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ABSTRACT

Recent research explaining the influence of teacher knowledge and beliefs on instruction tends to illustrate what teachers cannot do by describing their limited beliefs about mathematics or gaps in mathematics knowledge. Those studies illustrating what teachers can do tend to focus on one teacher. This paper describes what nine secondary level mathematics teachers--who are unusual in their exceptional backgrounds in mathematics and commitment to the "Professional Standards for Teaching Mathematics"--can do. By identifying patterns in these teachers' strategies for responding to unanticipated student observations (conveyed through errors, difficulties, and alternative problem solving approaches), a conceptualization of teacher knowledge use is proposed that is based in mathematics problem solving strategies. The teachers' beliefs are related to these strategies and their intentions to implement certain aspects of the "Standards" in their teaching. The implications for the mathematics instruction of prospective teachers and for other areas of teacher preparation are discussed. Appendixes contain the teacher interview protocol and a description of teaching episodes. (Contains 4 tables and 29 references.) (Author/SLD)

The "*Standards*-like" Role of Teachers' Mathematical Knowledge in Responding to Unanticipated Student Observations

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Contents

INTRODUCTION

The research reported in this article fits into that genre of studies that attempts to relate teacher beliefs and knowledge about mathematics with instruction [Thompson, 1984; Leinhardt and Smith, 1985; Ball, 1990, 1991(a); Stein, et al., 1990; Borko and Livingston, 1990; Borko, et al., 1992]. In general, this genre tends to illustrate what teachers *cannot* do by describing their limited views of mathematics or gaps in mathematics knowledge, while those studies illustrating what teachers *can* do tend to focus on *one* teacher [Ball, 1991(b); Lampert 1989]. The present study describes what a *group* of teachers *can* do. In particular, it describes the thinking and instruction of nine secondary-level teachers who are unique in their exceptional mathematics background and commitment to the *Professional Standards for Teaching Mathematics* (NCTM, 1991). How do these teachers apply their knowledge of mathematics in their teaching? Are there common strategies among these teachers that we can describe and learn from? What are the teachers thinking and experiencing as they implement their strategies? These questions provide the basis for the present study's design and the specific research questions it addresses.

This study considers classroom situations (illustrated in the *Standards*) in which teachers apply their mathematics knowledge in a "*Standards*-like" response to students' mathematical observations. More specifically, it considers a conceptualization of teacher use of mathematics knowledge based in how it responds to students' *unanticipated* observations and how it is intended to generate *Standards*-like classroom discussion. Given a teacher attempting a *Standards*-like response to an unanticipated student observation, this study explores: (1) how a teacher can use her knowledge of mathematics to promote her own and her students' *Standards*-like roles; (2) how the teacher's thinking can be related to her decision to implement certain *Standards*-like instructional roles; and finally (3) the significance of describing these teachers' use of knowledge and thinking to prospective teachers' own attempts to respond to students' unanticipated perspectives in ways that promote the *Standards* visions of teaching.

THEORETICAL FRAMEWORK

Unanticipated Student Perspectives

Indirectly, the topic of an unanticipated perspective arising during mathematics problem solving has been studied by various educators [Schoenfeld, 1983; Lampert, 1989; Ball, 1991(a), Borasi, 1994]. Schoenfeld (1983) illustrates how a mathematician might pursue unanticipated perspectives during problem solving, as well as the sometimes surprising implications of these explorations to finding a problem's solution. Borasi (1994) takes these explorations to the teacher-student level and likewise illustrates how unanticipated perspectives conveyed through students' errors can generate mathematical ideas beyond the scope of the original problem. In effect, these educators illustrate that such twists and turns can be interpreted as a natural, versus obstructive, part of problem solving. For mathematics educators interested in conveying mathematics as a discipline in which conclusions are based on reasoning about ideas that arise during problem solving, creating classroom settings in which students can explore these twists and turns has become an important priority (Romberg and Carpenter, 1986; Richards, 1991; NCTM, 1989, 1991).

Unfortunately, from the teacher's standpoint, an unanticipated student perspective can present certain challenges (Leinhardt, et al. 1991). Lesson plans can be thrown off course and time taken away from the teacher's objectives. Multiple unanticipated perspectives can become difficult to manage or lead to conflict between the teacher and student or amongst students. If the student perspective is unfamiliar to the teacher, it can introduce uncertainty into the teaching process (Lampert, 1989; Borko, et al., 1992). In the face of such difficulties, it becomes important to explore various teachers' strategies for responding to unanticipated perspectives and the strategies' implications to classroom problem solving. Because the *Professional Standards for Teaching Mathematics* (NCTM, 1991) provided the grounding for the present study's teachers during their teacher preparation, this study explores these teachers' "*Standards*-like" strategies for responding to unanticipated perspectives and the strategies' implications to classroom problem solving.

"*Standards*-like" Visions of Teaching

The *Standards* visions of mathematics acknowledges mathematics' connections; its multiple and unpredictable paths; and the reasoning, justifying, and challenging (along these paths) that determine the reasonableness of a mathematical result. Teachers in *Standards* classrooms are encouraged to pose questions and tasks that elicit, engage and challenge student thinking; students are encouraged to listen and respond to, and question, their teacher and one another (NCTM, 1991; p. 35, p. 45). For both teachers and students, these roles exemplify the responsivity necessary to effect and define the multiple and unpredictable paths that can be generated during the course of classroom problem solving.

Given this conceptualization of mathematics and its classrooms, it follows that unanticipated student perspectives can arise. Whether students are able to articulate these perspectives and whether the perspectives are valued depends significantly on the teacher. For example, "reacting directly to what students say," the teacher can encourage student thinking "by building on the descriptions they have given" (NCTM, 1991; p. 38). In so doing, a student's idea can be used to direct the class's exploration of a problem or even formulate new questions for study (NCTM, 1991; p. 30). Students, in turn,

> . . . are more likely to take risks in proposing their conjectures, strategies, and solutions in an environment in which the teacher respects students' ideas, whether conventional or nonstandard, whether valid or invalid. Teachers convey this kind of respect by probing students' thinking, by showing interest in understanding students' approaches and ideas, and by refraining from ridiculing students. (NCTM, 1991; p. 57).

By asking questions that compel a student to respond to his or her own question (NCTM, 1991; p. 47), teachers also can convey the expectation that the *student*–not solely the teacher–is expected to determine the validity of his or her own thinking. A student's peers can help in this endeavor, by supporting or challenging each other's ideas and reasoning together (NCTM, 1991; p. 47). During such explorations, teachers ask students to explain or justify their answers, and students support their ideas and solutions in response to one other's challenges or arguments (NCTM, 1991; p. 37, p. 47).

Thus, part of the *Standards* vision of teaching involves being able to follow students' arguments as they explore their thinking, and summoning appropriate representations to support or challenge this thinking so that students can reason for themselves (see also Grossman, et al., 1989; McDiarmid, et al., 1989). The *Standards* describe a number of factors that can influence a teacher's decision to pursue a student perspective during classroom problem solving. Two of these–a teacher's knowledge of mathematics and her beliefs about mathematics problem solving and her students (NCTM; 1991; p. 36)–are used in the present study to characterize the "*Standards*-like" strategies used by its participating teachers in responding to students' unanticipated perspectives.

Identifying "*Standards*-like" Use of Knowledge and Beliefs

Several educators have described the significance of teacher subject matter knowledge in responding to a student perspective (whether anticipated or not) [Dewey, 1990; Hawkins, 1974; Schön, 1987]. Dewey (1990) explains that a teacher's "own knowledge of the subject matter may assist in interpreting the child's needs and doing, and determine the medium in which the child should be placed in order that his growth may be properly directed" (p. 201). In the more recent literature on teaching, researchers have given form–in the conception of a teacher knowledge base–to these educators' ideas about the subject matter knowledge necessary for teaching (Shulman, 1986; Grossman et al., 1989; Ball, 1990). The domain of *pedagogical content knowledge*, for example, identifies knowledge of students' interpretations of subject matter as essential to teaching, as well as knowledge of representational forms that respond to these interpretations (Shulman, 1986). In the present study, the focus is on how its participating teachers *use* this kind of knowledge in classroom teaching in responding to students' unanticipated perspectives. In particular, the present study explores a teacher's "*Standards*-like" use of knowledge–identified by her interest in "understanding students' approaches and ideas" (NCTM, 1991;

p. 57) and in "building on the descriptions" students give (NCTM, 1991; p. 38) so that students can reason about their ideas themselves.

Knowledge alone does not ensure teachers will display an interest in "understanding students' approaches and ideas" or in "building on the descriptions" students give. Some researchers describe how "teachers' subject matter knowledge interacts with their assumptions and explicit beliefs about teaching and learning, about students and about context to shape the ways in which they teach mathematics" (Ball, 1991; p. 2; see also Thompson, 1984; Ball, 1990; Grossman, et al., 1989). Thus, beliefs related to mathematics, students' ideas, and students' roles in problem solving also were elicited from this study's teachers and compared to those described in the *Standards* visions of teaching to characterize the thinking underlying these teachers' *Standards*-like use of knowledge.

RESEARCH DESIGN

The present research is part of a larger study whose goal is to describe *Standards*-like mathematics teaching. The purposefully selected sample for this study is a group of nine teachers who attained their Master of Arts in Teaching degrees for secondary mathematics at the University of Chicago in 1992. As a group, these teachers were exceptional, all having bachelors degrees from preeminent institutions and majoring in mathematics or mathematics-related fields. The *Standards* provided the basis for their preparation during their methods course and student teaching practicum. As determined from an interview based on a framework developed for the larger study, the teachers' beliefs about mathematics and its teaching continued to reflect their support of specific views expressed in the *Standards* at the time of the study (Fernández, 1997).[1]

As part of the larger study, naturalistic lesson observations were made for each of the teachers during their fourth year of teaching. All nine teachers were observed teaching in public schools–eight in high schools and one in a junior high school. An audio-recorder

1 This is a work currently in progress.

was used and field notes taken to document these lessons. At least four observations were made per teacher and all the teachers were observed teaching ordinary, secondary level topics (lines or systems of equations, functions, area or triangle theorems). Two of the nine teachers characterized the students observed as lower level students; six characterized them as average or middle level students; and one teacher characterized his students as high level students.[2]

Using a framework developed for the larger study, three *Standards*-like teaching episodes were selected for each teacher from the larger corpus of observations. Each episode was recorded separately onto a tape and the episode's dialogue transcribed, along with any overhead or blackboard writing. An interview protocol was designed in which each teacher listened to, and read through, each episode and was questioned about her knowledge of, beliefs about, and objectives for, mathematics and her students during the episode. This "stimulated recall format" enabled me to obtain the participating teachers' perspectives on what was happening in the episode and validate the observed patterns of interaction "in a form of triangulation" (Evertson and Green, 1986). The interview itself employed a structured, open-ended questioning format (see Appendix A), although clarification and elaboration probes were utilized when deemed necessary (Patton, 1990). The teachers' responses to the interview questions also were recorded and the transcribed episodes and interview responses provided the study's raw data.

The larger study's 27 episodes were then examined for unanticipated student perspectives. Whether a student perspective is unanticipated is considered from the teacher's standpoint. For example, a student's mathematical observation about a problem that the teacher had not considered or that the teacher did not expect to investigate characterizes a student perspective as unanticipated. The teachers' interview responses were used to validate the unanticipated nature of the perspectives, as well as the teacher's interest

2 Although an attempt was made to observe all the teachers teaching the same content topic and teaching average or middle level students, scheduling difficulties made this impossible to achieve.

in or attempt to build upon the student perspective through her knowledge. The teachers' "*Standards*-like" uses of knowledge were then conceptualized using modified analytic induction (Bogdan and Biklen, 1992). The interviews also were coded inductively for the teacher thinking related to their uses of knowledge, as well as thinking concerning the resulting discussion. A case study using this episode and interview information was written for each teacher. Across-case analysis focused on finding patterns in the teachers' *Standards*-like use of knowledge and in their thinking related to these uses of knowledge.

RESULTS

Of the 27 initial episodes, 20 contained unanticipated student perspectives and the teacher's *Standards*-like use of knowledge.[3] The students' unanticipated perspectives were all conveyed through student errors, difficulties or alternative student-initiated approaches to problem solving. Student errors are mistakes in reasoning, computation or interpretation. Student difficulties are displayed obstacles to problem solving (typically conveyed through a student's question).[4] Alternative student initiated approaches to problem solving are valid (non erroneous) problem solving methods the teacher did not plan to use.

Five overall strategies describing teachers' *Standards*-like use of knowledge emerged as a result of the coding techniques described above (Table 1). Four of these strategies are suggestive of certain techniques generally used in mathematics problem solving (eg. Polya, 1973; Schoenfeld, 1983), while one provided an opportunity to examine the influence of a teacher's *Standards*-like thinking given a specific gap in the teacher's knowledge. In this section, I describe each strategy by discussing some detailed examples (from the episodes) and associated teacher thinking (from the interviews). Using the characteristics summarized under "*Standards*-like Visions of Teaching," I also describe how each strategy influenced the teacher's and students' *Standards*-like roles in the episodes

3 See Appendix for a description of the episodes not described in the paper's text.
4 A student's silence is coded as a difficulty.

(Table 2). Following this, patterns in the teachers' beliefs are summarized (Tables 3 and 4). All this information is relayed as it responds to the following questions:

(1) How can a teacher's use of mathematics knowledge be conceptualized in response to an unanticipated student perspective?

(2) What are the intended implications to the classroom discussion of these uses of knowledge and how do these relate to the *Standards*?

(3) What are some actual implications to the classroom discussion of these uses of knowledge and how do these relate to the *Standards*?

(4) What is the teacher thinking associated with these uses of knowledge?

Categories of Knowledge

Generating Counterexamples

In the *counterexample* category, the teachers generate counterexamples in response to students' unanticipated perspectives. In mathematics problem solving, counterexamples are used to disprove false generalizations (Cooney, Davis, Henderson, 1975). In keeping with the *Standards* philosophy, this study's teachers pose counterexamples to challenge their students and enable the students (versus the teacher) to examine their perspective. For example, in Nancy 1-C,[5] a student attempting to relate -f(x) to a general function f(x) makes the false generalization that -f(x) "flips over the line that the vertex is on." In response, Nancy applies her knowledge to ask the students to consider -f(x) when f(x) = x^2 + 3 (a counterexample to the student's observation). Nancy's interview reveals her intention to enable her students to reason through this false generalization themselves:

> I'm constantly throwing out *counterexamples* of things for kids. . . I was just trying to come up with something that didn't have the property she was talking about . . . I think it's something I do in a lot of classes is try to model what you do when you're stuck or how can you talk about what you just saw . . . those are things that I think are really important.

5 The designation "Nancy 1-C" consists of a teacher name (Nancy), episode number (1), and knowledge code ("C" for Counterexample). Depending on the context, the designation either refers to the corresponding episode or episode interview. Pseudonyms are used for both teachers' and students' names.

The participating teachers use their knowledge to generate counterexamples in five episodes, four in response to student errors and one in response to a student difficulty (see Table 1). In Nancy's episode, her students discuss her counterexample primarily in cooperative learning groups. In the remaining four episodes, the teacher's counterexample is discussed in whole class discussion, thereby providing more relevant opportunities to describe a counterexample's possible *Standards*-like impact on classroom discussion. For instance, in Marlen 2-C, Marlen and her seventh grade students are trying to approximate equations for lines 1 and 2 (see Diagram 1).

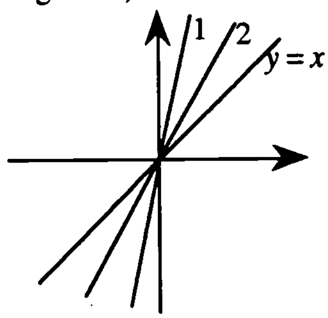


Diagram 1

Having asked the students to present their solutions in pairs (one equation for Line 1 and one equation for Line 2), Marlen has recorded the following student solutions:

Line 1	$y = 10x$	$y = 8x$	$y = 9x$	$y = 10x$
Line 2	$y = 5x$	$y = 5x$	$y = 6x$	$y = 3x$

At this point, one student (Raul) observes, "Ain't no right or wrong answer." Presumably, the generation of infinitely many solutions leads Raul to the false generalization that *any* pair of equations will describe lines 1 and 2. Thus, Marlen introduces a counterexample based in Raul's false generalization to help Raul examine his error:

T Could it be? Raul, let me ask you this. Could it be 3 x and then 5 x?
Raul (Silence)
S Yeah.
T Could this be acceptable? (T writes: $\begin{array}{l|l|} \text{Line 1} & y = 3x \\ \text{Line 2} & y = 5x \end{array}$ in blue marker)
S No.

T	If you say there's no right or wrong answers, could that be acceptable?
All	No!
T	Could "1" be $3x$ and "2" be $5x$?
S~	No, that wouldn't make sense because . . . (S~ stops)
T	Someone tell me why. Myra. Why? Why could the blue not be correct. Or why could it?
Myra	(Silence)
T	Jane.
Jane	Um, well, the, number 1 is higher than number 2. So I think it'd be um, different.
T	And what are you talking about "it?" Are you talking about the picture or the number?
Jane	Well, in the picture, number 1 is higher than number, is higher than number 2.
T	What do you guys think?
All	Yeah.
T	How many people say this $\begin{pmatrix} \text{Line1} & y = 3x \\ \text{Line2} & y = 5x \end{pmatrix}$ could be okay?
All	(No students raise their hands)
T	How many people say it couldn't be okay?
All	(Most students raise their hands)

Although Marlen is unable to involve Raul in the discussion, her counterexample motivates one student (S~) to explain why Marlen's counterexample equations don't describe lines 1 and 2. In so doing, this student introduces the idea to justify. This is picked up by Marlen, who elicits a justification and a subsequent resolution to Raul's error.

The five counterexample episodes illustrate additional *Standards*-like qualities in the interactions between the teachers and students: in all five episodes, the student with the unanticipated perspective helps to direct the lesson when the teacher's counterexample turns the perspective into a topic of discussion (see Table 2). In four of the five episodes, students play an active role in clarifying the confused students' errors and difficulties. In three episodes, justification is elicited or generated and in one episode, students challenge one another over the meaning of the student's error and the teacher's counterexample. By using their knowledge to generate counterexamples in response to students' errors and difficulties, these teachers are able to promote certain *Standards*-like qualities in the discussions between themselves and their students and amongst students.

The Follow Through

In Ed 1-FT, Ed (the teacher) and his students are learning to take determinants of 2x2 matrices so that Cramer's rule for 2x2 systems of linear equations can be derived. Given a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, Ed defines the determinant to be *ad* - *bc*. After Ed finds the determinant of matrix $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ to be -2, his student (Allie) articulates a difficulty:

Allie	I'm sorry. Does it matter what, are we going to be using *ad* minus *bc* or *bc* minus *ad*?
T	What would, what would change Allie if you did that? If this is -2 (pointing to matrix $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$), what would be different if we did it the way you said it?
Allie	Well, if you did 2 times 3 minus 4 times 1, for example, it'd be 2.
T	It'd be 2.
Allie	Ohh. Okay.
T	So it will determine the sign. So it is important the order in which you go.
Allie	All right.
T	Okay. So it will determine the sign.
Allie	It's the same answer, just different signs.

In this exchange, Ed illustrates a practice in mathematics problem solving which can be aptly described as a *follow through*. In a follow through, the idea is to explore the implications of the posed mathematical thought by continuing or following through with it. Schoenfeld (1983) alludes to follow throughs in his treatise on problem solving and demonstrates how they can lead to dead ends, as well as mathematically useful ideas. When a teacher enables a student to follow through on an error or difficulty, it provides an opportunity for the student to reason through the validity of her idea (as illustrated above). Ed elaborates on this and the thinking related to his use of the follow through strategy:

> So Allie has a question. . . she asks, . . ."Does it matter what, what order, um, we're gonna take this determinant in?" And so at that moment, this role. . . for me, is instead of just telling Allie the answer is to try and teach her a little bit about how she could try and think it through herself. And so, instead of just saying. . . "Well the sign, the sign will change" . . . when she has a question, to try and do the *follow through*. So that's what I'm trying to model for her. And that's what I see my role is. . . to model that mathematical behavior that I'd like her to use. I ask her to actually do it.

In episodes Ed 1-FT, Nancy 3-FT, Marianne 1-FT and Rosie 1-FT, the teachers' knowledge that a follow through lead to a *contradiction* (as in proofs by contradiction) is integral in enabling students to clarify errors or difficulties for themselves (see Table 1). In Dana 2-FT, a different kind of follow through is illustrated: rather than enabling the student to follow through on an error until a contradiction arises, Dana (the teacher) follows through on an idea resulting from a student's error. In her episode, Dana's students are trying to determine whether the information given in Diagram 2 implies the lines are parallel.

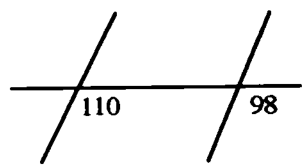


Diagram 2

Dana's intention (articulated in the interview) is for students to see that unequal corresponding angles imply the lines are not parallel. Sam however misses this point:

T	Do it Sam.
Sam	They're not parallel.
T	They're not parallel. What line of reasoning did you use.
Sam	Well, cause 98 and a hundred ten don't equal one eighty.
T	98 and a hundred what?
Sam	Don't equal one, they're more than one eighty.
T	Right. But why would you add those up? Why would that matter?
Sam	Cause, um, 98 is a vertical angle.
T	Oh, like up here maybe. 98 [writes 98 "vertical" to the original 98]. And if these lines were parallel, what would be true.
Sam	98 and one ten would equal.
T	Yeah. Maybe I'll just write 98 doesn't equal one ten therefore the lines aren't parallel.

In this episode, it is the teacher following through on a student's idea (98 is a vertical angle) that enables the student to solve the problem. Although Sam may not have clarified his misunderstanding, Dana uses her knowledge of mathematics to follow through on his idea and introduce alternate interior angles as a tool for solving the problem:

> I was confused on where he was going. But I didn't want to just turn him off because I didn't quickly get it . . . And so a lot of times when they've given a wrong answer, I try to talk them through it to kind of change their answer around . . . when he said "vertical" I was thinking, "Oh good we can at least get it, a correct answer, out of this because if the lower one was 98 then it's match is 98. And if these lines were parallel these two angles would have to be the same because they're alternate interior."

After this exchange, Dana enables another student to reason through the problem using corresponding angles. In this way, Sam sees the solution he was having trouble with carried out and justified. Dana's follow through sustains Sam's participation in the problem's discussion and because of Sam and Dana, a different method is generated for solving the problem (see Table 2).

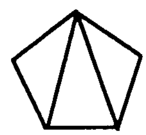
Table 2 summarizes some of the *Standards*-like characteristics resulting from the teachers' using follow throughs. In three of the five episodes, the student with the unanticipated perspective continues participating in the classroom discussion and in two of the five, students play a role in clarifying their peers' errors. In two episodes, the confused students are able to clarify their misunderstandings with help from the teacher or their peers. Thus, the teachers' use of a follow through enables students to continue playing an integral role in the problem under discussion.

Doing or Thinking About a Simpler or Related Problem

The strategy of doing or thinking about a simpler or related problem when a given problem is too difficult is a familiar one (Polya, 1973; Schoenfeld, 1983). The premise underlying this strategy is to be able to generate ideas from the easier case that can be applied to the more difficult one so that the original problem can be solved. As with the first two categories, a teacher's suggesting a simpler problem places the responsibility for addressing the error or difficulty in the student's domain. In some cases, using this strategy can illustrate mathematics' connections to other problems or to different modes of representation. Four of the present study's teachers employ this technique in response to some of their students' errors or difficulties (see Table 1).

In Ann 1-S/R, Ann and her students are trying to find the angle sum measure of a pentagon. However, several of Ann's students are having difficulties remembering the angle-sum measure formula for a polygon. Having spent considerable time deriving this formula in the days prior to the episode, Ann expresses her surprise at her students' memory loss during the interview. In the episode, she reintroduces the idea underlying the previous days' derivation to see if this will jog her students' memory:

T Okay. Let's see if this will jog your memory. We drew in diagonals [draws diagonals in the pentagon]. How many triangles do I make?



S~ Oh.
S I don't remember.
S~ Three.
T Three triangles, right? So we said n minus 2, right? [T writes $s(n) = (n - 2)$] We have 5 sides, we're making 3 triangles and each triangle is worth how much?
S Uh.
T How much in each triangle. How many angles, what's the angle sum of each triangle?
S One eighty.
T One eighty, yeah. So this is our formula [$s(n) = (n - 2)180$]. Remember that's the one we . . . developed in class. . . So let's find out, for a pentagon, what is s of 5. What's it gonna be? The sum of the angles in a pentagon. Plug it in.

By introducing a related geometric representation of the angle-sum measure formula, Ann hopes to jog her students' memory for the algebraic representation and enable them to re-derive it. Unfortunately, the students play a minimal role in resolving their difficulty in this episode. Nevertheless, it is important to note the connection made by Ann and two other teachers who employ this strategy to respond to their students' perspectives (see Table 2).

One of Nancy's episodes presents an interesting application of knowledge: the reader may have noted that Nancy's third episode is coded under the "follow through" (Nancy 3-FT) and "simpler or related" (Nancy 3-S/R) category. This is because Nancy uses these strategies in succession to assist a student (Aisha) who first makes an error and then displays a difficulty. In this episode, the students are working on the problem of drawing the graph of f(x+3) given f(x) [see Diagram 3]:

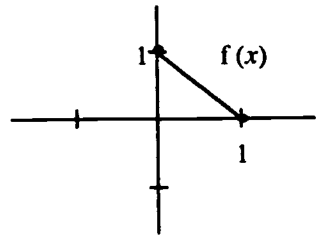


Diagram 3

One student observes that f(x+3) will "start at negative 3" and the dialogue develops as follows:

T	It starts at negative 3, so where's it gonna go?
Aisha	Um, negative, um, just like negative f of x, at 3.
T	It's gonna look like this [pointing to a graph of -f(x) on the overhead]?
Aisha	(Silence)
T	Tell me where, tell where the ends are gonna be. Tell me where these two points are gonna be [pointing to (1,0) and (0,1)] on the graph of f(x+3).

Nancy does a follow through in response to Aisha's error. When Aisha is silent, Nancy asks her to consider the simpler problem of translating the endpoints (1,0) and (0,1):

> That's something I started to try to do, say, "If you can't do the whole thing at once because it is hard, take one thing and see what happens to this." . . . if you're stuck in the middle, move the endpoints. You know, pick an easier case, pick something you can actually quantify like the point zero, one or the point one, zero. Can you do that part? Which is maybe an easier problem to go with or at least something specific to grab on to.

In this episode, Nancy's knowledge that mathematics can be done by applying something you know in an unknown situation enables one of her students to make the correct translation. Nancy's suggestion to focus on the simpler problem of translating the endpoints also generates justification and student challenges over how to solve the problem (see Table 2).

Understanding and Incorporating a Student's Method

One of the most straightforward uses of teacher knowledge is in understanding a student perspective. At some level, this use of knowledge is illustrated in all the categories above since understanding enables a teacher to generate appropriate counterexamples, follow through to contradictions or solutions or suggest simpler or related problems. In responding to an alternative student-initiated method, understanding can facilitate a teacher's incorporating the student's method into a lesson.[6] In keeping with the *Standards*, incorporating alternative student-initiated methods conveys that there exists more than one *correct* path to a problem's solution and that this path need not be the teacher's (NCTM, 1991; p. 37).

The present study's teachers use their understanding of mathematics to incorporate alternative student-initiated methods in six episodes. For instance, in Lydia 3-U/I, Lydia and her students are working on solving the system $\begin{cases} 8x - 4y = -11 \\ 6x + 2y = -2 \end{cases}$. Lydia intends her students to solve the system by eliminating the y variables when a student (Bill) suggests the following:

Bill	Okay, well. What you have to do is, um, multiply the first equation by negative six.
T	Okay. What do you, what is your goal here?
Bill	The goal is to get rid of one of the variables. So multiply the top equation by negative six. And the bottom equation by eight. [T writes $-6(8x - 4y = -11)$ and $8(6x + 2y = -2)$.
T	Okay. Good. What if, let's say we wanted to get rid of the y's Bill?
Bill	Well then we would get rid of the y's.
T	That, that's fine. That's exactly right.
Terry	No, but how?
Mike	You could do it that way [referring to Bill's method].
Terry	He's just saying, "I got negative six." How did you get negative six?
Bill	Okay. What you want to do is, linear equations that you add them together.
Terry	It's confusing to have
Bill	(Interrupting) What you, what you need to do is
Gary	(Interrupting) Do it the normal way!

6 Incorpating a student's method is much like enabling the student to *follow through* on his or her method. The title *Understanding and Incorporating* is used to underscore the role of teacher understanding in incorporating students' alternative approaches in this category and the upcoming one.

Bill	So you want the x's to have a negative number and their opposite a positive number. So they cancel each other out. So you, when you add them together. . .
Terry	So why did you . . .
Bill	(Interrupting) So obviously the goal is $48x$. You take the two variables with a x and you try to make them like. . .
Terry	Why didn't you just do the y's?
Mike	It doesn't matter which ones you do.
Terry	I know it doesn't matter but why?
Mike	Well he wants to do the x's. There's nothing wrong with that.

Lydia's incorporation of Bill's method generates an argument between two camps led by Bill and Mike (arguing that eliminating the x or y variables works equally well), and Terry and Gary (arguing that eliminating the y variable is easier). As the discussion develops, Gary's thinking reveals that his opinion is embedded in a memorized method for solving the problem which he has inappropriately adapted:

Gary	So, say I picked the y.
T	Uh-huh.
Gary	I would change 2 to a negative and then it would be negative 2. So then I'd multiply
T	(Interrupting) Why are you changing?
Gary	Because that's what you told me to do! I swear it! You said change the bottom to a negative and then it would be like negative 2. And then I'd multiply that by like $6x$ plus $2y$ equals negative 2. And then
Mike	That's right but it's supposed to be positive 2. Cause that's a negative 4 up there, so you want
Gary	(Interrupting) Oh, oh. I didn't. I thought. All right. So then I'd multiply it by 2, the bottom one by 2 cause I wanted to get rid of the y. And then one of them, that 2 would cancel out with the negative 4. And then you
T	Good. Yeah. [T writes $\begin{matrix} 8x - 4y = -11 \\ 2(6x + 2y = -2) \end{matrix}$]
Mike	All you'd have to do was $12x$ plus $4y$ equals negative 4.
Gary	Then what are you doing (talking to Bill)? He did it a really long way!
T	All he did, all he did was try and get rid of the x's first.
Gary	Well he's confusing me! It's not helping!

By incorporating Bill's method, Lydia generates opportunities for students to articulate difficulties they are having with the method of elimination and for other peers to help. In the episode's development, Mike suggests working out both methods so their equivalence (number of steps involved and the solutions obtained) can be demonstrated. In the end, Terry and some students maintain their preference for eliminating *y*, while Gary expresses wanting to understand Bill's method. By enabling her students to discuss two methods for solving this system, Lydia has validated the opinions in both camps and even encouraged a member from one camp (Gary) to seek understanding for the other group's method.

All the episodes in this category produce (at least) one method that is an outgrowth of the student perspective and the teacher's use of knowledge that directs the lesson (see Table 2). As illustrated in Lydia 3-U/I, justification is elicited or generated in all six episodes and connections are made or elicited in four of the six. In three of the six episodes, students attempt to help peers with errors or difficulties that are related to the alternative method while in four of the six, students have disagreements over the two methods. The remaining *Standards*-like qualities substantiate the value in incorporating student-initiated alternative methods into lessons.

"Understanding" and "Incorporating" an "Alternative" Method

Although a teacher's understanding of a student perspective can facilitate her decision to incorporate the perspective into the lesson, it is not a necessary criteria. In this episode, the teacher (Marianne) articulates a particular gap in her knowledge (she does not understand her student's perspective). She nevertheless adheres to the *Standards* philosophy that students be given an opportunity to justify or explain their thinking, thereby providing an opportunity to explore how a teacher's *Standards*-like thinking can influence the *Standards*-like qualities generated in classroom discussions in the absence of knowledge. The problem under discussion in this episode is the following:

> Nina is six years older than her brother Ramon.
> Three years ago, she was twice as old as Ramon. How old is each now?

As the episode starts out, Marianne asks Hal to explain the solution he has written on the blackboard (note: his "solution" yields a correct answer):

$$\begin{array}{c} 6 \\ \underline{\times 2} \\ 12 \\ \underline{+3} \\ \text{Nina} = 15 \\ \text{Ramon} = 9 \end{array}$$

Hal	It says she's twice as old than him. They say she's twice as old than him. So I did 6 times 2. It was, she was 12, he was 6 three years ago. So plus 3, is 15 and 9.
T	Tell me again. Go through the logic again. Where'd you, how'd you start with this 6.
Hal	Because it says she 6 six years older than him and, and, like 3 years ago, she was twice as old as, so when I find out, how, how old they were 3 years ago, so I . . .
T	(Interrupting) How'd you find out is what I want to know.
Hal	6 times 2. Because then she's twice as old than him. That's 12.
T	But again Hal. It never said she was 6 years old. It says she's 6 years older than he is.
Hal	It's the easiest way to do it. No that's how I figured it. Cause 6 times 2 is 12.
T	I'm trying to figure out why that works!
Hal	It's just the opposite. You don't understand! It's just like in my brain, but I don't know how to explain this!
S	Is that the right answer or not?
S	That is the right answer.

In this exchange and in her interview, Marianne articulates her failure to understand Hal's method; what she does "understand" is that his method is unclearly stated and requires a justification or explanation:

> I still don't understand why his method worked. . . you know, it could be [done using] a system [of equations], maybe it didn't have to be. But for me, I kind of needed to make it one. . . So that makes it hard, cause it's harder to explain. . . and he wasn't very good at saying *why* he got it . . . he certainly could show us *how*. But, you know . . . how come he decided. "Okay, I'm gonna take 6 times 2 and add it." To say why that was sound. . . wasn't something he could quite put a finger on and I just really couldn't either.

Because of her interest in Hal's justifying his method, Marianne "incorporates" his method into the lesson by providing Hal with nine opportunities to explain his thinking. Unfortunately, Hal's "explanations" all amount to his repeating the sequence of steps in his blackboard solution. As Hal attempts to explain his method, Marianne's *students* also begin eliciting a justification for his method. Unable to understand his thinking, Marianne decides to introduce the conventional systems approach for solving the system.

Hal's explanations do not make his thinking clear. Throughout his explanations however, he insists that he does *not* "assume" Nina is 6 years old. Marianne describes how Hal's insistence gave rise to her feeling that he had solved the problem using a method she could not comprehend:

> . . . a lot of kids, myself included, used to be able to do math without really having to understand it that well. And it was kind of on that level. That, like. . . he couldn't explain it, but he could make it work for himself.[7]

This episode illustrates the complicated role that knowledge and beliefs can play in a teacher's decision-making. Although the teacher's knowledge helps her understand the student perspective is unclear, it fails her in clarifying this perspective. Nevertheless, the teacher's underlying belief that there is something sound about Hal's method and her hope that he justify his method can be interpreted as influencing this episode's *Standards*-like qualities (Table 2): Marianne's desire that Hal justify his thinking compels her to "incorporate" his method which initiates the student demand for justification. Despite their inability to understand Hal's method, the thinking underlying Marianne's actions helps to explain some of the *Standards*-like qualities generated in this episode.

Teacher Thinking Related to Knowledge Categories

Although these teachers' backgrounds in mathematics and ability to apply mathematics appear to play an integral role in the episodes above, these episodes also illustrate the significance of a teacher's thinking or beliefs. The influence of this thinking is especially visible in Marianne 3-"U/I" in which the teacher's knowledge is somewhat lacking. In discussing the knowledge strategies they used above, the teachers expressed

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7 In fact, during her interview, Marianne realizes that Hal could have *informally* reasoned through this problem: if Nina and Ramon are *always* 6 years apart and at some point she is twice as old as he, then Nina was 12 and Ramon 6 at that time. Letting N and R be Nina and Ramon's ages at the time in question, if $N - R = 6$ and $N = 2R$, then $N = 12$ and $R = 6$.

certain goals or intentions (see Table 3).[8] These goals, stated primarily in terms of characteristics the teachers hope to instill in their students, reveal *Standards*-like aspects of the teachers' beliefs about mathematics, teaching and learning.

As encouraged in the *Standards* (NCTM, 1991; p. 37, p. 57), four of this study's teachers hoped to convey that their students' ideas are "valued" or "validated." Two teachers convey this goal in attempting to incorporate students' alternative approaches to problem solving and Dana expresses it in following through on her student's observation. In Rosie 2-C, Rosie articulates her hope that using a counterexample to discuss a student's (Jan's) error conveys to students the value she places on *all* their perspectives:

> I hope what this conveys to the kids is that I *value* their suggestions even when they're wrong. Because one thing that I'm really conscious of in the class is that I want my kids to feel free to make mistakes. . . . Um, and I also think it makes Jan feel pleased that she suggested it even though it was wrong. And the next time something like that comes up she'll be bold enough to speak again and suggest something that maybe will be right!

Another part of the teacher's role in a *Standards* classroom is to enable her students to "explore," "invent," and "conjecture," as well to encourage "curiosity and spontaneity" during problem solving (NCTM, 1991; p. 1, p. 3, p. 115). Five of the present study's teachers articulated a related goal of enabling students to "experiment" or "play" with mathematics. In following through on Sam's idea in Dana 2-FT, Dana expresses wanting her students to learn "it's worthwhile to try a lot of things" during problem solving. This goal imparts the participating teachers' emphasis on the exploratory aspect of mathematics, particularly with regard to student errors and difficulties.[9] Marlen's reflections in generating her counterexample for Raul also illustrates this emphasis:

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8 It is important to note that only "goals for" the episodes are included in Table 3, not "observations about" the episodes. For example, if a teacher states that she was trying to enable her students to actively participate in the classroom discussion, this is a goal. If a teacher states that her students were actively participating, this is not a goal but an observation.

9 Five of the six teachers who articulate this goal are applying knowledge categories that respond to student errors or difficulties.

. . . I don't know how much a 7th grader can, they're picking this up, as far as what goes on behind the scenes. But that it's okay to come up with a *hypothesis* and then check it several times and then ask questions to see if that really holds true for every case. Because we had Raul thinking everything and that's a natural assumption, but then you have to keep checking that and see if there's a pattern there. So I try to teach them to look at patterns and make generalizations for patterns.

In the quotation above, Marlen's association of Raul's error with a "hypothesis" discloses a pattern in the teachers' thinking that also is worth noting. In Rosie 2-C, Rosie calls her student's error a "theory." In Nancy 2-C, Nancy calls her student's error a "rumor." Although the goal of conveying mathematics' "experimental" or "playful" side was not explicitly stated in all these teachers' interviews, the teachers' language supports this goal by eliciting a dimension to errors and difficulties that is more conducive to exploring, rather than overlooking, them.

Interestingly, some of the teachers' goals correspond to the *Standards*-like qualities generated in the episodes (thereby revealing another layer in the interaction between the teacher and students). In discussing Jan's erroneous clue in Rosie 2-C, for example, Rosie describes wanting her students to learn "that in mathematics, a new idea is thought to be true because we can all agree on it and make some plausible argument for why it is true." Thus, part of what she is trying to teach her students through her counterexample is that justification is part of the problem solving process. In giving her students a simpler or related problem in Dana 1-S/R, Dana describes wanting to convey to her students that "connections" can alleviate the feeling that in mathematics, "every problem is an individual mountain to conquer." The goal for students to make or see justifications was articulated by four teachers, as was the goal to make or see connections (see Table 3). The last goal, mentioned in *every* teacher's interview, was that of ensuring students actively participated in the process of examining the unanticipated perspective presented.

The Traditional/*Standards* Conflict

Despite their support for certain aspects of the *Standards*, some of the teachers' reflections contain views that are more conflicted than those expressed above. These views or "Traditional/*Standards*" conflicts refer to situations in which the teachers describe struggles within themselves, between themselves and their students, or amongst students over whether to emphasize certain "traditional" or "*Standards*-like" aspects of mathematics problem solving. For example, in Kama 1-U/I, Kama describes a struggle over conveying a more conventional problem solving method following her student Anna's less conventional one (the fear being that the teacher's method could invalidate the student's):

> I mean the question is, "Does that, does that take away from Anna's explanation by throwing in my own afterwards? Or not?" And I guess because at the end I am saying, "This is two different ways". . . and I'm not sure that it was useful for them to have it [the teacher's explanation] there. I'm not sure they heard it or they cared. . . . and so maybe I wouldn't have done that. Or maybe I would. I don't know.

Although Kama had encouraged and valued the student's method (=*Standards*-like), she nevertheless feels a responsibility to "cover" a conventional method (=Traditional). In the end, Kama is not sure her students paid attention to her explanation.

Traditional/*Standards* conflicts arose in eleven of the teachers' episode interviews (see Table 4). As in Kama's episode, four conflicts concern alternative problem solving approaches and how these play out during classroom problem solving. In Lydia 3-U/I, Lydia relays how her students' lack of support for Bill's alternative approach may not be interpreted as a matter of preference, but as a personal attack on a student whose thinking is unconventional:

> I remember worrying in this one that. . . Bill would feel that um, they were saying, "No way, you can't do it his way." I remember feeling, you know, protective of him and that that was okay to do it that way. And uh, I didn't want him . . to be afraid . . of bringing a method or an answer to the group again. . . I didn't want him to feel attacked. . . So that, that was a part of the. . . that worrying that he was never gonna say anything again. Are they picking on him? Or is it, is it clear that it's just . . . they wanna be able to do it another way. . . and to convey that to him also. That they weren't picking on him. It was um, they were just seeing things a different way.

Although Lydia values students having and acting on their ideas (=*Standards*-like), she worries that in expressing these ideas, these students might receive the message that certain valid approaches are in fact *not* valid (=Traditional). Moreover, she worries that this might discourage Bill from expressing himself in the future.

In two conflicts, the teachers identify their lack of success at trying to implement certain aspects of the *Standards*. In Marianne 3-"U/I," Marianne describes how her student's focus on explaining "how" and not "why" he did a problem a certain way results in her opting for a conventional approach . In the end, she expresses her disappointment "that we couldn't see a way to explain his method in class. . . and that, that he didn't come up with a better explanation for it. That nobody did." Likewise, Rosie intends her students to use her counterexample to reason through their error, but in the end, she corrects the error herself for lack of time.

In the three remaining conflicts, the teachers describe a desire to undo certain traditional patterns of interaction that students have learned in math classrooms. Two teachers describe how students' fear of being wrong in mathematics inhibits their attempts to "test" ideas during problem solving. This traditional student thinking comes into conflict with both teachers' support for experimenting and testing ideas in mathematics. As Nancy explains:

> They're a little bit afraid to say a lot for fear of being wrong. They don't, they don't like to discuss things. . . They'll spit out an idea and then kind of sit there and let it hang for a minute and then if it . . .still sounds okay, they might add something to it. And if not, they jump back and say, "No, no I'm just kidding. That wasn't what I meant." . . . And so I find myself constantly saying, "Go ahead and try this, if you think something, you know make up something. Look at it. See if it'll test what . . . you know." I find myself doing that a lot.

It is important to emphasize that these teachers' Traditional/*Standards* conflicts describe what they were feeling or thinking, and not necessarily what was happening, at the time of the episode. Nevertheless, acknowledging these conflicts is critical for a complete and accurate description of the teachers' thinking and their roles during these "best case" situations. Despite their success at generating certain *Standards*-like qualities in their lessons, these inner conflicts reveal the difficulties these teachers experience in their attempts to implement certain aspects of the *Standards*.

SUMMARY

The conceptualization of knowledge use reported in this article was motivated by a search for a teacher's interest in "understanding students' approaches and ideas" and in "building on the descriptions" students give so that students can reason about their ideas themselves. The resulting conceptualization provides classroom-instantiated teaching strategies that can generate *Standards*-like discussion in responding to unanticipated student perspectives, as well as the teacher thinking associated with these strategies. This study's episodes illustrate how a teacher's using these strategies can turn a student's unanticipated perspective (particularly an invalid perspective) into a *worthwhile* topic of classroom discussion, as well as enable students to generate their own ideas for problem solving. The episodes also illustrate how these strategies can encourage student participation during classroom problem solving, particularly in the form of students helping or challenging one another. The justification and connections that were elicited or generated as a result of the teachers' using these strategies also substantiate the value in applying these strategies in response to students' unanticipated perspectives.

Although it has been noted that creating classroom settings in which students can explore their perspectives has become an important priority, it has also been noted that most studies relating teacher thinking to instruction describe teachers' limited views of mathematics or gaps in mathematics knowledge. The exceptional background of the present study's teachers complements this research by describing the teachers' *Standards*-like thinking and uses of knowledge and their influence on classroom discussion, thereby furnishing much-needed "images of the possible." (Shulman, 1983). The question remains however, of how other teachers can be encouraged to explore and apply similar strategies in their own teaching. In this last section of the paper, I describe how this study's results might be used in enabling prospective teachers to examine their own uses of knowledge and beliefs to respond to students' unanticipated perspectives in ways that promote the *Standards* visions of teaching.

The strategies used by this study's teachers in responding to their students' unanticipated perspectives are suggestive of certain techniques generally used in mathematics problem solving. It follows that one place for prospective teachers to acquire knowledge about these (and similar) strategies is during their own study of mathematics. In this area of teacher preparation however, Ball (1990) found that majoring in mathematics does not ensure the knowledge necessary for the kind of teaching described in this article and offered as one explanation the "rules based" understanding that dominates the study of mathematics. The present research suggests one way in which the study of mathematics might be enhanced for prospective teachers, namely, by articulating and discussing how the strategies involved in doing mathematics can be applied in certain classroom situations to encourage student involvement in problem solving. For example, the strategies of generating counterexamples, following through on mathematical ideas, doing simpler or related problems, or pursuing alternative problem solving approaches are concrete instances of what is involved in doing mathematics. Rarely, however, are these or other strategies made explicit in mathematics courses (Schoenfeld, 1983). Making such strategies explicit in prospective teachers' own study of mathematics and providing them with opportunities to apply and discuss these strategies during student teaching might assist them in developing their own repertoire of strategies for promoting *Standards*-like discussions.

The usefulness of the teachers' beliefs (as articulated in their goals) to teacher preparation is less direct. In the research domain, the present study illustrates a *positive* influence of beliefs on instruction in contrast to the negative influence illustrated in prior research (Thompson, 1984; Ball, 1991; Stein, et al., 1990). In the actual practice of teacher education, however, the usefulness of the teachers' beliefs comes up against a significant obstacle: the pervasive belief among teachers that teaching primarily involves telling students what they need to know and giving them practice in it (McDiarmid, et al., 1989). Educators propose enabling prospective teachers to identify and examine these traditional beliefs and the influence they have on prospective teachers' instruction (Grossman et al., 1989). Unfortunately, this exercise does little to expose the traditional teacher to untraditional beliefs and how these influence teaching. The beliefs of the present study's teachers can provide this exposure, as well as relations between these beliefs and the teachers' *Standards*-like strategies. For example, illustrating how a decision to generate a counterexample can convey a teacher's value for student thinking, or how a decision to follow through can relay mathematics' experimental side, or how all of this study's strategies were intended to enable students to examine their own thinking is integral in understanding the influence of the participating teachers' *Standards*-like strategies. In another example, this study's teachers articulate how introducing simpler or related problems can convey mathematics' connections. Finally, the teachers' intentions to have their students justify their ideas is especially underscored in the absence of one teacher's knowledge of her student's thinking (in the "understanding/ incorporating" category). These illustrations can serve as a contrast to prospective teachers' traditional beliefs, providing concrete examples of what can happen in untraditional classrooms and enhancing understanding of what to expect in these untraditional situations.

In addition to conceptualizing knowledge and describing beliefs, part of the thinking underlying these teachers' strategies provide conceptualizations that can be useful in encouraging attempts to apply *Standards*-like knowledge in teaching. For example, understanding that a student's error or difficulty can be thought of as a "theory" or "rumor" might facilitate a teacher's decision to explore the error or difficulty with students. The Traditional/*Standards* conflict furnishes another important conceptualization. In describing the conflicts they experience as they attempt to implement certain aspects of the *Standards*, this study's teachers articulate the kinds of doubts and difficulties that can arise during such attempts. Some of these conflicts acknowledge actual difficulties that arise during implementation and some acknowledge conflicts in the teachers' thinking. Others acknowledge failed attempts. In describing these conflicts, these teachers validate the existence of comparable conflicts in other teachers (particularly teachers who possess traditional beliefs). It is hoped that framing these conflicts according to the traditional and *Standards*-like characteristics they reflect will enable prospective teachers to discuss similar conflicts constructively so that their future attempts to implement certain aspects of the *Standards* in their own teaching will be encouraged.

CONCLUDING REMARKS

In keeping with the *Standards* philosophy, it is important to emphasize that this study's focus is on illustrative, not prescriptive, purposes. As described above, all the categories developed for the uses of knowledge and teacher thinking in promoting the *Standards* are illustrative in nature. In discussing the educational significance of my work, I've tried to focus on how these results can help prospective teachers examine their own beliefs and instructional practices by illustrating what alternative situations are like–in thinking and practice–for the teachers in this study. While the situations explored in this study (unanticipated student observations) are traditionally considered obstacles to problem solving, these episodes also illustrate how such twists and turns can be interpreted as a natural part of mathematical problem solving. By illustrating how this study's teachers use their knowledge and thinking in responding to the "unanticipated," it is hoped that other teachers will be encouraged to examine their own uses of knowledge and thinking in responding to students during these traditionally problematic, but presently promising, situations.

Table 1

Teacher Knowledge Categories Used in Responding to Unanticipated Student Perspectives

Teacher Knowledge Category	Form of Unanticipated Student Perspective	Episode Appearing In
Counterexample	Error	Ann 3-C, Marlen 2-C, Nancy 1-C, Rosie 2-C
	Difficulty	Nancy 2-C
Follow Through		
to Contradiction	Error	Marianne 1-FT, Rosie 1-FT, Nancy 3-FT(=Nancy 3-S/R)
	Difficulty	Ed 1-FT
to Solution	Error	Dana 2-FT
Simpler or Related Problem	Difficulty	Marlen 3-S/R, Ann 1-S/R, Nancy 3-S/R(=Nancy 3-FT), Dana 1-S/R
Understanding and Incorporating an Alternative Method	Alternative Method	Ann 2-U/I, Ed 3-U/I, Kama 1-U/I, Kama 3-U/I, Lydia 1-U/I, Lydia 3-U/I,
"Understanding" and "Incorporating" an "Alternative" Method	"Alternative" Method	Marianne 3-"U/I"

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Table2

***Standards*-like Qualities or Activities Generated in Episodes**

Standards-like Qualities Generated[1]	Episodes Generated In	Relative Frequency[2]
Unanticipated student perspective directs the lesson (p. 30)	All	20/20
Method, problem or idea is generated as an outgrowth of student perspective and teacher knowledge (p. 13, p. 25, p. 30)	Rosie 2-C, Rosie 1-FT, Dana 2-FT, Nancy 2-C, Ann 2-U/I, Ed 3-U/I, Kama 1-U/I, Kama 3-U/I, Lydia 1-U/I, Lydia 3-U/I	10/20
Justification or explanation elicited or generated (p. 47)	Ann 3-C, Nancy 1-C, Nancy 2-C, Marlen 2-C, Ed 1-FT, Rosie 1-FT, Marlen 3-S/R, Nancy 3-S/R, Ann 2-U/I, Ed 3-U/I, Kama 1-U/I, Kama 3-U/I, Lydia 1-U/I, Lydia 3-U/I, Marianne 3-"U/I"	15/20
Connections elicited or generated (p. 45)	Dana 1-S/R, Marlen 3-S/R, Ann 1-S/R, Ann 2-U/I, Ed 3-U/I, Lydia 1-U/I, Lydia 3-U/I	7/20
Continued whole class participation or inclusion of student with unanticipated perspective (p. 57)	Nancy 2-C, Dana 2-FT, Ed 1-FT, Marianne 1-FT, Dana 1-S/R, Nancy 3-S/R, Ann 2-U/I, Lydia 1-U/I, Lydia 3-U/I, Kama 1-U/I, Kama 3-U/I, Marianne 3-"U/I"	12/20
Student(s) attempt to help student with error or difficulty (p. 30, p. 47)	Ann 3-C, Marlen 2-C, Nancy 1-C, Rosie 2-C, Dana 2-FT, Rosie 1-FT, Marlen 3-S/R, Nancy 3-S/R, Kama 1-U/I, Lydia 1-U/I, Lydia 3-U/I, Marianne 3-"U/I"	12/20
Student with error or difficulty publicly self-resolves with help from teacher or students (p. 47)	Nancy 2-C, Ed 1-FT, Marianne 1-FT, Dana 1-S/R, Nancy 3-S/R	5/20
Student(s) challenge or disagree amongst themselves or with teacher (p. 47)	Rosie 2-C, Nancy 2-C, Rosie 1-FT, Nancy 3-S/R, Ann 2-U/I, Kama 3-U/I, Lydia 1-U/I, Lydia 3-U/I, Marianne 3-"U/I"	9/20

1 Following each *Standards*-like quality is the page number in the *Standards* in which the stated quality is supported.
2 To calculate this column's entries: ratio numerator equals number of episodes in corresponding "Episodes Generated In" column and ratio denominator equals total number of episodes.

Table 3

Teachers' Goals for Students and Discussion

Teachers' Goals	Episode Interview[1]	Teacher Count[2]
Student ideas are "validated" or "valued"	Rosie 2-C, Dana 2-FT, Ann 2-U/I, Ed 3-U/I	4/9
Students learn to "experiment" or "play" with mathematics or to "try" things in problem solving	Marlen 2-C, Nancy 1-C, Nancy 2-C, Dana 2-FT, Rosie 1-FT, Marianne 1-FT	5/9
Students make or see justifications	Rosie 2-C, Rosie 1-FT, Kama 1-U/I, Kama 3-U/I, Lydia 3-U/I, Marianne 3-"U/I"	4/9
Students make or see connections	Dana 1-S/R, Marlen 3-S/R, Ann 2-U/I, Ed 3-U/I	4/9
Students actively participate in examining the unanticipated perspective presented	All	9/9

[1] This column lists the episode interview in which the teacher articulated the stated goal.

[2] In this column's ratio, the numerator gives the number of teachers articulating the stated goal and denominator is the total number of teachers.

Table 4

Traditional/*Standards* Conflicts

	Standards-like	Traditional	Conflict	Episode
Alternative approaches	Teacher values alternative approaches	Some students want to be shown one correct method	These students will find alternative approaches confusing	Ann 2-U/I
	Teacher encourages a student's approach	Teacher feels a need to cover a conventional approach	Teacher's approach will invalidate the student's	Kama 1-U/I
	Teacher values alternative interpretation	Some students feel problems should have one correct and quick solution	These students feel two approaches is too time consuming	Kama 3-U/I Ed 3-U/I
	Teacher values student's alternative approache	Some students convey that a particular method is correct and alternative approach is not	Student with alternative approach is intimidated or discouraged from participating	Lydia 3-U/I
Unsuccessful Attempts	Teacher wants student to "justify" his thinking	Student "justification" is to restate steps in his method	Unable to understand student, teacher feels pressed to present conventional solution	Marianne-3"U/I"
	Teacher generates counterexample so students can self resolve	Teacher eventually gives student answer for lack of time	Student will not really understand teacher's answer	Rosie 2-C
Traditional Interaction	Teacher would like students to "test" their ideas	Students are afraid to discuss their ideas for fear of being wrong	Enabling students to understand that testing ideas is part of mathematics problem solving	Nancy 1-C, Marianne 1-FT
	Teacher would like students to investigate their difficulties	Students expect teacher to address their difficulties	Enabling students to "follow through" on their difficulties	Ed 1-FT
	Teacher encourages student to explain to another student	Some students value only teacher's explanation	These students won't take their peers' explanations seriously	Lydia 1-U/I

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Appendix A
Interview Protocol

Interview Introduction

As you know, earlier this fall I observed you teaching your students on the topic of [fill in topic]. And as you also know, it is not always possible to tell, just from observation, what a teacher is thinking during her exchanges with students. So what I'd like to do during this interview is give you an opportunity to articulate your thinking during certain classrom episodes that I've selected from the observations I made.

Let me explain how the interview is going to be conducted. I am going to play three episodes for you on this recorder, one at a time. Beginning with the first episode, I'll begin by giving you background on the lesson the episode was selected from to help remind you of the overarching lesson. Then I'll give you a transcription to help you follow along as I play the episode for you. Remember the transcription may not be literal and is only intended to help you hear some of the voices that are hard to pick up on the tape and to remind you of any blackboard or overhead writing (show her transcription format). After I play the episode once, I'll ask you if you'd like to hear it again.

Once we've finished listening to the first episode, I am going to ask you some questions about it and I'll record my questions and your responses on the recorder and I'll take some notes. If you don't understand a particular question, please let me know after I ask it and I'll try to rephrase it.

Once we've covered all the questions for the first episode, we'll repeat the protocol I've just described for the remaining episodes.

Do you have any questions about this protocol before we start?

Questions

0. Do you remember the episode you just heard? Would you like to listen to any of the tape preceding this episode or following to help you remember?

1. Can you describe the episode's mathematical situation? That is, describe the mathematics problem(s) and the discussion surrounding this problem.

2. a. Describe your role during the mathematical situation.
   b. Describe your thinking concerning the role you just described.

3. a. Describe your students' role during the mathematical situation.
   b. Describe what you believe is your students' thinking concerning the role(s) you just described.

4. Describe how your students influenced you during this episode's mathematical situation.

5. In this question, I'd like to ask you about some of your *objectives* during this episode's mathematical situation.
   (a) First tell me what you were hoping to convey to your students about mathematics and what you wanted your students to learn about mathematics given your role in the episode's mathematical situation.
   (b) What do you think your students learned about how mathematics is done in this episode?

6. a. How successful do you think you were in achieving your objectives?
   b. How are you assessing this success or lack of success?

c. Given your assessment of this episode, can you think of anything you would have done differently? Describe.

[Write down overall category responses for this question.]

7. (To be asked after all episodes only)
Compare and contrast these episodes for me. Be sure to refer to the episodes by their episode numbers (Episode #1 or Episode #2) when you're talking about them so I know what you are referring to when I listen to the tapes.

Appendix B

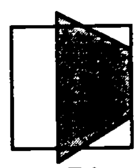
Description of Episodes

This appendix describes the 20 episodes containing the unanticipated student perspectives and the teachers' *Standards*-like use of knowledge. It is arranged according to the knowledge categories given in the paper. Each description includes the student unanticipated perspective (error, difficulty, alternative approach to problem solving), as well as the teacher's responding use of knowledge and how it was conceptualized (see Table 1). In addition, a description of the *Standards*-like qualities generated (see Table 2) and the length of the episode's surrounding problem is given.

Counterexample Category

<u>Ann 3-C</u> [5 minutes, 38 seconds]

The teacher and her students are working on the problem of finding the area of a trapezoid by partitioning it into squares and counting the squares. She does the problem twice, once using small squares and then using larger squares. The teacher asks the students whether using smaller squares will make the area more or less exact and the students respond, "Less." (student error) In response, the teacher superimposes a huge square over the trapezoid (counterexample) and asks whether this square will make the area more or less exact (see Appendix Diagram 1):



Appendix Diagram 1

In attempting to help her peers, one student tries to justify that with the larger square, "you can't fit as many wholes." (*Standards*-like qualities)

<u>Marlen 2-C</u> [4 minutes, 30 seconds]

See pages 9 to 10 of paper for an episode description.

<u>Nancy 1-C</u> [2 minutes]

See pages 8 of paper for a description of the student's error and the teacher's counterexample. After the teacher presents her counterexample, she gives the students some time in cooperative groups to discuss it.

Following the group session, one student gives the correct answer which the teacher then substantiates with a justification (*Standards*-like qualities).

<u>Rosie 2-C</u> [7 minutes, 12 seconds]

The teacher and her students are working on finding a "clue" for (*) in Appendix Diagram 2:

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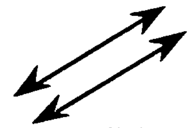
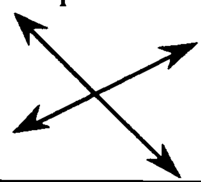
Equations	Clues	Graphs
$6x + 8y = 10$ $9x + 12y = 15$	coefficients and constants are in the same ratio	 coincident
$y = \frac{4}{5}x + 2$ $y = \frac{4}{5}x + 5$	coefficients in the same ratio but constants are not	 parallel
	*	

Appendix Diagram 2

One students suggests, "Coefficients and constants are all in different ratios." (student error) The teacher decides to point out the error in this response by showing the students that "every system has to fit somewhere." Thus, she introduces the system $\begin{cases} 3x + 2y = 7 \\ 9x + 4y = 21 \end{cases}$ (counterexample).

Following this, one student poses the problem of finding solutions to this system to see where it fits in the table. After finding the system's solution, the students get into an argument over the meaning of the error and the counterexample (*Standards*-like qualities).

<u>Nancy 2-C</u> [3 minutes, 22 seconds]

The teacher and her students are looking at a list of polynomial functions and determining whether they are even or odd. One student asks the question, "Like if there's even powers, does it come out even?" (student difficulty) The teacher explains that the function must by a polynomial and all its powers even for it to be even and then illustrates how $f(x) = x^3 + 3x^2$ has one even power in it but the function is not even (counterexample).

Following this, the teacher and the student have a disagreement about her clarification and counterexample. The student eventually understands with the teacher's justifying remarks, and the student's difficulty is turned into the "result" that "if a polynomial has all even powers, it is even." (*Standards*-like qualities)

Follow Through Category

<u>Marianne 1-FT</u> [2 minutes, 45 seconds]

The teacher and students are working on the following problem:

> Rich counts out \$61 in one-dollar and five-dollar bills. He has seventeen bills in all. How many one-dollar bills does he have? How many five dollar bills does he have?

Using a guess and check method (suggested by the student), the student's first guess is 9 \$5 bills and 8 \$1 bills yielding \$53. The student next suggests making the \$1 bills "higher." (student error) The teacher responds, "if you give him more ones, is he going to ultimately end up with more or less money?" (follow through)

The student sees Rich will end up with less money if they use her response and goes on to guess the correct answer. (*Standards*-like qualities)

Rosie 1-FT [3 minutes, 53 seconds]

The teacher and her students are trying to determine which solutions of $x+y = 7$ will satisfy $xy = 12$. After guessing (4,3) and(3,4) the teacher asks the students how they know there aren't any others and one student responds, "because you know all the multiples of 12." (student error) So the teacher asks the students to find all the multiples of 12 (follow through).

The problem of finding the multiples of 12 is discussed. Students argue over whether negative and rational solutions are allowed and eventually, students see there are infinitely many multiples of 12. The teacher explains that since there are infinitely many, they can't possibly be checked. (*Standards*-like qualities)

Nancy 3-FT (=Nancy 3-S/R) [3 minutes, 45 seconds]

See pages 14 to 15 of paper for an episode description.

Ed 1-FT [2 minutes, 20 seconds]

See pages 11 of paper for description of student difficulty and teacher follow through. Because of the follow through, this student continues participating and self resolves. (*Standards*-like qualities)

Dana 2-FT [2 minutes, 33 seconds]

See pages 12 to 13 of paper for an episode description.

Simpler or Related Problem

Marlen 3-S/R [5 minutes, 36 seconds]

Students are having trouble finding the rise and run of a horizontal line (student difficulty). Teacher reminds students that to find the slopes of other lines, the students first picked two points to find the rise and run (simpler or related problem).

The teacher thus, makes a connection to an earlier method and the students are able to explain and justify why the slope of a horizontal line is zero. (*Standards*-like qualities)

Ann 1-S/R [4 minutes, 44 seconds]

See pages 14 for an episode description.

Nancy 3-S/R

See Nancy 3-FT(=Nancy 3-S/R).

Dana 1-S/R [1 minutes, 32 seconds]

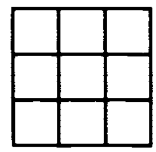
Student has trouble finding the complement of angle whose measure is $3k^0$. (Student difficulty) Teacher asks student how to find the complements of angles measuring 10^0, 45^0 and "[]"^{0}. (simpler or related problems)

As a result, the student continues participating and is able to self resolve her difficulty. Also, the teacher makes connections to earlier problems. (*Standards*-like qualities)

Understanding/Incorporating a Student Method

Ann 2-U/I [2 minutes, 25 seconds]

The students are working on the problem of how many square inches there are in a square yard. (see Appendix Diagram 3)



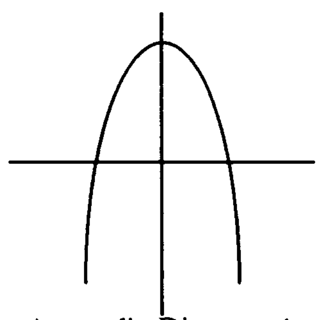
Appendix Diagram 3

The teacher wants to do the problem by finding the number of square inches in a square foot and multiplying by 9, but two students insist on finding the number of inches on the side of a square yard and squaring this result.

In this episode, the teacher and students have a disagreement over how to do the problem, but the students continue participating and contributes a different method. The teacher connects the two methods and explains how they're different and the same. (*Standards*-like qualities)

Ed 3-U/I [6 minutes, 10 seconds]

The teacher and students are discussing different ways that a line can intersect a parabola. In his interview, the teacher says he is looking for the example of a line touching a parabola at a tangent, when one student suggests the intersection between the *y* axis and the parabola in Appendix Diagram 4.



Appendix Diagram 4

Thus, the student contributes a method which the teacher explains and connects to previously learned material. (*Standards*-like qualities)

Kama 1-U/I [2 minutes, 13 seconds]

A student wants to know why the graph of $x = -4$ is vertical. One student explains that if she went to the point $x = -4$ on the x axis and graphed a horizontal line, then the line $x = -4$ would be the x axis which she knows is wrong. The teacher re-explains her method and follows it up with the explanation that every point on the line x = -4 is of the form $(-4,y)$.

Thus, a student generates a method for thinking about the problem which both she and the teacher justify. Also, the students help the first student with his difficulty (*Standards*-like qualities)

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Kama 3-U/I [4 minutes, 58 seconds]

The teacher and students are working on the following problem:

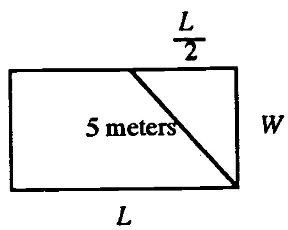
> Ray and Nancy Anne took turns driving on a 580-mile trip that took 10 hours to complete. Ray drove at a constant speed of 55 miles per hour and Nancy Anne drove at a constant speed of 60 miles per hour. Ray says he drove exactly half of the trip. Is he telling the truth? How do you know?

One student interprets "half" to mean "half" the time, that is, 5 hours and concludes that Ray could not have driven the half the trip since Ray's distance would have been $55(5) = 275$ and Nancy Anne's distance would have been $60(5) = 300$ which together do not total 580 miles. The teacher then elicits the solution using half the distance.

Again, the student generates an unanticipated method for solving the problem which he justifies. In the end, there is a disagreement over which interpretation is meant. (*Standards*-like qualities)

Lydia 1-U/I [1 minutes, 17 seconds]

The perimeter of the triangle in Appendix Diagram 5 is 12 meters. The teacher and students are working on writing an expression for this perimeter using the side lengths given in the diagram.



Appendix Diagram 5

One student suggests $L/2 + W = 7$, which the teacher accepts. Another student, however, objects saying she doesn't know where that equation from. The first student justifies her response [$L/2 + W = 7$ is equivalent $L/2 + 5 + W = 12$].

In this episode, the first student contributes an unanticipated method which she must justify following a disagreement with the second student. The teacher maintains a low profile and enables both students to help each other in making sense of both methods and connecting them. (*Standards*-like qualities)

Lydia 3-U/I [7 minutes, 38 seconds]

See pages 16 to 18 of paper for student's unanticipated method and teacher's response. In this episode, the student contributes a method, which he justifies and which he and other students try to connect to the anticipated method. Also, a disagreement develops between students advocating each method. (*Standards*-like qualities)

Marianne 3-"U/I" [9 minutes, 12 seconds]

See pages 18 to 20 of paper for student's "alternative" method and teacher's response. In this episode, both the teacher and students attempt to elicit justification from this student. A disagreement develops between this student and his peers, although the students and the teacher try to explain to him why his method doesn't make sense to them. Nevertheless, the student does continue participating in the discussion. (*Standards*-like qualities)

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