In scientific contexts, terms play a functional role. Their appropriateness is especially justifiable when they facilitate the formation of pertinent research questions. This interplay between the creation of terminology on one hand and the resulting implications on the other should specifically be clarified for the field of *mathematical beliefs*. The reported observations are constituted by a phenomenological character, and while the answers should be understood only as initial explanatory attempts, they should show the categorical prolificness of the terminology coinage. For conciseness, consequences for the everyday mathematical life of students, teachers and others cannot be discussed.

In the literature, there are a great number of papers to be found concerning beliefs in mathematics as well as beliefs in the learning and teaching of mathematics (e.g., Thompson 1992), featuring in particular teachers' beliefs. In current literature, however, there is still no consensus on a unique definition of the term 'belief'. There is also no evidence that enlightening contributions can be expected in the near future in the pedagogical related sciences, especially since considering culturally differing positions cannot be ignored (e.g., Alexander & Dochy, 1995). Our mathematical didactic focus on beliefs apparently indicates a demand that is otherwise not pivotal in social sciences. This sobering conclusion however is opposed by the observation that more than a few scientific papers have included substantial results concerning mathematical beliefs without first explicitly defining the term or specifically referring to an existing definition. Possibly concepts that incorporate the various potential definitions of beliefs are being tacitly underlain from each of the respective researchers, even if only implicitly (cp. to the comparative discussion in Furinghetti & Pehkonen, 1999).

However, there are many papers focusing on processes of learning and teaching which totally overlook beliefs, although some overlapping with belief features is evident. It might be that related theoretical frameworks, i.e. theories, are widely accepted and strongly established, e.g., theory of attitudes, theory of attributions or motivations, and these theories are then applied directly without mentioning beliefs. Inasmuch interesting developments that could enrich the theoretical discussion about beliefs unfortunately do not provide direct contributions to the technical advancement of the research of beliefs.

The author must ask to what extent the search for an authoritative definition is not a question posed improperly. Perhaps this can be presented analogously with an example from mathematics. At no point is it defined in arithmetic what is to be understood by a number, and yet man has successfully worked with numbers for many centuries in spite of this. Only Dedekind's noted work (1995, originally published 1888) *What are numbers and what should they be?* led the way to an axiomatic definition of numbers. In other words, the naïve number term is anchored in the perception of the (number) fields. The definition of a (number) field is by no means monomorphical, and so there are a number of non-isomorphical realizations and models.

Finally it should be noted that several authors have themselves legitimately modified the 'definitions' of what they understood to be beliefs over time, e.g., Schoenfeld (1985, 1998).

Theoretical Framework and Significance

The integral motivation for the struggle for a definition of beliefs is the effort to separate beliefs from cognition. Schoenfeld (1985) points out that the purely cognitive components of his framework for the analysis of mathematical behavior did a poor job of predicting the problem-solving processes of students. A significant contribution to this topic was presented in the works of Abelson (1979) as well as in those of Calderhead (1996); the former included quasi characterizing stipulations that do not provide the aspired limitations between beliefs and cognition, but are all in all considered constitutive (see also Nespor, 1987). In this sense we attempt to list relevant characteristics of beliefs and to understand the demands as a whole as a definition. The author is aware of the fact that all variables (for example the context dependence) will never be able to be made explicit with a conceptual construction this interwoven. On the other hand, highly dimensional models are seldom explanatory, thus we must reduce the number of variables.

Our starting point is Schoenfeld's (1998) definition approach in which he conceives mental constructs representing the codification of people's experiences and understandings as beliefs. It is our task to define what should be considered mental constructs. In the following, we aim to present constitutive characteristics, which are possibly central for all belief definitions.

As in the theory of attitudes (Eagly & Chaiken, 1992) we first logically speak of **belief objects**. Abelson (1979) uses the term 'content set.' Basically anything that shares a direct or indirect connection to mathematics can function as a belief object. We will provide several typical examples without an attempt at completeness:

(a) mathematical facts (= objects) (e.g. binomial theorem, the definition of a square, the number Pi etc.), mathematical procedures, domains within mathematics (e.g. geometry, calculus etc.), mathematics as whole, mathematics as a discipline (school mathematics, mathematics at university, industrial mathematics, mathematics within society etc.).

(b) relations where mathematics or a subunit of mathematics (see (a)) is a substantial part (mathematics and application, mathematics and history, usefulness of mathematics).

(c) relations where mathematics as well as the individual is a substantial part (self-concept as a learner of mathematics, self-concept as a teacher of mathematics, personal anxiety and mathematics etc.).

(d) the learning of mathematics itself, the learning within a specific domain, the learning of special content or topic etc.

It is noticeable that the belief objects have various 'sizes', so that we refer to the **breadth** of a belief object.

The belief object O is associated to the actual – what we are traditionally calling - 'beliefs' whereas a large variation breadth – from a single belief to complex network of beliefs - must be presumed here. When Schoenfeld (1998) refers to 'mental constructs', one can understand by that the individual statements, suppositions, commitment and ideologies, but also attitudes, stances, comprehensive episodical knowledge, rumors, perceptions and finally even mental picture. *It is essential that they can be allowed sufficient stability.* We will refer to the multitude of these mental associations as the **range R_O of a belief related to the object O**. We remark that Rodd (1997) differentiates between epistemic and attitudinal beliefs. Regarded mathematically, we associate not only a classical set to the belief object O, but indeed a fuzzy set R_O, i.e. for the elements of this set of mental constructs we allow various degrees of membership and so R_O turns into a fuzzy set (Zimmermann, 1990). In other words, we assign a **membership degree** $\mu(x) \in [0,1]$ to each element x within the range of beliefs. This approach takes into account the fact that beliefs can be held *with varying degrees of certitude*. On the other hand, *activation levels of a belief* can also be modeled using the membership degree. To insure completeness, it is often remarked that beliefs in differing contexts have differing strengths. To determine the underlying influencing variables (when, why, how much etc.), however, is a central question of research.

When we likewise accept pictures or perception as mental associations, we also make possibly the integration of the known term formation 'concept image' into our terminology. Unfortunately, it has far too rarely been noticed that Tall & Vinner's (1981) concept of concept images contains constitutive elements of belief definitions. One notes, *"We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over years through experiences of all kinds, changing as the individual meets new stimuli and matures."* or even more explicit, *"the visual representations, mental pictures, the impressions, and the experiences associated with the concept name"* (Vinner, 1991). In a rough approximation, the so-called 'concept definition' (using Vinner's terminology) plays the role of a belief object.

It is folklore and all various definitions in common that beliefs rely heavily on evaluative and affective components. For this reason we require as a further module one or more **evaluation map(s)** e_λ, defined on the range of a belief R_O and with a linguistic value scale (in fuzzy theory we speak of a 'linguistic variable' - see Zimmermann, 1990, p. 132). Possible values could be 'important' or 'minor', 'good' or 'bad' etc., however a continuous scale can also be assumed under other intentions.

There is no need to mention that every belief definition must take two basic variables into account, namely the person P who has professed the belief or to whom the belief is attributed. Finally beliefs are dependent on the time of constitution. By a **belief B** we understand a quadruple (O, R_O, μ, e_λ), whereby O is the debatable belief object, R_O represents the range of mental associations (what traditionally is called belief), μ models the activation levels or differing strengths of a belief and the evaluation map(s) is (are) represented by e_λ. Furthermore B should fulfill the following characteristics in a probabilistic sense:

(1) For each person $P' \neq P$ the range R_O of beliefs on the same belief object O is not necessarily consensual (nonconsensuality).

(2) Beliefs are likely to include a substantial amount of episodic material from either personal experience, from folklore or from propaganda which influences the evaluation map e_λ (episodic material and its evaluative impact).

(3) The range R_O of a belief is a priori not necessarily bounded (unboundedness).

(4) Beliefs are often anchored in authorities (external anchoring).

(5) Beliefs are directly or indirectly linked to the self-concept of the believer P at some level (self-linkage).

We stress: (1) Beliefs of different persons on the same beliefs objects are not necessarily consensual (nonconsensuality). Sementically, 'beliefs' as distinct from knowledge carries the connotation of disputability, and the believer is aware (or will become aware) that others may think differently. (2) It is known that e.g., especially knowledge systems are not necessarily dependent on episodical material and that the knowledge possibly carries a stamped date, which does not contridict the first point. (3) The 'openness' and 'unboundedness' apply to the amount R_O. This can be accounted for by the situation in which the process of the integration of episodic material can never be perceived as fully completed. (4) This condition can also be found in part in Abelson's (1979) work when he postulates that belief systems are in part concerned with the existence or nonexistence of certain conceptual entities. Here, these authorities might be virtual authorities in a platonic sense, might be teachers, colleagues, friends, parents etc. It should be mentioned that for transforming a belief into knowledge, the warrants of the beliefs are crucial (see Rodd, 1995). (5) This property is in some sense dual to (4): Abelson (1979) pointed out that knowledge systems usually exclude the Self, while beliefs do not.

We will forego a detailed definition of a **belief system**, but at any rate, several objects play a role in belief systems. The rational network of the objects in question then transfers itself onto the structure of beliefs resp. their ranges. Nespor also suggested that beliefs loosely bounded networks with highly variable and uncertain linkages to events, situations, and knowledge systems (Calderhead, 1996).

However, we are interessed in the question of to what extent sets of beliefs with respect to different belief objects are structured. The question of the structure of belief networks appears to us to be of greater importance. It can be assumed that via the internal network structures of beliefs enlightenment can be attained of the cognitive memory patterns and their links. At the same time, this should enable the localization of weaknesses in the acquisition of knowledge. In this sense, beliefs also have diagnostic characteristics and therefore understanding structures of belief networks is of central importance.

Possible Categorizations of Beliefs and Belief Systems

With reference to the above definiton of beliefs, we would like to present possibilities of structuring beliefs.

The personal parameter P as a variable - group-specific differentiation

Beliefs are often specified and then researched according to the various groups of subjects. Accordingly, various results are collected when surveying different groups (e.g., students, teachers, professors, etc.) about the problem field of beliefs on math. Building on this background, the central question arises as to what effect beliefs have on teaching and learning processes. Only a few isolated empirical-based results are available here, although there seems to be positive confirmation in that research.

Belief objects O as a variable - different belief dimensions

As previously mentioned concerning beliefs on mathematics (as a science, as a university subject, as a school subject, as an engineering discipline, etc.), the learning or teaching of mathematics as such also entails value judgements by the learner or the teacher and is thus in this sense directed introvertedly. A distinction according to these aspects leads to a preliminary categorization for terminology clarification. These specifications take the possible diversity of potential belief objects O into consideration. There are numerous indications that beliefs to single objects (e.g., mathematics) can hardly be discussed successfully when one ignores the relation to other objects (e.g., mathematics teaching). Thom´s quotation[1] (1973) which demonstrates in exemplary fashion that cross-links between the above-mentioned fields cannot be ignored is sufficiently well known.

The evaluation map e_λ - Green's dualistic categories

In his book *Activities of Teaching,* Green (1971) is also concerned with the question of which role beliefs play in the learning process. Alongside the obvious postulate that beliefs distinguish clusters, Green distinguishes beliefs according to two features. He refers to quasi-logical and quasi-psychological dimensions of beliefs and allocates them two polar states; in view of their quasi-logical character, beliefs can be, *primary* or, *derivative.* The quasi-psychological role can be either, *psychological primary* or alternatively be more *peripheral.* In view of the definition that we initially provided, Green differentiates with reference to possible evaluation maps e_λ. In one case, it deals with the quasi-logical scale with two possible values, namely primary or derivative. In the other case, the map e_λ measures quasi-psychological situations.

At a first glance this 2 x 2 typification appears quite convincing. However, it proves to be problematic and finally open-ended for the identification of beliefs. Only a few papers in the literature have previously offered convincing interpretations and contributions (cf. Cooney et al. 1998, Jones 1990), regarding which criteria should be correlated to each respective 'value'. An open question of research is the possible interaction patterns of the accordingly categorized beliefs.

Subject-Specific Structuring of Beliefs

At this point, we would like to return to the discussion on the differentiation of beliefs according to breadth of the belief objects O. In specialized literature, the word 'belief' is employed at times as a synonym for the terms 'philosophy' or 'ideology', in particular when a discussion focuses on general attitudes or beliefs, e.g., on mathematics as a discipline (McLeod, 1989). In view of the belief object, i.e. here of mathematics in general, we use the term *'global beliefs'*. This Top-Down-approach compliments a Bottom-Up-Analysis when referring to detailed aspects of mathematical objects. Analogous to the term subject-matter-knowledge used by Even (1993), we use the term *'subject-matter-beliefs'* which refers to the amount and organization of knowledge and beliefs per se in the mind of the subject (see also Lloyd & Wilson, 1998). However, any investigation of beliefs in the field of this subject matter will soon indicate that these two poles, namely global beliefs versus subject-matter-beliefs, are too distant to cover all aspects. We therefore propose the use of a middle 'intermediate level' which we give the term *'domain-specific beliefs'*.

Our research (Törner, 1999) shows that mathematical domains such as geometry, stochastics or calculus are always associated with specific beliefs. For example, in the case of calculus, beliefs represent views on the role of logic, application, exactness, calculation, etc. Similar dimensions are also relevant in other fields; however, there is the impression that subjective realizations differ. Domain-specific beliefs should be classed hierarchically higher than e.g. notions of the term 'derivative' or the term 'function', although on the whole they still touch on basic views on mathematics. Thus the following **research question** arises:

> *Which dependency or implication structure exists between global beliefs, domain-specific beliefs and subject-matter-beliefs? Do the sum of the beliefs from the individual fields of mathematics constitute beliefs on mathematics as a whole, or do general views tend to imprint subjective perceptions in the individual domains more?*

Sources of Information and Mode of Inquiry

In previous research, we asked six preservice upper-secondary-school teachers (in their post-graduate phase) to express their experiences with calculus lessons in the form of freely written essays. At the time of composing these essays, the students were still participants in a didactics of mathematics university course. Therefore, we had to rely on voluntary participation of the essayists and accept anonymous participation in completing the related questionnaire. The essay themes were "Calculus and me - how I experienced Calculus at school and university," "How I would have liked to have learned Calculus," and "How I would like to teach Calculus." We were subsequently presented with a total of 3 x 6 = 18 qualified, but partially anonymous statements (two to four pages each). These served as the basis of our survey in which we examined the beliefs imminent in the essays (comp. Törner, 1999). The individual results revealed the following main statements which can be allocated the status of a belief-characteristic: (1) Calculus is (reduced in school down to) calculating (not necessarily meaningfully) with functions. (2) Differential calculus is a craft - integral calculus is an art. (3) Logic is a central guideline for mathematics and in particular for calculus. (4) Exactness as a property of mathematics can be demonstrated in calculus in particular. (5) Calculus has the special task of preparing pupils for subsequent university courses. The next two statements with belief character address learning aspects. (6) Mathematical elegance and abstractness - a liking of mathematicians - mean a loss of descriptiveness and understandability. (7) The recognition of application links facilitates learning.

In the following, we assess statements (4) and (5) in the students' essays in view of a possible interrelation with general views on mathematics, whereby we must limit ourselves to the aforementioned essays as subsequent research was restricted by the partly anonymous nature of the essays.

Results

For conciseness, only a few aspects of the evaluation are listed here as an exemplary discussion. Lars made the most prominent statement on the aspects of logic in its relation to calculus. It is remarkable that his global view on mathematics is structurally dominated. We quote some excerpts: ... *'logical material' can easily be worked with ... when you have acquired the rules (e.g., the transformation of fractions into decimal numbers).* According to Lars, calculus has a similar pattern, *as ... mathematical-logical thought was developed and deepened here ...* The university seminar he visited on this strengthened his belief: ... *This began in calculus with the foundations of logic (which I found to be very helpful).* The consequence for him is a rigorous orientation to the aspects of logic: ... *if it were possible to do something on logic in school as early sixth grade* (with the eleven to twelve-year-olds). Without going into details, the beliefs on logic from Lars can be psychologically evaluated as central as well as primary.

Another student (Sascha) also underlines the central role of logic in calculus lessons. His view of mathematics was indirectly influenced by his assessment of lessons at secondary school in Germany in the mathematics courses in the Oberstufe, and it is his opinion that *the schools should pay greater attention to the demands of the mathematics students to make studying the topics later at university possible, even attractive (in the calculus course) with the aid of e.g., formal logic....*

Concerning assessment of exactness as an important feature of mathematics, which is experienced particularly, well in calculus, two further students, Nicolas and Lars, state their positions. Whereas Nicolas views exactness as an unavoidable difficulty, which can be didactically mastered, Lars views the aspect of exactness more fundamentally. Mathematics demands in his words ... *utmost precision and a lot of effort..., therefore one should start operating with exact terms as soon as possible. Calculus is suitable for this pursuit. ... For example the ε-δ-definition for constancy can be considered one of the greatest achievements in the cultural history of mathematics....*

Interpretation of results

The students' quotations show that domain-specific beliefs cannot be dislocated principally from global views on mathematics. A number of obvious conclusions can be made to this effect. Our mathematics lessons (and partially our university courses) do not necessarily induce a pluralistic worldview of mathematics. There are a number of reasons for this: mathematics is often taught in modules and for this reason often perceived as such. Also, from a learning psychology viewpoint, the perception of unity is more dominant than perception of broad variation. Thus, global beliefs are oriented towards a more structural-axiomatic organization of mathematics, which in turn leads to aspects of logic being allocated a central role. In this sense, a perceptive student can experience a reinforcement of his or her assessment due to the content and the methodology of the university calculus course. Under the "axiom" that school mathematics classes are a preparation for the university, school lessons are also viewed one-sidedly.

There is an impression that Perry's stages theory presented by Ernest (1991) in another context offers a possible explanation to understanding the strict dependency in Lars' beliefs: they can be understood as a dualism. From the author's viewpoint, there is evidence here of a multifaceted, pluralistic working with and understanding of mathematics. Central mathematization patterns have to balance scales with the multifaceted nature of mathematical phenomena and have to enrich each other in their interdependent nature. This ideal state could then be described in the wording of Perry as 'relativism'.

Note

1. In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics.

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TEACHER CHANGE IN THE CONTEXT OF COGNITIVELY GUIDED INSTRUCTION: CASES OF TWO NOVICE TEACHERS

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Abstract: Two teachers who joined a five-year Cognitively Guided Instruction [CGI] project during their first year of teaching seemed similar at the beginning of the project but exhibited different patterns of change in instruction and beliefs across four years of implementing the principles of CGI in the classroom. Change was documented in the areas of discourse, children's thinking, and instructional planning through analysis of transcribed annual interviews, teachers' written responses to a variety of instruments, and classroom observations with post-observation interviews. By the end of the project, Ms. A provided students with opportunities to solve a variety of problems but she did not use what children shared to make instructional decisions. In contrast, Mrs. D's instructional planning appeared to be driven by her knowledge about individual children's mathematical thinking. Their Belief Scale scores also differed significantly.

Instructional decision making for teachers who implement the principles of Cognitively Guided Instruction [CGI] (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989) is largely dependent on research-based knowledge of children's mathematical thinking. CGI teachers focus on students' understanding and then adjust instruction to match the thinking of individual students. This study focused on how two first-year teachers changed as they gained experience in implementing the principles of CGI across a four-year professional development program (National Science Foundation Grant ESI-9450518). Specifically examined were changes relative to discourse, children's thinking, and instructional planning using the theoretical framework provided by Fosnot's (1996) principles of learning derived from constructivism: learning is not the result of development, learning *is* development, disequilibrium facilitates learning, reflective abstraction is the driving force of learning, dialogue within a community engenders further thinking, and learning proceeds toward the development of structures.

Method

Project

Through the CGI project (Jan 95-Dec 99), 5 teams of mathematics educators (originally 2 teacher educators and 6 teachers on each team) learned to use CGI as a basis of mathematics instruction. Workshops were held in May 1995 (3 days), July

1995 (10 days), June 1996 (8 days), June 1997 (7 days), June 1998 (4 days), and June 1999 (2 days). Between workshops, teachers implemented CGI in their mathematics instruction, each team met approximately once a month, each teacher was visited approximately once a month by one of the team's teacher educators, and project staff visited each teacher once each semester to provide general support.

Instrumentation

Data sources were transcribed annual interviews, written responses on several annually administered instruments, field notes taken during classroom observations on two consecutive days in Spring 1998, Fall 1998, and Spring 1999, and transcribed debriefing interviews that followed each observation.

Subjects

At the beginning of the project, both Ms. A and Mrs. D were first year teachers with K-6 licensure. Ms. A was teaching third grade and Mrs. D was teaching second grade. Ms. A changed to first grade in the second year and continued to teach at that grade level through the remainder of the project; Mrs. D continued to teach second grade. Both of their principals participated in part of the summer workshops. During the year, Ms. A had limited within-school support from her principal and peers at her school. Mrs. D enjoyed ongoing enthusiastic support from her principal, assistant principal, and peers, with collaborative support each year from another teacher at the school who was also in the project. By the project's end, Ms. A appeared to be at level 3 on the Fennema, Carpenter, Franke, Levi, Jacobs, and Empson (1996) 'levels of instruction' scale. She provided students with opportunities to solve a variety of problems, but she did not use what children shared to make instructional decisions. Mrs. D seemed to be at level 4b: instructional planning appeared to be driven by her knowledge about individual children's mathematical thinking.

Analysis

Data from the teachers' reflections on teaching, classroom observations, and debriefing interviews were grouped into three categories (i.e., discourse, children's thinking, and instructional planning) using Fosnot's principles and the principles of CGI. The extent to which the teachers attended to the three components simultaneously in creating coherent mathematics instruction was determined through classroom observation data.

Results and Discussion

Discourse

At the beginning of the project, Ms. A's mathematics lessons involved giving students a problem and then telling them the steps that they needed to follow to solve it. By the end of the project, she encouraged students to construct their own solution strategies and share them with the class. However, the general pattern of sharing seemed to involve multiple monologues rather than a dialogue between the teacher and a student or among students. At the beginning of the project, Mrs. D also taught by demonstrating procedures and explaining steps, but by the end of the project, her references to discourse became characterized by increased emphasis on student-to-student discussions of important mathematical ideas facilitated by teacher input in the form of questions. Specific purposes for asking varying types of questions became more clearly defined and detailed as she applied her increased knowledge of problem types and students' solution strategies. She described her class as a learning community where she was a partner with her children in making sense of mathematical ideas.

Children's Thinking

Across the project, Ms. A referred to children collectively and appeared to focus little if at all on the importance of mathematical thinking of individual students. Her reflections relative to assessing children's understanding centered on strategies that are global in nature (i.e., "seeing" how children solve problems in general and "reading" the looks on their faces). She seldom referred to explicit solution strategies used by individual children nor did she seem to consider the effect of different problem types on children's understanding. By the end of the project, Mrs. D assessed children's thinking based on the solution strategies they constructed and the content of their explanations. At the beginning of the project, she believed that students struggle with mathematical ideas only when instruction has been insufficient; by the end, she realized that, by letting students struggle to develop their own solution strategies and to make sense of strategies shared by others, her students were developing confidence in their ability to understand mathematical concepts. She appeared to recognize how knowledgeable and capable her students were and, subsequently, set higher expectations of their mathematical growth.

Instructional Planning

Ms. A's instructional planning at the end of the project was similar to her pre-CGI planning. In general, she planned lessons that focused on having children memorize basic facts and learn to do standard arithmetic algorithms. She viewed CGI in a limited way; she "taught CGI" once a week and "taught math" the rest of the week. In addition, she selected students to share strategies dependent more on whether they had been on task than the strategies used and how they might be helpful to other students. Mathematics lessons followed the textbook and the state-curriculum's organization rather than important levels of thinking that students must go through in order to develop concept understanding. At the beginning of the project, Mrs. D focused on "students' prior knowledge" and "learning styles" as her basis for instructional decision making. Individual instruction was provided for students who struggled during the whole-group instruction. By the end of the project, she based instructional decisions on specific knowledge of the children's levels of thinking. She differentiated instruction within lessons for individual students by varying numbers in problems, types of problems assigned, and the choice of manipulatives available for student use. She also grouped students who were at similar levels of thinking and encouraged students to pose and solve their own problems. She intentionally challenged students to struggle with making sense of mathematical concepts because she saw the need to create an instructional environment that continually encouraged children to move toward more efficient strategies.

Mathematics Instruction

Ms. A and Mrs. D changed their mathematics instruction across the project, but in different ways.During observational visits, Ms. A posed CGI-type problems during at least one lesson, but there often seemed little "connection" between the 2 consecutive lessons. For example, Ms. A followed a lesson on open and closed plane figures with a lesson based on two-digit addition and subtraction without regrouping. During the latter lesson she focused mainly on the students' factual knowledge with little apparent use of the information gained as children shared their solutions. She seldom sought further clarification concerning the level of their mathematical thinking. During another visit, Ms. A indicated that she taught the day's lesson because it was the next one in the textbook. Mrs. D typically planned instruction based on her knowledge of student thinking. For example, during one lesson, a student shared the following solution to a word problem involving the sum of 5 and 7: "I know that 5+2 is 7, 5 and 5 more is 10, and 10+2=12." Ms. D had not yet introduced the concept of place value, but because of this student's solution strategy, she posed a problem the next day that involved the sum of 6+4+7 to see if the children might begin to think in terms of 10 as a unit. This use of children's thinking when planning the next mathematics lesson was evident across Mrs. D's observed lessons.

Summary

Although the two teachers were similar in their teaching experience at the beginning of the project, attended the same workshops each summer, and had supportive classroom visits each year, they changed across the project in different ways. Ms. A's instructional activities often were inconsistent with the principles of CGI, while Mrs. D clearly and effectively based mathematics instruction on CGI principles. *Why did these two teachers conclude the project with such different CGI implementation levels?* This question may be answered, in part, by differences in the amount of support each received. Ms. A's principal supported her use of CGI, but she was mainly concerned about how CGI might help improve student scores on standardized testing. Most teachers at Ms. A's school were not interested in observing in her classroom or learning about CGI, and they were concerned about whether her students would have the procedures and facts that "first graders are supposed to learn." This view by her colleagues may explain (a) why Ms. A distinguished between teaching CGI and teaching mathematics and (b) what appeared to be a lack of confidence in her own knowledge and competencies. In contrast, Mrs. D received strong support for her CGI implementation and, in addition, a system-level supervisor arranged for some of Mrs. D's lessons to be videotaped for system-wide professional development sessions.

Consideration also needs to be given to what appear to be critical differences in beliefs about teaching and learning. Mrs. D's baseline total scale score on the Beliefs Scale (Peterson, et al., 1989) was 27 points higher (more constructivist) than Ms. A's score. Belief Scale scores at other times during the project support the notion that Mrs. D's "buy in" to the project stabilized early. Ms. A's scores increased some but then declined across the project. In fact, Ms. A's total scale score in Spring 1999 was 7 points below Mrs. D's baseline score (May 1995).

In applying Fosnot's principles to the data, differences in teacher change may be related to the amount of support each experienced. Mrs. D received ongoing direct support and encouragement in her school setting and as a result continued to develop as she learned; i.e., *learning* is *development*. Ms. A, however, may have seen little need to learn about children's thinking because of the lack of support and encouragement that she experienced. As a result, she may not have developed to the level that she might have if she had experienced support similar to that which Mrs. D enjoyed. Once Ms. A added problem-solving activities to her instructional planning and having children share their solution strategies, she seemed content with her instructional planning and may have seen little need to struggle with finding out about individual children's mathematical thinking. Thus, she did not experience *disequilibrium, which in turn would have facilitated further learning.* Without ongoing collegial support and interaction, Ms. A may not have been challenged to a reflective abstraction level that could have served as an impetus for further learning (i.e., *reflective abstraction is the driving force of learning*). Further, she did not have an opportunity for *dialogue that might have engendered further thinking.* In total, these limitations may have affected the *development of structures* that were needed for Ms. A to reach a higher level of CGI implementation. For example, implementing CGI principles only once a week would limit the extent to which most teachers could learn about children's thinking, let alone plan instruction based on the students' mathematical understanding.

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CHANGES IN TEACHERS' BELIEFS: FOUR YEARS OF CGI IMPLEMENTATION

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We examined changes in teachers' beliefs about teaching and learning across a five-year cognitively guided instruction (CGI) project. The CGI Beliefs Scale (Peterson, Fennema, Carpenter, & Loef, 1989) was administered six times during the project; we had complete data on 16 teachers. For each administration, four subscale scores (i.e., Role of Learner, Relationship Between Skills and Understanding, Sequencing of Topics, and Role of the Teacher) and a total scale score were determined. Scores were subjected to repeated measures analysis of variance and non-linear regression analysis. For each subscale and the total scale, time effects were significant at an alpha = 0.0001 level, and hyperbolic least-squares regression models across time fit the data with r_{corr} values in the range of 0.97.

During the initial year of implementation of principles of CGI, teachers' beliefs declined, though not all the way back to the baseline. By the end of the second implementation year teachers' beliefs *recovered* to the same level evidenced immediately after the initial workshops. Little change occurred in scale and subscale scores after the second implementation year. During the entire four-year implementation period, teachers were supported extensively. It seems possible that without this continuous support, teachers might have given up on CGI and abandoned it as an organizing scheme for their mathematics instruction. This study strongly supports the need for long-term, intensive support for teachers. The hyperbolic models derived from the data in this study may prove helpful in assessing project impacts and in beginning to correlate levels of beliefs to implementation.

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ELEMENTARY PRESERVICE TEACHERS' CHANGING BELIEFS: A CONTRAST OF TRADITIONAL AND NON-TRADITIONAL STUDENTS

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This study was designed to explore changes in preservice elementary teachers' beliefs about mathematical instruction and to contrast changes in beliefs between traditional students and non-traditional students within the same teacher education program. Within the mathematics methods course, the students were instructed in a constructivist approach to teaching mathematics. To assess changes in the preservice teachers' beliefs about teaching and learning mathematics, the CGI Beliefs Scale (Peterson, Fennema, Carpenter, & Loef, 1989) was administered at the beginning, and again at the end, of the mathematics methods course.

Though the two groups started at essentially the same level, and both the traditional students and the non-traditional students increased their scores over time ($p<.0001$), the increase was greater for the non-traditional group. One possible explanation for the observed differences between the traditional and the non-traditional students is that the material in the methods course may have been more meaningful to the non-traditional students by virtue of their increased exposure to children and children's thinking before the course began. For traditional students, the relatively small and undifferentiated changes across time seem to indicate a need to significantly increase traditional students' experiences with children prior to their enrollment in a mathematics methods course.

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FUTURE TEACHERS' BELIEFS AND CONCEPTIONS ON THE USE OF TECHNOLOGY, MATHEMATICS, AND MATHEMATICAL PROBLEMS VS EXERCISES

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Recent changes in curricula have considered that problem solving and the use of new technology can be utilized as a frame in the teaching and learning of mathematics. In this context, the role of teachers is now seen as a dynamic, critical, reflexive, agent who now has a greater responsibility in the teaching-learning process of mathematics. This study aims to report some conceptions and beliefs endorsed by future teachers concerning the use of new technology, the notion of mathematical problem vs. exercise and the nature of mathematics. We designed a questionnaire with three open-ended questions. The study sample consists of 43 students of the last semester of the undergraduate degree in mathematics at the University of Granada (Spain), the main job perspective for these students was teaching. The questions were:

1. When calculators, computers or other technologically advanced instruments are used, is mathematics being done? Explain your answer.
2. For you, what are the five main characteristics of mathematics?
3. State the main criteria that differentiate a problem from an exercise in mathematics. Illustrate with an example, if necessary.

The analysis of the written responses provided by the subjects helped us identify that mathematics is being done when new technology is used, since it lends itself to conjecturing, exploring, doing calculations etc.; that mathematics is logical-formal; and that there is a clear distinction between problem and exercise in mathematics. Moreover, the group of subjects who believe that mathematics is not being done when new technology is used state that the main characteristics of mathematics are its "aesthetic and beautiful" character as well as its practical usefulness, without ignoring its logical-formal characteristic.

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THE IMPACT OF AN INTEGRATED CONTENT/METHODS COURSE ON PRE-SERVICE TEACHERS' BELIEFS

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There is substantial evidence that teachers' beliefs about mathematics impact their teaching of mathematics. Given this, it seems imperative that teacher education programs assess their effectiveness, at least in part, on how well they nurture beliefs that are consistent with their philosophy of learning and teaching.

To explore this perspective a study was conducted with a cohort of pre-service elementary teachers participating in an alternative certification program that was grounded in a *Standard*'s based philosophy and used a *model/experience/reflect* framework. Within the program students were required to take 6 hours of mathematics and 6 hours of mathematics education. One faculty member taught the 12 hours as a seamless course over three semesters. The mathematics was taught in a manner that *modeled* mathematics teaching that was consistent with our philosophy. The teachers *experienced* teaching in this manner in their field experiences. Finally teachers *reflected* on each of the experiences to encourage assimilation.

Before and after year-one in the program the students completed a 30-item Mathematics Belief Instrument (MBI). It was predicted that the teachers would change in a positive direction on all parts of the MBI. To test this hypothesis a t-test for paired samples was computed for each part. Using a one-tailed test, the difference between the responses at the beginning and at the end of the first year were significant for all three parts: Part A, $t(13)=1.80$, $p<.05$; Part B, $t(13) = 4.40$, $p<.001$; and Part C, $t=1.79$, $p<.05$. In addition, qualitative data in the form of weekly teaching logs were used to identify teacher behaviors that were consistent with the MBI. Results suggest that the program was successful in changing teacher *beliefs* and *behaviors.*

CASE STUDY OF ADJUSTMENT: A GRADUATE TEACHING ASSISTANT'S STRUGGLES

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This paper seeks to provide further evidence of the problems graduate students face as they are teaching. In order to accomplish this, this study presents a singular case study of the graduate teaching instructor of Mr. M culled from an on-going investigation of the struggles graduate teaching assistants face when front-line instructors. Drawing upon a multitude of data sources such as daily journal entries, a hour-long interview, quizzes, tests, and copies of student work and reactions, the data revealed that Mr. M. experienced conflict in reference to: (1) constructing a non-threatening classroom versus losing control; (2) wanting students to ask questions versus answering "dumb" questions; (3) wanting an open classroom versus students talking out of turn; (4) teaching toward the bottom of the class versus boring the top; (5) ensuring students do homework versus not enough time to grade; (6) the responsibilities of teaching versus the requirements of being a student; (7) the lecture style versus active learning; and (8) implementing "difficult" quizzes to motivate students versus assessing to give a grade. Each of these conflicts reveals and repeats themselves at various times throughout his journal entries.

Beyond struggles that posed conflict for Mr. M. in terms of choices that he needed to make during instruction, a variety of problematic scenarios arose as part of teaching the course. In particular, five specific scenarios influenced Mr. M as he was teaching and included: (1) being accused by a student of telling, during a test, the student an answer was correct when it was not correct; (2) belief that two brothers were cheating off each other; (3) student wearing a t-shirt advertising his homosexuality; (4) being accused of not informing students that they needed to check their work when solving radical equations; and (5) watching students give up on the course and fail multiple times. Each of these situations greatly disturbed Mr. M and caused him to question his beliefs as a teacher and the interaction of his own moral character. This paper provides further evidence of the problems graduate students face as they are teaching and pose recommendations for the correction of the problems.

EFFECTS ON BELIEFS ABOUT PROBLEM SOLVING AND ITS INSTRUCTION AS A RESULT OF CAPSTONE COURSES

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This study focused on evaluating the effects of capstone courses in problem solving on students' beliefs about the role of problem solving in teaching mathematics. The two capstone courses were designed for prospective secondary school teachers and aimed to develop a deeper understanding of problem solving across the strands of mathematics (Roddick, Becker, & Pence, 2000).

Six people who had taken both courses in problem solving during the academic year 1998-99 were the participants in the study; three of these participants were teaching in some capacity during the course of the study. Both written and interview data were collected over a period of 12 months.

The six participants in this study seemed to fall on a continuum when examined from the perspective of changes in their beliefs about problem solving and its role in instruction. Two participants fell on one end of the continuum indicating minimal change, while one was classified towards the opposite end, indicating substantial change. The other three fell in between these extremes.

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EXAMINING HOW BELIEFS SHAPE INSTRUCTION: CASE STUDIES OF TEACHING ASSISTANTS IN A 'REFORM' CALCULUS COURSE

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Mathematics educational reform calls for increased emphases on conceptual understanding and communication of mathematics. Such reform often calls for new teaching practices, different from those that the teachers themselves experienced or were taught. Research has documented the significant influence that teachers' beliefs have on their practice (Cooney & Shealy, 1997; Thompson, 1992). This body of research has established the role that beliefs play in shaping how practices such as problem solving and sense-making are carried out in classrooms. It has not, however, analyzed the influence of beliefs on the moment-to-moment decisions that teachers make in their classrooms.

This study examines how graduate student teaching assistants' (TAs') beliefs about teaching, learning and the nature of mathematics shapes their interactions with calculus students as they implement reform-oriented discussion sections. The sections were designed so that TAs facilitate discussions and assist students working in collaborative small groups to solve challenging problems. Course materials are designed with the intention that TAs' interactions with students will focus on sense-making and conceptual understanding.

Fine-grained analyses of videotaped classes and interview transcripts are used to examine how TAs' beliefs influence their moment-to-moment decisions about asking questions and offering assistance. Different degrees to which each TA emphasize sense-making and conceptual understanding of the mathematical ideas are traced to different beliefs held by each TA.

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THE QUESTION IS SIMPLE: ARE MATH TEACHERS TRADITIONAL?

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They are everywhere. They fill up our professional literature, they haunt our memories... They are traditional mathematics teachers. As we approach the new millennium, a "simple" question remains: are most mathematics teachers still typically traditional?

Part of a larger project on teachers' beliefs, a survey with a random sample of 198 teachers from the southern part of Rio de Janeiro, Brazil, explored such a question. A questionnaire with 6 Likert scales represented traditional teachers as those who believe that: knowing math means being able to perform rapid calculation and apply correct procedures; one learns math through memorization and individual work; and one teaches math following the textbook and demonstrating examples of assigned tasks (based on Schram & Wilcox, 1988; Schram, Wilcox, Lappan, & Lanier, 1989). The result was that teachers disagreed with the statements presented (independent of their educational background and the level in which they teach).

Are most teachers not traditional? Are we, mathematics educators, presenting a skewed picture of teachers? The presentation will explore different venues for understanding such results.

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EXAMINING TEACHER BELIEFS RELATED TO ALGEBRA COMPETENCY IN THE CONTEXT OF THE "ALGEBRA FOR ALL" MATHEMATICS REFORM DEBATE

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The study of algebra in school has been a long and heated debate for almost a century (Schoenfeld, 1994; Stanic, 1987). What is it about this domain of mathematics that makes for such strong passions, especially when reform policies advocating algebra for all students are concerned (Chazan, 1996; Davis, 1994; Moses, 1994; Noddings, 2000)? Proponents of algebra requirements argue on the grounds of literate citizenry, labor productivity, and the need for participation in the global economy. Counter to these pressures are concerns that "coercing" some students into algebra courses because of a state exam or graduation requirement may do harm to their self-esteem, leading to school failure and dropout. This is a particularly important issue for high school mathematics teachers who face a growing trend in district, state, and national mathematics reform policies that mandate and/or recommend the study of algebra for all students (NCTM, 2000). Interestingly, little is known about how teachers view specific mathematical content such as algebra (Kieran, 1992). Yet, arguably, the debate around "algebra for all" is grounded in beliefs about the domain, how one learns the domain, and its usefulness in preparing for the future.

This case study of one comprehensive urban high school responding to a new district graduation requirement of three years of college preparatory mathematics examines how the mathematics teachers viewed issues related to their students' algebra competencies. It focuses on their beliefs about: algebra as a mathematical domain; developing student competency in that domain; the utility of that domain for their students' future endeavors. Analysis of teacher interviews and department meeting field notes yielded three main results. First, faculty conversations highlighted the role of abstraction as a key component of algebra competency. Second, this component distinguished algebra from other domains such as geometry and statistics. This has direct implications on curricular paths to advanced mathematical study. Third, teacher beliefs about the role of abstraction were strongly connected to beliefs about the utility of the domain for future educational, career, and life endeavors of their students. The beliefs about content, student competency, and the utility of mathematics raise complex and serious questions about implementing mathematics reform policies that advocate all students study algebra. Implications of this research on curriculum and instruction practices, professional development, and mathematics reform policy are discussed.

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AN INVESTIGATION OF BEGINNING SECONDARY MATHEMATICS TEACHERS' TEACHING: UNPACKING THE COMPLEX RELATIONSHIP BETWEEN BELIEFS AND PRACTICE

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Attempting to understand teachers' beliefs has become an important area of study for mathematics teacher education (Thompson, 1992). Mathematics teachers' beliefs about mathematics, and mathematics teaching and learning, have been shown to critically influence what happens in the classroom (Cooney, 1994). This study aims to contribute to the literature on secondary mathematics teachers' beliefs by shedding new light on how those beliefs relate to practice.

This interpretive case study (Merriam, 1988) explores two beginning (less than three full years of teaching experience) secondary mathematics teachers' beliefs about mathematics and the teaching and learning of mathematics, and how these beliefs are related to their developing practices within the context of their classrooms and schools. Data were collected between September 1999 and January 2000. Primary sources of data included interviews (10 with each participant) and classroom observations (15 with each participant).

The relationship was theorized to involve multiple factors including (1) teachers' beliefs about mathematics, teaching and learning, (2) teachers' content and pedagogical content knowledge, and (3) teachers' perceptions of aspects related to the physical setting (e.g., school and classroom). Preliminary results suggest that all of the theorized factors play a role in shaping these beginning teachers' decision-making and practice. The poster will describe a model of this relationship.

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THE RELATIONSHIP OF MATHEMATICS TEACHERS' BELIEFS AND EXPERIENCES ON THE USE OF TECHNOLOGY IN THE CLASSROOM

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Guidelines from the National Council of Teachers of Mathematics (NCTM) and other research in mathematics education encourage the use of technology in mathematics classrooms for teaching and instruction. But research has shown that the belief structures of classroom teachers do not, at times, coincide with these teachers' classroom practices.

This dissertation pilot project attempts to answer important questions relating teachers' philosophical inclinations of mathematics and their perceptions of the role of technology in the classroom. The project seeks to measure the variation in teachers' perceptions of the nature and role of mathematics and the variation in their personal use of technology in the classroom. Are their opposing views? What are the characteristics that define them?

The study involved 17 mathematics educators enrolled in a technology post-graduate course. Each of the subjects was given a basic demographics questionnaire, an attitudinal technology survey (ATS), an experiential technology survey (ETS), and a mathematics belief survey (MBS). The MBS results were analyzed for correlation with both the ETS and ATS instruments. Further clarifications of these relationships were analyzed through a pair of structured interviews conducted by this researcher.

Results of this study indicate that a teacher's perception of the nature and role of mathematics must be taken into account whenever reform efforts involving technology are attempted. Those with Aristotelian tendencies support the use of technology with less reservation than do those with Platonic tendencies.

BELIEFS AND PREFERENCES OF TEACHERS AND STUDENTS ABOUT TEACHING AND LEARNING MATHEMATICS

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Two questions motivate our research:

1. What are students' attitudes, beliefs and motivations relating to their study of mathematics, and how does this contrast with their instructors' ideas about attitudes, beliefs and motivations?
2. How do students prefer to be taught mathematics, and how does this compare to what their instructors think teaching should be?

Relationships between beliefs and achievement are a perennial concern for mathematics students. We extend these studies to the freshman calculus level, investigating students' preferences about being taught mathematics and the effects of contrasting beliefs of instructors and of students. We have a thousand students and two dozen instructors and gathered data through two parallel surveys of 15 items, adapted from the Views About Mathematics Survey used by Carlson and Buskirk (1997).

Our poster presentation will present descriptive statistics, factor analysis, and student-instructor comparisons for the data collected. We will identify students' beliefs relating to their study of mathematics and compare these findings with the beliefs of their instructors. Investigating mismatched beliefs in students and teachers can inform change in our own classroom demeanor to better communicate with students. Results of this study will also be valuable in developing and orienting new instructors.

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MATHEMATICAL TASKS CHOSEN BY A PROSPECTIVE TEACHER IN HIS PROFESSIONAL SEMESTER

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The *Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics, 1991) maintain that teachers must help students develop conceptual and procedural understanding of mathematics. Conceptual knowledge is characterized by Eisenhart et al. (1993) as the knowledge of the underlying structure of mathematics-the relationships and interconnections of ideas that explain and give meaning to mathematical procedures. Procedural knowledge is defined by Hiebert and Lefevre (1986) to be made up of two parts: the formal language of mathematics and the rules, algorithms or procedures used to perform mathematical tasks. Research findings show that procedural knowledge is emphasized in most lessons. This study examines the tasks chosen by a prospective teacher in his profession semester in light of his belief system as revealed by his writings and categorized using ideologies identified by Ernest (Ernest, 1991). These tasks were studied to determine whether the focus of the task was conceptual or procedural. Sources of data were audio and videotapes of teaching episodes, the prospective teacher's lesson plans and the researcher's field notes. Data were analyzed using multiple sorts to define which tasks were emphasized procedural knowledge and those that emphasize conceptual knowledge. The belief system was categorized using Ernest's ideologies. The findings were that the prospective teacher's beliefs were consistent with those of an industrial trainer and the tasks he chose were largely conducive to the student's learning mathematical procedures.

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Teacher Education

THE NOTION OF NETWORKING: THE FUSION OF THE PUBLIC AND PRIVATE COMPONENTS OF TEACHERS' PROFESSIONAL DEVELOPMENT

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Abstract: Krainer's (1999) story of Gisela depicts an individual whose early teaching experiences in the Austrian schools were marked by isolation. Krainer's analysis involved the notions of action, reflection, autonomy, and networking as these constructs apply to an individual working cooperatively with critical friends who discuss problems they encounter in their classrooms. Gisela's story is about an individual working in a community of professionals. But the story is also about Gisela's individual propensities that can be characterized by a scheme developed by Cooney, Shealy, and Arvold (1998). In an effort to fold these two theoretical perspectives together, the authors introduce the notions of horizontal and vertical networking to capture socialization processes that also honor an individual's ability to reflect and adapt. Other research on teacher change is also considered.

Perhaps at no time in the history of research in mathematics education has there been such an emphasis on the professional development of teachers. We now see teachers as cognizing agents whose beliefs and knowledge are recognized as an essential and contributing determinant to what gets learned in classrooms. In this paper we present the case of Gisela and analyze her professional development through various lenses in an effort to further our understanding of teachers' professional development more generally. We begin with a brief glimpse of Gisela's journey.

Gisela's Professional Journey

Gisela entered the teaching profession in 1971 as a mathematics and geography teacher in the Austrian schools having supported her university studies by giving private lessons. Her studies were primarily subject matter oriented with minimal attention on how to teach mathematics. The culture of the university and its emphasis on scholarship had a significant impact on Gisela's orientation toward teaching. She described her early years of teaching mathematics as traditional and isolated in that teachers rarely discussed problems they encountered in their classrooms.

In 1985, Gisela became interested in a series of professional development seminars in which teachers engaged in various sorts of research activities aimed at improving their teaching. In 1989 Gisela moved to a new high school with a small mathematics faculty but in which communication among colleagues was encouraged. Two years later, Gisela assumed administrative responsibilities in addition to her teaching responsibilities. Since 1995 Gisela has actively promoted professional development activities among her growing mathematics faculty. Most of the professional seminars were conducted by the second author. (See Krainer, 1999.)

Gisela's Individuality

Gisela's journey was influenced by several factors. She was motivated to take advantage of the professional development programs supported by the Austrian Ministry of Education in an effort to improve the teaching of mathematics. She welcomed the opportunity to interact with other "critical friends" (Krainer, 1998) and to diminish the extensive isolation she had experienced in her early days of teaching. Krainer's (1999) analysis of the case of Gisela is predicated on four dimensions: action, reflection, autonomy, and networking. These dimensions are defined in the following way.

> *Action*: The attitude towards, and competence in, experimental, constructive and goal-directed work.
>
> *Reflection*: The attitude towards, and competence in, self-analysis and one's ability to reflect on his/her actions.
>
> *Autonomy*: The attitude towards, and competence in, self-initiating, self-organized and self-determined work.
>
> *Networking*: The attitude towards, and competence in, communicative and co-operative work with increasing public relevance. (Krainer, 1994)

These four dimensions provide a basis for describing ways that teachers develop within a group setting in which critical friends assist in analyzing and reflecting on each other's lessons (Krainer & Goffree, 1999). Imagine a teacher who works in isolation to realize his/her individual goals. But then imagine that the teacher is challenged to reflect on his/her actions and to share his/her individual experiences and beliefs with other colleagues in an effort to explore alternative methods of teaching. That transition is indicative of Gisela's professional experience. Reflecting and networking are fundamental precepts of Krainer's professional development programs but they are not necessarily part and parcel of teachers' experiences. The notion of autonomy can be conceived in two different ways. On the one hand, autonomy can be a limiting factor in a teacher's ability to reflect and network if the teacher insists on "going it alone" rather than working in concert with colleagues. On the other hand, the individual's efforts to reform must necessarily be self-initiated and self-sustained. Gisela's early teaching experience could be described as almost completely autonomous in an isolated way. Although she was probably reflective in those early years, circumstances prevented her from networking with other teachers and benefiting from their insights.

What was there about Gisela that enabled her to develop professionally when other teachers in the Austrian system were much more inclined to isolate themselves? To account for Gisela's individual propensity to reflect, adapt, and grow professionally, we can turn to the scheme developed by Cooney, Shealy, and Arvold (1998). This scheme consists of four positions that describe a teacher's way of knowing. Fundamental to this scheme is an individual's orientation toward seeing authority as the ultimate determiner of truth.

Isolationist: An individual who tends to have beliefs structured in such a way that beliefs remain separated or clustered. Evidence exists of closed-mindedness.

Naïve idealist: An individual who tends to be a received knower in that he/she absorbs what others believe to be the case but without critical analysis.

Naïve Connectionist: An individual who realizes conflict or differences in beliefs between him/herself and others but fails to resolve or account for those differences.

Reflective Connectionist: An individual who realizes conflict or differences among beliefs and who attempts to resolve these differences through reflection and critical analysis.

These four positions provide a means of conceptualizing the structure of teachers' beliefs and their potential for reforming beliefs and practice. Interestingly, Krainer keyed on Gisela's unhappiness with being a "lone fighter" and her sense of isolated autonomy, precepts that are fundamental to Cooney, et al.'s scheme as well. Gisela's propensity to reflect *and* to make connections with critical friends are trademarks that characterize the position of a reflective connectionist. Had Gisela not been a reflective connectionist, perhaps she would not have overcome the isolated autonomy that characterized her early years of teaching. Thus, reflecting and networking are fundamental precepts both of Krainer's dimensions of professional development, and of Cooney et al.'s stages of professional growth. In the following, we mainly focus on the notion of networking, differentiating between horizontal and vertical networking.

The Notion of Networking

Generally speaking, it is quite natural for people to network in the sense of forming communities. We can envision two ways in which this networking occurs. To use the metaphor of tree, one can conceive of networking in which the roots of the tree network in a horizontal way, perhaps spreading widely but not deeply. The roots of a pine tree come to mind. One can also imagine a different type of networking in which the roots develop more vertically but not necessarily as much horizontally as, for example, with a fir tree. What seems obvious is that some kind of networking is necessary in order for the organism to survive and thrive. We examine each of these kinds of networking.

Horizontal Networking

Horizontal networking is common among a considerable number of teachers as they strive to form relationships with like-minded peers. The commonality may be an interest in school affairs, technology, or other facets of school life. This is a powerful kind of networking as mutual interests are shared and individuals strive to become critical friends. Gisela demonstrated this kind of networking when she became an administrator at her school and wanted the entire mathematics faculty to become a community as they engaged ideas of reform. Although originally skeptical, the teachers found the seminars invigorating and profitable. They were surprised that their colleagues could offer so many insights about teaching and learning, insights that they had not considered (Krainer, 1999). In some cases, their horizontal networking developed into vertical networking, e.g., when they began to question their traditional approach to assessment Some teachers, however, honored their colleagues' views in a reflective way, but provided no strong evidence that they connected those views intensively to their own teaching. In a sense, they demonstrated a form of naïve connectionism. The group as a whole showed a general tendency of increased reflection and networking, and a movement from naive to reflective connectionism. There were, however, obvious differences among individuals.

Even for Gisela, there was some pain involved in the process of sharing. In one of the seminars the group decided to investigate what their students believed about mathematics. Subsequently, students in a class taught by Gisela were asked to draw a picture and provide a brief explanation of the picture. One of Gisela's 17 year-old female students drew a picture of a thick book with a locked clasp and called it the "The Great Book of Mathematics." She added the caption: Where is the key? She provided the following explanation (translated from German): "The sealed, closed book is not accessible to all. It is only possible to open the book with the key. But even when the book is open one may not understand the content. You either understand it or not! In order to understand it, you have to read it from beginning to end." Gisela appreciated the student's creativity yet was upset that a student in her class held such a view. It was a rather painful experience for Gisela but, nonetheless, it opened her eyes as to what her students were thinking about mathematics. In this case it would have been easy for Gisela, who doubled as a school administrator as well as a mathematics teacher, to withdraw from the community and resort to a kind of vertical networking void of sharing experiences with others. Individuals who share their challenges with others not only increase their own capacity for dealing with these challenges but also contribute to the learning of others. Thus, horizontal networking is an important characteristic of a reflective connectionist and a good starting point for vertical networking.

Vertical Networking

Vertical networking involves accommodation and a certain willingness to change and adapt one's teaching. It is the second defining characteristic of the reflective connectionist. Vertical networking can be difficult and tension-filled. The fact that Gisela dwelled on the student's drawing suggests that she reflected on the significance of the girl's response and found that reflection to be somewhat painful. It would have been easier for her to reject the girl's view as idiosyncratic. Rather, Gisela probed her own consciousness for how she might have contributed to that girl's meaning. The act of reflecting is not always a pleasant experience if it brings to the forefront the essence and consequences of what one believes. It is obvious that someone who seriously reflects on his/her own practice has a better starting point for learning from reflections of other people and to contribute to their learning. Thus, horizontal and vertical networking are closely interconnected, in sum being the characteristic of a reflective connectionist.

The case of Sue (Cooney, 1994) illustrates vertical networking quite dramatically. Her view of teaching mathematics was shaken and then reconstructed during her graduate program. She then attempted through horizontal networking to influence her fellow teachers the year following her graduate study. But they would have none of it, at least initially. But Sue was committed, stayed the course, and eventually began to influence her colleagues to adopt a more reform-oriented approach to teaching. For Sue, her propensity to be reflective invigorated her own vertical networking which then lead, eventually, to horizontal networking.

Reflecting on the Notion of Teacher Change

We often wonder how it is that preservice teachers seem so enthusiastic in our methods classes yet demonstrate so little of what they learned during their student teaching. See, for example, Cooney, Wilson, Chauvot, and Albright (1998). Cooney and Wilson (1995) described two preservice teachers' beliefs as they progressed through their reform-oriented teacher education program. Harriet showed evidence of isolationist thinking as it appeared that she did not attend to much of what was emphasized in her teacher education program. Kyle, on the other hand, did attend to the inquiry orientation of his program but failed to resolve differences between his beliefs and his experiences as a first year teacher. The analysis provided by Cooney, Shealy, and Arvold (1998) provides a theoretically descriptive account for how teachers like Harriet and Kyle make sense of their individual worlds. What is perhaps missing from this analysis is attention to the powerful horizontal networking that the teachers experienced during their teacher education program. Even an isolationist like Harriet defined her being in terms of the group: She knew how to teach, the others did not.

The situation with preservice teachers is often mirrored with inservice teachers. For example, Mr. Burt (Wilson and Goldenberg, 1998), a middle school teacher, accepted certain tenets of reform but only in rather transparent ways. He was interested in networking with others in the sense of sharing ideas about reform but failed to incorporate them in fundamental ways into his teaching. Jaworski's (1998) research on the development of teaching that involves teachers in cycles of reflective activity suggests that teachers can engage in vertical networking but, perhaps, only when accompanied with substantial horizontal networking with critical friends. However, we also have cases of teachers who show deep reflections on their own teaching, being convinced of their personal approach, but who have tremendous problems in making their approach understandable to other colleagues. Consequently, they often remain isolated and have little or no impact on the professional development of others.

Could it be that horizontal networking is a precondition for vertical networking? Or is it the case that vertical networking is a precondition for horizontal networking? Or is each a precondition for the other? The strength of Gisela's reflective connectionist orientation seems to confirm the latter hypothesis. What is clear is that teachers need some kind of networking. Unlike trees, people can consciously select and promote both kinds of networking. The problem occurs when people rely too much on only one kind of networking. Horizontal networking provides the community that is so vital yet so often missing with teachers. Yet, like pine trees, when the brutal winds of public opinion or controversy over reform occur, the pines usually fall from want of deeper roots. Gisela had deep roots. Consequently, she had the strength to persevere even in the face of disinterest or when faced with the unpleasant evidence that her students had not perceived mathematics in the way she had hoped. Fir trees usually remain standing tall in stormy times. But this characteristic, applied to people, can be limiting if the result is a certain intransigence in which the individual chooses to isolate him/herself from the system as a whole. Let us find ways to promote the kind of networking that allows teachers to develop the strength to stand tall with their own philosophy of teaching rooted in the evidence of student learning, but at the same time to share their actions and reflections with others in order to learn from them, and, equally important, to give others the chance to learn from them. Teacher educators and researchers play an important role in this process. On the one hand, we have to foster teachers' struggle for reflection and (horizontal and vertical) networking. On the other hand, we have to increase our own reflection and networking on teacher education, as a field of both practice and inquiry.

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THOUGHT AND ACTION IN CONTEXT: AN EMERGING PERSPECTIVE OF TEACHER PREPARATION

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Abstract: This teaching experiment examined one high school preservice teacher's thoughts and actions when given a task of planning a lesson to teach rate of change to algebra 1 students. Pre and post interviews were used to probe the preservice teacher's knowledge of school mathematics and her ideas of pedagogy. Results indicated that her knowledge of ratio and proportion was incomplete and sometimes incorrect, even though she was a successful mathematics student at the university. Despite incomplete understanding, she planned to involve her students in collecting, analyzing, and interpreting data with rich choices of problem types and strategies. This protocol holds pedagogical promise for teacher educators' as a powerful learning and assessment tool for prospective teachers.

Introduction

The preparation of teachers has been marked by extraordinary changes over the past two decades, generating considerable interest and study by mathematics educators. Schulman (1986) described the knowledge base of teaching that grounded a number of investigations (for example, Ball, 1990; Cooney, 1994; Simon, 1995). Some of these findings suggest that knowledge of mathematics, particularly for elementary teachers, may be an inhibiting factor in their professional development. Some studies of preservice teachers' pedagogical and pedagogical content knowledge focused on the knowledge of children's thinking (see Tirosh, 2000). Others examined tasks, activities, and representations used by preservice teachers to teach school mathematics (Blanton, 1998). More recently, Ma (1999) extended the notion of mathematical understanding to include knowledge of subject matter along with the explanations and approaches to teaching mathematical ideas to elementary school children. It is from this comprehensive perspective that this study asks the question: How do prospective high school teachers envision teaching ratio and proportion in general, and rate of change in particular to Algebra 1 students? Over the past 25 years, a number of studies have provided insights into the components of students' proportional thinking (for example, Noelting, 1980; Lachance & Confrey, 1996). Few studies have examined how prospective secondary mathematics teachers think about teaching ratio and proportion.

Theoretical Perspective and Methodology

Schoenfeld (1999) noted that we are considerably distant from possessing a theoretical perspective for education that unifies how we think and act. He questions:

Is it possible to build robust theories of how we think and act in the world – theories that provide rigorous and detailed characterizations of "how the mind works," in context? (p.5)

The inability to link cognition and context theoretically creates some obstacles for researchers. For this study, we recognized the lack of a unifying theory, but viewed our research questions with the contextual binocular vision of Ma (1999). One lens focused on the preservice teachers' knowledge of ratio, proportion, and rate of change in school mathematics. The other lens provided insight into their planned approaches to teaching ratio, proportion, and rate of change. It is with these two lenses that we viewed prospective teachers' thoughts and actions within the context of lesson planning. This qualitative research study derives its tradition from that of a teaching experiment. Individually, ten preservice teachers engaged in activities that were designed to access their thinking about teaching mathematics. The context of this investigation was purposefully naturalistic in terms of the developed protocol, with interviews before and after the lesson planning activity.

Preservice Teachers

Due to limitation of space, data from one preservice teacher is reported and analyzed here. Planning to teach high school mathematics, Chris, age 20, transferred from a community college to the university. At the time of study, she was taking her first mathematics education methods course with 40 hours of school internship and had completed six rigorous mathematics courses beginning with the engineering calculus sequence. Her 3.8 GPA indicated successful college experiences as a student.

Protocol

The research protocol contained the following components: a) pre-plan interview (10 minutes); b) lesson planning activity (30-45 minutes), and c) post-plan interview (30 minutes). In the pre-plan interview, the prospective teacher was asked to recall his or her personal experiences and the connections of ratio and proportion to other school math topics. They were asked for a definition of "rate of change" and if needed, given examples of rate of change for clarification. Then the following activity was posed: *Plan a lesson to introduce rate of change to a class of Algebra 1 students. Connect the lesson to ratio and proportion.* A number of instructional materials and a methods textbook were supplied in a quiet corner to simulate the conditions of a teacher's planning activity. As much time as needed was given to complete the activity before the preservice teacher explained her/his plan for the lesson.

Plan of Analysis

All transcripts, videotapes, artifacts, and field notes were reviewed to select a subject with a strong academic record and a comprehensive lesson plan to gather as much information as possible about prospective teachers' thoughts and actions. Chris's transcripts were initially coded to determine major categories of data. From these categories a second sort of the transcripts coded specific actions and thoughts described within each category. These thoughts and actions were matched to the initial categories and then reviewed to develop initial conjectures about high school teacher preparation.

Two lenses of thought and action were used to analyze the data of the lesson planning activity and the supporting mathematical knowledge of the preservice teacher. The lesson plan analysis identified three major categories and subcategories in the initial sort of the data. These categories were *activities* (students', teacher's), *tools* (students', teacher's) and *problems* (contexts, types, strategies). Chris's knowledge of school mathematics that emerged from the pre- and post-plan interviews, as well as, the planned lesson supported all of the major categories of the lesson planning activity. The aspects of Chris's school mathematical knowledge that were examined were rate, ratio, fraction, proportion, and slope.

Making Sense of the Data

Table 1 contains Chris's 1) Lesson Plan Activities and Tools, 2) Problems Posed in the Lesson Plan, and 3) School Subject Matter Knowledge. Chris's selected student activities for her plan to engage and involve her students in data collection. The post interview revealed that this activity was suggested to Chris from the array of instructional materials available in the lesson planning corner. She portrayed her role of teacher as leading and guiding throughout the process, while her students had access to a number of tools such as tape measures, stopwatches, and charts. Graph tools for both teacher and students were added by Chris as she gained new mathematical understanding of school mathematics during the post-plan interview. The problems that Chris selected for her lesson indicated a flexibility of thought in posing problems that initially drew upon the students' data. From these rate problems, Chris then planned to pose rate comparison problems and then to find missing x and y values of different ratios. The number of different types of problems was enhanced with four different solution strategies found within her plan: equivalent *fractions, cross multiplication, finding unit of rate,* and *within strategies.* The two problem contexts selected were distance/time and item/cost and would be familiar contexts for algebra 1 students.

Chris's successful academic record in college mathematics supported the planning of a lesson to involve her students mathematically. The involvement was inextricably linked to learning school subject matter as the plan moved from collecting data to analyzing the data to solving problems of rate of change posed by the teacher. However, Chris's knowledge of school mathematics was not fully developed in several areas. Her rendering of a ratio was in reality a fractional representation and, therefore, she considered all ratios to be fractions. In addition, Chris had never considered slope as a ratio until the post interview where she assimilated the new information

Table 1. Thoughts and Actions for Teaching Rate of Change to Algebra I Students

Lesson Plan Activities and Tools

Teacher Activities	Teacher Tools	Student Activities	Student Tools
• Gives explanations/ directions • Asks questions • Gives examples • Poses problems	• Graphs!	• Collect data • Answer questions • Calculate answers • Graphs rates!	• Tape measure • Stopwatch • Graphs! • Table/chart

Problems in Lesson Plan

Problem Types	Problem Strategies	Problem Contexts
• Missing value (x) • Missing value (y) • Finding rate (y/1) • Comparing rates	• Set up 2 equivalent fractions • Cross multiplication • Finding unit of rate • Within	• Distance/time • Item/cost

School Subject Matter Knowledge

Definition of Ratio	Examples of RoC	Connections	Unresolved
• Ratio is a fraction • Rate is a ratio • Ratio is a comparison	• Acceleration • Speed • Displacement	• Remembers rate of change from physics • Remembers how to write a ratio • Discovers slope is a ratio!	• Meaning of proportion

Note: "!" Indicates idea emerging in interview

and introduced graphing tools into her lesson plan. While Chris's knowledge of slope seemed to be enhanced, the researchers were not able to resolve Chris's understanding of proportion.

As we examined Chris's representations of teaching, depth of mathematical knowledge, and connections to school mathematics, we saw pieces of a puzzle that were emerging to create a dimensional and connected picture. Beginning her study of teaching, we noted that Chris had many strengths. Some of the pieces were assembled, but not all were connected. Her memories of school mathematics were incomplete and initial applications were drawn from more recent experiences in college physics.

Implications

We conjecture that with modification, the lesson planning task has potential as a powerful learning and assessment tool for teacher educators. The preservice teachers all responded positively to the individual experience that was adapted to their personal beliefs, philosophy, and knowledge of teaching secondary mathematics. Thinking aloud with verbal and written communication was an important tool to give voice to and validate the preservice teachers' ideas. The interview dialogues were non-judgmental and provided information on an as-needed basis to the preservice teachers. Probing questions on the interviewer's part, assisted these future teachers in thinking beyond numerical answers to deepen, and in some cases change, their understanding of school mathematics. The lesson planning corner provided pertinent school subject matter information needed to plan the lesson and the instructional materials suggested possible lesson planning activities for students and teachers. As teacher educators, we can assess our prospective high school mathematics teachers on an individual basis, understanding more clearly how their thoughts about school mathematics determine their actions in the classroom.

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AN ANALYSIS OF PROFESSIONAL DEVELOPMENT UTILIZING REFORM CURRICULA

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Abstract: The goal of this study was to analyze the planning and implementation of a workshop that was part of a larger, long-term professional development project for middle and high school mathematics teachers using reform curricula. This workshop focused on developing teachers' understanding of the reform curricula's approach to the teaching of algebra. Teachers worked on problems from the curricula and reflected on the algebraic ideas developed from these problems. Transcripts of planning/ debriefing meetings before, during and after the workshop, and pre- and post-workshop interviews with the facilitator and participants were analyzed. Tensions were revealed particularly around goals and content, shedding light on the issues that must be addressed when planning and implementing professional development.

Objectives or Purpose of Study

This study analyzed the planning and implementation of a two-day workshop for middle and high school teachers on the development of algebraic concepts. The workshop was set in the context of a larger National Science Foundation Local Systemic Change (LSC) professional development project for secondary school mathematics teachers implementing the Connected Mathematics Project (CMP) and the Core-Plus Mathematics Project (CPMP) curricula. The goal of the four-year project is to support teachers as they learn about the content and pedagogical approach of these reform curricula through professional development utilizing CMP and CPMP and through the creation of a collaborative community of learners. We chose to focus on the two-day algebra workshop because we felt it was representative of the professional development offerings in the larger project and the smaller scale allowed us to analyze the situation in greater depth. Specifically, we focused on the tensions that arose during the planning and implementation of this workshop and their resolutions.

Perspective or Theoretical Framework

Regarding the structure of professional development, researchers have found that modeling best teaching practice has a greater impact on the participants than just sharing information (Cobb, Wood, & Yackel, 1990). Best teaching practice in this context refers to that which is aligned with the recommendations of the National

Council of Teachers of Mathematics (1991, 1998). This includes providing tasks that require participants to investigate and discuss results and allowing for multiple approaches and a variety of representations. Furthermore, programs that allowed participants to investigate and construct their own learning were more successful (i.e. improved students' achievement in mathematics) than those programs that simply talked about the content (Kennedy, 1999). The algebra workshop, along with professional development opportunities throughout the larger project, was based on these premises.

Successful professional development is clearly dependent upon effective planning. Our planning of the two-day workshop was guided by the Professional Development Design Process for Mathematics and Science Education Reform (Loucks-Horsley, Hewson, Love, & Stiles, 1998, p. 17). This process entails setting goals, planning the implementation, implementing the plan and reflecting. Professional developers are advised to reflect on the following as they are setting goals and planning: knowledge and beliefs about learning, strategies for best meeting objectives, context and background of participants, and critical issues that may jeopardize the success of a program. Finally, the authors explain that it is inevitable in the process of negotiating all of these factors that tensions will arise and compromises must be made. This study looks in detail at the tensions that arose in one professional development setting and the subsequent resolutions.

Methodology

Data Collection

There are two groups involved in the research of this professional development experience: four mathematics educators (including the researchers and the workshop facilitator) who developed the workshop and 18 middle and high school teacher participants. Three planning meetings of the development team over the three months prior to the workshop and five debriefing sessions at half-day intervals during the workshop were audio-taped, transcribed and coded. Additional pre- and post-interviews were completed with the workshop facilitator to determine how she had implemented the ideas generated in the development team planning meetings. Questions asked of the workshop facilitator focused on the rationale for including particular activities, the thinking process that took place during individual planning, and adjustments she made.

A pre- and post-workshop phone interview with each participant included questions that attended to both the participants' beliefs and mathematical knowledge. All interviews were audio-taped, transcribed, and analyzed. A follow-up mailed survey was sent to each of the participants to spark further reflection on the algebraic content of the workshop. An end-of-workshop evaluation was completed by each participant prior to leaving the workshop and an outside evaluator submitted an evaluation of the professional development as well.

Analysis

We began our analysis by using preliminary categorizations based on our initial research questions. These categorizations were: mathematical ideas; adjustments made during the workshop; pedagogical tools--related to decisions made about how to help teachers understand the content and what artifacts to share with the teachers; and effectiveness--used for both evidence of teacher growth and the developers', participants', and outside observers' perceptions of growth. The transcripts were coded individually by the research team members who then met to identify themes across the different types of transcripts and to refine categories. A finer categorization crossing all initial categories emerged around predominant tensions that arose as developers planned and implemented the workshop.

Results and Evidence

As in all instances of professional development, time was a tension. We discuss this briefly, and then focus our attention on the more substantial tensions of the goals and content of the workshop. Although we address goals and content separately for ease of discussion, there is overlap and interaction between the two. We conclude this section with a discussion of the effectiveness of our responses to these tensions.

Time Tension

The first time tension involved planning for the workshop. Given the number of people involved in planning and their involvement in several projects, the end result was three collaborative planning meetings during a three-month period with individual planning in between. Another time tension involved balancing the amount of time needed to do justice to the topics with the amount of time the teachers were willing to give out of their busy schedules. In particular, we wanted to reach teachers who had not been willing to participate in week-long workshops in the past. Two was arrived at as the optimal number of days to do minimal justice to the content and encourage maximum participation. Once the length of the workshop was determined, the tension of how to best use that time emerged. Our response to this tension is discussed in both of the following sections.

Goal Tension

Much of the development team's discussion, especially during and after the workshop, focused on the tension between what we wanted the teachers to learn and what seemed realistic for them to learn in two days. In this context, we had difficulty deciding which goals were most important. When asked in the pre-workshop interview what she hoped participants would take away from the workshop, the facilitator responded with a litany of goals. When asked what she would be satisfied with them attaining, she replied, "All of the above. Nothing less. ... All of those are really important and I think that the materials are so rich that if those things weren't realized, I'd feel like

I hadn't done a very good job." In the end, the development team agreed that the main goal of the workshop was for the teachers to come away with an understanding and appreciation of the approach to the development of algebra taken by the CMP and CPMP curricula.

In addition, the interviews with the participants prior to the workshop identified discrepancies between their goals and those of the workshop developers. For example, many of the teachers were looking for ideas for hands-on activities and opportunities to share ideas with other teachers. One resolution of this tension was to have teachers who had taught the unit under discussion share their experiences. As the facilitator explained in the post-workshop interview, "I think that one of the things that I get out of workshops is that people like to listen to other people. They want to hear experienced teachers share their own experiences. And so [having them share experiences] gets to that desire."

Content Tension

In the process of considering which algebraic concepts to include in the workshop, the development team came to the conclusion that the content had to be narrowed to one key concept. After much deliberation, we chose "rate of change" because it is a fundamental concept of both CMP and CPMP, and arguably of algebra in general. Once this was agreed upon, making decisions about what problems to look at and which activities to engage the participants in raised additional issues.

The first was choosing between problems rich with opportunities for mathematical learning versus problems that were narrower in their scope and therefore predictable. For example, one unit was left out because it may have detracted from the mathematical focus of the workshop. As the facilitator said in her post-interview: "That particular unit, one of the difficulties with it is that they have some really wonderful experiments that would be engaging, but the trouble is, because they're experiments, the data doesn't come out nice. And so what's supposed to be a linear model turns out very often not to be. So you're getting to some important ideas, but you're not getting to this rate of change idea."

The second issue dealt with balancing coverage of content with time for reflection. Since it was difficult for us to predict how well the reflection segments of the workshop would turn out, the facilitator made allowances for variances in the agenda. In the pre-workshop interview she reported that "I've tried to build in some flexibility. Some places where I can remove things if we get into conversations that seem so valuable that I don't want to cut them off."

Discussion/Conclusion

In many ways, the workshop would be considered a success. The most significant change was in the teachers' conceptions of algebra. The language they used for describing algebra during the workshop changed from "equations, variables, rules for manipulating variables and solving for a variable" to "identifying a pattern and coming up with a representation of that pattern, predicting change, and understanding that there are multiple representations for situations and thinking about how these representations are related." The comments on the post- interview corroborated this shift away from symbol manipulation to a broader perspective of algebraic reasoning. Our conscious decision to spend time addressing teachers' conceptions of algebra was worthwhile in this sense.

An area in which the workshop was less successful was in the teachers' mastery of content knowledge. For example, in the post-workshop interviews, although the teachers had a strong understanding of linear relationships, less than half of the participants correctly identified a quadratic relationship. During the workshop, little time was spent on attaching labels to particular relationships and investigating the differences between the rate of change in quadratics versus exponential relationships. Although the teachers had learned much over the two days, their understanding was still far from ideal. Perhaps the facilitator stated it best in the post-workshop interview when she said, "I felt like they had come a long way, but they weren't quite where, at the point I would have liked them to be."

Researchers have discussed the inevitability of tensions that occur during the planning and implementation of professional development (Loucks-Horsley et al., 1998). Our study identified some critical components of the goal and content tensions and the effects of possible resolutions. Teasing out what helps make professional development effective is complicated at best. If professional development providers are conscious of the aspects of the goal and content tensions in advance, both the planning and implementation of professional development sessions will be more efficient, and perhaps more effective.

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LEARNING MATHEMATICS WHILE LEARNING TO TEACH: MATHEMATICAL INSIGHTS PROSPECTIVE TEACHERS EXPERIENCE WHEN WORKING WITH STUDENTS

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Abstract: The question of how preservice teachers might learn more mathematics while they learn to teach is explored in this paper. It is proposed that in the context of field-related experiences during their mathematics methods courses, preservice teachers face and learn from the puzzling mathematical situations that arise in their work with students. The contexts in which these situation arise—(a) selecting and designing mathematical tasks; (b) analyzing students' work and (c) offering mathematical explanations to their students—are described and illustrated. Two contrasting orientations—inquirying and intimidated—towards learning mathematics while teaching are also discussed.

Introduction

The typical structure of teacher education programs seems to assume that learning mathematics occurs prior to learning mathematics pedagogy. In most teacher preparation programs, mathematics courses are typically taught independently of pedagogical concerns. Mathematics education courses, in turn, tend to place mathematical inquiry in the background while focusing on the theories and methods of teaching and learning. Research, however, suggests that neither of these structural arrangements have been very successful in helping preservice teachers construct interconnected knowledge of mathematics and mathematical pedagogy (see Brown & Borko, 1992; Lappan & Theule-Lubienski, 1994). That is, even though preservice teachers may extend their mathematical understandings, they do not necessarily translate this understanding into pedagogical practice. The construction of mathematical understanding, on the other hand, is very difficult to promote in courses which focus on instructional methods and practices (Simon, 1994).

Such results or lack thereof, however, are not surprising when considered from the perspective of learning in authentic contexts of practice (see Brown, Collins, & Duguid, 1989; Lampert, 1985; Schon, 1983). Current teacher education practices, however, have become responsive to the idea of learning through engagement in authentic teaching activity. Course-related field experiences have, for instance, become more popular and often take the form of classroom observations, interviews with students, and even teaching episodes. These field-related assignments, however, are often thought of as opportunities to develop prospective teachers' pedagogical content knowledge. A reasonable question to ask, in light of the issues discussed, is how can such experiences be also construed as opportunities to study and investigate subject matter?

Data Sources and Analysis

To explore the question of how field-related experiences could become occasions for prospective teachers to investigate mathematics I draw upon data of my own teaching of elementary mathematics methods. For the past five years I have incorporated different kinds of field-related experiences into the mathematics methods courses I teach. In this paper I focus on three types of interactive field-related experiences I have offered at one time or another: (a) a mathematics letter writing exchange, (b) a mathematical interview, and (c) a teaching session. For each of these interactive activities preservice teachers are asked to identify and write about an interesting, surprising, puzzling event. These written reflections are used to describe the nature of preservice teachers' developing mathematical understanding and dispositions towards mathematical inquiry. Furthermore, the data of preservice teachers with contrasting experiences are used to describe and analyze typical orientations towards the learning of mathematics while teaching. In addition, I draw upon data from a "teaching experiment" in my most recent methods course to further examine the question raised in this paper. In this assignment I explicitly asked my students to keep a journal of "mathematical insights" throughout the semester. This journal was defined as a notebook for collecting insights experienced during regular on-campus classes, field visits, work with students, readings, and independent study.

Results

In the context of their interactive experiences with students, I have noticed that prospective teachers in my mathematics methods courses have engaged in mathematical explorations of their own. These explorations have typically occurred in the following contexts: (a) when selecting and designing mathematical tasks; (b) when analyzing students' work, particularly when dealing with students' incorrect work; and (c) when offering mathematical explanations to their students. I will illustrate these with examples from each of the three interactive teaching activities mentioned.

Designing Mathematical Tasks

There are several ways in which preservice teachers engage in mathematical inquiry in this context. Some examples include figuring out the mathematical demands of the tasks and questions they are going to pose to their students. Other explorations take place when adapting or re-scaling mathematical tasks to make them more accessible to students of differing abilities and different grade levels. Preservice teachers also engage in mathematical inquiry when they design extensions and related questions to the tasks they offer to students. An example can be found in "Camilla's" work. She chose to re-scale one of the problems we had worked on in our class by changing its fractional numbers from thirds to halves. By doing so, she made an interesting discovery, that is, that her students were able to obtain the correct answer to the new version of the problem while using an erroneous solution method. This unexpected outcome launched Camilla into her own investigation as to why such method worked for halves and not thirds, and into the reasons why such a "minor" change could alter the original problem.

Analyzing Student's Work

This is another context that provides preservice teachers with multiple opportunities for mathematical inquiry. Some examples include learning from students their different strategies for thinking about a particular problem or concept. Other opportunities arise when in the process of planning or having actual interactions with students they have to assess the validity and generalizability of students' methods. Furthermore, in this context they often need to provide counter scenarios or examples to disprove or challenge a student's erroneous conceptualization. This happened to "Daniela" when interviewing a 2nd Grader about her strategies for sharing cookies among different number of people. She found her student theorizing that if the number of cookies was even, it could be shared "evenly" among people, and that if the number of cookies was odd, it could not. Daniela, therefore, found herself in a position to challenge her student's theory and have the student realize that this theory does not work for all cases.

Providing Mathematical Explanations

When working with young students, preservice teachers often find themselves providing definitions for seemingly simple ideas, or ideas they have forgotten and taken for granted. For instance, when interviewing students about their mental strategies for doubling whole numbers, and for sharing different numbers of cookies between even and odd number of people, many of my students had to define what these two terms meant. Another example from "Thea" in the context of her letter exchanges with a fourth grader will better illustrate. Thea found herself having to formulate a mathematical explanation when her student was trying to figure out "how to tell whether or not one fraction is bigger than another." Initially her student thought that the fraction with the bigger denominator was the biggest fraction. Thea devised an explanation using drawings (as opposed to symbolic manipulation such as finding a common denominator) to show the student that this was not the case. Her student, however used the drawings to show that the fraction with the largest denominator was actually bigger by drawing two different sized wholes for the given fractions. Faced with this puzzling work, Thea realized that in her explanation it was assumed that the two fractions were parts of the same or equal sized wholes. Further analysis of this led Thea to ask an important mathematical question, What happens when you compare fractions that belong to unequal-sized wholes? This work led Thea to later make an important mathematical discovery, that "when you use common denominators to determine which fraction is bigger you are ensuring that each 'whole' is the same size."

Preservice Teachers' Reactions to Puzzling Mathematical Situations

The above examples show preservice teachers who have a common orientation towards the puzzling mathematical situations they face while teaching. Their orientation can be characterized as an "inquirying or adventurous" approach towards learning mathematics from their experiences with students. Other preservice teachers, however, have not reacted with such open mind. In contrast to this inquirying orientation, other preservice teachers tend to have an "intimidated or evasive" response to the mathematics that arise in their work with students. An example from Terry, who also received some very interesting responses in her student's letters, will illustrate. Terry asked her student to use a page with eight 4x6 (dotted grid) rectangles to draw 1/4 in as many ways as possible. Her student replied by sending not only a few typical drawings for 1/4, she also sent unusual partitions (equal sizes but different shapes) for that fraction. In addition, the student asked whether such uneven shapes were allowed. Terry's reaction to her student's interesting work, however was quite different from Thea. While Thea explicitly raised questions about her student's work and continued to investigate it, Terry made no further attempts to investigate the questions the student raised and did not provide a response to her student's query in her response letters.

Discussion

From the teacher development perspective (see Brown and Borko, 1992; Kagan, 1992) the explanation for why prospective teachers like Camilla, Daniela, and Thea seem to be more naturally inclined to investigate puzzling episodes in their teaching have to do with their level of development and concern as learners of teaching. According to developmental theories of knowledge growth of teachers, prospective teachers move through stages of development and concerns which explains why they attend to certain things and ignore others. It may be possible that concerns for their own understanding of the subject matter is not high in the priority list of preservice teachers like Terry. A different perspective, however, which takes into consideration the context and structure of the task these preservice teachers were asked to perform, in turn, suggests that the explanation may instead lie in the inexplicit nature of these tasks. Studies of structural constraints in tasks posed to prospective teachers often report this to be the case (e.g., Richert, 1992; Grossman, 1992). It is plausible that if a more explicit structure for investigating mathematical insights were provided, everyone would have shown an "inclination" to do so.

The teaching experiment I mentioned earlier offers some insight into this issue. Preliminary analysis of the data suggests that while field related experiences may offer preservice teachers multiple opportunities to explore mathematical ideas, it is necessary to provide the structure and the expectation to do so. Many of the self-initiated entries preservice teachers made in their "Math Insights" journal focused initially on pedagogical concerns such as issues of gender, calculator usage, and on students' abilities and behaviors. Yet with prodding and specific prompts their journal entries eventually began to focus on the mathematical questions they encountered when working with students. In a class of 27 preservice teachers more than half of their entries focused on self-initiated explorations that arose during their fieldwork with students. Most importantly, this exercise brought about some important realizations for prospective teachers about the role and the importance of their mathematical dispositions and understandings for their learning to teach. In short, this experiment highlights the importance of making the learning of mathematics an explicit and important goal of pedagogical courses. If this is not emphasized, preservice teachers are not likely to focus their attention onto (or be able to recognize and learn from) the mathematics that arise in the context of their teaching.

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LEARNING TO TEACH MATHEMATICS WITH LITERATURE CONNECTIONS

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Abstract: This study provides insight for a deeper and better understanding of how preservice elementary teachers think about making connections between mathematics and literature for the purpose of designing and implementing meaningful mathematics instruction.

Connecting mathematics with other subject areas and developing mathematical communication through reading, writing, and speaking have continued to be emphasized as desirable goals in mathematics instruction (NCTM, 2000). Teaching mathematics by making connections with children's literature is an increasingly popular way to not only develop understanding of mathematical ideas, but also to develop mathematical communication through reading, writing, and speaking. Linking literature with mathematics instruction has even permeated textbooks (including textbooks used in mathematics methods courses for preservice elementary teachers), with lists of books that may be used with various mathematics topics. This implies that future elementary school teachers need to be prepared for designing, implementing, and assessing the results of such lessons. The purpose of this study was to understand how preservice elementary teachers think about using children's literature as a part of mathematics lessons and to what extent they could apply what they learned about teaching mathematics using children's literature. This paper will describe the connections between literature and mathematics made by preservice elementary teachers when planning for and using literature to teach mathematics based on their experiences in a mathematics methods course and related field placement.

Theoretical Framework

Current curricular trends include making connections between and among various disciplines and within a discipline. Using literature in mathematics instruction provides natural opportunities to make such connections (e.g., Smith, 1995). In addition, literature connections with mathematics instruction may provide opportunities for a broader and deeper understanding of mathematical ideas (e.g., Wickett, 2000). Many logical arguments have been put forth regarding the benefits of linking literature with mathematics (Kolstad & Briggs, 1996; Schiro, 1997). Research has provided some encouraging evidence that using literature in mathematics lessons can increase problem solving skills, student interest, and achievement (Burnett & Wichman, 1997; Hong, 1996; Jennings, Jennings, Richey & Dixon-Krauss, 1992) and reduce math anxiety (Burnett & Wichman, 1997). In addition, the growing number of examples describing how teachers have linked literature with the teaching of mathematics in their classrooms has provided a great repertoire of ideas for successful lessons (e.g., Wickett, 2000). Preservice elementary teachers need to be aware of ways to develop mathematical understanding and ways to link mathematics with other subject areas; using literature appears to be one viable way to accomplish these goals.

Literature may be used in several ways as a part of mathematics lessons (Schiro, 1997; Welchman-Tischler, 1992). Schiro (1997) distinguished using literature for mathematical literary criticism and editing as a way to integrate mathematics and literature from using literature as a springboard into mathematics activities. Welchman-Tischler (1992) described several ways to use literature as a springboard into mathematics activities (e.g., providing a context or posing a problem). Many examples exist of teachers using literature as a springboard into mathematics activities, while there are few examples of teachers using literature in a more integrated way. Preservice elementary teachers not only need to be aware of different ways to use literature as a part of mathematics lessons, but ought to have experiences using literature in mathematics lessons and time to reflect upon those experiences.

Thompson (1992) discussed how teachers' knowledge and beliefs interact with classroom practices. Preservice teachers' knowledge and beliefs about connecting mathematics and literature may influence how they design and implement such lessons. Thus, it is important to find out what preservice teachers know and think about using literature as a way to develop mathematical understanding and to connect mathematics with other subject areas.

Methods

Participants

Fifty-one preservice teachers were enrolled in two sections of an elementary mathematics methods course. They ranged in age from 20 to over 40 years old. Experiences in the mathematics methods course included exposure to literature related to mathematics topics through book talks, and modeling and discussing examples of using literature in mathematics lessons. Each preservice teacher was placed with one or two peers in an elementary classroom (Kindergarten through fifth grade) for a seven-week field placement, two hours per week. Most taught a lesson each time they were in the field. Some preservice teachers chose to team-teach and others taught solo lessons. All were encouraged to try at least one mathematics lesson involving literature. Eight students in one section and 15 students in the other section taught at least one lesson using children's literature.

Data Sources and Analysis

Sources of data included preservice teachers' written reflections on class discussions and activities, copies of their lesson plans, their written reflections on lessons they taught, and fieldnotes from classroom observations of ten preservice teachers teaching mathematics lessons using literature. Qualitative methods were appropriate for investigating how preservice teachers thought about and applied what they learned about teaching mathematics using children's literature. The data were examined for emerging patterns regarding how literature was used in mathematics lessons and to what extent the mathematics of each lesson was connected to the piece of literature used. Categories were developed to classify lessons according to the way the literature was used (e.g., as an introduction to the lesson or as the main focus of the lesson) and linked to mathematical ideas involved in the lessons (e.g., strong link, weak link, no link). A strong link involved connecting an obvious mathematical idea contained in the literature with related mathematics activities (e.g., reading a book about measuring length in inches to launch a lesson activity measuring lengths of various items in inches). A weak link involved using literature with potential for an obvious mathematical connection to the mathematics lesson but that connection was not emphasized and sometimes ignored after the book was finished being read. An example of no link between using a piece of literature and teaching a mathematics lesson involved a mismatch between the mathematics contained in the literature and the mathematics content of the lesson (e.g., reading a poem with an obvious connection to counting as an "introduction" to a lesson on measuring length). Comments made in reflections on lessons taught and notes from classroom observations of lessons taught were used to substantiate and clarify different ways the preservice teachers thought about and used literature in their mathematics lessons.

Results and Conclusions

Reflecting on experiences with connecting mathematics and literature in the methods course, typical comments from preservice teachers included "I used to think math and reading are two separate subjects, but there are so many books and novels that use mathematical concepts," and "I did not realize how many math ideas you could get from one book ... I had read the book before, but didn't associate it with math." These comments indicate a new awareness of how mathematics and literature might work together in the classroom. In addition, many preservice teachers thought it would be easy and fun to plan and implement mathematics instruction using literature after experiencing some examples of such instruction in the methods course. One student commented, "I learned there is math in numerous children's books and these can be very easily incorporated into the teaching of math. Using books to teach math is a fun way for the students to learn the abstract concepts in math." Other comments such as "it was good that we went through the different steps in using a book with math. The modeling was very effective" indicated that for many of the preservice teachers, it was important to have direct experiences with how to use literature in mathematics lessons.

The preservice teachers designed mathematics lessons using literature in a variety of ways. These included using the literature as an introduction to the lesson to establish a context, to pose problems, to introduce a manipulative, or to further develop a concept (Welchman-Tischler, 1992). Some chose to use literature with an explicit connection to the mathematics of their lesson (e.g., a book about fraction concepts with a lesson on fraction concepts), while others chose to use literature with an implicit connection to the mathematics of their lesson (e.g., a book about seasons to motivate a lesson on measuring temperature).

The preservice teachers had varying degrees of success with using literature in ways that really connected with the mathematical ideas of the lesson. Some preservice teachers used the literature throughout the lesson by having students act out the story, highlighting the mathematics. Some preservice teachers selected literature that connected well with lesson objectives. For example, "the literature connection went very well. They all enjoyed the story and it made an excellent introduction to measurement" was part of a reflection on a lesson that closely matched the lesson activities with the context of the story. Another reflected "I wouldn't read the book if I were to do this again. Although it was a simple, informative book, it wasn't a story and the children got fidgety. Also, it was rather long and when I tried to cut it by skipping a couple of pages, some of the children took exception to that." This reflection revealed an awareness that even though the book related well to the planned lesson objectives, sometimes it might not work well to use literature in a lesson and there was a need to think carefully about the selection of a book and how it would be used in an appropriate way. Others made weak connections between the literature and the lesson objectives, while a few choose a piece of literature that did not support the lesson objectives.

Some preservice teachers used the literature as a basis for exploring mathematical ideas while also incorporating language arts activities such as acting out the mathematics of the story or using the context for writing and solving related problems. Discussion and activities around the mathematical ideas of the lesson took place before, during and/or after the reading of the literature.

Most preservice teachers used the literature as a springboard into mathematical activities. There was some discussion before, during, or after the reading of the literature regarding the mathematical ideas of the lesson. However, the mathematical activities after reading and discussing the literature focused on the mathematics and did not directly engage students in continuing to think about the mathematical connections to the literature.

Some preservice teachers read a piece of literature to the students and had very little or no discussion about the literature and connections to the mathematics of the lesson. These preservice teachers sometimes commented that the literature they chose did not really fit the lesson as they originally thought it would. In other cases, the preservice teachers in this situation did not seem to have a clear idea about how to connect using the literature with the mathematics of their lesson. One preservice teacher commented that she chose to read a "math poem" at the beginning of the lesson because it would be "fun," and it would "help to get the children settled down after coming back from recess," even though the poem did not connect with the mathematics of the subsequent lesson. She reflected on this experience and realized that it would "make more sense to choose something related to the math in the lesson."

Although some preservice teachers may benefit from exposure to various pieces of literature with suggestions for making mathematical connections, other preservice teachers need more structured examples and models for making such connections for well-designed lessons. In addition, preservice teachers need opportunities to try out their ideas in real classrooms and reflect on those experiences. Learning about how to effectively use literature in a mathematics lesson involves learning from what does not work as well as learning from what does work. Implications of this study indicate that modeling effective teaching practices in methods courses and having opportunities to try out and reflect on their ideas is a necessary part of the learning process.

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UNDERSTANDING TEACHER LEARNING AS CHANGING PARTICIPATION IN COMMUNITIES OF PRACTICE

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Abstract: This study investigates teachers' learning trajectories as they studied their own students' mathematical thinking. Teachers examined student work in monthly workgroups. Within a community of practice perspective, we analyzed different forms of participation across workgroup and classroom communities. We identified two forms of peripheral participation and one form of full participation, which we call generative. Teachers who moved towards generative participation experienced the workgroup and classroom communities as tightly linked and interactive. Their participation in the classroom altered the kinds of questions they raised in the workgroup, which in turn changed their participation in the classroom. In contrast, peripheral participants, while contributing to the development of the workgroup community, experienced their classroom community either as separate or as partially connected to the workgroup community. Implications for supporting teacher learning are discussed.

Teacher learning in mathematics is more than a matter of expanding knowledge and developing new pedagogical practices (Franke, et al., 1998). It is also an enterprise that consists of crafting and recrafting an identity about what it means to teach and learn mathematics. This study advances our understanding of teacher learning as developing identities through participation in communities of practice.

Theoretical Framework

We draw from the work of Lave, Wenger, and Rogoff to interpret learning within a community of practice perspective in which the construct of participation plays a central role. Lave and Wenger (1991) define a community of practice as "a set of relations among persons, activity and world, over time and in relation with other tangential and overlapping communities of practice" (p. 98). Lave (1996) describes learning as "changing participation in changing 'communities of practice'" (p. 150). Learning is not a process of acquiring or transmitting knowledge. Rather learning is apparent in the way participation transforms within a community of practice. The shifts in participation do not merely mark a change in a participant's activity or behavior, however. A shift in participation involves a transformation of roles and the crafting of a new identity, one that is linked to but not completely determined by new knowledge and skills (Lave, 1996; Rogoff, 1997; Wenger, 1998).

Method

The study took place at an elementary school in a small district in a large urban area in California. Eighteen teachers participated. Data were collected during the 1997-1998 school years. The student body, roughly 1300 students, was 90% Latino. Over 90% of the student body was on free or reduced lunch.

Teachers met in one of two monthly workgroups facilitated by the authors. Prior to each workgroup, they posed a similar problem (provided by the authors) to their students. The workgroup discussions centered on the student work those problems generated. The problems were drawn from research on the development of children's mathematical thinking and focused initially on whole number operations (Carpenter et al., 1999). The goal of the workgroup sessions was to create frameworks for understanding children's thinking that reflected teachers' practical knowledge and current research knowledge (Richardson, 1990, 1994). The authors visited each teacher's classroom at least once, and usually twice, between each workgroup meeting for purposes of data collection, professional support and continuity.

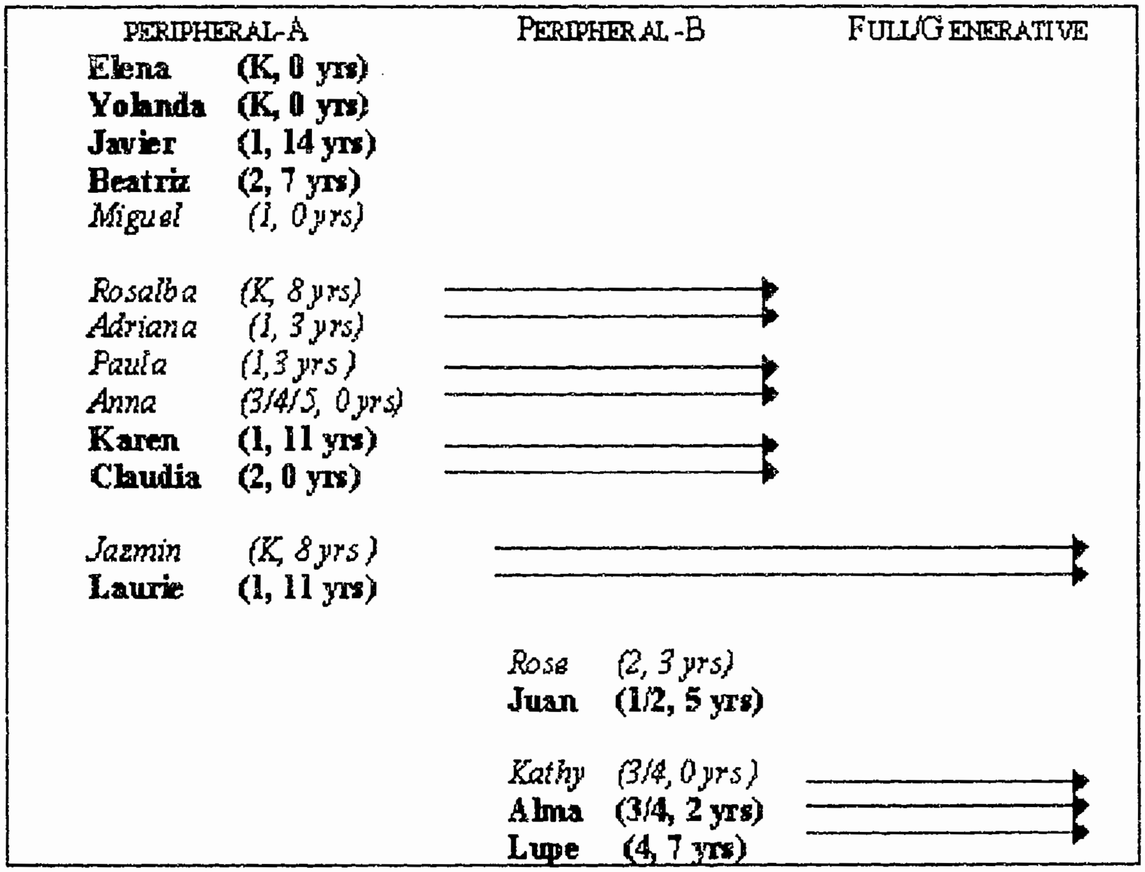
Data included initial and final teacher interviews, transcripts of workgroup conversations, student work, fieldnotes from classroom visits, and teachers' written reflections during workgroup meetings. The data provided the basis for longitudinal case studies that unveil the diversity and shifts in individual participation across the workgroup and classroom communities.

Results

The workgroups developed as communities where teachers brought student work from their classroom, made public their classroom practices, and investigated the teaching and learning of mathematics. We traced different trajectories of participation across one year of workgroup meetings (the teachers have continued to meet, now into the fourth year). The forms of participation are described in relation to the goal of creating classroom communities where building on student thinking is central. These trajectories of participation allow us to examine the diverse ways teachers participated in the workgroups and in their classrooms and the implications for changes in teacher learning about student thinking and pedagogy. We identified three broad forms of participation that emerged and shifted across the year in predictable and meaningful patterns: two forms of peripheral participation and one form of full participation, which we call generative (see Figure 1). We describe the forms of participation by tracing how teachers used the workgroup problem, how they interacted with their colleagues in the workgroup and with their students in the classroom, and how they reflected on their experience during the year.

Peripheral Participation -- A

A number of teachers' participation remained peripheral in relation to the evolving purpose of the workgroup meetings. They attended workgroup meetings but did



Note: Names are pseudonyms. Grade level and years of teaching experience are designated in parentheses.
Italic = Teachers meeting in one workgroup;
Bold = Teachers meeting in second workgroup.

Figure 1. Shifts in participation for individual teachers during first year.

not buy in to the broader goals of developing their classroom practices in relation to students' mathematical thinking. Peripheral-A teachers posed most if not all of the workgroup problems to their students, but the problems remained separate from the rest of their curriculum. They adapted the problems to make them easier for their students. The patterns of participation with their students remained the same throughout the year. Their lessons would begin with examples of how to solve particular problems followed by time for student practice. Teachers rarely elicited student thinking. When they did, they reported their frustrations about students' inability to communicate their thinking. The teachers did not experiment with classroom participation structures that would allow students to solve problems using their informal strategies or to develop students' ability to articulate their thinking.

Peripheral-A teachers proceduralized problem solving by modeling solution strategies. They viewed students' informal strategies as unusual or more applicable to their bright students. They described the workgroup experience as not being very helpful either because they were unable to get the ideas they expected to or because the conversations did not add to what they already understood about teaching. The student work they shared, for the most part, fit with their expectations for student performance. They reported that many of their students were confused and unsuccessful in solving the workgroup problems. They also reported surprise when one of their struggling students was able to solve the problem. Their interactions with us outside of the workgroup were pleasant, and they rarely had questions about their students' thinking or asked about how they could develop a dimension of their practice.

Peripheral Participation -- B

Another group of teachers also participated peripherally, but in a different way than the first group of teachers. This second group may have adapted the workgroup problem or posed it more than once, but the adaptations served as "warm ups" to expose students to the problem type. Other times, teachers were curious to see if their students would be able to solve the problem. The adaptations were only loosely tied to student thinking. In the classroom, teachers allowed students to solve problems using their own strategies, but they continued to model the problem first or step in with a strategy if a student was having difficulty. Teachers began to experiment with classroom participation structures. For example, teachers may have posed a problem and asked students to share their strategies.

In the workgroup context, peripheral-B participants regularly shared more than one student strategy. They did not question their practice to the extent of a generative participant (see below), though they were willing to experiment with the problem and devote a larger part of their curriculum to problem solving. At a general level, they expressed interest in the goals of the workgroup and found the experience to be helpful. In their interactions with us, they may have asked us for problems to pose and sought out suggestions about ways to structure their classroom discussions.

Full or Generative Participation

The generative teachers (Franke et al., 1998) were not all at the same level of mastery in appropriating and making use of a framework for children's thinking. Rather, they were developing and shaping their identities as learners in both the classroom and workgroup settings. They used their emerging understanding of student thinking to purposefully adapt the workgroup problem several times. One adaptation would lead to another, each a means to gather information about student thinking. Generative teachers consistently brought student work to the meeting, sharing work that related to questions they were developing about mathematics or about student understanding. The student work and the workgroup conversations raised questions about a dimension of student thinking that teachers investigated further in their classroom.

In the classroom, generative teachers developed and experimented with ways to elicit students' strategies. They allowed and expected all of their students, not just their most advanced students, to use informal strategies. Generative teachers learned about strategies that were unfamiliar to them through their students' explanations. If a student was having difficulty, they tried to find a way for the student to use what s/he knew rather than impose a teacher strategy. In their informal interactions with us, teachers were eager to share "amazing" strategies. They wrote new problems to pose to their students but asked for our suggestions. We puzzled together over strategies they did not follow, and they had a growing concern with student understanding and next steps for instruction that would build their students' thinking.

Discussion

Wenger (1998) claims that, "By its very practice, the community establishes what it means to be a competent participant, an outsider or somewhere in between" (p. 137). Generative participants were defining and establishing what full participation in communities centered on student thinking and professional growth would involve. Peripheral participants were possibly on an inward trajectory towards full participation to the extent that they were developing an ability and desire to identify and support their students' mathematical ideas. Yet peripheral participants continued to divide their workgroup math from other math practices in their classroom. Generative participants allowed their students' mathematical thinking to drive their experimentation in the classroom and their participation in the workgroup meetings.

As facilitators, our goal was to help teachers develop strong connections between the workgroup and the classroom. As we traced what happened over the course of the year, it was clear that all teachers did not gain the same benefit from their participation. What can we learn from this analysis? First, the data speak to the difficulty of transforming teachers' knowledge and classroom practice within the course of one year. Clearly, change takes time, and schools and districts must be mindful of the length of time teachers need to engage in sustained work before significant changes become visible. As we have continued to work with the teachers, we have been able to document the benefits of continued professional development for both teacher and student learning. Second, our experience spending time with teachers across the classroom and workgroup communities underscores the difficulty of drawing on the workgroup experience in the context of classroom structures that conflict with the goal of eliciting and building student thinking. Teachers and students must learn new ways of experiencing mathematics class, which is not a small or trivial task. For many teachers, however, student work became an intriguing and powerful artifact that provided a purpose and direction to their emerging thoughts and practices (Wertsch, 1998). Finally, our vision of mathematics education remains a strong challenge to the status quo in schools, and developing school cultures that support teacher and student learning merits continued study.

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ELEMENTARY PRESERVICE TEACHERS' EXPERIENCES IN THE FIELD: THE INTENDED VS. THE REALIZED EXPERIENCE

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Abstract: This paper focused on elementary preservice teachers' experiences in a field-based mathematics methods course, in particular, their experiences during one of the field placements. The theoretical perspective of symbolic interactionism was used to focus on the meanings the preservice teachers had for the field experience. The findings suggest that how the preservice teachers defined their situation affected what they did during the field experience. In short, the tutoring perspective they used overshadowed the messages they were hearing in the mathematics methods course.

Introduction

Research findings suggest that many teacher education programs have minimal impact on preservice teachers' views about teaching and learning, resulting in most classroom teachers teaching the way they were taught (Anderson and Bird, 1994; Borko, Eisenhart, Brown, Underhill, Jones, and Agard, 1992; Zeichner and Tabachnick, 1981). Thus, scholars have suggested that researchers and educators place greater attention on the preservice preparation process (Brown & Borko, 1992; Frykholm, 1999). In particular, Frykholm (1999) suggested that "we need focused research on the impact of methods courses and field experiences on student teachers and on the connections between the two experiences" (p. 103). Early field experiences are also integrated into many teacher education programs. This situation invites us to explore the connections between the methods course and these field experiences as well as the impact field-based methods courses have on our preservice teachers.

The purpose of this research was to understand elementary preservice teachers' experiences in a field-based mathematics methods course. In particular, the purpose was to examine how the preservice teachers interpreted their field experiences and the connections they made between the field experiences and the methods class.

Theoretical Perspective

Symbolic interactionism, which served as an orienting theoretical perspective for this study, asserts that we use perspectives as a means to understand the world around us. A perspective is a point of view that guides our perceptions of reality or helps us to make sense of the world around us. We adopt our perspectives from the people with whom we interact (i.e., our reference groups). We use perspectives as filters because they force us to focus on certain things in a situation while ignoring other things. Charon (1998) described perspectives as sensitizing "the individual to see parts of reality, they desensitize the individual to other parts" (p. 3). Therefore, no one perspective can capture the whole reality. In fact, depending on the perspective we use, we will see the world in a particular way. When a different perspective is used, a different world will be seen, and perhaps a new way of looking at things will be revealed.

Furthermore, symbolic interactionism emphasizes that the meaning of an object in our environment is not inherent in the object itself but emerges through interaction with the object and others (Blumer, 1969). (Objects in symbolic interactionism can be physical objects, such as manipulatives, or they can be ideas, such as using questions to probe children's mathematical thinking.) Most importantly, people act toward objects on the basis of the meaning that those things have for them, not on the basis of the meaning that those things have for someone else.

In the past twenty years, mathematics education researchers have focused on teachers' beliefs as a way of understanding their actions in the classroom (e.g., Cooney & Shealy, 1997; Thompson, 1984). This research does not consider that how a person *defines* the situation at hand can have a major impact on his/her actions in that situation. Beliefs and past experiences play a role in our present behavior; however, according to symbolic interactionism, the most important way our beliefs and our past influence our action is in helping us to define our environment. We then act according to this definition. In other words, we are not *controlled* by what happened to us in the past or by our beliefs; we use these to interpret the current situation and then we act accordingly.

Methods and Data Sources

The case study method was chosen because it allowed the researcher to pursue the participant's perspective by permitting an up-close view of the preservice teachers' experience in the field-based methods course (LeCompte & Preissle, 1993; Merriam, 1988). The case studies focused on five female elementary preservice teachers, Elise, Susan, Allison, Jackie, and Mary during the first of two mathematics methods courses. (All names are pseudonyms.) Four semi-structured interviews (audiotaped and transcribed), course assignments, journal writings, and observations of the participants during the on-campus classes and weekly field experiences informed the case studies. The on-campus classes were audiotaped and field notes were taken to enhance data collection. Field notes were also taken during observations in the field.

The research cycled between data gathering and analyzing, with the initial analysis phase informing subsequent data collection. In particular, data collection and analyses were guided by analytical induction techniques (LeCompte & Preissle, 1993). Analytic induction involves scanning the data for themes and relationships, and developing and modifying hypotheses on the basis of the data.

Two field experiences were integrated into the first methods course. Once a week the preservice teachers worked in pairs with an elementary school student at a nearby school, Edison Elementary. With this field experience, the methods instructor intended to provide opportunities for the preservice teachers to focus on a child's mathematical thinking in order to evaluate the child's conceptual understanding. During the other field experience, each preservice teacher spent all day in a classroom for four weeks. This field experience was not mathematics-specific as was the Edison experience. This paper focuses on the preservice teachers' interpretation of the Edison field experience and how that interpretation affected their actions in the field placement.

Findings

Despite the teacher educator's explicit intent that the preservice teachers use the Edison field experience to focus on children's mathematical reasoning, all five of the participants defined their situation at Edison as tutoring: helping individual students with procedural skills. This definition prevailed in spite of Dr. Mathis telling "us that we would be called the 'math tutors.' But that we're not, that this wasn't about tutoring" (Elise). The situation fit with their past experiences of tutoring because they focused on the fact that they were working with one student who they assumed needed help mastering a skill in mathematics. This tutoring perspective influenced what the preservice teachers perceived they were to do with their elementary school students and what they perceived to have learned from the experience. Most importantly, they did not see that the methods course had informed much of their interactions during the field experience.

The preservice teachers distinguished between tutoring and teaching. They saw teaching as introducing a new topic to students. As a tutor, they were not introducing a new topic, but were "reviewing stuff that they've already been taught….Stuff that the teacher had said she struggled in, that they had already been taught in class….You're teaching to an extent, but it's more so reviewing" (Allison). Elise explained that she had tutored in the past and thus, saw the Edison experience as tutoring: "Just through doing it, and experiencing it, it was yeah, this really is tutoring….We are not introducing new lessons to them….You are only working with one student. You are not teaching a whole class."

Elise and Susan were so focused on the idea that Edison was tutoring that they could not see until late in the semester that the Edison student actually needed to be challenged. They were confused because their experiences with their Edison student were not consistent with their idea of tutoring: the student did not need a great deal of help with his class work. Elise even confessed, "I'm not sure what he needs help with because he seems to understand what we go over....Susan and I both wonder why he was chosen to be an Edison Buddy."

Tutoring for these preservice teachers entailed having the student practice procedures. This perspective was evident in the tasks they asked the students to do.

Consider the following typical episode from a day during the Edison field experience, in which Mary and Jackie asked their student to add 28 and 22. The student wrote down 40. Notice the questions Mary and Jackie asked the student.

$$\begin{array}{r} 28 \\ \underline{+22} \\ 40 \end{array}$$

Mary: How did you do this one?

Child: (No response)

Jackie: Which numbers did you add first?

Child: 2 plus 2.

Mary: Why did you add that first?

Child: (No response)

Mary: Let's look at it again. Start with 8 and 2. What do you get?

Child: 10.

Mary: Ok, so write down the zero...

One of the ideas endorsed in the methods course was asking a student questions, whether the child answered correctly or incorrectly, to probe his or her mathematical reasoning. In particular, the preservice teachers were encouraged to ask students how and why questions. While they readily "adopted" this questioning strategy, as this excerpt illustrates, they did not use appropriate how and why questions to probe the child's conceptual understanding of the mathematics. Rather their questions mainly centered on steps to a procedure.

Moreover, instead of using ideas from the class or posing problems and using the child's responses to inform their instruction, Mary and Jackie waited for the classroom teacher's input as to what they should do with their student. Tutoring required them to know what the student was having trouble with and as far as they were concerned, it was not their role to determine this but the teacher's obligation to inform the tutor. They "needed more help to know what to plan" (Jackie) and they felt that the teacher, who was with the student every day, was in a better position to tell them what the student needed help with. Mary explained,

> The Edison experience could have been [very beneficial] but I don't know if it really was for us. If we had a teacher that, it plays with the teacher's interactions, letting us know what is going on and stuff like that, to where it could have been beneficial but it really wasn't.... I think if we had gotten more feedback from the teacher who was with her everyday, rather than once a week...that we could have helped her out more.

Consequently, they blamed the classroom teacher for the experience being less than beneficial. In fact, when reflecting on what she learned from the experience, Mary stated that "If you [as a classroom teacher] have someone help one of your students, make sure you communicate with them about what the child needs. Do not leave them guessing."

Early in the Edison field experience, each of the preservice teachers interviewed a second grade student using a standard protocol. This experience gave them the opportunity to compare the mathematical thinking of second grade students across a specific class. Later in the semester, Jackie claimed,

> I liked [the second grade] interview because we had to ask more questions than we did even just at Edison. Because we had to ask how you did it, why you did it. So I asked a lot more questions and found out a lot about how she was thinking, more so than I did with [our Edison student].

What was enlightening about her comments was the fact that probing the student's mathematical thinking was precisely the intent of the Edison experience. Instead, the preservice teachers interpreted Edison to be about tutoring or practicing procedures with their students.

Discussion

The preservice teachers in this study acted in the field experiences in ways that were meaningful to them. They used a tutoring perspective, which influenced what they perceived they were to do with the students even in light of the explicit intentions of the teacher educator. This perspective limited the preservice teachers' actions to an emphasis on procedures and appeared to prevent the preservice teachers from using this experience to further understand course content. What is important to note is that the preservice teachers felt their experience at Edison would have been different had it not been about "tutoring." Future research should investigate how preservice teachers interact with students during field experiences when they define their situation differently from tutoring.

The study provides insight into the connections preservice teachers make and do not make between methods course content and a concurrent field experience. In particular, this study highlights the need for educators and researchers to examine how preservice teachers define their current situation. If ignored, we may slip into believing that our preservice teachers are interpreting our courses in ways that we are intending, when actually this may not be the case.

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COLLEAGIAL INTERACTION AND TEACHER DEVELOPMENT

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Abstract: This study sought to examine the impact of the use of peer teaming and peer supervision as a professional development strategy on the practices of two 7th grade mathematics teachers. It became evident that although teachers were provided the opportunity to interact, neither one of them felt it necessary to provide professional suggestions that could impact the peer's practice. Teaching was perceived by the teachers as an individual practice whose direction and content was determined by personal preferences of each teacher, and depended upon his own personal judgment. Findings of the study suggest that while peer teaming and supervision techniques hold some promise for motivating change in teachers' practices, questions concerning the substance of change are of concern.

Teaching has been described as an isolated and private profession. Lortie (1975) proposed that teaching is marked more by separation, both physical and intellectual, than by interdependence. Other researchers have made similar assertions as they studied the lives of teachers (Hargreaves, 1994; Romberg, 1988; Noddings, 1993). In a general sense, it is reported that teaching is viewed and practiced as a solitary occupation that occurs behind closed doors.

The increased interest in teacher development has lead to augmented efforts to highlight the potential of peer collaboration and dialogue on improving teaching. Eisener (1983) and Rosenholz (1989) suggested that in order for teachers to grow as professionals, schools should be transformed into communities in which self renewal through collaborative networks supports instructional improvement. Many researchers have proposed models of professional development built around collegial interaction (Hargreaves, 1994; Little, 1987, 1990).

Calls to restructure schools to facilitate more professional and collegial environments have thus increased (Fullan, 1993). More and more professional development activities are designed to provide opportunities for teachers to engage in collaborative investigations of school curriculum and pedagogical innovations. Renewal programs such as peer supervision, and team planning are now established to foster teacher development (Hawkey, 1997; Wasley, 1991; Zahorik, 1987).

Surprisingly, in spite of the widespread agreement on the impact of collegial interaction on teacher change, there is limited literature on the dynamics of such interaction. The literature on the role of interaction between teachers and its contribution to professional development of teachers is slim (Mitchell, 1997).

Examining the potential of professional discourse on teacher change was the primary aim of the current study. The goal was to investigate the context and the content of two teachers' interactions in order to develop an understanding of the distinctive contributions that teacher peers make in the process of improving teaching.

Research Questions and Setting

Two questions framed the research effort:

1. How is collegial interaction carried out among teams of teachers?
2. What do colleagues learn from each other and how is that knowledge manifested in their practice?

The research was conducted in a Midwestern public school district serving nearly 600 middle level students. The middle level mathematics teachers of the school district were involved in the first year of implementing a reform-based textbook. The mathematics coordinator of the school district had sought incentive funds to plan a full year of professional development activities for the teachers as they taught the new textbook. The project had several components. It supported bi-monthly workshops for teachers. Release time (one hour a week) was provided for the teachers to form peer meetings and to discuss issues that concerned the implementation of the new textbook. Teachers had release time two instructional periods a month to observe each other's teaching and to provide one another with feedback on their instruction. I served as the external evaluator to the project. I was asked to investigate the impact of the program on teachers' practice. The data for this research report comes from a larger study of, 10 pairs of teachers involved in the project.

Data Collection

Data collection techniques were participant observation and informal and semi-structured interviews with the teachers. I also participated in the staff meetings scheduled for the mathematics faculty, and the professional development meetings planned for the mathematics teachers. The peer teachers were observed, 10 times during their team planning time. Teachers' classrooms were observed 5 times during the entire data collection period. In addition, as teachers engaged in peer observations I attended those sessions with the teachers and participated in their post observation discussions. The data was collected over a period of 7 months.

The Participants

The team of teachers I discuss in this paper consisted of Gary and Ben. Each teacher had over 10 years of classroom experience. Both teachers taught 7th grade mathematics. Ben and Gary knew each other well and were familiar with one another's classroom practices.

Opportunities for interaction between the teachers were frequent, as their classrooms were adjacent to one another. Both teachers also served on several staff committees together. This allowed them further "dialogue" time in addition to the monthly mathematics faculty meetings, staff meetings scheduled once a semester, and the planning meetings that occurred weekly.

Findings

Teachers' Interactions

The primary purpose of the peer teaming was to provide the teachers with an opportunity to exchange constructive ideas concerning the implementation of the new curriculum, and to allow them time to examine teaching and learning issues in collaboration with colleagues. It was envisioned that the time would be spent on sharing professional information about those teaching strategies that contributed to successful implementation of the program. However, such conversations did not occur between Ben and Gary. Although problems associated with teaching certain activities were discussed, these discussions were contrived and did not focus on analyzing elements that contributed to difficulties or successes they experienced in class.

The exchange of ideas was, for the most part, limited to "coverage" of the materials. Issues that concerned individual teacher's pedagogy were not addressed. Moreover, discussions about student's learning, or the strategies students used in the course of completing activities were limited. The teachers made references to those activities that were well received by the students, however, aspects of students' cooperation, nature of the students' interactions, and reasons that they felt contributed to the success of the activity in class were not discussed. In a general sense, implementation tips that were shared by teachers were limited to quick references to whether an activity was completed within a specified amount of time or not. Ben and Gary's interactions and dialogues also centered around discipline issues, and students' personal problems. They exchanged "stories" about particular students or the events that had caused disruption in class.

In a general sense, the teachers' collegial interactions were affective based. Teachers provided encouragement and emotional support for one another, supported one another's pedagogy even if they fundamentally disagreed with each other's practice. At the initial level of contact, teachers came together to describe their daily experiences, and to share stories. In further contact, teachers both volunteered and solicited narrative of their daily experiences. However, this exchange did not intend to originate particular discussions about teaching or learning mathematics.

Teachers' reactions to, and reflections on, one another's practice followed a general pattern consisting of two levels. At the first level of collegial interaction and on a public sphere, the teachers provided emotional support for their peers as they listened to one another's stories or observed each other's classrooms. As the teachers observed their peer's teaching, or heard their stories, often times they began to reflect on their practice as they examined specific aspects of their own work in light of what they had seen or heard. This reflection happened in private and in their conversations with the researcher. Teachers did compare and contrast their teaching styles. They also identified areas in peer's practice that they felt were in need of improvement. However, neither one of the teachers took the initiative to offer his perspective to the peer, to openly question their choice of pedagogy, or to solicit specific information on the theoretical basis for their practice. Public discussions concerning peer's pedagogy were avoided.

Ben and Gary exhibited a genuine acceptance of the pedagogical decisions made in their classrooms. This "accepting approach" to peer teaching served as a major barrier in their ability and willingness to directly confront the peer's practice or to even attempt to address aspects of the peer's teaching that needed refinement. Even when there was an opportunity to offer constructive criticism, both teachers were fearful that they might jeopardize the relationship. In a general sense, Gary and Ben avoided professional discussions. Their dialogues did not intend to engage either one in intellectual deliberation about learning and teaching issues.

Influence of Peers on Improving Practice

Ben and Gary entered the collegial team process without a vision of how the peer could help improve their teaching. Neither Ben nor Gary entered the peer teaming process with the expectation to benefit from collegial interactions. The peer teaming procedure was assumed more as an opportunity to socialize rather than discuss professional issues. The notion of influencing practice was not deeply rooted in their conceptualization of the professional exchange as each teacher recognized his peer as one who knew what was most appropriate to do in his classroom. Neither one of the teachers felt it necessary to provide professional suggestions that could potentially impact the peer's practice. In fact, neither one of the teachers expected the peer to offer professional perspectives that lead to a change in their intellectualization of teaching. Teaching was perceived by both Ben and Gary as an individual practice whose direction and content was determined by personal preferences of each teacher, and depended upon his own personal judgment. In spite of this, they continued to come together, visited one another's classrooms, and maintained a positive outlook on the value of such collegial interaction in the workplace.

Observation and Feedback

Gary and Ben's observations and feedback were marked by non-specificity, and positive feedback. Their post observation discussions had an incidental nature and were very informal. Neither one of the teachers provided written feedback on what they had observed during the session. Moreover, neither one of the teachers tried to question how the peer introduced mathematical concepts to students, how they synthesized ideas in class, or ways in which they responded to students' questions. Although numerous occasions for providing critical feedback on specific aspects of the peer's practice were present, instances of such exchange did not occur between them. In spite of their visible differences in how they implemented the same lesson, neither one was influenced by, or tried to influence, the activity of the peer.

Inherent in Ben and Gary's analysis of each other's teaching was an understanding of, and a compassion for, the "uniqueness" of what they did as teachers. Although they shared instructional materials, they were hesitant to recommend to each other even those pedagogical practices that they believed crucial in facilitating learning. Moreover, they did not attempt to convince one another of the validity of their chosen instructional strategies. In effect, neither Ben nor Gary tried to impact their peer's practice. In a general sense, the collegial proximity for them did not contribute to refinement of peer practice. Observation and feedback or critique of one another's teaching did not appear to be a natural part of their professional lives.

Final Comments

The notion of teacher growth in the presence of collegial interaction is an intuitively sensible assumption. However, this research provided some evidence that collegial interaction may not lead to positive teacher development. Although the teaming process was intended to invade the isolation among teachers, this physical proximity did not naturally provoke intellectual collaboration.

To initiate and sustain a culture in which teachers work with peers to improve both self and peer's practice, the teachers need to first believe that they have the right, and the potential, to influence the profession. This requires them to adopt a new paradigm on the very nature of the profession and how the roles and responsibilities of colleagues are defined in advancing that profession both at local and global levels. The teachers need to also learn how to engage in collaborative reflection on both self and peer practice in ways that improve teaching and facilitate teacher growth.

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PERFORMANCE-BASED ASSESSMENT OF SECONDARY MATHEMATICS STUDENT TEACHERS

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Abstract: We describe the use of performance-based assessment of student teachers and discuss its influence on student teachers, their cooperating teachers, and their university supervisors. We present action research that describes the development, implementation, evaluation, and refinement of a high-inference, criterion-referenced student teacher rating scale that is based on performance attributes. We report feedback from constituents who have used the process and present an analysis of quantitative data to determine patterns in attribute ratings as well as in student teaching grades.

Statement of the Problem

As part of an effort to reform teaching and learning, national organizations have proposed significant changes in teacher preparation, including a move toward more reliable methods for evaluating teachers' performance. One such method, performance-based assessment (PBA), has been a focal point of new standards for the certification of novice teachers, such as those proposed by the Interstate New Teachers Assessment and Support Consortium (INTASC) (1995). The National Council for Accreditation of Teacher Education (NCATE) has adopted the INTASC principles as benchmarks to be used in NCATE accreditation (Wise, Leibbrand, & Williams, 1997). As a result, many programs are focusing attention on PBA (Diez, 1997; Kain, 1999).

In this report, we describe action research undertaken to study the impact of PBA of secondary mathematics teacher candidates. We have developed criterion-referenced assessments, linked to national standards, that are designed to provide reliable mechanisms by which to judge teacher candidates' readiness for success. We will describe the development, implementation, evaluation, and refinement of an instrument used to assess student teacher performance. This instrument identifies mathematics student teacher attributes at three performance levels and includes standardized grade criteria linked to a candidate's level of achievement within these performance levels. Cooperating teachers, university supervisors, and student teachers use this instrument for midterm and final evaluations during a 12-week student teaching experience. We also describe and analyze evidence we have collected related to the instrument.

Literature Review

One factor underscoring the need to reconsider methods of student teacher assessment is the routine inflation of student teachers' grades (Brucklacher, 1998). Hartsough, Perez, and Swain (1998) describe several types of rating bias that may contribute to this problem. For example, the halo effect occurs when a rater who is impressed with a student teacher's overall performance rates the student teacher highly in all categories regardless of observed weaknesses. Another type of bias is due to logical error. This may occur when a rater is convinced that two attributes are related and, consequently, rates a student teacher highly on both attributes, despite differing performance in these areas. At Illinois State University, a review of student teaching grades across programs for two consecutive semesters revealed that 88% of all assigned grades were "A's" (Kinsella & Stutheit, 1998).

To avoid these biases, performance assessments have been developed, including professional teaching portfolios and final exhibitions, that incorporate a checklist approach (Kain, 1999; University of Indianapolis, 1999). More comprehensive schemes aimed at preservice or novice teachers have also been devised (e.g., California New Teacher Project, Beginning Teacher Support and Assessment Program [Yopp, 1999], Classroom Observation and Assessment Scale for Teacher Candidates, Competency Based Teacher Education Scale, Preservice Teacher Rating Scale [Hartsough, Perez, & Swain, 1998]). Users of these performance assessments have documented substantial contributions to preservice teacher development, including increased self-reflection and improved performance. Most of these instruments, however, are not content specific. They are designed for teachers at any grade level, teaching in any content area. Through our performance attributes and associated criteria for grade assignment, we present a framework for evaluating the performance of preservice mathematics teachers.

Modes of Inquiry

Our collection and analysis of data have resulted from two years of an iterative process involving the development, implementation, evaluation, and revision of a student teacher performance assessment instrument. Prior to using the criterion-referenced instrument, each university supervisor relied on his or her own knowledge, experiences, and preferences to grade student teachers, primarily from a norm-referenced perspective. (University supervisors assign two grades to each student teacher, such as A/A, A/B, B/B, and so on.) To move toward a more consistent evaluation scheme, as well as one with a more justifiable frame of reference, mathematics faculty who supervise secondary mathematics student teachers developed a high-inference, criterion-referenced rating scale. In the first year, we developed 12 student teacher attributes (e.g., content mastery, planning, organization, and rapport with students) with behavior descriptors in categories of "exceptional," "satisfactory," and "inadequate" (www.math.ilstu.edu/~day/PME2000/paper/appendixA.html).

Upon completion of a pilot semester (spring 1999), we reviewed categorical ratings and final grades for student teaching, analyzed feedback from supervisors, cooperating teachers, and student teachers, and refocused our effort to align our program with national standards. Based on our analysis of the data, we made several changes the second year (spring 2000), including the consolidation of some attributes and the addition of new ones. These revisions resulted in 11 student teacher attributes (e.g., assessment: multiple forms, conceptual/procedural balance, communication, and professionalism) with accompanying behavior descriptors in categories of "exceeds expectations," "meets expectations," and "does not meet expectations" (www.math.ilstu.edu/~day/PME2000/paper/appendixB.html). We also conducted a half-day training seminar for all university supervisors before the second year of implementation. After reviewing second year data, we may revise the attributes and performance descriptors again. We also have obtained funding to bring the cooperating teachers to campus in early spring 2001 to discuss PBA.

Results

Student Teaching Grades

Review of student teaching grades for the five years prior to the use of PBA and grades for the two years applying PBA shows that student teaching grades may be beginning to change. The percentages of A/A's earned by student teachers during 1999 (64%) and 2000 (50%) are lower than the average percentage of A/A's earned over the previous five years (70%), although the percentage of A/A's for 1999 is fairly similar to the percentages in 1995 (64%), 1997 (67%), and 1998 (61%). The remainder of the grade distributions for 1999 and 2000 are similar to those for the previous years. Although the pattern of A/A grades for 1999 and 2000 may indicate some change, it is premature to draw conclusions based on two years of implementation of PBA.

Attribute Rating Assignment

To assess whether student teachers, cooperating teachers, and university supervisors were in agreement in their attribute rating assignments for student teachers, we compared raters' final evaluations of each student teacher. In the first year, we obtained cooperating teacher and university supervisor evaluations for 19 of the 26 student teachers. For a variety of reasons, 7 of the 19 sets of evaluations contained only one evaluation form. For the remaining 12 students, there was agreement among the raters on 104 of 144 possible attributes, constituting a rater agreement of about 72%. Of the 40 disagreements among raters, a significant majority (72.5%) occurred because the cooperating teacher rating was higher than that of the university. This may indicate that the tendency to inflate student teacher grades is more severe among practicing school teachers than it is among university faculty.

In the second year, we obtained evaluations for all 20 student teachers. In addition to being rated by a university supervisor, each student teacher was self-rated and rated by at least one cooperating teacher. For all 20 student teachers, the average overall rater agreement (the number of attributes for which there was agreement among all raters) was 54%. We also computed pair-wise rater agreements between the student teacher and all cooperating teachers, between the student teacher and the university supervisor, and between all cooperating teachers and the university supervisor. These percentage agreements were 73%, 63%, and 65%, respectively.

Evaluation Questionnaires

At the end of each year, questionnaires were sent to all student teachers, their cooperating teachers, and the university supervisors. Response rates were 31.7% and 80% for 1999 and 2000, respectively. Here we provide some of the feedback that goes beyond general positive comments.

A common first-year expression of concern about the *student teacher attributes descriptions* was that the three levels of performance (exceptional, satisfactory, unsatisfactory) did not adequately address performance differences. Respondents suggested that one or more additional levels be included on the scale of attribute performance. Responding to the revised instrument in 2000, with three re-named levels of performance (exceeds expectations, meets expectations, does not meet expectations), only 1 of 39 responses mentioned levels.

The 1999 feedback on the *student teaching grading criteria* focused on how a student teaching grade corresponded to the attribute categories. Several people requested more specificity about how many "exceptional" attributes constituted a "significant majority" as stipulated for an A/A student teaching grade. Others appreciated the leeway of the high-inference grading scale. One experienced university supervisor commented on how these criteria caused him to give a lower grade than he would using his previous grade-assignment scheme.

In 2000, one supervisor and one classroom teacher commented on grade assignment. This supervisor suggested that "cooperating teachers are concerned with lower-than-usual grades" and the classroom teacher expressed concern about the apparent lack of ISU policy regarding the use of performance assessment: "ISU is putting their math graduates at a disadvantage." A classroom teacher suggested that letters of recommendation and a student's professional portfolio make better assessment instruments than do letter grades. Three student teachers commented on the university supervisor role in the grading process. They noted that a supervisor cannot expect to see all the various attributes within a limited number of classroom visits or that, due to the high-inference nature of the grading scale, each university supervisor may evaluate differently. One of these three suggested that one grade in the double-letter grade be assigned by the university supervisor and the other by the cooperating teacher.

Year 1999 comments on the *mid-term and final evaluation instruments* offered suggestions for format improvements. One respondent also offered a suggestion about the process:

> I think it would be helpful to be observed and evaluated by different people. More opinions would give a more holistic and unilateral [sic] way to grade. Even with this rubric, all teachers grade differently and have different expectations and standards.

In 2000, a university supervisor suggested that cooperating teachers meet with supervisors and program personnel before student teaching. A student teacher/cooperating teacher pair asked whether the evaluation process was based on achievement throughout the semester or on the final achievement reached by a student teacher. The cooperating teacher said that, "If the student teacher has improved and is doing great work at the end, they should receive a high grade." This clearly had been an issue among that triad of raters.

Each year we solicited *general comments* on student teacher evaluations. A 1999 student teacher raised the issue of accountability, describing a situation in which a colleague staged the replay of a previously taught lesson on the day that the supervisor was visiting. Year 2000 respondents offered a variety of comments. A university supervisor suggested that student teachers should not evaluate their own performance: "This puts him/her in a difficult position as he/she has little or no basis for comparison." This comment suggests that the university supervisor may have been operating from a norm-referenced perspective. Student teachers' comments focused primarily on the roles of the university supervisor and the cooperating teacher. These comments focused on grade assignment, training of university supervisors, and improved communication.

Conclusions

Two years experience implementing PBA has led us to some preliminary conclusions. First, university supervisors have addressed grade inflation by decreasing the number of A/A's assigned to student teachers during the past two years. However, the same is not true for student teachers and cooperating teachers who have a tendency to rate the student teacher more highly and with greater agreement than does the university supervisor. One explanation for the differential behavior of university supervisors and others may be that only the university supervisors were trained in the use of performance assessment. Another explanation may be that the daily contact between student teachers and cooperating teachers generates more information on which to base rating decisions. However, cooperating teachers and student teachers also have a greater personal investment in the process and may be less able to take an objective view. Second, we have observed that it is difficult for some users to make the transition from a norm-referenced perspective on grading to a criterion-referenced perspective. This was apparent in the cooperating teacher comment that the Mathematics Department's use of PBA was a disadvantage for student teachers in the job market, and in the university supervisor opinion that student teachers are unable to rate themselves accurately without having a basis for comparison. Finally, despite concerns of fairness to student teachers with respect to who assigns grades and the potential for job placement difficulties due to lower grades, many participants claimed that PBA is a valuable tool for making justifiable, equitable decisions about rating student teachers.

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ı *and final evaluation instruments* offered
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ved and evaluated by different people.
istic and unilateral [sic] way to grade.
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ested that cooperating teachers meet with
:e student teaching. A student teacher/
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·tudent teacher has improved and is doing
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ıts on student teacher evaluations. A 1999
:ability, describing a situation in which a
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fered a variety of comments. A university
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·rsity supervisor may have been operating
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·lusions

ıA has led us to some preliminary conclu-
dressed grade inflation by decreasing the
s during the past two years. However, the
›operating teachers who have a tendency
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lifferential behavior of university supervi-
ity supervisors were trained in the use of
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g teachers and student teachers also have
and may be less able to take an objective
ifficult for some users to make the transi-
grading to a criterion-referenced perspec-

ASE STUDY IN SUPPORTING ESERVICE TEACHERS' ONAL GROWTH

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to document the effectiveness of a CD-ROM
preservice elementary teachers' developing
teach mathematics for understanding. In
in which to examine the use of cases in
:rs' professional development, including their
ıt and pedagogy. Our analysis will document
teachers analyzed and critiqued the classroom
In particular, we will point to significant
ıerged as a result of the preservice teachers'

ument the effectiveness of a CD-ROM based
teachers' developing understandings of what
:rstanding. In doing so we provide a context
n supporting elementary preservice teachers'
nderstandings of issues of both content and
991; Copeland & Decker, 1996; Fennema, et
Ball, 1990; Merseth & Lacey, 1993; Schifter,
l, 1992). As part of our analysis we present
:hematics methods course that incorporated a
n the coursework. In particular, our analysis
of one of the cases[1]. The case is developed
lists, totals, and average which was taught in a
resources on the CD include edited classroom
post-interviews with the teacher with scrolling
‹, the teachers' lesson plan, and a seating chart
students. The focus of the analysis is on the
achers explored the case and how those issues

Setting

:us of this analysis is the first in a two-course
› intended for elementary preservice teachers
associated field experience. As a result, the

instructor[2] uses a series of three CD-ROM based cases to create classroom contexts that can be explored by the preservice teachers. A primary goal of the course is to support the preservice teachers as they begin to tease out the complexities involved in teaching mathematics at the elementary-school level. As such, the National Council of Teachers of Mathematics Standards [NCTM] documents, numerous articles from the NCTM practitioner journals, *Making Sense: Teaching and Learning Mathematics for Understanding* and three CD-ROM based cases comprise the basis of the resources.

Data

Data for this study were collected during the fall semester of 1999 and consist of daily videotape-recordings from one camera while the case was being utilized. Additional documentation consists of copies of all the preservice teachers' written work, surveys completed by the preservice teachers, a survey completed by the instructor, daily field notes that summarize classroom events, notes from the teacher's daily planning, and taped interviews conducted with the instructor after each class.

Methodology

The general methodology falls under the heading of a teacher development experiment (TDE) (cf. Simon, 2000). This methodology is derived from the constructivist teaching experiment (cf. Cobb & Steffe, 1983; Steffe & Thompson, 2000) and whole-class teaching experiments (cf. Cobb, 2000) by acknowledging that a team of "knowledgeable and skillful researchers can study development by engaging in fostering development through a continuous cycle of analysis and intervention" (Simon, p.336, 2000). The distinction between the TDE and the teaching experiment is that the TDE is concerned not only with the mathematical development of the participants (i.e. the preservice teachers), but also their professional development. In this way, the TDE can be characterized as a "whole-class teaching experiment in the context of teacher development" (p. 345).

Results of Analysis

During the exploration of both the second and the third CD-ROM based case, the instructor focused the preservice teachers' explorations on four main themes: (1) mathematical content (2) planning for instruction, (3) facilitating the lesson, and (4) understanding students' thinking. These themes were chosen in support of the instructor's goal of concurrent development of the preservice teachers' understanding of (1) the mathematics they will teach and (2) effective strategies for teaching mathematics for understanding. Both the text resources and the CD-ROM based cases highlighted these aspects of teaching. (For example, in the text The Nature of Classroom Tasks coordinates with planning for instruction, understanding students' thinking, and math content. On the CD-ROM the issues matrix highlights excerpts from the classroom focused on each of the four themes.) The issues that emerged around these themes as the preservice teachers investigated the case provided a basis for significant discussion of issues of both content and pedagogy.

Mathematical Content

Issues surrounding the mathematical content were addressed as the preservice teachers engaged in the same mathematical tasks in which the seventh-grade students participated. In particular, the students were asked to first generate a list of features that they consider when purchasing sneakers. They were then asked to work in groups to rank order the list from least to most important. The final aspect of the task was to take the six ranked lists that had been generated by the groups and compile them into one list that summarized the data in the individual lists.

The preservice teachers' solution methods included (1) placing each quality in the rank in which it appeared most often, (2) adding the ranks of each quality across the lists and re-ranking according to the sums, and (3) finding the average rank for each quality and then re-ranking by average. The preservice teachers then compared and contrasted the solutions as the instructor attempted to help them tease out the mathematical differences in each method. Questions from the instructor such as, *"What if you hadn't found the average and you'd just taken your sum. Do you think it would have changed the order any?"* offered opportunities for the preservice teachers to clarify their understandings of the mathematics of the lesson as they worked to understand the differences in the solution methods.

Planning for Instruction

In discussing planning for instruction, the preservice teachers noted the importance that the video teacher attributed to the anticipation of student responses as she planned her lesson. For the preservice teachers, this was an aspect of the classroom with which they had no experience. In addition, they had no basis for making these anticipations. They therefore questioned how they could incorporate this strategy into their own planning. However, as they reflected back on the mathematical exploration from the lesson, they realized that their own activity could provide a basis for these anticipations. The instructor facilitated this process by making it an explicit topic of conversation.

> *Instructor:* Okay, so here's what I want you to do. I want you to take your student hat off a minute and put your teacher hat on. And I want you to think like teachers now for a minute. If we gave this task to seventh-grade students, all right, seventh-grade students, what do you think they would do?

In the ensuing discussion, the students pointed to their ways of reasoning on the task as they began to make reasoned anticipations about the seventh-grade students. In doing so, they attempted to anticipate how they might orchestrate a whole-class discussion that incorporated these diverse solution strategies.

Facilitating the Lesson

As the preservice teachers discussed facilitating the lesson, they experienced a perturbation in that they believed all students' solutions should be valued equally. The teacher on the CD-ROM explained in her interview that she selected and sequenced students to share their solutions in order to advance her mathematical agenda. This caused a discussion about the role of student solutions in whole-class discussions. The preservice teachers had not reasoned about how to effectively build from their students' contributions. They had only considered that they should ensure that all students have a chance to share.

This appeared to be a pivotal episode for the preservice teachers as they acknowledged that they might need to make judgments about the worth of a student's contribution. This highlights a tension in teaching by making explicit the importance of the teacher's understanding of (1) individual student's offered solutions and (2) the relation of the offered solution to the overall mathematical agenda. The importance of the teacher's attention to both of these aspects of a student's contribution brings to the fore the complex nature of the practice.

A second issue that emerged while discussing facilitating the lesson concerned the classroom norms and student participation in whole-class discussions.

> *Nance:* She asked "why" a lot just to have the students reflect on the like it's kind of along with classroom norms, like justifying why they came up with their solutions.
>
> *Steph:* And after a while they started doing that on their own. There was evidence that they, before she needed to ask could you explain that, they would just go ahead and someone would raise their hand and question one of the students.

The preservice teachers noted that the process of negotiating the classroom norms evolved in the course of this one lesson. They saw evidence of the importance of making the norms explicit and then following through in action. They judged the results of such norms as positive and something that they would want to work to achieve in their own classrooms.

A third issue that emerged concerned the fact that at the end of the lesson, the teacher did not identify the "best" method for solving the task. Some of the preservice teachers repeatedly referred to this as giving the lesson "lack of conclusion" and "uncertainty." However, other preservice teachers disagreed. In the course of the ensuing discussion, the preservice teachers noted the similarities in the method involving sums and the method involving averages. They pointed to the mathematical significance of the difference and discussed possible goals that could be achieved by allowing students to solve the task with either method. Through this discussion, they were beginning to find merit in highlighting the diversity in students' ways of reasoning.

Understanding Students' Thinking

The fourth theme, understanding students' thinking, was especially problematic in that the preservice teachers questioned the video teacher's ability to adequately assess the students since all of their activity was conducted in groups. The preservice teachers felt that without an individual written assignment, there was no way to effectively assess. They viewed this as a weakness of the lesson. In discussion, the instructor pointed to the problems inherent in their suppositions. They were implying that a written assignment had to be given every day in order for a teacher to be able to make a valid assessment. The preservice teachers had difficulty reasoning about how a teacher might use the results of interacting with students to conduct formative assessments that would then inform instructional next steps.

In contrast, the preservice teachers commented favorably on the teacher's interactions with the students as they worked in groups and discussed in a whole-class setting.

> *Rosa:* She'd just ask questions that would make them, to see if they really understood and she would ask a lot of different kids so she'd make sure that the whole class understood, not just the vocal ones.

Here it appears that Rosa is acknowledging the importance of using classroom discussions as a means of assessing students' understandings. However, the preservice teachers characterized this aspect of the teacher's practice as focusing on "classroom norms." They were unable to make the link between supporting the development of a classroom participation structure that valued students' thinking and how that would contribute to ongoing assessment of students' understanding.

Conclusion

In this paper, we have highlighted the opportunities that emerged for the instructor as she built on the issues raised by the preservice teachers while still working to advance her agenda. In particular, the instructor noted that the case provided an exemplary way for her to focus on mathematical content in the context of teaching for understanding. As preservice teachers begin to tease out the complexities of teaching, it is important to situate that development in a deep understanding of the mathematics that they will teach. This implies a concurrent development of (1) an understanding of mathematical content and (2) an understanding of effective pedagogical strategies for the classroom. Cases which provide the resources necessary to focus on both of these aspects can play a pivotal role in preservice teachers' development by creating opportunities for grounded discussions of issues of both content and pedagogy. It is important to note, however, that we are *not* attributing the professional growth and development of the preservice teachers to the case described. What we *do* claim is that the CD-based case provided the instructor with the means of support necessary to achieve her goals for the course in the context of deliberately facilitated discussions.

Notes

1. The CD-ROM based case that is the focus of this study was developed by the first author in collaboration with Janet Bowers of San Diego State University, and Helen Doerr and Joanna Masingila of Syracuse University with funding from the National Science Foundation under Grant No. REPP 9725512.
2. The second author was the instructor of record in the methods class that is the focus of this study. The first author is the teacher on the video in the CD-ROM case.

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CHANGING BELIEFS AND TEACHING PRACTICES: AN ILLUSTRATION OF A REFLECTIVE CONNECTIONIST

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Abstract: Carrie entered her preservice elementary teacher education program professing a strong dislike for mathematics and a desire to avoid teaching the subject. Her beliefs about mathematics were in strong contradiction to her more general beliefs about teaching and learning. As a result of a field experience in which she observed an exemplary mathematics teacher, Carrie was able to modify some of her beliefs about mathematics and to craft her mathematics teaching practices to be consistent with her beliefs about teaching and learning. Theoretical work from the beliefs literature is used to explain how Carrie was able to change her beliefs and actions. In particular, the notion of a reflective connectionist, proposed by Cooney, Shealy, and Arvold (1998) is elaborated.

The literature abounds with examples of preservice and inservice mathematics teachers who hold less than desirable beliefs about mathematics and who, despite a teacher education course or inservice experience, fail to show evidence of substantial change in their beliefs. In this article, I present a preservice teacher, Carrie, who showed evidence of some dramatic changes in her beliefs in a relatively short period of time. I contend that Carrie is an example of reflective connectionist as described by Cooney, Shealy, and Arvold (1998), and I elaborate on what it might mean to be a reflective connectionist.

Theoretical Framework

Cooney et al. (1998) presented the case of a preservice secondary mathematics teacher who held a coherent set of beliefs about teaching mathematics and who modified his beliefs in the face of new experiences. They characterized the teacher as a *reflective connectionist* because he was able to mold his beliefs based on a careful analysis of the views of others (such as mentor teachers, peers, teacher educators). Because he engaged in this type of analysis, he was very committed to his beliefs and seemed inclined to act on those beliefs in the classroom. To amplify the notion of a reflective connectionist, I drew on the work of Green (1971) and Raths, Harmin and Simon (1987, as cited in Seah & Bishop, 2000). I propose that a reflective connectionist holds what Green termed an *ideal belief system* and goes through a process described by Raths et al. as *valuation.*

Green contended that the purpose of teaching is to modify not only the content but also the structure of students' belief systems. He argued that the goal of teaching should be to help students develop belief systems in which "the number of core beliefs and belief clusters are [sic] minimized, the number of evidential beliefs are [sic] maximized, and the quasi-logical order of ... beliefs is made to correspond as closely as possible to their objective logical order" (p. 52). For a thorough description of each of these characteristics, see Green (1971).

Raths, Harmin and Simon (1987, as cited in Seah & Bishop, 2000) proposed that attitudes, beliefs, and interests go through a process of valuation in order to become part of one's system of operating. They suggested that one must choose freely from among alternatives after careful consideration of the consequences of each alternative, that one must cherish one's beliefs and affirm them to others, and that one must act in concert with one's choices repeatedly to form a pattern in one's life.

Methods

Data Collection

Carrie was one of four preservice teachers that I studied during a field-based mathematics methods course. Carrie and her peers were placed in a fourth-grade classroom with a teacher that was locally recognized as an exemplary mathematics teacher who was committed to teaching in a manner consistent with current reform recommendations. Data collected during this study included two individual interviews, Carrie's journal, four audiotapes of Carrie conducting task-based interviews with individual children, three audiotapes of Carrie teaching mathematics to a small group, one videotape of her teaching a small group, field notes on eight observations of the classroom teacher, and audiotapes and field notes from eight discussions among the four preservice teachers, the mentor teacher, and me. See Mewborn (1999) for more details of the study. I kept anecdotal records of informal conversations with Carrie during the first two years of her teaching career, and I conducted a study of her teaching during the third year. Data from this study included weekly classroom observations, two individual interviews, and classroom artifacts such as lesson plans, classroom displays, and student work.

Data Analysis

The data were analyzed from the perspective of the interpretive paradigm for teacher socialization (Zeichner & Gore, 1990) in an attempt to understand the nature of a social setting at the level of subjective experience. I hoped to gain an understanding of mathematics teaching and learning from Carrie's perspective. The data were analyzed using the methods of grounded theory (Glaser & Strauss, 1967) and grounded interpretivism (Addison, 1989). Grounded theory and interpretive research methods are both constant comparative methods that emphasize the importance of context and social structure in research settings.

A Portrait of Carrie

Carrie's beliefs about teaching and learning were tightly clustered around her core belief that teachers should treat children with love, compassion, and respect. She believed that school should be a place where children interact with adults who love and care about them as human beings, and she saw school as a place that could rectify the unpleasantness in some children's lives. From the beginning of her teacher education program, Carrie exhibited a strong care ethic and demonstrated concern for the societal and family situations that impact negatively on children's lives. Her core belief was held with passionate conviction and formed a central aspect of who Carrie was as a person and a teacher. She held beliefs about students, learning, and teaching that were derived from her primary belief about respecting children.

With regard to students, Carrie believed that children must have confidence in themselves as learners if they are to be successful in school and in life. She believed that learning is a process of understanding, not a means to a correct answer, and she believed that teachers should be role models for their students. These beliefs and her core belief about respecting students were all logically related and tightly clustered. Carrie's beliefs were organized in a manner consistent with Green's (1971) description of an ideal belief system.

However, as a preservice teacher Carrie held a cluster of beliefs about mathematics as a discipline, about herself as a learner of mathematics, and about herself as a teacher of mathematics that was in sharp contradiction to her general teaching and learning belief cluster. Carrie disliked mathematics and did not see herself as a competent mathematics student, despite earning good grades in four years of college preparatory mathematics in high school. Carrie did not see mathematics as engaging, interesting, logical, or meaningful. She described mathematics as "alphabet soup with numbers" and "a headache that won't go away." For her, learning mathematics was like "putting a puzzle together and finding one piece is missing." Carrie's experiences and beliefs about mathematics conflicted with her beliefs about teaching and learning, in general, and her belief that a teacher is a role model, in particular. This conflict was a significant source of concern for her because she knew that she was not able to model excitement and enthusiasm for learning mathematics. She did not want her students to have the same types of mathematical experiences that she had, so she was very concerned about her ability to teach mathematics.

Carrie's initial reaction to this conflict was to say that she did not want to teach mathematics because she did not want to do a disservice to students. However, she soon realized that this was not a realistic solution as most elementary teachers are expected to teach mathematics. From this point forward, Carrie displayed a genuine desire to learn about teaching mathematics in a way that would help her overcome her negative experiences and enable her to be a competent teacher. She was open, willing, and eager to learn. In her first interview, Carrie said, "That's why I volunteered for this study, because I want to see someone who really loves math teach it so that maybe I can see math differently."

During the initial study, Carried was involved in a field experience in which she worked with a skilled mathematics teacher, and she saw that it was possible for students to experience mathematics as a dynamic, creative, fun, interesting discipline with opportunities for individual exploration and interpretation. At the end of the field experience Carrie said she had learned that "teaching math is nothing more than exploring math *with* your students. I've learned that 'wrong answers' are such a gift in the classroom because they open the doors for so much more understanding and exploration of math." As a result of this experience, she was able to shape her mathematics teaching practice to be consistent with her other beliefs.

For example, Carrie manifested her belief in children's need for self-confidence by placing a lot of emphasis on children's mathematical thinking. She took every opportunity to praise and reward children for their thinking. Carrie thought it was important to find something of value in each child's thinking, regardless of the correctness of the response. She manifested her belief in learning as a process by insisting that children explain their answers and by seeking and rewarding multiple solution strategies. She enacted her belief that a teacher is a role model by revealing her mathematical thinking to students. She also believed that it was important to "be human" and admit to making errors or admit to not knowing all of the answers.

Discussion

The structure of Carrie's belief system enabled her to change her actions and beliefs. She held one core belief about respecting children and held it with passionate conviction so that it was necessary for her to resolve any beliefs that conflicted with this core belief. She had a minimum number of belief clusters, and she was able to find a way to connect some of her beliefs about mathematics to her core belief cluster. All of her beliefs were held evidentially, which allowed her to modify beliefs in the face of new evidence. In particular, Carrie's beliefs about mathematics were held evidentially because they were based on her experiences as a learner. Thus, when she saw evidence, in the form of her mentor teacher, that mathematics could be taught differently, she was able to use this evidence to modify her beliefs about mathematics. Then she was able to fold her belief cluster about mathematics into the cluster about teaching and learning, thus minimizing the conflict she had previously experienced.

The process by which Carrie altered her actions and beliefs is consistent with the description given by Cooney et al. (1998) and Raths et al. (1987, as cited in Seah & Bishop, 2000). A key element of the process proposed by both sets of authors is the notion of *choice*. Carrie deliberately chose her teaching actions from among alternatives presented by her past experience and her teacher education program (particularly the field experience). She was able to integrate her experiences as a learner with her experiences in her teacher education program, analyze the merits of various positions and shape her mathematics teaching practice to be consistent with her general beliefs cluster. During this process she also changed some of her beliefs about mathematics. She cherished her core belief, so she was able to become committed to beliefs that were consistent with that belief. Specifically, she was able to act consistently on beliefs within her core cluster, including her new beliefs about how mathematics should be taught.

The extent to which Carrie changed her beliefs about mathematics as a discipline and herself as a learner of mathematics is still an open question. She certainly changed her *actions* as a mathematics teacher, and it is plausible that she changed her beliefs about mathematics teaching. However, there is no evidence that Carrie now sees mathematics (beyond school mathematics) as a dynamic, meaningful, sensible discipline or that she sees herself as a competent learner of mathematics. It seems likely that because she was able to act as a mathematics teacher in a manner consistent with her beliefs about teaching and learning, she gave herself license to lock away her other beliefs about mathematics because they were no longer a source of conflict for her. This explanation resonates with Guskey's (1986) proposal that change in action precedes change in belief, a proposal that is worthy of further exploration by the mathematics teacher education community.

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USING CLINICAL INTERVIEWS TO PROMOTE PRESERVICE TEACHERS' UNDERSTANDING OF CHILDRENS' MATHEMATICAL THINKING

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Abstract: The purpose of this research paper is to share results of a "teaching experiment" in which preservice teachers were provided with opportunities to learn to use the clinical interview method with children. Our central hypothesis was that the clinical interview can be a fundamental method for helping teachers—both preservice and in-service -- develop their own personal and instructionally relevant theories of how children interpret mathematics and then devise better ways of teaching it. Results indicate that preservice teachers do develop a deeper understanding of the ways in which children build mathematical ideas as a result of clinical interviewing children.

Theoretical Framework

Teaching mathematics well is a difficult task. In addition to managing the behavior of some 20 or more children, the teacher should understand the mathematics to be taught, have a grasp of useful pedagogical techniques, and have insight into students' minds (Ginsburg, 1998; Schorr and Lesh, 1998). For example, to help first graders understand the notion of "equivalence" as it is involved in a statement such as 2 + 3 = 5, the teacher must understand that equivalence refers to a special mathematical relationship, that certain kinds of manipulatives (for example, a balance) may be used as a "model" of that relationship, and that first graders are likely to provide *their own distinctive interpretations* of the "equals sign." Children tend to believe that the "equals sign" refers not to a relationship-- the teacher's view-- but to an act of adding. For first graders, the "equals sign" usually does not mean "the same as," but is instead interpreted as "makes" or "get the answer" or, as one child put it, "the end is coming up." The teacher who has insight into children's minds can appreciate the sense in their interpretations-- after all, the "equals sign" can legitimately refer to the outcome of an operation-- and can deal with them constructively. (For example, the teacher can help the child to understand that the "equals sign" has at least two legitimate meanings, both of which can be useful.)

By contrast, the teacher who lacks understanding of children's minds is left in a kind of pedagogical delusional state: the teacher understands equivalence in a certain way, thinks that concept is being taught to the child, but the child is in fact learning an entirely different concept of which the teacher is unaware. In this case, there is a wide gap between the mind of the teacher and the mind of the child. The teacher tends to deal with what is seen as the child's failure to learn equivalence by "shouting louder"--

that is, by redoubling efforts to teach the concept (as interpreted by the teacher)-- and remains unaware that the child is in fact attempting to learn something else entirely. In our view, such gaps between teachers' minds and students' minds are widespread, and characterize teaching from preschool through university. As Piaget (1976) pointed out, it takes a psychological equivalent of the Copernican revolution for the adult to realize first that the child's thinking does not necessarily revolve around or take a form similar to that of the adult's, and second that children's minds, although often radically different from the adult's, can nevertheless make their own kind of sense.

This research focuses on one technique for reducing these wide mind gaps, namely the "clinical interview" method. The clinical interview, as originally developed by Piaget (1976), is a flexible and deliberately non-standardized method of questioning, which aims at providing insight into children's ways of thinking-- into their personal "constructions"-- which are often different from the adult's. In the clinical interview, the adult poses a specific task to the child, and usually begins with some predetermined questions. However, the adult is free to modify the questions as necessary, depending on the child's apparent understanding of the questions, the child's motivation, and particularly the child's response to the initial question. The interviewer has the freedom to rephrase the questions to ensure that the child understands them, to follow up on interesting remarks, to clarify responses, and even to challenge them so as to establish the child's degree of conviction. The clinical interview method has been used as the basis of a good deal of research on children's understanding of school mathematics for many years, perhaps beginning with Davis & Greenstein (1964), and is now receiving increasing recognition as a major tool for psychological research into cognitive functioning (Ginsburg, 1998).

Our central hypothesis is that the clinical interview can be a fundamental method for helping teachers—both preservice and in-service, achieve the needed Copernican revolution in their thinking about children's thinking. The clinical interview method helps teachers to both develop their own personal and instructionally relevant theories of how children interpret mathematics and then devise better ways of teaching it. This becomes especially important if we intend to move away from teaching techniques that simply enable students to repeat, without understanding, various specific algorithms, rules or procedures. As Cohen points out, "The teaching that reformers seem to envision would require vast changes in what most teachers know and believe" (Cohen & Barnes, 1993, p. 246). "Teachers who take this path must... have unusual knowledge and skills... They must be able to comprehend students' thinking, their interpretations of problems, their mistakes...they must have the capacity to probe thoughtfully and tactfully. These and other capacities would not be needed if teachers relied on texts and worksheets" (Cohen, 1988, p. 75).

This research builds upon previous work which shows that it is possible for teachers to learn the clinical interview method and to develop useful forms of it for practical implementation in the classroom (Ginsburg, Jacobs, & Lopez, 1998). That research showed that elementary level teachers from very different types of schools—inner city, suburban, and private— were able to become rather good interviewers and develop distinctive styles of interviewing appropriate for their classrooms. For example, some teachers developed forms of interviewing individual students; others developed methods for interviewing groups of students; others integrated interviewing into their teaching; and another teacher taught her students to interview each other. Almost all teachers said that the process of learning and implementing clinical interview methods was an extremely valuable experience and indeed changed their whole approach to understanding children and teaching them. Colleagues have shared similar anecdotal results: the student teachers or in-service teachers with whom they have worked report that conducting clinical interviews can be a transforming educational experience. The goal of the present research is to provide data concerning what preservice teachers learn from clinical interviewing.

Methods and Procedures

The research reported in this paper took place over a 15 week period as part of a Math Methods course for elementary and middle school preservice teachers at Rutgers University. As part of the course, the prospective teachers were provided with opportunities to learn the clinical interview method, interview children, and reflect on the interviews during classroom sessions. They were also provided with opportunities to deepen their own understanding of the mathematics that they were expected to teach, and consider the pedagogical implications of teaching mathematics in a thoughtful manner.

More specifically, during weekly class sessions, the prospective teachers would investigate a particular mathematical idea by solving a problem or series of problems related to the idea, generally in a group setting. They would then reflect on their own solutions and the solutions of others in the class. Next, they would watch an interview involving a child or series of children grappling with the same or similar mathematical ideas. During and after the interview, they would share reflections about the questions posed and the interview techniques used. They would also discuss the child's mathematical thinking, and the pedagogical implications of teaching the ideas in a thoughtful manner. Afterwards, they were encouraged to actually interview a child about the same ideas, and share the results during the next class session. As part of their final written project, they had to interview a child (either the same child, or a different child) about a particular mathematical idea, record significant aspects of the interview, and discuss the overall interview and the implications for teaching.

The following example will illustrate the process. Several class sessions were devoted to the development of ideas relating to numerical operations. In one particular session, preservice students worked in small groups to consider the development of a base 5 number system and then use their system to solve a series of addition and subtraction problems. After sharing their results with other groups, the class spent some time discussing their mathematical thinking, and sharing some of the misconceptions that occurred over the course of the session. Next, they observed a series of videotaped interviews[1] in which they could begin to consider children's understanding of place value and written calculation. One such interview involved a second grade girl who had not memorized the basic addition combinations, but was quite capable of solving computational problems by using different counting strategies, some of which involved using her fingers. The students then reflected on a series of questions including the following: How did this girl figure out the different combinations? What strategies did she use? What do you think she needs to learn next? A second interview involved a second grade child, who could perform simple subtraction with regrouping easily enough, yet appeared to have a weak notion of the meaning of place value. After watching the interview, the class discussion for the prospective teachers revolved around the following questions: What do you think this student understands about place value? Given this understanding, what is the meaning for her of written calculation procedures involving regrouping? How does she use the chips to represent numbers? What kind of instruction would be useful for her?

After solving the problems, watching the interviews, and having the class discussions, the prospective teachers performed their own interviews.

Results and Conclusions

To document and discuss insights attained by the teachers as a result of clinically interviewing children, this paper focuses on specific examples of interviews conducted by prospective teachers. One preservice teacher wrote the following about a second grade student that she had interviewed (after watching the videotaped interviews described above). As background, it is important to note that the child being interviewed was in a basic skills math program due to overall poor performance in mathematics.

> *Conducting this interview left me surprised at how much Jake really knew. Because I had been privy to his "academic standing" in school, I had expected his knowledge to be rudimentary at best. At the very least this interview taught me not to accept labels as a defining standard for any child. In the future I will not assign ceilings to a child's capabilities. They are so often arbitrary, as well as debilitating. Beyond that, I was astounded to see that the war between his outlook of his "life math" and his "school math" was so pervasive. It was as if he had on and off switches that programmed him to either flourish and find solutions or render him rigid; searching his memory bank of useless, incoherent symbols that would somehow mysteriously pop into his head, gifting him with the appropriate response....This interview has done me a greater service than just highlight some of the common issues we*

face in trying to teach students math. It has blatantly pointed out that the major hurdle in teaching is connecting to the students.

As documentation to support her comments, this teacher provided several excerpts, one of which included the following:

When I asked Jake about school [math], a somewhat blank, confused look came over him. "Uh, I don't know. Um, oh, I forget. Oh, uh, no, now I remember. We are doing teens." I then asked what was he doing with the teens. He said they were "like adding and taking away." I asked a few additions facts. He got the correct answers, but did not seem as sure of himself.... At one point, I asked him what 12 take away 8 was. He tried to unobtrusively look down at his fingers. He had all 10 splayed and then curled down four fingers from each hand, leaving only the 2 thumbs. He looked up at me, smiled triumphantly and said "4"!". Then he lowered his eyes and sheepishly informed me that his teacher gets mad and hollers at him if he uses his fingers. It broke my heart. I told him I thought it was okay to use his fingers, and in fact sometimes I use my fingers too. I then asked him to explain how his fingers helped him to figure out 12 minus 8. "Well, I have 10 fingers, and 2 more invisible ones is 12. Then I put down 8. Then I had my 2 thumbs and my 2 invisible ones...so I have 4!"

This teacher went on to share her own interpretation of Jake's informal strategies for doing addition and subtraction. She (and many of the other preservice teachers) noticed that the children that they interviewed often used informal strategies that they already knew in order to solve new problems rather than traditional algorithms. They noted that they would never have know this had they not had the chance to interview actual students who could share their thinking about mathematical tasks. Taking the time to interview students was really important for this preservice teacher. She realized that although Jake did not perform well when asked to solve computational problems in school, he appeared to be quite good at using his own invented strategies. In her journal reflections, she noted that prior to this course "numbers were numbers to me-nothing more. I've used them when I had to-for the typical things such as checking accounts, budgeting and of course sales-but even still they were just a necessary evil". After reflecting on her own thinking, the thinking of her classmates, and most particularly, the thinking of children, she said the following:

I would not be rigid in my approach to teaching. I would use manipulatives freely and frequently; this includes allowing them [the children] to use their fingers if they feel so inclined. I would encourage them in finding their own solutions to the problems, thus giving them the ownership that creates a freedom of exploration over their own work and processes. It does no one any favor to demand a one-way problem and solution strategy useful only in repetitious math drills.

Other teachers also reported that the clinical interviewing process had been helpful. The excerpt below is representative of the comments of others:

> *After spending a great deal of time on the clinical interviewing process, I was exposed to the reality that many classrooms do not lend to a student enough time to frame solid thoughts, or invite students to use their own logical methods of calculation. I have also come to understand that being a teacher is not solely comprised of facilitating information but rather being a part of teaching that invites reciprocal processing. The reciprocal process allows students and teachers to learn from each other by sharing. During the time I spent with clinical interviewing, I felt that the role of the preservice teacher and student became inter-changeable and communal. Each question that I posed to Joseph, the student, he offered an answer with reasons to his approach. This in turn allowed me to witness his implementation of different cognitive processes .... The constant feedback in the interview had a positive impact on both my becoming a better teacher and my teaching practices. Moreover, the student was able to gain insight into his own mathematical thinking. Through clinical interviewing, I was able to see the challenges Joseph experienced that would have not been seen in a traditional classroom setting. This ...gave me a chance to see how a student's mind works.*

The point of sharing these reflections is not merely to confirm that indeed, preservice teachers enjoyed the clinical interview process, but rather, to suggest that as a result of the experience, these prospective teachers will spend more time considering children's thinking, and how children build mathematical ideas, when they actually become teachers.

Space limitations of this paper do not allow a more complete description of these or other preservice teacher's comments. The documentation is provided to suggest that using the clinical interview method had an impact on the prospective teachers' approach to understanding children and on how they intend to teach them.

Note

1. The tapes and corresponding guide are part of a series entitled "Children's Mathematical Thinking-Videotape Workshops for Educators" developed by Herbert Ginsburg, Rochelle Kaplan, and Rebecca Netley. They are distributed by the Everyday Learning Corporation.

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FIELD EXPERIENCES AS OPPORTUNITIES FOR MATHEMATICAL CONVERSATIONS

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Abstract: Field experiences are a substantial part of secondary teacher preparation in mathematics, but it is not clear what mathematical knowledge is gained from the experience. Some argue that generic managerial functions are emphasized more than the learning and teaching of specific content. This study investigated the nature of mathematical discussions between mathematics teachers and student teachers in six high schools and one middle school. Mathematics was discussed in the context of the student teachers' lessons, and the conversations focused on pedagogical content knowledge. Mentor teachers offered advice on how they would teach a specific topic or on student difficulties with the topic. Discussion of content knowledge was motivated by student teachers' questions or mathematical difficulties. We found little evidence of collaborative mathematical exploration between the mentor and student teacher.

Preservice teachers often claim that student teaching was one of the most valuable parts of their education. The opportunity to be in the context of a high school and to teach students seems to heighten their awareness and perhaps their urgency to learn. Field experiences provide an opportunity for both mentor teachers and student teachers to have significant mathematical conversations, but mathematics is only one area that receives attention in the field. We studied mathematical conversations, what motivates them, and related implications.

Purpose of Study

Despite the numerous opportunities for conversations that field experiences offer, researchers (Sudzina, Giebelhaus, & Coolican, 1997; Wideen, Mayer-Smith, & Moon, 1998) remind us that taking advantage of those opportunities is difficult. There are different goals between classroom teachers, student teachers and university supervisors. There are perceived gaps between theoretical constructs developed in teacher education programs and practical constructs valued by mentor teachers. In a recent study of four teachers mentoring preservice mathematics teachers, only one teacher was identified as spending a significant amount of time discussing mathematics or mathematical issues (Wilson, Anderson, Leatham, Lovin, & Sanchez, 1999). Mentors often spend time discussing managerial issues, school policies, and student behavior. This study was designed to gain insight into the following questions.

- What is the nature of mathematical conversations that occur in student teaching situations?

- What motivates these conversations?
- What can be concluded from these conversations?

Theoretical Framework

Our work with teachers has been influenced by Charon (1998) who advocated symbolic interactionism and the power of considering a person's viewpoint in a given situation or event. Mentor teachers, student teachers, and university supervisors have different roles during field experiences, and they also bring different perspectives to teacher development. There is a tension in preparing teachers between training teachers to become good managers of the learning environment and educating teachers to understand learning and learners (Goodman, 1984). Shulman (1986) argued that it is not only important for teachers to gain general pedagogical knowledge but also content knowledge and pedagogical content knowledge. Teachers need to know mathematics, but they also need specialized knowledge about how students learn mathematics and effective ways to teach mathematics. We believe that mathematical conversations and analysis of those conversations are important parts of reflection on mathematics teaching, because the conversations add to the knowledge that teachers use to connect mathematical ideas and make teaching decisions.

Methodology and Data Sources

The Partnerships in Reform In Mathematics Education (PRIME) project involves a collaborative effort between university staff and mentor teachers who are working with preservice mathematics teachers in six high schools and one middle school. In the first year, the research focused on representing and understanding mentors' views on the process of mentoring.

We found little evidence of mathematical conversations during the first year, but that was not the purpose of our data collection at that time. One teacher explained that student teachers learn their mathematics at the university, and they need to learn other lessons in the field prompting us to investigate mathematics during the second year. Research suggests that it is more common for mentor teachers to discuss management with student teachers than other issues (Goodman, 1984; Zeichner, 1985), but we wanted to know more about what, when and why mathematics was discussed.

During the second year, 32 mentors and 24 student teachers participated in the study. Mentors and student teachers completed an initial survey about student teaching and possible topics of discussion between mentors and students. We held a working session with mentor teachers and student teachers to discuss mathematical conversations and to engage in mathematical conversations in small groups. Twenty-two teachers completed a questionnaire about mathematical conversations and their views about the mathematical preparation of their student teachers. Similar data were collected from their student teachers. Fourteen teachers participated in follow-up interviews. Our data also included evaluation forms from student teachers and mentors, notes from mentor workshops, and field notes from supervisors. This paper focuses on the 14 mentors who participated in an interview and their student teachers.

As we analyzed our data, a working definition of mathematical conversations evolved. A conversation that was likely only to occur in a mathematics classroom was classified as mathematical. We recognize that our attention to these conversations does not mean that these conversations are necessarily typical.

We worked with self-report data and interviews in order to represent the perspectives of mentors and student teachers during the student teaching field experience. Classroom observations would add a viewpoint on mathematical conversations that is not represented in this paper.

Mentors' Viewpoints

Views on Mentoring

The initial survey contained 53 topics, chosen from research studies, that mentors and student teachers discuss. Mentors were asked to rate each on how frequently they would discuss that topic and then select the five most important and five least important topics. The three most popular were (1) classroom management, (2) strengths/weaknesses of the day's lesson, and (3) alternative ways of presenting mathematical content. Other choices included student learning and understanding. The student teachers selected (1) strengths/weaknesses of the day's lesson and (2) student understanding. Third place was shared by "alternative ways of presenting mathematical content", "classroom management", "students' learning processes", and "student motivation." The agreement on classroom management, student learning, teaching strategies, and performance critiques illustrated common expectations for field work. This standard view of field experiences did not focus on mathematics even though it may have been part of the discussion. It is interesting that both teachers and mentors selected the "nature of mathematics (i.e. what is mathematics)" as a least important topic. Although it is a possibility that the participants misunderstood the phrase, we have not identified epistemological discussions between student teachers and mentors.

Views of Mathematics Learning and Mathematics Teaching

Most mentors viewed learning mathematics as an activity that required thinking and involvement of the participant. Half of the mentors chose a jigsaw puzzle explaining that learning mathematics is about putting the pieces together. Other characteristics included persistence, revelation, and the lack of memorization. Four mentors chose similes that emphasized beginning with a foundation and building complexity. One person noted that you must be active to do mathematics. This was echoed in the mentors' choice of what was not like mathematics. Half of the mentors said mathematics is not like watching a movie because learning mathematics is not passive.

The mentors agreed that mathematics teaching is not telling mathematics but is nurturing and supporting students to achieve goals, similar to the roles of a coach or facilitator. Two of the mentors thought that teaching was like gardening. To them, the teacher provides nourishment, and the students' mathematical abilities grow. Eight of the mentors indicated that teaching mathematics is not like news broadcasting because they are not reporting facts and must interact with their audience. Others felt that being an entertainer was a poor simile because although they hoped their students were enjoying their lessons, the goal of entertaining was secondary to helping them learn.

Views on Mathematical Preparation of Student Teachers

There are many topics vying for the attention of both the teacher and the student teacher. We know from our survey that mentors are interested in classroom management issues and also value a good knowledge of mathematics. Six of the mentors specifically stated that you need to know more mathematics than you are teaching. Two of those mentors explained that you can always learn more mathematics and implied that you should continue to learn mathematics throughout your career. One of the six mentors admitted that she did not often discuss mathematics with her student teacher because the student seemed proficient. In contrast, two mentors explained that student teachers really did not need to know much more than high school mathematics to teach what they taught. In general the mentors were satisfied with the mathematical preparation of the student teachers.

A good knowledge of mathematics is necessary for making connections, but it is probably not sufficient. Nine of the fourteen mentors discussed a need for connecting mathematics. Although they did not refer to the structure of mathematics, they noted that student teachers did not have a good knowledge of the "mathematics curriculum." For example, they wanted student teachers to understand what ideas were prerequisite to what they were teaching. They also wanted their mentees to understand that the concepts they were currently teaching should build toward future topics. Some mentors also mentioned connecting mathematics to applications.

Mathematical Conversations

The Motivation for Having Mathematical Conversations

Half of the mentors reported that the described conversation originated with the student teacher. In 13 of the 14 conversations, the mentor appeared to be providing advice or information for the student teacher. We found that teaching episodes were excellent stimuli for the discussions. In general, mentors were preparing students to teach a lesson in the future (i.e. later that day, the next day, the next year). In some cases, mentors corrected a student teacher or supplied missing knowledge. Often mentors explained how they had taught the lesson in the past.

The Nature of Mathematical Conversations

Mentors reported having mathematical conversations at least once a week with 12 out of 14 reporting conversations at least 3 times a week. The student teachers reported similar frequencies. Mentors were asked to describe a recent mathematical conversation. Six conversations were classified as pedagogical content knowledge (PCK), four as content knowledge (CK) and three were both. One generic conversation about pupils' tendencies toward mi. conceptions was classified as pedagogical knowledge (PK).

Six of the conversations referred to mathematical skills (i.e. determining angles, simplifying radicals, defining absolute value, determining degrees of polynomials, factoring special products, graphing quadratic equations), and three involved mathematical concepts (i.e. exponential growth, logarithms, x^0). Four mentors did not specify a topic but referred to modeling a lesson, misconceptions, critiquing a worksheet and discussing two-column proofs. Only one conversation appeared to be a mutual exploration of a mathematical topic (i.e. random numbers).

Four of the seven CK conversations addressed a deficient knowledge of the student teacher in order to prepare the student teacher to teach the topic at some point in the future. Two CK conversations discussed the value and placement of a particular concept (i.e. exponential growth, logarithms) in the curriculum. In the CK discussion about random numbers, the student teacher shared a calculator algorithm.

The PCK discussions addressed deficient knowledge on the part of the student teacher. Mentors were offering their advice to the student teachers in most cases. Five of the nine PCK conversations were about how to teach a particular topic based on the mentor's experience. Three of the PCK conversations implied ways to teach based on the mentor's knowledge about student learning. The remaining conversation focused on preparing a comprehensive worksheet.

Conclusions

This study suggests that mentors did discuss mathematics with their student teachers, but the conversations were focused on the teaching of mathematics and often on preparing for the next lesson. The typical conversations were centered on mentors providing advice on teaching the content. Advice was based on the mentors' experiences and motivated by the teaching of their mentees. A notable exception to our generalization was a discussion, dominated by the student teacher, on using a calculator to generate random numbers. Technology represents an area where student teachers felt competent and could draw from their experiences. Mathematical conversations may fail to meet their potential if they are used only as a tool for fixing urgent problems in a particular lesson.

We found evidence that mentors rely heavily on their experiences to guide their advice to preservice teachers. It is difficult to know to what extent a mentor's recall of experiences incorporates theories related to learning or teaching mathematics. Advice based on experience of what works is eagerly embraced by student teachers and may not be integrated with theory. This practice possibly perpetuates the status quo, but it also may lead to inappropriate application of the advice. In contrast, when teachers share the reasons that inform their practice, student teachers can move beyond imitation and begin to build their own theories needed to drive their own practice. Although field experiences provide an excellent context for such discussions, other responsibilities seemed to have a higher priority. Mentors' attitudes toward theory may be influential. If the perceived gap between theory and practice can be minimized, the experiences student teachers gain in the field could complement and enhance their experiences at the university, creating a better understanding of learning and teaching mathematics.

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USING CLASSROOM ARTIFACTS FOCUSED ON THE MATHEMATICAL THINKING OF CHILDREN AS A SOURCE OF CURRICULUM FOR MATHEMATICS METHODS

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Many programs for inservice teacher education have incorporated artifacts of teaching practice into their curriculum for teacher development. Narrative cases, samples of student work, and video tapes presenting images of children as they explore the mathematics of number and operation have all been used to examine teachers' decision making, to develop teachers' content knowledge and to help teachers articulate and question their beliefs about the learning process.

There is also an increasing interest in using artifacts of teaching practice as a mechanism for teacher preparation for the preservice population. Examining classroom scenarios, assessing student work, and analyzing student dialogue are all seen as vehicles to support the learning of prospective teachers; yet, how can preservice students with little classroom experience as teachers benefit from the study of such classroom artifacts? How do their ideas about what mathematics is affect the ways they interpret student work or student mathematical thinking? What are the potentials for using these kinds of materials with pre-service students? What are the barriers to the use of these types of materials and what are ways of working with or around those barriers?

The organizers of this discussion group have drawn from these materials in their work with preservice teachers. While the presenters have used the print and video cases differently in their mathematics methods courses, they have all found them to be effective tools in supporting the learning of prospective teachers as well as useful in highlighting many of the issues associated with effective mathematics instruction. In this session, we will share what we are learning, offer examples of our own students' writing to examine what they have learned, and share our current questions about using artifacts of teaching practice with the preservice population.

The session will be highly interactive. The organizers of this discussion group will share their experiences by offering examples of their own students' work for examination.

Participants will have the opportunity to:

- read, view, and discuss cases in both small and whole group settings.
- read and reflect on preservice teachers' work and writings in both small and whole group settings.
- see a variety of ways that the materials have been used in mathematics methods courses.
- raise questions and concerns about using such materials with preservice teachers in an open and respectful discussion, especially in light of the fact that methods courses are already jammed with too many topics to cover in a limited amount of time.

The goal of this discussion group will be to enhance each participants' understanding of the potential uses of classroom artifacts in preparing pre-service teachers for effective mathematics instruction. This goal is closely related to goal number 3 of the PME/NA, "to further a deeper understanding of the psychological aspects of teaching and learning mathematics."

FACILITATING DEVELOPMENT AS A RESEARCHER IN THE MATHEMATICS EDUCATION COMMUNITY

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The purpose of this discussion group is to provide opportunities for established members to advise future members of the mathematics education community regarding (a) seeking academic positions, (b) the transition from student to professor, particularly starting a research program and obtaining grants, and (c) balancing teaching and research.

For the first session, we propose a four-person panel discussion. Two junior faculty will briefly discuss how they determined if a job position would be well suited to their needs. For example, the junior faculty will discuss questions such as: (a) How does one compare teaching and departmental responsibilities across institutions? (b) Was the availability of research money a factor in their decisions? and (c) How were their decisions influenced by the availability of faculty to work with? Karen King of the Department of Mathematics at Michigan State University and Eric Knuth of the Department of Curriculum and Instruction at the University of Wisconsin have agreed to act as panel members for this part of the discussion.

Two senior faculty will discuss how departments determine whether a faculty candidate is well suited to their institutions. In particular, they will address the following questions: (a) How important is it that the applicant have publications? (b) Is K-12 teaching experience essential or are other K-12 experiences acceptable? and (c) What role did the applicants' research program play in the department's selection of potential candidates? Martin Simon of the Department of Curriculum and Instruction at Penn State University and Marilyn Carlson of the Department of Mathematics at Arizona State University have agreed to act as panel members for this part of the discussion.

Both senior and junior faculty have also been asked to discuss: (a) balancing teaching, research, and departmental responsibilities; and (b) obtaining funding, e.g., grants, for their research programs.

In the second session, in the first 30 minutes, two doctoral students will facilitate a discussion of the issues raised during the panel discussion. One student will begin by summarizing the previous session. The second student will propose discussion questions that resulted from this summary. In the remaining time, doctoral students will have an opportunity to identify and connect with other students with similar research interests. At the conclusion of the session, attendees will be surveyed to determine which issues should be addressed at subsequent meetings.

We believe that the formation of this discussion group will support the general goals of PME-NA. In particular, this discussion group will promote contact among individuals and the exchange of information in the psychology of mathematics education.

MULTIPLICATION WITH RATIONAL NUMBERS: TEACHING PROSPECTIVE K-8 TEACHERS

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The majority of current K-8 teacher education candidates have not experienced *Standards*-based math instruction, emphasizing number and operation sense, in their K-12 schooling. They must, therefore, construct meaning for multiplication with positive rational numbers during their teacher education coursework. Meaning about multiplication can be constructed through multiple perspectives (the Conceptual Field theory). In work and research by the author with prospective K-8 teachers over 6 years, four perspectives on -- or conceptual dimensions within -- multiplication have required special instructional attention for the majority of teacher education candidates:

1. *Understanding and Modeling the Types of Relationships Modeled by Multiplication*: modeling quantitative relationships such as of 1 or 1 quantities of , which includes understanding what these relationships mean and having a method for quantitatively determining them;

2. *Connecting Non-Whole Rational Number and Whole Number Multiplication Meanings*: connecting the types of relationships described above with *multiplication* -- i.e., with whole number meanings for multiplication); interpreting multiplication expressions in terms of the relationships involved -- using two interpretations (by commuting the two factors); translating multiplication expressions into real world situations through translating the understood relationships into situations within real world contexts;

3. *Interpreting Referents*: understanding the referents involved in the factors and products of multiplication situations, particularly understanding the numerical and referent meaning of the product in non-whole positive rational number situations -- countering such misconceptions as x as representing pizza multiplied by pizza or interpreting the product of x as , rather than as ;

4. *Understanding Multiplication as an Invariant Operation*: recognizing real world situations that could be modeled by whole number and non-whole number positive rational number multiplication as multiplication situations and being able to describe multiplication as one invariant operation with one invariant meaning across whole and non-whole positive rational numbers.

IMPLEMENTATION OF A REFORM MATHEMATICS CURRICULUM: A CASE STUDY

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An urban school district in the Midwest selected a reform mathematics curriculum for city-wide adoption, following a year of piloting across grades Kindergarten through Five. A grant from a State-funded agency was used to conduct staff development sessions, work with local administrators and community members, and monitor progress of the implementation. The purpose of this study was to determine the effects on attitudes and practices of teachers and achievement levels of students as a result of the new program. Questionnaires, written cases of student work, interviews, test results, and site visits were used to generate data.

The program has directly or indirectly affected the teaching and curriculum of 104 teachers. Of these teachers, 21 were involved in the piloting process, and 48 of them attended staff development sessions in the grant-funded program. Evolutionary modifications to the staff development process occurred as administrators and teachers met jointly at in-service sessions to discuss implementation of the reform mathematics program. Together, they decided to establish several internal staff development teams that could support other teachers. Each team consisted of three teachers; a total of ten teachers were designated to serve in this capacity. The most consistent district-wide problem was that of assessing student performance.

The case studies written by project teachers show students thinking about mathematics in nontraditional ways. Teachers were consistently surprised at the level at which their students functioned when pressed to communicate and reason in a mathematics class, and the staff development has influenced their thinking on the process of teaching and learning mathematics itself. State Proficiency Test scores at the building with complete implementation have increased over the past two years, and an experimental open-ended question administered in two buildings showed the reform classroom to perform significantly higher on the item. Involvement of middle grades teachers and the success of this project have led the district to consider adoption of a reform middle school mathematics curriculum as well.

LEARNING TOGETHER: TEACHERS' USE OF COMMON PLANNING TIME

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This study was designed to investigate the ways a group of middle school mathematics teachers made use of time scheduled each week for common planning meetings. The teachers who engaged in these meetings were participants in the fourth year of a national, multiyear mathematics reform project, QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning). In particular, the study examined the topics and issues addressed by teachers in these meetings and, the amount of time spent on each, and the ways topics and issues were addressed. In this report we focus on the way time was used to address issues related to mathematics content. We identified three categories of interest. "Scope and Sequence" refers to the time teachers spent discussing mathematics topics taught and the order in which they are taught. "Talking about Tasks" refers to the time teachers spent discussing specific tasks or activities that were or would be used in instruction. "Working through Tasks" refers to the time teachers spent actually working through a task as a student would.

In the 15 common planning time sessions on which we have data, about 13% of the lines of transcript were coded "Scope and Sequence", with equal time spent talking about the past and looking to the future. Approximately 80% of the lines of transcript were coded as "Talking about Tasks." The majority of this time was spent describing and reflecting on mathematical tasks teachers had already used in class, often focusing on problems that students had with the tasks. Only three instances of "Working through Tasks" were found. This constituted a little over 5% of the lines of transcript. In both instances, a mathematics educator who frequently met with the teachers initiated and led the work, involving the teachers in topics with which students struggle.

LEARNING TO LEARN TOGETHER

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While theoretically and practically we can argue for the development of school learning communities we have a vague notion of what constitutes a school learning community, little data to support their development and an idea that may sound deceptively simple. Our goal is to bring together the voices of those involved in the development of a particular learning community and begin to articulate the theories and practice that drive the development of the community and outline the strengths and challenges of the pursuit. The conversation will be supported by data collected over the three-year period on the learning of all participants.

Approximately 40 teachers from all four tracks, administrators and support staff, including the principal, assistant principal, and special education teachers, participated in monthly workgroup meetings focused on the development of children's mathematical thinking. The research became a significant part of the learning context. The research methods were integral to the professional development work and often defined the interactions that occurred throughout the school community. The research agenda also placed the researchers in the role of learners. Each member of the community began to learn together as we struggled to make sense of student's mathematical thinking and how to foster that thinking. The key became experimentation with the workgroup problem; experimentation allowed all of us to learn about students' understanding and consider ways to build on that understanding through next tasks, questions asked, or discussions facilitated. This experimentation connected ideas and practice, made reflection explicit and fostered communication across members of the community, all in the service of supporting the students' learning.

RESULTS FROM AN INVESTIGATION OF THE SCHOLARLY ACTIVITY OF RECENT GRADUATES OF DOCTORAL PROGRAMS IN MATHEMATICS EDUCATION

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Purpose and Background

The scholarly activity of graduates of doctoral programs in mathematics education from 1993 to 1995 was investigated using the results of a survey of about 200 individuals and follow-up interviews with a subset of 12 of the identified graduates.

A steady increase in the number of programs and graduates from these programs (NRC, 1999) is evidence of a growing interest in mathematics education doctoral programs. This was reflected at the National Conference on Doctoral Programs in Mathematics Education held in 1999 and sponsored by NSF. Discussions at the conference focused on how doctoral programs should be structured to help define the field of mathematics education as the "changing of the guard" takes place at the doctoral level (Reys, Glasgow, Ragan, & Simms, under review). The examination of scholarly activity by graduates offers a lens through which doctoral programs may be evaluated.

Analysis and Results

The survey was designed to examine the educational background, work experiences following degree completion, and current activity of graduates. Comparisons were made across differing types of current employment positions and across differing sizes of doctoral programs. Follow-up interviews were conducted to gain insight on the graduates' views of scholarly activity and the impact on their activity by current employers and by their doctoral program preparation. The resulting description of scholarly activity may be used to evaluate the need for change in the preparation of future graduates of doctoral programs in mathematics education.

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TENSIONS INVOLVED IN PROVIDING PROFESSIONAL DEVELOPMENT

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We examined our own decision-making in the planning and implementation of professional development that focused on understanding children's mathematical thinking. Data included transcripts of planning meetings and workshop sessions, field notes from classroom visits, surveys completed by the teachers, and debriefing interviews with the teachers at the end of the year-long project. Below we identify the significant tensions that we had to negotiate:

- **Workshop Facilitators:** Given the limited workshop time, we struggled with the balance between addressing substantive issues and establishing a supportive learning community. Final interviews suggested that teachers held various perspectives with some requesting more presentations of information and others preferring more informal sharing time.
- **In-Class Support Providers:** During classroom visits, we viewed ourselves as resources who could help teachers by observing children, posing reflective questions, and jointly brainstorming next steps. In contrast, the teachers sometimes viewed us as evaluators with our own agenda. We struggled with how to change this perception.
- **Researchers:** We often found the goals of researchers and professional developers to be incompatible. For example, as researchers, we wanted to collect baseline data in order to document teachers' growth. As professional developers, we wanted to use our initial interactions to begin to build a supportive, *non-evaluative* community that encouraged teachers to take risks. We struggled with these discrepant goals and learned later that a few teachers even declined to join the program because of our baseline data requests.

We hope to begin a dialogue about these tensions as we feel they are not unique to our project.

FROM TEACHER RESEARCH TO SCHOOL CHANGE: INVOLVING MORE TEACHERS IN TEACHING MATHEMATICS FOR UNDERSTANDING

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From 1998-2000 our research group, composed of one university researcher and three teacher researchers, has documented the use and effects of constructive assessment and teaching methods in mathematics in order to learn what happens when the whole school adopts new assessment methods and more teachers start to use methods designed to build student understanding of mathematics. We began by working with the principal to establish a set of assessment benchmarks to be used school-wide and to organize a program of workshops and peer coaching to introduce our colleagues to teaching methods based on the work of Constance Kamii (1985, 1989) and the Cognitively Guided Instruction Program (1994). We gathered and analyzed data in relation to two main questions: To what extent will each of the 26 teachers participate? What changes will we see in student performance in mathematics?

All of the nineteen K-3 teachers have been actively involved in grade level grouping for mathematics, in the staff development meetings, and in coaching. Three fourth grade teachers participated,but there was not as much evidence of change in their teaching of mathematics. The fifth grade teachers continued with their own methods but did participate in the assessment and showed some interest in the mental math. In relation to our benchmark assessments for conservation of number, missing addends, place value, timed addition and multiplication, and explaining multiplication, students have shown significant and steady growth through the three years. In relation to the state's standardized test the average percentiles for the second, third and fourth grades showed little change from the first to the second year and very large gains (ranging from 9 to 22 percentiles) in the third year, while the fifth grade scores improved each year.

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THE IMPACT OF SITE-BASED ELEMENTARY MATHEMATICS METHODS COURSEWORK

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The purpose of this study was to contribute to our understanding of environments and experiences that support elementary preservice teacher development in mathematics. We investigated the impact of teaching methods courses simultaneously with a practicum placement at an elementary school site. A total of 47 elementary education majors divided into two groups participated in the study. Twenty-two preservice teachers attended methods courses on a university campus and were assigned to a part-time practicum placement; 25 preservice teachers participated in both their practicum and their methods courses at the same school site.

Several sources of data including interviews, written reflections, and anecdotal records were collected throughout the study. The written mathematics reflections indicated that the university-based group was focused on the completion of their assignments and their grades in the mathematics methods course. Although the site-based group also indicated these concerns, they were more focused on the well-being of the children in their classroom placements, the effectiveness of their mathematics lessons on student learning, and assimilating to the school culture as a teacher. Due to their "real world" focus this group perceived their learning activities to be instrumental to reaching their future goals, and therefore they were more engaged in learning about how to be a better teacher of elementary mathematics.

DOCTORAL PROGRAMS IN MATHEMATICS EDUCATION IN THE UNITED STATES

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Purpose

This session will highlight a National Conference on Doctoral Programs in Mathematics Education sponsored by the National Science Foundation. The conference provided a forum for discussing key issues related to doctoral programs, including 'core' areas of preparation.

Results

A pre-conference survey collected information on the current nature of 48 doctoral programs in mathematics education in the United States, and these programs have produced about 80 percent of the doctorates awarded during the last 20 years. The survey provides a snapshot of information regarding faculty, students, and program characteristics. Programs range from those where candidates take virtually all graduate work in mathematics followed by research in mathematics education to programs where all coursework and research is in education. The single most startling finding is with respect to projected retirements that will soon be occurring within these institutions. For example, over 50% of the mathematics education faculty are eligible for retirement within 0-2 years, and nearly 20% are eligible within 3-5 years. Thus, a changing of the guard is eminent, and likely to change significantly the nature and structure of future doctoral programs in mathematics education. The study also revealed how difficult it is to identify people earning doctorates in mathematics education. For example, dissertations reported in JRME, ERIC and *Dissertation Abstracts* do not necessarily reflect research being done by mathematics educators.

IMPLEMENTING STATE MATHEMATICS STANDARDS IN HAWAI'I: A PROFESSIONAL DEVELOPMENT MODEL

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The state of Hawaii has expressed a clear commitment to changing the content of mathematics to be taught, but also has voiced the desire to influence teachers in their choices of methodology for teaching mathematics. Department of Education mathematics personnel joined with University of Hawaii mathematics and mathematics education faculty in an Eisenhower grant to develop and trial a professional development model to prepare teachers to deal with both the mathematics and the methodology changes written into the new standards document for mathematics.

Teachers were to learn (1) to judge the quality of their own children's work, (2) to write tasks for their own students, and (3) to engage in some of the mathematics content now introduced in the new standards but perhaps unfamiliar to teachers.

The project consisted of six full-day workshops for 177 participants from seven schools. The State Department of Education paid for substitute teachers. Data were collected from four distinct sources: (1) focus groups, (2) portfolios from each teacher, (3) surveys completed during the final day, and (4) observations and anecdotes collected and made by the university and Department of Education facilitators during workshop sessions.

Preliminary analysis indicates that teachers underwent a highly positive experience and feel better equipped to deal with implementing standards in their classrooms. Some samples of student work and teacher commentary on that student work will be made available for discussion during the session.

QUEST FOR A CONSTRUCTIVIST PRACTICE OF TEACHING ELEMENTARY MATH METHODS COURSE: EXPERIMENTS, QUESTIONS, AND DILEMMAS

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A typical elementary math methods course has a crowded agenda. There are a multitude of goals that need to be considered, prioritized, and realized. Given that mathematics education is in the process of reform, preservice teachers need to be equipped with the competencies and skills to help them understand the new vision of mathematics in elementary schools. The purpose of this presentation is to outline the experience of a new mathematics teacher educator in planning, and implementing an elementary math methods course.

In order to help preservice teachers develop an ownership of the vision of reformed math education, it is necessary that they construct their own knowledge and skills of mathematical pedagogy. In other words, as we model constructivist mathematics instruction in these courses, we need to practice a constructivist approach to teach the methods course itself. Here are some questions that need to considered for designing an elementary math methods course:

1. With so many students coming with unfavorable attitudes and dispositions towards math, how can we help them develop positive dispositions within the limited context of this course?
2. How can we reinforce the inadequate content knowledge of some students while teaching pedagogical knowledge at the same time?
3. What are effective ways of modifying unproductive beliefs of students about the nature of math and teaching math?
4. What are worthwhile types of pedagogical content knowledge that need to be covered in the course (that is, models, metaphors, analogies used to convey mathematical concepts)? How can this knowledge best be constructed by students?
5. How do we balance and relate teaching of content specific pedagogical knowledge (such as numbers, operations, geometry, fractions, etc.) with teaching for process skills (such as problem solving, reasoning, communication, connections, and representations)?

6. How do we best help integrate pedagogical content knowledge of smaller grain size to create lessons aligned with reformed vision rich in discourse and eliciting higher order thinking?
7. What are effective ways of modeling reform-oriented yet realistic instruction?
8. What are best ways of teaching how to teach children with diverse backgrounds and needs (e.g., children with limited English proficiency)?
9. What are effective ways of teaching how to make meaningful connections between mathematics and other subject matter areas?
10. What are effective ways of encouraging for continuous professional development after graduation?

The author of this presentation has engaged in an extensive process of designing and redesigning the course to meet the goals embedded in the above questions. The result is mixed success at best. The right direction seems to be a careful blending of content and pedagogical content knowledge supported with case analysis of teaching mathematics and guided field experiences. Goals, resources and outcomes of the course are presented in an integrated framework.

BUILDING A COLLABORATIVE MATHEMATICS EDUCATION COMMUNITY

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The gulfs between the worlds of educators, especially mathematics professors, mathematics education professors, and K-12 mathematics teachers, have been acknowledged for many years and sincere efforts have been made to bridge the gulfs yet most remain. Although the National Council of Teachers of Mathematics promotes communication, collaborative efforts are often limited to relatively small audiences and are often temporary in nature. Noddings (1992) noted collegiality as necessary to the mathematics education profession yet most reviews of the research literature, including hers, have been limited to discussion of mathematics teachers as individuals rather than as a community of life-long learners. Recent research such as that of Gutierrez (1996) does provide insights into school mathematics departments, but few studies include a wider audience of participants. Fullan (1999) points out that "contrary to myth, effective collaborative cultures are not based on like-minded consensus". They value diversity because that is how they get different perspectives and access to ideas to address complex problems. This presentation is based upon the assumption that collaboration is a celebration of commonalties and differences and that a concerted effort on the part of all participants will promote a healthy mathematics education community.

The research took place within a partnership based upon the premise that the creation of a sustainable learning community with a core of mathematics professors, mathematics education professors, and K-12 mathematics teachers will greatly benefit not only the educators but also their students and the community at-large. The research during the first year of the study addressed the successful and unsuccessful attempts at creating and sustaining a collaborative community. The participants in the research were two educators from each of the areas mentioned above and three graduate students. All participants agreed to replace hierarchical structures and positions of authority with collaborative efforts that supported both common and individual goals. Mathematical investigations and activities that centered on learning how to use communication and mathematics learning technologies for professional development and classroom instruction, became a central theme for building trusting relationships that supported collaboration. Our work promotes the exchange of scientific information by inviting interdisciplinary perspectives that deepen our understanding of what it means to teach and learn mathematics.

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PRE-SERVICE ELEMENTARY TEACHERS' CONCEPTIONS OF MATHEMATICS AND EXPERIENCES IN MATHEMATICS AND PEDAGOGY, AND THE INFLUENCES OF A MATHEMATICS METHOD COURSE

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This study investigated three preservice elementary teachers' conceptions of mathematics, their learning experiences in mathematics, and their conceptions of how to teach mathematics. More specifically, data address the influences of a mathematics methods course on two of the preservice teachers' views of mathematics and their ways of teaching children mathematics. The three pre-service teachers had their own conceptions of mathematics and teaching mathematics based on different experiences. Two preservice teachers who were enrolling the mathematics methods course could see mathematics and how to teach mathematics in a different way and change their conceptions of mathematics and ways of teaching mathematics as well as their attitudes toward mathematics. In contrast, the other preservice teacher did not have opportunities to think about mathematics and mathematics teaching in a different way from the way she learned mathematics. Consequently, her definition of mathematics was operational and she described teaching as procedure-oriented and traditional. Her way of teaching relied on what she experienced and saw when she was in mathematics classes.

Based on the data, this study extracted four aspects that influence preservice teachers' ways of teaching mathematics: 1) conception of mathematics, 2) disposition and attitudes toward mathematics, 3) experiences as a learner of mathematics and pedagogy, and 4) practical experiences such as field experience and student teaching. These aspects are intertwined and affect one another. Mathematics method courses are important in that they can positively influence each aspect, by providing preservice teachers with new experiences in mathematics and mathematics teaching. Based on what preservice teachers experienced previously, mathematics methods courses should provide opportunities necessary for the preservice teachers. Mathematics methods courses should not only teach how to teach mathematics but also challenge preservice teachers' traditional views of mathematics (e.g., a set of rules and algorithms). In addition, method courses should encourage preservice teachers to realize why they need an alternative way of teaching mathematics as opposed to traditional way that they were taught mathematics.

ATTITUDES ON MATHEMATICS TEACHING AND LEARNING BY PRESERVICE ELEMENTARY TEACHERS IN A PROJECT DRIVEN MATHEMATICS CLASS

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There have been many reforms in mathematics classrooms for preservice elementary teachers in recent years. Presumably, such teaching has had some impact on students' beliefs. This poster reports work from a preliminary study of students in an undergraduate mathematics course for elementary teachers. It describes instruction in the class and gives students' self-reported attitudes on what it means to do mathematics in school, what mathematics teaching is, and changes in their confidence in learning and teaching mathematics as a result of taking the class. Students had been asked to respond to several questions on the course in a reflective writing assignment. Questions were open-ended and students had a great deal of latitude in choosing on what to comment.

The class was the first of two courses in mathematics for prospective elementary teachers, but which must double as a methods course. The teacher used the Launch-Explore-Summarize (Lappan et al., 1998) guided discovery model for instruction. The class was project driven and was taught in a laboratory where activities were fully integrated within lecture/discussion. All new mathematics was introduced through group activities and with curriculum materials that could be used, as is, to teach mathematics in the K-8 classroom. Considerable attention was paid to verbal and written justifications for contributions to class discussions of mathematics and in problem/activity write-ups. Students were asked to wrestle with the tension between everyday and mathematical language on an ongoing basis.

Students reported increased confidence in their own mathematics learning, a change in beliefs from mathematics teaching as explaining and practicing to needing to struggle with problems, valuing groupwork and the use of manipulatives, and an appreciation for careful use of language.

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EFFECTS OF AN INNOVATIVE PROFESSIONAL DEVELOPMENT PROGRAM ON TEACHING GEOMETRY

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This poster presentation describes an innovative professional development model for elementary and middle school teachers and the evaluation of its effect on teacher change. In its fourth year, the ***PRO***gram for ***M***athematics ***A***nd ***S***cience ***E***ducation [**PROMASE**] has had 226 graduates with 97 completing a master's or specialist's degree in mathematics education. The PROMASE is a distance learning project focusing on improving teacher effectiveness in mathematics and science education and preparing teachers for the changes of the new millennium. Currently 100 teachers are enrolled in the mathematics education program. This poster presentation will concentrate on these teachers with a specific focus on teaching geometry. Furthermore it will present an analysis of the components of the PROMASE (using the principles and strategies of professional development described by Loucks-Horsley [Eisenhower National Clearinghouse, 1998] and describe the changes in teacher practices.

Responses were collected from class assignments to determine the teachers' geometric understanding as categorized by the van Hiele levels. For each level, further analysis of class work, informal interviews and individual classroom observations of teaching were done with a subset of teachers. The relationship between content understanding and pedagogical content knowledge will be presented.

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FOSTERING THE DEVELOPMENT OF ELEMENTARY SCHOOL TEACHERS AS LIFE-LONG LEARNERS IN MATHEMATICS TEACHING AND LEARNING

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Elementary school teachers must be empowered to adapt mathematics teaching and learning strategies to meet the needs of students in the New Millennium. This is a challenging endeavor given that most elementary school teachers are not mathematics specialists and do not actively pursue *individual* professional development activities in this field. A program that fosters the development of senior-year preservice elementary school teachers as *life-long* learners in this arena is currently in its third year of implementation, evaluation, and refinement.

The purpose of this poster presentation is to present the components (and related instructional activities) of a senior-level mathematics teaching methods course for pre-service elementary school teachers that contribute to the development of elementary school teachers as life-long learners in mathematics teaching and learning. These course-embedded components are (a) self-selected professional development activities; (b) active participation in local, regional, and national professional mathematics teaching organizations; and (c) domain-specific life-long learning strategy instruction. The conceptual framework that guided the development of the course components is grounded in theories of self-regulation and motivation and based upon research that suggests that self-regulatory skills can be enhanced through instruction (Smith, 1998). Instruction followed Zimmerman's (1998) cyclical model of self-regulation (forethought -> performance ->reflection -> forethought...) and activities were initially *teacher*- rather than *self*-directed.

During the course, students explored professional development resources, designed and engaged in their own professional development activities, and became actively involved in professional mathematics teacher organizations. Related student products and reflections will be shared as well as course-specific instructional strategies. Evidence that pre-service teachers who have completed this course continue to grow as teachers of mathematics by engaging in life-long learning activities during their first years of elementary school teaching will also be presented.

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RESEARCH-BASED CASES FOR MATHEMATICS TEACHER EDUCATION: THE COMET PROJECT

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In COMET (Cases of Mathematics Instruction to Enhance Teaching), which is funded by the National Science Foundation, practice-based materials for mathematics teacher professional development are being created. At the heart of the COMET materials is a set of narrative cases, each of which features an instructional episode drawn from classroom observations in the QUASAR project. Each case portrays the events that unfold in the classroom of a teacher in an urban middle school who is attempting to enact standards-based instruction and whose students are engaging with a cognitively challenging mathematics task. Told from the perspective of the teacher, the case makes salient the teacher's thoughts and actions as he/she interacts with students and with key aspects of mathematical content. Accompanying each case is a facilitation guide intended to support effective use of the materials by teacher educators. The guides include: a discussion of the key mathematical ideas in the lesson and the pedagogical moves which facilitated student learning of these ideas; a set of suggestions for launching a discussion and analysis of the case; and a list of suggested readings that provide the facilitator with additional insight into the mathematical and pedagogical issues raised in the case. COMET materials are organized into three content clusters: Algebra as the Study of Patterns and Functions, Geometry and Measurement in Two and Three Dimensions; and Rational Numbers and Proportionality.

COMET materials are grounded in research frameworks and data drawn from the QUASAR project and they provide a foundation for further research inquiry into teacher learning and professional development mediated by group interaction and discussion and the facilitation of a teacher educator. The materials allow teachers to explore the mathematical ideas associated with tasks that can be used with students, to consider student thinking about important mathematical ideas, and to analyze how teacher and student actions and interactions in the classroom support or inhibit learning. In this way, COMET provides sites for teachers and teacher educators to engage in critique, inquiry, and investigation into the practice of teaching.

Teacher Knowledge

WHAT CONSTITUTES A MATHEMATICALLY RICH AND MEANINGFUL TASK: PERSERVICE ELEMENTARY SCHOOL TEACHERS' PERCEPTIONS

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Abstract: This study examined the criteria that preservice teachers use to determine whether a given task is *rich* and mathematically meaningful. In addition, cases were used to explore the various perceptions that emerged and how they evolved over the course of a semester. Nineteen preservice elementary school teachers, all of whom were receiving either a major, minor, or emphasis in mathematics, participated in the study. Students submitted a variety of written artifacts in which they were asked to both generate and evaluate a variety of mathematical problems. Six archetypes of student responses, along with several general characteristics of student perceptions, were identified.

Significance of Problem and Theoretical Framework

Current reform movements in mathematics education emphasize the need for students to be engaged in solving rich, open-ended problems (NCTM, 2000). Since students learn from the tasks they are given in class (Doyle 1983; 1988), the nature of the task plays a critical role in determining the type of mathematical understanding that will occur in the mathematics classroom (Hiebert et al., 1997). "Rich" tasks need to extend beyond computational problems (Lampert, 1990) and should enable students to formulate an understanding of mathematical concepts that integrates both conceptual and procedural knowledge. Utilizing the work of Hiebert and Lefevre (1986, pp. 3-8), we view conceptual knowledge as knowledge rich in relationships, a connected web of knowledge. Procedural knowledge is characterized as symbolic representations (formal mathematical "language" of symbols), rules, algorithms, or procedures used to solve a mathematical task. In addition, drawing from ideas in constructivist theory (e.g., Glaserfeld, 1991; Noddings, 1990), we feel that in order for students to acquire this holistic mathematical knowledge, they must dynamically interact with and construct an understanding of a concept as they engage in different tasks.

The types of tasks that students engage in are for the most part determined by the teacher as she selects and/or creates these different problems. Consequently, if teachers do not have a view consistent with the current reform movement regarding the nature of a rich mathematical task, then it is unrealistic to expect them to carry out the intentions of mathematics education reform that emphasizes a deep conceptual and procedural understanding of mathematics. Few studies exist that specifically investigate how preservice teachers judge what constitutes a good mathematical problem. This study extends the current body of research by examining how preservice elementary school teachers judge mathematical tasks.

Research Methodology and Data Collection

The researchers used qualitative research methods, and in particular, case studies, to explore preservice elementary school teachers' perceptions of what constitutes a "rich" and mathematically meaningful task. Nineteen preservice elementary school teachers enrolled in an upper division mathematics class (investigating algebraic concepts and related pedagogical issues) at a Midwestern mid-sized university participated in the study. They were told that participation (or lack thereof) in the study would not affect their final course grade. All of the students in the class gave written permission to use their work in this study. The preservice teachers were of a traditional college-age and pursuing a major, minor, or emphasis in mathematics

Data for this study were collected in a variety of settings to enhance the validity of the research findings (Denzin, 1978). Artifacts of written work including copies of exams, homework assignments, and portions of a function portfolio submitted by students were gathered. In addition, since the first author was the instructor for the course, she was a complete participant observer and the second author was an occasional nonpassive participant observer (Spradley, 1980) of the students in the classroom. Furthermore, one class session in which students spent one hour and fifteen minutes discussing and evaluating two problems was videotaped and analyzed. Finally, the researchers discussed students' responses to questions on exams and assignments, and made field notes on these discussions. Triangulation (Denzin, 1978) was incorporated into the data collection process.

The researchers systematically and rigorously organized, synthesized, and categorized the data using the methods of Spradley (1980). This type of data analysis enabled us to make the transition from asking general questions about the study to asking and answering more specific questions that stemmed directly from the data. Categories of criteria used to evaluate problems emerged through our systematic analysis and reanalysis of the data. Cases were then utilized to present a descriptive, holistic view (Merriam, 1985) of the preservice teachers' perceptions and how they evolved/changed over the semester. Of the 19 subjects, six were chosen as cases for this study since they provided informative and representative snapshots of the analyzed data.

Data Analysis and Results

Through our data analysis, we identified a total of six different categories of criteria used to determine if a problem was rich. These criteria were used on multiple occasions and by at least five students. The categories included multiple representational modes, patterns, variables and change, real world context, "how" and "why" questions, and connections. Within each category, a number of subcategories were also identified. Typically, when the students evaluated a problem, they focused (nearly) all of their argument on the general characteristics of the problem and spent little time evaluating the mathematics inherent in the problem.

We now provide a brief synopsis of the six archetype cases that emerged as the preservice teachers' perceptions were analyzed over the course of a semester. First, many demonstrated minimal change; submitted problems and corresponding critiques of these problems continued to be either weak or strong throughout the semester. Jill represented one fifth of the preservice teachers who had a very limited understanding of the mathematical content in a problem and consequently could not provide "good" examples of rich problems nor give a substantial argument as to why a particular problem was rich and mathematically meaningful. Her criteria focused on general characteristics of a rich problem such as "making students think" and did not examine issues related to mathematical content. Tanya represented another fifth of the preservice teachers who experienced minimal change throughout the course of the semester. Tanya, a very strong mathematics student, consistently submitted very rich and conceptually-based problems that linked together a variety of ideas. She also explained with clarity why these problems were rich and meaningful. Through the course of the semester her problems maintained a high level of quality but her explanations became more specific and fine-tuned, focusing on both mathematical aspects as well as general characteristics of the problems submitted. One third of the students demonstrated inconsistency in change. In general, they showed improvement in particular problems and/or critiques as they received written and verbal feedback from the instructor of the course. However, no consistent general patterns of change were observed.

About one fourth of the preservice teachers showed evidence of observable change. One type of change was in the language that was used to critique and analyze problems. Sam exemplifies this type of change. He was a procedurally strong student used to doing well in mathematics classes, and he consistently submitted very procedural problems. However, as the class read and discussed various articles, he began to incorporate the language and vocabulary of the articles into his problems and justifications as to why they were rich and mathematically meaningful. However, no structural change occurred in his problems. Two students demonstrated change in the mathematical and general content of submitted problems. However, the strength of the problem submitted was dependent on the source of the problem since the students were allowed to adapt problems discussed in class, found in texts, or supplementary materials. For example, Jessica M. submitted problems that were entirely procedural and problems that integrated conceptual and procedural knowledge. However, her analyses of why these problems were "rich" and mathematically meaningful were very similar. She could not differentiate between the quality of problems she was submitting. The final change observed in two students dealt with the type of critique and analysis given of a particular problem. Jessica S. and Marie began the semester by submitting problems that were quite conceptual; however, they had difficulty articulating why these problems were mathematically rich and meaningful and tended to focus on more general characteristics of the problem and did not focus on mathematical content. By the end of the semester, their problems were even stronger and both could more clearly articulate why problems were rich. However, most of Marie's analysis continued to center on general characteristics of a problem though she became more specific in her mathematical analysis. Jessica S. focused on both general characteristics and mathematical characteristics, such as patterns.

Conclusions and Implications

We draw several conclusions from this study. First, students had difficulty evaluating problems from a mathematical viewpoint and tended to focus their analysis on general characteristics of the problem. In particular, few students mentioned or made meaningful connections between different function representations. Second, students with strong problems and critiques tended to cite criteria such as "extend and make predictions based upon patterns" and "make connections between different representations." Finally, some students demonstrated a mismatch of words and actions (e.g., a strong critique associated with a weak problem).

We believe that this study also has several important implications for the teaching and learning of mathematics. First, mathematical understanding is necessary but not sufficient for identifying mathematically meaningful and rich problems. Similarly, knowledge of mathematical content alone does not imply an ability to create or pose rich mathematical problems for students. Second, as teacher educators, we need to be sensitive to both the mathematical tasks created by preservice teachers and their analysis of these tasks. Furthermore, teacher education programs must devote time to explicitly explore the nature, the creation, and the analysis of rich and mathematically meaningful task. This preparation is important because teachers play a critical role in providing the necessary guidance and connections between these tasks (Lampert, 1991; NCTM, 1991). Finally, research has also shown that preservice teachers' beliefs about learning and teaching mathematics are formed during their own personal schooling and mathematical experiences (Ball, 1988). They bring these beliefs as well as their existing knowledge of mathematics with them as they enter teacher education programs. Thus, preservice teachers need to have experiences in which they are encouraged and expected to evaluate problems to determine if they are truly mathematically meaningful.

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TEACHERS' EVOLVING MODELS OF THE UNDERLYING CONCEPTS OF RATIONAL NUMBER

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Abstract: This multi-tiered teaching experiment describes three teachers' evolving models of the underlying concepts of rational number. The teachers participated in a 14 week workshop study where they grappled with the skills and concepts embedded within model eliciting problems. They created curriculum maps to illustrate their understanding of the interrelatedness of the skills and concepts found in the problems. Further, they implemented these problems in their middle school classrooms and documented student thinking to better understand the problems from their students' perspectives. The results illustrate the different ways the teachers revised and refined their ways of thinking about their own content knowledge, pedagogical content knowledge and knowledge of their students' mathematical thinking.

Objectives

This study focused on middle school teachers' cyclic models or conceptual schemes that evolve as they grapple with the concepts underlying particular model eliciting problems (see Lesh & Doerr, 1998) and how this impacts their content knowledge, pedagogical content knowledge and knowledge of student's mathematical thinking. Specifically we will focus on three teachers' mapping of curriculum, identification of concepts and skills, and their attention to student thinking in their classrooms.

Research Design and theoretical Framework

The study reported here is part (one tier) of a larger research project that utilized a research design referred to as a multi-tiered teaching experiment (Kelly & Lesh, 1999). This research design was chosen as it produces auditable trails of documentation that focus on multiple levels of development in this case at the student and teacher level. Within the multi-tiered teaching experiment, some of the most important key events focus on sequences of model eliciting problems in which participants are repeatedly challenged to reveal, test, and refine, or revise important aspects of their ways of thinking. Model eliciting problems are designed to produce constructions, explanations, or descriptions that are conceptual tools and are in themselves the most important goals of the problem solving episode (Lesh & Doerr, 1998). That is, to a large extent, the process is the product. Consequently, the product explicitly reveals significant information about the reasoning processes that produced it. In this way these problems, the central feature of the research design, promote learning; yet, at the same time, a byproduct of learning is that auditable trails of documentation emerge that reveal important aspects about the nature of the construct being developed. Thus, in the study reported in this proposal, we specifically focused on the teachers' evolving conceptions regarding the underlying concepts of particular model eliciting problems reported in the mapping of curriculum, as well as how they revise and refine their ways of thinking about their own content knowledge, pedagogical content knowledge and knowledge of their students' mathematical thinking.

This study examined three middle school mathematics teachers with varying levels of experience that were teaching in suburban schools. The three teachers involved in the study were enrolled in a masters degree program in Atlanta, Georgia with one of the researchers serving as the instructor. The teachers volunteered to take part in this study with the understanding that they would consider meaningful forms of mathematics instruction, specifically model eliciting problems, to supplement their mathematics curriculum. They agreed to identify underlying concepts and skills of each model eliciting problem to begin mapping their curriculum for one year. Moreover, they agreed to focus their attention on their students' mathematical thinking to better understand the nature of mathematics and their students' perceptions of the skills and concepts in particular problems.

The foundation and underlying premise of this investigation was that teachers need appropriate experiences and materials from which to build new models of instruction, learning and assessment (Schorr & Alston, 1999). They also must be afforded with opportunities to construct deeper understanding of the mathematical concepts they are expected to teach and an increased awareness of the ways in which children learn (Carpenter & Lehrer, 1999; Schorr, Maher, & Davis, 1997; Janvier, 1996; Cobb, Wood, Yackel, & McNeal, 1993.)

Teachers met weekly with researchers in workshop environments where they were presented with one model eliciting problem every other week for 14 weeks. They collaboratively grappled with solving the problems as well as identifying and visually mapping key concepts, skills and important mathematical ideas that were embedded within the rich problem. They used national, state and school standards to further document the types of concepts represented in each problem. They went on to categorize smaller problems sets and sets of symbolically represented problem hat were aligned with the model eliciting problems that could serve as follow up problems or homework sets. Again these smaller problems were combined within the mapping as well. After sharing their own ideas and representations, they agreed to use these problems in their own classrooms.

During classroom implementation with researchers present, teachers were encouraged to recognize and analyze student interpretations as they were continually revised and refined. This in turn aided their understanding of the skills and concepts embedded within the model eliciting problem. Independently they would reflect, revise and refine their own thinking about the mapping of the concepts and skills found within the model eliciting problems. They would bring their thoughts and ideas back to share with their colleagues in subsequent workshop sessions. Studying each other's mappings as a group afforded the opportunity to both consider the development of their ideas, to discuss students' thinking in regards to particular model eliciting problems as well as discuss the pedagogical implications of using model eliciting problems in their classrooms.

In this study, the researchers' goal was to simultaneously stimulate and document changes in teacher knowledge and knowledge of student thinking. This was accomplished through the use of model eliciting problems in which the teachers were repeatedly challenged to reveal, refine, revise, and extend important aspects of their ways of thinking. In turn this provided the researchers with an opportunity to stimulate changes in the teachers' classroom practices that were twofold (Carpenter & Lehrer, 1999). The first is by challenging them to construct a deeper understanding of their own curriculum. Secondly by helping them become more familiar with student-generated ways of thinking.

Data Collection and Analysis

The data for the study include: (a) the teachers' curriculum mapping about what concepts and skills they identify as important, (b) transcripts from two semi-structured interview sessions and informal questions in regards to what they perceive to be the main ideas or key concepts in model eliciting problems and students' mathematical thinking, (c) teachers' work from model eliciting problems that were collected during working group sessions, (d) transcripts from teacher workshop sessions, (5) teachers' reflections on their own and other teachers work, (e) student work from model eliciting problems that were analyzed during teacher workshop sessions, and (f) researcher field notes taken while working with teachers in classrooms and workshop sessions.

Each workshop session was video-taped and audio-taped. The audio-tapes were fully transcribed and the videotapes were used to record nonverbal communication important to the analysis, such as a teacher using his or her hands to explain his or her thinking to the rest of the group.

Codes emerged progressively throughout data collection. It appeared that the codes fit together to form a coordinated analysis. First, it appeared that the teachers were able to glean a deeper understanding of the underlying concepts and interrelationships of rational number by (a) solving the student model eliciting problems collaboratively, (b) constructing their own concept maps to use as their own curriculum guides, and (c) by trying to better understand their students thinking. Second, the teachers went through modeling cycles when identifying the underlying concepts of a particular model eliciting problem. Their modeling cycles went from more naïve conceptual understandings to more complex over time.

Results

Results indicated quite conclusively that for teachers the model eliciting task of constructing concept maps that illustrated underlying concepts and skills of particular model eliciting problems for students were a valuable form of information about the growth and acquisition of deeper understandings of the middle school curriculum they currently teach. Teachers constructed multiple modeling cycles in regards to their perceptions of the interrelationships of concepts and skills. These modeling cycles appeared to increase in stability and sophistication throughout the workshop sessions and throughout the 14 week investigation. The teachers' solutions illustrated that through the course of the investigation their understanding of the interrelatedness of middle school mathematics concepts became more sophisticated as these ideas were continually identified, tested and refined.

The way in which the teachers conceived of the interrelationships of rational number was determined by the ways in which their workshop group came to understand the different model eliciting problems, their ability to discriminate between interrelated skills, and their ability to document student thinking and student strategies. After the workshop session where small groups of three teachers solved the student model eliciting problems and documented underlying skills on their concept maps they went ahead and implemented these problems in their own classroom with the support of a researcher. It was at this time that the teachers were able to listen to student thinking and better grasp a concept from student perspectives as well as identify student misconceptions. To this end, the teachers were empowered to aid students in grasping a concept that they indicated would have been overlooked beforehand. Moreover, they gained a more sophisticated, deeper understanding of the underlying skills and concepts of rational number documented on their concept map as well as different strategies used by their students.

Project Implications of Teaching and Research

When helping teachers glean deeper understanding of mathematical concepts it appears that this inquiry process, using model eliciting problems, is effective for several reasons. First, teachers develop an understanding of the interrelatedness of mathematics curriculum when solving a model eliciting problem like constructing a concept map where the teachers focus is to not only attend to student thinking but to construct a tool that can be used subsequently for multiple purposes. Moreover, having teachers attend to students' mathematical thinking as well as the underlying concepts and skills of the student model eliciting problems may be a critical means to helping them build a deeper understanding of rational number and algebraic and functional reasoning to inform their teaching.

As teachers begin to use model eliciting problems or the ideas of models and modeling, they typically begin to analyze how their students think mathematically.

Thus, it is important that they have instilled a deeper understanding of mathematics themselves, but further they need to know how to use this information to inform their own teaching.

Model eliciting problems can be designed to be thought revealing activities and because of this they provide powerful tools to help teachers examine, test, refine and revise their own ways of thinking. They allow and recognize the need for teachers to develop their own ways of adapting and using successful teaching strategies as well as provide a means to deepen their own content knowledge, pedagogical content knowledge, and knowledge of student thinking.

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PROSPECTIVE ELEMENTARY TEACHERS' DOMINANT SITUATIONS AND KNOWLEDGE ABOUT REPRESENTATIONS OF RATIONAL NUMBERS

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Abstract: In this paper we examine 92 prospective elementary teachers' dominant situations and knowledge about representations of rational numbers. We found teachers' knowledge of story-problem representations for 3/5 very limited since only 142 of the 275 generated responses were categorized as well-posed story-problems with solution 3/5. The data also revealed that the dominant situation for 3/5 was the part-whole subconstruct followed by the measure subconstruct. Although most students could represent 3/5 pictorially with units of different sizes using a continuous model, most students were not able to conceptualize that a given shaded region could represent different fractions from the ones most naturally suggested by the given diagram.

Students' understanding of rational numbers is critical to the development of their mathematical competence. Mathematical competence includes both procedural and conceptual knowledge (Hiebert & Lefevre, 1986). However, as reported by national assessments (e.g., Kouba, Brown, Carpenter, Lindquist, Silver & Swafford, 1988) and more in-depth research studies (e.g., Markovits & Sowder, 1991; Vinner, Hershkowitz & Bruckheimer, 1981; Wearne & Hiebert, 1983) students' understanding of rational numbers seems to be quite limited. To improve students' performance and understanding, teachers need to put more emphasis on helping them develop their conceptual knowledge of mathematical ideas in general and rational numbers in particular. To teach rational numbers for understanding, teachers need to have a strong conceptual knowledge of the underlying mathematical principles. Therefore, it is imperative to examine prospective teachers' conceptual knowledge of rational numbers. Since conceptual knowledge is knowledge that is rich in relationships (Hiebert & Lefevre, 1986), it is necessary to examine teachers' understanding of connections between multiple representations. The purpose of this study is to investigate prospective elementary teachers' dominant situations and explicit knowledge about representations of rational numbers. To this end, the following research questions were formulated:

1) Do prospective elementary teachers know story problems whose solutions are 3/5?
2) What are prospective elementary teachers' dominant situations for 3/5?
3) What types of pictorial representations do prospective elementary teachers draw to represent 3/5?
4) What is prospective elementary teachers' understanding of the arbitrary nature of a unit?

Theoretical and Empirical Background

The concept of rational number is a complex and multifaceted construct. The theoretical background of this study is based on cognitive and conceptual analyses of rational numbers performed by other researchers (Behr, Lesh, Post, & Silver, 1983; Kieren, 1976; Marshall, 1993; Ohlsson, 1988) as well as on the concept of unitizing. Among the *subconstructs* (Behr et al., 1983), *situations* (Marshall, 1993), or *interpretations* of rational number identified by these researchers are: *part-whole*, *fractional measure*, *quotient*, *rate*, *ratio*, and *operator*. The *part-whole situation* involves the physical or mental partitioning of some continuous whole or a set of discrete objects into equal-size parts. The *fractional measure situation* addresses the question of "how much there is of a quantity relative to a specified unit of that quantity" (Behr et al., 1983, p. 99). This interpretation of a rational number focuses on the notion of a rational number as a number rather than on part-whole relationships. The *quotient situation* represents an indicated quotient or division. The *ratio situation* represents a relationship between two quantities of the same dimension or with the same units. The *rate situation* defines a new quantity as a relationship between two quantities with different units. The *operator situation* of rational number treats a rational number as a transformation: it operates on a value to produce another value. Its main function is that of a multiplier or divider. These and other researchers suggest that students need to have a complete understanding of the rational number subconstructs as well as an understanding of the connections among the subconstructs for students to have a complete understanding of rational number. The learning of whole numbers, fractions, proportions, and other mathematical constructs involves the construction and coordination of abstract units (Reynolds & Wheatley, 1996). Lamon (1996) defines unitizing as the "cognitive assignment of a unit of measurement to a given [or a certain] quantity" (p. 170). She cites research that supports the idea that "the nature of the unit largely accounts for the cognitive complexity entailed in linking meaning, symbols, and operations" (p. 170) for rational numbers. Several researchers (e.g., Simon, 1993; Post, Cramer, Behr, Lesh, and Harel, 1993) have studied some aspects of prospective elementary teachers' knowledge about the representations of the construct of rational number. Their findings suggest that prospective elementary teachers' knowledge of both pictorial and story-problem representations of operations with fractions tends to be very limited. This study extends their research by examining prospective elementary teachers' understanding of the arbitrary nature of a mathematical unit and of basic representations of fractions.

Methodology and Data Sources

A total of 92 prospective elementary teachers enrolled in three sections of a mathematical content course for elementary majors were the participants of the study. The students were asked to complete the four tasks described in the appendix. The tasks were selected to assess prospective elementary teachers' explicit knowledge or ability to create or describe representations for 3/5 and asses their understanding of the arbitrary nature of a unit. Within each of the three groups, the tasks were given one at a time. The students were given each subsequent task after all the students had completed the previous task. The purpose of the first task was to stimulate students to write story problems involving a variety of interpretations (e.g., part-whole, measure, quotient, and ratio). The purpose of the second task was to assess students' knowledge of pictorial representations as well as to investigate the extent to which prospective teachers use units of different sizes. The purpose of tasks three and four was to examine prospective elementary teachers' understanding of the arbitrary nature of the unit.

Results

A content analysis of each student's response for each task was performed using the aspects of the conceptual framework or the purposes of the tasks.

Prospective Elementary Teachers' Knowledge of Story-Problem Representations for 3/5

The participants generated a total of 275 responses of which 234 were problems and 41 were not problems (e.g., verbal statements, symbolic statements, pictures, etc.). Only 142 (51.6%) of the responses were categorized as *well-posed story problems with solution 3/5*. The remaining responses were not story problems, or were not well-posed story problems, or were well-posed story problems whose solution is not 3/5.

Prospective Elementary Teachers' Dominant Situations about Rational Numbers

Students generated 142 well-posed story problems whose solution is 3/5. Out of these 142 story problems, 90 (63.4%) involved part-whole relationships, 44 (31%) involved the measure situation of rational number, four (2.8%) interpreted 3/5 as a ratio, and three (2.1%) used a quotient interpretation for 3/5.

Prospective Elementary Teachers' Knowledge of Pictorial Representations for 3/5

Thirty-five (38%) students generated three different correct pictorial representations for 3/5 involving units of different size. Thirty-nine (42.5%) students generated three correct pictorial representations but only two representations involved units o different sizes. Seven (7.5%) students created three correct pictorial representations for 3/5 but they all involved a unit of the same size. Ten (11%) students created at least one incorrect or unclear pictorial representation for 3/5.

Prospective Elementary Teachers' Understanding of the Arbitrary Nature of a Unit

Students were asked what fraction or fractions could be represented by the shaded portion of a given diagram (See Appendix, Task 3). Twenty (21.5%) students believed that the given diagram could only represent 3/4, 12 (13%) students stated that the diagram could only represent 3/5 and 40 (43.5%) students thought that the given diagram could represent 3/4 or 3/5. A total of 14 (15%) students thought that the given diagram could represent fractions equivalent to 3/4 or to 3/5, or to both 3/4 and 3/5. To some degree, these findings are not surprising because the diagram suggests these two representations. However, it is worthwhile to mention that only three students indicated and provided a correct justification that the given diagram could represent fractions different from 3/4 or 3/5. To gain further insight into students' conceptualization of a unit, students were asked to complete Task 4. Seventy-five (81.5%) students indicated that the given diagram could represent 3/5. It is interesting to note that most of these students said or implied that the given diagram could represent 3/5 if the circle were completed or they assumed that the figure was a circle. That is, it seems that students could not conceptualize that the diagram could represent 3/5 by itself, without the need to have five parts. For example, the diagram could represent 3/5 of a pound of beef. It is interesting to notice that almost all the students (88 or 95.5%) indicated and justified correctly that the given diagram could represent 3/4. Fourteen (15%) students said that the diagram could represent 3/10 but none of them provided a clear and correct justification for that fact. Interestingly, only 17 (18.5%) students indicated and justified correctly that the diagram could represent 1 1/2.

Discussion and Conclusion

Rational numbers are an important component of the school curriculum. The research reported in this paper provides insight into teachers' dominant situations and explicit knowledge of representations of rational numbers. It was discouraging to find that only 142 out of the 275 teachers' responses could be categorized as well-posed story problems with solution 3/5. Many of the responses were categorized as statements, as no story problems, as no well-posed story problems, and as well-posed story problems but with solution different from 3/5. These findings seem to indicate that some teachers still have an underdeveloped understanding of the meaning of 3/5 and when it is appropriate to talk about 3/5. Not surprisingly, the dominant situations for 3/5 were the subconstructs of part-whole and measure. This is so because preservice teachers' experiences are probably dominated by these two subconstructs of rational number and the fact that the explicit notation of 3/5 suggested 3/5 (as opposed to 3:5 or 3÷5). A second important finding was that most students (74 or 80.5%) could represent 3/5 pictorially with at least two units of different size using a continuous model. However, the third and probably the most interesting and significant finding was that most teachers have not developed a complete understanding of the concept of fraction and its relationship to the arbitrary and abstract nature of the unit. To reiterate, none of the prospective teachers stated and provided a correct justification that the given diagram could represent 3/10 and only 17 students indicated and correctly justified that the given diagram could represent 1 1/2. The findings of this study indicate that prospective elementary majors need instructional interventions to develop a better conceptualization of a fraction and its relationship to the arbitrary and abstract nature of the unit. Examination of some mathematics textbooks for elementary teachers revealed that representations of fractions using composite units are lacking (Contreras, in preparation). It seems that textbook authors assume, at least implicitly, that prospective elementary teachers understand simple and composite representations of fractions and the arbitrary and abstract nature of the unit. In this paper we have examined prospective elementary teachers' knowledge of representations of rational numbers. Studies are also needed to examine both elementary and secondary teachers' mathematical knowledge and its relationship to instruction, especially cases of teachers who have strong mathematical knowledge. Contreras (1997) and Contreras and Martínez (1996) have done some work in this area but further studies are needed to gain a more profound understanding of teachers' knowledge.

Appendix[1]

1. Create three different story problems whose solution can be represented by 3/5 . Use a different meaning or interpretation of for each problem[2].
2. Think of the following teaching situation:

   In a mathematics course students are asked to draw a diagram to represent 3/5. Students are provided with graph paper.

   Draw three different correct representations that students could draw. Your representations should be as different as possible. [*Three 5 by 15 rectangular grids are provided*]
3. What fraction or fractions could the shaded portion of the figure represent? Explain your responses and draw diagrams if appropriate.

4. a) Could the following diagram be used to represent 3/5? Justify your answer [*The same diagram as in question 3 was shown*]. Provide diagrams if appropriate. [*Similar questions were asked for 3/4, 3/10, and* 1 1/2]

Notes

1. For space limitations, the instrument is presented in a simplified and compressed form.
2. A group of students was asked to provide 4 story problems for 3/5.

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A RECONCEPTUALIZATION OF TEACHER THINKING IN TERMS OF MEDIATING RESOURCES[1]

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Abstract: This paper defines and illustrates *resources* as a kind of cultural/ psychological tool that mediates teacher thinking. The focus is on resources that accord primacy to learning over teaching as the activity around which to organize instruction. Four major feature are posited: resources can exist anywhere in the material and social enactment of a task; resources are dynamic rather than fixed or static; resources do not exist independently of teacher action; and resources are a means to integrate the activities of teaching and learning. In closely coupled activity, teachers use resources to appropriate several aspects of students' strategies to conceptual frameworks integral to elementary math. When student productions become resources for further instruction, the result is growth in understanding.

Although learning can and does take place without teaching, teachers do not exist without students. Lave (1996) proposed that learners should "constitute the working conditions for teaching rather than the other way around" (p. 159). Research on teaching often treats teaching and learning as conceptually distinct activities. Lave's claim is a call to reconsider the interrelationships between the two, and to accord primacy to learning over teaching as the activity around which to organize instruction.

Schools are not currently organized to support learner-centered instruction in math, although there is a call for new teaching practices centered on students' understanding (National Council of Teachers of Mathematics [NCTM], 2000). Teachers are faced with the challenge of fostering and understanding student-generated strategies, and integrating those strategies with instructional goals. This challenge has created a need for schools and teachers to draw on and create new sets of knowledge resources to elicit, make sense of, and act on children's mathematical activity. In this paper, I consider the features of resources that support teachers' use of children's activity to advance mathematical thinking.

I use *resources* to refer to a kind of cultural/psychological tool (Wertsch, 1998) used to accomplish valued goals. Resources are the means by which teachers engage in tasks designed to move classroom activity towards those goals. As tools, their meanings and uses are continually in flux (Wenger, 1998). Potential resources include: the knowledge frameworks teachers use to interpret what children say and do; children's strategies; the materials, curriculum programs, and instructional goals set by policy; and the physical set up of the classroom.

A focus on the role of resources in the facilitation of instructional interactions shifts the burden of explanation for student learning (or lack of it) from a major emphasis on teachers' knowledge to "[teachers]-operating-with-mediational-means" (Wertsch, 1998, p. 26). The significance of this shift is to situate the teacher and his or her thinking (i.e., his or her "operating") in the contexts of the classroom and schooling, and to begin to understand how what teachers do is the result of the confluence and coordination of a variety of resources -- some gen rated through practical inquiry (Franke, et al., in press), some provided through policy decisions, some appropriated from professional development, and some contingent on the activity and purposes of specific locales.

There are several implications of this shift in perspective on teacher thinking. They are organized around one main claim, founded on the premise that children learning should be a major constituent of the context of teaching: To organize classrooms in which children learn math with understanding, teachers need resources that facilitate children's participation in legitimate mathematical activity and teachers' responses to that activity. The central idea is that teaching consists of a series of interactions in which resources are continually deployed, invoked or collectively produced to make sense of and direct activity. An important subclaim is that the more *closely coupled* teachers' use of resources is with students' mathematical activity, the more likely students are to learn and understand math (Empson & Junk, in preparation).

Example: Closely Coupled Activity

Closely coupled activity refers to teachers' use of resources to incorporate several aspects of students' thinking activity into instruction. In this example from a third-grade classroom, Shatysh has solved a problem that asked how much cake each person gets, if 12 children are sharing 20 little cakes. When the teacher[2] approaches, she does not understand what Shatysh has done. At the conclusion of the interaction, Shatysh's strategy has been refined as one where the sharing situation was transformed from 12 sharing 20 to 3 sharing 5. It will be used as a resource in later group discussion, as an example of a ratio-based strategy. To reach this point, the teacher appropriates several aspects of what Shatysh has done, including her initial answer of 5, her grouping of the children and cakes, and finally her distribution of groups of cakes to groups of children (which is not clear initially).

T: (looking at Shatysh's paper) OK, you got an answer of 5? 5 what? (pause) Tell me what you did here (gesturing to Fig. 1b)

S: First I did this one (pointing to Fig. 1a) I grouped the kids....

T: .... OK, and are these (circles in Fig. 1b) cakes or people?

S: Yeah, I put -- cakes -- I put the numbers in them. Like that's 1 group, and 2 groups (indicates a group of 5, and another group below it)

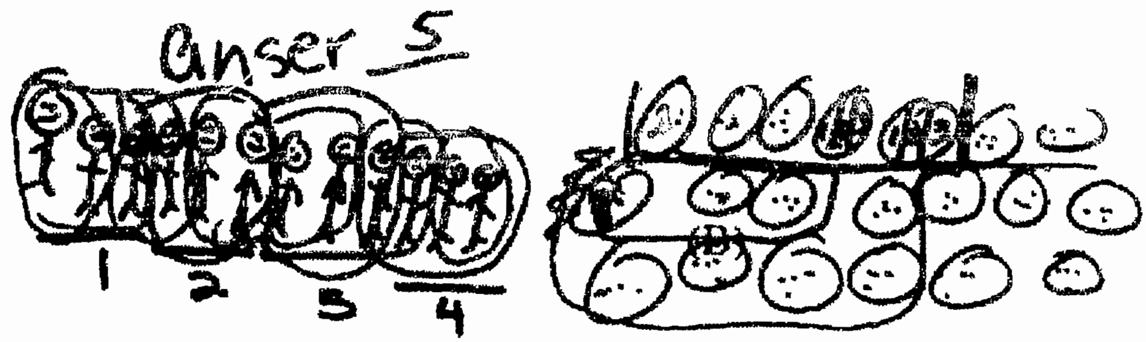

Figure 1. Shatysh's written work for 12 people sharing 20 cakes. (1a on left; 1b on right)

T: OK. How many cakes did this person (first person in Fig. 1a) get?

S: 5.

T: Where are their 5 cakes?

S: (counts out first 5 cakes in Fig. 1b)

T: (counts out loud as Shatysh points to each cake) 1, 2, 3, 4, 5. (low voice) All right. Where are *this* person's cakes? (second person in Fig. 1a; S counts out next 5 cakes).... But I don't see where all of these kids have cakes. (pause) Or are you having *two* kids share 5 cakes? (referring to circled groups of children in Fig. 1a). I don't--

S: 3.

T: 3 kids share 5 cakes?

S: Yeah. See, (points to children in group of three) 1, 2, 3.

T: Those 3 right there? (pointing to first 3 children in Fig. 1a)

S: Yeah.

T: So. (pause) So you have 3 kids sharing 5 cakes?

S: (nods).... I could split 'em up (i.e., share the 5 cakes among the 3 children).

T: (brief interaction where T underlines each group of 3 children, and numbers groups 1 to 4).... I see how you have the 3 circled now. I understand what you're doing. You went 5 (underlining group of 5 cakes)-- and you tried to make it so this would be split up. So this group (of 3 children) got 5 cakes...

S: (shows how she would share the 5 cakes among 3 children by giving out whole cakes to each person, then splitting last 2 cakes in 3 pieces.)

T: This child right here (points to first in group), how much cake does she get?

S: A whole.

T: A whole, plus what else? .... (S is not sure) *how much cake* is that little piece right there? .... How many of these little pieces does it take to make the whole cake?

S: 3.

T: OK, so what do we call it?

S: 1 third? (S writes "1/3 + 1/3 + 1" with some assistance.)

Shatysh's ratio strategy is a resource that was collectively produced; note however that Shatysh's activity motivated the defining logic of the strategy. Resources the teacher used in this production include her knowledge of ratios in equal-sharing situations, the questions that helped Shatysh name the fractional quantity, the task itself to motivate a range of student strategies.

Features of the Framework

There are four main implications that follow from the perspective that teachers' thinking is best characterized in terms of the use of mediating resources. First, rather than existing solely as representations internal to a teacher's thinking, *resources can exist anywhere in the material and social enactment of a task* (Newman, Griffin, & Cole, 1989). They may be internalized to teachers' ways of operating, or they may exist or emerge externally. There is no theoretical reason to separate these two categories of classroom activity, since both have a bearing on what students take with them from instruction.

Second, *resources are dynamic rather than fixed or static*. What they mean, how they are used, and the form they take are in constant negotiation (Wenger, 1998), which results in refinements and other kinds of changes. Thus, not only should teachers be provided certain resources -- the "products of the inquiry of others" (Lampert, 1998, p. 57) -- but more significantly their work should be organized to create, test, and improve resources. This implication is in line with other recommendations concerning teaching and inquiry (Cochran-Smith & Lytle, 1999). Further, an inherent ambiguity in what resources are and how they can be used means that the possibility of individual interpretation and improvisation always exists.

Third, *resources "do not really exist independently of [teachers'] action"* (Wertsch, 1998, p. 25). The full significance of resources emerges in interaction. Feiman-Nemser and Remillard (1995) pointed out that many perspectives on teacher knowledge "leave open the question of what it means to know and use such knowledge in teaching ... misrepresent[ing] the interactive character of teachers' knowledge and sidestep[ping] the issue of knowledge in use" (cited in Cochran-Smith & Lytle, 1999, p. 257). A focus on what teachers do and the means by which it is accomplished, rather than on what teachers say about what they do, ameliorates research problems posed by using measures that are proxies for knowledge-in-action.

Fourth, *resources are a means to integrate the activities of teaching and learning, for purposes of professional development and research.* The emphases on the use of children's activity to inform teachers' practice and on how teaching actions can elevate and enfold children's thinking has the potential to support an inquiry-oriented feedback loop in instruction. There has been recent interest in helping teachers engage in practical inquiry in ways that lead to generative understanding of student thinking and mathematical content (Franke, et al., in press), where *generative* refers to the ability to continue to expand knowledge and deepen understanding. Possible explanations for how some teachers become generative learners may be found through examining teachers' use of resources in practice, especially how teachers appropriate aspects of students' activity to ongoing instruction. In closely coupled activity, teachers use resources to appropriate several aspects of students' strategies to conceptual frameworks integral to elementary math (e.g., knowledge of equivalence or development of children's equal-sharing strategies). When student productions become resources for further instruction, the result is in an iterative process of forward movement.

Conclusions and Future Questions

Understanding teacher thinking in terms of mediating resources can shed new light on how we address the problem of educational change, by allowing us to ask: What uses of resources *depend* on teacher learning? What uses of resources *result* in teacher learning? Further, what kinds of inquiry cycles or feedback loops couple teachers' practice with student outcomes? How do policy decisions regarding instruction position children as certain kinds of learners? Only a small fraction of research has been devoted to analyzing domain-specific aspects of teachers' thinking (beyond content knowledge) in a theoretically informed way (e.g., Carpenter, Fennema & Franke, 1996). Just as Glaser (1984) argued for domain-specific analyses of problem solving, one can argue for similar analyses of teachers' thinking as it is situated in the work of teaching fractions, geometry, whole-number operations, and so on. I believe that all of these questions are best answered by concentrating on a single, albeit complex, aspect of teacher activity -- the use of learners as context for teaching -- in a single content domain, such as fractions. This narrowing of the questions provides a descriptive scale that is in accord with the problem space(s) within which teachers operate on a day-to-day basis, but admits analysis of contextual influences from several realms.

Notes

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2. Debra L. Junk, a member of the research team. We co-taught a unit on fractions as part of pilot work.

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MODELS OF CURRICULUM USE IN THE CONTEXT OF MATHEMATICS EDUCATION REFORM

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Abstract: This paper examines how elementary-school teachers use a reform-based mathematics curriculum. Our analysis identified four processes that together describe the teachers' use of the new materials: (a) *reading* the curriculum, (b) *evaluating* the curriculum, (c) *adapting* the curriculum, and (d) *implementing* the curriculum. All of the teachers studied engaged with each of the four processes. However, the teachers engaged with these processes in different ways resulting in a number of what we call *models of curriculum use*. We believe that understanding how and why teachers implement mathematics reform in the ways that they do is a key to promoting more effective curriculum implementation on a large scale.

A central part of teachers' work is to interpret curriculum materials and to decide how to use these materials in the classroom. This process is particularly critical in light of recent reform efforts in mathematics education. While there is a proliferation of reform documents and recommendations, the fact remains that reform often reaches teachers through new curricula that teachers are expected to implement. In addition, teachers' experiences using reform-based curricula can influence their beliefs about reform and their willingness and ability to make changes in their practices.

Despite the importance of curriculum in the reform process, relatively little attention has been paid to trying to understand the ways in which teachers interpret and implement new curricula. Instead, researchers studying the impact of reform-based curricula often focus on improvements in student learning of mathematics. Much less attention has been given to the teacher's role in the implementation process. In contrast, we argue that understanding the process through which teachers use new curricula is critical for the success of mathematics education reform and for designing curricula that support both teachers' and students' needs in this time of change.

In this paper, we explore this issue by examining the process through which elementary-school teachers use a reform-based mathematics curriculum. We investigate how teachers use these new materials and the patterns of use that appear. In addition, we consider the relationship between teachers' curriculum use and their instructional practices. We believe that understanding how and why teachers implement mathematics reform in the ways that they do is a key to promoting more effective curriculum implementation on a large scale.

Curriculum Materials and Teachers' Instructional Practices

Our research is framed by three perspectives concerning the relationship between curriculum materials and teachers' instructional practices. First, policy-makers' decisions about which mathematics curricula to adopt at the state or local level affect teachers' instructional practices (Spillane & Zeuli, 1999). That is, teachers tailor their instruction, to some extent, to the curricular choices of the district within which they teach. At the same time, prior research on teachers' use of curriculum materials supports the notion that there is no such thing as a "teacher-proof" curriculum. While most classroom teachers are guided by a set of published curriculum materials, these materials are not used blindly (Ben-Peretz, 1990).

A second, and more recent, perspective explores the ways in which teachers respond to curriculum materials specifically in the context of mathematics education reform. In particular, researchers find that teachers adapt and revise materials in light of their students, their school, and their own teaching style and goals (Ball & Cohen, 1996). In addition, teachers may transform a novel lesson into a lesson with which they are more familiar, even if in doing so they bypass the intended purpose of the new lesson (Brown & Campione, 1996; Sherin, 1996). However, in other cases, the process of implementing a new lesson or unit provides valuable learning opportunities for the teacher, particularly as students offer novel explanations (Hufferd-Ackles, 1998). One goal of our research is to extend prior studies by examining more closely the kinds of adaptations that teachers make as they implement reform-based curricula.

Third, most of the above work on the relationship between teachers' practices and their use of new curricula has been in the form of case studies of teachers implementing reform (e.g. Cohen, 1990; Wilson, 1990). As a result, teachers' use of reform-based curricula can seem individual and idiosyncratic. In our work, we attempt to move beyond such results by explicitly looking for patterns among teachers' use of reform curricula. We want to understand more fully what influences how teachers use new curricula as well as the process through which curriculum materials are translated into instructional practices.

Research Design

Our research is part of a larger project that is investigating the experiences of teachers piloting the research-based reform mathematics curriculum Children's Math Worlds (Fuson et al., 2000). This paper focuses on six teachers who used the curriculum during the 1998-99 school year. The teachers teach grades one through three and have between one and fifteen years of teaching experience.

The data for this study consist of between 15 and 30 videotaped observations of each teacher's classroom. In addition, post-observation interviews were conducted in which the teachers were asked about their interpretation of the day's lesson. Half of the teachers also participated in "mathematics story interviews" in which they were asked to describe their previous experiences teaching and learning mathematics, as well as their future plans for mathematics teaching (Drake & Hufferd-Ackles, 1999). The data were analyzed using an iterative and grounded approach. The interviews were coded in order to identify patterns of curriculum use. These patterns were then confirmed, revised, or disconfirmed based on fine-grained analysis of classroom videotapes. Where appropriate, the identified patterns were revised a second time using the mathematics story interviews.

Results and Conclusions

As expected, none of the teachers simply used the curriculum as written. Instead, the teachers interpreted and transformed the materials as they prepared to teach and then carried out the lesson. In examining how this occurred, we found that the teachers' curriculum use could be characterized by four processes: (a) *reading* the curriculum, (b) *evaluating* the curriculum, (c) *adapting* the curriculum, and (d) *implementing* the curriculum. *Reading* the curriculum consists of reviewing any of the materials provided for the teacher or the students. In *evaluating* the curriculum, teachers consider their own understanding of the activities, as well as their students' ability to engage with the activities provided. *Adapting* the curriculum is defined as making significant changes in a lesson such as inserting or deleting an activity. Lastly, *implementing* the curriculum involves using the materials during instruction.

All of the teachers engaged with each of the four processes. Nevertheless, the teachers engaged with these processes in several different ways resulting in a number of what we call *models of curriculum use*. For example, Jill's model indicates that she first read the description of the lesson, then implemented the lesson, and afterwards evaluated and adapted the lesson for future implementation. In contrast, Laura adapted the lesson as she read the materials. Laura then implemented the revised lesson and finally evaluated the lesson following instruction.

With these models in mind, we then set out to examine the relationship between teachers' curriculum use and their instructional practices. Here we describe three key results. First, the order of the four processes in teachers' models seems to influence their ability to implement the curriculum effectively. In particular, we found that several teachers evaluated and adapted lessons prior to teaching, while others evaluated and adapted either during instruction or following the lesson. Those teachers who evaluated and adapted lessons in the midst of instruction seemed better able to engage students in discussions of mathematical ideas than the teachers who evaluated and adapted lessons prior to or following instruction. This finding is in line with other research that emphasizes the need for teachers to be able to make adaptations of lessons *during instruction* in order to implement reform successfully (i.e. Hammer, 1997).

Second, the teachers had different meanings for the different processes in their models of curriculum use. These varied meanings impacted teachers' practices in different ways. For instance, while all six of the teachers adapted the curriculum lessons and activities, each teacher tended to adapt in a particular way. Marta's adaptations generally involved either eliminating an activity or transforming a reform lesson into a traditional lesson covering the same content. For Sandra, adapting a lesson involved selecting among the different ways that the curriculum suggested to engage students with a particular concept:

> I'm thinking...[to] maybe do the blackboard structure at the beginning for a little while, then maybe do the routines and check-up...and then maybe go back to [the blackboard structure.]

Finally, for Carrie, adapting meant creating activities or materials to serve as transitions or linkages between curricular lessons:

> Then the math stories...as I was doing it, I was thinking I should just have them put numbers on the story...three over here, four over here, and seven over here. I'm trying to transition them into a number sentence.

Each teacher was clearly adapting, but the resulting instructional practices in each of their classrooms were quite different.

At the same time, the relationship between teachers' models of curriculum use and their classroom practices was not unidirectional. As teachers learned through their instruction about their students and about the curriculum, the order and meaning of the processes in their models often changed. In two cases, the change in the teachers' models was particularly pronounced. As these teachers developed greater knowledge about and comfort with their students and the curriculum, they began to shift from evaluating and adapting *after* instruction to evaluating and adapting *before and during* instruction. As suggested above, this change in the timing of the evaluation and adaptation processes had significant effects on the instructional practices of these teachers, particularly on their ability to hear and respond to students. It is interesting to note that both of these teachers were first-year teachers. As a result, it is possible that some of the shift in their models of curriculum use was due to their gaining general pedagogical experience, in addition to their greater knowledge and understanding of the curriculum and of their students.

Implications

Understanding how teachers make use of reform-based curricula is a critical issue for mathematics educators and researchers today. First, such research will add to our understanding of teacher cognition by illuminating the knowledge that teachers bring to bear to the task of interpreting curriculum materials. In addition, understanding how teachers make sense of new materials will have important implications for the design of reform-based curricula. For example, curriculum designers might want to make explicit the expectation that teachers will evaluate and adapt the materials presented to them. Furthermore, continuing to examine the relationship between models of curriculum use and teachers' instruction should help us to better understand how to foster successful implementation of mathematics education reform. In particular, effective models of curriculum use could be used in the future as the basis for professional development programs.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE OF MATHEMATICS AND THEIR ABILITY TO UNDERSTAND NONSTANDARD REASONING

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Abstract: This study examines eight PSTs' ability to make sense of nonstandard explanations of division of fractions problems after receiving six weeks of instruction on this topic. The PSTs have extreme difficulty understanding the nonstandard explanations. Three categories are proposed to describe PSTs' newly acquired knowledge. This study suggests that PSTs' knowledge of the mathematics they are going to teach is problematic, and not easily remedied.

Past studies of prospective secondary mathematics teachers [PSTs] have shown that PSTs have critical gaps in their understanding of secondary school mathematics topics (Ball, 1990b; Cooney, Wilson, Albright, & Chauvot, 1998; Even, 1993). In particular, Ball (1990a, 1990b) showed that PSTs have difficulty creating representations for division of fractions or articulating the connection between division of fractions and whole number division. A question currently facing teacher educators is the following: How difficult is it to repair the gaps in PSTs' knowledge of a mathematics topic such as division of fractions so that PSTs will have the mathematical understanding necessary to engage in instruction that focuses on student reasoning and sense-making? One might assume that since many PSTs are mathematics majors in college, helping them fix the gaps in their knowledge of division of fractions—a mathematics topic that is typically taught for the first time in elementary school, but is nonetheless reviewed repeatedly in junior high and high school—would be a relatively quick and easy task. However, the research reported in this paper suggests otherwise. Surprisingly, even after the eight PSTs in this study learned multiple ways of representing and solving division of fractions problems, most still lacked the mathematical understanding to make sense of nonstandard explanations of division of fractions. This study examines the mathematical understandings of the eight PSTs for clues as to why understanding nonstandard reasoning was so difficult for them, even after receiving substantial instruction to develop rich conceptions of division of fractions. In particular, this paper identifies characteristics of the PSTs' mathematical knowledge that seemed to influence their ability to make sense of nonstandard reasoning about division of fractions problems.

Method and Data Sources

This study was part of a larger qualitative study designed to investigate the changes that occurred in PSTs' knowledge of and beliefs about mathematics as the PSTs participated in a semester-long capstone mathematics course specifically designed to challenge and invoke change in their mathematical knowledge and beliefs (Siebert, 2000). Eight subjects were recruited to participate in the study, representing approximately a fourth of the class. All eight subjects were either juniors or seniors majoring in mathematics, and typically had two or fewer math courses remaining to take before graduation.

The first unit of the capstone course focused on division of fractions, and lasted approximately six weeks. During this unit, the instructor engaged the PSTs in developing meaning for division of fractions and in identifying and articulating the important mathematical concepts underlying this topic. PSTs were required to draw pictures to both solve division of fractions problems and to justify the invert and multiply (IM) rule, write story problems for division of fractions number sentences, analyze and discuss videotaped and written accounts of student thinking about division of fractions, and create conceptual analyses of division of fractions.

During the division of fractions unit, I interviewed the subjects for 45 minutes each week to determine what changes they were making in their knowledge of division of fractions and their beliefs about mathematics. I also videotaped the class sessions and collected copies of their written assignments. The qualitative analyses of the data was guided by my desire to create coherent, theoretical models of the PSTs' knowledge of and beliefs about mathematics. Like Simon and Tzur (1999), I assumed that the subjects in my study were acting in ways that were sensible to them. Thus, in my accounts of their knowledge and beliefs, I tried to capture the sensibility and coherence the PSTs perceived in what they did and said.

Results

The unit on division of fractions was very successful in helping the PSTs develop new understandings of division of fractions. When the PSTs began the unit, none of them could provide an explanation for why the IM rule worked or draw a picture of $x \div 1/y$, where x and y are nonzero whole numbers. The PSTs also had difficulty writing story problems for division of fractions number sentences. By the end of the unit, however, all eight PSTs could use pictures to solve division of fractions problems, illustrate the IM rule using pictures, create story problems for division of fractions number sentences, and identify and explain several of the important concepts underlying division of fractions.

To illustrate the type of understanding the PSTs had developed for the topic of division of fractions, consider, for example, Josie's explanation for why the IM rule works for the division of fractions problem $5 \div 2/3$. Josie recognized that one way to think about this problem was to ask the question, "How many 2/3s are in 5?" To answer this question, Josie first found how many 2/3s were in 1 by drawing and reasoning about the diagram in figure 1:

Josie discovered that there are 1 1/2, or 3/2, two-thirds in 1. She then made the following argument for why 5 ÷ 2/3 = 5 × 3/2:

> There's three-halves of two-thirds in one.... If you have five wholes, um, you multiply by the three-halves, because that's how many you have in one, so it's five times that amount.... I'm thinking of it visually, like, OK, if I had three-halves of something, one and a half of something, and I have five of those [three-halves of] something, then how many will I have? It's a multiplication problem.

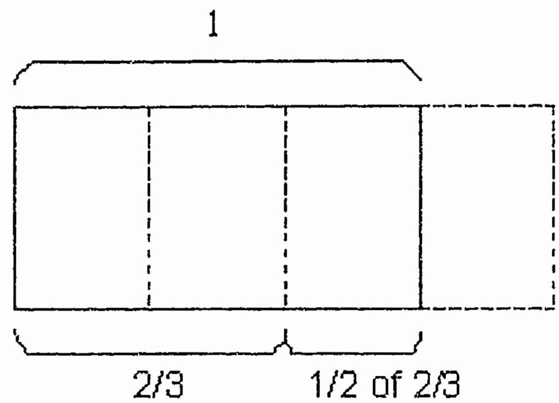

Figure 1. Diagram of how many 2/3s are in 1.

Josie's argument for why the IM rule works consisted of first noting that 3/2, the reciprocal of 2/3, is the number of 2/3s in 1. Since there are 5 ones in the dividend, and there are 3/2 two-thirds in each 1, it follows that there are 5 × 3/2 two-thirds in 5. Thus, 5 ÷ 2/3 = 5 × 3/2. This line of reasoning about the IM rule was common in the class, and all eight PSTs demonstrated this reasoning during the interviews.

Despite the progress the PSTs made during the unit, when I asked them in the last week of the unit to grade hypothetical students' explanations for the division of fractions problem 1/4 ÷ 5/8, only one of the PSTs recognized that the following two explanations were correct:

> Carlos: There are eight 1/8s in 1. There are only one-fifth as many 5/8s in 1 as there are 1/8s in 1. So there are eight-fifths 5/8s in 1. There are only one-fourth as many 5/8s in 1/4 as there are in 1. So the answer is 1/4 × 8/5 = 2/5.

> Kristin: There are two one-eighths in 1/4. I need five one-eighths to make 5/8. So 1/4 has two of the five one-eighths that I need to make 5/8. So 1/4 ÷ 5/8 = 2/5.

This result was surprising to the course instructor and myself; both of us had felt this task was well within the grasp of the PSTs, particularly since the ideas in these explanations had arisen in class during the six weeks of instruction. The PSTs' poor performance on this task revealed important characteristics of their newly acquired knowledge of division of fractions, namely that this knowledge was often rigid, fragile, and missing pieces of conceptual understanding.

Rigidity in PSTs' Solution Methods for Division of Fractions

Six of the eight PSTs had difficulty interpreting the first three sentences in Carlos's explanation. This difficulty seemed to be caused at least in part by Carlos's use of a different multiplicative comparison than what the PSTs were accustomed to using to solve division of fractions problems. To arrive at a fact like "there are eight-fifths 5/8s in 1," the PSTs had learned to draw a picture to determine how many copies of 5/8s fit in 1 whole, as Josie did above. This involved making a comparison between the relative sizes of 5/8 and 1. Carlos's reasoning, in contrast, compared the relative sizes of 1/8 to 5/8, and used this relationship to derive the number of 5/8s in 1. Three of the six PSTs who struggled to make sense of the first three sentences of Carlos's explanation explicitly noted that Carlos was making a multiplicative comparison between the number of eighths and the number of five-eighths in one, but they could see no way for Carlos to know that there were 1/5 as many 5/8s in 1 as there were 1/8s in 1 *without first actually finding the number of 5/8s in 1*:

> Roberto: But there are only one-fifth as many five-eighths, to me it's like, OK, how did he know that? How do you know that without, without doing a proportion? So he said, well, how many five-eighths, how many five-eighths are in one? Well, eight-fifths. Then you could see that the proportion of eight-fifths to eight equals one-fifth. So then you say, yeah, there's one-fifth as many.

The PSTs' skill at solving this problem in a particular way seemed to get in their way of seeing a different approach to the problem.

Fragility of Their Knowledge of Division of Fractions

As the PSTs engaged in making sense of nonstandard reasoning, their knowledge of division of fractions seemed to be challenged in a unique way. Occasionally, their attempts to make sense of nonstandard reasoning seemed to cause them to temporarily lose sight of correct reasoning about division of fractions. For example, while trying to make sense of Kristin's reasoning, two of the PSTs spontaneously began to investigate how many 2/8s were in 5/8, which would yield the answer to 5/8 ÷ 1/4, not 1/4 ÷ 5/8. Consider, for example, Julie's reasoning about Kristin's explanation:

Julie:	I don't understand where Kristin is getting fifths [in the answer]. There's no explanation. I was trying to reason through it, and think there are five parts here [in the 5/8], but if you're grouping it in [groups of] two [one-eighths], then there would be one of two left over, instead of one of five.
Interviewer:	Why are you grouping it in two?
Julie:	Because it takes two eighths to get into a fourth. So how many

eighths do you need out of 5/8s to get in, that will go into a fourth?

One might argue that perhaps Julie didn't know how to reason about a problem where the dividend is smaller than the divisor, but Julie had demonstrated earlier in the interview that she could interpret remainders in division of fractions problems, thus indicating she understood that when the divisor is greater than the dividend, the meaning for division switches from "how *many* of the divisor fit in the dividend" to "how *much* of the divisor fits in the dividend." A more plausible explanation is that she got caught up in making sense of Kristin's comparison of 2/8 and 5/8 and lost track of how the division problem should be interpreted. The task of coordinating the acts of (a) making sense of someone else's reasoning and (b) maintaining a correct understanding of the problem seemed to put too great of a strain on her understanding of division of fractions.

Missing Pieces in Conceptual Understanding

Six of the PSTs had difficulty making sense of the last two sentences in Carlos's explanation. This difficulty may have been caused by their tendency to view multiplication as repeated addition. The PSTs developed their explanations for the IM rule by examining division of fractions problems involving whole number dividends. Like Josie, they tended to justify the step of multiplication in the IM rule by reasoning that for each 1 in the dividend, they had a certain number of the divisor, and instead of repeatedly adding this number to itself, they could multiply that number by the dividend. For $5 \div 2/3$, this meant that for each 1 in 5, there were 3/2 two-thirds. To get the answer, one could multiply 5 times 3/2 instead of adding 3/2 five times. When the PSTs were asked to justify the IM rule for non-whole number dividends, they had difficulty justifying the step of multiplication; they often tried (unsuccessfully) to use the repeated addition meaning for multiplication in their explanations, despite that this interpretation for multiplication breaks down when neither factor is a whole number. It is likely that Carlos's explanation for why he multiplied did not match with their understanding of what multiplication means, and thus did not make sense to them.

Conclusion

By studying how PSTs' respond to nonstandard reasoning, this research offers insight into important characteristics of PSTs' newly acquired mathematical knowledge, namely that this knowledge can be rigid, fragile, and missing pieces of conceptual understanding. The difficulty PSTs experienced in making sense of nonstandard explanations, despite receiving significant instruction and developing new understandings, suggests that helping PSTs fix the gaps in their understanding of the mathematics they will teach can be a formidable task.

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TURNING POINTS: STORIES OF THREE VETERAN TEACHERS REFORMING THEIR MATHEMATICS INSTRUCTION

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Much research has focused on the fact that mathematics reform requires significant teacher learning (e.g., Ball, 1997). Additional work has suggested that teachers at certain stages of their careers may be more or less interested in this kind of learning and reform (Huberman, 1988). In this paper, I find that it may be the combination of key events at particular stages in teachers' careers that leads certain teachers to reform their mathematics instruction. More specifically, veteran teachers who have recently experienced "turning points" (McAdams, 1993) in their understandings about mathematics or mathematics teaching are especially likely to become invested in mathematics reform projects.

In the three cases presented here, these mathematical turning points followed teachers' active and positive participation in literacy reform efforts. As a result of these experiences, the three veteran elementary teachers began to view mathematics as a language and mathematical knowledge as a form of literacy. This new understanding of mathematics influenced the teachers' practices in a number of ways including focusing them on mathematical vocabulary and explanations.

It is important to understand the nature and implications for practice of the mathematics and literacy connections that experienced elementary teachers are making. Transferring principles of literacy instruction to mathematics reform may be the impetus that gives experienced elementary teachers both the confidence and the motivation to reform their mathematics instruction. At the same time, these teachers may miss or misunderstand those aspects of mathematics reform that are subject-specific, moving instead towards more generally reform-oriented teaching practices.

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FROM ONE COMMUNITY OF LEARNERS TO ANOTHER: THE INFLUENCE OF TEACHER PROFESSIONAL DEVELOPMENT ON PRACTICE

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This paper focuses on the teacher learning stimulated by involvement in professional development (PD) sessions. In earlier work, I studied changes teachers made as they developed what I call a *math-talk learning community* (Hufferd-Ackles, 1999). I identified key dimensions along which teachers made changes, and the trajectory of change in each. I adapted these math-talk trajectories into PD materials to be used during after-school meetings with teachers implementing a reform-based curriculum (Table 1). In the sessions, we looked at five written trajectories and corresponding video excerpts and discussed them. Eight teachers (grades 1-3) participated in the PD sessions at a large urban elementary school. Transcripts of classroom video, video from bi-weekly PD sessions, and audio-taped teacher interviews were analyzed using an iterative approach.

Table 1: Example trajectory; Overview of the Questioning Trajectory, the Teacher's Changing Role

Level 0	Level 1	Level 2	Level 3
Teacher asks questions of specific students (mostly short-answer form). Questions often function to keep students listening.	Teacher asks students questions about their thinking, focuses less on answers. Teacher rarely elicits additional strategies.	Teacher asks many probing questions to prompt fuller explanations and elicit multiple strategies. Student- student talk is facilitated.	Teacher expects students to ask one another questions about their work. Questions function to guide the discourse.

The analysis has identified two ways that PD sessions have influenced teacher learning. First, teachers did in fact learn about facilitating discourse in the PD sessions. Participants learned: a) about alternative roles for the teacher that allow discourse to develop in the classroom, and b) about student capacity to contribute to classroom discourse. Second, participating in the PD sessions seemed to foster changes in teachers' classroom practice. Participants: a) began to allow students to contribute to discussions by encouraging student questioning and fuller explanations of work, b) changed their physical location during math discussions to remove themselves from the center, c) did less direct-teaching of new topics and allowed more student exploration.

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AN EXPLORATORY STUDY OF PRESERVICE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE OF FOUR ARITHMETIC OPERATIONS

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Mathematical knowledge is a component of teacher knowledge and is defined as knowledge of and about mathematics. Problem solving is a component of the nature of mathematics and teachers are expected to model and emphasize problem solving, including problem formulation. Teachers' knowledge of problem-solving is important for it has an impact on student mathematical development. We investigate prospective teachers' knowledge of the relationship between four arithmetic operations and real-world problems. We asked eight prospective secondary mathematics teachers to produce two real-world problems to be solved with a numerical statement (a total of 9). 136 mathematical problems were created: 75 were appropriate (a real-world problem that can be solved using the operation proposed); 44 were partial (problem is incomplete, or can be solved with an equivalent mathematical operation, or is only partially a real-world situation); and 17 were inappropriate (problem is not a real-world situation or cannot be solved with the statement). Examples for 1/4*1/3 are: "Oranges are on sale: 1 lb for 331/3. How much does 1/4 lbs cost?" (appropriate), "For a school project Kim needs to find out how many feet are in 1/3 of 1/4 mile. What is her first step?" (partial); "If one container is 1/4 full of mud and the other container is 1/3 full of mud, what is the product of the two containers of mud?" (inappropriate). It is discouraging to see about one appropriate problem per student for each statement. We observe the need of instructional interventions in teacher education programs and weaknesses in participants' knowledge of the relationships between the operations and real-world applications. Students verbalized statements or verbalized a context with no meaning. Division and multiplication of fractions were the most difficult statements.

THE IMPACT OF AN EXPERIEMENTAL COURSE: MATH FOR BIOLOGY

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Members of the biology faculty at UNI believe the mathematical skills of their majors have declined. They report that many students were not succeeding in entry-level science courses due to their mathematics skills. These students need a review of elementary and college algebra. Placing this review in the context of biological science applications appealed to both biology and mathematics faculties. The course, Mathematics for Biology, was designed and implemented using these premises. We studied the impact of the course on the students enrolled.

Thirty-four students enrolled in two sections of the course offered in the spring semester of 2000. Pre- and post- attitude surveys and content tests were administered to all of the students attending the course. Twenty-nine students took both surveys and tests. Using the remaining data, we found that the students did not have a significant change in attitude toward mathematics, but that their content knowledge dramatically improved. We were disappointed in the lack of improvement in attitude toward mathematics, especially since a number of students reported positive changes to Dr. Veenstra. Using qualitative data collected for six case studies, we found several students reported a different view of mathematics at the end of the course. This appeared to indicate they did think more positively about mathematics, as it related to science, but not about mathematics in general.

The impact this course had on the students may become evident as they continue in their biology majors. It may be that it is not possible to improve both the content knowledge and mathematical attitudes of biology majors with a one-semester mathematics course.

TRANSITION TOWARD ALGEBRA[1] [T^2A]: ENHANCING TEACHERS' KNOWLEDGE OF ALGEBRA

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The purpose of this poster session is to present the project, Transition Toward Algebra, as a model for professional development for mathematics teachers of grades five to nine. T^2A took place during three years, and sixty-three teachers participated. T^2A was developed to help teachers experience and think through effective teaching strategies that promote a rich and positive environment for maximizing the opportunities for student learning and transition toward algebra. Teams of teachers from schools were sought in order to promote collegiality, collaboration, and support among them throughout the school year.

Through baseline data collected prior to the project, we found that the teachers' initial views of algebra were often limited. An important project goal was to expand the teachers', and in turn students' views of algebra and algebraic thinking. T^2A presented three views of algebra: (1) Algebra as the study of patterns and relationships; (2) Algebra as a tool for problem solving; and (3) Algebra as generalized arithmetic. Many of the explorations, problems, and activities involved an overlap of the three views.

Through the engagement in and discussion of mathematics experiences, the teachers have expanded their views of algebra and algebraic thinking as well as their understanding of the importance of engaging students in problem solving and exploration of mathematical ideas through a variety of representations. The teachers' expanded views have helped them understand the importance of engaging their students in discovering relationships, translating among mathematical representations, and problem solving.

Note: T^2A Project was funded by the Eisenhower Mathematics and Science Education Program

TEACHERS MAKING SENSE OF THEIR STUDENTS' ALGEBRAIC THINKING THROUGH DISCUSSIONS OF STUDENT WORK

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Frameworks of children's mathematical thinking have provided a basis for teachers' practical inquiry into the teaching and learning of mathematics (Franke, Carpenter, Fennema, Ansell, & Behrend, 1996). Due to the paucity of similar frameworks in algebra and because I sought a more concrete link between professional development and classrooms, I relied upon student work to provide examples of students' algebraic thinking. In this year-long project, I engaged fifteen teachers in workgroups. For each monthly workgroup the teachers brought samples of their own students' work (generated from pre-selected problems) for discussion. The teachers began to make sense of their students' algebraic thinking by creating their own frameworks of their students' strategies. At first the frameworks were simplistic, containing only two categories for "concrete" and "abstract" strategies. However, as the teachers began to discuss the strategies in more detail, the relationships among them, and the mathematics underlying them, the teachers elaborated the frameworks to reflect the complexity of their students' strategies and of the mathematics involved in utilizing them.

TEACHING TASKS AS TOOLS FOR ASSESMENT AND TOOLS FOR CHANGE

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Ma (1999) emphasized the importance of acquiring a profound understanding of fundamental mathematics (PUFM) and how lesson planning facilitates its development. Ratio and proportion concepts, in particular, are fundamental to many Algebra I concepts and historically have been difficult to master. While simultaneously focusing on practical and theoretical implications, as suggested by Schoenfeld (1999), I searched for ways to assess and develop teacher knowledge in the context of a ratio and proportion lesson-planning task. Through naturalistic methods for data collection, sophomores, majoring in mathematics education, age 19-20, while enrolled in their first "methods" course at a large public university, were interviewed, given a lesson-planning task and then interviewed again. I constructed a theoretical framework that blends some of the perspectives of the cognitive (Strike & Posner, 1992) and social (Vygotsky, 1986) sciences. Specifically, I am assuming that signs of cognitive conflict regarding a concept imply that a particular concept is in a prospective teacher's zone of proximal development (ZPD). Analysis of the data involved PUFM coding and sorting, and looking for evidence of cognitive conflict and metacognitive thinking. Preliminary results show that the *interview-task-interview* process gives access to each prospective teacher's ZPD and can serve as a tool for change in the development of teacher knowledge, which will enable teacher educators to plan future tasks to enhance the development of teacher knowledge.

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PRESERVICE TEACHERS' LEARNING TO TEACH MATHEMATICS: THE IMPORTANCE OF THE INTERCONNECTION AMONG MANIPULATIVES, TALK, AND SYMBOLS

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What does it mean for preservice teachers to learn to teach children mathematics through the interaction between actions and manipulatives, the talk describing these actions, and mathematical symbols describing the talk and actions?

During interviews to assess a preservice teacher education course, students demonstrated how they would use manipulatives to teach concepts in mathematics, selecting from those use in the course (e.g., base 10 blocks, pattern blocks, algebra tiles, etc.). They were to demonstrate how manipulatives could model a particular mathematical concept, the language associated with using manipulatives, and how mathematical symbols described the actions with the manipulative.

Some preservice teachers 'took for granted' the multiple meanings of mathematical expressions, thereby failing to demonstrate their meanings with manipulatives. One, for example, trying to demonstrate 8÷2 did it within the concept of sharing, both in terms of her actions and language. However, she could not demonstrate division as repeated subtraction with the manipulatives, repeatedly returning to the idea of division as sharing. Yet she was able to calculate 8÷2 using repeated subtraction.

Using interviews for assessment reveals both understandings and misunderstandings to the interviewer and provides an occasion for the preservice teachers to reflect on their teaching. The interviews showed that preservice teachers need more help to develop awareness of multiple meanings of mathematical expressions.

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Technology

CHILDREN'S MEANING-MAKING ACTIVITY WITH DYNAMIC MULTIPLE REPRESENTATIONS IN A PROBABILITY MICROWORLD

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Abstract: This paper describes how fourth grade children used representations in a computer microworld as both *objects to display* and interpret data, and as *dynamic objects of analysis* during experimentation. The children used the multiple representations in both static and dynamic form to develop their probabilistic reasoning and explore concepts such as the law of large numbers. The results are from a 6-week teaching experiment investigating children's development of probabilistic reasoning with a newly developed probability microworld.

Design of the *Probability Explorer* Microworld

One of the most promising uses of computer technology in mathematics education is the ability to view multiple representations of phenomenon. Many software applications for middle and secondary school students include dynamic multiple representations. Although several software packages exist that allow elementary students to represent data in multiple forms, the representations are often only used as *objects of display* rather than *dynamic objects of analysis*.

My intent was to create a microworld containing multi-linked representations that update simultaneously as random events are simulated. The representations available in the *Probability Explorer* microworld at the time of the study included: 1) moveable iconic representations of every trial; 2)"stacking columns" to create pictographs; 3) a data table displaying results as a count, fraction, decimal and percent; 4) a pie graph; and 5) a bar graph.

Theoretical Framework

The theoretical foundation for the software design, instructional approach, and methods of inquiry is based on a constructivist theory of learning and research on students' use of microworlds (Steffe & Wiegel, 1994; Battista, 1998; Olive, 1994). I believe the tools available in a computer environment, meaningful instructional and playful activities, students' understanding, and their social and computer interactions all operate *interactively* as potential meaning-making agents for students' construction of concepts. Children interacting in a "fertile computer environment" (Battista, 1998) can learn through developing theories-in-action as they generate intuitive-based theories and modify them as they reflect upon experiences that either confirm their intuitions or challenge their theory through perturbations (Land & Hannafin, 1996).

Methods of Inquiry

The author conducted a 6-week (12 hours) constructivist teaching experiment (Steffe, 1991), including pre and post task-based interviews, with three fourth grade children. The purpose of the study was two-fold: 1) to study how children develop probabilistic reasoning in a dynamic multi-representational environment; and 2) to gather evidence of how children used microworld tools to inform future development of the software and instructional activities.

Each teaching session was video and audio taped to capture group interactions and a PC-to-TV converter was used to videotape children's computer interactions. The author was the lead teacher-researcher (T-R) while another T-R facilitated activities and a non-participant observer kept written records. After each session, the author critically reviewed the videotapes and developed subsequent tasks in the children's zone of potential construction (Olive, 1994).

The tasks included simulations with both equiprobable (e.g., coin flips, die toss) and unequiprobable (e.g., marble bag containing 1 white and 3 black marbles) events. In addition, the children also used a "weighting" tool in which the probability of an event could be changed (e.g., changing the likelihood of heads and tails for an unfair coin). The main tasks were planned and posed by the lead T-R, but many tasks were created on-the-fly based on the children's interactions, and "what if" or "can I try" questions. After completion of the teaching experiment, all sessions were transcribed and annotated while watching each videotape. The initial annotation included social and computer interactions and preliminary conjectures of children's meaning-making. Subsequent analysis revealed meaning-making themes for each child.

Results

One theme was the children's investigation of what Carmella termed as the "evening out" phenomenon (EOP). Although the children used many of the representations as *objects to display the data*, their observations during the simulation process facilitated their use of the representations as *dynamic objects of analysis*. It was the dynamic process that engaged students in exploring and developing their understanding of the EOP (i.e., law of large numbers).

The children used all the representations available in the microworld to re-present and make sense of randomly generated data after a simulation was complete (representations as *objects of display*). They often discussed the range of results by comparing the "highest" number of outcomes to the "smallest" number in the stacking columns. They also used the graphs and data table to make sense of the data for both a small and large number of trials.

During the first teaching session, the children discovered they could display the graphs during a simulation and watch the graphs do the "hula." This led them to spontaneously run a large number of trials and comment on the distribution of data during the simulation process (representations as *dynamic objects of analysis*). They discussed that results were not always as they expected with a small number of trials, but the pie graph "would hardly move at all" and "even out" (equiprobable outcomes) in a large number of trials.

Example Investigation

During the third teaching session, the children investigated a with-replacement experiment with a bag of 2 white and 2 black marbles (Figure 1a). For 10 trials, the children predicted many "5 and 5's" and "6 and 4's," ran several trials of 10, stacked their results, and viewed the pie graph and table after each simulation. When Amanda got 9 white and 1 black marble (Figure 1b), she squealed "oh my, this is very unlikely, usually you get 6 and 4, 5 and 5 and usually they are pretty much the same number." Carmella and Jasmine gathered around Amanda's screen, observed the representations as *objects of display*, commented on how different the 9-1 result looked from results of 5-5, 6-4 and 7-3.

Carmella and Jasmine jumped to running 500 trials "to see if the pie will even out." Based on their prior experiences with the dynamic simulation process, they anticipated the "even out" results. Amanda ran a large number of trials and predicted "I think maybe they [the number of black and white marbles] are going to be far apart at one point and then get very close." Carmella and Jasmine watched their graph during the simulation and noticed the large variation in the beginning (more black than white marbles in 50 trials) and how the "moon [white] made a comeback" and "evened out" as the trials approached 500 (Figure 1c).

In this episode, the children used the multiple representations as both *objects of display* and *dynamic objects of analysis*. Their use of representations as *objects of display* allowed them to critically analyze the results and make connections between the representations. However, their analysis of the representations during simulations, and their prediction based on prior dynamic visualizations, demonstrates how the use of representations as *dynamic objects of analysis* can facilitate their reasoning about powerful probabilistic concepts.

Children's Individual Use of Static and Dynamic Representations

The dynamic link between numerical and graphical results during the simulation process facilitated Carmella's theory-in-action about the EOP. After her initial experiences with equiprobable situations, Carmella used the representations to understand that a large number of trials would not result in an "even" distribution of outcomes, but rather, should approach a distribution close to the "weights" (i.e., theoretical probability) in unequiprobable situations. In addition, her use of representations as both *objects of display* and *dynamic objects of analysis* helped her think about the deviation from the "expected" based on theoretical probability. For example, with 10 trials of a fair coin toss, she considered a deviation of two (from the expected 5-5) greater than the same deviation with 100 trials (48-52). The iconic

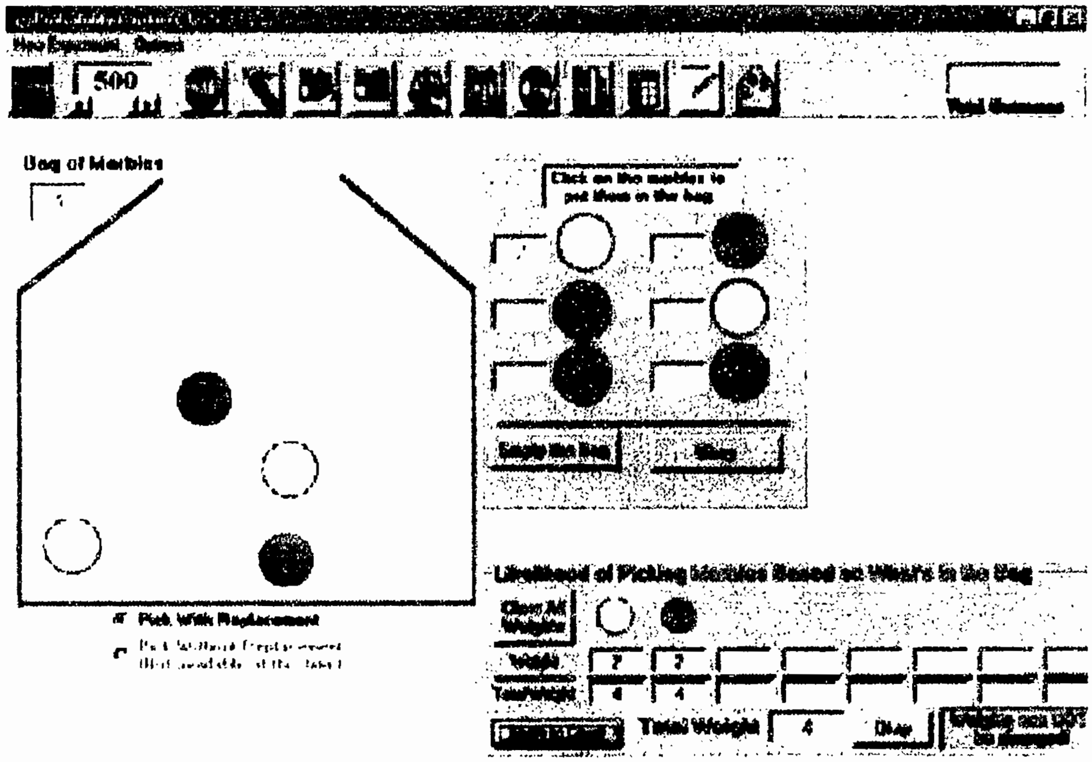

Figure 1a. Equiprobable marble bag experiment screenshot

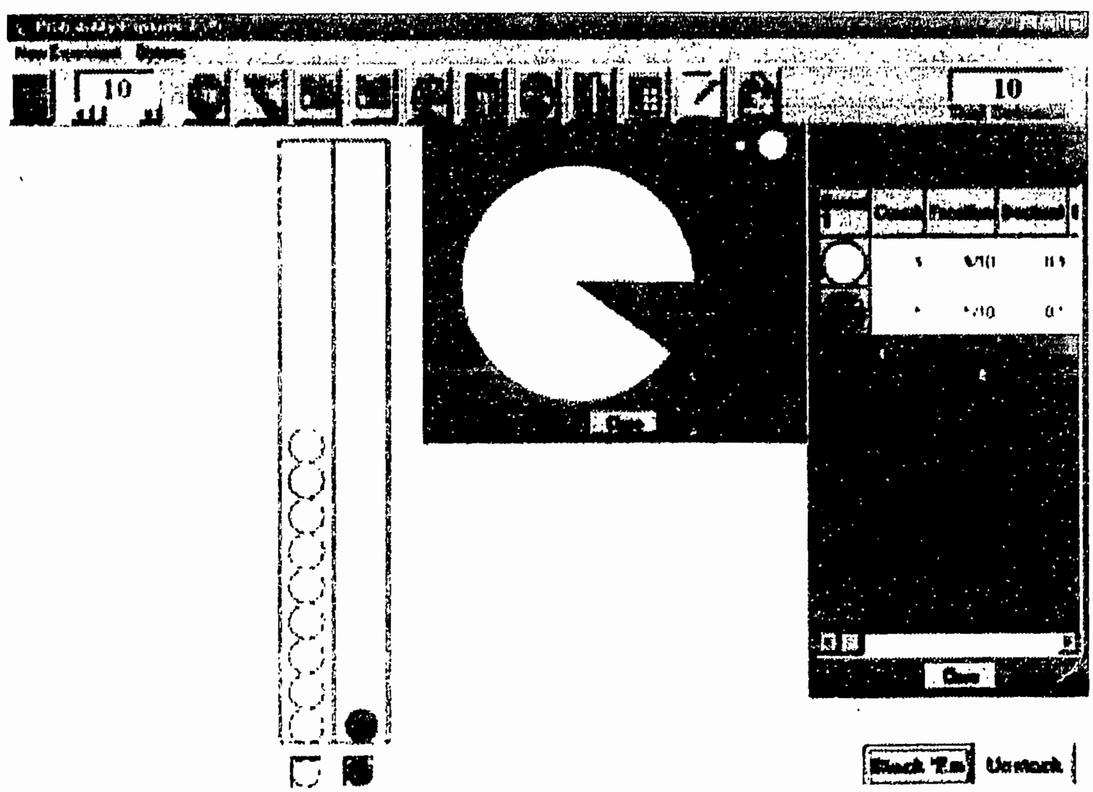

Figure 1b. Amanda's "unlikely" result screenshot

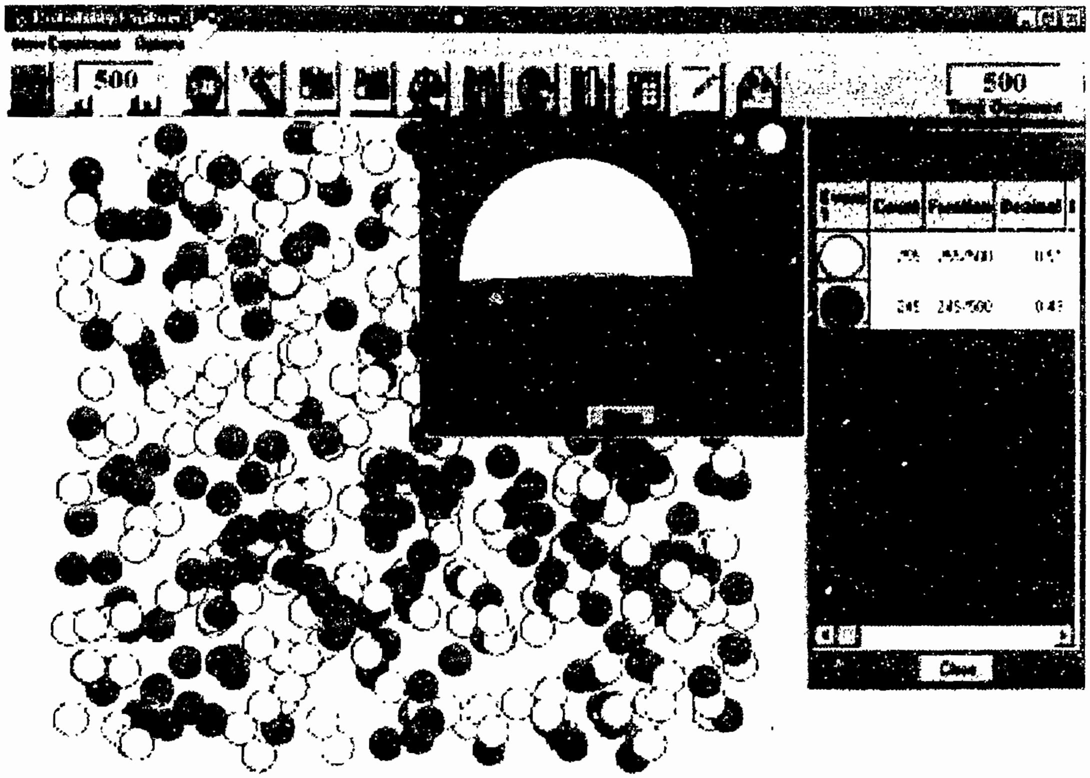

Figure 1c. Tendency towards "even" screenshot

stacked data representation and the numerical displays in the data table focused her on the absolute variability in the range with a small number of trials. However, the dynamic pie graph facilitated her recognition that the important visual variability in the range was relative to the total number of trials (e.g., "the slices stay mostly the same" as the number of trials increased).

Jasmine used all the representations available in the microworld as *objects of display.* She once commented that she knew "five ways to tell" what the results were from an experiment (data table, pie graph, bar graph, stacking columns, and manually sorting and counting). From her first experience with the representations as *dynamic objects of analysis*, she made clear and accurate connections between the motion of the graphs, the random simulation process, updated results, and the tendency towards "settling down" near an even distribution (for equiprobable outcomes). The pie graph representation was a major cognitive prompt for Jasmine in a variety of contexts and induced perturbations that helped her understand the EOP and part-whole relationships. Her use of the pie graph as a *dynamic object of analysis* helped develop her understanding of the EOP similar to Carmella's. In addition, she also used the pie graph as an *object of display* to help her transition to part-whole reasoning by thinking about the whole pie as a set number and then reasoning about the number associated with each slice relative to the whole (e.g., 1/10 is bigger than 1/20 because a pie cut in 10 has bigger slices).

Amanda's use of the representations as *objects of display* was limited by her weak conceptual understanding of numbers and graphs. Her reasoning with the representations was not always probabilistic (e.g., she often focused on the "bumpiness" of the lines separating pie segments rather than the relative size of the slices). Amanda never enacted a theory-in-action about the EOP like the other children. Although she made reference to the pie graph "staying in the same place" during a simulation with a large number of trials, she made only occasional references to the actual simulation process or the theoretical probability in her analysis of the *dynamic objects*. She rarely reflected on the relationships between multiple representations and between empirical results and theoretical probability. Thus, for Amanda, the representations, as static and dynamic objects, were often a deterrent in her development of probabilistic reasoning.

Conclusion

As shown with Carmella and Jasmine, dynamically linked multiple representations have the *potential* to facilitate children's probabilistic reasoning and encourage them to develop theories-in-action about the law of large numbers. The representations were mostly not helpful for Amanda. Thus, I added a feature in the microworld to allow the user to change the speed of a simulation. Slowing down the simulation process should allow students to attend to the changes in the representations and reflect on the effect of the number of trials on the results, rather than just watching the dynamic "motion."

Using *Probability Explorer* can substantially extend typical experiences with physical objects and lead children to play, experiment, predict, and discover probabilistic ideas by using multiple representations as both static and dynamic objects. This study begins the research on children's use of dynamic representations in developing probabilistic reasoning.

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PRE-SERVICE ELEMENTARY TEACHERS' PERCEPTIONS OF THEIR READINESS TO TEACH VIA TECHNOLOGY

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Educators are being urged to use technological resources to help learners construct understanding of disciplines and interrelationships between disciplines as well as between disciplines and their lives. However, relevant literature and my own experiences indicate that while most teachers desire to use technological resources as teaching tools, many are or perceive themselves to be ill-prepared to do so (Roblyer & Erlanger, 1999; Dusick, 1998).

In light of this, it appears that preparation programs have emphasized skills and practices that are inconsistent with teachers' needs. Nevertheless, teacher educators need to take into account pre- and in-service teachers' perceptions of their needs regarding their preparation to teach via technology. For without taking into account teachers' perspectives, teacher educators may continue to provide developmental opportunities that are inconsistent with teachers' needs. Accordingly, the purpose of this research was to gain insight into pre-service elementary teachers' perceptions of their preparedness to teach via technology. These insights were then used as the basis for recommendations for teacher preparation programs.

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THE CALCULATOR AS AN INSTRUMENT OF VALIDATION OF MATHEMATICAL KNOWLEDGE: A CASE STUDY

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The discussion that arises when demonstration in mathematics is dealt with often causes opposing positions regarding the wisdom of its use. If, in addition, we consider the contribution of new technology when teaching, it is necessary to reflect on how the form of validating mathematical knowledge is affected, if it is at all altered, or not. In this paper we describe a case study with teachers; our objectives were to inquire into teachers' conceptions on mathematical demonstration, and the value they place on it in an educational environment. Also, we were interested in their conceptions about what new technology can contribute to this topic and how they examined and validated their statements through the use of TI-92 calculators. Out of the five teachers who participated in the study, only one declared that it was possible to do formal demonstrations using technology. Indeed, she defended the statement that, in fact, the example she presented showed mathematical reasoning as used in the formal discipline.

According to the theory considered for this study, the idea of *argumentation*, proposed by Duval (1999) is present in the opinions of the subjects in terms of it being a type of reasoning that does not have the level of rigor required as a demonstration, although it is a type of justification of important and valid interest in junior high school mathematics education. Also, it can be concluded from the participants' responses that they understood that the work with argumentation requires careful handling and that it will not lead rapidly, nor directly, to formal justification.

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HOW DO STUDENTS USE GRAPHICAL CALCULATORS TO MODEL AND SOLVE SPEED PROBLEMS

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This work focuses on how students in an introductory calculus course modeled problems of distance vs time relationships, using calculators with curve fitters to infer speeds with the model they found. The research questions that guided the study were: What specific difficulties in representation did students deal with when they used the calculators? How did they select a model among the various functions of the calculator to relate distance and time? The study was conducted with a 12th grade high school group during a semester-long introductory calculus course. Teams of cooperative learning groups of 3 or 4 students were formed. Each team had one TI-82 calculator, which they were taught to use while they solved. Observing students using calculators revealed that some of the difficulties they have do not necessarily concern dealing with new representations, but sometimes old, well-identified problems emerged; for example, it is assumed that there was a lack of understanding of letters as representations of numbers and of multiplication of signs or by zero.

With the introduction of technology, the manner of representing mathematical concepts increases. Could this increase the risk of problems in communication? This may well be, as long as some students seem to see calculator notation as if it were formal algebraic notation, in spite of warning them that each software or calculator has a different way of expressing mathematical concepts. This is the same warning contained in *Principles* (NCTM, 1998). It would help if the notations used by technology would adjust more to those most frequently used in algebra. In modeling, some students had problems in identifying the variables of the model. In spite of the fact that some students encountered many difficulties, others were able to model and infer the speed of that model using a variety of strategies.

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EXPERIENCING A TECHNOLOGY-RICH MATHEMATICS CLASSROOM FROM A DISTANCE

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By using video confei :ncing we provided our preservice teachers (PTs) a window into a middle school mathematics classroom where every student had a laptop computer to take both home and to school, offering the PTs opportunities to learn about how teachers teach and students learn with technology. Our approach was modeled after constructivist philosophy (von Glasersfeld, 1995). We investigated the advantages and disadvantages of the long distance interactions as perceived by the PTs. We also investigated what the PTs learned about the use of technology for teaching and learning mathematics.

Conclusions drawn from PT's responses on questionnaires, interviews, a "town meeting" and our observations indicated that they overwhelmingly viewed their experiences as positive. The PTs liked that they could discuss, observe, question and wonder throughout the videoconference because they could control whether the cooperating inservice teachers and their students could see or hear them. As one put it, "I like how we can turn off the microphone and discuss what is happening together." They thought one of the biggest advantages of observing over the Internet was they could "see a variety of classes in many places without the time or expense of driving there."

Our PTs observed the classrooms in action, asking and responding to pedagogical and content questions before, during and after lessons. They became active learners, responsive problem solvers, and critical thinkers (Oppong & Russell, 1997). After the video conferencing interactions, they were convinced they needed to know more about teaching with technology. The PTs commented, "It helped to see how a teacher would actually go about teaching using computers." According to them, the most important thing they learned was that "technology is not just good for typing papers and surfing the net but a powerful tool for students to use in their learning."

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STUDENTS EXPLORING GEOMETRICAL CONCEPTS USING TEACHERS' WEB PAGES

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Adapting to changes of the new millennium implies better understanding of the psychological aspects of teaching and learning mathematics using technology. Through the use of technology we can enhance the abilities of the human brain to detect patterns, make approximations, self-correct and learn from experience via analysis of external data and self-reflection. This proposal is two-sided. I will report some results and views from my students when they worked with geometrical explorations designed in my Web Pages. I will also present how I implemented these activities in the Web pages.

DEVELOPING A VIRTUAL COMMUNITY OF TEACHERS: THE EFFECTS OF LIVE CONTACT AND ON-LINE CONVERSATIONAL DYNAMICS

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I was invited by an educational non-profit organization based in the southwestern United States to investigate and assess a distance-learning course they ran in spring 2000. The students were algebra teachers from various sites in Texas, who had regular, asynchronous class discussions over the Internet supplemented by two videoconferences. The subject matter was "contextual learning," a constructivist theory of learning based on modeling and applications.

I was given access to all public and private electronic messages. I surveyed the students electronically, and I performed hour-long interviews with ten students at course end with follow-up interviews planned for six and twelve months later. After the follow-up interviews, I intend to study longer-term effects of the course on their teaching, use of technology and continuing relationships with former classmates.

Preliminary results show that live interactions have a profound effect on student dedication, on-line interactions and on recall and perceptions of the class discussions. Students with live interactions with fellow students stayed in the course despite enormous personal obstacles, whereas all isolated students dropped out of the course. All students who completed the course had a live partner. Students had nearly no recall of the ideas and postings of their fellow students unless they discussed their postings live with their partner. Even then, they remembered other students in unsubtle stereotypes. All partnered students credited the course with increasing the closeness of their relationship with their live partner, usually dramatically. We also note other factors influencing the dynamics of the virtual community, including class organization, the effectiveness of the moderators, and real-life political factors local to the students.

A DYNAMIC SOFTWARE VISUALIZATION TOOL FOR CALCULUS INSTRUCTION AT THE COLLEGE ENTRY-LEVEL

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In November of 1999, I carried out an exploratory study designed to evaluate the potential of a particular dynamic software visualization tool in developing students' intuitive understanding of the derivative function, at the college level. This study provided evidence that dynamic graphics (as opposed to static calculator-based sketches) can be very valuable in shaping students' insight about specific abstract concepts, such as the notion of the derivative function. It also demonstrated that, at least at the college level, teachers can begin implementing dynamic software-based activities without investing extensive class-time hours or requiring software-specific knowledge from their students.

The purpose of this presentation is to allow educators and students to directly experience and evaluate the dynamic visualization tool used in the above study: a "Derivatives Microworld" created using The Geometer's Sketchpad software package. An analysis of the specific reform-oriented goals underlying the design of the activity, as well as a discussion of the benefits and potential drawbacks of this and other dynamic software visualization tools will be provided.

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APPRENTICESHIP IN A TECHNOLOGY-RICH CLASSROOM

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We report on a project to establish an apprenticeship/participation program in developmental mathematics classes at Xavier University of Louisiana. The medium is computer-based lessons written in a computer algebra system (CAS) that produces graphic animations from student responses. Use of the CAS provides a context in which students, quite naturally, talk with and instruct each other. One of the project goals is to have a direct and positive effect on the social and academic maturation of the students. This goal is supported by peer collaboration, which requires increased and shared responsibilities among students.

We purposely chose more sophisticated software than was strictly necessary for generating the graphic animations. This added versatility to the teaching tool; the instructor can modify exercises on the spot. It also added depth and value to the students' learning. For the project, several students were recruited to serve as interns/student leaders. During the course of the semester several other students stepped forward to emulate what they saw the interns doing and to actively seek out students with whom to share newly discovered knowledge. A typical activity involved interns collecting friends for an out-of-class session at which they made mini-presentations to each other.

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USING COMPUTER-BASED LABORATORIES TO TEACH GRAPHING CONCEPTS AND THE DERIVATIVE AT THE COLLEGE LEVEL

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University freshmen often do not understand velocity or its relationship to distance, and do not connect the slope of a distance graph to rate of change or to velocity. This poses a problem in introductory calculus, where the example of distance and velocity is often used to introduce the derivative, which is usually depicted as the slope of a tangent line. Computer-based labs with motion sensors improve students' understandings of graphs of motion events, but are expensive and inconvenient. This study examines a web-based Java simulation that appears to produce the same educational benefits with lower cost and greater convenience.

Volunteers from large lecture sections of first-semester calculus were tested before and after two and a half hours of computer-based instruction. About half of the 40 subjects used motion sensors for the instruction; the other half used the Java applet. Tests included both achievement and attitude items. ACT and department placement test scores were used as covariates. Following the post-test, eight subjects were interviewed. Both groups significantly increased performance on the achievement items; the interviews indicate that this change was due to the instruction. No difference was found between treatment groups. This study suggests that the Java applet provides the beneficial effect of motion sensor instruction at a lower cost and greater convenience, thereby making it available to more students.

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WHY I HATE COMPUTERS: PRESERVICE TEACHER ATTITUDES TOWARDS TECHNOLOGY IN MATHEMATICS CLASSROOM

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I have observed that many preservice education majors enter college lacking the ability to appropriately use technology in the mathematics classroom. The purpose of this case study was to investigate present attitudes towards technology of preservice teachers through the thoughts and feelings of one elementary education major that is intimated by technology in a mathematics classroom. The participant in this study was taken from an elementary education mathematics content course at Illinois State University. The data sources for this pilot study consisted of a survey, pre-interview, classroom observation and post interview, all conducted on separate days. The subject describes herself as one who is "frustrated" by technology. However, while she feels some frustration towards technology, she experiences conflict in that, at the same time, she sees the benefits of technology. This conflict is as a result of a dualistic view towards technology. On one hand, technology is equated with mathematics, a subject she dislikes. On the other hand, she has experienced many educational benefits of technology; for example, the benefit of using a word processor versus using a typewriter, and research on the Internet versus driving to the library. In these cases her attitude towards technology is favorable since the benefits of technology outweigh the frustration of trying to learn how to use the technology. However, in her mathematics experience, the benefit of technology in the classroom as a learning tool did not seem evident to her. Since such conflict exists, it is important that teacher-training experiences bring students through this conflict by creating academically rich and positive experiences with technology in the mathematics classroom.

Whole Number

THE APPLICATION AND DEVELOPMENT OF AN ADDITION GOAL SKETCH

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Abstract: Siegler's latest strategy-choice model includes an addition goal sketch, which is presumed to develop as a child practices concrete counting-all and to subsequently affect strategy choice by identifying legal and illegal strategies and suppressing the latter. A study of 20 kindergartners was undertaken to examine key assumptions of this model. Participants were individually interviewed to determine their own strategy use and their ability to evaluate legal and illegal strategies. Consistent with Siegler's model, children who had just learned a concrete counting-all procedure were prone to use illegal strategies. Inconsistent with the model, more advanced adders did also. Furthermore, children who had not yet invented counting-on did not view this strategy as legal as Siegler has suggested. Implications for computer simulations of development are discussed.

Rationale

A key aspect of the latest version of Siegler's (e.g., Shrager & Siegler, 1998) strategy-choice model is an addition goal sketch.

Recognition of Novel Legal Strategies

According to Siegler (e.g., Crowley, Shrager, & Siegler, 1997), a goal sketch permits children to evaluate new versions of strategies they invent or even novel strategies they have never used themselves. Consistent with this claim, Siegler and Crowley (1994) found that 5-year-olds who themselves had not yet invented counting-on from the larger addend (COL; e.g., for 3 + 5, counting: 5; 6 [is one more], 7 [is two more], 8 [is three more]) rated this strategy and the familiar, concrete counting-all strategy (CCA; e.g., for 3 + 5, counting out three items to represent the first addend, then five more items to represent the second addend, and finally counting all the items put out to determine the sum) as "smart."

Siegler and Crowley's (1994) finding contradicts Baroody's (1984) case study observations of Felicia. This 5-year-old typically used a verbal counting-all strategy—either beginning with the first addend (CAF ; e.g., for 3 + 5, counting: "1, 2, 3, 4 [is one more], 5 [is two more], 6 [is three more], 7 [is four more], 8 [is five more]") or beginning with the larger addend (CAL; e.g., for 3 + 5, counting: "1, 2, 3, 4, 5; 6 [is one more], 7 [is two more], 8 [is three more]") for single-digit combinations. However, when presented with larger challenge items, she immediately and consistently used either a COL strategy (e.g., 5 + 22: "23, 24, 25, 26, 27") or a COL-like strategy (e.g., 32 + 6: "31, 32, 33, 34, 35, 36, 37, 38"). When single-digit combinations were reintroduced, Felicia reverted to using CAF or CAL. Moreover, when COL was modeled for her with these smaller combinations and she was asked to evaluate the strategy, the girl declared, "You can't do it that way." These results were replicated on several more occasions.

The reason for this discrepancy may be that Siegler and Crowley (1994) modeled a conceptually less-advanced form of COL than did Baroody (1984). More specifically, the former demonstrated an *indirect-modeling* version of counting-on (IM), a strategy only somewhat more sophisticated or (in the case of CAF or CAL users) less advanced than their own. (Indeed, Fuson and Secada,1986, found that learning an IM strategy did not promote the learning of COL.) In contrast, Felicia (Baroody, 1984) evaluated a COL strategy conceptually more advanced than her own. A main purpose of the present study, then, was to determine if other children with less-advanced addition strategies can typically recognize COL as a legal strategy or not.

Recognition of Illegal Strategies

Siegler and Crowley (1994) also found that their 5-year-old participants, whether they had already invented COL or not, rated an illegal strategy (representing one addend twice) as "not smart." Crowley et al. (1997) have further noted that "children never discover illegitimate addition" (p. 469). Shrager and Siegler (1998) attribute the ability to distinguish between legal and illegal strategies and choose only the former to an addition goal sketch. Unfortunately, this generalization was based on a small and select sample (eight above-average calculators). Our previous research indicates that children of all abilities do sometimes invent and use illegal addition strategies. One such strategy is to create a nondistinct representation of the two addends. For 2 + 4, for instance, Jonna (almost 8-years old and diagnosed as having learning difficulties) represented the first addend by counting, "One, two" (accompanied by raising two fingers), looked at the expression and represented the second addend by counting, "three, four" (accompanied by raising two fingers), and then determined the sum by counting the four extended fingers. In effect, the "one, two" was used to represent the first addend and the first two counts needed to represent the second addend.

The second primary goal of the present study was to gauge whether children could identify a genuine and more subtle error (Jonna's error) than the artificial and obvious error of counting the same addend twice. A secondary goal was to find additional evidence of illegal strategy use.

The Development of an Addition Goal Sketch

Crowley et al. (1997) argued that an additive goal sketch is constructed from the repeated application of the most basic informal counting strategy, concrete counting-all (CCA) which is learned by rote by imitating parents, siblings, or peers. As a child practices CCA, a metacognitive component is increasingly relieved of the burden of micromanaging the execution of this strategy and increasingly takes on a monitoring function. As monitoring requires using only the key elements of CCA, only these are reinforced. The other steps (elements) fade from memory leaving a goal sketch of the strategy. A key implication of this view is that children who have just learned CCA do not have a goal-sketch and, thus, are likely to invent illegal strategies as they are legal strategies. Another secondary goal of the present study was to check whether this was true.

Method

Participants

Twenty kindergartners, ranging in age from 5 years and 4 months (5-4) to 7-2 (median age = 6-1) participated in the study. Gender was equally represented; 14 participants were Caucasian; 3, Asian; 2, African-American; and 1, Indian.

Procedure

The participants were individually interviewed or tested over five sessions. Preliminary testing of prerequisite counting and number skills was done in Session 1. In Session 2, children were administered a nonverbal addition and subtraction task (for details, see Huttenlocher, Jordan, & Levine, 1994) and change add-to word problems to assess their informal understanding of arithmetic; to determine which children already knew informal, counting-based strategies; to teach those who did not have such a strategy CCA; and to gauge the general level of a child's addition strategies. The word problems and a child's answer to each were translated into a number sentence to familiarize participants with symbolic addition. In Sessions 3 and 4, children were administered the strategy-evaluation task (see Siegler & Crowley, 1994, for details). In each session, five strategies were modeled in the following order: CCA, IM (which Siegler & Crowley, 1994, incorrectly called COL), counting out an addend twice (E1), COL, and Jonna's error (E2). For each strategy, a child was asked to rate it as "very smart," "smart," or "not smart." To analyze the data, these responses were coded as 2, 1, and 0, respectively, and the average a child's rating for each strategy was determined. In Session 5, the children's strategy use was assessed by presenting them 10 written addition expressions, accompanied by a verbal description. These included small items (1 + 3, 4 + 2, 3 + 2), medium items (3 + 5, 6 + 4, 5 + 6, 8 + 4) and large items (16 + 1, 11 + 2, 3 + 12). The last type was included to give children who knew a COL strategy a particularly strong incentive to use it. The tasks were typically presented in the format of a game to maximize motivation.

Results

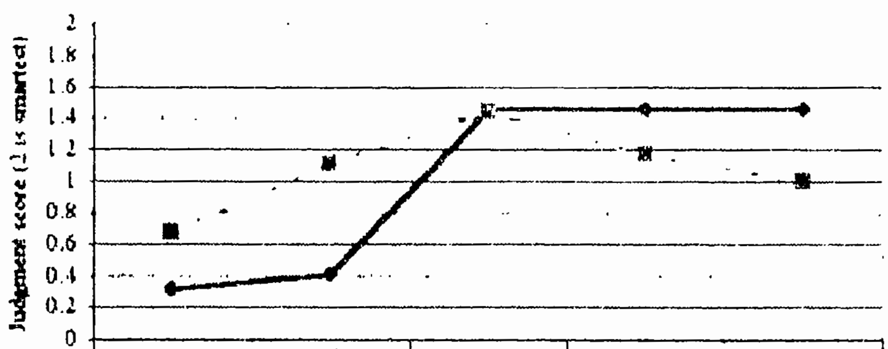
Eleven children used COL at least once or, in two cases, relied solely on strategies more advanced than COL (e.g., retrieval); nine children exhibited no use of COL. Both quantitative and qualitative analyses were performed.

Quantitative Data

Figure 1 summarizes the quantitative results of the study. A 2 (group: use: COL user or non-COL user) x 5 (demonstrated strategy: E1, E2, CCA, IM, COL) repeated-measures ANOVA revealed a significant main effect for demonstrated strategy F (4, 72) = 8.627, $p < .001$, no effect for group, and (unlike Siegler & Crowley, 1994) a significant interaction effect for the two factors, F (4, 72) = 3.291, $p = 0.016$. Tests for simple main effects were conducted to follow up the significant interaction. The COL users judged the legitimate strategies significantly more favorably than the illegitimate strategies (F [4, 80] = 1.969, $p = 0.05$), but the non-COL user's judgments of the five strategies did not differ significantly. The non-COL users (incorrectly) judged the Jonna's error strategy (E2) more favorably than did the COL users (F [1, 90] = 6.621; $p = .05$), but the judgments of other specific strategies were not significantly different between the groups.

Qualitative Data

Like the quantitative analysis, qualitative analyses indicated that, consistent with the case of Felicia (Baroody, 1984), children who have not yet invented COL may have trouble recognizing this strategy as legal. In fact, five of these children favored CCA over COL; three rated the former as "very smart" and the latter as "not smart" in



	E1	E2	CCA	IM	COL
COL Users	0.32	0.41	1.46	1.46	1.46
Non-COL Users	0.67	1.11	1.44	1.17	1.00

Demonstrated Strategy

Figure 1. Mean Evaluation Scores for Five Addition Strategies by Group (COL Users vs. Non-Col Users)

both sessions. For example, Brianna, who had to be taught the CCA strategy in Session 2 and who rated this strategy as "very smart," rated the IM strategy as "kinda' smart" and the COL strategy as "not smart." Beth, who was an accomplished user of CCA or its shortcuts considered the CCA and the IM strategy as "smart" but Jonna's procedure (E2) and representing one addend twice (E1) as "not smart."

Consistent with Crowley et al.'s (1997) prediction, children who had just learned CCA were prone to invent illegal, as well as legal, strategies. Inconsistent with their predictions, children who adopted more advanced strategies and presumably had constructed a goal sketch also used illegal strategies. For example, in Session 5, Beth devised a relatively advanced strategy, which involved keeping track. For example, for 4 + 2, she counted, "one, two, three, four" (as she extended four fingers consecutively) and then "five, six" (as she extended two more fingers consecutively). Yet for 5 + 6 and 3 + 12, she devised a strategy similar to Jonna's. For example, for the former item, she counted, "one, two, three, four, five" (while extending five fingers in turn), next counted, "six" (as she extended one more finger), and then announced an answer of "six." In brief, she devised an illegal strategy to deal with relatively large and challenging problems, despite fairly clear evidence of conceptual or metacognitive knowledge of addition.

Conclusions

Some scholars argue that computer simulations based on information-processing theory have important advantages over verbal (constructivist and social-learning) theories in that they require explicitly delineating questions, assumptions, and predictions (e.g., Klahr & MacWhinney, 1998), and that this permits more precise theorizing and theory testing. In fact, the validity of a computer simulation such as SCADS depends on the completeness and accuracy of the developmental theory and data used to design it. The results of the present study indicate that even this latest and most sophisticated version of the strategy-choice model does not accurately represent the development of addition strategies. In particular, SCADS does not adequately account for how children construct an understanding of addition and how they this knowledge to invent addition strategies. For example, many, if not most, children invent CCA, because preschoolers begin to construct a concept of addition *before* they devise counting strategies for solving word problems or symbolic expressions (e.g., Huttenlocher et al., 1994). That is, contrary to Shrager and Siegler's (1998) assumption, the development of addition does not begin when parents or others teach CCA to a child. Inconsistent with Shrager and Siegler's (1998) assumption that the same goal sketch underlies the invention of all strategy improvement beyond CCA, the results of this study indicate that there is a qualitative or conceptual leap that must be made to move from counting-all to counting-on (e.g., Fuson, 1992; Steffe, von Glasersfeld, Richards, & Cobb, 1983). Furthermore, non-COL users were less able to distinguish between legal and illegal strategies than Siegler and Crowley (1994) evidence suggests and, contrary to Shrager and Siegler's (1998) claim, some non-novices, do invent illegal strategies.

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STUDENTS' UNDERSTANDING AND USE OF MULTIPLE REPRESENTATIONS WHILE LEARNING TWO-DIGIT MULTIPLICATION

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Abstract: We report the results of implementing a two-digit multiplication unit that relied on modeling areas of rectangles. We worked in one urban and one suburban fourth-grade classroom to determine whether such an approach could support diverse students as they learned a general computation method invented by urban students. Both classrooms outperformed U.S. fifth-graders in traditional curricula, and the suburban classroom was comparable to Japanese and Chinese classrooms. Results also suggested ways to make the unit more accessible to low-achieving students.

Introduction

We report the results of implementing a two-digit multiplication unit in urban and suburban fourth-grade classrooms. The work is part of Children's Math Worlds (CMW), a project that develops instructional materials for elementary school mathematics and that conducts research on teaching and learning as teachers use those materials in their classrooms. A main objective of CMW is to make the goals of the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) accessible to all students. The standards and principles about number and operations, representation, problem solving, communication, and equity are most relevant to the study reported here.

CMW combines a Vygotskiian (1978, 1986) view of teaching with a constructivist view of learning. In particular, the project investigates means by which teachers can help students take what they already know and construct culturally adapted conceptions of mathematics. Central to all CMW units (including the two-digit multiplication unit) are activities in which students use drawn representations of situations to solve problems and explain solutions to others in the class. In the course of such activities, teachers help students connect their experiences and understandings to traditional mathematical symbols, words, and procedures. Equity Pedagogy (Fuson et al., 2000) describes in more detail the principles that guide our design efforts.

The work reported here integrates and extends three areas of research: that on multiplication and division, that on place value and its role in multi-digit addition and subtraction, and that on students' understanding of representations. Extensive research has already been done in each of these literatures individually. Although existing research has investigated ways in which students might conceptualize multiplication as a model of equal groups, multiplicative comparison, Cartesian product, and rectangular area situations (see Greer, 1992 for a review), to the best of our knowledge, existing research has not examined the following question: Can whole classrooms of diverse students, including inner-city students, master two-digit multiplication using a modeling approach?

Methods and Data

We based our two-digit multiplication unit on modeling areas of rectangles both because area is a core meaning for multiplication and because this approach allowed us to help students build a general computation method based on their prior experiences modeling areas of smaller rectangles with single-digit numbers. We used a progression of three area representations that built on students' strategies for tallying unit squares and directed their sense-making toward our target computation method, a method invented by urban fourth- and fifth-grade students. We chose the target method because it shows all four quantities and all four sub-products involved in two digit-multiplication (i.e., 42 x 36 = 40 x 30 + 40 x 6 + 2 x 30 + 2 x 6). The rectangles afforded drawn representations of the quantities involved in the target method and thus could potentially support students' understanding that the product of 2 two-digit numbers is the sum of four sub-products.

We use the example 13 x 14 to outline the progression of activities linking area representations to the target method. The first area representation (see figure 1 below) showed all of the unit squares in a 13 by 14 rectangle. We wanted teachers and students to propose and discuss strategies for grouping and counting the total number of unit squares, and then build on those contributions that led to the second area representation (figure 2). This transition was important because the second area representation supported connections among area, base-10 place value, and the target method. (Note that for problems with larger factors, such as 23 x 34, students constructed figure 2 by breaking apart the factors into 10 + 10 + 3 and 10 + 10 + 10 + 4 and by drawing a representation that grouped unit squares into six "100 squares," tens, and ones.) To prepare students for work with larger numbers, we had them abbreviate their work in figure 2 to create a third area representation (figure 3). Finally, we had students connect the third area representation to the target method for multiplying two-digit numbers (figure 4). As students moved to problems with larger factors and products over 1,000, they found figure 2 cumbersome to work with and increasingly relied on just figure 3 and figure 4, and finally just on figure 4. We used different colors to help students connect the four sub-products of the area representations to each other and to the computation method.

We piloted the unit in one urban and one suburban classroom, each with 25 to 30 students. The teachers in these classrooms faced challenges common in the United States. Many of the urban students, and their parents, were recent immigrants and struggled with English as a second language. Nearly a quarter of the suburban stu-

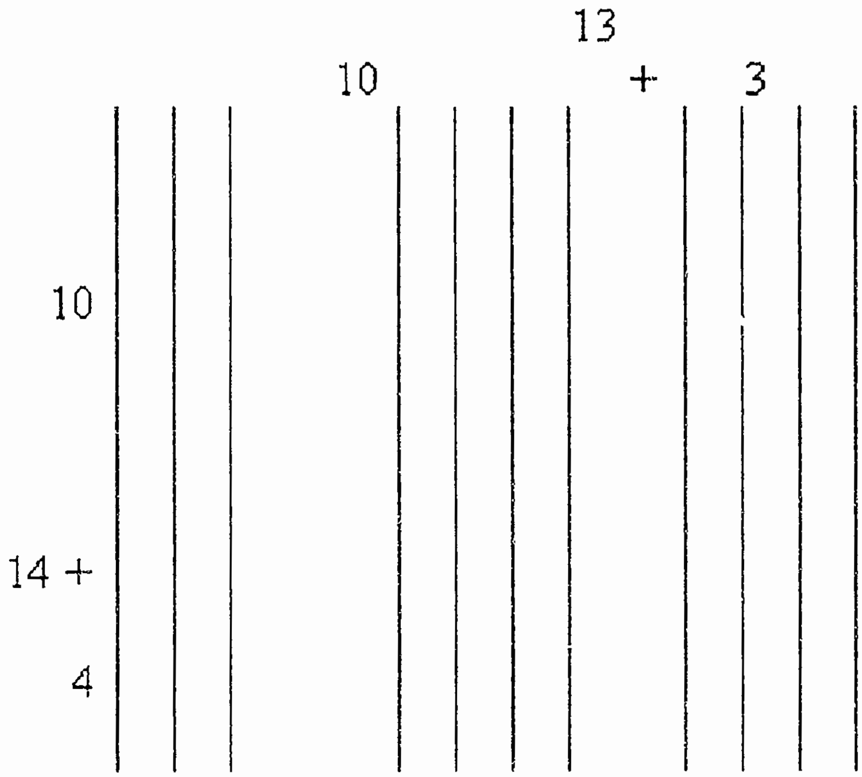

Figure 1

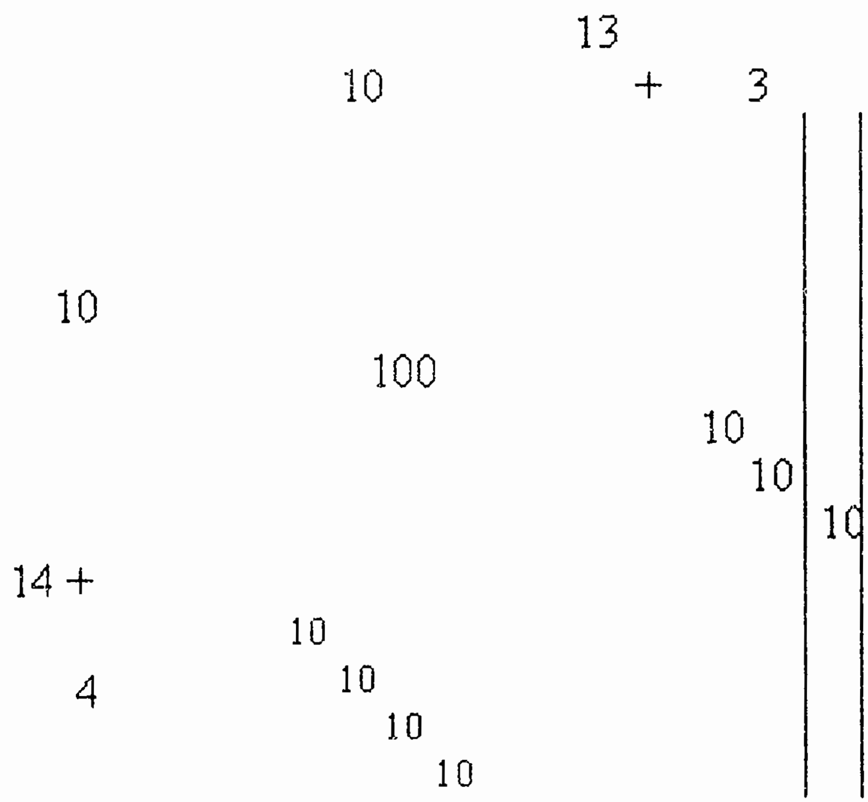

Figure 2

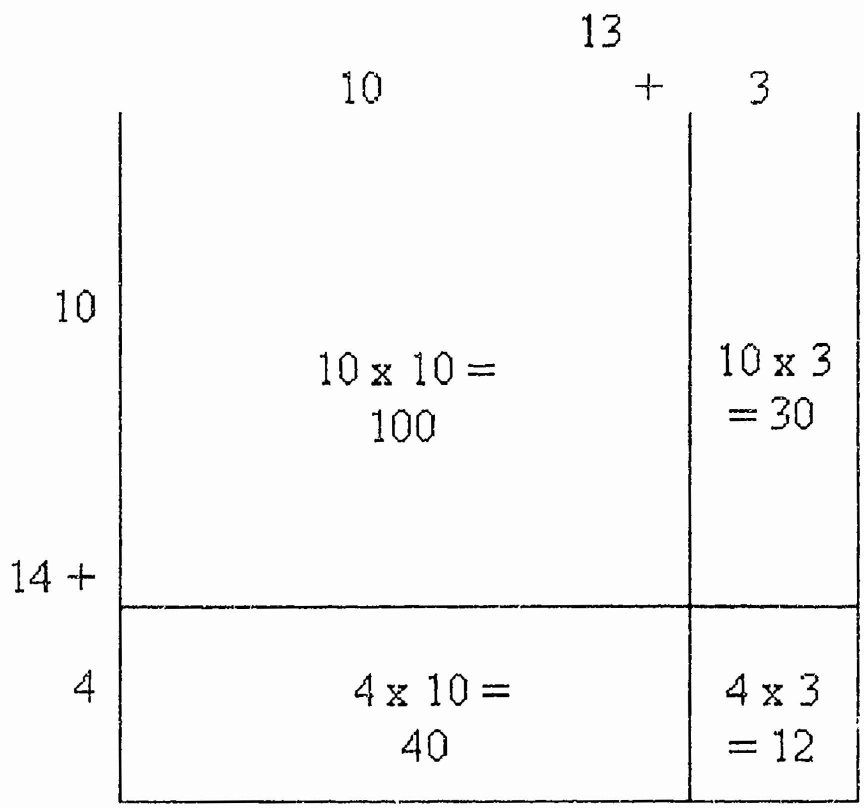

Figure 3

$$\begin{aligned} 13 &= 10 + 3 \\ 14 &= 10 + 4 \end{aligned}$$

$$\begin{aligned} 100 &= 10 \times 10 \\ 30 &= 10 \times 3 \\ 40 &= 10 \times 4 \\ 12 &= 4 \times 3 \end{aligned}$$

$$100 + 30 + 40 + 12 = 182$$

Figure 4

dents were main-streamed students with learning disabilities, and about the same proportion were bilingual.

To gather data on implementation and learning, we worked intensively with both teachers in and out of class. We observed lessons in both classrooms at least twice a week. Videotapes and field notes from classroom observations provided data on how the teachers used the materials, and hence how students actually experienced the unit. We met with teachers after school to discuss aspects of the unit that students understood, aspects that were hard for students, and ways in which the materials could be adapted to better support students' learning. We also gathered students' written work, primarily tests, and conducted in-depth, videotaped interviews with students at the end of the unit. For these forty- to fifty-minute interviews, we selected a cross-section of students from low- to high-achieving and asked them to work problems similar to those that they had done in class and for homework. These data provided access to students' strategies, understandings of the three area representations, and connections among the area representations and the expanded algorithm (figure 4) (or abbreviations of the algorithm).

Analysis and Results

To assess how well diverse students mastered two-digit multiplication using our modeling approach, we first analyzed the accuracy with which students multiplied two-digit numbers at the end of the unit. To put our results in context, we say more about where each class of students began at the start of the school year.

Many students at the urban school began the year still having difficulties with place value and multi-digit addition and subtraction. For example, many students lined up left-most digits when adding and could not borrow across zero correctly when subtracting. Students could perform some single-digit multiplication either by recall when the factors were small (i.e., 2 x 3, or by counting repeated groups (often on their fingers).

When analyzing the accuracy of the urban students' two-digit multiplication at the end of the unit, we found the following percent correct by item: 17 x 12 (94%), 45 x 26 (61%), 37 x 24 (56%), and 92 x 78 (56%). We traced many of the errors to near, but faulty, products of single-digit numbers (i.e., 6 x 4 = 20) and to faulty place value when multiplying multiples of 10 (i.e., 30 x 20 = 60). We gave students additional practice with single-digit multiplication and place value, re-tested, and got the following results by item: 26 x 7 (83%), 65 x 43 (61%), 40 x 9 (87%), 50 x 6 (91%), 80 x 70 (65%), and 423 x 3 (87%). By way of comparison, Stigler, Lee, and Stevenson (1990) reported international performance by fifth-grade students on multiplication problems. Percentages for Japanese, Chinese, and U.S traditional students on 30 x 60 were 73%, 74%, and 35%, respectively. Fuson and Carroll (1999) reported percentages for U.S traditional and U.S. reform (Everyday Mathematics) fifth-grade students on 45 x 26 as 54% and 78%, respectively.

Students at the suburban school began the year much better prepared than the urban students. About half had been in third-grade classes that used CMW materials the year before. These students already had a good start on single-digit multiplication, could add and subtract multi-digit numbers accurately, and were used to working with drawn representations of situations. When analyzing the accuracy of students' two-digit multiplication at the end of the unit, we found the following percent correct by item: 17 x 12 (88%), 45 x 26 (80%), 37 x 24 (80%), and 92 x 78 (64%). Many of the errors were similar to those made by the urban students: near, but faulty, products of single-digit numbers and faulty place value when multiplying multiples of 10. We note that these results compare favorably with the international comparison of fifth-grade students cited above.

In analyzing students' understandings of the area representations and connections to the computation method, we found that some low-achieving students (mostly urban) began with a shaky understanding of basic geometry concepts. A number of urban students did not understand that opposite sides of rectangles have the same length, and some urban and suburban students were confused by the distinction between area and perimeter. By the end of the unit, both teachers reported that such students had developed a much better understanding of these properties and concepts.

When analyzing the extent to which students used the area representations as supports for multiplication methods, we found that many could use the rectangles to find products, but that this became harder as the numbers got larger. By the end of the unit, many students were using at least two methods in class, because they learned the traditional algorithm at home. In such cases, we found that students could only explain why the expanded algorithm worked. We also found cases in which students' strategies were closely tied to particular features of the area representations. For example, some students could count unit squares along edges in figure 1 to determine correct sub-products, but could not use figure 3 to determine correct sub-products. High-achieving students at both schools could articulate connections among all four representations.

Analyses of our data suggest several refinements that should make the unit more accessible to low-achieving students. One set of refinements revolve around the sequence of activities. Introducing the unit with activities to insure that students understand basic properties of rectangles and placing greater emphasis early in the unit on problems of the form 30 x 20 = 60 should reduce many of the errors that we saw during our first implementation. A sec nd set of refinements revolve around the design of the representations. Students seemed to have a better grasp of the connections between figure 1 and figure 4 than between figure 3 and figure 4. Eliminating figure 2 and redesigning figure 3 so that it contains unit squares along the top and left hand edges of each region should help students connect initial to subsequent methods for computing products of two-digit numbers.

Conclusions

We are taking steps toward curricula that provide all students the opportunity to achieve at those levels articulated by the National Council of Teachers of Mathematics. The results of our pilot study suggest that diverse students, including inner-city students, can master two-digit multiplication using a modeling approach if activities and representations are carefully designed and students are expected to understand and explain their computational methods.

Note

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SUPPORTS FOR LEARNING MULTI-DIGIT ADDITION AND SUBTRACTION: A STUDY OF TAIWANESE SECOND-GRADE LOW-MATH ACHIEVERS

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Abstract: Taiwanese second-graders were randomly chosen from ten classes and three schools in Taiwan and tested to compose a low-math-achiever (LMA) and high-math-achiever (HMA) group (each n =37). All children were given an IQ test (Raven), a place-value task, a written test of 2- and 3-digit addition and subtraction problems, and a strategy interview of mental math problems. LMAs compared to HMAs had a less mature place-value concept, used less sophisticated and slower procedures in solving the addition but not the subtraction mental math problems, and performed more poorly on the multi-digit addition but not subtraction problems. The base-10-structured methods were favored by both groups when interviewed about their strategies. A "Linguistic and Visual Support" Model was proposed to explain the unusual findings of subtraction easier than addition.

Rationale

In the past two decades, Asian students have been the "winners" of various international studies in math. These results sometimes lead to the stereotype that all Asian students are math wizards who do not need to struggle with math. However, little is known about Asian low-math achievers. The results of most current cross-national studies have failed to explain why Asian low-math achievers do not do as well as their normally-achieving peers, given the facts as most cross-cultural studies have suggested (e.g., Stevenson & Stigler, 1992) that they all learn from a more centralized math curriculum, use a more regular number-word system, and live in a society where effort is stressed more and parental supports are more available to a child's education. This study initiates an examination of characteristics of Chinese low-math achievers.

The theoretical framework was based on a Vygotskiian socio-cultural perspective (1978, 1986) which postulates that the formation of minds requires study of the sociocultural setting in which activities take place. Thus, solution strategies were examined to ascertain how well all children could use the semiotic tools used in their culture in the context of math problem solving: the regular Chinese number words that name the ten (e.g., 12 is said as "ten two" and 32 is said as "three ten two") and the 10-structured methods of adding and subtracting taught in the classroom. "Make-a-ten" methods were predominantly used in the class to solve addition problems. These methods varied in which number made a 10: making the big number to 10 [e.g., $7+8 = (8+2)+5=15$] or making the small number to 10 [e.g., $7+8 =5+(3+7)=15$]. For subtraction problems, the "make-a-ten" method was taught in class. This method splits the teen number into ten and some [e.g., for 15-8: ten five (15) = ten and five, and the 8 is taken from the 10 leaving 2 (many students just know 10-partners of all numbers) which is combined with 5, the other part of 15, to make 7]. Korean students (Fuson & Kwon, 1992a) use two related methods that involve going up over ten or down over ten. Other methods that have been proposed by American researchers (e.g., Fuson, 1988) such as doubles, or counting on were not emphasized in the instruction either.

Methods and Data Sources

Second-grade Chinese children from three schools in Taiwan chosen to span a range of typical schools (one in a city and two in rural areas) were followed throughout their second-grade year. The subjects were randomly chosen from ten classes randomly chosen from the three schools. Children from the top and bottom 10% of each class based on a composite score of their math screening test given in the beginning of second-grade and their first-grade math GPA made up the high-math-achiever (HMA: n=37) and low-math-achiever (LMA: n=37) groups. A nonverbal IQ test (Raven) was used to choose children with IQs in a normal range. Children included in this study were also required to have no hearing or visual impairment and no emotional problems. This information was gathered from students' psychological reports and class teachers' reports.

The math tasks included a place-value task (the Kamii task), a written test of 2- and 3-digit addition and subtraction problems (for details, see Fuson & Kwon, 1992b), and an interview about their strategies for solving two single-digit mental math problems (8+7, 14-6). The Kamii task required children to show how many objects the number "1" means in 16. Students were given the 3-digit problems before they had studied the topic in school. The goal was to examine whether they could transfer their understanding and procedures in solving 2-digit to 3-digit problems. The 2-digit addition problems required a trade from the ones (27+57 and 54+19), and the 3-digit problems required a trade from the tens (571+293 and 284+681); the subtraction problems were the inverse of the addition problems.

Results

Fewer LMAs than HMAs had a solid understanding of place value. In the fall, only 26 (70%) LMAs answered that "1" in 16 was worth "ten" instead of one, whereas 36 (97%) HMAs answered correctly without a prompt. Five LMAs needed further prompts by the researcher ("Is this a 10 or a 1?") to be able to come up with the right answer. However, six LMAs were too adamant to change their answer even after the prompt. In the winter, still one LMA answered incorrectly and two LMAs needed a prompt, while all HMAs answered correctly.

LMAs were as accurate as HMAs in solving mental math problems with totals in the teens when there was no time constraint. However, more LMAs adopted a less efficient strategy to solve the problems, such as finger counting or "counting on" methods. For the addition problem (8+7), there was a significant difference in the strategy use for the two groups (see Table 1).

Table 1. Frequencies of Use of Different Strategies for Solving 8+7.

Category	I			II				III		I	II	III
Methods	1	2	3	4	5	6	7	8	9	Subtotal 1**	Subtotal 2-6**	Subtotal 7-9**
HMAs N= 37	7	16	4	4	1	2	3	0	0	7	27	3
LMAs N= 37	2	15	0	0	0	1	16	2	1	2	16	19
Total	9	31	5	3	1	3	19	2	1	9	43	22

Notes:

Category I means "Automatic Procedures". 1) direct retrieval

Category II means "More Sophisticated and Faster Procedures" 2) making the big number ten; 3) making the small number ten; 4) Chinese imaginary abacus method; 5) finger abacus method, 6) doubles

Category III means "Less Sophisticated and Slower Procedures" 7) finger counting on from the big number, 8) finger counting on from the small number, 9) counting all.

** means a significant difference between HMAs and LMAs on a χ^2 test at $p < 0.01$.

Nineteen (26%) LMAs, but only three (4%) HMAs used the less sophisticated and slower procedures, especially counting. More HMAs than LMAs (27 vs. 16) adopted more sophisticated and faster procedures (many used the "make-a-ten" methods). Methods 4 to 6 were not emphasized in class and more HMAs than LMAs used these methods (7 vs. 1). Additionally, fewer LMAs than HMAs rapidly retrieved the answer rather than using a solution method (2 vs. 7). However, the groups did not show significant differences in solving the subtraction problem (14-6) (see Table 2). Interestingly, most students in both groups commonly used the "make-a-ten" method to solve this problem. For both 2-digit and 3-digit problems, LMAs did significantly worse than the HMAs in the addition but not in subtraction problems (see Table 3).

Discussion

First, Chinese LMAs' place-value concept was less mature and took longer to develop than did that of the HMAs. Although they did read any 2-digit numbers with the Chinese words "ten something" (42 as "four ten two"), the verbal label "ten" did not necessarily create an automatic "magic" for all Chinese children to be aware of the meaning of the word in relation to the place value of that number and understand the number sense (e.g., embeddedness of number relations). The result supports

Table 2. Frequencies of Use of Different Strategies for Solving 14-6.

Category	I			II		III	IV	I	II	III	IV
Methods	1	2	3	4	5	6	7	Subtotal 1	Subtotal 2-5	Subtotal 6	**Subtotal 7**
HMAs N= 37	0	27	3	4	2	1	**0**	0	36	1	**0**
LMAs N= 37	1	30	4	0	0	1	1	1	34	1	**1**
Subtotal	1	57	7	4	2	2	1	1	70	2	**1**

Notes

Category I means "Automatic Procedures" 1) direct retrieval

Category II means "More Sophisticated and Faster Procedures" 2) up over ten, 3) down to ten, 4) Chinese imaginary abacus method, 5) finger abacus method

Category III means "Less Sophisticated and Slower Procedures" 6) counting down.

Category IV means "Others" 7) don't know

Table 3. Percentage Correct for Written Multi-Digit Problems for the Two Achievement Groups.

Addition	HMAs	LMAs	Subtraction	HMAs	LMAs
26+57*	100 (0)	87 (35)	83-57	97 (16)	89 (32)
54+19#	100 (0)	89 (32)	73-19	97 (16)	92 (28)
571+293**	97 (16)	76 (44)	864-571	89 (32)	78 (42)
284+681**	95 (92)	73 (45)	965-284	87 (35)	76 (44)
Total**	98 (9)	81 (31)	Total	93 (20)	84 (32)

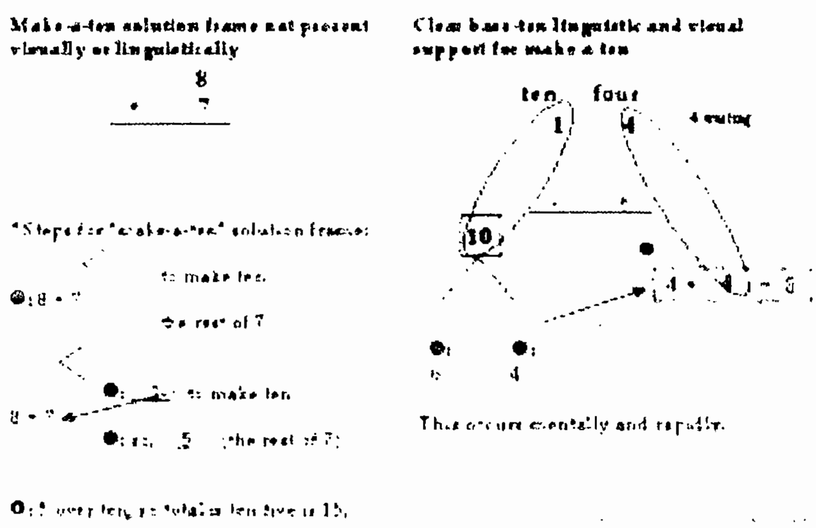
Table 3 Notes: # means a marginal difference on a Fisher's exact test at $0.05 < p < 0.10$; *means a significant difference on a t-test at $p < 0.05$;
** $p < 0.01$. Numbers are in percentage and the numbers in the parentheses are standard deviations.

the notion proposed by Vygotsky that psychological functioning occurs first in the interpersonal level and then gradually shifts to the intra-personal state. Although the Chinese number words correspond well with the Arabic number system, it is only through the enculturation process that Chinese children gradually come to understand the link between the language (number words) spoken and the object meaning of the base-10 place value of the number system used in the society. It is a gradual process, and LMAs seem to need more time than their high-achieving peers to develop this concept. Thus, the gap Ho and Fuson (1998) identified in Chinese kindergarten children between those who understood the ten in teen numbers and those who did not is not closed by grade 2. This suggests that LMAs may need to be provided more and longer explicit teaching before their place-value concept is consolidated.

The subtraction-superior-performance findings seem to contradict the common assumption that subtraction is more difficult and error-prone (e.g., Fuson, 1984). Based on the analyses of strategy data, I propose a "linguistic and visual support" model to justify this unusual finding (see Table 4). In the process of solving the subtraction problem, the base-10 structure is explicitly accessible via both visual and linguistic supports. In 14-6, the thinking procedure would be: ten minus six, four; four plus four, eight. "Ten" is first seen within and read for the ten four (14), that then sustains and connects linguistically and visually with the make-a-ten method. However, such supports are less transparent in the addition problem. If 8+7 is solved by a "ma. e-a-ten" method, the procedure will be as: eight plus two, ten; ten plus five, ten five. The base-10 linguistic support is less clear (it must be generated during the solution) along with no visual base-10 support in this case.

The educational implications of this study are as follows. First, given adequate supports, especially in both linguistic and visual domains, it is possible for Chinese LMAs to do as well as their HMA peers. This finding is encouraging because it suggests that given the right scaffolding, children can learn to SEE and HEAR and

Table 4. Contrast of Base-Ten Linguistic and Visual Support for Problems of 8 + 7 and 14 - 6.



BE AWARE of patterns, structures, and relationships, and then use these as tools to solve problems. This echoes the *Principles and Standards for School Mathematics* of NCTM (2000) for how to learn and teach children mathematics. Second, following this paradigm, the question of whether children use a less regular number system or not seems not as important as asking the question: "How do we use (linguistic) support to help our children learn and do math?" Finally, as Vygotskiian theory indicates, "...we are empowered as well as constrained in specific ways by the mediational means of a sociocultural setting" (Wertsch, 1992, p.42). Thus, teachers should be aware of potential means of support when teaching children in math or any subject.

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THE ROLE OF THE FORMAT OF ARITHMETICAL TASKS IN CLASSROOM INTERACTIONS

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Abstract: The analysis of two students' solutions of simple addition/subtraction tasks presented in different formats indicates that the students became more aware of their mental actions and the meaning of the standard algorithms.

This paper analyzes the mathematical activity and classroom interactions of two students when they solved addition/subtraction tasks presented in different formats. The first section lays out the theoretical rationale for the analysis of the classroom mathematical activity in terms of the actions and interactions of the teacher and the students. The second briefly describes the methodology of a year-long teaching experiment with fourth graders. The third presents teaching episodes that illustrate the mathematical actions and interactions among the students and the teacher in the context of addition and subtraction tasks.

Theoretical Rationale

One of the goals of the reform movement (National Council of Teachers of Mathematics [NCTM], 1991, 2000) is to promote a shift from a *passive* teaching-learning paradigm to an *active* one. This shift views the classroom mathematical activity in terms of the students' and teacher's self-awareness of their mathematical actions and interactions. Furthermore, the students' conceptual development is viewed as the result of a continual and orchestrated collaboration among the students and the teacher. In this process, the teacher needs to take into account, among other things, the students' current understanding of mathematics, the nature of the instructional tasks, the nature of the teachers' questions, the format of the tasks, and what is expected from the classroom mathematical activity (Cobb, Yackel, and Wood, 1992; Gravemeijer, 1994). When these elements are coordinated, they sustain a classroom discourse that is based on understanding and consensual collaboration.

Before we continue, let us clarify the sense in which the words actions, interactions, and classroom activity are used in this paper. Actions are considered to be all acts made by one person that may or may not have the potential of being shared with others. Interactions are considered to be all the personal acts that are shared with others. Finally, classroom activity is considered to comprise all actions and interactions of the participants.

Methodology

Teaching Experiment

The teaching experiment consisted of daily teaching episodes in a fourth grade classroom. These episodes were characterized by the teacher-student and student-student discussion of students' solutions to arithmetical tasks. Tasks were presented verbally or on paper and in each episode the teacher tried to infer and interpret the students' mathematical actions so that the interactions among the classroom participants could be sustained.

Data Collection

Daily lessons were videotaped and field notes were kept. In addition to maintaining records of the mathematical activity of the students, task pages and scrap papers were also collected.

Format- and Diagram-Mediated Interactions

In this section, we analyze the students' re-conceptualization of whole numbers in terms of different units of ten and how this new way to "see" numbers influenced the generation of mental strategies that provided a better understanding of the standard algorithms.

At the beginning of the school year, the teacher observed that the students could read three-digit numbers but had no sense of the relationship between different units of ten. For example, students could interpret 678 only as 6 hundreds, 7 tens, and 8 ones but they could not conceptualize it in any other way. Conceptualizing 678, for example, as 67 tens and 8 ones, or as 6 hundreds and 78 ones was a foreign idea for the students. Further, it was observed that the students carried out numerical computations only by following the standard algorithms in a rote manner.

To provide students with opportunities to develop a better understanding of the standard algorithms, the teacher began to emphasize mental computation. To avoid suggesting mental habits already formed by rote memorization of the algorithms, some of the tasks purposely displayed the numerals in unconventional ways. For example, in matrices or on the surface of plane geometric figures.

It was also observed that when students were presented with paper-and-pencil tasks, they were inclined to compute using only the standard algorithms as computational devices. But when tasks were presented verbally, students solved them mentally by using novel strategies for decomposing numbers into different units.

Let us first introduce a classroom episode that indicates students' initial adherence to the standard addition algorithm as the only means for adding numbers. Students were asked to mentally add 159 and 199. The task was intended to help students develop mental strategies for adding. After discussing possible ways for rounding these numbers to the nearest ten and hundred to estimate the sum, the teacher asked the students to find the sum mentally. Attempting to give students a hint, the teacher asked, "Could we start adding the hundreds?" To this question one of the students promptly responded, "You could, but it would screw up the whole problem." This answer indicates that this student was relying on the standard algorithm for addition. This algorithm starts every addition by first adding the digits indicating the units of one, then adding the digits indicating the units of ten, and so on, in strict sequential order from right to left. In the absence of paper and pencil, some students carried out the algorithm by arranging the numerals vertically in their minds while gesturing in the air with their fingers as if they were using the standard algorithm on paper.

Because of daily emphasis on mental computation, students began to decompose numbers into units that served their self-scripted goals to add and subtract. The diversity of solutions presented by the students to the class allowed the teacher to orchestrate discussions about computational strategies. For example, when the teacher asked the students to mentally add the numbers 759 and 684, Preston, the same child who weeks before argued that starting with the hundreds "would screw up the whole problem", took a different approach. He explained his mental strategy at the same time that he drew Figure 1 on the board. The letters *Th*, *H*, *T*, and *O* stand for thousands, hundreds, tens, and ones, respectively. This notation was collectively selected and commonly used by the students to facilitate the explanation of mental computations in terms of different units of ten.

Preston: Well, you take 7 hundreds and 6 hundreds, add them; that would be 13.

Teacher: 13 what?

Preston: 13 hundreds.

Teacher: Okay.

Preston: And you take 5 hundreds, and 8 hundreds.

Teacher: 5 what and 8 what?

Preston: Oh, yeah! 5 tens and 8 tens.

Teacher: Okay.

Preston: That would be 13 tens. Then, take 9 tens with 4 tens ... 9 ones with 4 ones [correcting himself], that would be 9...10-11-12-13. Yeah! 13. Hey! 13 hundreds, 13 tens, and 13 ones.

Teacher: Magic! 13 of everything! Now that you have 13 of everything, what do you have to do?

Preston: You take 10 hundreds and make 1 thousand and have 3 hundreds left. You take 10 tens and make 1 hundred and have 3 tens left. You add 1 hundred to the 3 hundreds to make 4 hundreds. Then, you put together the 3 tens and 1 ten to make 4 tens. Then, you have left only 3 ones. So, we have 1 thousand, 4 hundreds, 4 tens, and 3 ones. That is one thousand four hundred forty-three.

Both the diagram and the dialogue indicate that it was not difficult for Preston to begin adding first the hundreds and then the other units of ten in a decreasing order. His explanation gives us an illustration of the influence of both the re-conceptualization of numbers into different units and the verbal format of the tasks on the mathematical activity of the students.

When the teacher posed the addition problem to be solved mentally, Preston reciprocated her action by devising a mental strategy that was based on both the decomposition of the two numbers and the use of a numerical diagram. It is difficult to infer whether the diagram was the result of Preston's progressive tinkering with the numbers while building it up or whether it was simply a representation of his elaborated mental strategy. The conceptual understanding manifested in both Preston's diagram and dialogue with the teacher indicates a drastic conceptual shift in how he thought about what he did when he added whole numbers.

It is worth noting that not all the students shifted their ways of thinking about numbers at the same rate. Nonetheless, all the students started to generate and imitate

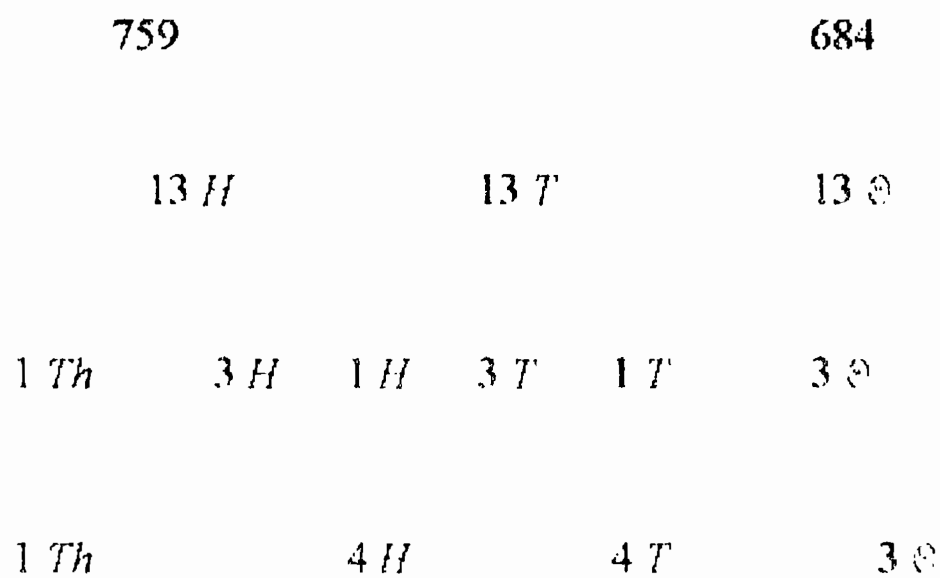

Figure 1. Preston's unit-decomposition strategy to add two numbers

novel solutions like Preston's and soon they started to use diagrams to generate results or to explain their strategies. It is also worth noting that these diagrams, as well as other kinds of symbolizing, were widely used by the students while interacting with the teacher and the other students in the class (Sáenz-Ludlow, 1995, 1996, 1998).

The teacher also presented another set of tasks designed to increase the students' understanding of the mathematical processes that are built into the computation contained in the standard algorithms for addition and subtraction. Students were asked to express in writing the strategies they used. Figure 2 shows the general format in which these responses were obtained.

XXX ± YYY

Estimate the answer	Show a strategy to find the exact answer	Show another strategy to find the exact answer

Verification!

Does your answer make sense?

Figure 2. General format for addition and subtraction tasks

The following episode indicates how the teacher's actions to modify the format of simple subtraction tasks influenced the students' actions. Again, the students' solutions were mediated by the emergence of numerical diagrams.

Figure 3 shows Hollis' actions to find the difference between 374 and 92. These actions were manifested by what he did to symbolize his strategies. The continuous lines are part of Hollis' solution. The curved dotted lines are introduced to indicate the direction in which he moved his fingers when explaining his solution, and the horizontal dotted arrows are introduced to indicate the direction he followed when operating with the numbers.

Hollis: Let me show you what I did.

Teacher: Okay.

Hollis: First I made 300 out of 374, and I made 100 out of 92, and I know that 300 take away 100 is 200.

Teacher: Okay.

Hollis: Here I took 374 and made 2 hundreds, 17 tens, and 4 ones; and I took 92 as 0 hundreds, 9 tens, and 2 ones. Then, I did this, 2 hundreds take away 0 hundred is 2 hundreds; 17 tens take away 9 tens is 10-11-12-13-14-15-16-17 [showing eight fingers] 8 tens; 4 ones take away 2 ones is 2 ones.

Teacher: Okay. What else did you do?

Hollis: Oh, yeah! From 4 ones I took away 2 ones and that is 2. From 7 tens I cannot take away 9 tens, so I go to the 3 and borrow 1. [He moves his fingers from one digit to the other.]

Teacher: [interrupting] go to the 3 what and borrow 1 what?

Hollis: Well 3..., 3 hundreds and borrow 1 hundred, and I know that 1 hundred is 10 tens, and 10 tens plus 7 tens is 17 tens. 17 tens take away 9 tens is 8 tens. And 2 hundreds take away 0 hundreds is 2 hundreds. I have left two hundred eighty-two.

Teacher: How do you know that 282 is the correct answer?

Hollis: I added 282 and 92, I learned that last year. You see here 2 ones and 2 ones is 4 ones, 8 tens and 9 tens is 17 tens, 7 tens and carry 1 hundred. 2 hundreds and 1 hundred is 3 hundreds. That is three hundred seventy-four.

The format of the task and Hollis' diagram along with his gesture-mediated actions indicate several aspects of his thinking. First, in both cases, he rounded the numbers to hundreds but not necessarily to the nearest hundred. Second, he was willing to generate a strategy by positioning the numbers horizontally and then decomposing the number 374 into 2 hundreds, 17 tens, and 4 ones. Third, he clearly understood 92 as 0 hundreds, 9 tens, and 2 ones. This decomposition allowed him to find the subtraction by operating first with the hundred units, then with the ten units, and finally with the one units at the same time that he generated a numerical diagram. Fourth, he resorted to the vertical algorithm for subtraction as another way of subtracting the numbers, thereby following the convention of "borrowing" and operating with the digits from right to left. His new conceptualization of the algorithm was indicated when he explicitly referred to different units of ten in his explanation. Fifth, Hollis reconstructed the initial number (374) by adding the difference between the two given numbers (282) to the number subtracted (092). In so doing, he not only

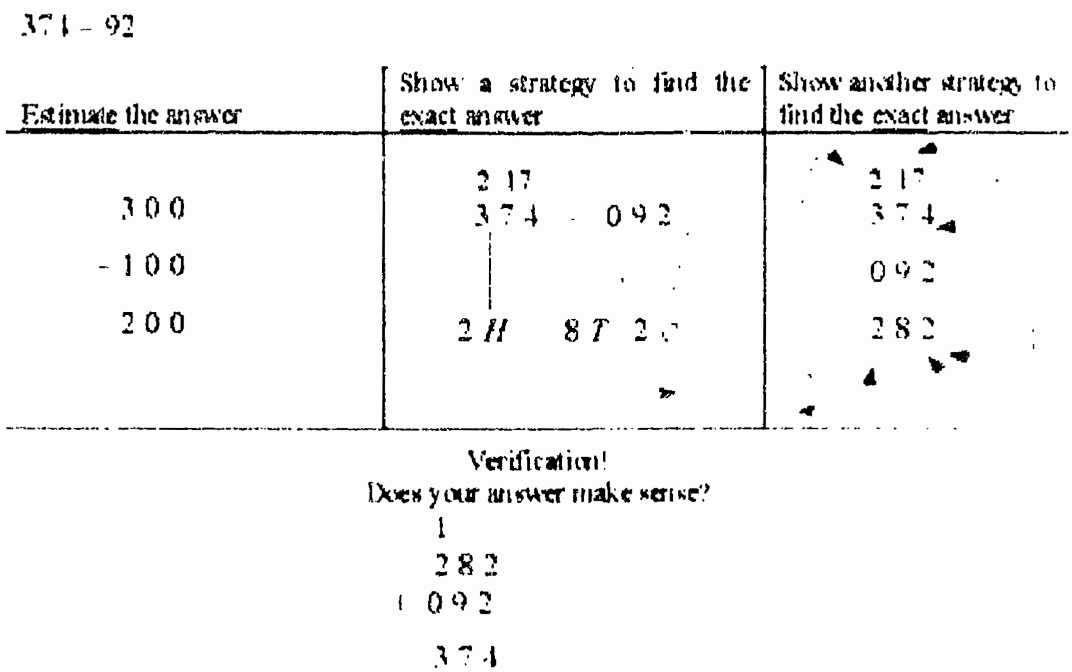

Figure 3. Hollis' strategies to subtract two numbers and to verify the result

used the standard addition algorithm but he also explicitly used different units of ten along with using the convention of "carrying." It can be said that Figure 3 indicates Hollis' modified understanding of the subtraction and addition algorithms.

In summary, the students' cognitive actions seemed to depend, in part, on the pedagogical actions of the teacher to present tasks in different formats. The teacher's actions were product of her ongoing interpretation of students' re-conceptualizations of whole numbers. The strategies used by Preston and Hollis reflected the mental actions that reciprocated the pedagogical actions of the teacher to present tasks in different formats. The students also generated numerical diagrams that mediated the communication of their solutions to the class. It is argued here that the interactions sustained among the students and the teacher fostered the numerical activity of the students. This argument is supported by the fact that the students did not show this type of creativity at the beginning of the school year when they used only the standard algorithms in a rote manner. What is indicated in the analysis of the solutions of these two students is a conceptual shift in their conceptualization of numbers and a re-organization of their understanding of the standard algorithms for addition and subtraction.

Note

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MULTIPLICATIVE REASONING

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Although many researchers agree that multiplicative concepts are difficult for students to understand (Clark & Kamii, 1996; Vergnaud, 1983), very little research has been done to determine how to help students overcome these difficulties. Two papers on multiplicative reasoning will be presented by the group leaders. The group participants will then have an opportunity to discuss these studies, as well as share their own experiences and insights on multiplicative reasoning.

The theoretical framework for these studies is based on a constructivist view of learning. Children's cognitive development is interpreted using Piaget's concept of schema, assimilation, and accommodation, as well as Vygotsky's concept of the zone of proximal development. According to Resnick, children who do well in mathematics are able to make the connection to real life; whereas, "weaker mathematics learners seem very prone to allow mathematical symbols to become disassociated from their quantity and situational referents. . . . [they] try to memorize and apply the rules that are taught, but do not attempt to relate these rules to what they know about mathematics at a more intuitive level." (1986, p. 191)

Greer (1992) lists ten different multiplication contexts. They are equal groups, equal measures, rate, measure conversion, multiplicative comparison, part/whole, multiplicative change, cartesian product, rectangular area, and product of measures. However, many multiplicative situations cannot be described by this model, e.g., cartesian product. In order for students to be solid multiplicative thinkers, it is important that they understand all of the models of multiplication.

The first study was conducted with an intact sixth grade general math class. Students were interviewed individually and were asked to perform seven tasks. The tasks were model multiplication sentences using 24 chips, write a multiplication word problem, give a definition of multiplication, draw a "picture" to explain multiplication, use chips to model multiplication, and replication of Clark & Kamii's (1996) fish task.

In general, the authors found that the students, although proficient in their knowledge of the basic facts of multiplication, were unable to place multiplication in context. Less than 40% of the sixth graders were able to generate a problem which was multiplicative in context. However, most of the students used multiplication to solve the problems they generated.

The second study was conducted with an intact seventh grade remedial mathematics class and was designed to correct the misconceptions that students have about multiplication. The teaching experiment was focused on the different models of multiplication, and on students' understanding of these models. The students were interviewed prior to the teaching experiment and were asked to complete six tasks, similar to those from the first study. The purpose of the teaching experiment was to present multiplication in context and from a different perspective. Each lesson in the teaching experiment was focused on a different model of multiplication. After the experiment was completed, the students were re-interviewed to determine if and how their ideas about multiplication had changed.

The main implication of these studies is that the method of teaching multiplication must change to look not only at content, but also at the pedagogy of instruction. Use of concrete materials at a younger age and for a longer period of time will help the children become familiar with the representations of many of the models of multiplication. Furthermore, multiplication should be taught in context so that the facts and algorithms are not disconnected from the meaning of multiplication.

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INTEGERS VERSUS FRACTIONS: A STUDY WITH EIGHTH GRADE STUDENTS

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In this research (Novoa, 2000), integers are examined in relation to fractions' numerical domain, as they are taught at school. The study is based on Filloy's (1990) methodological-theoretical framework through the development and integration of three components: models of teaching, models of cognitive processes and models of formal competence. We report on results derived from an experience involving 41 eighth grade students who answered an exploratory questionnaire. Nine students were selected to participate in individual clinical interviews.

Study's Evidence. When localizing fractions on the number line, some misinterpretations found were an integer plus a decimal fraction, or an integer plus a unitary fraction. Many students order the negative integers on the basis of their absolute value, and when ordering unitary fractions they only consider the denominators. Some of them compare fractions taking only the numerators into account or they identify negative integers as positive unitary fractions having such numbers as denominators. In operations with integers, most students apply the law of signs (minus times minus becomes plus). Fixation on this law hinders subtraction operation. When doing a subtraction, the students can become aware of the order of integers. Nevertheless, when operating with fractions they focus attention on algorithms and miss which fraction is larger. In regard to problem solving, the students build their own syntax altering the algorithm for subtraction.

Study's Implications. It is important that students realize that they are working with three different numerical domains: natural numbers, integers and fractions. They must learn to add and subtract integers before the law of signs is introduced.

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AN ADDITIONAL EXPLANATION FOR PRODUCTION DEFICIENCIES

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Counting out or producing a specified number of objects is a relatively difficult counting task. A common error among young children is a failure to stop the counting process after reaching the specified number (the target). Such non-stop errors have been attributed to two types of memory failures, namely a failure (a) to register or remember the target or (b) to stop counting at the designated target because of a working-memory overload (Resnick & Ford, 1981). However, Baroody (1987) found that production errors occurred even when children could remember the target and hypothesized that a conceptual deficit might be a cause of such errors. Specifically, he argued that production requires what Fuson (1992) calls the "cardinal-count concept" (e.g., understanding that the cardinal term "five" predicts or is equivalent to the outcome of counting a set of five objects).

A case study (Madison, age 2 years, 11 months) was conducted to examine the possibility that production errors are also due to a conceptual deficit. Production errors made by Madison did not appear to be the result of memory failures. That is, testing indicated that he could both register and retain the target and use it to stop a verbal count. However, he exhibited--at best--a shaky understanding of the cardinal-count concept.

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In the proceedings the authors works are assembled by topic area. The codes in the parentheses indicate the topics or types of work: pl-Plenary, wg-Working Group, AMT-Advanced Mathematical Thinking, AT-Algebraic Thinking, A-Assessment, D-Discourse, E-Epistemology, FG-Functions and Graphs, GT-Geometric Thinking, PR-Probability and Statistics, PS-Problem Solving, RN-Rational Number, RM-Research Methods, SC-Social and Cultural Factors, TB-Teacher Beliefs, TE-Teacher Education, TK-Teacher Knowledge, T-Technology, WN-Whole Number

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