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ABSTRACT

Hypotheses that do not have the standard bilinear form theta=CBM=0 occur naturally in the analysis of repeated measurement designs. An expanded class of the tests of the form psi=Tr(G theta)=0, called extended linear hypotheses, provides a richer class of parametric functions. A method to analyze double multivariate and mixed multivariate models (MMM) using the Statistical Analysis System (SAS) is demonstrated. The analysis is extended to extended hypotheses, and a new approximate test of extended linear hypotheses for MMM designs is developed that does not require multivariate sphericity, but only a general Kronecker structure. Two appendixes provide SAS programs for these methods. (Contains 4 tables and 26 references.) (SLD)

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Full Rank Multivariate Repeated Measurement Designs and Extended Linear Hypotheses

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Abstract

Hypotheses that do not have the standard bilinear form $\Theta = \mathbf{CBM} = \mathbf{0}$ occur naturally in the analysis of repeated measurement designs. An expanded class of tests of the form $\psi = Tr(\mathbf{G}\Theta) = 0$, called extended linear hypotheses, provide a richer class of parametric functions. In this paper we show how to analyze double multivariate (DMM) and mixed multivariate models (MMM) using SAS. We extend the analysis to extended hypotheses and develop a new approximate test of extended linear hypotheses for MMM designs that does not require multivariate sphericity, only a general Kronecker structure.

Key words: double multivariate model, multivariate mixed model, multivariate sphericity, multi-response, growth curve model

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1. Introduction

In the analysis of repeated measurements, one may be interested in modeling changes in the matrix of means, testing hypotheses regarding the location parameters, and modeling the covariances among the repeated measurements. Standard statistical analysis of such designs under multivariate normality use the multivariate analysis of variance (MANOVA) model or the general growth curve model, the GMANOVA model developed by Potthoff and Roy (1964), Timm (1980) and von Rosen (1991). These models may not be used to test all hypotheses about the parameter matrix **B** of means since certain linear hypotheses about **B** may not have the simple bilinear form

$$H{:}\Theta = \mathbf{CBM} = \mathbf{0}. \tag{1.1}$$

For example, if

$$\mathbf{B} = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix}$$

for a simple 2 x 2 repeated measurement design, the hypothesis

$$H{:}\mu_{11} - \mu_{22} = 0 \text{ and } \mu_{21} - \mu_{12} = 0 \tag{1.2}$$

may not be put in the form of (1.1). To solve this problem, Mudholkar, Davidson and Subbaiah (1974) introduced the extended linear hypothesis

$$H_{\Gamma}{:}Tr(\mathbf{G}\Theta) = 0 \tag{1.3}$$

for all $\mathbf{G}\varepsilon\Gamma$ where Γ is a well defined set of $p \times g$ matrices and $Tr(\cdot)$ denotes the trace of a matrix. More recently, Hecker (1987) introduced the completely general MANOVA model (CGMANOVA) to analyze hypotheses like (1.2). In this article, we use the full rank model to analyze hypotheses involving means for the double multivariate model (DMM) and the multivariate mixed model (MMM) using SAS (1990) software, analyze hypotheses of the form (1.3) for a repeated measurement design, and develop a new approximate test for extended multivariate MMM hypotheses without assuming multivariate sphericity.

2. Repeated Measurements Designs

Models for the analysis of repeated measurements are wide and varied. For a comprehensive overview, see the recent works of Diggle, Liang and Zeger (1994), Lindsey (1993), and Longford (1993). In this section we review the standard repeated measures design under multivariate normality where a vector of p-variates is observed over q occasions discussed by Boik (1988, 1991).

For a multivariate repeated measures design, we observe p-variates over q conditions for N subjects where $i = 1,2,\ldots,N$; $j = 1,2,\ldots,p$, and $k = 1,2,\ldots,q$. The data matrix $\mathbf{Y}$ of order $N \times pq$ for the design is displayed in Figure 1.1

Subject	1	2	...	p
s_i	$\mathbf{y}_i = \begin{pmatrix} y_{i11} \\ y_{i11} \\ \vdots \\ y_{i1q} \end{pmatrix}$	$\mathbf{y}_i = \begin{pmatrix} y_{i21} \\ y_{i22} \\ \vdots \\ y_{i2q} \end{pmatrix}$	...	$\mathbf{y}_i = \begin{pmatrix} y_{ip1} \\ y_{ip2} \\ \vdots \\ y_{ipq} \end{pmatrix}$

Figure 1.1 Data layout for multivariate repeated measures design.

Using matrix notation, the double multivariate linear model (DMM) for the design may be expressed:

$$\mathbf{Y} = \mathbf{XB} + \mathbf{U} \tag{2.1}$$

where $\mathbf{Y}_{N\times pq}$ is the data matrix, $\mathbf{X}_{N\times k}$ is the full rank design matrix, rank (**X**) = k, $\mathbf{B}_{k\times pq}$ is the matrix of fixed unknown location parameters, and $\mathbf{U}_{N\times pq}$ is a matrix of random errors. Each row of **U** is assumed to be independently, normally distributed with common covariance matrix $\mathbf{\Sigma}_{pq\times pq}$. Using the standard $\mathrm{vec}(\cdot)$ operator that stacks columns of a matrix, we write that

$$\mathrm{vec}(\mathbf{U}) \sim N[\mathbf{0}, (\mathbf{\Sigma} \otimes \mathbf{I}_N)] \tag{2.2}$$

where $\otimes$ denotes the standard, right Kronecker matrix product ($\mathbf{A} \otimes \mathbf{B} = \{a_{ij}\mathbf{B}\}$), Graham (1981).

Given (2.1) and (2.2), we are primarily interested in estimating **B** and testing hypotheses of the general form:

$$H{:}\,\Theta = \mathbf{CB}(\mathbf{I}_p \otimes \mathbf{A}) = \mathbf{0} \tag{2.3}$$

where $\mathbf{C}_{g\times k}$ is the hypothesis test matrix of full row rank, rank (**C**) = g, and $\mathbf{A}_{q\times u}$ is a within subjects matrix of full column rank u, rank (**A**) = u. Without loss of generality, **A** is assumed to be semi-orthogonal, $\mathbf{A}'\mathbf{A} = \mathbf{I}_u$.

The least squares (LS) and maximum likelihood (ML) estimator of Θ is

$$\hat{\Theta} = \mathbf{C}\hat{\mathbf{B}}(\mathbf{I}_p \otimes \mathbf{A}) \text{ where } \hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}. \tag{2.4}$$

The estimator $\hat{\Theta}$ is unbiased and the covariances of the elements are given by

$$\mathrm{cov}(\mathrm{vec}\,\hat{\Theta}) = \mathbf{\Omega} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}' \tag{2.5}$$

where $\mathbf{\Omega} = (\mathbf{I}_p \otimes \mathbf{A}')\mathbf{\Sigma}(\mathbf{I}_p \otimes \mathbf{A})$. Given (2.2), the

$$\mathrm{vec}\,(\hat{\Theta}) \sim N(\mathrm{vec}\,\Theta, \mathbf{\Omega} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}') \tag{2.6}$$

so that the distribution of $\hat{\Theta}$ depend on $\mathbf{\Sigma}$ through $\mathbf{\Omega}$. When $\mathbf{\Omega}$ has special structure, mixed model procedures may be used to test (2.3).

When $\mathbf{\Omega}$ is known, the likelihood ratio (LR) statistic for testing (2.3) is

$$X^2 = \mathrm{vec}'(\hat{\Theta})[\mathbf{\Omega} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}']^{-1}\,\mathrm{vec}\,(\hat{\Theta}) \tag{2.7}$$

and $\mathrm{vec}'(\hat{\Theta}) = (\mathrm{vec}\,\hat{\Theta})'$. X^2 has a chi-square distribution with noncentrality parameter δ:

$$\begin{aligned} &X^2 \sim \chi^2(pgu, \delta) \\ &\delta = \mathrm{vec}'(\Theta)[\mathbf{\Omega} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}']^{-1}\,\mathrm{vec}\,(\Theta). \end{aligned} \tag{2.8}$$

However, Ω is usually unknown. Then standard tests of (2.3) depend on the characteristic roots of $\mathbf{HE}^{-1}$ where

$$\begin{aligned}\mathbf{E} &= (\mathbf{I}_p \otimes \mathbf{A})'\mathbf{Y}'(\mathbf{I}_N - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{Y}(\mathbf{I}_p \otimes \mathbf{A}) \\ \mathbf{H} &= \hat{\Theta}'(\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}\hat{\Theta}\end{aligned} \tag{2.9}$$

are the error and hypothesis sum of squares and cross-products matrices. An unbiased estimator of Ω is $\hat{\Omega} = \mathbf{E}/(N-k)$. Given (2.2), **E** and **H** have independent Wishart distributions:

$$\begin{aligned}\mathbf{E} &\sim W_{pu}(N-k, \Omega, 0) \\ \mathbf{H} &\sim W_{pu}(g, \Omega, \Omega^{-1}\Psi)\end{aligned} \tag{2.10}$$

with noncentrality matrix $\Psi = \Theta'(\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}\Theta$. The standard multivariate test statistics: Wilks's Λ, Roy's largest root, the Bartlett-Lawley-Hotelling trace and the Bartlett-Nanda-Pillai trace criteria all depend on the characteristic roots of $\mathbf{HE}^{-1}$. The Bartlett-Lawley-Hotelling trace statistic, also called Hotelling's generalized T_o^2 statistic, has a simple form. Using the fact that the $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\,\mathrm{vec}\,\mathbf{B}$ and the $Tr(\mathbf{AB}) = \mathrm{vec}'(\mathbf{A}')\,\mathrm{vec}\,\mathbf{B}$ implies that the $Tr(\mathbf{AZ}'\mathbf{BZC}) = Tr(\mathbf{Z}'\mathbf{BZCA}) = \mathrm{vec}'(\mathbf{Z})(\mathbf{A}'\mathbf{C}' \otimes \mathbf{B})\mathrm{vec}(\mathbf{Z})$, it is easy to show that

$$\begin{aligned}T_o^2 &= (N-k)Tr(\mathbf{E}^{-1}\mathbf{HE}^{-1}\mathbf{H}) = (N-k)\sum_i \lambda_i \\ &= \mathrm{vec}'(\hat{\Theta})(\hat{\Omega} \otimes \mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}\,\mathrm{vec}(\hat{\Theta})\end{aligned} \tag{2.11}$$

where λ_i are the characteristic roots of $|\mathbf{H} - \lambda\mathbf{E}| = 0$ or of $\mathbf{HE}^{-1}$.

Pillai (1960) tabled percentage points of $U = \sum_i \lambda_i = T_o^2/(N-k)$. Alternatively, one may also employ an F-distribution approximation, SAS Institute (1990, p. 19) developed by Pillai and Young (1971). The statistic is

$$F = \frac{2(sn+1)}{s^2(2m+s+1)}U \sim F\left[s(2m+s+1), 2(sn+1)\right] \tag{2.12}$$

where $s = \min(g, pu)$, $m = (|g-pu|-1)/2$, and $n = (N-k-pu-1)/2$ has an approximate F-distribution when H in (2.3) is true. For large departures from H, T_o^2 appears to be best, this is not the case for small departures, Seber (1984, p. 415). Power for each of the tests for specific alternatives is easily approximated using a SAS IML routine developed by Muller, LaVange, Ramey and Ramey (1992).

If the matrix Ω satisfies the multivariate sphericity assumption:

$$\Omega = \Sigma_e \otimes \mathbf{I}_u = \{\sigma_{ij}\mathbf{I}_u\} \tag{2.13}$$

where Σ_e is the covariance matrix of the vector of p-variates, Boik (1988) showed that (2.13) is the necessary and sufficient condition for obtaining exact MMM tests from the DMM. Thomas (1983) developed a test for multivariate sphericity. Boik (1988) improved the accuracy of the test by using Box's correction factor.

Boik (1991) using the generalized trace operation $T_p(\mathbf{W}) \equiv \{Tr(\mathbf{W}_{ij})\}$ where $\mathbf{W}_{pu \times pu} = \{\mathbf{W}_{ij}\}$ and $\mathbf{W}_{ij}$ is a $u \times u$ submatrix of **W** developed by Thompson (1973), shows how one may obtain MMM tests from the DMM. The tests depend on the characteristic roots of

$$\mathbf{H}^* = T_p(\mathbf{H}) \text{ and } \mathbf{E}^* = T_p(\mathbf{E}). \tag{2.14}$$

For additional detail, see Boik (1988) and Timm (1980).

One can also develop a general MMM directly from (2.1). The model for the i^{th} subject is

$$\mathbf{y}_i = (\mathbf{I}_{pq} \otimes \mathbf{x}_i')\,\mathrm{vec}(\mathbf{B}) + \mathbf{u}_i \tag{2.15}$$

where $\mathbf{x}_i'$ and $\mathbf{u}_i'$ are row vectors of $\mathbf{X}$ and $\mathbf{U}$, respectively. To construct a mixed model, we let

$$\mathbf{u}_i = \mathbf{Z}_i \lambda_i + \mathbf{e}_i$$

where $\mathbf{Z}_i$ is a $pq \times h$ matrix of covariates of rank h, λ_i and $\mathbf{e}_i$ are independently normally distributed:

$$\begin{aligned} \lambda_i &\sim N_h(\mathbf{0}, \mathbf{D}_h) \\ \mathbf{e}_i &\sim N_{pq}[\mathbf{0}, (\boldsymbol{\Sigma}_e \otimes \mathbf{I}_q)] \end{aligned} \tag{2.16}$$

where $\mathbf{D}_h$ and $\boldsymbol{\Sigma}_e$ are positive definite matrices. Then from (2.2) and (2.16), we have that

$$\begin{aligned} \mathbf{y}_i &\sim N[(\mathbf{I}_{pq} \otimes \mathbf{x}_i')\,\mathrm{vec}(\mathbf{B}), \boldsymbol{\Sigma}_i] \\ \boldsymbol{\Sigma}_i &= \mathbf{Z}_i\, \mathbf{D}_h\, \mathbf{Z}_i' + (\boldsymbol{\Sigma}_e \otimes I_q). \end{aligned} \tag{2.17}$$

Reinsel (1982) in the development of a multivariate growth curve model with a multivariate random-effects covariance structure for subjects proposed a model similar to (2.17). To obtain Reinsel's model, we associate $\mathbf{Z}_i \equiv \mathbf{I}_p \otimes \mathbf{1}_q$ so that $h = p$ and

$$\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma} = (\mathbf{D}_p \otimes \mathbf{1}_q \mathbf{1}_q') + (\boldsymbol{\Sigma}_e \otimes \mathbf{I}_q) \tag{2.18}$$

where $\mathbf{D}_p$ is a $p \times p$ positive definite matrix. The structure of Reinsel's matrix is more commonly called multivariate compound symmetry, a sufficient but not necessary condition for exact MMM tests. When $\boldsymbol{\Sigma}$ satisfies (2.13), we have Scheffé's MMM which also requires homogeneity of the covariance matrices. Given (2.13), one may directly test (2.3) using (2.17), the mixed MANOVA model with homogeneity of the $\boldsymbol{\Sigma}_i$.

Rearranging rows and columns in (2.18), observe that (2.18) is a special case of the more general linear form

$$\boldsymbol{\Sigma} = \mathbf{G}_1 \otimes \boldsymbol{\Sigma}_1 + \mathbf{G}_2 \otimes \boldsymbol{\Sigma}_2 \tag{2.19}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ commute with each other. Tests for matrices that have the linear structure given in (2.19) where the matrices $\mathbf{G}_i$ are known and may or may not commute are reviewed by Krishnaiah and Lee (1980). With the rows and columns rearranged, $\boldsymbol{\Omega}$ has the form

$$\boldsymbol{\Omega} = (\mathbf{A} \otimes \mathbf{I}_p)' \boldsymbol{\Sigma} (\mathbf{A} \otimes \mathbf{I}_p) \tag{2.20}$$

so that the sphericity condition becomes

$$\boldsymbol{\Omega} = \mathbf{I}_u \otimes \boldsymbol{\Sigma}_e. \tag{2.21}$$

Given the general form (2.19) with $\boldsymbol{\Omega}$ defined in (2.20), we see that $\boldsymbol{\Omega}$ may be factored:

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}_u \otimes \boldsymbol{\Sigma}_e \tag{2.22}$$

where $\boldsymbol{\Sigma}_u$ and $\boldsymbol{\Sigma}_e$ are positive definite matrices and $\boldsymbol{\Omega}$ does not satisfy (2.21), the multivariate sphericity condition. Given that $\boldsymbol{\Omega}$ can be factored and estimated, we develop a large sample test of (2.3) given (2.22) as an alternative to $\hat{\varepsilon}$ – adjusted MMM tests which on the whole are less efficient than DMM tests, Boik (1991). Ameniya (1994) has developed a test procedure for testing that of subset of $\boldsymbol{\Omega}$ is zero, using the Bartlett-Lawley-Hotelling criterion.

3. Extended Linear Hypotheses

Not all multivariate hypotheses have the general bilinear form given in (2.1). More generally, one wants to consider all parametric functions $\psi = \mathbf{d}'\Theta\mathbf{f}$ for arbitrary vectors $\mathbf{d}'$ and $\mathbf{f}$. However, $\psi = Tr(\mathbf{f}\,\mathbf{d}'\Theta) = Tr(\mathbf{G}\,\Theta)$ where the rank $(\mathbf{G}) = 1$. This motivated Mudholkar et al. (1974) to consider the more general decomposition of (2.1):

$$H_\Gamma: \bigcap_{\mathbf{G}\in\Gamma} \{H(\mathbf{G}): Tr(\mathbf{G}\,\Theta) = 0\} \tag{3.1}$$

where Γ is the set of all $m \times g$ matrices defined:

$$\Gamma = \{\mathbf{G}: \mathbf{G} = \sum_{i=1}^{\nu} \lambda_i\,\mathbf{G}_i, \lambda_j's \text{ real}\}.$$

The hypotheses in (3.1) are called the extended linear hypotheses of order ν for known matrices $\mathbf{G}_j, j = 1,2,\ldots,\nu$.

To test (3.1) when $\nu = 1$, Mudholkar et al. (1974) developed a union-intersection (UI) test of

$$H: \psi = Tr(\mathbf{G}\Theta) = 0 \tag{3.2}$$

using an argument involving symmetric gauge functions, Krishnaiah, Mudholkar and Subbaiah (1980). Their test statistic is

$$U(\mathbf{G}) = Tr(\mathbf{G}\hat{\Theta}) / \{Tr(\mathbf{G}\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}'\mathbf{G}'\mathbf{E})\}^{1/2} \tag{3.3}$$

where $\mathbf{E}$ is defined in (2.9), and $\mathbf{C}$ is given in (2.3) for the overall test. Letting $\mathbf{c}_o$ be the critical value for the F-approximation in (2.12), $1-\alpha$ simultaneous confidence intervals for ψ are:

$$\psi \in \hat{\psi} \pm c_o\left\{Tr(\mathbf{G}\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}'\mathbf{G}'\mathbf{E})\right\}^{1/2} \tag{3.4}$$

for all matrices $\mathbf{G}_{m\times g}$ and $\hat{\psi} = Tr(\mathbf{G}\hat{\Theta})$.

To test any intermediate extended linear hypothesis requires finding the supremum in (3.3). Given $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_\nu$ where $\mathbf{G} = \sum_{i=1}^{\nu} \mathbf{G}_i$ for some vector $\lambda = (\lambda_i)$ it is easily shown that

$$\{U(\mathbf{G})\}^2 = \sup_{\lambda}\{U(\mathbf{G})\}^2 = \tau\,\mathbf{T}^{-1}\,\tau \tag{3.5}$$

where the elements of τ and $\mathbf{T}$ are:

$$\begin{aligned} \tau_i &= Tr(\mathbf{G}_i\hat{\Theta}) \\ t_{ij} &= Tr(\mathbf{G}_i\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}'\mathbf{G}_j'\mathbf{E}) \end{aligned} \tag{3.6}$$

Mudholkar et al. (1974). Hence, the intermediate hypothesis is rejected if $\{U(\mathbf{G})\}^2$ exceeds the square of the critical value for the test of H in (3.2).

To illustrate several extended linear hypotheses, consider a one-way MANOVA design involving three groups and three repeated measures so that the parameter matrix is

$$\mathbf{B} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \tag{3.7}$$

where the primary hypothesis of interest is the equality of group means:

$$H_{G^*}: \begin{pmatrix}\mu_{11}\\ \mu_{12}\\ \mu_{13}\end{pmatrix} = \begin{pmatrix}\mu_{21}\\ \mu_{22}\\ \mu_{23}\end{pmatrix} = \begin{pmatrix}\mu_{31}\\ \mu_{32}\\ \mu_{33}\end{pmatrix}.$$

To test H_{G^*}, one may select $\mathbf{C} \equiv \mathbf{C}_o$ and $\mathbf{A} \equiv \mathbf{A}_o$ to test the overall hypothesis where for example,

$$\mathbf{C}_o = \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix} \text{ and } \mathbf{A}_o = \mathbf{I}_3 \tag{3.8}$$

so that we may represent H_{G^*} in the general form $\mathbf{C}_o\mathbf{B}\,\mathbf{A}_o = 0$. Upon rejecting H_{G^*}, suppose we are interested in the contrast

$$\psi = \mu_{11} + \mu_{22} + \mu_{33} - (\mu_{12} + \mu_{21})/2 - (\mu_{13} + \mu_{31})/2 - (\mu_{23} + \mu_{32})/2 \tag{3.9}$$

which compares the diagonal means with the average of the off diagonals, a situation that may be of interest in a repeated measures design. Contrast (3.9) may not be expressed as a bilinear form, $\psi = \mathbf{c}'\mathbf{B}\mathbf{a}$. Letting

$$\mathbf{G} = \begin{pmatrix}1 & .5\\ -.5 & .5\\ -.5 & 1\end{pmatrix},$$

observe that ψ in (3.9) may be expressed as an extended linear hypothesis since

$$\psi = Tr(\mathbf{G}\,\Theta) = Tr(\mathbf{G}\,\mathbf{C}_o\mathbf{B}\mathbf{A}_o) = Tr(\mathbf{A}_o\mathbf{G}\mathbf{C}_o\mathbf{B}) = Tr(\mathbf{G}^*\mathbf{B})$$

where $\mathbf{C}_o$ and $\mathbf{A}_o$ are defined in (3.8) and $\mathbf{G}^* = \mathbf{A}_o\mathbf{G}\mathbf{C}_o$. Thus, in terms of the original parameters, we have that

$$\mathbf{G}^* = \begin{pmatrix}1 & -.5 & -.5\\ -.5 & 1 & -.5\\ -.5 & -.5 & 1\end{pmatrix}$$

where each of the coefficients in each row and column of $\mathbf{G}^*$ sum to one. A contrast of this type involving rows and columns in a standard two-way design is called a generalized contrast by Bradu and Gabriel (1974) and Boik (1993).

Testing that the contrast in (3.9) is zero is achieved using a single matrix $\mathbf{G}$. Alternatively, suppose we are interested in the following hypothesis

$$H: \begin{array}{ccccc}\mu_{11} & = & \mu_{21} & & \\ \mu_{12} & = & \mu_{22} & = & \mu_{32}\\ \mu_{23} & = & \mu_{33} & & \end{array} \tag{3.10}$$

which is not a general linear hypothesis of the form $\mathbf{CBM} = \mathbf{0}$, but an extended linear hypothesis. To see this, observe that hypothesis (3.10) is equivalent to testing

$$Tr(\mathbf{G}\,\Theta) = Tr(\mathbf{G}_i\,\mathbf{C}_o\,\mathbf{B}\,\mathbf{A}_o) = Tr(\mathbf{A}_o\mathbf{G}_i\mathbf{C}_o\mathbf{B}) = Tr(\mathbf{G}_i^*\ \mathbf{B}) = 0$$

for $i = 1, 2, 3, 4$, where

$$\mathbf{G}_1 = \begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}, \mathbf{G}_2 = \begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 0\end{pmatrix}, \mathbf{G}_3 = \begin{pmatrix}0 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}, \mathbf{G}_4 = \begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}.$$

To test (3.10) requires obtaining the supremum in (3.3) as outlined in (3.5).

The examples we have illustrated have assumed $\mathbf{A}_o = \mathbf{I}$. We now consider the test of parallelism for a repeated measures design so that we have

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \mathbf{0}$$

$$\mathbf{C}_o \, \mathbf{B} \, \mathbf{A}_o = \mathbf{0}.$$

Following the overall test, suppose we were interested in the sum of the following tetrads:

$$\psi = (\mu_{21} + \mu_{12} - \mu_{31} - \mu_{22}) + (\mu_{32} + \mu_{23} - \mu_{13} - \mu_{22}). \tag{3.11}$$

Such a contrast may not be expressed using the simple bilinear form, $\psi = \mathbf{c}'\mathbf{B}\mathbf{a}$. However, selecting

$$\mathbf{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we see that the contrast

$$\psi = Tr(\mathbf{G}\Theta) = Tr(\mathbf{G}\mathbf{C}_o\mathbf{B}\mathbf{A}_o) = Tr(\mathbf{A}_o\mathbf{G}\mathbf{C}_o\mathbf{B}) = Tr(\mathbf{G}^*\mathbf{B})$$

where

$$\mathbf{G}^* = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

is a generalized contrast matrix.

As our last illustration, we let $\mathbf{C}_o \equiv \mathbf{I}$ and for a repeated measures design suppose we are testing for equality of conditions:

$$\mathbf{H}_{C^*}: \begin{pmatrix} \mu_{11} \\ \mu_{21} \\ \mu_{31} \end{pmatrix} = \begin{pmatrix} \mu_{12} \\ \mu_{22} \\ \mu_{32} \end{pmatrix} = \begin{pmatrix} \mu_{13} \\ \mu_{23} \\ \mu_{33} \end{pmatrix}.$$

Expressing the hypothesis in the linear form, we have that $\mathbf{C}_o = \mathbf{I}_3$ and

$$\mathbf{A}_o = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Following the overall test, one may be interested in the contrast

$$\psi = (\mu_{11} - \mu_{12}) + (\mu_{22} - \mu_{23}) + (\mu_{31} - \mu_{33}). \tag{3.12}$$

This contrast is again an extended linear hypothesis and tested with

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ or } \mathbf{G}^* = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

in the expression $\psi = Tr(\mathbf{G}\Theta) = Tr(\mathbf{G}\mathbf{C}_o\mathbf{B}\mathbf{A}_o) = Tr(\mathbf{G}^*\mathbf{B})$.

4. SAS Illustrations

DMM/MMM - Example

To illustrate the application of the DMM and the MMM, we use the dental data analyzed by Timm (1980), Thomas (1983), and Boik (1988, 1991) using SAS. The data were obtained from Dr. Thomas Zullo in the School of Dental Medicine at the University of Pittsburgh. The data matrix **Y** consists of nine subjects randomly assigned to two orthopedic treatments. At each of three activator treatment conditions (occasions), three dependent variables were observed representing three measurements associated with the adjustment of the mandible.

The parameter matrix **B** has the matrix form

$$\begin{array}{ccccccccccc} & 1 & & & 2 & & & 3 & & & \text{variable} \\ & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & \text{conditions} \\ \mathbf{B} = & \multicolumn{9}{c}{\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16} & \mu_{17} & \mu_{18} & \mu_{19} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} & \mu_{25} & \mu_{26} & \mu_{27} & \mu_{28} & \mu_{29} \end{pmatrix}} & . \end{array} \tag{4.1}$$

The first hypothesis of interest for the DMM is whether the profiles for the two groups are parallel. Is there an interaction between treatment groups and conditions? The hypothesis is

$$\begin{aligned} H_{GC}\colon & (\mu_{11}-\mu_{13}, \mu_{12}-\mu_{13}, \ldots, \mu_{17}-\mu_{19}, \mu_{18}-\mu_{19}) \\ & = (\mu_{21}-\mu_{23}, \mu_{22}-\mu_{23}, \ldots, \mu_{27}-\mu_{29}, \mu_{28}-\mu_{29}). \end{aligned} \tag{4.2}$$

The matrices **C** and **A** are

$$\mathbf{C} = (1,-1) \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \tag{4.3}$$

in the general form $\mathbf{CB}(\mathbf{I} \otimes \mathbf{A}) = \mathbf{CBM} = \mathbf{0}$. Normalizing **A** so that $\mathbf{A}'\mathbf{A} = \mathbf{I}$ and nesting differences in conditions for each variable, we may write **M**:

$$\mathbf{M}' = \begin{pmatrix} .707 & 0 & -.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.408 & .816 & -.408 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .707 & 0 & -.707 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.408 & .816 & -.408 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .707 & 0 & -.707 \\ 0 & 0 & 0 & 0 & 0 & 0 & -.408 & .816 & -.408 \end{pmatrix} \tag{4.4}$$

so that the normalized first difference contrasts are followed by the normalized second difference contrasts. To test for vector differences in conditions the hypothesis becomes

$$H_C\colon \begin{pmatrix} \mu_{11} \\ \mu_{21} \\ \mu_{14} \\ \mu_{24} \\ \mu_{17} \\ \mu_{27} \end{pmatrix} = \begin{pmatrix} \mu_{12} \\ \mu_{22} \\ \mu_{15} \\ \mu_{25} \\ \mu_{12} \\ \mu_{28} \end{pmatrix} = \begin{pmatrix} \mu_{13} \\ \mu_{23} \\ \mu_{16} \\ \mu_{26} \\ \mu_{19} \\ \mu_{29} \end{pmatrix}. \tag{4.5}$$

To test (4.5), the hypothesis matrices are:

$$\mathbf{C} = \mathbf{I}_2 \text{ and } \mathbf{A} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{4.6}$$

To test for differences in Groups, H_{G^*}, the overall hypothesis is

$$H_{G^*}\colon \mu_1 = \mu_2 \tag{4.7}$$

and the hypothesis test matrices are

$$\mathbf{C} = (1,-1) \text{ and } \mathbf{A} = \mathbf{I}_3 \tag{4.8}$$

Tests (4.5) and (4.6) do not require parallelism of profiles. Given parallelism, tests for differences in conditions and groups are written by averaging over the other factor:

$$H_G\colon \begin{pmatrix} \sum_{j=1}^{3} \mu_{1j}/3 \\ \sum_{j=4}^{6} \mu_{1j}/3 \\ \sum_{j=7}^{9} \mu_{1j}/3 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{3} \mu_{2j}/3 \\ \sum_{j=4}^{6} \mu_{2j}/3 \\ \sum_{j=7}^{9} \mu_{2j}/3 \end{pmatrix} \tag{4.9}$$

$$H_C\colon \begin{pmatrix} \sum_{i=1}^{2} \mu_{i1}/2 \\ \sum_{i=1}^{2} \mu_{i4}/2 \\ \sum_{i=1}^{2} \mu_{i7}/2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2} \mu_{i1}/2 \\ \sum_{i=1}^{2} \mu_{i5}/2 \\ \sum_{i=1}^{2} \mu_{i8}/2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2} \mu_{i3}/2 \\ \sum_{i=1}^{2} \mu_{i6}/2 \\ \sum_{i=1}^{2} \mu_{i9}/2 \end{pmatrix}$$

which simplify the interpretation of the tests. The corresponding hypothesis test matrices are:

$$\begin{aligned} \mathbf{C}_G &= (1,-1) \;\text{ and }\; \mathbf{A} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \\ \mathbf{C}_C &= (1/2,-1/2) \;\text{ and }\; \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}. \end{aligned} \tag{4.10}$$

Normalization of the post matrix **M** associated with (4.10) permits one to obtain the MMM from the multivariate tests given parallelism and multivariate sphericity, Timm (1980) and Boik (1988). This is not the case for the tests H_{G^*} and H_{C^*}.

Utilizing SAS, the code for these hypotheses is provided in Appendix A, Program 1.sas. Reviewing the output for the SAS run, one may construct Table 1 using the Bartlett-Lawley-Hotelling trace criterion and associated F-ratios.

Table 1

DMM Output, Zullo Data

Hypotheses	$U = T_o^2 / \nu_e$	df	F	df	p-value
GC	0.7153	(1, 2, 4.5)	1.3114	(6, 11)	0.3292
G*	1.3684	(1, 3.5, 3)	1.2163	(9, 8)	0.3965
C*	29.3173	(2, 1.5, 4.5)	24.4311	(12, 20)	<0.0001
G	0.1314	(1, 0.5, 6)	0.6132	(3, 14)	0.6176
C	28.6021	(1, 2, 4.5)	52.4371	(6, 11)	<0.0001

Because the test of parallelism is not significant, tests of conditions and groups given parallelism are appropriate. From these tests, one may obtain the MMM tests of H_C and H_{GC} which are valid given multivariate sphericity. The MMM test of group differences given parallelism is identical to the DMM since the test of H_G does not depend on the sphericity assumption.

To test for multivariate sphericity, we may use the procedure developed by Thomas (1983). Thomas showed that the LR statistic is

$$\lambda = |\mathbf{E}|^{N/2} / |u^{-1} \sum_{i=1}^{u} \mathbf{E}_{ii}|^{Nu/2} \tag{4.11}$$

where $u = \text{rank}(\mathbf{A})$ or $u = \text{rank}(\mathbf{M}) / p$. The asymptotic null distribution of $-2 \ln \lambda$ is a central chi-square distribution with degrees of freedom

$$f = p(u-1)(pu+p+1)/2. \tag{4.12}$$

Letting $\alpha_i (i = 1,2, \ldots, pu)$ be the eigenvalues of $\mathbf{E}$, and $\beta_i (i = 1,2, \ldots, p)$ the eigenvalues of $\sum_{i=1}^{u} \mathbf{E}_{ii}$, a simple form of (4.11) is

$$W = -2 \ln \lambda = N[u \sum_{i=1}^{p} \ln(\beta_i) - \sum_{i=1}^{pu} \ln(\alpha_i)]. \tag{4.13}$$

When p and/or u are large relative to N, the asymptotic distribution of W may be poor. To correct for this, Boik (1988) using Box's correction factor for the distribution of W showed that the

$$\begin{aligned} P(W \le W_o) &= P(\rho^* W \le \rho^* W_o) \\ &= (1-\omega) P(X_f^2 \le \rho^* W_o) + \omega P(X_{f+4}^2 \le \rho^* W_o) + O(\nu_e^{-3}) \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} \rho &= 1 - p[2p^2(u^4-1) + 3p(u^3-1) - (u^2-1)]/12uf\nu_e \\ \rho^* &= \rho\nu_e / N \\ \omega &= (2\rho^2)^{-1} \left\{ \left[\frac{(pu-1)pu(pu+1)(pu+2)}{24\nu_e^2} \right] - \left[\frac{(p-1)p(p+1)(p+2)}{24u^2\nu_e^2} \right] - \left[\frac{f(1-\rho)^2}{2} \right] \right\} \end{aligned} \tag{4.15}$$

$\nu_e = N -$ rank($\mathbf{X}$), and f is defined in (4.12). Hence, the p-value for the test of multivariate sphericity using Box's (4.16) correction becomes

$$P(\rho * W \geq W_o) = (1-\omega)P(X_f^2 \geq W_o) + \omega P(X_{f+4}^2 \geq W_o) + O(v_e^{-3}).$$

Using (4.14) and (4.15), the IML code is given in Program 1.sas for the test of multivariate sphericity. From the program output,

$$W = -2\ln\lambda = 74.367228 \text{ with } df = 15$$

and the p-value for the test is 7.365×10^{-10}. With Box's correction, ρ*W=54.742543 with p-value of 2.7772×10^{-6}. Hence, the MMM assumption is not satisfied with Zullo's data. Thus, one should use the DMM for these data.

Continuing, assuming the test of multivariate sphericity was not significant, we show how one may analyze the DMM data using the MMM. In addition, if one could not test for sphericity, we illustrate how one may construct the ε-adjusted MMM tests statistics developed by Boik (1988, 1991) which extends the Greenhouse-Geisser type procedure to the MMM. While one may directly derive the MMM test matrices from the normalized DMM tests, as illustrated in Timm (1980), it is more convenient to use the SAS PROC GLM to obtain the test statistics and hypothesis test matrices.

For the MMM analysis, the data must be reorganized as a mixed model. Subject is a random factor nested with the fixed group factor, and groups are crossed with the fixed condition factor. To perform the tests in SAS, we must use the RANDOM command to calculate the expected mean squares to locate the appropriate error sum of squares matrices for the multivariate tests. From the expected mean square output, we see that the subject within group effect is the error term to test for group differences given parallelism. The commands for the analysis become:

```
model y1-y3 = group subj(group) cond cond*group;
random subj (group);
contrast 'Group' group 1-1/e=subj (group);
manova h = cond group*cond/printh printe;
```

as illustrated in Program 1.sas. The option e on the contrast statement defines the error matrix for the test of groups. The "overall" error matrix is being used to test for conditions given parallelism and the test for parallel profiles, the condition by group interaction. The MANOVA table for the multivariate mixed model analysis is shown in Table 2.

Table 2

MMM Analysis, Zullo Data

Hypothesis	$U = T_o^2 / v_e$	*df*	F	*df*	p-value
Group/Parall	0.1314	(1, 0.5, 6)	0.6132	(3, 14)	0.6176
Cond/Parall	13.7514	(2, 0, 14)	66.4651	(6, 58)	<0.0001
Group x Cond	0.1907	(2, 0, 14)	0.9218	(6, 58)	0.4862

That the multivariate mixed model results may be obtained from the DMM, observe that for the test of conditions given parallelism that the hypothesis test matrix

$$\mathbf{H}^* = \begin{pmatrix} 152.93 & & (Sym) \\ 95.79 & 62.73 & \\ 14.75 & 8.56 & 1.59 \end{pmatrix}$$

is the average of $\mathbf{H}_{11}$ and $\mathbf{H}_{22}$ where

$$\mathbf{H}_{11} = \begin{pmatrix} 148.02 & & Sym) \\ 96.32 & 62.67 & \\ 13.38 & 8.71 & 1.21 \end{pmatrix} \mathbf{H}_{22} = \begin{pmatrix} 4.89 & & Sym \\ -0.53 & 0.06 & \\ 1.36 & -0.15 & 0.39 \end{pmatrix}$$

are the submatrices of H for the test of H_C given parallelism. The degrees of freedom for the mixed model analyses are $v_h^* = v_h u = v_h$ rank (M) / $p = v_h$ rank(A) $= 1 \cdot 2 = 2$ and $v_e^* = v_e$ rank(M) / $p = v_e$ rank(A) $= 16 \cdot 6/3 = 32$. The results for the test of cond x group interaction follow similarly, Timm (1980) and Boik (1988). From the MMM results, one can also obtain the univariate split-plot univariate F-ratios, a variable at a time.

When multivariate sphericity is not satisfied, one cannot perform a MMM analysis. Boik (1988, 1991) developed an ε-adjustment for the tests of conditions and conditions x group interaction following the work of Box. Because the multivariate tests may be approximated by a Wishart distribution when the null hypothesis is true:

$$\mathbf{E}^* \,\dot{\sim}\, W_p(v_e^* = \varepsilon u v_e, \boldsymbol{\Omega}, \mathbf{0})$$
$$\mathbf{H}^* \sim W_p(v_n^* = \varepsilon u v_h, \boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}, \mathbf{0})$$

Boik showed how one may use an estimate of ε to adjust the approximate F tests for the multivariate criteria where

$$\hat{\varepsilon} = \frac{Tr\left[\left(\sum_{i=1}^{u} \mathbf{S}_{ii}\right)^2\right] + \left[Tr\left(\sum_{i=1}^{u} \mathbf{S}_{ii}\right)\right]^2}{u\left\{\sum_{i=1}^{u}\sum_{j=1}^{u}\left[Tr\left(\mathbf{S}_{ij}\right)\right]^2 + Tr\left(\mathbf{S}_{ij}^2\right)\right\}} \tag{4.17}$$

$u = \text{rank}(\mathbf{A})$ and v_h and v_ε are the hypothesis and error degrees of freedom for the DMM tests. Boik (1991) proposed an alternative adjustment factor that improves the approximation when $\boldsymbol{\Omega}$ has the general Kronecker structure given in (2.22).

Using Wilks's Λ criterion, Boik (1988) shows how one may make the $\hat{\varepsilon}$ – adjustment to the associated F-statistic. The result applies equally to the other criteria using Rao's F-approximation. In Program 1.sas, we use T_o^2, the Bartlett-Lawley-Hotelling trace criterion. Then

$$\begin{aligned} F &= 2(sN+1)T_o^2 / s^2(2M+s+1) \\ &\dot{\sim}\, F\left[s(2M+s+1), 2(sN+1)\right] \end{aligned} \tag{4.18}$$

where

$$s = \min\,(v_h^*, p),\ v_h^* = \hat{\varepsilon} u v_e,\ v_h^* = \hat{\varepsilon} u v_h$$
$$M = (|v_h^* - p| - 1)/2,\ N = (v_e^* - p - 1)/2.$$

From the program output, $\hat{\varepsilon} = 0.7305055$ and the $\hat{\varepsilon}$ – adjusted MANOVA Table 3 for the MMM tests results.

Table 3

$T_o^2, \hat{\varepsilon}$ – adjusted Analysis, Zullo Data

Hypothesis	F	*df*	p-value
Cond/Parall	65.090	(4.38, 30.31)	< 0.0001
Group x Cond	0.903	(4.38, 30.31)	0.4830

The p-value for the unadjusted MMM was 0.4862 for the test of parallelism. The comparison using Wilks's Λ criterion is 0.463 versus 0.4687, for the adjusted MMM and the unadjusted test, respectively. The nominal p-values for the two procedures are approximately equal, 0.483 vs. 0.463, for T_o^2 and Λ, respectively, for the test of parallelism.

It appears that we have three competing strategies for the analysis of a DMM. The DMM, the MMM, and the $\hat{\varepsilon}$ – adjusted MMM. Given multivariate sphericity, the MMM is of course most powerful. When multivariate sphericity does not hold, neither the adjusted MMM or the DMM analysis is most powerful. Boik (1991) recommends using the DMM. The choice between the two procedures depends on the ratio of the traces of the noncentrality matrices of the associated Wishart distributions which is seldom known in practice.

Extended Linear Hypotheses - Example

Krishnaiah et al. (1980) developed a "Roots" program to test extended linear hypotheses using three test criteria: $T_{\max}^2$, Roy's largest root and T_o^2. They consider hypotheses using only a single matrix **G**. We illustrate their procedure using Roy's largest root test and T_o^2 using PROC IML and data from Timm (1975, p. 454) shown in Table 4 involving three groups and repeated measurements data.

Table 4

Repeated Measurements, Timm (1975, p. 454)

		Conditions		
	Subjects	1	2	3
Drug group 1	1	2	4	7
	2	2	6	10
	3	3	7	10
	4	7	9	11
	5	6	9	12
Means		4	7	10
Drug group 2	1	5	6	10
	2	4	5	10
	3	7	8	11
	4	8	9	11
	5	11	12	13
Means		7	8	11
Drug group 3	1	3	4	7
	2	3	6	9
	3	4	7	9
	4	8	8	10
	5	7	10	10
Means		5	7	9
Grand mean		5.333	7.333	10

Using PROC GLM we test the standard MANOVA test for groups, conditions, and parallelism (group × condition interaction) using Program 2.sas in Appendix B. Finding the differences in groups to be significant, suppose we are interested in the extended linear hypothesis $\psi = 0$ where ψ is given (3.9). Using the PROC IML, we verify the calculations of the overall test and estimate

$$\begin{aligned}\hat{\psi} &= Tr(\mathbf{G}\hat{\Theta}) = Tr(\mathbf{G}\mathbf{C}_o\hat{\mathbf{B}}\mathbf{A}_o) = Tr(\mathbf{A}_o\mathbf{G}\mathbf{C}_o\hat{\mathbf{B}}) \\ &= Tr(\mathbf{G} * \hat{\mathbf{B}}) = -2.5\end{aligned}$$

with the matrices $\mathbf{G}, \mathbf{G}^*$, and $\hat{\mathbf{B}}$ defined:

$$\mathbf{G} = \begin{pmatrix} 1 & .5 \\ -.5 & .5 \\ -.5 & 1 \end{pmatrix}, \mathbf{G}^* = \begin{bmatrix} 1 & -.5 & -.5 \\ -.5 & 1 & -.5 \\ -.5 & -.5 & 1 \end{bmatrix} \text{ and } \hat{\mathbf{B}} = \begin{pmatrix} 4 & 7 & 10 \\ 7 & 8 & 11 \\ 5 & 7 & 9 \end{pmatrix}.$$

Solving $|\mathbf{H}-\delta\mathbf{E}_o^{-1}|=0$ with $\mathbf{H}=\mathbf{G}\mathbf{W}_0\mathbf{G}'$, $\mathbf{W}_o=\mathbf{C}_o(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}_o'$ and $\mathbf{E}_o=\mathbf{A}_o'\mathbf{E}\mathbf{A}_o=$ $\mathbf{A}_o'\mathbf{Y}'(\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{Y}\mathbf{A}_o$, we solve the characteristic equation using the IML routine EIGVAL for a symmetric matrix. Because of rounding, the last characteristic root is essentially zero and, $\hat{\delta}_1=6.881$ and $\hat{\delta}_2=2.119$. By (3.4),

$$\hat{\sigma}^2_{\text{Trace}}=Tr(\mathbf{G}\mathbf{W}_o\mathbf{G}'\,\mathbf{E}_o)=\sum_i\hat{\delta}_i$$

for the Bartlett-Lawley-Hotelling trace criterion. The corresponding value of $\hat{\sigma}$ for largest root criterion is

$$\hat{\sigma}_{\text{root}}=\sum_i\hat{\delta}_i^{1/2}$$

as shown by Krishnaiah et al. (1980).

Since $\hat{\psi}=-2.5$, the extended trace and root statistics for testing $\psi=0$ are:

$$|\hat{\psi}|/\hat{\sigma}_{\text{Trace}}=0.8333$$
$$|\hat{\psi}|/\hat{\sigma}_{\text{root}}=1.3320.$$

Evaluating (2.13) with $\alpha=0.05$ for the trace criterion and a similar result for Roy's largest root test, the critical values for the criteria are 1.331991 and 0.9891365, respectively. As shown in the program output, approximate confidence intervals for ψ are (-53.77, 48.78) and (-6.50, -1.50) indicating nonsignificance of the contrast ψ for both criteria.

We next illustrate testing (3.10) using the PROC IML by evaluating the supremum in (3.5) as outlined in (3.6). With $\tau_i=Tr(\mathbf{G}_i\hat{\Theta})$ and $t_{ij}=Tr(\mathbf{G}_i\,\mathbf{W}_o\,\mathbf{G}_j\,\mathbf{E}_o)$, we have in the Output that

$$\tau'=(-3,-1,1,2)$$

$$\mathbf{T}=\begin{pmatrix}29.6 & 26 & -13 & -7\\ 26 & 27.2 & -13.6 & -7.6\\ -13 & -13.6 & 27.2 & 15.2\\ 7 & -7.6 & 15.2 & 10.4\end{pmatrix}$$

so that $\tau'\mathbf{T}^{-1}\tau=2.2081151$. Comparing this with the U(G) critical value $(1.3319921)^2=1.774203$, we see that the test is significant. To find confidence intervals for contrasts involving $\psi_i=Tr(\mathbf{G}_i\hat{\Theta})$, one would find a $1-\alpha$ simultaneous confidence interval for each contrast as illustrated above. Finally, 95% confidence intervals are constructed for ψ in (3.11) following the test of parallelism, and for ψ in (3.12) following the test of equal condition vectors, Program 2.sas.

5. An extended linear MMM hypothesis test

As the within subject design in a repeated measurements experiment becomes more complex, we often find that the multivariate sphericity condition is not satisfied. For complex designs, the more general form of the hypothesis in (2.3) becomes

$$\mathrm{H}{:}\,\Theta=\mathbf{C}\mathbf{B}(\mathbf{L}\otimes\mathbf{A})=\mathbf{0} \tag{5.1}$$

where $\mathbf{L}_{p\times p}$ is an orthogonal contrast matrix of rank p, $\mathbf{L}'\mathbf{L} = \mathbf{I}_p$. Given (5.1), the covariance structure for (2.1) is:

$$\Omega = (\mathbf{L}\otimes \mathbf{A})'\Sigma(\mathbf{L}\otimes \mathbf{A}) \tag{5.2}$$

and $\hat{\Theta} = \mathbf{C}\hat{\mathbf{B}}(\mathbf{L}\otimes\mathbf{A})$. Using the DMM, one may test (5.1) or (3.2) for arbitrary Ω.

We can also test (5.1) or (3.2) given multivariate sphericity. Suppose, however, that Ω does not satisfy the multivariate sphericity condition, but has general Kronecker structure:

$$\Omega = \Sigma_e \otimes \Sigma_u \tag{5.3}$$

where $\Sigma_e(p\times p)$ and $\Sigma_u = \mathbf{A}'\Sigma\mathbf{A}(u\times u)$ are arbitrary positive definite matrices, a structure commonly found in three-mode factor analysis models, Bentler and Lee (1978).

To test the null hypothesis that Ω has the structure given in (5.3) versus the alternative that the rows $\mathbf{y}_i$ of $\mathbf{Y}$ have a general structure, under multivariate normality, we obtain a likelihood ratio test using the normal likelihood, Timm (1975, p. 558). The likelihood ratio statistic for testing the covariance structure is given by

$$\lambda = \frac{|\hat{\Sigma}_\Omega|^{N/2}}{|\hat{\Sigma}_\omega|^{N/2}} = \frac{|S|^{N/2}}{|\mathbf{A}'\hat{\Sigma}_e\mathbf{A}|^{Np/2}|\hat{\Sigma}_u|^{N(q-1)/2}} \tag{5.4}$$

where $\hat{\Sigma}_e$ and $\hat{\Sigma}_u$ are the maximum likelihood (ML) estimates of Σ_e and Σ_u under the null hypothesis, $\mathrm{S} = \sum_{i=1}^{N}(\mathrm{y}_i - \overline{\mathrm{y}})(\mathrm{y}_i - \overline{\mathrm{y}})'/N$ is the ML estimate of the covariance matrice and $\overline{\mathrm{y}} = \sum_{i=1}^{N}\mathrm{y}_i/N$.

To test for the structure, the statistic -2lnλ is asymptotically distributed as a chi-square distribution with ν degrees of freedom where $\nu = pq(pq+1)/2 - [p(p+1)+q(q+1)+1]/2 = (p-1)(q-1)[(p+1)(q+1)+1]/2$. However, to solve the likelihood equations to obtain $\hat{\Sigma}_e$ and $\hat{\Sigma}_u$ involves an iterative process as outlined by Krishnaiah and Lee (1980), Boik (1991),and Naik and Rao (1996). Naik and Rao provide a computer program using the SAS IML procedure to obtain the ML estimates.

Given that Ω satisfies (5.3) and Σ_e and Σ_u are estimated by $\hat{\Sigma}_e$ and $\hat{\Sigma}_u$, we may develop a test of the extended linear hypothesis

$$\mathrm{H}: \psi = Tr(\mathbf{G}\Theta) = 0. \tag{5.5}$$

Following Mudholkar et al. (1974), a test statistic for testing (5.5) is

$$X^2 = \{U(\mathbf{G})\}^2 = [Tr(\mathbf{G}\hat{\Theta})]^2 / Tr[\mathbf{GC}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}'\mathbf{G}(\hat{\Sigma}_e \otimes \hat{\Sigma}_u)] \tag{5.6}$$

since

$$\hat{\psi} \sim N(\psi, \hat{\sigma}^2_{\hat{\psi}} = Tr[\mathbf{GC}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}'\mathbf{G}(\hat{\Sigma}_e \otimes \hat{\Sigma}_u)] \tag{5.7}$$

for fixed $\mathbf{G}$. The statistic X^2 converges to a chi-square distribution with one degree of freedom.

The statistic in (5.7) is an alternative to Boik's $\hat{\varepsilon}-$adjusted multivariate test procedure for the more general hypothesis given in (5.1) when multivariate sphericity is not satisfied. While we have provided a asymptotic test of (5.1) given (5.3), the more difficult problem is the estimation of Σ_e and Σ_u, a solution to the likelihood in (5.4) over all positive definite matrices Σ_e and Σ_u. Naik and Rao (1996) have developed an alternative Satterthwaite type approximate for the MANOVA model when multivariate sphericity is not satisfied.

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Appendix A

```
/* Program 1.sas */
/* Example from Timm(1980) and Boik(1988,1991) - Zullo Dental data */
options LS=78 ps=60 nodate nonumber;
title 'Output: Double Multivariate Linear Model and Multivariate Mixed Model';
data dmlm;
 infile 'mmm.dat';
 input group y1 - y9;
proc print data=dmlm;
proc glm;
   class group;
   model y1 - y9 = group/nouni;
   means group;
/* Multivariate test of group differences for mean vectors */
   manova  h=group /printh printe;
proc glm;
   class group;
   model y1 - y9 = group / nouni;
/* Multivariate test of Parallelism */
   contrast 'Parallel'
          group 1 -1;
   manova        m=(.7071 0 -.7071 0 0 0 0 0 0,
              0 0 0 .-.408 .816 -.408 0 0 0,
              0 0 0 0 0 0 .7071 0 -.7071,
              -.408 .816 -.408 0 0 0 0 0 0,
              0 0 0 .7071 0 -.7071 0 0 0,
              0 0 0 0 0 0 -.408 .816 -.408)
                      prefix = parl/ printh printe;
proc glm;
   class group;
   model y1 - y9 = group /noint nouni;
/* Multivariate test of Conditions as vectors */
   contrast 'Mult Cond' group 1 0,
               group 0 1;
   manova        m=(1 -1 0 0 0 0 0 0 0,
              0 1 -1 0 0 0 0 0 0,
              0 0 0 1 -1 0 0 0 0,
              0 0 0 0 1 -1 0 0 0,
              0 0 0 0 0 0 1 -1 0,
              0 0 0 0 0 0 0 1 -1) prefix = diff/printh printe;
proc glm;
   class group;
   model y1- y9 = group/noint nouni;
/* Multivariate test of Conditions given Parallelism */
   contrast 'Cond|Parl' group .5 .5;
   manova    m=(.7071 0 -.7071 0 0 0 0 0 0,
            0 0 0 -.408 .816 0 -.408 0 0 0,
            0 0 0 0 0 0 .7071 0 -.7071,
           -.408 .816 -.408 0 0 0 0 0 0,
            0 0 0 -7071 0 -.7071 0 0 0,
            0 0 0 0 0 0 -.408 .816 -.408) prefix=cond/printh printe;
proc glm;
   class group;
   model y1 - y9 = group/noint nouni;
```

```
    contrast 'Group|Parl' group 1 -1;

    manova      m=(.577 .577 .577 0 0 0 0 0 0,
                0 0 0 .577 .577 .577 0 0 0,
                0 0 0 0 0 0 .577 .577 .577) prefix=Ovall/printh printe;
/* Multivariate Mixed Model Analysis */
data mix;
    infile 'mixed.dat';
    input group subj cond y1 y2 y3;
proc print data=mix;
proc glm;
    class group subj cond;
    model y1 - y3 = group subj(group) cond cond*group;
    random subj(group);
    contrast 'Group' group 1 -1/e=subj(group);
    manova h = cond group*cond/printh printe;
/* Test for Multivariate Spericity and calculation of Epsilon for MMM */
proc iml;
print 'Test of Multivariate Sphericity Using Chi-Square and Adjusted Chi-
Square Statistics';
e={ 9.6944  7.3056  -6.7972 -4.4264 -0.6736   3.7255,
   7.3056  8.8889  -4.4583 -3.1915 -3.2396   2.9268,
  -6.7972 -4.4583  18.6156  2.5772  0.8837 -10.1363,
  -4.4264 -3.1915   2.5772  5.3981  1.4259  -1.8546,
  -0.6736 -3.2396   0.8837  1.4259 18.3704   -.7769,
   3.7255  2.9268 -10.1363 -1.8546 -0.7769   6.1274};
print e;
n=18;
p=3;
t=3;
k=2;
u=6;
q=u/p;
nu_e=n-k;
nu_h=1;
e11=e[1:3,1:3];print e11;
e22=e[4:6,4:6];print e22;
dn=(e11+e22)/2;
b=eigval(dn); print b;
a=eigval(e); print a;
b=log(b);
a=log(a);
chi_2=n#(q#sum(b)-sum(a));
df=p#(q-1)#(p#q+p+1)/2;
pvalue=1-probchi(chi_2,df);
print chi_2 df pvalue;
c1=p/(12#q#nu_e#df);
rho= 1-c1#(2#p##2#(q##4-1)+3#p#(q##3-1)-(q##2-1));
ro_chi_2=(rho#nu_e/n)#chi_2; print rho;
c2=1/(2#rho##2);
c3=((p#q-1)#p#q#(p#q+1)#(p#q+2))/(24#nu_e##2);
c4=((p-1)#p#(p+1)#(p+2))/(24#q##2#nu_e##2);
c5=df#(1-rho)##2/2;
omega=c2#(c3-c4-c5); print omega;
p1=1-probchi(ro_chi_2,df);
```

```
p2=1-probchi(ro_chi_2,df+4);
cpvalue=(1-omega)#p1+omega#p2;
print ro_chi_2 cpvalue;
s=e/nu_e;
s11=s[1:3,1:3]; s12=s[1:3,4:6];
s21=s[4:6,1:3]; s22=s[4:6,4:6];
enum=trace((s11+s22)*(s11+s22))+(trace(s11+s22))##2;
eden=q#( trace(s11)##2+trace(s11*s11)+trace(s12)##2+trace(s12*s12)+
      trace(s21)##2+trace(s21*s21)+trace(s22)##2+trace(s22*s22));
epsilon=enum/eden;
nu_h=nu_h#q; nu_e=nu_e#q; s0=min(nu_h,p);
Mnu_h=nu_h#epsilon; Mnu_e=nu_e#epsilon; ms0=min(mnu_h,p);
m0=(abs(mnu_h-p)-1)/2; n0=(mnu_e-p-1)/2;
denom=s0##2#(2#m0+ms0+1); numer=2#(ms0#n0+1);
df1=ms0#(2#m0+ms0+1); df2=2#(ms0#n0+1);
print 'Epsison adjusted F-Statistics for cond and group X cond MMM tests';
print epsilon;
f_cond = df2#13.75139851/(ms0#df1);
f_gXc  = df2#0.19070696/(ms0#df1);
print f_cond f_gXc df1 df2;
p_cond=1-probf(f_cond,df1,df2);
p_gXc=1-probf(f_gXc,df1,df2);
print 'Epsilon adjusted pvalues for MMM tests using T0**2 Criterion';
print p_cond p_gXc;
```

Appendix B

```
* Program 2.sas */
/* Data from Timm(1975, page 454) */
options ls=78 ps = 60 nodate nonumber;
title 'Output: Extended Linear Hypotheses' ;
data timm;
  infile 'exlin.dat';
  input group y1 y2 y3 x1 x2 x3;
proc print data=timm;
proc glm;
    class group;
    model y1-y3 = group/nouni;
    means group;
/* Multivariate test of Groups     */
    manova h=group/printh printe;
proc iml;
use Timm;
a={x1 x2 x3};
b={y1 y2 y3};
read all var a into x;
read all var b into y;
beta=inv(x`*x)*x`*y;
print beta;
n=nrow(y);
p=ncol(y);
k=ncol(x);
nu_h=2; u=3; nu_e=n-k; s0=min(nu_h,u); r=max(nu_h,u); alpha=.05;
denr=(nu_e-r+nu_h);
roy_2=(r/denr)*finv(1-alpha,r,denr);
rvalue=sqrt(roy_2);
m0=(abs(nu_h-u)-1)/2; n0=(nu_e-u-1)/2;
num=s0**2*(2*m0+s0+1); dent=2*(s0*n0+1);df=s0*(2*m0+s0+1);
t0_2=(num/dent)*finv(1-alpha,df,dent);
tovalue=sqrt(t0_2);
print s0 m0 n0;
e=(y`*y-y`*x*beta);
co={1 -1 0, 0 1 -1};
ao=i(3);
eo=ao`*e*ao;
bo=co*beta*ao;
wo=co*inv(x`*x)*co`;
ho=bo`*inv(wo)*bo;
print,"Overall Error Matrix", eo ,"Overall Hypothesis Test Matrix",ho;
/* c`c=eo where c is upper triangle Cholesky matrix */
c=root(eo);
f=inv(c`)*ho*inv(c);
eig=Eigval(round(f,.0001));
vec=inv(c)*eigvec(round(f,.0001));
print,"Eigenvalues & Eigenvectors of Overall Test of Ho (Groups)", eig vec;
/* Extended Linear Hypothesis following Overall Group Test */
m={1 .5, -.5 .5, -.5 -1};
g=ao*m*co;
print, "Extended Linear Hypothesis Test Matrix",m g;
psi=m*bo;
```

```
psi_hat=trace(psi);
tr_psi=abs(psi_hat);
h=m*wo*m`;
eo=inv (eo);
c=root(eo);
f=inv(c`)*h*inv(c);
xeig=Eigval(round(f,.0001));
print, "Eigenvalues of Extended Linear Hypothesis", xeig;
to_2=tr_psi/sqrt(sum(xeig)); print, "Extended To**2 Statistic", to_2;
print, "Extended To**2 Critical Value", tovalue;
root=tr_psi/sum(ssq(xeig)); print, "Extended Largest Root Statistic", root;
print, "Extended Largest Root Critical Value", rvalue;
print psi_hat alpha;
ru=psi_hat+rvalue*sum(ssq(xeig));
rl=psi_hat-rvalue*sum(ssq(xeig));
vu=psi_hat+tovalue*sqrt(sum(xeig));
vl=psi_hat-tovalue*sqrt(sum(xeig));
print 'Approximate Simultaneous Confidence Intervals';
print 'Contrast Significant if interval does not contain zero';
print 'Extended Root  interval: ('rl ',' ru ')';
print 'Extended Trace interval: ('vl ',' vu ')';
/* Multiple Extended Linear Hypothesis using To**2  */
m1={1 0,0 0,0 0}; m2={0 0,1 0,0 0}; m3={0 0,0 1,0 0}; m4={0 0,0 0,0 1};
print,"Multiple Extended Linear Hypothesis Test Matrices", m1,m2,m3,m4;
g1=ao*m1*co;    g2=ao*m2*co;    g3=ao*m3*co;    g4=ao*m4*co;
t1=trace(m1*bo); t2=trace(m2*bo);  t3=trace(m3*bo);  t4=trace(m4*bo);
tau=t1//t2//t3//t4;
t11=trace(m1*wo*m1`*eo);
t21=trace(m2*wo*m1`*eo); t22=trace(m2*wo*m2`*eo);
t31=trace(m3*wo*m1`*eo); t32=trace(m3*wo*m2`*eo); t33=trace(m3*wo*m3`*eo);
t41=trace(m4*wo*m1`*eo); t42=trace(m4*wo*m2`*eo); t43=trace(m4*wo*m3`*eo);
t44=trace(m4*wo*m4`*eo);
r1=t11||t21||t31||t41;
r2=t21||t22||t32||t42;
r3=t31||t32||t33||t43;
r4=t41||t42||t43||t44;
t=r1//r2//r3//r4;
print tau,t;
to_4=tau`*inv(t)*tau;
print, "Extended Linear Hypothesis Criterion To**2 Squared", to_4;
print, "Extended To**2 Critical Value", t0_2;
/* Multivariate test of Parallelism */
data timm;
 infile 'exlin.dat';
 input group y1 y2 y3 x1 x2 x3;
proc glm;
   class group;
   model y1-y3 = group/nouni;
   manova h = group m = ( 1 -1  0,
                    0  1 -1) prefix = diff/printe printh;
proc iml;
use Timm;
a={x1 x2 x3};
b={y1 y2 y3};
read all var a into x;
```

```
read all var b into y;
beta=inv(x`*x)*x`*y;
n=nrow(y);
p=ncol(y);
k=ncol(x);
nu_h=2; u=2; nu_e=n-k; s0=min(nu_h,u); r=max(nu_h,u); alpha=.05;
denr=(nu_e-r+nu_h);
roy_2=(r/denr)*finv(1-alpha,r,denr);
rvalue=sqrt(roy_2);
m0=(abs(nu_h-u)-1)/2; n0=(nu_e-u-1)/2;
num=s0**2*(2*m0+s0+1); dent=2*(s0*n0+1); df=s0*(2*m0+s0+1);
t0_2=(num/dent)*finv(1-alpha,df,dent);
tovalue=sqrt(t0_2);
print s0 m0 n0;
e=(y`*y-y`*x*beta);
co={1 -1 0, 0 1 -1};
ao={1 0, -1 1, 0 -1};
eo=ao`*e*ao;
bo=co*beta *ao;
wo=co*inv(x`*x)*co`;
ho=bo`*inv(wo)*bo;
c=root(eo);
f=inv(c`)*ho*inv(c);
eig=eigval(round(f,.0001));
vec=inv(c)*eigvec(round(f,.0001));
print,"Eigenvalues & Eigenvectors of Overall test of Ho (Parallelism)", eig vec;
/* Extended Linear Hypothesis following overall Parallelism test */
m={0 1,1 0};
g=ao*m*co;
print, "Extended Linear Hypothesis Test Matrix", m g;
psi=m*bo;
psi_hat=trace(psi);
tr_psi=abs(psi_hat);
h=m*wo*m`;
eo=inv(eo);
c=root(eo);
f=inv(c`)*h*inv(c);
xeig=eigval(round(f,.0001));
print, "Eigenvalues of Extended Linear Hypothesis", xeig;
to_2=tr_psi/sqrt(sum(xeig)); print, "Extended To**2 Statistic", to_2;
print,"Extended To**2 Critical Value", tovalue;
root=tr_psi/sum(ssq(xeig)); print, "Extended Largest Root Statistic", root;
print, "Extended Largest Root Critical Value", rvalue;
print psi_hat alpha;
ru=psi_hat+rvalue*sum(ssq(xeig));
rl=psi_hat-rvalue*sum(ssq(xeig));
vu=psi_hat+tovalue*sqrt(sum(xeig));
vl=psi_hat-tovalue*sqrt(sum(xeig));
print 'Approximate Simultaneous Confidence Intervals';
print 'Contrast Significant if interval does not contain zero';
print 'Extended Root  Interval: ('rl ',' ru ')';
print 'Extended Trace Interval: ('vl '.' vu ')';
/* Multivariate test of Conditions as vectors */
data timm;
  infile 'exlin.dat';
```

```
 input group y1 y2 y3 x1 x2 x3;
proc glm;
 class group;
 model y1-y3 = group/noint nouni;
 contrast 'Mult Cond' group 1 0 0,
                group 0 1 0,
                group 0 0 1;
 manova                    m=(1 -1  0,
                          0  1 -1) prefix = diff/ printe printh;
proc iml;
use timm;
a={x1 x2 x3};
b={y1 y2 y3};
read all var a into x;
read all var b into y;
beta=inv(x`*x)*x`*y;
n=nrow(y);
p=ncol(y);
k=ncol(x);
nu_h=3; u=2; nu_e=n-k; s0=min(nu_h,u); r=max(nu_h,u); alpha=.05;
denr=(nu_e-r+nu_h);
roy_2=(r/denr)*finv(1-alpha,r,denr);
rvalue=sqrt(roy_2);
m0=(abs(nu_h-u)-1)/2; n0=(nu_e-u-1)/2;
num=s0**2*(2*m0+s0+1); dent=2*(s0*n0+1); df=s0*(2*m0+s0+1);
t0_2=(num/dent)*finv(1-alpha,df,dent);
tovalue=sqrt(t0_2);
print s0 m0 n0;
e=(y`*y-y`*x*beta);
co=i(3);
ao={1 0, -1 1, 0 -1};
eo=ao`*e*ao;
bo=co*beta*ao;
wo=co*inv(x`*x)*co`;
ho=bo`*inv(wo)*bo;
c=root(eo);
f=inv(c`)*ho*inv(c);
eig=eigval(round(f,.0001));
vec=inv(c)*eigvec(round(f,.0001));
print, "Eigenvalues & Eigenvectors of Overall test of Ho (Conditions)", eig vec;
m={1 0 1, 0 1 1};
g=ao*m*co;
print, "Extended Linear Hypothesis Test Matrix", m g;
psi=m*bo;
psi_hat=trace(psi);
tr_psi=abs(psi_hat);
h=m*wo*m`;
eo=inv(eo);
c=root(eo);
f=inv(c`)*h*inv(c);
xeig=eigval(round(f,.0001));
print, "Eigenvalues of Extended Linear Hypothesis", xeig;
to_2=tr_psi/sqrt(sum(xeig)); print, "Extended To**2 Statistic", to_2;
print, "Extended T0**2 Critical Value", tovalue;
root= tr_psi/sum(ssq(xeig)); print, "Extended Largest Root Statistic", root;
```

```
print, "Extended Largest Root Critical Value", rvalue;
print psi_hat alpha;
ru=psi_hat+rvalue*sum(ssq(xeig));
rl=psi_hat-rvalue*sum(ssq(xeig));
vu=psi_hat+tovalue*sqrt(sum(xeig));
vl=psi_hat-tovalue*sqrt(sum(xeig));
print 'Approximate Simultaneous Confidence Intervals';
print 'Contrast Significant if interval does not contain zero';
print 'Extended Root  Interval: ('rl ',' ru ')';
print 'Extended Trace Interval: ('vl ',' vu ')';
```

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