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ABSTRACT

This paper asserts that the key to algebra reform is to integrate algebraic reasoning across all grades and all topics, to "algebrafy" school mathematics. The distinction is made between algebra "the institution" and algebra "the web of knowledge and skill," which is also clarified. Finally, suggestions are made as to how the educational community might work towards a genuine algebra for all. The appendix provides concrete, classroom-based illustrations of the different aspects of algebra at the elementary grade level. (Contains 11 references.) (CCM)

# Transforming Algebra from an Engine of Inequity to an Engine of Mathematical Power By "Algebrafying" the K-12 Curriculum[1]

**James J. Kaput, University of Massachusetts-Dartmouth**

We begin with two assumptions. First, just as algebra has acted as a constricted gateway to significant mathematics and all that follows from mastery of that mathematics, algebra *reform* is the gateway to K-12 *mathematics reform* for the next century. Second, by acknowledging the several different aspects of algebra and their roots in younger children's mathematical activity, a deeply reformed algebra is not only possible, but very achievable within our current capacity for change.

The key to algebra reform is integrating algebraic reasoning across all grades and all topics - to "algebrafy" school mathematics. This integration solves three major problems:

1. It opens curricular space for 21st century mathematics desperately needed at the secondary level, space locked up by the 19th century high school national curriculum now in place.
2. It adds a new level of coherence, depth, and power to school mathematics, both as curriculum and as a habit of mind.
3. It eliminates the most pernicious curricular element of today's school mathematics - late, abrupt, isolated and superficial high school algebra courses.

A strands approach to algebra that begins early also fits well with an inclusive, big-idea strands oriented approach to the curriculum at large, contrasting with the layer cake-filter structure that delays and ultimately denies access to powerful ideas for all but the few. An algebrafied K-12 curriculum helps democratize access to powerful ideas.

Our discussions of algebra must be as honest and clear as possible. To this end, it helps to distinguish *Algebra the Institution* from *Algebra the Web of Knowledge and Skill* that we want students to develop in school, so that criticisms of the former are not heard as statements about the latter. Algebra the Institution is a peculiarly American enterprise embodying the standard courses, textbooks, tests, remediation industry, and their associated economic arrangements, as well as the supporting intellectual and social infrastructure of course and workplace prerequisites, cultural expectations relating success in algebra to intellectual ability and academic promise, special interests, relations between levels of schooling, and so on. Exhortation for and legislation of Algebra For All tacitly assume the viability and legitimacy of this Institution. But

***this* algebra is the disease for which it purports to be the cure!**

It alienates even nominally successful students from genuine mathematical experience, prevents real reform, and acts as an engine of inequity for egregiously many students, especially those who are the less advantaged of our society.

Our challenge is to create an implementable alternative to this inimical Institution, to transform an engine of inequity to an engine of mathematical power. This paper will first contextualize our situation historically, second, clarify what we mean by Algebra the Web of Knowledge and Skill - what is algebra? - and third suggest how we might work towards a genuine algebra for all. An Appendix provides concrete, classroom based illustrations of the different aspects of algebra at the elementary grade level.

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## Much More Math for Many More People

We sometimes lose sight of the magnitude of the changes that are taking place across the generations. Centuries-long trends, both in the evolution of the subject matter and in the demographics of those who need to learn that subject matter are not only continuing, they are accelerating. Shop-keeper arithmetic was sufficient for more than 90% of the population until this century. Nowadays, we want 100% of the population to graduate from secondary school of one kind or another, and hear calls for all students to obtain at least a two year college education (although we have difficulty raising actual secondary graduation rates above 75%). Contrast these expectations with the fact that **the percentage of the 17-18 year old population cohort taking AP calculus today (about 3.5%) equals the percentage of the population graduating from high school a century ago!** So, to contextualize the contemporary calls for "Algebra for All" we ask what percent of the population was expected to learn algebra a century ago? Perhaps 2% - mainly boys who were socially, economically, culturally and ethnically very similar to their teachers. And we further ask if either the curricula, the texts, or the pedagogies have changed in a way that might deliver algebra to a mightily more diverse 98% who were previously absolved of algebra learning? (We are assuming all students are still in school in 8th or 9th grade.) The answer to these questions is plainly NO! And this is one reason for the current attention to algebra reform.

Another reason for deep algebra reform has to do with the future. We have every reason to believe that the mathematics of the next century will be more different from today's than today's is different from that of the 18th century. The primary reason for this is the emergence of the computer medium, a medium in which new mathematical forms flourish. For example, Dynamic Geometry systems such as the Geometer's Sketchpad and Cabri Geometry are simply not possible in the static inert medium of pencil and paper, but, when coupled with appropriate educational activities, provide entirely new mathematical experiences for the student (as do similar systems for sophisticated users and builders of mathematics). Of more direct relevance is the iterative mathematics supported in the computer medium that takes the form of dynamical systems, for example, and their use in modeling nonlinear phenomena. This mathematics and the nonlinear science that grows with it is exploding in importance as we end the 20th century. Indeed, the nature of science is undergoing a profound transformation, whereby the complex phenomena that were in principle ignored by classical methods are now the focus of intense study across many different sciences - physical sciences, life sciences and social sciences, as well as applications in engineering, economics, and elsewhere. The rapid iterative computation that computers make possible is coupled to highly visual displays of complex data that results, as in the now-familiar fractal and Mandelbrot graphics, for example. Similar things could be said about changes in statistics and probability, as bootstrapping and other resampling methods are rapidly replacing complex and cumbersome traditional formulas, and new interactive visual displays of data are becoming mainstream tools.

A direct implication of these changes is the need for more room at the secondary level for the mathematics for the next century that is already blooming profusely, mainly outside of schools. An accompanying implication is the need to do much more mathematics in K-8. Jointly, these implications suggest that we can no longer afford the inefficient, curriculum and resource gobbling high school (Institutional) algebra courses that dominate curricula and expectations today, and that we must instead integrate a larger, more modern and powerful algebra throughout K-8 mathematics. I will now try to be a bit clearer on what kind of algebra we mean.

## Five Forms of Algebraic Reasoning

In my view, algebraic reasoning is a complex composite of five interrelated forms of reasoning. The first two of these underlie all the others, the next two constitute topic strands in the curriculum, and the last reflects algebra as a web of languages - its linguistic side. All five richly interact conceptually as well as in activity - to understand *this* algebra is to make connections, abstractions and generalizations. All five can and should be started early.

1. (Kernel) Algebra as Generalizing and Formalizing Patterns & Constraints, especially, but not exclusively, Algebra as Generalized Arithmetic Reasoning and Algebra as Generalized Quantitative Reasoning
2. (Kernel) Algebra as Syntactically-Guided Manipulation of Formalisms
3. (Topic-strand) Algebra as the Study of Structures and Systems Abstracted from Computations and Relations
4. (Topic-strand) Algebra as the Study of Functions, Relations, and Joint Variation
5. (Language aspect) Algebra as a Cluster of (a) Modeling and (b) Phenomena-Controlling Languages

Figure 1 is intended to provide an image of how the forms of reasoning overlap and interrelate.

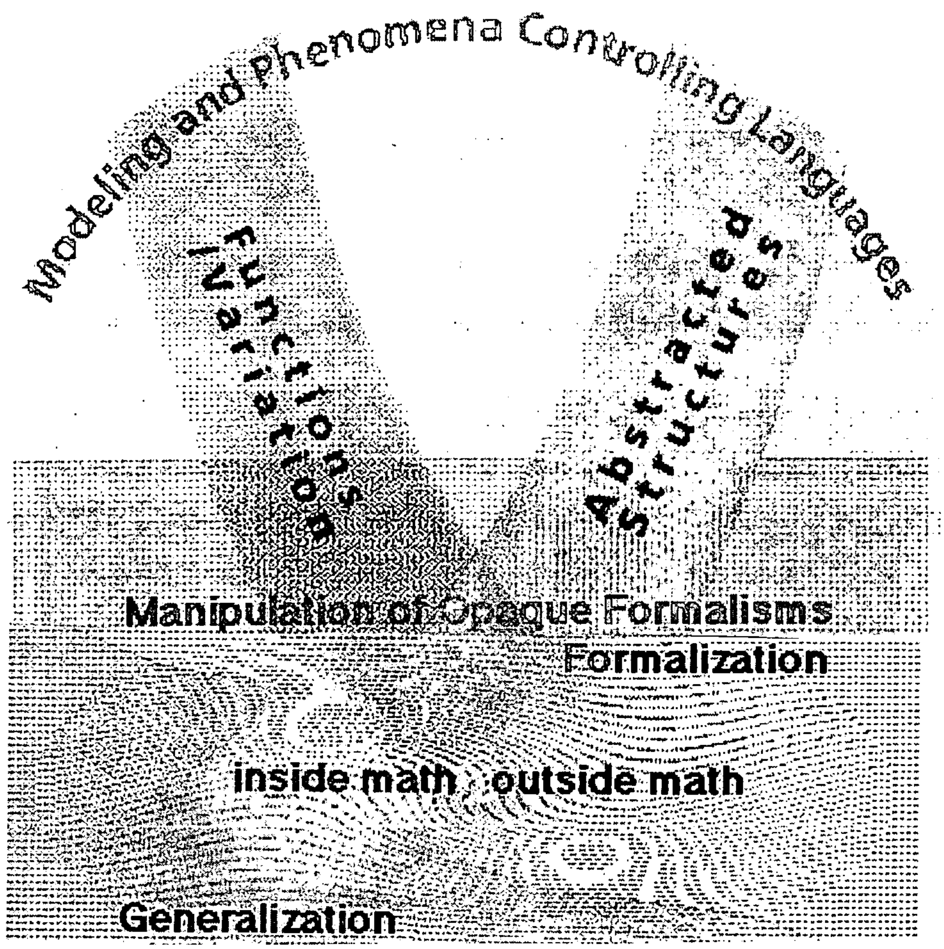



Figure 1: Five Aspects of Algebra

Forms (1) and (2) underlie all the others, with (1) based both within and outside of mathematics, and (2) done in conjunction with (1). It is difficult to point to mathematical activity that does not involve generalizing and formalizing in a central way. It is one of the features of thinking that makes it mathematical. Also, the actions one performs with formalisms identified as (2), the kinds of manipulations that dominate current algebra courses, should typically occur as the result of prior formalizing of situations and phenomena, so that they can be related to those situations and phenomena. And the formalisms may be of many different types, not merely variables over sets of familiar numbers (or transcendentals over some field). Furthermore, it is possible that the manipulation can yield general patterns and structures at another level of generalizing and formalizing - which is the essence of the third, structural, form of algebraic reasoning. In order to use or communicate generalizations, one needs languages in which to express them, which leads to (5).which in turn permeates all the others. While (3) is a school mathematics topic strand occurring nowadays mainly at the advanced levels, it is also an important growing domain of mathematics in its own right - abstract algebra. On the other hand, topic strand (4) functions, is more a school mathematics domain, and lives in the world of mathematics more as a general purpose conceptual tool rather than a branch of mathematics. One hint of the breadth of this school algebra is the fact that (3) and (4) lie on opposite sides of a deep boundary in mathematics separating algebra and analysis. Both appear in school algebra.

Traditional school algebra focuses on (2) at the expense of all the others. And while calls for a functions approach to algebra went ignored for almost a century, some of our contemporaries tend to see (4) as all of school algebra. But the list suggests that algebra is more than functions, although the idea of function is an extremely powerful organizer of mathematical activity across topics and grade levels. But so are all the other forms of algebra listed, which is exactly why algebra can play the key role across K-12 mathematics that I and others suggest. This wider view of algebra emphasizes its deep, but varied, connections with all of mathematics.

## Connections to the Framework's View of Algebra

The analysis of algebra offered here is fully compatible with that offered in the Algebra Framework appearing elsewhere in this document. My call for Integration is another way of emphasizing the role of Contextual Settings offered in the Framework. And there is a close connection between the five forms described above and the four Organizing Themes described in the Framework. The most obvious connections are between my two Topic Strands, Structures and Systems (3), and Functions, Relations and Joint Variation (4), and, respectively, the Themes of Structure and of Functions and Relations. The two Themes of Modeling, and Language and Representation, are embodied in my "Web of Modeling and Phenomena Controlling Languages" (5). I chose to identify two essential aspects, or forms of algebraic reasoning as "kernels" that underlie the rest - Generalizing and Formalizing, and the Manipulation of Formalisms. These are embodied in each of the Framework Themes as well. Given the complexity and richness of algebra both as a tool of thought and as a object of study we should expect differences in descriptions, and perhaps be surprised at the similarity of the two offered here. But, of course, real differences and diversity can be expected to appear in their realization in curriculum and their implementation in the classroom. We can expect and should welcome wide variation in how algebra can be integrated in the K-12 curriculum for all students.

## Algebra Before Acne and the Role of Teachers

The language aspect of algebra supports both early and integrated algebra. Early because people require repeated use of a language over an extended amount of time to become fluent in its use. Indeed, "algebra before acne" is more than a flippant phrase - language learned before puberty is learned without an accent and is deeply integrated with one's patterns of thinking. And algebra learning must be integrated with the learning of other mathematics because, to learn a language people need to use it to express something significant to them, such as the quantitative relationships arising inside mathematics (for example that occur in arithmetic and geometry) and outside mathematics when we use it to model our world.

As the examples in the Appendix indicate, appropriate instructional materials can "seed" each aspect of algebra listed in relatively ordinary elementary mathematical activity. One key is that teachers need to be able to identify and nurture these roots of algebraic reasoning in forms that appear very different from what is deemed "algebra" under the auspices of the Algebra the Institution. For example, generalization is initially expressed using ordinary language, intonation, and gesture rather than through the use of formal symbolism. These identification and nurturing skills require teacher-development focused on student thinking rather than skills with traditional formalisms. The other key is that these beginnings *go somewhere important mathematically,* both in terms of growth in notational competence and in terms of the significance of the big ideas that these notations are used to express, which in turn will require carefully designed classroom materials to help guide the way.

The new approaches will begin in familiar circumstances, but will lead to new tools, unprecedented applications, populations of students traditionally not targeted to learn algebra - introduced by teachers traditionally not educated to teach algebra - neither the old algebra nor some new version. This route involves generalizing and expressing that generality using increasingly formal languages, where the generalizing begins in arithmetic, in modeling situations, in geometry, and in virtually all the mathematics that can or should appear in the elementary grades.

**The next generation of mathematics education reform begins with current reforms and the elementary school teachers and classrooms of today, but its ultimate success depends on success in this algebrafication of school mathematics. Mathematical power as defined in the Standards lies in this direction.**

## Appendix: Classroom-Based Examples of Early Algebraic Reasoning[2]

### Introduction

The following examples are based in actual student work and language, with a few examples of student MIS-understanding based in the traditional curriculum. They focus on the different forms of algebraic reasoning outlined in the paper above, taking illustrations from across many grade levels and mathematical topic areas. Most of the illustrations are adapted from a book being prepared by the author, *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*. We emphasize where we need to go rather than where we are, or have been. One key aspect of the examples below that contrasts with traditional symbol manipulation algebra is the frequent opportunity to reflect on or articulate their knowledge to others as opposed to concentrating on remembering procedures that they can only know as strings of symbols - the intensive study of the last three letters of the alphabet. A second key aspect is the way teachers build on students' naturally occurring linguistic and cognitive powers in ways that put a premium on active sense-making and understanding.

### 1. Algebra as Generalizing and Formalizing Patterns & Constraints

Generalization and formalization are an intrinsic feature of much mathematical activity, and the mathematical systems and situational contexts in which generalization and formalization can be

[2] would like to thank Virginia Bastable, Deborah Schifter, Dolores Strom, Rich Lehrer, Guershon Harel, Margie Pligge, Mary Spence, Cornelia Tierney, and Steve Monk for sharing their vivid examples.

done are everywhere. Indeed, it is difficult to point to mathematical activity that does not involve generalizing and formalizing in a central way. Perhaps pure computational arithmetic of the sort that dominates elementary school mathematics, the kinds of counting and sorting involved in combinatorics, and pure spatial visualization are candidates of mathematical activity that do not emphasize generalizing and formalizing. Also, the actions one performs with formalisms, identified as the second kernel aspect of algebra, are typically not generalizing and formalizing per se, but typically occur as the direct or indirect result of prior formalizing. It is also possible that the manipulation can yield general patterns and structures at another level of generalizing and formalizing - which is the essence of the third, structural, aspect of algebra in our list.

Generalizing involves deliberately extending the range of one's reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where patterns across and relations among cases or situations become the focus, rather than the cases or situations themselves. Appropriately expressed, the patterns, procedures, relations, structures, etc., can become the objects of reasoning or communication. But in order to use or communicate generalizations, one needs languages in which to express them, which, for a young child who does not yet possess a formal language, may mean using spoken natural language. Here, intonation and gesture may be used to communicate the intention that a statement about a particular case be read or heard as representing a general class of statements. In this case identifying the intended generality may require a skilled and attentive ear, the ear of teachers who have had experience in listening carefully to children.

We distinguish two sources of generalization and formalization: (a) reasoning and communicating in mathematics proper, usually beginning in arithmetic, and (b) reasoning and communicating in situations that are based outside mathematics but are subject to mathematization, usually beginning in quantitative reasoning. In a sense, this is a bogus distinction if one believes, as I do, that all mathematics arises from experience and becomes mathematical upon appropriate activity and processing. However, if the starting point for the generalizing and formalizing is in *previously mathematized experience*, then I would argue that it falls in "mathematics proper," whereas, if it starts in a situation experienced as yet-to-be mathematized, then I would say that its source is outside "mathematics proper," and is based in phenomena or situations.

The distinction is especially problematic in the early years, where mathematical activity takes very concrete forms and is often tightly linked to situations that give rise to the mathematical activity. Nonetheless, even here a distinction seems worthwhile. A student who is generalizing patterns in sequences of numbers in a hundreds table or multiplication table is working with objects and relations already conceived as mathematical. On the other hand, consider a student who is comparing differences in prices between cashews (expensive) and peanuts (cheap) for two different brands, A and B. If the A-brand cashew-peanut difference is bigger than the B-brand difference, and she claims that a small increase in the price of B-brand peanuts will not change the outcome of the comparison, then I would regard her as generalizing from her conception of the *situation* rather than from within mathematics proper. Later, she might model the same situation using algebraic differences and inequalities, writing something analogous to

$$A\text{-}a > B\text{-}b \text{ implies } A\text{-}a > B\text{-}b\text{+}x \text{ if } x>0 \text{ (and perhaps } x<B\text{-}b).$$

In this case, she would be reasoning within mathematics if she took the necessity of the implication as following from properties of the number system she was working in. If she worked with the inequality apart from modeling any situation and tried to prove the implication using number system properties, then she would clearly be working strictly within mathematics. On the other hand, we would expect that her conceptions of the differences and inequalities were rooted in conceptual activities based in situations experienced as meaningful.

**Examples of Early Generalizing and Formalizing:**

The following situations were observed and documented by Virginia Bastable and Deborah Schifter (Bastable & Schifter, in preparation). The first involves a third grade class in which the teacher asks how many pencils are there in three cases, each of which contains twelve pencils. After the class arrived at a repeated addition 12+12+12 solution, the teacher showed how the result could be seen as a 3X12 multiplication. She expected to move on to a series of problems of this type, but a student noted that each 12 could be decomposed into two 6's, so the answer could be described as 6+6+6+6+6+6+6, or six sixes, which could be written as 6X6. Another student observed that each 6 could be thought of as two threes, which led to 12X3 as another way of expressing 36. "And at this, Anna exclaimed, 'Wow, we have found a lot of things that equal thirty-six. Oh look! This one is the backwards of our first one, 3 x 12.'" Anna's observation led to an extended investigation, described by Bastable & Schifter as follows.

> The children then continued to find more ways to break apart and group the numbers to total 36. Looking at the column of twelve 3s, Steve offered that if you circle three 3s, you end up with four groups, giving you 4x9=36. At this, Joe declared, "And so we can add another one to the list because if 4 x 9 = 36 then 9 x 4 = 36 too."
>
> Anna objected to this last claim, asking, "Does that always work? I mean, saying each one backwards will you always get the same answer?" [Virginia] Brown [the teacher] responded, "That's an interesting question. What do you think?" Anna replied, "I'm not sure. It seems to, but I can't tell if it would always work. I mean for all numbers."
>
> For homework, Brown asked the class to think about ways to prove or disprove Anna's question. The next day various children explained their thinking by noting number pairs such as 3 x 4 and 4 x 3. While some children used manipulatives to illustrate their examples, Anna was not totally convinced. "But I'm still not sure it would work for all numbers." The teacher decided to table the question but to continue to explore multiplication by introducing arrays.
>
> Two weeks later, Brown reminded the children of Anna's question: "Can anyone think of a way to use arrays to prove that the answer to a multiplication equation would be the same no matter which way it was stated?"
>
> The class thought about this for a while--some alone, others with partners --until Lauren timidly raised her hand. "I think I can prove it." Lauren held up 3 sticks of 7 Unifix cubes. "See, in this array I have three 7s. Now watch. I take this array," picking up the three 7-sticks, "and put it on top of this array." She turned them ninety degrees and placed them on seven 3-sticks she had previously arranged. "And look, they fit exactly. So 3x7 equals 7x3 and there's 21 in both. No matter which equation you do it for, it will always fit exactly."

At the end of Lauren's explanation, Jeremy, who had been listening intently, could hardly contain himself. He said that Lauren's demonstration had given him an idea for an even clearer way to prove it. "I'll use the same equation as Lauren, but I'll only need one of the sets of sticks. I'll use this one." He picked up the three sticks which had seven in each stick. "When you look at it this way," holding the sticks up vertically, "you have three 7s." Then he turned the sticks sideways. "But this way you have seven 3s. See? . . . So this one array shows both 7x3 or 3x7."

Anna nodded her head. Although Lauren and Jeremy had demonstrated with a 3x7 array, the representation convinced her of the general claim. "That's a really good way to show it, and so was Lauren's. It would have to work for all numbers."

This example illustrates students actively generalizing before they have a formal language in which to express their generalizations. They are using a variety of notational devices in combination with natural language in a social context. Importantly, the questions of certainty and justification arise as an integral aspect of generalizing, and are interwoven in the use of the different notations. The basic issue voiced by Anna was the range of the generalization - does it hold for all numbers? The students are using the cubes and sticks to generate their ideas, to show one another their thinking, and to justify their claims. The mathematical claims are clearly theirs rather than the teacher's. It would trivialize this account and its contents to think of it merely as the children developing the concept of commutativity of multiplication (of natural numbers), because the very idea of multiplication is being built (although only two aspects, repeated addition and array models), as is the idea of mathematical justification and proof. While the episode began in a concrete situation, it quickly became a *mathematical* exploration - pencils and cases were simply a stepping off point that (inadvertently) led the students to grouping and decomposition of whole numbers. Eventually, it led back to concrete arrays being used to exhibit equivalence of alternative groupings. The invariance of the "amount," first under alternate groupings of 21 and then under alternate orientations of the same grouping (offered by Jeremy), is made concrete in the physical arrays. And in the end, the generality is not only realized, but is made explicit - "It would have to work for all numbers." It is easy to imagine that this property might be given a more formal expression later, first perhaps as "box times circle = circle times box" where numerals could be written in the spaces, and then even later as aXb = bXa.

Another aspect of this situation deserves attention: the fact that the generalizing took place in what most teachers would regard as the normal course of mathematical concept development in an "ordinary" mathematics classroom ("ordinary" in the sense of fitting the NCTM Professional Teaching Standards - this is clearly an excellent teacher doing a good job). It was certainly not traditional symbol manipulation algebra - it was a series of arithmetic lessons where generalizing rather than computing was at the center. But why is generality so important? Because that's what makes these concepts mathematical!

The Bastable/Schifter paper (in preparation) from which the above case was taken includes several examples of such episodes across grades 1-6 involving properties of numbers (odd-even, zero) and operations, extensions to other number systems beyond the natural numbers, and so on. Many important questions remain unanswered about these activities and how to organize them, including the roles of language and special notations, how to discern generality in students' informal utterances, what the interplay between generalizing and justification might be, what the role of concrete situations may be, and so on.

## 2. Algebra as Syntactically-Guided Manipulation of (Opaque) Formalisms

The tremendous power of algebraic symbolism and its syntax that we use to guide our manipulation of it is behind the prodigious development of modern science and technology. When dealing with formalisms, whether they be traditional algebraic ones or more exotic ones, the attention is on the symbols and syntactical rules for manipulating them rather than what they may stand for. However, it is also possible to act on formalisms "semantically," where your actions are guided by what you believe the symbols to stand for. Indeed, most fruitful use of symbols involves alternating between actions on symbols without reference to what they might stand for and then interpreting the results semantically - that is, in terms of what the symbols stand for. This dual status of "looking at" vs. "looking through" symbols is reflected in Figure 2.

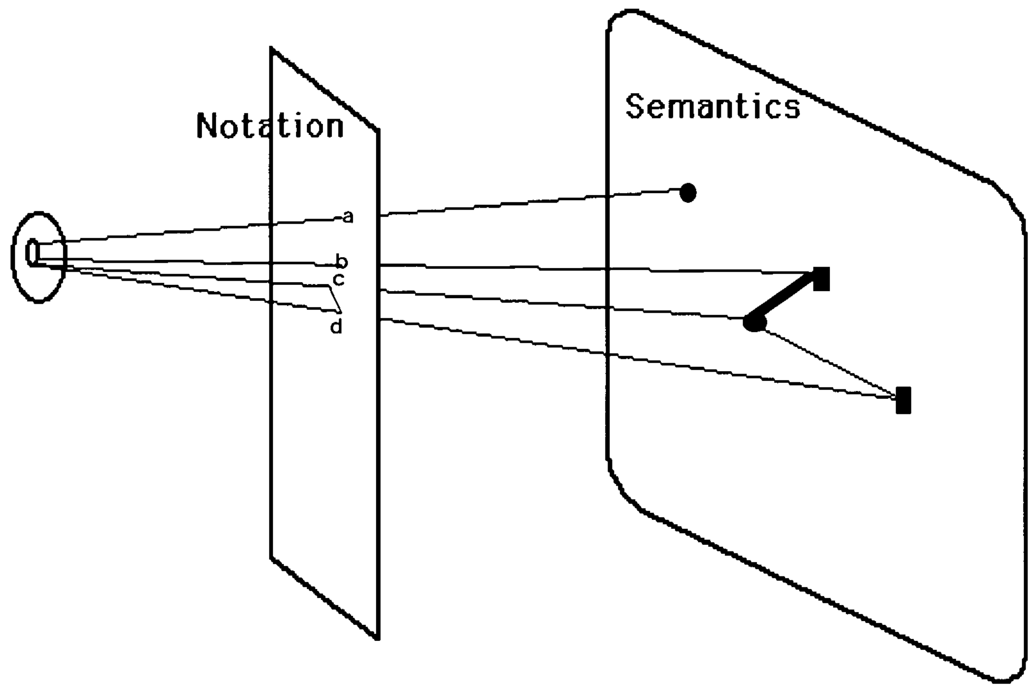



Figure 2: Looking AT vs. Looking THROUGH Symbols

As is widely appreciated, much of the traditional power of formalisms arises from the internally consistent, referent-free operations that they afford. These free the user to operate on relationships far more complex than could be managed if the user needed to attend to what the symbols and transformations stood for. One suspends attention to meaning and focuses on the symbols themselves. To paraphrase Bertrand Russell, "Algebra allows you to think less and less about more and more."

The problem has been that our traditional algebra curriculum has concentrated on the "less and less" part, resulting in alienation from meaning and even from mathematics itself for many students, who practice endless rules for symbol manipulation and come to believe that this is what mathematics is. The power of using the *form* of a mathematical statement as a basis for reasoning is lost as students lose the connection with the quantitative relationships that the symbols might stand for. Research has provided many examples of the difficulties that students have been led into, quite often due to over-generalizing patterns such as linearity - for example, believing that (a rules and interpret them in both realms to their own designs and to those of others. Students gradually move towards more abstract substitution rules that they can apply to arbitrary strings of symbols, sequences of their own initials, for example. Here we see students coming to work comfortably within a world of opaque symbols that are not at all based on or refer to numbers. This form of operative reasoning is closely related to the third aspect of algebra described below. The essential difference for our purposes is that this second aspect occurs all through mathematics, independently of topic or whether modeling is involved, whereas the third aspect is a mathematical domain or topic in its own right. Indeed, the next is what professional mathematicians often call "algebra."

### 3. (Topic-strand) Algebra as the Study of Structures Abstracted from Computations and Relations

Acts of generalization and abstraction based in computations - where *the structure of the computation rather than its result* becomes the focus of attention - give rise to abstract structures traditionally associated with the phrase "abstract algebra," which, in turn, is traditionally regarded as fancy university level mathematics. This side of algebra, beginning with computations on familiar numbers, has some roots in the 19th century British idea of algebra as universalized arithmetic, but has deeper roots in number theory and the solution of equations. As a school subject, it can draw on structures arising elsewhere in students' mathematical experience - in matrix representations of motions of the plane, in symmetries of geometric figures (see below), in modular arithmetic, in calendar arithmetic, in manipulations of letters in words, or other, fairly arbitrary and even playful contexts - as in the previous example. These structures have three purposes, (1) to enrich understanding of the systems that they are abstracted from, (2) to provide intrinsically useful structures for computations freed of the particulars that they once were tied to, and (3) to provide the base for yet higher levels of abstraction and formalization.

Spence & Pligge (in preparation) cite the powerful understanding exhibited by students using pre-formal language, especially natural language, quoting the following 5th grader, "Alicia" who had just completed the "Sums of Even and Odd" subsection of the *Maths in Context* unit, "Patterns and Symbols." On the previous page, students play a game of "once, twice, go". Two players, on a signal, display a certain number of fingers from one hand. One player wins if the sum is even and the other player wins if the sum is odd. They also use arrays of dots to represent odd and even numbers as well as various sums of such. They are then asked to explain the patterns in these sums. Alicia's highly articulate response follows, indicating the power of natural language in the voice of a fifth grader, to express and justify general relationships, which we would recognize as "an even plus an odd is always odd," and "an odd plus an odd is always even."

> An even number and an odd number is always odd. Even always has pairs. Odd always has an extra. Putting them together will still leave that extra, so it's always odd.
>
> An odd number and an odd number is always even. Odds always have 1 left over, so 2 left over form a new pair.

### Example of Abstract-Structural Algebra in 2nd Grade: Dihedral Groups

Strom & Lehrer (in preparation) describe a second grade class that used a quilting activity based in the Education Development Center-IBM curriculum unit, <u>Geometry Through Design</u> to engage students in a series of ideas that are customarily associated with courses in abstract algebra offered to university mathematics majors. The activity begins with students designing a "core square" that is then flipped or rotated to produce four versions of itself in a 2X2 array, the foundation design to

+ b)$^2$ = a$^2$+b$^2$ for any a and b. Reflection and testing would convince most students that this pattern does not hold for real numbers except when a or b is zero. It and the coming examples illustrate what happens when students do *not* construct relationships among pieces of mathematical knowledge.

### Examples of Common Symbol-String Misunderstandings

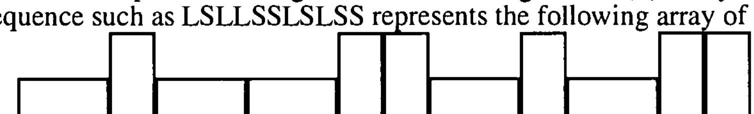
Following is an example offered by Guershon Harel (in preparation) of a high school student (mis)solving the inequality $(x-1)^2 > 1$ by assuming that equality behaves essentially the same way as inequality.

> Patti's solution to this inequality was $x > 1$. When she was asked to explain how she arrived at this solution, she responded:
> The solution to the equation $(x-1)(x-1)=0$ is $x=1$, x = 1 (She wrote down these three equalities).
> Then she crossed out the three equality signs, wrote above them the inequality sign as follows:
> $(x-1)(x-1)\overset{>}{=}0$ is $x\overset{>}{=}1, x\overset{>}{=}1$,
> and said:
> "x is greater than one."
> Following this, Patti was asked to solve $(x-1)(x-1)=3$; she wrote:
> $(x-1)=3, (x-1)=3$
> Patti's mathematical behavior suggests that she was not thinking about the situations [or quantities] that these strings of symbols may represent; rather, the strings themselves were the situations she was reasoning about. That is, Patti's thinking was in terms of a symbolic, superficial structure shared by the three strings... . From her perspective, these strings share the same symbolic structure and, therefore, the same solution method must be applicable to them all.

For such students, who appear to be in the majority, not only is the surface shape of a symbol string a call to perform a certain procedure, dealing with symbol strings is what mathematics is all about. For them, "understanding" is remembering which rules to apply to which strings of symbols. Given superficial similarities between symbol-strings, it is common and expectable that inappropriate procedures will be recruited. Another common example involves "cross multiplying" - often, whether two fractions are separated by an equal sign or a plus sign, the same procedure is called. Understanding algebra absolutely requires being able to connect your knowledge of procedures with other things that you know.

### Example of Meaningful Operations on Opaque Symbols

Curricular materials jointly produced by NCRMSE and the Freudenthal Institute include a unit aimed at fifth graders entitled "Patterns & Symbols" (Roodhardt, Kindt, Burrill and Spence, in press). Among the activities it contains involves transformations on sequences of the letters "S" and "L" where the letters represent rectangular blocks standing on end (S) or laying on their side (L). Hence a sequence such as LSLLSSLSLSS represents the following array of blocks:



Students work with various transformation rules, e.g., SS-->L and LL-->S to act upon such arrays, interpreting their results in terms of strings and vice-versa. For example, what happens if you repeatedly apply these rules to the above array of blocks? The students make up their own

CC: "Ok, now instead of just doing one flip, I'm gonna try four flips. Do you think it will look the same or different?"
Class: "Same."
CC: "One. Two. Three. Four." [Carmen flips the square 4 times, counting as she does so.]
Br: "Brrr-di-doo di-doo! The same!"
Class: "Same, same!" [more trumpet sounds and clapping.]

So the class found that four up-down flips returned the core square to its original position. But one student, Br, saw more: he conjectured that flipping the square any even number of times would "make the square look the same" as when they started. Carmen decided to take Br's lead:

CC: "Um, what do you think about this idea of Br's? Br's idea is that: I could do <u>any even</u> number of flips on this core square—"
Br: "Can't do eleven, but you can do twelve."
CC: "—- meaning two, four, six, eight, ten, twelve. And she kept going. Any <u>even</u> number of flips, and it would look the same."
Br: "One's odd; two's even; three's odd; four's even."
CC: "And then she said for you, 'One's odd; two's even; three's odd; four's even.' Ok, that's her idea. Ke—- has a question for you, Br—-, about your idea."
Ke: "Well, you can go besides by one's, by two's. But if you go by one's it'll just, like, the square will be on the other side; by two's, you could go up, like, as far as you wanted and it would still be the same as when it was started, if you go by two's. If you flip it two times."
CC: "So if I flip it two times, what will happen, Ke—-?"
Ke: "It will be the same."
CC: "Ok, so you're saying I could do what Br—was saying, count by two's, as high as I wanted—-"
Ke: { If you stopped at any number, but you counted by two's,} "then it would be the same as it is now."
CC: "So, no matter how big that number got, if I just counted by two's and then stopped at that number, and then I flipped it that many times, it would look just like this?"
Class: "Yeah. Yes!"

So the class decided that performing any even number of flips, even very large even numbers ("as high as you wanted"), on the square would return it to its original position. Then Carmen asked what would happen if she flipped the square an odd number of times. Notice how she helps act out the children's flips, helping coordinate the flips with the counts, so that the students can focus on the key issue at hand.

CC: "Well, what about five times?"
Ke: "No, odd."
Br: "No, that 's odd."
Ke: "That would be on the back side."
CC: "So I would see the back side, not the front side?"
Class: "Yeah."
CC: "Let's try that and see. One. Two. Three. Four. Five." [Carmen flips the square five times, counting as she goes.]
Class: "Odd, odd, odd."
CC: "Do you know how you can tell it's the flip side?"
Ka: "Yeah, cause you can see the x."
Ju: "The triangles are on the bottom, and on the other side, those were on the top."
CC: "In her repeating design, yep, she had the two greens on the top. Good job!!"

be repeated to produce a quilt. While the students cannot be said to be "doing group theory," they are working in what we could regard as the concrete group of rigid motions of the square. They confront many of the issues that university students confront when dealing with this group, such as: What is the operation? When do I know two elements are the same? and the question discussed below, What is the result of repeatedly multiplying an element by itself, and will I ever get the identity element, the same as the result of not doing anything at all? (This last question leads to the standard group theory question, "What is the order of the element of the group?)

Prior to the episode described here, the students had determined that an "up-flip" (bringing the bottom forward and upward) led to the same result as a "down-flip" (bringing the top forward and downward. Hence they had decided on the convention to call these two actions on the (two-sided) physical square by the same name, an "up-down flip." Hence they had previously dealt with the question of when two elements are the same. We pick up the discussion when they were trying to determine how many up-down flips are needed to return a core square to its original position (which was identified using a small "x" in the upper left corner). Notice how "CC," Carmen Curtis, the teacher, scaffolds the discussion, while the students actually drive it forward, with their own extensions of the ideas and their own conjectures.

CC: "How many up-down flips could I do, or should I do on Katie's core square so that it will look exactly like it does now?"
Pa: [looks at core square, but makes no response]
CC: "If I do one up-down flip, will it look the same?"
Pa: [shakes his head, no]
CC: "So how many should I do? I've got to get it exactly the way it looks now."
Pa: "Two?" [quietly]
CC: "Two? Let's try it. Watch. Memorize Katie's core square. This is what it looks like. One." [Carmen flips the core square.] "Two." [She flips it again.]
Br: "Yep! Brrrr-di-doo-doo!" (trumpet sound)
CC: "You know how I can tell it's in exactly the same position?"
St: "Cause there--the two green triangles are at the top."
CC: "Yeah. And, there's her little x, to mark the top of the core, so I know this isn't the flip side. So two up-down flips gets it back right to where it started from."

Hence the class discovered that it takes two up-down flips to return the core square to its original position. But now the question became what if you repeated the flip more times?

CC: "Yeah. And, there's her little x, to mark the top of the core, so I know this isn't the flip side. So two up-down flips gets it back right to where it started from."
Na: "And zero, um, zero flips."
CC: "Zero flips. Yeah, not flipping it."
Br: "And four!"
CC: "Four? Let's try that."
Br: "Four, six, eight, ten!"
CC: "Why would two, four, six, eight and ten flips make—"
Br: "Because, um, because, like, one's an odd number, and two's an even number. So if you just flipped it once it would be—"
St: "Different!"
Br: "It'd be the back. So try it four times."
CC: "Ok. This is Katie's beginning position, the x is in the top left. I'm gonna do up-down flips, four of them. Watch what one up-down flip makes it look like." [Carmen flips the square.] "Does it look the same or different?"
Br: "Different."

Good job indeed! These second graders not only dealt with concrete forms of issues and concepts important in elementary group theory, in episodes not shown here (see Lehrer & Strom, in preparation) they also dealt head-on with questions of argumentation, moving between the particular and the general, and, specifically, the eternal problem of induction from examples. Clearly, the highly skilled orchestration over a long period of time by Carmen Curtis, is critical to this class's culture of careful inquiry and open discussion. The individual quilts not only focus the mathematical thinking and provide a common set of tangible discussible objects, they also make tangible each student's ownership of the problem. Even more, by design, their quilts share an underlying structure that can gradually be defined and elaborated through their discussion. As with the previous examples, we can see this kind of algebra foundation-building as within the reach of most students and teachers - despite the fact that the mathematics that they are building toward is currently regarded as appropriate for university level mathematics majors!

## 4. (Topic-strand) Algebra as the Study of Functions, Relations, & Joint Variation

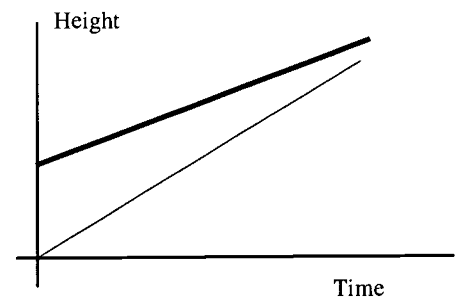
The idea of function has perhaps its deepest conceptual roots in our sense of causality, growth, and continuous joint variation - where one quantity changes in conjunction with change in another. As illustrated below in the examples taken from Tierney & Monk (in preparation), these ideas can be fruitfully approached in the early grades, where familiar quantities change over time and are represented pictorially and with time-based graphs. Such quantities can include heights of plants or people as they vary over time, or the day's temperature, or numbers of people in town who are eating or who are asleep at various times throughout the day. Similarly, students can appreciate the total cost of, say beans, as a function of the number of packages of beans purchased, where the students themselves package the beans, and develop their own ways of describing the cost of different numbers of bags of beans. The two ideas of correspondence and variation of quantities that underlie the concept of function are extremely powerful because they cut across and unify many different kinds of common mathematical experiences that can readily be introduced in elementary school classes, including those involved with counting, measuring, and estimating. But because the idea of function embodies multiple instances all collected within a single entity - represented as a list, a table, or a graph of some kind - it also involves generalizing, answering the question, What is it that all these instances have in common? The traditional place where functions have been introduced in American schools has been in precalculus courses at the high school level, and the traditional notation for representing them has been symbolic, as algebraic formulas. But we now know that such delay is unnecessary and inappropriate, as our next example illustrates.

### Example of Functional Thinking in the Context of Modeling Using Graphical Language

This example involves fourth grade students who analyze graphs of plant height over time, graphs which represent functions - functions of time. It is chosen from a virtually unlimited set of possibilities and actually embodies both the modeling and the language aspects of algebra examined in the next two sections and could fit in both of those sections just as appropriately as it fits here. This again illustrates how the different aspects of algebra tightly interconnect. We will see several big ideas associated with interpreting functions come alive - without numerical values and without formulas.

Here students are studying height vs time functions to compare both changes in the plants' heights and the rate of change of height. In the graphical symbol system, they are able to use their knowledge and language for thinking and talking about vertical height of familiar objects to begin to make important distinctions between these two ways of looking at functions - their value vs. the rate of change of value, height vs growth rate.

The students are using a unit called "Changes Over Time" (Tierney, Weinberg, Nemirovsky) from the curriculum series *Investigations in Numbers, Data, and Space.* The children have grown actual plants from seed and have recorded and graphed their plants' heights each day for two weeks. In this episode they are interpreting qualitative graphs of plant height over time in which no quantities are shown on the axes; only the shapes of graphs are provided. (These were provided by the instructor.) In this particular class, all the students interpreted steeper graphs as meaning the plant was growing faster and higher graphs as showing a taller plant. Now they are working on a problem that has two graphs, one that is above the other, but not as steep, and the other that is lower but steeper as indicated in the figure above. This "crossing-difference" between the respective height and steepness properties of the two graphs provokes disagreement based in distinguishing height versus change of height; change versus rate-of-change. Some students focus on change-in-height depicted by the graph while others focus on current height and the growth of the plant that yielded the beginning height before the part of the story shown on the graph. We quote Tierney and Monk (in preparation).




When the teacher asks which plant is growing faster, Michelle compares the growth of the plants by comparing the changes in height in a fixed amount of time. She is comparing rates of growth.

Michelle: The light line. It started really small and got bigger and bigger and took the same amount of time to get to the same height.

However, Sean and James respond directly to the shape by interpreting it in terms of comparative change.

Sean: The light one [grows faster], because it always going up. The dark one is kind of steady and kind of going across.

James: The dark one is slightly going up and its not going fast.

When Darius questions him, James changes to Michelle's approach:

Darius: It is going fast

James: It didn't grow high in a short time.

When the teacher questions him, James bases his answer on the shape of the line, describing it in a language appropriate for the plant it depicts:

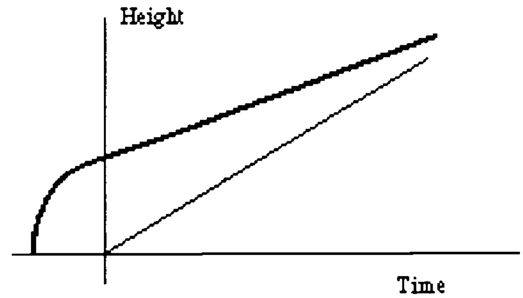
Teacher: Tell me about the changes

James: The dark line is only growing a little bit over a long time. The light line, the changes are bigger over the same amount of time.

Then Bobby and Sarah speak of the plant before and after the time depicted on the graph.

Bobby: The dark line grew faster at the beginning, before the graph.

Sarah: I chose the dark line. The light line takes time to grow up. It's going to take it a long time to catch up with the black line.




The teacher asks Bobby to come up to the board and draw the dark line as he thinks it might have been before the graph began. He starts at left end of the dark line and extends it leftward, making a line that curves down to the horizontal axis, almost vertically.

Bobby: [Moving his finger along the line he drew] It grew fast, then still fast, then started to get steady.

We see above part of a spirited discussion of the properties of these functions and how they may relate to what the functions stand for. Different ways of comparing changes and interpreting graphs that occur among students right up through university calculus courses appear here among fourth graders. Their understanding of how plants grow that was refined in the previous two weeks is now a resource to make sense of the graphs and relations among sizes of plants, changes in size and rates of change. Hence to decide which plant is growing faster, some children focus on its visual aspects, such as steepness, while others focus on its implicit quantitative information. Bobby's view that there is a part of the graph that was missing from the original one the class was given raises an important issue in the use of symbols. How does the representation (or model) relate to the thing it is assumed to represent? "Is it a complete record, the only source of available information about the event, or is it like an illustration that tells part of the story to be supplemented by other things we know and believe?" (Tierney and Monk, in preparation). This is a subtle and deep matter: How do the ways we organize the representation system into parts and wholes relate to the way we organize that part of our experience that the representation is being taken to represent? Does the whole graph stand for a whole situation or only part of one? Importantly, elementary school students, in an extended exploration in investigative classroom culture, can begin to make sense of this important issue.

## 5. Algebra as a Cluster of Modeling and Phenomena-Controlling Languages

Many have argued that modeling situations is the primary reason for studying algebra. Quantitative reasoning as well as the use of functions and relations can be regarded as modeling - building, usually in several cycles of improvement and interpretation, mathematical systems that act to describe and help reasoning about phenomena arising in situations. In modeling, we begin with phenomena and attempt to mathematize them. But computers now enable us to turn things around in interesting ways. For example, we can now use mathematics to simulate phenomena within the computer and can even drive physical devices such as motorized cars on a track using data from a computer. In fact, computer languages amount to an algebra-like language within which we can create or experience explorable and extensible mathematical environments. But manipulable graphs likewise can act to create phenomena. Ricardo Nemirovsky and colleagues at TERC have turned around the now standard acronym (MBL - for "Microcomputer-Based Laboratory") to LBM ("Lines Become Motion"). Whether these computer environments are used to model phenomena or to create/control phenomena, they change in fundamental ways how we relate the particular to the general, and how we can state and justify mathematical conjectures. But even more importantly, they change how we may relate to the mathematics itself. I will illustrate a new level of intimacy between students' activity and the mathematical notations that they use and interpret.

### Example of Mixed Modeling & Phenomenon-Controlling Interactions: Rates, Totals and Graphs for Physical and Computer-Based Motion

Background

Fifteen students were in a 5 week summer program for economically disadvantaged children in an urban school setting and had recently finished either 3rd or 4th grade. They were involved in an extended exploration of first their physical movement and then related issues in a computer simulation developed as part of the SimCalc Project headed by the author. Two teachers and the principal from the school were assisted by the author and two project staff members. Students' work began with the students marking out a fifty foot path in the gymnasium with masking tape, and marking it at two foot intervals, with double marks at the tens places. Over a period of three days they studied their own motion using a combination of the marked masking-tape path and stop-watches. While they were unable to quantify their velocity numerically, they were quite able to distinguish three values of their own speed, "slow," "medium," and "fast." They also accepted the fact that one person's "medium" might be close to someone else's "slow" or "fast." They timed one another's "trips" down the "path" and recorded these in three tables, one each for "slow," "medium," and "fast." The fact that they were able to move, measure, count, and record their data was a source of delight and fascination for them. They then moved to a motion simulation software system, "Elevators," part of the SimCalc *MathWorlds* software system in which a number of the activities that they had engaged in physically were re-enacted by elevators that they, the students, were able to control using velocity vs time graphs. We will describe how they related their physical and kinesthetic experience to their computer-based experience with elevator simulations over an eight day period spread over two hot (and un-airconditioned) summer weeks.

Building Understanding in Students' Nerves, Bones and Muscles

On the third day, after they had recorded the lengths of time they took to move the entire 50 foot pathway at their "slow," "medium," and "fast" paces, and had discussed how these rates differed from student to student, they engaged in planned movements. The student was to move (either walking or running) with a stop-watch for given lengths of time at their various paces. For example, a student might be asked to go for two seconds each at "slow," then "fast," then "medium," or go fast for two seconds, slow for three, and medium for two seconds. Several students would repeat a given set of directions, stopping and standing at the end of their "trip" as a way of recording and comparing with their peers how far they went under the given conditions. The boys, more than the girls, tended to value higher speeds, hence distances, particularly early in the activities. The issue of competition arose, as some students came to realize that the purpose of the activity was not to go as far as possible, but to be as precise as possible in carrying out the motion-instructions. These activities helped the students develop a perspective on their own motion and a sensitivity to relationships among time, speed and distance - although they were not expected to quantify these relationships until much later.

On the third and fourth days, students engaged in a paired activity, in which one student of the pair was to move at a constant rate while the other moved according some directions similar to those that they had previously enacted. After a couple of trials where the students started at the same time and ended at the same time, the "constant speed" student was asked to travel at a perfectly constant rate in such a way as to end up at exactly the same place as the other student at the end of the given time interval. Thus, if successful, the constant-speed student would be moving at the average speed of his/her counterpart, whose speed would vary according to the given instructions (some of which were provided by other students). This activity brought forth the idea of constant speed in a very concrete and visible way that directly confronted students' tendencies to want to "catch-up," for example, with the constant-speed requirement. Since those students who were not in the currently moving pair lined the pathway, they observed the motion closely, and when a student who was supposed to be traveling at a constant speed speeded up or slowed down, usually in fear that they would not reach the endpoint simultaneously with their counterpart, the "audience" would shout its disapproval or corrections. This led to considerable concentration on the part of those moving to maintain their constant pace.

For these students (which eventually included all of them since they all cycled through variations of the same activity), the notion of constant rate was not merely a verbal description, or something that appeared on a graph, but something that they were acutely sensitive to in their own motion. They discussed and were concerned, for example, that in beginning a constant motion that they reached the constant speed as soon as possible, and when they reached the end they stopped abruptly. They also became aware of the lengths of time intervals, since other students were timing them as they carried out the publicly announced slow-medium-fast instructions. Hence the quantities and concepts associated with them were intimately tied to their own deeply felt kinesthetic sense.

Representing Quantitative Relationships in Dynamic Graphs

At first, the quantities and concepts of speed, time and distance were described using action, language and numbers. However, beginning on the fifth day, the students moved to computer simulations, where they were provided with a building and two elevator shafts as in Figure 3.

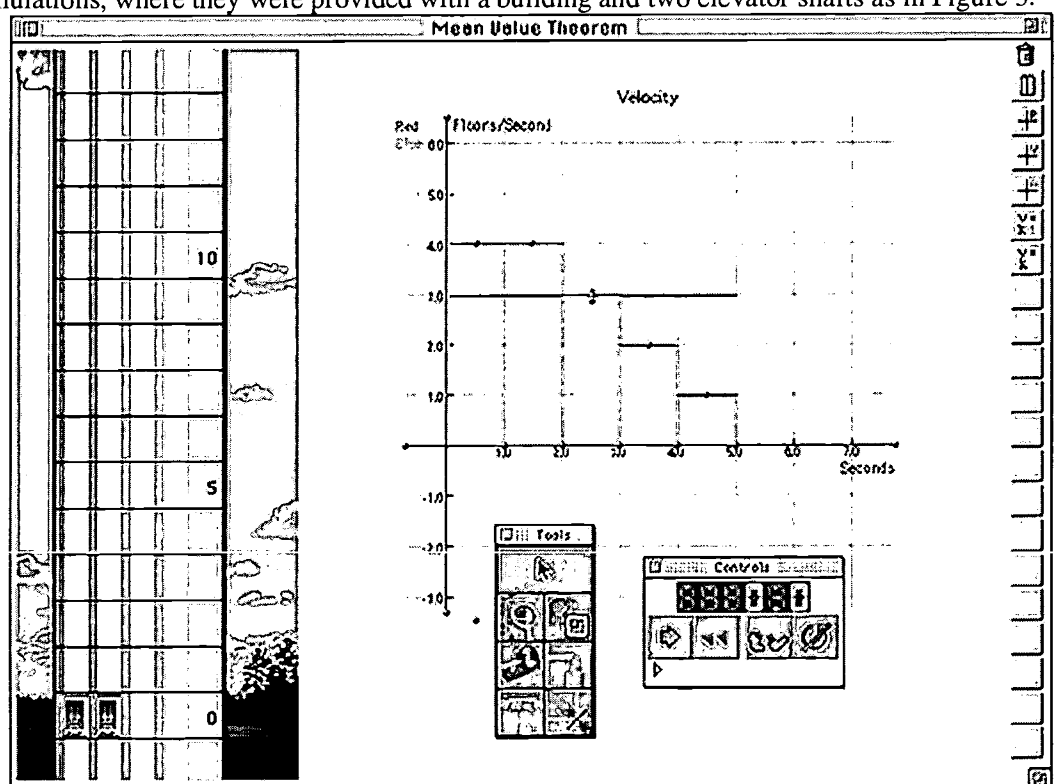



Figure 3. Constructing the Average Speed of An Elevator

They had three icons that they could drag from a "ToolBar," one each for slow, medium and fast. They could drop these on a velocity vs time graph as indicated in the figure, resulting in a horizontal line segment that represents either one, two or four floors per second in the simulation. They could then stretch these segments horizontally as needed to determine the length of time the elevator would travel at that indicated speed. Thus students could create and manipulate what mathematicians might call "piecewise constant velocity functions" which control the motion of the elevators - which are color coded to their respective graphs. Working in pairs at the computer, the students worked through a series of activities over the fifth and sixth days that paralleled and then built upon the work that they had done in the physical context.

Without the prior experience of physical motion that built the concepts and kinesthetic sense of what these velocity graphs were all about, this would have been an empty exercise in symbol manipulation, where in this case, the symbols happen to be coordinate graphs. The computer program allowed students to name and color the elevators, and they almost unanimously chose to name the elevators after themselves. They were quite clearly personally invested in the motions that they were creating, and, when pairs of motions were involved, such as the average speed activity, they took ownership of their respective elevators, often referring to them in the first person (as in ""I'm ahead" or "I'm slowing down). We feel that the structure of the activity and the fact that they had carried out similar activities physically were the foundations for their investment. Additionally, the fact that they worked on the computer in pairs allowed an extension of the conversational mode that had been employed in the physical context. However, the point of the computer side of the activity was to continue the process of quantifying, or mathematizing, motion. The physical and the cybernetic environments differ in a fundamental way: the physical is kinesthetically rich, but quantitatively poor (or, perhaps more accurately, inaccessible); the cybernetic is kinesthetically vacuous but quantitatively rich - the floors in the building are numbered and the graphs are also all numbered and labeled, with a close connection between them. Thus, for example, a student could create a "medium" velocity graph that will make the elevator go up at 2 floors per second for 4 seconds, and be asked "Where will the elevator finish its trip"? Rather quickly, the students come to see that the area of the rectangle under the flat-topped graph segment will give the answer, and, if the velocity is composed of several segments, they can predict the final position by adding up all the areas under the respective segments. Indeed, they quickly adopt this strategy to obtain average values of variable velocity motions.

In the next two days, the students returned to the physical context to deal with the issues of changing direction and negative velocity.- repeating the cycle of physical and then computer-based motion. Somewhat surprising was the ease with which students transitioned between horizontal and vertical motion. It appears that the natural motion-talk that occurred in both contexts served to link them. Discussion about "speeding up," "slowing down," "turning around," and so on applied in both realms, and was used to describe the graphs and their effects on the motion.

In summary, the development of understanding in this situation involved intimate connections between students physical action and the motion simulations that were mediated by the students' own talk about both sides of the connection. The graphical notation, while not quite standard in its dynamical form used here, is a powerful form of expression for the students, serving both as a modeling language and as a phenomenon-controlling language. They are not only beginning algebra, but, a moment's reflection reveals that they are beginning calculus as well!

### Reflections on the Five Examples

The five aspects of algebra that we have examined are not well represented in standard algebra courses. And we deliberately chose illustrations of them that exhibit young students led by sensitive teachers making sense of complex situations while simultaneously building big mathematical ideas. Some may be tempted to say that this is not algebra, to which we would reply

"True, if we identify 'algebra' with Institutional Algebra - what occurs in Algebra I and Algebra II courses." But my point is that algebra must be much broader, deeper and richer than that algebra. It cuts across topics and adds a conceptual unity that our curriculum, especially in the earlier grades, has been absent. Hopefully, it has become apparent that this algebra is neither a mystery nor out of reach of most teachers and most students. Indeed, as with the other illustrations in this document, it may be more accessible than that algebra that everybody loves to hate - that "intensive study of the last three letters of the alphabet" (Williams, 1997).

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