DOCUMENT RESUME

ED 414 172		SE 060 871
AUTHOR	Malmstrom, Jay A.	
TITLE	The Return of Dangerous Dan: Further Adventures in Recreational Mathematics.	
PUB DATE	1997-05-00	
NOTE	10p.	
PUB TYPE	Guides - Classroom - Teacher (052)	
EDRS PRICE	MF01/PC01 Plus Postage.	
DESCRIPTORS	*Arithmetic; Higher Education; *Induction; Learning Activities; *Mathematical Concepts; Mathematics Education; *Modular Arithmetic; Secondary Education	
IDENTIFIERS	Combinatorics; Graph Theory; Sequences (Mathematics)	

ABSTRACT

This paper describes six tricks on different mathematical concepts for mathematics classrooms. The mathematical concepts emphasized in these activities include arithmetic, modular arithmetic, limit cycles, graph theory, pairings, combinatorics, cyclic groups, induction, and sequences. (ASK)

Jay A Malmstrom
Oklahoma City Community College
Oklahoma City, OK 73159
email: malmstrm@qns.com

The Return of Dangerous Dan: Further Adventures in Recreational Mathematics

Trick 1: Tapping The Hours

Mathematical Concepts: Arithmetic, Modular Arithmetic

Requirements: A standard analog clock.

Mechanics: Ask a member of the audience to select a number between 1 and 12. While you tap on numbers on the clock, have that person silently count up to 20, starting with the selected number. The first eight numbers tapped may be any numbers. The ninth number tapped must be 12. Move backwards from 12 until your are told to stop - this will be the selected number.

Mathematics: Let N be the number selected, $1 \leq N \leq 12$. Counting from N to 20 will involve tapping a total of 20 - N + 1 or 21 - N numbers.

1. When you tap the 12, you will have tapped 9 numbers leaving 12 - N taps.
2. Going counter-clockwise 12 - N numbers from 12 can be thought of as subtracting 12 - N from 12.
3. $12 - (12 - N) = 12 - 12 + N = N$.

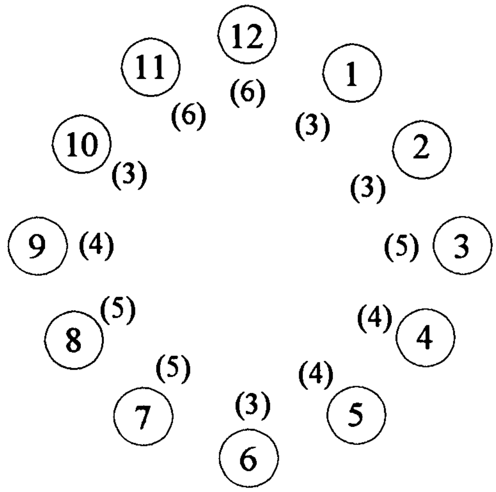
We could also use ideas from modular (clock) arithmetic to explain this trick.

1. Moving counter-clockwise from 12 can be thought of as going from 0 through the negative numbers.
2. Moving counter-clockwise 12 - N places would bring us to $-(12 - N) = N - 12$.
3. $N - 12 \equiv N \pmod{12}$.

Trick 2: The Disappearing Clock

Mathematical Concepts: Modular Arithmetic, Limit Cycles

Requirements: A standard analog clock (preferably printed on a sheet of paper) shown below.



Mechanics: Have everybody pick a number to start on then define a legal move in the following manner:

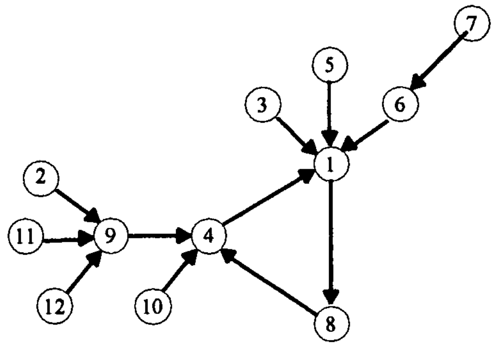
1. From the number selected, move clockwise a number of spaces equal to the number of letters in the word for the number you selected; e.g. if you had selected the number 9, you would move clockwise four places (since nine has four letters) to the number 1.
2. Repeat this process for the number you landed on; e.g. you started on the number 9, you moved clockwise four places to the number 1, now move clockwise three places to the number 4.
3. Nobody should be on the number 7, so have everybody remove this number from their clock.
4. Repeat steps 1 and 2, having people remove, successively, the numbers 8,9,10,11,12,1,2,3.

Results: Everyone should end up on the number 6.

Mathematics: Think of the rule for moving around the clock as a function, f (x), defined in the following manner.

x	f (x)	x	f (x)	x	f (x)	x	f (x)
1	8	4	1	7	6	10	4
2	9	5	1	8	4	11	9
3	1	6	1	9	4	12	9

Notice that 7 is not in the range of f. When we start the next cycle of this trick, everybody will be starting from 1, 4, 6, 8, or 9. You can also look at f using a directed graph:



Notice that the numbers 1, 4, and 8 form a cycle. If you continued the trick without removing any of the numbers, everybody would end up on either 1, 4, or 8. This is what is known as a limit cycle. The system is exhibiting a property called coalescence. This happens when repeated application of a function results in the number of possible outcomes being narrowed down no matter what the initial starting point was. If you continue the process of removing the number furthest from a possible limit cycle and furthest (counter-clockwise) from the 6, you will eventually force everybody onto the 6.

Trick 3: The Break in the Chain

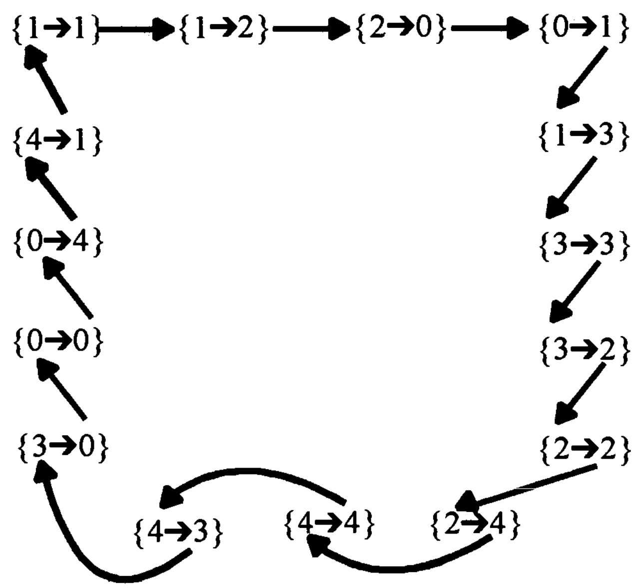
Mathematical Concepts: Graph Theory, Pairings, Combinatorics

Requirements: A standard set of double-six dominoes.

Mechanics: Before facing the audience, remove a non-double domino from the set, glance at the numbers, and pocket the domino. Ask for a volunteer to supply a slip of paper and write the numbers from the pocketed domino on the slip of paper. Hand the slip of paper to the volunteer. Then have somebody else lay out all the remaining dominoes in a chain, matching the numbers on the dominoes.

Results: When all the dominoes are laid out, the numbers on the end of the chain will match the numbers on the paper.

Mathematics: There are seven integers (0, ..., 6) used in constructing a standard set of double-six dominoes. Each integer is matched up exactly once with each of the integers 0, ...6 including itself. This means each number will appear a total of eight time on seven dominoes. In laying out the dominoes, you would expect to see four pairs of each integer, e.g. four pairs of 3's. If the dominoes are laid out in a chain, one of these pairs would appear as the integers at the ends of the chain. There are two cases to consider when a domino is missing from the set i) the missing domino is not a double or ii) the missing domino is a double. In the first case, there will be two integers which now appear only seven times in the set of dominoes. Every other integer can be paired off four times, leaving two numbers which can be paired off only three times leaving one occurrence each of two integers. These integers would have to be at the ends of the chain. In the second case, one of the integers would be paired off three times while all the rest could be paired off four times. Since no extras remain, any integer could appear at the ends of the chain. Both of these cases can be illustrated by looking at the diagram shown below. This diagram represents a possible layout for a set of double-four dominoes.



The lay out is shown in the form of a cycle since you could start the chain with any integer from 0 to 4. If a non-double domino is removed from the set, the cycle would be broken and that the integers at the end of the chain would correspond to the integers on the missing domino. If a double domino is removed from the set the cycle could be maintained and the integers at the end of the chain would match - and not necessarily match the integers on the missing domino.

You can generalize this trick by considering a standard set of double-N dominoes. In this set the integers 0, ..., N would appear on the dominoes. Each integer would show up on N + 1 dominoes for a total of N + 2 appearances in the set. To make the trick work, each of the integers must be paired off, so N + 2 must be even. This means N must also be even.

If you think of the integers as vertices and the dominoes as edges of a graph, then laying out the dominoes in a cycle is the same as trying to find an Euler circuit around a graph. Laying out the dominoes in a cycle is the same a finding an Euler circuit for a graph. Recall, a graph has an Euler circuit if and only if all the vertices of the graph have an even number of edges. Since each integer appears on $N + 1$ dominoes, the number of edges for each vertex is $N + 2$ (the double domino would be a loop connecting a vertex to itself and thus count as two edges). $N + 2$ is even if N is even and odd if N is odd so the dominoes can be laid out in a cycle if and only if N is even.

Trick 4: The Row of Thirteen

Mathematical Concepts: Modular Arithmetic, Cyclic Groups.

Requirements: A standard set of double-six dominoes.

Mechanics: Select the thirteen dominoes from the set as follows: the double blank - {0,0} and twelve other dominoes whose total number of spots corresponding to the integers from 1 to 12, e.g. : {0,1} {1,1} {1,2} {0,4} {2,3} {1,5} {3,4} {4,4} {3,6} {4,6} {5,6} {6,6}. Lay out the selected dominoes, face down, in numerical order from left to right, placing the {0,0} on the right end of the row. The layout will look like this:

0	1	1	0	2	1	3	4	3	4	5	6	0
1	1	2	4	3	5	4	4	6	6	6	6	0

Now ask someone to move from 1 to 12 dominoes, one at a time from the left end of the row to the right end of the row. Demonstrate what you want done by moving k dominoes, $1 \leq k \leq 12$, from the left end to the right end. Remember how many dominoes you turned over during the demonstration.

Results: After the volunteer moves n dominoes, $1 \leq n \leq 12$, from left to right, the layout will look like this:

k+n +1	...	...								...	...	k+n

From the right side of the layout, count over $k + 1$ dominoes and turn this domino over. The numbers on this domino will tell you how many dominoes the volunteer moved.

Mathematics: The dominoes are laid out in order: 1 to 12 then 0. Think of the numbers as being in a cycle of length 13. Moving the dominoes one at a time maintains the relative position of the dominoes in the cycle. Counting $k + 1$ dominoes from the right end can be thought of as subtracting $k + 1$ from $k + n + 1$. $k + n + 1 - (k + 1) = k + n + 1 - k - 1 = n$.

Trick 5: The Magic of Nine

Mathematical Concepts: Partitioning

Requirements: Several circular game pieces from a game such as checkers.

Mechanics: Layout several disks in the shape of a nine. Remember how many disks there are in the tail of the nine. Have a volunteer start counting from the start of the tail of the nine. Tell the volunteer to count out a number of disks, k, that is greater than the number of disks in the tail of the nine. Starting from the disk they landed on, have the volunteer count backwards around the circle of the nine a total of k disks. Have them place a marker under the disk. The marker should be placed so that its edges are not visible.

Results: You will turn around and immediately select the disk under which the slip of paper is hidden.

Mathematics: Assume that the tail has n disks in it. The volunteer selected a number, k, $k > n$. You can find a number, $m > 0$, such that $k = n + m$, i.e. you have partitioned n. In counting $n + m$ disks from the start of the tail, a volunteer will end up m disks, counterclockwise in the circle of the nine, from the point where they left the tail. In counting back, clockwise around the circle, the first m moves will bring the volunteer back to the place where they entered the circle. The volunteer now has n moves remaining -exactly the same number of moves as there are disks in the tail. The marker will always be n disks clockwise from the disk where the volunteer entered the circle.

Trick 6: The Lazy Magician

Mathematical Concepts: Modular Arithmetic, Induction, Sequences.

Requirements: A standard deck of cards.

Mechanics: Ask the volunteer to select 10 cards at random from a standard deck of cards. Have the volunteer mix the cards and hold the cards facing them. The volunteer then selects one of the cards and makes a note of the card's position, $1 \leq n \leq 10$, from the left (top of the pack). Have the volunteer collapse the cards into a pack with the card on the left on top of the pack. Ask the volunteer to move 5 cards from the top to the bottom, one at a time. Now have the volunteer move n cards from the top to the bottom, one at a time. Now have the volunteer move the top card to the bottom of the pack and discard the next card. The volunteer will repeat this process until only one card remains.

Results: The card selected by the volunteer will be the only card remaining.

Mathematics: After moving 5 cards and then n cards from the top of the pack to the bottom, the selected card will end up as the 5^{th} card from the top of the pack. Reason - Moving the cards from top to bottom moves cards from the bottom of the pack up to the top of the pack. So we use subtraction to find a card's new position. Since there are 10 cards in the pack, we do the subtraction modulo 10. Starting from the n^{th} position: $n - 5 - n = -5 \equiv 5 \pmod{10}$. The 5^{th} card will be the only card remaining after the rest of the trick is performed. To show why this is so, I will generalize the trick and use induction. Suppose you start with two cards (k = 2): move top card to the bottom, discard the next card. The top card is the only card remaining. Start with four cards (k = 4): move top card to bottom, discard the next card. Three cards remain, the "top" card is now the third card. Move the new top card to the bottom and discard the next card. Two cards remain, the "top" card is now on top again. We are now in the case, k = 2. By induction, if the pack contains $k = 2^m$ cards, the "top" card will be the only card remaining after the trick is over.

We started with k = 10 cards, $10 = 2^3 + 2$. Move the top card to the bottom and discard the next card. We are left with nine cards. The 5th card is now the 3rd card. Move the new top card to the bottom and discard the next card. We are left with eight (2^3) cards. What started as the 5th card is now the top card. What started as the 5th card will be the only card remaining after three more repetitions.

We can generalize this trick further in the following manner: if the pack contains $2^k + m$ cards ($0 \leq m \leq 2^k - 1$), then the card in 2m + 1 position will be the only card remaining at the end of the trick. This trick also gives rise to the following sequence:

1,1,3,1,3,5,7,1,3,5,7,9,11,13,15,1, ...

Starting with a value 1 for the $n = 2^k$ position in the sequence, the terms in the sequence will increase in value by 2 until we reach a value of $2^{k+1} - 1$ for the $(2^{k+1} - 1)^{th}$ position in the sequence.

We can also use this trick to illustrate the matrix representation of a transformation. Think of the initial position of the cards as a vector: $v_{10} = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 0]$. The first time that the top card is moved to the bottom and the second card discarded, the vector, $v_9 = [3\ 4\ 5\ 6\ 7\ 8\ 9\ 0\ 1]$ would be the result. This transformation could be represented by the following matrix:

$$L_{10} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $L_{10}: \mathbf{R}^{10} \mapsto \mathbf{R}^{9}$ and $L_{10} * v_{10}{}^{T} = v_{9}{}^{T}$. Eachsuccessivemovemovement ofthetop cardto thebottomfollowedbydiscardingthesecondcardcouldberepresentedbyasimilartransformation, $L_n: \mathbf{R}^{n} \mapsto \mathbf{R}^{n-1}$, $L_n * v_n{}^{T} = v_{n-1}{}^{T}$, $2 \leq n \leq 10$. The entire trick could be represented by the followingexpression:

$$\left(\prod_{n=1}^{10} L_n\right) * v_{10}{}^{T} = v_1{}^{T}$$

where:

$$\left(\prod_{n=1}^{10} L_n\right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

andv_1 = [5].

Selected References On Discrete Mathematics, Recreational Mathematics and Self-Working Magic:

Fulves

<u>Self-Working Card Tricks</u>: Dover

<u>Self-Working Close-up Card Magic</u>: Dover

Gardner

<u>Mathematical Carnival</u>: MAA

<u>Mathematical Circus</u>: MAA

<u>Mathematics, Magic, and Mystery</u>: Dover

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NCTM

<u>Discrete Mathematics Across The Curriculum K-12</u>

Simon

<u>Mathematical Magic</u>: Dover

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Signature:	Printed Name/Position/Title: Jay A Malmstrom, Professor of Mathematics
Organization/Address: Oklahoma City Community College, 7777 S May Ave, Oklahoma City, OK 73159	Telephone: (405)682-1611x7365 FAX: (405)682-1611x7496
	E-Mail Address: malmstrm@qns.com Date: 10 Oct 97

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