DOCUMENT RESUME

ED 404 183 SE 059 726

AUTHOR	Reynolds, Theodora H.
TITLE	Addressing Gender and Cognitive Issues in the Mathematics Classroom: A Constructivist Approach.
INSTITUTION	Columbia Univ., New York, NY. Esther A. and Joseph Klingenstein Center for Independent School Education.
PUB DATE	17 Apr 95
NOTE	62p.
PUB TYPE	Guides - Classroom Use - Teaching Guides (For Teacher) (052) -- Reports - Research/Technical (143)
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	*Calculus; Cognitive Processes; *Constructivism (Learning); *Females; High Schools; *Mathematics Instruction
IDENTIFIERS	*Limits (Mathematics); Mathematics Activities

ABSTRACT

This project was designed to address gender issues in mathematics education. The author's findings led her to conclude that students construct their own understanding and knowledge as a result of activities in the classroom. Constructivism, a theory of knowledge based on the idea that people construct their own realities by interacting with the world, offers a framework for designing lessons that incorporate suggestions from the research for improving mathematics education. Constructivist guidelines were used to design a unit to illustrate the use of constructivist teaching at the upper high school level. This project begins with a review of recent articles on issues of gender in the mathematics classroom. The review is followed by discussion of some cognitive research on how students learn mathematics, and this is followed finally by a series of lessons on limits. Since the unit is meant to be constructivist, any teacher using it will necessarily adapt and change it to meet the students' needs. The pacing and depth of the unit depends on the abilities and interests of students in the class. The annotated bibliography contains 53 references and a summary of each. (PVD)

Addressing Gender and Cognitive Issues in the Mathematics Classroom: A Constructivist Approach

Theodora H. Reynolds
Klingenstein Project
April 17, 1995

Introduction

When I first began teaching mathematics twenty years ago, I stood in front of the classroom and lectured. I wrote a lot on the blackboard for students to copy and occasionally students wrote homework solutions on the board and explained them to the class. My classes were very teacher and procedure dominated. Over the years my classes have come to include more student involvement, and I have attempted to incorporate new ideas and suggestions from conferences and reading. The more studying I did about math education, however, the more frustrated I became. I felt that my classes were not reflecting the results of research, especially research about how girls learn best. Therefore, I decided to study the research and work to provide a module of a calculus topic which reflects the results of research. This module is meant to serve as a guide for how I believe math teachers can develop lessons based on research and theories about how students learn mathematics.

This project begins with a review of recent articles on issues of gender in the mathematics classroom. Then I discuss some cognitive research on how students learn mathematics. Constructivism, a theory of knowledge, offers a framework for designing lessons which incorporate suggestions from the research for improving math education. Finally, I have constructed a module investigating the concept of limits to illustrate how the research can be incorporated into an advanced placement calculus curriculum. The module can be adapted for use in a precalculus or analysis course as well.

Gender Issues

Research on gender issues in mathematics first became well known in the 1970's. Since then the focus has broadened and changed as more has been learned and educators have made efforts to respond to research observations. By the mid-1970's, research evidence supported the claims that there were differences in boys' and girls' learning of mathematics and that those differences were most noticeable in situations which required complex reasoning. The observed differences increased at the beginning of adolescence (Fennema, 1993). The Fennema-Sherman studies of the mid-70's reinforced these findings. They also found that once required courses were completed girls did not continue in mathematics to the level boys did. This result led to the suggestion that encouraging girls to take more higher level courses would lead to a diminishing of gender differences (Fennema, 1993). Studies completed in the 1980's indicate that gender differences have declined and that more girls are taking higher level math courses; however, there are still indications that male and female students have different attitudes about mathematics, its importance and their own abilities, and that interactions between teacher and student tend to differ based on the sex of the student. In addition, studies show that certain types of assessment and content lead to gender differences. I will look at some research which gives indications of strategies teachers can use to make their classrooms more gender equitable.

The Fennema-Sherman studies beginning in the 1970's investigated affective and attitudinal variables. Confidence in mathematical ability and perception of the usefulness of mathematics were found to correlate with gender differences. Boys feel that math is more useful than girls do, and boys have more confidence in their ability to learn math. The gender differences appear even when there is no difference in math

achievement (Meyer, 1990). Marsh (1989) found that math achievement is closely related to previous success; achievement in the senior year is related to achievement in the sophomore year.

Several studies in the 1970's and 1980's investigated to what students attribute their success or failure. In 1974 Weiner developed the attribute model shown below relating the four types of reasons individuals give to explain success and failure (taken from Meyer, 1990). According to this model ability and task difficulty are stable and

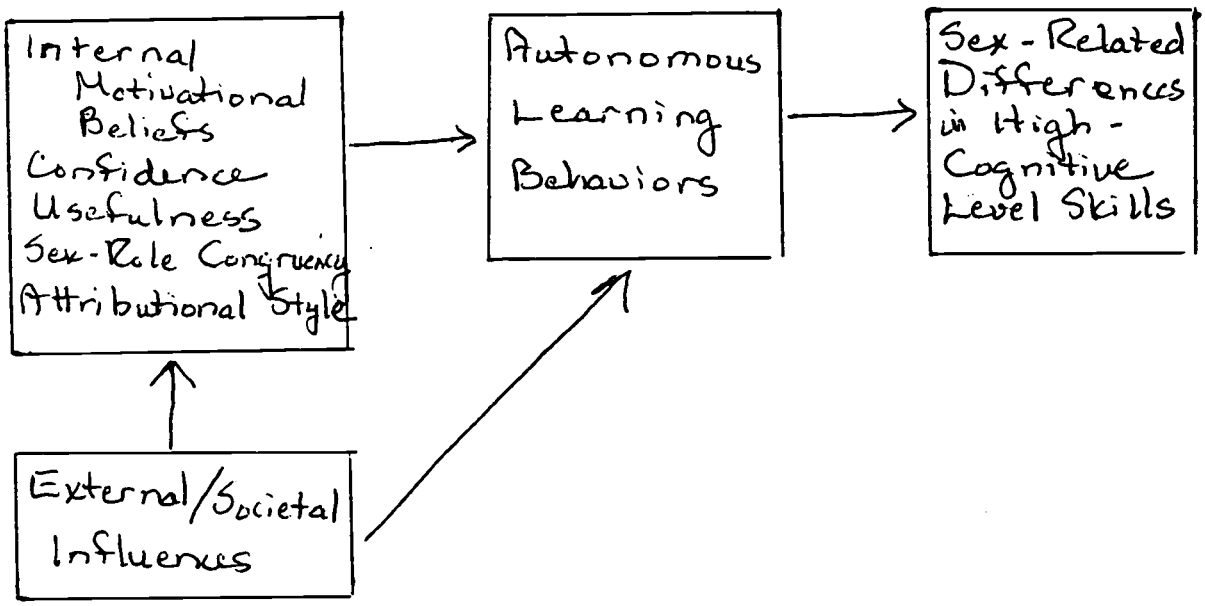
Loss of Control

Stability	Internal	External
stable	Ability	Task Difficulty
Unstable	Effort	Luck

unchanging whereas effort and luck are unstable; ability and effort are internal, under the control of the individual, whereas task difficulty and luck are external, and uncontrollable. Attribution styles have been found to differ by gender in several studies completed in the 1970's and 1980's. Boys attribute their success to ability while girls attribute success to effort. Boys attribute failure to bad luck, and girls attribute failure to difficulty of the task and lack of ability (Meyer, 1990). Girls internalize failure saying "It's my fault" whereas boys say "Well if I just studied a little more or the question was different I would have done better." Girls feel they are stupid when they fail.

Gender differences are largest on high complexity tasks, and two models have

been developed which help to explain the relationship between internal belief systems and gender differences. Fennema and Sherman created the Autonomous Learning Behavior model which describes the development of learning behaviors leading to autonomous learning and consequently success on high complexity tasks. According to their model, societal and external influences impact on both internal motivating beliefs and autonomous learning behaviors which are necessary for the completion of high complexity tasks. The diagram below taken from Meyer (1990) illustrates the relationships in their model.



The Model of Academic Choice developed by Eccles et al. relates the impact of factors involved in making academic choices. Two strands which affect academic choices are the expectation of success and the individual's determination of the value of the task (Meyer, 1990).

Corinna Ethington (1992), using the Eccles's model, studied the impact of psychological variables on current achievement of students. Her study included a statistical analysis of the following variables and their relationships to each other: prior

achievement, perceptions of parents' attitudes, family help, stereotyping, self-concept, expectancies, goals, difficulty of subject, value of subject, and current achievement. Her analysis showed that for males only prior achievement and value have a significant direct effect on current achievement, but for girls stereotyping, family help, difficulty, prior achievement, and value all have a direct impact on current achievement. This research indicates that both internal and external influences are different for girls and boys.

According to the Autonomous Learning Behavior model what occurs in the classroom can have a great impact on the development of autonomous learning behaviors. Consequently, much work has been done recently on what happens in the classroom. No significant differences are noted in classroom interactions based on the sex of the teacher even though female teachers tend to be more student-centered and supportive of students than male teachers. Both male and female teachers, however, interact differently with boys than with girls. Research in the math classroom reflects the comments of Myra and David Sadker (1994) that in general boys receive more attention in the classroom than girls. Boys call out more, and volunteer more. Teachers call on boys more, ask them higher level questions, give them more wait time, and discipline them more (Koehler, 1990, Hart, 1989, Leder, 1990a, Leder 1990b). Boys also give the impression of being more mathematically competent, whether they are or not, because of the way students participate in typical classes (Jungwirth, 1991).

Laurie E. Hart (1989) studied classroom interactions of high ability seventh grade students based on the students' confidence level and gender. According to her study, the average student has only one or two public interactions with the teacher in the classroom during the day. Most of these interactions are initiated by the teacher and involve low-cognitive-level tasks. Classes in which more interactions involved boys than

girls were more teacher dominated and contained more overt sex role stereotyping by teachers than classes in which more interactions were with girls. Self-confidence increased more in classes in which girls participated more. High confidence students engaged in more mathematical activities than low confidence students during class time, but engagement time for high level tasks did not change significantly based on either sex or confidence level.

Elizabeth Fennema argues that, because of the impact teachers have on the development of autonomous learning, teacher attitudes toward students are important. In research with Peterson, Carpenter, and Lubinski (1990), she found that first grade teachers believed boys' successes and failures are caused by ability and girls' successes and failures are caused by effort. The teachers believed the best boys were more competitive, logical, and independent than the best girls. Fennema et al. argue that these and similar attitudes of teachers may possibly be reflected in development of student beliefs, since teacher expectations have been shown to correlate with student achievement (Fennema, 1990).

Classroom situations in which girls are more successful are varied. At the Potsdam Campus of the State University of New York a predominately male faculty runs an extremely successful program for both males and females. Students, there, respond to a mixture of challenge and caring from father figures. Professors encourage independence through coaching (Rogers, 1990). Zelda Isaacson (1990) in describing a successful program for math-anxious women in their twenties and thirties returning to school talks of inspiring confidence in a relaxed, supportive, non-competitive environment which encourages collaboration, cooperation, and creativity. Fennema (1993) argues that classroom environments that lead to success for girls in mathematics encourage students in autonomous learning behaviors.

Such studies on classroom interactions suggest that teachers can make changes in their classrooms to help ensure that all students are heard and valued. Teachers can strive to give equal attention to both boys and girls. They can avoid stereotyping and encourage female role models. They can pay attention to whom they are asking higher-level-cognitive questions so that both genders are challenged. They can monitor the length and frequency of wait time to ensure that girls are given the same opportunities as boys for reflection before answering. As the Sadkers (1994) indicate effecting such changes in the classroom is not easy because society has conditioned all of us to gender inequities, but the attempt should be made.

Belenky, Clinchy, Goldberger, and Tarule in *Women's Ways of Knowing* maintain that women have their own ways of knowing. Many women see knowledge and learning as connected to their lives and experience while men treat knowledge as separate from themselves. The authors compare the bull sessions of male college students which revolve around debates of issues with the gossip sessions of female students which center on discussions of experience. Carol Gilligan in *In a Different Voice* compares the relationship oriented ways that girls approach moral decisions to the rational approaches of boys. This area of feminist psychological research suggests that work in the classroom should emphasize connections in learning. In fact, in *Women's Ways of Knowing* the authors argue that constructivist teaching is one way to help schools meet the needs of female students.

Several studies of math education have investigated specific topics or strategies which impact male and female success differently. Since females as a group have poorer spatial skills than males, it was suggested that differences in math achievement could be explained by differences in spatial ability. Lindsay Tartre (1990) reports, however, that spatial ability needs to be looked at within genders. Males, whether they

have high or low spatial ability, can handle math problems. Low-spatial-ability males make use of hints. High-spatial-ability females were successful on the tasks, but females with low spatial ability could not do the problems even with the help of hints. Tartre's research suggests that low-spatial-ability females may not benefit from practice in spatial ability. Joan Ferrini-Mundi (1987), on the other hand, in a study of college calculus students found significant sex differences which favor men in spatial ability and women in calculus achievement. Her work indicates that female students benefit from practice with spatial problems. Both projects show that differences in spatial ability are not responsible for the demonstrated differences in mathematics achievement between the sexes. They do suggest, however, that exposing female students to spatial training will not hurt students and may help.

Research has also been undertaken to study the impact of testing style on gender differences. In a study of university students in the United Kingdom, Johnston Anderson (1989) found that while males and females scored the same on open-ended problem solving tests, females' scores were consistently only 80% to 86% of the men's scores on multiple choice and true-false tests. Since the students were equal on all other measures, Anderson concluded that women and men approach objective tests differently. In looking at timed tests as opposed to untimed tests, Miller, Mitchell, and Ausdall (1994) concluded that males scored significantly better than females on timed tests. When similar untimed tests were given, female scores improved significantly, while male scores did not. The female scores did not equal the male scores even when untimed, however. This research suggests that if timed objective tests remain important as evaluative tools, then teachers should work with female students to try to help them learn to approach the timed tests more successfully.

An analysis of the longitudinal High School and Beyond study indicates that sex

differences are small and appear to be decreasing. Stereotypes that males do math and females do English are diminishing. The study found the achievement of girls in math only slightly below that of boys. Significant sex differences still appear in academic attitudes where girls had poorer math attitudes than boys, however. The biggest gender difference occurred in the better grades girls received overall (Marsh, 1989).

Even if gender differences are decreasing, they still exist in the learning of complex concepts, in personal attitudes, in classroom interactions, and in the choice to continue in math. Therefore, as teachers we can learn from the research and try to establish classroom environments which encourage gender equity. The research I have outlined suggests several avenues to explore from the concrete to the more nebulous. Some ideas to investigate include:

1. monitoring classroom interactions to ensure girls and boys receive the same opportunities to participate
2. monitoring classroom interactions to ensure girls and boys receive the same feedback
3. incorporating more wait time and allowing it equitably
4. using some untimed tests
5. exposing students to multiple choice and true-false assessment forms in a learning environment
6. incorporating activities which encourage the development of spatial ability
7. removing, if possible, and addressing, if necessary, stereotypical gender references
8. including more cooperative learning opportunities
9. decreasing the use of competition as a motivator
10. creating more student-centered classrooms

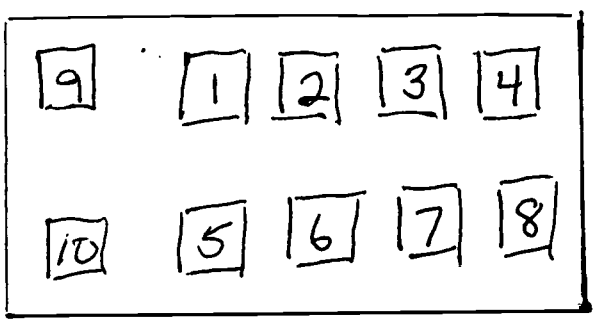
11. emphasizing the connections in mathematics
12. coaching rather than telling
13. encouraging autonomous learning
14. challenging students with complex mathematics.

The list is overwhelming. The question remains how can a teacher incorporate all of these ideas? What kind of structure will address these issues? In order to find a framework to structure these ideas, I turned to research on how students learn mathematics - not just how women learn, but how all students approach mathematical tasks. The next two sections discuss some of that research and an epistemological theory of knowledge which, I believe, gives structure to the type of classroom instruction gender equitable teaching requires.

Learning of Mathematics

Much research in math education is now being done from a cognitive science perspective. Attempts are being made to determine how students learn mathematics. Research procedures usually involve individual interviews and videotapes. Often a student or teacher is asked to reflect on her thinking in an effort to add meaning to observed events. A few classes or students are observed in depth and then researchers look for universals which help explain how students learn. The emphasis is on finding generalizations for cognitive procedures which seem to apply to all students. Rarely are students singled out by gender, race, or socioeconomic background. Much of the work I read has been done at the elementary level. Researchers often compare two classrooms: one in which traditional, teacher-dominated, procedure-oriented teaching occurs and another in which a different concept of learning is used as a basis of instruction. I will examine several typical traditional classrooms and interactions and then review some examples of alternative techniques and their results.

The literature includes many descriptions of students and teachers misunderstanding each other. There is the kindergarten boy, Nicholas, who is asked to draw on a piece of paper a representation of how he ordered ten objects. Nicholas turned in the paper shown below.



The teacher wrote on the paper the word "backwards". When Nicholas brought the

paper home, his mother asked him about it. He responded that he had drawn one through four then ran out of room so drew five through eight below one through four and again ran out of room so he placed nine and ten where he had room with the nine first. When the mother suggested the boy take the paper back to the teacher and explain, he did not bother saying it did not matter (Brooks, 1993). I claim it does matter. The boy had in fact completed the assigned task correctly. He had ordered the numbered objects logically, just not according to the teacher's model.

Maher and Davis (1990) discuss a similar situation which occurs in a class where Brian and Scott are working together on the following problem:

> At Pizza Hut each large pizza is cut into twelve slices. Mrs. Elson ordered two large pizzas. Seven students from Mrs. Elson's class are to eat one piece from each of the pizzas. What fraction of the two pizzas was eaten?

Brian and Scott have worked on the problem together and separately for a considerable period of time. Brian has used two pattern block hexagons to represent each pizza and a pattern block triangle to represent one piece. The boys are convinced that they have a correct solution and Brian tries to explain it to the teacher.

> Brian: So there's 24 slices in both pizzas, so Mrs. E. wants 7 students... she took 7 students to Pizza Hut, so ... she's gonna give'em one slice from each pizza so we should have, uh, 14 out of 24, right, slices.
> Teacher: All right, now let me ask you this. How do you get 24 slices in the one pizza, and 12 slices in the other?
> Note that this is not what Brian had actually done or said.
>
> . . .
>
> Brian: In all.
> Teacher: Brian.
> Brian: There's 12 slices in the one pizza, and 12 slices in the other.
> At this point the teacher interrupted Brian and told him how he (they) should think about the pizzas. [The teacher expected the students to use one pizza of twelve slices as the unit not two pizzas with 24 slices as the unit.]
>
> . . .
>
> Brian seemed ready to abandon his solution. The teacher directed the students by correcting their work; she discarded Brian's solution (which had, in fact, been correct).

As the tape continues the boys give in to the teacher's method which she uses to lead them to the answer 14/12ths. Unfortunately, not only has she undermined the boys' reasoning, especially Brian's, but Brian's solution was correct and hers was not. Unfortunately, I believe in a busy classroom all too often the teacher fails to take the time to listen to students and to understand their representations and solutions to a problem.

Not only do teachers and students miscommunicate in many classrooms, but also students frequently do not understand algorithms which they appear to be able to use. Students can apply the algorithms correctly, but they do not understand what the algorithm represents or they can not determine which algorithm is appropriate in a given situation. The lack of understanding of an algorithm is apparent in Wheatley's (1992) discussion of Talitha's work when she was asked verbally to find "235 minus 341". Talitha was in the top fifteen percent of her fifth grade class. She had worked with borrowing and could compute problems like 2,005 minus 1,237. She had never worked with negative numbers, however. Her work on 235 minus 341 is shown below.

$$
\begin{array}{rrrr}
 & 11 & 12 & \\
 & \cancel{1} & \cancel{13} & \\
 & \cancel{2} & \cancel{3} & 5 \\
- & 3 & 4 & 1 \\
\hline
 & 8 & \cancel{9} & 4 \\
 & & 8 &
\end{array}
$$

Talitha subtracted 1 from 5 and got 4. She then knew that she could not subtract 4 from 3 so she borrowed a 1 from the 2 making it a 1 and getting 9 by taking 4 from 13. Then

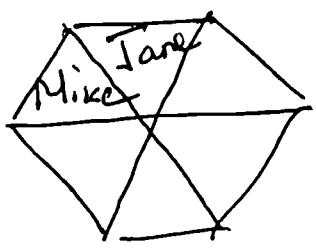
when she attempted to take 3 from 1 she realized that she could not so she borrowed from the tens place making the 13 a 12 (hence the 9 became an 8) and getting as her final answer 884. When asked about her answer later Talitha thought maybe she should have borrowed from the 5 ones rather than the 13 tens. Talitha did not understand the concept behind borrowing. She had not internalized that when one borrows a 1 from the hundreds place you are really rewriting 100 as 10 tens. Therefore, you can not turn 1 ten into 10 hundreds. Talitha could apply the borrowing algorithm correctly in a typical subtraction problem, but did not understand how the process worked. In addition, she did not think of mathematics as the process of making sense. She did not observe that 235 minus 341, no matter what it might be, could not possibly be as big as 884.

Davis and Maher (1990) illustrate the situation where a student can find the answer using common sense and representations, but uses the wrong algorithm and becomes confused. Ling Chen, another fifth grader was faced with the following problem:

> Jane has one third of a candy bar. She gives half of what she has to Mike. How much of a candy bar does she give to Mike?

Ling wrote the fractions correctly and drew the picture below to illustrate her solution.

Jane has $\frac{1}{3}$ of a candy bar. She gives $\frac{1}{2}$ of what she has to Mike. How much of the candy bar does she give to Mike?



$\frac{1}{6}$ of the c.b.

She divided the candy bar in thirds and then divided the thirds in half to get sixths;

consequently, Mike got one-sixth of the candy bar. When the teacher asked Ling if she could do the problem with numbers, after working for a while, Ling wrote

$$\frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$$

Ling knew the algorithm for dividing fractions and she executed it correctly. Unfortunately, division of fractions was the wrong algorithm for this problem. When the discrepancy in her answers was pointed out, she still believed one-sixth was the correct answer so then she tried

$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$$

to get one-sixth. When that did not work she tried

$$\frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

She now has said that multiplication and division of fractions are the same. She believed that this must be the correct numerical solution because it got the right answer. Ling can solve problems successfully using representations, but she does not understand the link between algorithms and problems yet.

These examples illustrate the importance of teacher interactions in the classroom and the difficulties even good students have in understanding and applying algorithms. Students appear to make the most progress when they represent problem situations in their own way. Remember that Brian solved the pizza problem correctly with his own method and Ling solved the candy bar problem quickly and accurately with

her own presentation. As teachers we need to provide classrooms which encourage student investigation and understanding. Next I will consider two situations in which students were encouraged to think and to explore.

Wheatley (1992) gives the protocol below as an example of a situation in which a student needed time to reflect on his reasoning.

Interviewer: What is 21 take away 19?
Jim: One ... no, TWO!
Interviewer: What is 31 take away 28?
Jim: 12
Interviewer: What is 31 take away 29?
Jim: 11
Interviewer: What is 31 take away 30?
Jim: [long pause] One?
. . .
Interviewer: What is 31 take away 29?
Jim: Two ... I've got it now.

Wheatley claims, and I agree, that Jim took the long pause to think about an obvious dilemma he faced if he continued to follow the mental image that would give him 10 when he subtracted 30 from 31. Jim had time to reflect and he thought through the two conflicting images of 31 take away 30 until he felt comfortable with the logic of his answer. The time allowed Jim to think through his answer, to make sense of the math. The willingness of the interviewer to allow Jim to correct his own false image appears to have given Jim confidence in his mental image.

The second example comes from Pirie and Kieren (1992). In this classroom scenario the students are reviewing orally responses to seat work on generating fractions of an hour from a certain number of given minutes.

Teacher: What about 15 minutes? Miguel?
Miguel: Two eighths.
Teacher: [A little surprised by the answer] Two eighths? Two eighths of what?
Miguel: Two eighths of [hesitation] a pizza.
Teacher: Well, we're dealing with time here.
Miguel: Oh yeah! Two eighths of an hour.

Teacher: Any other fraction of an hour which tells us about fifteen minutes? Bonnie?
Bonnie: One-fourth hour.
Teacher: Good. Let's start a list. [Writing] Two eighths, one fourth ... Others?
Sarah: Three 12ths.
Teacher: Why is that?
Sarah: Well 5, 5, and 5 are 15. They are one 12th so three 12ths of an hour.
Teacher: Yes, we could see that on the clock - in five minute pieces. One, two, three 12ths. [Writes 3/12.] Any other?
Pete: Four 16ths.
Teacher: [Adding 4/16 to the list] Other?
Dan: How about seven 28ths.
[The list now reads: 2/8, 1/4, 3/12, 4/16, 7/28]
Leanne: [interrupting] Wait a minute. Some of those fractions are less equivalent that others. [Pause] No, no they're all equivalent to one fourth.
Jose: No, Leanne. Go! Don't stop! That's a good idea. See: two 8ths and one 4th and three 12ths - they're really equivalent. But two 8ths and three 12ths aren't as equivalent and three 12ths and seven 28ths, they're hardly equivalent at all!

This class is very different from Brian and Scott's. Here when Miguel gives a different answer than the one the teacher was expecting it is accepted. In fact, it is used as a stepping stone to new discoveries about equivalent fractions. In this class students are encouraged to question and investigate. All answers are accepted, but note when Miguel says two eighths of a pizza he is made to see that they are talking about hours so that he can correct his own answer.

Such examples taken from cognitive research suggest that students are most successful when they have opportunities to construct their own understanding. Mechanical learning of algorithms and rote procedures does not ensure that students understand the concepts behind the procedure or can determine when to apply the procedure. "It is not enough to absorb and accumulate information. Children must be given the opportunity to assimilate mathematical knowledge - to construct and complete mathematical understandings. This requires that instruction build upon children's

existing knowledge...Such an approach is important for fostering self-regulation and a positive disposition toward mathematical learning and problem solving - as well as meaningful learning" (Baroody, 1990, 63).

Many researchers have their own suggestions of how to design the most effective classrooms for encouraging the type of learning suggested by cognitive research. Alan Schoenfeld (1988) argues that math learning is contextually bound and that math classrooms should create a culture which is similar to the culture of mathematicians. Such a culture is based on sense making and uses algorithms and procedural math as a tool. Brown, Collins and Duguid (1989) argue for an even more situated learning. They believe that mathematics is learned best in an apprenticeship situation. Students learn by working collaboratively as though they were apprentice mathematicians solving real problems.

Work in developing new forms of curriculum is underway at the early elementary levels. Constance Kamii (1985) has worked with teachers in the first grade who teach through the use of games to stimulate the students' natural desire to understand and to move to higher levels of understanding through carefully designed game situations. Fennema, Carpenter, Franke, and Carey (1993) are involved with the Cognitively Guided Instruction program which encourages students to design their own understanding and requires them to share their ideas and interpretations with each other. This sharing of understanding helps other students to develop new understandings. Grayson Wheatley (1992) is associated with the Mathematics Learning Project in which learning is problem-centered. A problem is presented and then students work in groups of two. After about twenty minutes, solutions or attempts are shared in whole class settings and the class determines which of the solutions they believe are correct. In these programs collaborative learning is an essential part. In

addition, teachers spend time listening to their students to determine each individual student's level of understanding. Curriculum is often adapted to meet the needs of students.

Traditional mathematics teaching has used a "procedures-first" approach. The teacher explains a procedure and then students practice the procedure usually individually in a worksheet format. Wheatley (1992) argues that this traditional format does not encourage students to make sense of mathematics and consequently capable students such as Talitha do not understand the procedures which they can execute so well. With the use of manipulatives Wheatley maintains that teachers often use an "abstract-first" approach. In this situation the teacher explains the abstract mathematical concept and then illustrates it by showing the students how to use manipulatives to achieve an answer. Once again Wheatley argues that students do not have an opportunity to create understanding. The understanding is created by the teacher from the teacher's understanding of the concept not from the students' understanding.

If, as cognitive research suggests, students learn best by working through concepts themselves what behaviors should we as teachers encourage in our classrooms? As a starting point, I suggest that teachers who wish to encourage the learning of all students without regard to gender should strive to include the following behaviors in their classrooms:

1. attention to where the students are in their understanding
2. respect for student opinions about mathematics
3. reflection about mathematics
4. sharing of ideas
5. listening to others
6. writing about mathematics

7. making connections
8. persistence
9. interactions involving mathematics
10. collaboration.

Constructivism is a theory of knowledge which encourages many of these behaviors. In the next section, I will argue that using a constructivist approach teachers can design lessons which respond to the results of cognitive research.

Constructivism

Constructivism has been described as consisting of two hypotheses:

1. Knowledge is actively constructed by the cognizing subject, not passively received from the environment.

2. Coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower. (Lerman, 1989, 211)

The constructivist theory of knowledge is based on the idea that each person constructs her own reality by interacting with the world. Each interaction changes the world the individual knows; consequently, each interaction changes the individual's construction of reality. As a basis for a mathematics pedagogy, constructivism demands that teachers recognize that each student constructs her own mathematical knowledge and that her knowledge is constantly being adjusted to incorporate new mathematical experiences. As teachers it is our responsibility to help our students develop "good" mathematical constructions. This requires teachers to understand where their students are in mathematics so that they can provide appropriate mathematical experiences.

Constructivism developed from Piaget's theories of cognitive-development psychology. Piaget's concept of abstract reflection is a process of internalizing mathematical operations as we manipulate objects in our environment. We are constructing the mathematical results through our use of the objects. The cognitive structures we develop while working with the objects and the abstract reflection we use to form the cognitive structures are both constructivist according to Nel Noddings (1990).

In 1973 Stanley Erlwanger's studies of Benny had an enormous impact on the development of the pedagogy of constructivism. Erlwanger reported that, taught in a

behavioristic traditional manner, Benny created his own understandings of how to get the answers on the answer sheet. Benny's constructions led to results such as the conclusion that 3/2 and 2/3 both equal .5. Constructivists use Benny as an example of how poorly students can learn mathematics if they are not taught in a manner which encourages mathematical thinking (Noddings, 1990). In addition to providing a constructivist framework to explain Benny's learning, Erlwanger showed the value of a new kind of research and methodology which investigated the understanding of one student in depth. Both the research and the methodology served to advance the theory of constructivism (Steffe, 1994).

In the early 1980's, von Glaserfeld influenced the theoretical development of constructivism. According to Steffe and Kieren, von Glaserfeld and Richards argued in 1980 that "in constructivism one is not studying reality, but the construction of reality" (Steffe, 1994, 721). Jere Confrey (1990) describes the dual nature of constructivism: we construct our own knowledge of reality and in the process of construction that reality continuously changes.

> Put into simple terms, constructivism can be described as essentially a theory about the limits of human knowledge, a belief that all knowledge is necessarily a product of our own cognitive acts. We can have no direct or unmediated knowledge of any external or objective reality. We construct our understanding through our experiences, and the character of our experiences is influenced profoundly by our cognitive lenses. To a constructivist, this circularity is both acceptable and unavoidable. One's picture of the world is not, however, static; our conceptions can and do change. The essential fact that we are engaged in living implies that things change. (Confrey, 1990, 108)

If students construct their own understanding and their own reality, the question remains how should teachers structure classrooms to help students construct "good mathematics." Clearly, "acceptance of constructivist premises about knowledge and knowers implies a way of teaching that acknowledges learners as active knowers"

(Noddings, 1990, 10), but what way of teaching?

Jere Confrey (1990) identifies necessary beliefs and skills of a constructivist teacher.

> [T]he most basic skill a constructivist educator must learn is to approach a foreign or unexpected response with a genuine interest in learning its character, its origins, its story and its implications. Decentering, the ability to see a situation as perceived by another human being, is attempted with the assumption that the constructions of others, especially those held most firmly, have integrity and sensibility within another's framework. The implications for working with students are stunning. (108)
>
> When one applies constructivism to the issue of teaching, one must reject the assumption that one can simply pass on information to a set of learners and expect that understanding will result. Communication is a far more complex process than this. When teaching concepts, as a form of communication, the teacher must form an adequate model of the students' ways of viewing an idea and s/he then must assist the student in restructuring those views to be more adequate from the students' and the teachers' perspective. (109)

A basic concern of a constructivist teacher is understanding the students' perspective of the material. Since constructivists believe each person constructs his or her own learning, constructivist teachers need to structure activities which encourage students to investigate mathematical concepts. Such teachers strive to adjust classwork to meet students where they are in their understanding. Consequently constructivist lessons include opportunities for students to examine their own reasoning as well as ways for teachers to become aware of students' perceptions and reasoning so that activities can be adjusted accordingly. Learning for each student involves a constant readjustment of his developing structures of understanding. Constructivist teachers can not just impart knowledge with a lecture and assume students will construct "good mathematics". Learning is interactive. Students construct understanding by reflecting on new ideas and concepts. Carefully structured activities can help students in the

reflection process. As teachers we need to create classrooms which encourage students in abstract reflection.

Jacqueline Grennon Brooks and Martin G. Brooks in their book, *In Search of Understanding: The Case for Constructivist Classrooms* (1993), provide guidance for teachers who wish be constructivist by describing what they believe are the essential characteristics of a constructivist teacher. Their work is intended for teachers of all disciplines and the examples they use come from a variety of subjects. Their blueprint for a constructivist classroom can easily be applied to the math classroom. They claim that "when teachers recognize and honor the human impulse to construct new understandings, unlimited possibilities are created for students" (21). Such classrooms allow students to make connections and follow paths of interest by freeing them from fact-driven curriculums. Grappling with big ideas of interest encourages students to realize that learning is complex and not easy just as the real world is complex and filled with multiple perspectives. In constructivist classrooms students are encouraged "to clarify for themselves the nature of their own questions, [and] to pose their questions in terms they can pursue" (30).

The Brooks (1993) give five guiding principles for establishing a constructivist classroom. Topics to be studied should be of "emerging relevance to students". The emerging relevance may occur either because the questions are intrinsically important to students or because teachers present them in such a way that their importance to the students emerges through classwork. Learning itself should be structured around "primary concepts." Information should cluster around a concept so that the structure is holistic rather than separate isolated snippets of information. When concepts are presented holistically students will break them apart to assist in the analysis. Big ideas and large concepts give students more opportunities to find ways to become attached to

the information. From my perspective the most important of the Brooks's principles of constructivism is the importance and value of the student's point of view. Constructivist teachers actively seek their students' points of view. They accomplish this by listening, by questioning and by avoiding inserting their own point of view. Listening to students is crucial for effective constructivist teaching because of the importance of the constructivist belief that learning is constructed from present knowledge. Once students' viewpoints are known, curriculum is developed to address student perceptions. Constructivist education constantly adapts to adjust to where students are in their understanding. Assessment also changes in the constructivist classroom. Teachers assess students continually. By listening to student points of view and by asking students to explain their work, teachers have constant feedback about student learning. Consequently, traditional testing is not necessarily used. Teachers make use of authentic assessment and non-judgmental feedback instead.

Nel Noddings (1993) suggests that constructivist pedagogy, which requires teachers to understand and value student purposes, also requires an ethic of care. Because the constructivist teacher is dedicated to understanding student concerns and to responding to them, the constructivist teacher cares for her students. Caring about students helps teachers to connect mathematics to what is important in students' lives. Caring constructivist mathematics is connected to real life not just to "real" mathematics.

If we examine the ideas of constructivism, we can see that the constructivist classroom has many of the following characteristics:

1. considering problems of emerging interest to the students
2. studying major concepts, the big picture
3. seeking students' points of view
4. valuing students' points of view

5. teaching through questions rather than answers
6. listening to each other
7. allowing time for ideas to form
8. interacting in groups
9. encouraging persistence
10. striving to make sense
11. making connections
12. communicating both verbally and in writing
13. adjusting curriculum to students' understanding
14. using authentic assessment
15. giving non-judgmental feedback
16. reflecting
17. caring
18. encouraging autonomy
19. honoring creativity.

These characteristics can serve as guidelines for a teacher who wishes to use a constructivist pedagogy. The ideas on this list include all the behaviors that were generated from a study of the cognitive science research on how students learn mathematics. Therefore, this list responds to the concern for addressing in the classroom how students learn. In addition, there is considerable overlap with the list of classroom issues generated from the research on gender. Many of the gender issues seem to disappear in a constructivist setting. Classrooms are more cooperative and student-centered in constructivist settings. Teachers will need to monitor their interactions with students to ensure that both genders receive comparable treatment, but since constructivist teachers are constantly monitoring student understanding the

teachers should be aware of gender inequities. Overall, I believe that a constructivist framework offers the best way to ensure that all students receive the best mathematics education possible. "A constructivist framework challenges teachers to create environments in which they and their students are encouraged to think and explore. This is a formidable challenge. But to do otherwise is to perpetuate the ever-present behavioral approach to teaching and learning" (Brooks, 1993, 30).

A Constructivist Unit

Believing that a constructivist pedagogy provides the best framework for equitable and effective math education, I have developed a unit on limits from a constructivist perspective. Changing theory into practice is difficult for me. Therefore, using the guidelines given on the preceding pages I have designed a series of lessons on limits which I believe reflect a constructivist approach to education. This unit is meant to serve as one example of how to structure constructivist learning at the upper levels of high school. The unit was designed with the Advanced Placement curriculum in mind, but it could be used as a model for any class with access to graphing calculators. Since the unit is meant to be constructivist, any teacher using it will necessarily adapt and change it to meet her students' needs and interests. The pacing and the depth of study depend on the abilities and interests of students in the class. Consequently, each teacher must decide which units to emphasis and how to pace the lessons.

Developing an Understanding of the Meaning of a Limit

To examine functions exhibiting a variety of behaviors as the domain values approach a specific point.

To appreciate the need for a concept, the limit at a point, to help determine the behavior of a function near a point.

To understand the importance of range values when determining the limit of a function at a point.

To work with the definition of the limit to determine whether a function has a limit.

To find limits of functions algebraically, numerically, and graphically as appropriate.

To investigate one-sided limits.

To examine the advantages of defining a limit as approaching infinity.

To extend the concept of limit to include the limit of a function as x approaches infinity.

Examining functions which exhibit a variety of behaviors as the domain values approach a specific point.

Before beginning this unit, students need to know the definition of an open neighborhood. They also need familiarity with a graphing calculator which they may use as a tool whenever they wish.

This first section begins with a worksheet. Although the worksheet could be worked on individually, I recommend groups of three or four working together. Working in a group adds the element of discussion which often aids in an investigation. Each student should write her own response to any questions, however. For an AP class or

above average students, I would give them the first worksheet in groups with no introduction. An essential part of the exercise for the teacher is to interact with the groups using questions (no answers) to encourage their explorations. At this point, getting the correct solution is not as important as thinking about how functions might behave at a point.

Worksheet

Determine the behavior of each function for x values in a neighborhood of $x = 0$. Write one or two sentences for each function describing its behavior and how you determined it. Be sure to note what happens at $x = 0$.

1. $f(x) = 1 - x$

2. $f(x) = (x - x^2)/(x + 1)$

3. $f(x) = (x - x^2)/x$

4. $f(x) = (1 - x)/x$

5. $f(x) = |x|/x$

6. $f(x) = \sin x$

7. $f(x) = \sin (1/x)$

8. $f(x) = x \sin (1/x)$

9. $f(x) = (\sin x)/x$

10. $f(x) = \begin{cases} 3x + 1, & x > 0 \\ 4, & x = 0 \\ 2 - x, & x < 0 \end{cases}$

11. $f(x) = \begin{cases} (x - x^2)/x, & x \neq 0 \\ 4, & x = 0 \end{cases}$

12. $f(x) = \begin{cases} (x - x^2)/x, & x \neq 0 \\ 1, & x = 0 \end{cases}$

13. $f(x) = \ln |x|$

14. $f(x) = \begin{cases} 3x + 1, & x > 0 \\ 4, & x = 0 \\ 1 - x, & x < 0 \end{cases}$

After students have had time - one or two days depending on class length - to work with each of the functions, I suggest a class discussion of observations. The purpose of the discussion is to encourage students to see, categorize, and question some of the behaviors a function may exhibit at a point. It is a time to raise questions based on observations, not to answer them. The structure of the discussion will depend on student input; however, some possible exploratory questions follow.

Which functions were most interesting/confusing? Why?

Did any of the functions behave in unusual ways? How?

Did any of the functions surprise you with their behavior? What were you expecting? Why?

What questions do you have after looking at these functions?

Such questions are intentionally vague in order to encourage students to create their own understanding and questions rather than enforcing the teacher's view.

Before asking questions along the outlines above, in some classes you may want to talk about each of the functions around $x = 0$. Different groups can report on different functions. When students discuss a function they need to explain the method used to achieve their answer. Some of the functions if graphed on a graphing calculator may allow students to describe what is happening near $x = 0$ without the students realizing that the function does not have zero in its domain, i.e., $f(x) = (\sin x)/x$, or that $f(0)$ is not the same number as the value the function is near.

Appreciating the need for a way to help determine the behavior of a function at a point.

After discussing all of the functions, students should be encouraged to generate specific questions. Depending on the make-up of the class, such questions could be:

brainstormed

developed in groups

worked on individually perhaps as homework.

Brainstorming, although the fastest method, is the least attractive option, because some students, especially girls, may refrain from posing questions which seem to them silly. Those may, in fact, be the most important questions.

Perhaps the best method to generate questions, if time allows, is to have students think of questions for homework and then get together at the beginning of class in small groups to generate several questions which reflect the crucial issues found in the homework questions. Having students generate questions individually at first allows the slower worker an opportunity to think through relationships before they are suggested by a group member. But having groups work together helps to refine the questions and carry analysis to a higher level. Each group would supply at least one question to a class list of questions which students felt needed to be investigated. Some typical questions that might arise are:

What is the effect of x in the denominator?

Why do some functions have asymptotes and others holes and still others no holes?

Why do some functions jump?

How can you tell what's going to happen without a graph?

The specific questions are not as important as the process of asking them. The teacher can always insert a few questions, but the student is more involved if she feels her questions are helping to guide the inquiry.

At the end of class, each group would turn in their individual questions as well as their group questions since the questions allow the teacher to gain knowledge about the

understanding of individuals. In addition, important student questions may be overlooked in the class discussion. Having a chance to look at each group's list allows the teacher the opportunity to revisit issues the following day or to add another question. The questions may also serve to facilitate review of the concepts.

Once the questions are generated, it will probably become obvious that what happens to the function at a point and what happens to it around $x = 0$ may be two different things. We can find out about $f(0)$ by substitution, but we need a new method to investigate the behavior of $f(x)$ when x is near zero. The goal of the unit to this point is to raise the awareness of the behavior of functions at or near a point. Students should have become aware that $f(0)$ is not always the value that the graph is near for domain values near 0. In fact, the function may not even be defined at $x = 0$. Consequently, mathematicians need a way to talk about and distinguish what is happening to a function near a value of its domain.

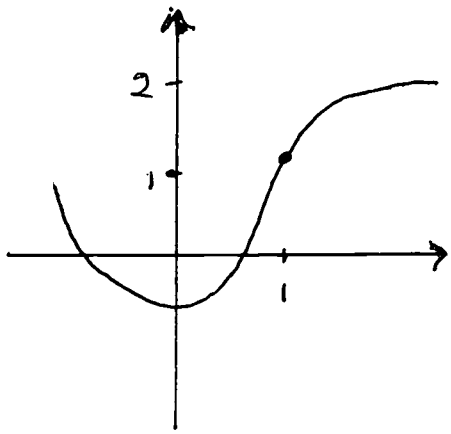
If students have not generated an intuitive understanding of the complexities of functional behavior, then it may be necessary to revisit some of the worksheet functions or similar functions before going on to a discussion of the limit at a point. In another class it may be more appropriate to go on to limits, giving a big picture, and then return to specific examples of why the limit concept is helpful. The teacher needs to determine the students' readiness and develop the lesson starting from where the students are.

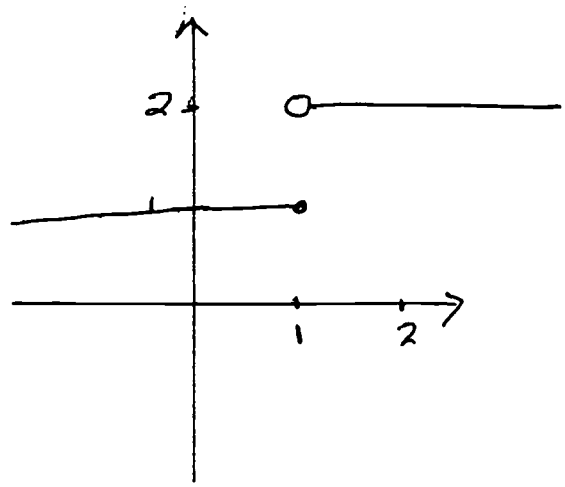
Understanding the importance of range values when determining the limit.

We now need to investigate what we mean when we say the limit of a function at a point. Some of the functions we have looked at have a limit as x approaches zero and some do not. If we tentatively say that the limit of $f(x)$ as x approaches c, a point, is the finite number $f(x)$ gets very, very close to as x gets very, very close to c from both

directions then we can intuitively guess which of our functions have a limit at x = 0. We need to revisit the functions and discuss why some have a limit and some do not - still intuitively. For homework or group work I would then give the students the two graphs shown below and ask them to write down in their own words why one function has a limit at x = 1 and the other does not.

1,





My hope is that they would begin to see the importance of the range in determining whether the function has a limit. After a discussion of the homework problems we can now introduce the $\varepsilon-\delta$ definition of the limit.

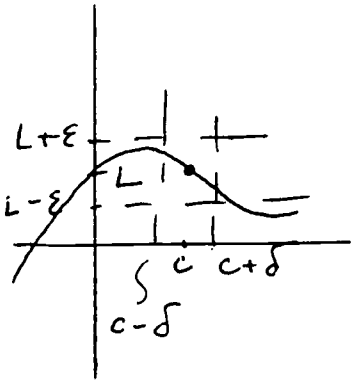
We call L the limit of the function f(x) as x approaches c,

written $\lim_{x \to c} f(x) = L$

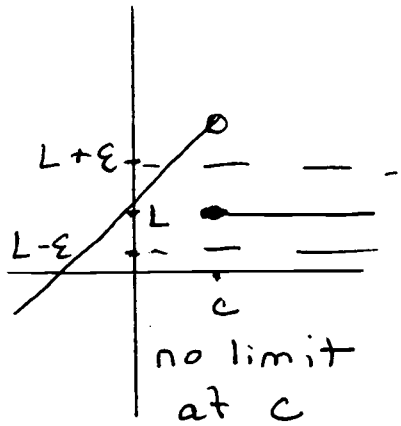
if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

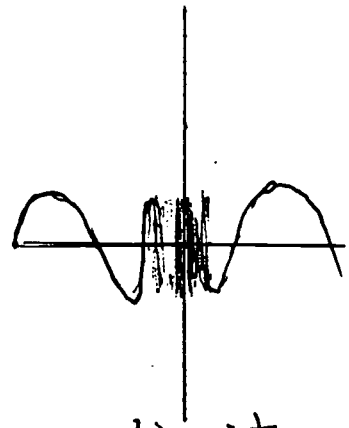
It may be necessary to review what the inequalities mean. In addition, several pictorial examples of functions with and without limits help to interpret the definition.



limit exists at c



no limit at c



no limit at x=0

Working with the definition of the limit.

Using the graphing calculator, we can now revisit the original functions. By using the window (or range) option, we can choose an epsilon and either find a delta or convince ourselves that no delta exists. Only one or two examples are necessary to illustrate why a function does not have a limit at a point. In groups, students could explore several of the functions we intuitively said had no limits to see if they could determine why those functions do not satisfy the definition of the limit as x approaches 0. f(x) = sin (1/x) is an excellent choice for the groups to examine. No matter what you choose as a limit if $\varepsilon < 1/2$, the graph goes off the screen for all values of delta you choose. Another good choice is f(x) = |x|/x; you can lose half the graph with $\varepsilon < 1$. Comparing the graph of f(x) = (sin x)/x with the two preceding functions shows the difference between a function having a limit at a point and a function which fails to have a limit at that point. The graphing calculator allows students to make the range boundaries on the graph of f(x) = (sin x)/x as close to 1 as they wish and still find domain values for x around 0 so that the value of the function is within the range values chosen.

To reinforce the analysis by the groups I suggest putting diagrams on the board, or, if available, using an overhead graphing calculator for groups to present their own analysis of the functions in terms of the definition. The overhead graphing calculator

allows students to quickly investigate ideas or questions suggested by other classmates while leading the discussion. After examining several of the original functions in class, I would assign the remaining ones to be discussed for homework. For each problem there should be a sketch of the graph and and a few sentences describing why the function has a limit at 0 or why it does not.

After this introduction to the definition of a limit at a point, I would return briefly to a more traditional approach to learning limits. Using a function such as $f(x) = 1 - x$, and choosing a finite value for epsilon, say 1/100, I would show the students how to find a delta which would satisfy the definition of the limit. The graphing calculator would allow us to confirm the choice of delta and to illustrate that any smaller delta would also work. Several other simple examples of the definition of the limit would be investigated in class. This is a good time to use an example such as the limit of $f(x) = 4x + 2$ as x approaches 5, since up until this point we have worked only with functions when x was approaching zero. All students would be required to work through several similar problems on their own. Whether much work is done using a generic epsilon to find a delta would depend on the level of the class. In BC Calculus, students would be required to work through some epsilon-delta problems and limit theorems would be proven; in regular calculus or precalculus classes such rigorous mathematics would only be briefly discussed unless students showed interest.

Finding limits of functions algebraically, numerically, and graphically.

Once limit theorems have been discussed, I would spend some time working with students to find limits algebraically. A standard calculus text offers examples and explanations of the necessary manipulations, and I would refer to the text for examples and some homework problems. After students feel comfortable finding limits

algebraically using limit theorems, I would ask them to consider the following two problems to see if they can solve them without graphing.

1. $\lim_{x \to 0} (\sin 3x)/x$

2. $\lim_{x \to 1} \dfrac{x^5 - 6x^4 + 5x^3 + 2x^2 + 3x - 5}{3x^3 - x^2 - 9x + 7}$

These are good problems to have small groups discuss, because they can both be done using techniques from previous courses (trig identities and synthetic division) as well as by substituting numbers close to 0 and 1, respectively. I would require that each group submit a written justification of their solution - particularly because of the tendency of students to assume that sin (3x) = 3 sin x. These problems were chosen to illustrate the technique of approximating limits numerically with a calculator so if no group uses that technique either the teacher has to suggest it or supply other problems to try and encourage it. Both problems can be done using all three techniques: algebraic, numerical, and graphical. They serve as a vantage point to discuss the various ways to determine the value of a limit, the dependability of each method, and when a particular way might be most helpful.

Investigating one-sided limits.

Now, the students have enough experience with two-sided limits so we can begin looking at one-sided limits. Note, in some classes questions about one-sided limits may have already arisen. When such questions occur, a constructivist teacher should encourage students to follow their interests. In that situation, a teacher may wish to adapt this material for earlier use.

I suggest using small groups again and asking students to investigate the

behavior of the function, $f(x) = x^x$. The directions below could be given to each group to guide their investigation.

> Without using the graphing capabilities of your graphing calculator, determine as much as you can about the behavior of the function, $f(x) = x^x$. Among the characteristics to speculate on are the range and domain and a possible sketch of the graph. Please share with me your written description of the function's behavior and a possible sketch before you check your results against the graph on the calculator. As you write about the function you may discover some questions you can not answer, simply include those questions and why you have them in your write-up. You may be able to answer them after you look at the graph on the calculator.

The function, $f(x) = x^x$, provides a wonderful opportunity for students to review implicit domains, ranges, and roots of negative numbers. It also can serve as an introduction to calculating limits using logarithms as a tool when such limits appear in the curriculum. I have chosen to use it here because of its behavior as x approaches zero.

Once the students have investigated the graph of $f(x) = x^x$, they should be able to tell that the limit of x^x as x approaches 0 through positive values is one. Students should then be encouraged to look at this situation in terms of the definition of the limit at a point. Possible questions are:

Does the definition of the limit we have allow for this situation where one only has x values on one side of c?

If so, why?

If not, could you design a definition which would work in this case?

There are at least two ways to handle this problem. (Students may suggest others to investigate.) One can consider only values of x in the domain of the function when using the limit definition, or one can define a new kind of limit - a one-sided limit. Both ways work in this case, and restricting the definition to the domain of the function makes sense as the easier solution. In fact, that restriction is included in definitions of

the limit in the field of mathematics known as analysis. But if students examine the behavior of the function $f(x) = |x|/x$ as x approaches 0, they will probably see the benefits of defining special case limits known as right-handed and left-handed limits. Students can be asked to write a definition for right-handed limits based on the definition we have given for the limit. For homework, students could find or create examples of functions which have unequal one-sided limits at a point and functions which are undefined at a point, but have equal one-sided limits.

Using the functions students have found for homework, we can begin to answer some of the questions which the class generated as we began the study of limits:

Why do some functions jump?

Why do some functions have open holes?

How can you tell whether piecewise functions will meet?

These questions can be answered with the use of right and left hand limits, and students should be encouraged to write several sentences to answer each of them. Now is a good time to return to the original worksheet, and ask the students to respond to it again using limits to describe what is happening at $x = 0$.

Allowing the limit at a point to approach infinity.

Included on the original worksheet are the functions $f(x) = (1 - x)/x$ and $f(x) = \ln |x|$. When asked to describe the behavior of those functions using limits as x approaches 0, students will almost surely say that those functions do not have a limit at $x = 0$. The limit is a finite value. Comparing these functions to the behavior of $f(x) = \sin (1/x)$, however, can remind students that not having a limit can mean different things about the behavior of the function at that point. Asking students to describe in words the differences in the two situations - one where the function oscillates rapidly and

the other where the functions go off to positive or negative infinity - should lead into the idea of talking about a function approaching infinity as x approaches a point c. Again, the value of the concept of right and left hand limits can be seen. The following definition can be derived for right-handed limits which approach positive infinity.

We say the limit of f(x) approaches positive infinity as x approaches c from the right, written $\lim_{x \to c^+} f(x) = +\infty$,

if for every $M > 0$ we can find a $\delta < 0$ such that

$f(x) > M$ whenever x $(c, c + \delta)$.

In groups of four students, pairs can be asked to construct ten problems for the other pair. These problems should illustrate all the different types of limits at a point that have been discussed. The pairs can then exchange problems and answer each other's problems. The advantage of having students make up the problems is that it forces them to think about the structure of functions both algebraically and graphically. Having the pairs work on each other's problems encourages clarifying discussions when a problem is not understood or well defined. As always this work should be collected so that the teacher can evaluate whether the students understand and can use the definitions of various types of limits at a point.

Examining the limit of a function as x approaches infinity.

So far we have had students examine the behavior of functions at a point to help justify definitions of the limit. Now I suggest we reverse the procedure to help determine whether students can use their constructions of the limit concept to interpret the definition of the limit of a function as x approaches infinity. I would give the definition of the limit as x approaches infinity and several examples of functions, some of which have

limits and some of which do not, to groups of three or four. Students would have an opportunity to discuss the definition and examples to try and create an understanding of this new concept. The worksheet below would serve as instructions for the group work and a homework assignment.

Worksheet

Consider the definition of the limit as x approaches infinity and each of the examples. Draw a diagram which helps you interpret the definition. In your group discuss the definition, your illustration and each example. Identify why the limit exists or fails to exist for each function given.

Definition of the limit of a function as x approaches infinity:

The function f(x) has a limit L as x approaches infinity,
written $\lim_{x \to \infty} f(x) = L$,
if for every $\varepsilon > 0$ there exists an $M > 0$ such that
$|f(x) - L| < \varepsilon$ whenever $x > M$.

Consider the following examples:

1. $\lim_{x \to \infty} (1 - x) / x = -1$
2. $\lim_{x \to \infty} 1 / x = 0$
3. $\lim_{x \to \infty} (3x^2 - 2x + 1) / (4x^2 + 2) = 3 / 4$
4. $\lim_{x \to \infty} \sqrt{3x + 2} / \sqrt{x - 1} = \sqrt{3}$
5. a. $\lim_{x \to \infty} e^x$ no limit
   b. $\lim_{x \to -\infty} e^x = 0$
6. a. $\lim_{x \to \infty} \sin x$ no limit
   b. $\lim_{x \to \infty} (\sin x) / x = 0$
7. $\lim_{x \to \infty} (4x^2 - 2x + 1) / x$ no limit
8. $\lim_{x \to \infty} \sqrt{2x^2 + 1} / x = \sqrt{2}$
   b. $\lim_{x \to -\infty} \sqrt{2x^2 + 1} / x = -\sqrt{2}$
9. $\lim_{x \to \infty} (1 + 1/x)^x = e$

Homework

Write a brief essay which explains your interpretation of the definition of the limit of a function as x approaches infinity. Please include a sketch illustrating your explanation and a discussion of several functions to illustrate when the limit exists and when it fails to exist. Diagrams may be helpful.

As the groups are working to interpret the new definition, the teacher needs to walk around and keep an eye on each group to be certain that progress is being made. Working with the groups is tricky because the goal is to have each group, ideally each individual, construct its own meaning for the limit as x approaches infinity. Therefore, as teachers when we work with groups, we need to strive to understand the students' constructions and not impose our own. We should question groups to ascertain their interpretations before guiding them with suggestive questions. Our questions should be open-ended such as "Have you considered ... ?" "What If ... ?" Group work is the perfect time for the teacher to offer extra support to students who have found the limit concept especially challenging. By this time I would expect the students to automatically investigate the given functions with a graphing calculator, but if this does not happen in all groups, such a suggestion should be made.

The homework essay is important for two reasons. It forces the students to think through individually the implications of the new definition. Writing reinforces and clarifies the thought processes. In addition, the essay gives immediate feedback on each student's interpretation of the new concept. The essays allow the teacher to assess whether students have grasped the new concept and whether the class is ready to move on or needs to spend time as a class talking through the new concept. In order for the teacher to have a night to read the essays, the next class could be spent learning how to determine limits algebraically as x goes to infinity. The class, as a whole, could justify that the limit of $1 / x^p$, where $p > 0$, is zero. Then, it might be fun to allow students to

work with the examples on the worksheet. The limits were given to them there, but now the group challenge would be to see which limits they could determine using algebraic manipulations. A super challenge would be to see if they could identify and justify any short cuts for evaluating limits as x goes to infinity. I would expect some group to identify the algorithm that the limits of quotients of polynomials can be determined by inspection of the highest powered terms, but groups might construct their own equally valid procedures. After allowing time for the groups to play with limits as x approaches infinity, the groups should have an opportunity to share their results and to engage in a class evaluation of each procedure to determine which could be generalized. An open discussion encourages students to look more closely at the work done in their own groups. If the group can convince the whole class that a procedure or algorithm works then the group has had an opportunity to internalize their own learning and increase the understanding of classmates.

Note

A natural progression from this discussion of limits would be an investigation of asymptotes - horizontal, vertical, and oblique - and their relation to limits. Such a topic would make an exploratory lesson for advanced students, especially those in AP Calculus courses. In classes in which some students need more time working with limits, students who were comfortable with limits could investigate asymptotes and then share their learning.

Assessment

The question arises about how to assess learning which occurs in a format such as that described above. First, it must be noted that constant informal assessment is

part of the structure of these lessons. Students are asked to write frequently about their understanding and to explain group conclusions to the class. In addition, an important part of this type of constructivist learning is continual evaluation of student understanding by the teacher. Each written assignment should be read and notes should be kept. During classwork the teacher should observe and listen carefully to student interchanges. Teacher knowledge about where students are and what they understand is invaluable in pacing, structuring and modifying lessons.

But more is needed. Students deserve the opportunity to show what they know and to extend their understanding through review and consolidation. A variety of assessment vehicles allow students to continue to grow in understanding and to prepare for other possible math courses. Assessment can and should range from multiple choice and short answer to open-ended essay questions to applications to portfolios. The important thing is that students have an opportunity to show what they know - not what they do not know.

Multiple choice problems are an important assessment tool especially if students will be taking an advanced placement exam. Research indicates that females do less well on multiple choice and true-false type problems than males. Therefore, exposing students to such questions in the classroom setting is important. In order to remove some of the anxiety that AP and multiple choice exams engender, I suggest introducing students to such questions in a group format. Working together on challenging multiple choice problems, students should have some success. As individuals feel more comfortable with a particular format they have a better chance of success on their own. Also, students can learn that some problems are very hard and that it is not at all unusual for groups working together to have difficulty solving them.

Tests and examinations can be designed to allow students to show their learning.

Traditional test questions such as compute the limit can be interspersed with questions on definitions and brief essays describing the explorations students have undertaken. In constructing these evaluation instruments the graphing calculator should be considered. When computing limits for example, some might be done more easily with the graphing calculator than with algebraic manipulations. In fact, students can be asked to discuss limits for types of functions which are entirely new to them when they have the graphing calculator as a tool. Students could be asked to find the limit as x approaches infinity of $f(x) = (1 + 3/x)^x$ even though they had never investigated such a function.

Tests can be given in an untimed format or as take-homes. If students have ample time to reflect, the questions on evaluation instruments can address higher level cognitive skills . Given in a group format tests allow students an opportunity to pool their efforts much as they would in the working world. Testing and assessment no longer mean one student writing madly or panicking for 40 to 50 minutes. Tests are just an additional way to help students construct good mathematics and to help the teacher understand how successful the student has been.

In addition to tests or instead of tests, students could be asked to write papers or give reports. Possible topics are why a specific definition works or what limits tell you about a particular function. Each student or groups of students could investigate different functions and then give oral reports. The possibilities are endless. Students could investigate why limits are important. Applications in medicine and science are especially interesting. One possibility is to have students look at the absorption of a drug over time. Projects investigating applications open new horizons to students.

Portfolios are another way for students to show what they have learned. I suggest that students keep portfolios which include all their written responses to group work or homework questions. Not only do portfolios remind students and teachers of

the student's growth in understanding but they also serve as a vehicle for review. Looking back at one's own observations helps one to remember. In addition, self-assessment can be a part of the portfolio. Students benefit from thinking about their own learning and determining themselves what they have or have not learned well.

Assessment, no matter what form it takes, should be an opportunity for students to reflect and justify as it helps them prepare for new concepts. The assessment tools used should blend naturally with the structure of the class as one of exploration and construction of knowledge. Students should be able to show what they know as they continue to expand their understanding.

Conclusion

This project was designed to help address gender issues in mathematics education. A review of the recent research on gender issues indicated that while gender differences appear to be decreasing differences still exist in the learning of complex concepts, in personal attitudes toward math, in interactions in the classroom, and in the level of mathematics studied. While the research suggested several strategies teachers can implement in the classroom, the startegies seemed isolated and not part of an overall structure to improve learning in the classroom. Therefore, an investigation was undertaken to examine research on how students of both genders learn mathematics. This research led me to the belief that each student constructs her own understanding and knowledge as a result of activities in the classroom. Constructivism is a theory of knowledge which addresses those beliefs. Consequently, a review of constructivist literature led me to a list of guidelines for an effective classroom which, I believe, meets both concerns brought up in studies of gender and concerns brought up in cognitive research on how students learn. Finally, using the constructivist guidelines I designed a unit to illustrate the use of constructivist teaching at the upper high school level. I believe that constructivist teaching with units like the one on limits will help address the issues of gender in mathematics education and at the same time improve mathematics understanding for all our students.

Annotated Bibliography

Anderson, Johnston. Sex-Related Differences on Objective Tests Among Undergraduates. *Educational Studies in Mathematics 20*: 165-177. 1989.

In a study of university students in England majoring in mathematics males performed significantly better than females on objective multiple choice, true-false, and relationship analysis questions even though there was no significant gender difference on more general examination papers.

Baroody, Arthur J. and Herbert P. Ginsburg. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (pp. 51-64). Reston, VA: National Council of Teachers of Mathematics. 1990.

Research by cognitive psychologists during the last 20 years is highlighted in this chapter. The chapter serves as background for the research which follows.

Bauerfeld, Heinrich. Classroom Cultures From a Social Constructivist's Perspective. *Educational Studies in Mathematics 23*: 467-481. 1992.

This article gives examples of a constructivist orientation in presenting various mathematical concepts.

Belenky, Mary Field, Blythe McVicker Clinchy, Nancy Rule Goldberger, and Jill Mattuck Tarule. *Women's Ways of Knowing: The Development of Self, Voice, and Mind.* United States: Basic Books. 1986.

The authors studied women's ways of learning. They classified the ways of knowing into five categories: silence, received knowledge, subjective knowledge, procedural knowledge, and constructed knowledge. They identify two types of procedural knowledge: separated and connected. Women, they claim, are more likely to reason from connections than men. They conclude that constructivist teaching will help address connected learning in the classroom.

Brooks, Jacqueline, and Martin G. Brooks. *In Search of Understanding: The Case for Constructivist Classrooms*. Alexandria, VA: Association for Supervision and Curriculum Development. 1993.

In Search of Understanding argues that constructivist teaching recognizes and encourages student thinking. The authors claim students are responsible for their own

learning and teachers should serve as facilitators. Constructivist teaching will encourage students to construct their own knowledge. Part II explains and illustrates five principles the authors feel are basic to constructivist teaching: posing problems of emerging relevance to students, structuring learning around primary concepts, seeking and valuing students' points of view, adapting curriculum to address students' suppositions, and assessing student learning. The third section of the book gives recommendations on how to be a constructivist teacher, and how constructivist teachers handle situations. This book is a practical guide for the teacher who is interested in establishing a constructivist classroom. The classroom examples from different grade levels and disciplines help one see how constructivism can and should work. The principles can serve as guidelines for the experimenting teacher.

Brown, John Seely, Allan Collins, and Raul Duguid. Situated Cognition and the Culture of Learning. *Educational Researcher* (pp. 32-42). January - February 1989.

The authors review recent research on situated versus non-situated cognition. They conclude that learning is situated in the context and culture in which it is learned. Therefore they argue for apprentice type learning in the classroom.

Burton, Leone, ed. *Gender and Mathematics: An International Perspective*. London: Cassell Educational Limited. 1990.

This book contains articles which reflect the international debate in mathematics education as it relates to gender issues. The articles fall into four major categories: Gender and Classroom Practices, Gender and the Curriculum, Gender and Achievement, and Women's Presence in mathematics education. The section on Gender and Classroom Practices looks at research which has been done on the climate in the math classroom and its effects on those who study mathematics. In the Gender and the Curriculum section the debate between the view of mathematics as abstract, objective and unbiased and the view of mathematics as creative, dynamic and relative is discussed in terms of curriculum and pedagogy. In the sections on Gender and Achievement and Women's Presence the researchers report on the achievement of women in terms of perception of mathematics and different mathematical skills, and the influences which have impacted on women who stay in mathematics. *Gender and Mathematics* is interesting because it adds an international flavor to the debate on gender issues in mathematics. The articles are written by researchers from countries all over the world.

Confrey, Jere. What Constructivism Implies for Teaching. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (pp. 107-122). Reston, VA: National Council of Teachers of Mathematics. 1990.

Jere Confrey explains constructivism and develops a model of constructivist instruction which promotes: student autonomy, reflective processes, use of case histories, negotiation of solution paths, group discussion of paths, and adherence to mathematics.

Davis, Robert B. and Carolyn A. Maher. What Do We Do When We "Do Mathematics"? In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (pp. 65-78). Reston, VA: National Council of Teachers of Mathematics. 1990.

This chapter investigates how students learn mathematics through the examination of classroom videotapes and interviews with students. The emphasis is on the reasoning underlying student work.

Davis, Robert B., Carolyn A. Maher, and Nel Noddings, eds. *Constructivist Views on the Teaching and Learning of Mathematics. Journal of Research in Mathematics Education Monograph, 4.* Reston, VA: National Council of Teachers of Mathematics. 1990.

This monograph contains a discussion of constructivism and how a constructivist approach can and should be used in the mathematics classroom. The first section of the book "Constructivism: Promise and Problems" starts with a philosophical discussion of the meaning of constructivism and its theoretical impact on teaching. The section entitled "The Nature of Mathematics and How It Is Learned" consists of several articles which suggest that mathematics is learned through individual constructions. Each student constructs her own understanding in the classroom. "Constructivism in the Classroom" includes discussions of what constructivism means in the classroom and how it works. The last section on "Children and the Education of Teachers" reviews teacher development in terms of constructivism. This monograph is excellent for mathematics teachers wanting to learn more about the implications of constructivism in mathematics education. By starting with a philosophical background the editors and authors provide justification for the detailed discussions of what is and is not constructivist teaching. The analysis of classroom interchanges allow the reader to understand how constructivism can be used effectively in the mathematics classroom.

Davis, Robert B. and Carolyn A. Maher, eds. *Schools, Mathematics, and the World of Reality*. Needham Heights, MA: Simon & Schuster, Inc. 1993.

This book continues the analysis of constructivist teaching in mathematics which was begun in *Constructivist Views on the Teaching and Learning of Mathematics*. Research studies begun in the monograph have become longitudinal. Noddings connects constructivism to the ethic of care. Chapters are included on constructivism

and its implications for technology use. Issues concerning appropriate assessment for the constructivist classroom are also addressed.

Ethington, Corinna. Gender Differences in Mathematics: An International Perspective. *Journal for Research in Mathematics Education 21, no. 1*: 74-80. 1990.

In a study of mathematics achievement across eight countries no gender differences were found across countries. Gender differences were a function of country and content within the country.

Ethington, Corinna. Gender Differences in a Psychological Model of Mathematics Achievement. *Journal for Research in Mathematics Education, 23*, no. 2: 166-181. 1992.

The relationship among psychological influences on mathematics achievement such as self-concept, stereotyping, value, prior success, difficulty, goals, and current achievement are investigated for both boys and girls. The analysis shows that girls have more indirect influences than boys. The diagrams clearly illustrate the results.

Fennema, Elizabeth. Teachers' Beliefs and Gender Differences in Mathematics. In E. Fennema & G. C. Leder (Eds.), *Mathematics and Gender* (pp. 169-187). New York: Teachers College Press. 1990a.

Fennema investigates the impact of teacher beliefs on students. She finds that teachers tend to stereotype males and females in terms of the students' abilities to do mathematics. Fennema argues that more research is needed on teacher beliefs and their impact on students.

Fennema, Elizabeth and Gilah C. Leder, eds. *Mathematics and Gender*. New York: Teachers College Press. 1990b.

This collection of articles on the most recent research in gender and mathematics discusses trends in research results as well as specific reports of research. The effects of spatial visualizations on math ability are investigated as they relate to verbal ability. Spatial ability appears to relate more closely with success in math for women; however, the difference in spatial skills between genders does not explain the discrepancy in math achievement between genders. Internal influences on males and females are investigated, and models are proposed to explain the differences. The Autonomous Learning Behaviors Model and Attribute Theory are especially helpful in interpreting the research results. Studies investigate the impact of teacher beliefs and student-teacher interactions on issues of gender. Such interactions are found to be extremely complex,

but teacher attitudes are important. This collection of articles on gender issues explains the complexity of the issues which impact the classroom and which need to be studied.

Fennema, Elizabeth, Penelope L. Peterson, Thomas P. Carpenter, and Cheryl A. Lubinski. Teacher's Attributions and Beliefs about Girls, Boys, and Mathematics. *Educational Studies in Mathematics 21*: 55-69. 1990c.

In this study of first grade teacher beliefs, the teachers were found to attribute boys' successes and failures to ability and girls' to effort. Teachers believed their best boys were more adventuresome, more logical, more competitive, and more independent than the best girls. The beliefs of teachers about male and female students could lead to different development in autonomous learning behaviors.

Fennema, Elizabeth, Thomas P. Carpenter, and Susan J. Lamon, eds. *Integrating Research on Teaching and Learning Mathematics*. Albany: State University of New York Press. 1991.

Several of the articles in this book are concerned with what teachers know and how it affects what students learn. Other articles deal with research on learning and how that can be used to improve teaching. Many of the research projects are looked at from the point of view of what has not been studied and what implications that has as well as what has been studied and what conclusions can be drawn. The research is almost all on teaching and learning at the elementary and middle levels.

Fennema, Elizabeth. Mathematics, Gender and Research. In *Woodrow Wilson Gender Equity in Mathematics and Science Congress (WW-GEMS)*. Princeton, NJ: The Woodrow Wilson National Fellowship Foundation. 1993.

This manuscript reviews the development of research on gender issues in mathematics. It provides an excellent synopsis of the trends in gender research since the 1970's with special emphasis on Elizabeth Fennema's conclusions.

Fennema, Elizabeth, Thomas P. Carpenter, Megan L. Franke, and Deborah A. Carey. Learning to Use Children's Mathematics Thinking: A Case Study. In R. B. Davis & C. A. Maher (Eds.), *Schools, Mathematics, and the World of Reality* (pp.93-118). Needham Heights, MA: Simon & Schuster, Inc. 1993.

The authors describe the changes in one teacher's attitudes and teaching as she used the Cognitively Guided Instruction program for a number of years. The structure for the CGI program is based on research about how children think.

Ferrini-Mundy, Joan. Spatial Training for Calculus Students: Sex Differences in Achievement and in Visualization Ability. *Journal for Research in Mathematics Education, 18*, No. 2: 126-140. 1987.

The spatial ability of college calculus students and the possibility of teaching spatial visualization were investigated. Sex differences were found favoring women in calculus achievement and men in spatial ability. There were indications that spatial ability can be enhanced in women with practice, especially while studying solids of revolution.

Fey, James T. and Christian R. Hirsch, eds. *Calculators in Mathematics Education, 1992 Yearbook*. Reston, VA: National Council of Teachers of Mathematics. 1992.

Although this yearbook contains no research on the issue of gender in mathematics, it does make a strong argument for the use of calculators in mathematics classes. Calculators can be used to involve students in their own learning. Because of the rapid improvement in the capabilities of graphing calculators the specific recommendations for class work are out of date. The authors clearly point out the possible pitfalls of calculator use by students, however, so many of the articles are still timely.

Gilligan, Carol. *In a Different Voice: Psychological Theory and Women's Development*. Cambridge, MA: Harvard University Press. 1993.

This book, which first appeared in 1982, described the differences in the way men and women deal with moral issues. According to Gilligan, men attack moral issues from a rational framework while women see moral issues in a relational framework. This book started a whole branch of feminist studies about how women reason.

Grossman, Herbert, and Suzanne H. Grossman. *Gender Issues in Education*. Boston: Allyn and Bacon. 1994.

This is a comprehensive review of research on gender issues. Both gender and ethnic issues as they relate to education are addressed. Each statement based on research is carefully acknowledged. The bibliography at the end of each chapter is arranged under the same topic names as the sections of the chapter so that the interested reader can easily determine where to find more or corroborating information. The authors are writing to the classroom teacher. They include self quizzes throughout each chapter so that the reader can assess his or her attitudes on the issues under discussion. In addition, the authors have recommendations at the end of each chapter for ways to teach which respond to the issues identified by the research they cite. The

Grossmans have written a helpful and general book on gender and ethnic issues for teachers.

Hackett, Gail, and Nancy E. Betz. An Exploration of the Mathematics Self-Efficacy/Mathematics Performance Correspondence. *Journal for Research in Mathematics Education, 20*, No. 3: 261-273. 1989.

This study showed that self-efficacy rates of males and females were similar. Self-efficacy was related to choice of mathematics-related majors and there was some correlation with performance so the authors argue teachers should work to improve students self-efficacy.

Hart, Laurie, E. Classroom Processes, Sex of Student, and Confidence in Learning Mathematics. *Journal for Research in Mathematics Education, 20*, No. 3: 242-260. 1989.

Interactions between seventh grade high achieving, both high and low confidence students and their teachers were observed. High and low confidence students had similar interactions with teachers; however, boys and girls were found to interact differently in the classroom. This article contains a nice breakdown of teacher types and interactions.

Isaacson, Zelda. They Look at You in Absolute Horror: Women Writing and Talking about Mathematics. In L. Burton (Ed.), *Gender and Mathematics: An International Perspective* (pp. 20-28). London: Cassell Educational Limited. 1990.

Women with bad past experiences in mathematics return to a cooperative, relaxed, supportive environment and experience success.

Jungwirth, Helga. Interaction and Gender - Findings of a Microethnographical Approach to Classroom Discourse. *Educational Studies in Mathematics 22*: 263-284. 1991.

This article discusses how students interact with teachers. Gender specific behaviors change student teacher interactions. Boys give the impression that they know the material while girls leave the impression that they do not whether either boys or girls know the material.

Kaiser-Messmer, Gabriele. Results of an Empirical Study into Gender Differences in Attitudes Towards Mathematics. *Educational Studies in Mathematics 25*: 209-233. 1993.

A West German study of boys and girls 14-19 shows that attitudes about gender have changed but girls, even though they express open-mindedness about mathematical opportunities, still lack strong self-images in math.

Kamii, Constance with Georgia DeClark. *Young Children Reinvent Arithmetic*. New York: Teachers College Press. 1985.

This book describes Georgia DeClark's experiences teaching first grade math using only games. Constance Kamii believes that students create their own understanding through their experiences, and DeClark puts Kamii's theories into action.

Koehler, Mary Schatz. Classrooms, Teachers, and Gender Differences in Mathematics. In E. Fennema & G. C. Leder (Eds.), *Mathematics and Gender* (pp. 128-148). New York: Teachers College Press. 1990.

Koehler reviews studies about how teachers interact with boys and girls in the classroom. Her own study investigates the effects of teacher assistance especially on girls.

Leder, Gilah C. Gender and Classroom Practice. In L. Burton (Ed.), *Gender and Mathematics: An International Perspective* (pp. 9-19). London: Cassell Educational Limited. 1990a.

In a study across grades 3, 6, 7, and 10 Leder finds that teachers interact differently with males and females. Boys are called on more, seek attention more, and receive more wait time. Differences in behavior between the sexes precede differences in performance.

Leder, Gilah C. Teacher/Student Interactions in the Mathematics Classroom: A Different Perspective. In E. Fennema & G. C. Leder (Eds.), *Mathematics and Gender* (pp. 149-168). New York: Teachers College Press. 1990b.

Leder looks at interactions by gender in Australia and compares her results to others results in the United States. She finds that the same type of unequal interactions are found in both school communities.

Lerman, Stephen. Constructivism, Mathematics and Mathematics Education. *Educational Studies in Mathematics 20*: 211-233. 1989.

Radical constructivism is supported as a new paradigm to describe the development of mathematical understanding. Two basic hypotheses are included: knowledge is constructed, and understanding is a process of organizing ones's experiential world, not a preexisting world.

Maher, Carolyn A. and Robert B. Davis. Building Representations of Children's Meanings. In Davis, R. B., C. A. Maher, & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (pp. 79-90). Reston, VA: National Council of Teachers of Mathematics. 1990.

This chapter in a monograph on constructivist teaching looks at the difficulties facing teachers as they attempt to understand students' reasoning.

Marsh, Herbert W. Sex Differences in the Development of Verbal and Mathematics Constructs: The High School and Beyond Study. *American Educational Research Journal, 26*, No. 2: 191-225. 1989.

Results from the HS&B study indicate that stereotypical sex differences are declining; however, better math skills were associated with poorer verbal attitudes and better verbal skills were associated with poorer math attitudes.

Meyer, Margaret R. and Mary Schatz Koehler. Internal Influences on Gender Differences in Mathematics. In E. Fennema and G. C. Leder (Eds.), *Mathematics and Gender* (pp. 60-95). New York: Teachers College Press. 1990.

Meyer and Koehler review research on attribution as it relates to gender issues. They discuss both the Autonomous Learning Behavior model and the Model of Academic Choice. Results from longitudinal studies and the Sherman-Fennema scale studies are considered.

Miller, L. Diane, Charles E. Mitchell, and Marilyn Van Ausdall. Evaluating Achievement in Mathematics: Exploring the Gender Biases of Timed Testing. *Education, 114*, No. 3: 436-438. 1994.

SAT type tests were given both timed and untimed. Females although still scoring worse than males closed the gap when the tests were untimed.

Moreno-Armella, L., and G. Waldegg. Constructivism and mathematical education. *International Journal of Mathematics Education, 24*, No. 5: 653-661. 1993.

Two philosophical theories of mathematical knowledge are investigated. One leads to the imparting of mathematical knowledge and the others leads to the construction of knowledge. The authors argue that constructing knowledge is more successful.

Noddings, Nel. Constructivism in Mathematics Education. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (pp. 7-18). Reston, VA: National Council of Teachers of Mathematics. 1990.

Noddings provides background on constructivism and argues that despite some epistemological weaknesses constructivism offers mathematicians a strong pedagogical basis.

Noddings, Nel. Constructivism and Caring. In R. B. Davis & C. A. Maher (Eds.), *Schools, Mathematics, and the World of Reality* (pp. 35-50). Needham Heights, MA: Simon & Schuster, Inc. 1993.

Noddings argues that constructivism's pedagogical demand for tailoring inquiry to the student's need suggests an ethic of care as part of constructivist teaching. Care involves commitment by both the carer and the caree; therefore, caring connects the constructivist teacher and student to real life.

Pirie, Susan, and Thomas Kieren. Creating Constructivist Environments and Constructing Creative Mathematics. *Educational Studies in Mathematics 23*: 505-528. 1992.

The authors define teacher beliefs which they claim are essential for creating constructivist mathematics classrooms. The beliefs are illustrated in several classroom activities on the study of fractions.

Pirie, Susan, and Thomas Kieren. Growth in Mathematical Understanding: How Can We Characterize It and How Can We Represent It? *Educational Studies in Mathematics 26*: 165-190. 1994.

This article outlines a theory of the development of mathematical understanding which includes stages of development and folding back between stages. The theory is illustrated with classroom examples.

Rodgers, Marie. Mathematics: Pleasure or Pain? In L. Burton (Ed.), *Gender and Mathematics: An International Perspective* (pp. 29-37). London: Cassell Educational Limited. 1990.

Rodgers studied females working on the A level and O level exams and found that fun, laughter, and a good experience associated with a teacher were important to the females' success. Present teaching encourages girls' tendency to serialization in the study of maths.

Rogers, Pat. Thoughts on Power and Pedagogy. In L. Burton (Ed.), *Gender and Mathematics: An International Perspective* (pp. 38-46). London: Cassell Educational Limited. 1990.

Rogers looks at the highly successful mathematics program at SUNY Potsdam where a male department enrolls a high percentage of female and male majors.

Sadker, Myra, and David Sadker. *Failing at Fairness: How America's Schools Cheat Girls*. New York: Charles Scribner's Sons. 1994.

The Sadkers indict American schools for the sexism which they claim pervades our schools. This is an easy to read popular book which is based on research. The citations are found in the notes. There is a wonderful bibliography of recommended reading at the end which includes books about girls and women.

Schoenfeld, Alan H. Problem Solving in Context(s). In R. I. Charles & E. A. Silver (Eds.), *The Teaching and Assessing of Mathematical Problem Solving* (pp. 82-92). Reston, VA: National Council of Teachers of Mathematics. 1988.

Schoenfeld argues that learning is both cultural and cognitive. Mathematics is the act of sense making so mathematics classrooms need to teach both mathematical tools and a predilection to analyze.

Simon, Martin A. Reconstructing Mathematics Pedagogy From a Constructivist Perspective. *Journal for Research in Mathematics Education 26*, No. 2: 114-145. 1995.

Simon looks at three constructivist teaching situations. From an analysis of them he develops a schematic model, the Mathematics Teaching Cycle, of teachers' decision making.

Steffe, Leslie P., and Beatriz S. D'Ambrosio. Toward a Working Model of Constructivist Teaching: A Reaction to Simon. *Journal for Research in Mathematics Education 26*, No. 2: 146-159. 1995.

The authors argue that there is a type of teaching which is constructivist. Their model of constructivist teaching involves reflection, autonomy, and situations that encourage assimilating generalization.

Steffe, Leslie P., and Thomas Kieren. Radical Constructivism and Mathematics Education. *Journal for Research in Mathematics Education 25*, No. 6: 711-733. 1994.

In this article the authors describe the background and development of radical constructivist theory based on a reinterpretation of Piaget's genetic structures. Constructivism developed over a thirty year period of theorizing and research.

Strauss, Sheila. Girls in the Mathematics Classroom: What's Happening with Our Best and Brightest? *Mathematics Teacher*, pp. 533-537. Oct. 1988.

A mathematics teacher at Hunter College High School, NY, evaluates the effect on female success in calculus when the school goes coed.

Tartre, Lindsay A. Spatial Skills, Gender, and Mathematics. In E. Fennema & G. C. Leder (Eds.), *Mathematics and Gender* (pp. 27-59). New York: Teachers College Press. 1990.

This is a comprehensive investigation into the relationships among high and low spatial ability and gender. Spatial ability is separated into its different components in the analysis of student spatial ability and math achievement. Low spatial ability girls have the lowest achievement in related mathematics.

Webb, Norman L. and Arthur F. Coxford. *Assessment in the Mathematics Classroom, 1993 Yearbook*. Reston, VA: National council of Teachers of Mathematics. 1993.

This NCTM Yearbook discusses the reasons to change assessment in the mathematics classroom. Although the Yearbook does not discuss gender issues directly, changing assessment techniques is shown as a way to include more creative discovery in the mathematics classroom. Including writing, open-ended problems, and self-assessment as part of assessment in mathematics is recommended. Examples of

different types of assessment help the reader understand the recommendations of the authors of the various articles.

Wheatley, Grayson. The Role of Reflection in Mathematics Learning. *Educational Studies in Mathematics 23*: 529-541. 1992.

Reflective abstraction is important in constructivism. Wheatley argues problem-centered learning promotes autonomy and facilitates reflection.

SE059726

U.S. Department of Education
Office of Educational Research and Improvement (OERI)
Educational Resources Information Center (ERIC)

REPRODUCTION RELEASE

(Blanket Document)

I. DOCUMENT IDENTIFICATION (Class of Documents):

All Publications:
submitted by the Klingenstein Center

Series (Identify Series): Klingenstein Fellows Projects

Division/Department Publications (Specify):
Private & Independent School Education

II. REPRODUCTION RELEASE:

In order to disseminate as widely as possible timely and significant materials of interest to the educational community, documents announced in the monthly abstract journal of the ERIC system, *Resources in Education* (RIE), are usually made available to users in microfiche, reproduced paper copy, and electronic/optical media, and sold through the ERIC Document Reproduction Service (EDRS) or other ERIC vendors. Credit is given to the source of each document, and, if reproduction release is granted, one of the following notices is affixed to the document.

If permission is granted to reproduce and disseminate the identified document, please CHECK ONE of the following two options and sign at the bottom of the page.

☑ Check here
For Level 1 Release:
Permitting reproduction in microfiche (4" x 6" film) or other ERIC archival media (e.g., electronic or optical) *and* paper copy.

The sample sticker shown below will be affixed to all Level 1 documents

PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL HAS BEEN GRANTED BY
Sample
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

Level 1

The sample sticker shown below will be affixed to all Level 2 documents

PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL IN OTHER THAN PAPER COPY HAS BEEN GRANTED BY
Sample
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

Level 2

☐ Check here
For Level 2 Release:
Permitting reproduction in microfiche (4" x 6" film) or other ERIC archival media (e.g., electronic or optical), but *not* in paper copy.

Documents will be processed as indicated provided reproduction quality permits. If permission to reproduce is granted, but neither box is checked, documents will be processed at Level 1.

"I hereby grant to the Educational Resources Information Center (ERIC) nonexclusive permission to reproduce and disseminate this document as indicated above. Reproduction from the ERIC microfiche or electronic/optical media by persons other than ERIC employees and its system contractors requires permission from the copyright holder. Exception is made for non-profit reproduction by libraries and other service agencies to satisfy information needs of educators in response to discrete inquiries."

Sign here→ please

Signature: Pearl Rock Kane	Printed Name/Position/Title: Pearl Rock Kane - Director
Organization/Address: The Esther A. & Joseph Klingenstein Center for Independent School Education, Box 125 Teachers College, Columbia University 525 West 120th Street NY NY 10027	Telephone: 212-678-3156 / FAX: 212-678-3254
	E-Mail Address: prk4@columbia.edu / Date: 12/10/96