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ABSTRACT

This paper begins with a description of a set of difficulties in classes where students were given problems which asked them to explore and "make conjectures." After presenting background on debates about discovery learning, changing philosophical conceptualizations of the role of the empirical in mathematics, and innovations involving the use of geometry construction programs, the paper then focuses on two philosophical views on induction and the origin of conjectures. These philosophical views are used to distinguish two approaches to and rationales for exploratory laboratory problems in geometry classes. Concerns are then raised about one of these approaches (discovery problems) which may be carried out under the banner of current reforms but which may lead students to conceptualize classroom exploration as a search for the single idea on a teacher's mind. (Contains 45 references.) (Author)

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Abstract

This paper begins with a description of a set of difficulties in classes where students were given problems which asked them to explore and "make conjectures." After presenting background on debates about discovery learning, changing philosophical conceptualizations of the role of the empirical in mathematics, and innovations involving the use of geometry construction programs, the paper focuses on two philosophical views on induction and the origin of conjectures. These philosophical views are used to distinguish two approaches to and rationales for exploratory laboratory problems in geometry classes. Concerns are then raised about one of these approaches which may be carried out under the banner of current reforms but which may lead students to conceptualize classroom exploration as a search for the single idea on a teacher's mind.

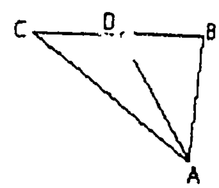
WHERE DO STUDENT CONJECTURES COME FROM? EMPIRICAL EXPLORATION IN MATHEMATICS CLASSES

Daniel Chazan

As a researcher on a project helping teachers integrate empirical exploration into their high school geometry classes, I had occasion to visit classrooms where new sorts of tasks were introduced, where students were asked to "make conjectures." Here is some of what I saw.

I saw teachers give students problems which they chose as well-structured, "simple" problems appropriate for introducing this new activity into their classes. An example of such a "simple" problem might be:

> Use the software to draw a median in a triangle. Measure the lengths of the new segments created along the side of the triangle (CD and DB). Repeat these steps on a series of triangles and make conjectures.[2]



The problems told students how to generate data and which data to collect. Then students were asked to examine their data and made generalizations, conjectures. I saw students dutifully collect the data but, then, not know what to do. They did not know what it meant to "make conjectures." They seemed unaware of what, to the teachers, were self-evident patterns in the data.

Teachers gave separate grades for data collection and conjectures. The data collection grades were high, but the conjecture grades were very low. An uncomfortable dynamic was created; after an exploration, the teacher would lead the students to see a pattern in their data or would say, "What you should have found was . . ." Sometimes, when students heard what their teacher was looking for, they would say that they had

[2]This problem has an added twist because students are being asked to "discover" a property that is the definition of a median.

noticed the pattern but didn't realize it was what the teacher wanted. One teacher began to elicit these patterns by asking, "What is the duh?" implying that some patterns were so self-evident that they might not seem worth mentioning.

> In one class, the teacher thought the students needed more direction and . . . was convinced that this difficulty [making conjectures] stemmed from an inability to see patterns in the numerical data. She therefore carefully structured their [students'] inquiry, rewriting problems, providing explicit directions and charts for recording data. She hoped that this structured approach would make patterns in the data apparent and lead students to conjectures. (Yerushalmy, Chazan, and Gordon 1987, p. 10)

This teacher had well-thought-out reasons for using charts to structure the activity for students. The charts reduced some of the students' anxiety with a new type of activity. They specified ". . . what needs to be done [in the sense of data to be collected] to create a successful solution" (Yerushalmy, Chazan, and Gordon 1990, p. 239). Yet, the charts didn't solve the problem. Students' difficulties making conjectures were greatest with problems that provided a chart indicating the measurements to be made: While the charts suggested which data to collect, there wasn't similar guidance for how to create generalizations. As a result, "in a class where tables and charts were used frequently, students ignored written instructions and turned directly to them [the charts], limiting their inquiry to the headings specified in the charts" (Yerushalmy, Chazan, and Gordon 1990, p. 239).

These initial observations intrigued me and led me to think about how to help students "make conjectures." My explorations led to readings in the philosophy of science about theories of induction. These readings made it clear that my question was not a technical one but one that must be addressed with respect to one's goals and purposes in having students "make conjectures." This deepening of the question sensitized me to different goals and purposes that teachers had for two different sorts of exploratory activities they used in their classes.

Empirical Exploration in Current Mathematics Education Reform

In current mathematics education reform efforts, there is little discussion of the goals of empirical inquiry or the relationship between inquiry and mathematical ideas which students are to develop. Too often, such inquiry, in and of itself, seems to be one of the goals of reform efforts.

One goal of involving students in empirical inquiry seems based both on an image of mathematical practice and on a desire to have students do mathematics. In presenting its view of mathematical practice, the National Council of Teachers of Mathematics (NCTM) suggests in its publication *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989, p. 7) that "making conjectures, gathering evidence, and building an argument to support such notions are fundamental to doing mathematics." In many cases, the *Standards*[3] illustrate the process of having students do mathematics with activities involving manipulatives—objects like base 10 blocks which are used as teaching tools (known in the '60s as concrete embodiments) or microworlds (like the geometry software described below), a popular type of design in the mathematics education software design community. Microworlds in mathematics are intended to allow students to explore models of mathematical concepts "empirically" by manipulating objects which appear on a computer screen.

Support for having students make conjectures based on empirical exploration is also marshaled from constructivist views of learning popular in the cognitive sciences (NCTM 1989, p. 10). Yet neither the 1989 *Standards* nor other reform documents (for example, NCTM 1991; National Research Council and Mathematical Sciences Education Board 1989) explicitly outline views of the relationship between students' empirical experimentation and the mathematical ideas that they develop (Schmittau 1991).

As Ball (1992) points out with reference to manipulatives, this omission may result in problematic pedagogy in the name of student-centered learning. Specifically, she argues that some uses of manipulatives assume "that mathematical truths can be directly 'seen' through the use of concrete objects" (Ball 1992, p. 17), while her classroom

[3]*Standards* refers to *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) and *Professional Standards for Teaching Mathematics* (NCTM 1991).

experience suggests that students may interpret these objects differently than is intended (Ball 1992, p. 18).[4] Similarly, experience with microworlds leads me to believe that this omission may result in the use of exploratory microworlds advocated by reform documents in ways which cause students to conceptualize exploration as figuring out what the teacher would like them to "find."

This paper concentrates on the relationship between empirical exploration by students and the conjectures they create.[5] I believe that the arguments made in the paper are broadly useful in conceptualizing empirical exploration in the mathematics class, including work with graphing calculators, manipulatives (or concrete embodiments), and microworlds. However, in order to ground the philosophical discussion in teaching practice, I will illustrate these issues in the specific context of instructional uses of geometry construction programs (like Schwartz et al. 1985; Laboratoire Structures Discretes et Didactique 1988; Jackiw 1991), microworlds for use in high school geometry classes which have received acclaim from the mathematics education reform community. The paper will not address exploration in other disciplines, like science, in which the role of empirical evidence is arguably different.

The first section of the paper situates the current discussion in a historical context by rehearsing formulations of the central questions in the 1960s' debates about discovery learning. The second section outlines philosophical issues surrounding the "empirical" in mathematics, while the third uses a description of The Geometric Supposers[6] to illustrate the capabilities of geometry construction programs. The fourth section contrasts two views of induction—the creation of general notions from specific examples—by illustrating them with two types of laboratory exploration problems posed by teachers

[4]For an example of a philosophically rigorous approach to this issue, see Nesher (1989).

[5]This paper focuses on the empirical and the particular in the context of students' creation or discovery of mathematical ideas. For an article which explores the teaching of deductive proof in light of quasi-empirical developments in the philosophy of mathematics, see Chazan (1990).

[6]This is the software that was used in the project in which I participated. I use it because it is the program with which I am most familiar. My remarks would apply equally well to other geometry construction programs.

using geometry construction programs. The fifth and final section presents a principled critique of one type of exploratory activity.

Changing Conceptions of Student Exploration

As constructivist views of learning began to gain ascendancy and while curriculum projects like the New Math were being developed, there was a debate about discovery learning. The U.S. Department of Health, Education, and Welfare gave Stanford University and the Social Science Research Council a contract to convene a conference. The central question of the debate was formulated differently by the various participants in the debate. Shulman and Keislar (1966) include various formulations in their volume of proceedings, *Learning by Discovery: A Critical Appraisal.* An unidentified participant offered:

> When people advocate discovery, they are advocating the withholding of answers from pupils. The teacher knows how an answer is obtained, but the students do not. Hence, the major question in the issue of learning by discovery is the extent to which you get better pedagogy by not telling the student what the teacher already knows. (P. 27)

The editors suggest that this central question is enacted on instructional, curricular, psychological, and research levels. For example, at the psychological level:

> "[The] question becomes, What is the transfer value of statements of principles given to a subject, as contrasted with individually derived principles?" (P. 181)

Other researchers lamented that the word "discovery" doesn't capture a single phenomenon. During the conference, Robert Davis presented a film in which he taught a junior high school mathematics class a unit on the multiplication of matrices. As a practitioner, he said:

> It is my present notion that there are many different kinds of discovery experience, and we confuse the issue badly when we treat discovery as a single well-defined kind of experience. (P. 114)

The editors summarize the discussion which followed the examination of this film by indicating ways in which discovery experiences might differ:

> There appeared to be two independent axes operating in this definition of teaching by discovery. One was the extent to which "messing around" was characteristic of the behavior of the student. The second was the extent to which students were called upon to invent or discover facts or generalizations in the subject matter, in contrast to being told the given statements directly. In messing around, the students may generate a wide variety of possible gap-fillers, all of which may be equally acceptable to the teacher . . . In contrast to messing around, the object of discovery was to come up with one right answer. (P. 129-30)

At the same time, in their conclusion, Shulman and Keislar use a quote from Dewey to subvert the notion of "defining" what is meant by discovery learning:

> "Intellectual progress usually occurs through sheer abandonment of questions together with both of the alternatives they assume . . ." We have seen in this volume a recurrent call for this kind of reformulation of the present issue . . . It is the hope of the authors of this chapter that a highlighting of the terms of the controversy will hasten the process of reformulations rather than impede it. (P. 198)

I do not know whether the hope of the authors was borne out; however, since the publication of their volume, there has been some reformulation of the question, or at least a change in the vocabulary. When describing pedagogy in which students "mess around," the terms "inquiry" and "exploration" are now used more often than "discovery."

In judging whether this "messing around" leads to "desirable pedagogy," researchers now consider a wider range of pedagogical goals; the desirability of pedagogy is not gauged solely by student achievement on timed, paper-and-pencil examinations. Alternative assessments are advocated to assess student learning (e.g., Webb 1992). Pedagogy is also scrutinized for the messages it presents about the origins of mathematical knowledge (e.g., Nickson 1992).

Little has been done, however, to flesh out Robert Davis's notion of different kinds of discovery learning. As a first step, this paper distinguishes between two pedagogical strategies for incorporating student empirical exploration into mathematics classes. Later, I critique one such strategy which can deteriorate into students trying to guess what is on the teacher's mind. I hope that this distinction will encourage others to pursue more careful characterizations of different strategies for incorporating empirical exploration into mathematics classes.

I now turn more specifically to an examination of the use of empirical exploration in geometry classes. I begin with philosophical views about the empirical in mathematics before introducing a description of geometry construction programs.

On the Empirical in the Philosophy of Mathematics

Traditional rationalists' views of mathematics, like the views presented by Nagel (1956), suggest that mathematics, unlike science, does not describe natural objects and is not dependent on nature. In such views, mathematics is not subject to falsification through experience; the "facts" of nature cannot call mathematical truths into question. For example, he argues that if experimentation revealed that, when adding gallons of alcohol, seven plus five is not twelve, one's confidence in arithmetic would not *and indeed should not* be shaken or influenced. Instead, one should question whether addition of whole numbers is the correct mathematical tool to model the combinations of two volumes of liquid alcohol. Scheffler (1965, p. 3) illustrates such a rationalist position by arguing that:

> A diagram may well be used to *illustrate*[7] a geometrical theorem, but it cannot be construed as *evidence for* the theorem. Should precise measurement of the diagram show that it failed to embody the relations asserted by the theorem, the latter would not be falsified. We should rather say that the physical diagram was only an approximation or suggestion of the truth embodied in the theorem.[8]

While not suggesting that mathematical objects are empirical in the same way as the objects of scientific study, Imre Lakatos disagrees and points to "A Renaissance of

[7]Author Scheffler's italics.

[8]Indeed, it was a combination of thinking of this sort and examples of misleading diagrams which led many textbooks in the late 1800s to avoid including diagrams.

Empiricism in the Recent Philosophy of Mathematics" (Lakatos 1986). Along with other philosophers like Putnam (1986), Kitcher (1986), and Tymoczko (1986); historians of mathematics like Grabiner (1986); and mathematicians like P. Davis (1986), Polya (1954), and Hersh (1986), he makes cogent arguments for the importance of the "empirical" and the particular in the practice of mathematics. Such a view calls for concomitant revisions in philosophical views of mathematical practice.[9]

To argue his point, Lakatos (1976) presents an imagined classroom in which the discussion mirrors the historical development of the theory of polyhedra. The text begins from a relationship between the vertices, edges, and faces of polyhedra noted by the mathematician Euler, $V - E + F = 2$. In Lakatos's "historical" classroom, this result is corroborated, proven (with a proof historically attributed to Cauchy), and then "empirical" counterexamples arise. These counterexamples force a reevaluation of the accepted theory. In the course of the discussion, the class takes up the following examples (see Figure 1) which suggest refinements both in their definitions of polyhedra and their proofs. In each of these cases, $V - E + F \neq 2$. Interestingly, Lakatos notes that the nested cubes counterexample suggested itself to one mathematician after examining a mineralogical collection in which there was a transparent crystal encasing a violet one (Lakatos 1986, p. 13, footnote 1).

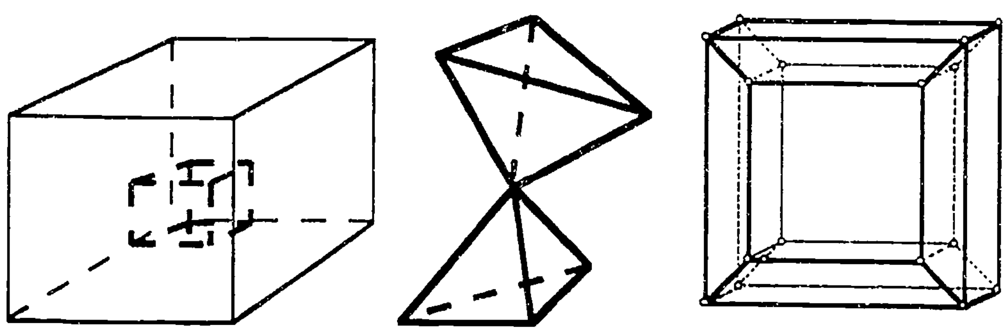


Figure 1. Nested cubes, the double prism, and the (thick) picture frame

[9]The introduction of computers into mathematical practice has helped fuel this reevaluation of the role of the empirical both in the discovery/creation and in the justification of mathematical ideas. See Tymoczko (1986) for an analysis of the ramifications of the proof of the Four Color Theorem for understanding justification in mathematics and Gleick (1987) for descriptions of the explorations behind recent developments in Chaos Theory.

While Lakatos does not describe these mathematical objects (the counterexamples) as empirical, sensory objects, he argues that proven mathematical results are elaborated and refined by a dynamic of proofs and refutations. Thus, in contrast to Nagel, Lakatos suggests that mathematical results are indeed open to falsification through experience with mathematical objects. He later names his view "quasi-empiricism."

If we find Lakatos's description of the role of experience in mathematics plausible, it is reasonable to investigate the role of these objects in the development of mathematical ideas. Within Lakatos's imagined classroom, there are different views about the relationship between empirical data (examples) and the development of conjectures. Beta, a student in the class, suggests that data precedes conjectures. He argues that his original conjecture ($V - E + F = 2$) was suggested to him by the facts appearing in a table of data about the vertices, edges, and faces of polyhedra. The teacher disagrees strongly and labels this view as "the myth of induction."

> Beta: Then what suggested $V - E + F = 2$ to *me*,[10] if not the facts, listed in my table?
>
> Teacher: I shall tell you. You yourself said you failed many times to fit them into a formula. Now what happened was this: you had three or four conjectures which were quickly refuted. Your table was built up in the process of testing and refuting these conjectures. *Naive conjectures are not inductive conjectures: we arrive at them by trial and error, through conjectures and refutations*. But if you—wrongly—believe that you arrived at them inductively, from your tables, if you believe that the longer the table, the more conjectures it will suggest, and later support, you may waste your time compiling unnecessary data. Also, being indoctrinated that the path of discovery is from facts to conjecture, and from conjectures to proof (the myth of induction), you may completely forget about the heuristic alternative: deductive guessing. (Lakatos 1976, pp. 73-74)

The teacher suggests an alternative view of the relationship between examples and conjectures. Rather than the examination of a collection of examples leading to a

[10]Author Lakatos's italics.

conjecture, conjectures lead to the examination of examples. Stronger conjectures are built out of the corroboration and refutation of prior conjectures.

This debate over the relationship between examples and conjectures presents an exciting challenge to philosophers of education who suggest that students' learning experiences should help them appreciate the processes used in the disciplines (for example, Scheffler 1965), to those advocating that teachers acquire pedagogical content knowledge (for example, Shulman 1987), and to mathematics educators who believe that "mathematical theory and practice should be reflected in the secondary school curriculum" (Hanna 1983, p. 3).[11] In particular, this debate suggests that mathematics education needs to develop its views of the "empirical" in mathematics teaching,[12] yet, for most practicing and prospective high school teachers taught a rationalist view of mathematics, the empirical and the particular is almost by definition nonmathematical.

This situation raises questions: As philosophers' and practitioners' views of the processes used in a discipline like mathematics change, how should these changes be examined critically with teachers interested in changing their teaching practice? More pointedly, are all approaches to teaching mathematics which involve students in empirical exploration created equal? Are there grounds for preferring some approaches over others?

Teaching With Geometry Construction Programs

Traditional histories of mathematics celebrate the Greek creation of systematized, nonempirical, Euclidean geometry (e.g., Boyer [1968] 1985). In line with the celebration of this progress away from the empirical, traditional geometry courses reify deductive reasoning as the only way to decide the truth of statements and dismiss empirical verification as a nonmathematical tool (e.g., Jurgensen, Brown, and King 1980, p. 27). This trend became so pronounced at the time of the development of non-Euclidean

[11]Edwin Moise (1975) argues that geometry is an especially important course in the secondary math curriculum because it plays a special role in helping students see how mathematics work. It is the only place in the high school curriculum where proof is emphasized.

[12]This seems especially important in light of current reforms. For example, in college-level courses, the Mathematics Association of America is now actively promoting "laboratory" calculus courses.

geometries that it was suggested that all diagrams should be omitted from geometry texts lest students be led erroneously to rely on diagrams (Greenberg 1980, p. 48).

The introduction of microcomputers into classrooms has led to a daring challenge to this view of the pedagogy of geometry. In the early 1980s, building on the ease and speed with which computers can draw diagrams, Judah Schwartz and Michal Yerushalmy created computer software called The Geometric Supposers to reintroduce the empirical into geometry classes. Their program is composed of three sets of electronic tools: a compass and straightedge, a ruler and protractor, and a repeat function.[13] The user chooses an initial shape (e.g., a right scalene triangle), creates a diagram with the compass and straightedge tools (e.g., draws a median in the original triangle and two medians from the foot of this median), makes measurements of the parts of the diagram (e.g., the areas of the four subtriangles), and can then repeat this construction and these measurements on a new initial shape (e.g., any other type of triangle).

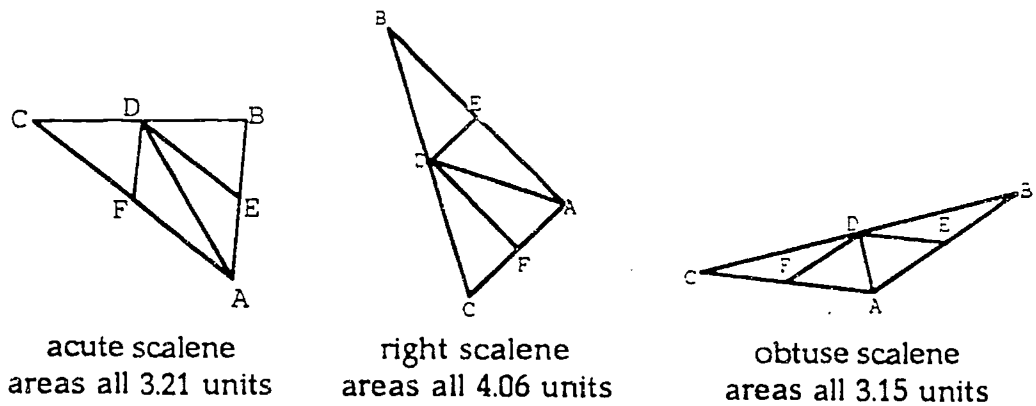


Figure 2. Drawings and measurements created by repeating a single procedure

With the computer, one can quickly produce a series of diagrams for any construction and a set of data about those diagrams. The availability of this empirical information makes possible a new pedagogy in geometry (Schwartz 1989).

Because geometry is usually taught to college-bound high school students in the United States, a large portion of the course is devoted to production by the students of short deductive proofs for statements provided in the textbook. Students know that these

[13]Other construction programs have similar capabilities but slightly different architectures. Recent versions of geometry construction programs make use of the "mouse" which was not available when the first geometry construction programs appeared.

statements have been "proven" year after year by others sweating at their desks. Teachers, texts, and students assume that the truth value of these statements has been established.

In traditional texts (like Jurgensen et al. 1980), these assignments are typically given with a diagram and a description of the "given," that which the student is to assume to be true from the start, and the "to prove," that which the student must show to be true. In textbooks, the ratio between diagrams and problems is usually one to two; for every diagram, there are two problems, an odd-numbered and an even-numbered problem.

With geometry construction software, the ratio of diagrams to problems is inverted and multiplied. Diagrams proliferate. Rather than have students write deductive proofs for statements from the text, constructions ("the givens") are described for students, usually in a worksheet format. Using the measurement capabilities of the software, students then create diagrams that fit these givens and explore these diagrams to develop conjectures about all the diagrams which share these givens. It is these conjectures, whose truth value is always in some doubt, which students then attempt to prove deductively (NCTM 1989, p. 158).

Two Pedagogical Approaches and Their Philosophical Bases

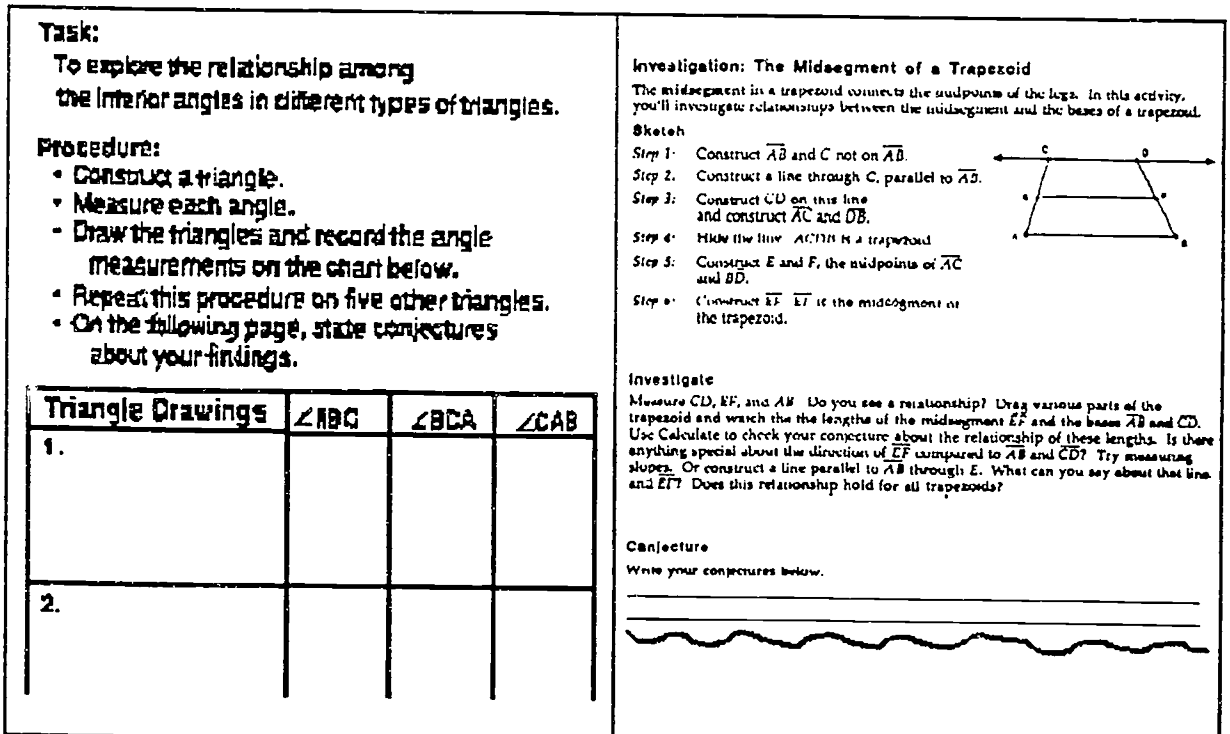
Are all pedagogical approaches which involve students in empirical exploration created equal? I will argue that they are not, but, in order to do so, I must distinguish between different approaches and their underlying theories about the origins of what Lakatos calls "naive conjectures." In *Philosophy of Science: The Link Between Science and Philosophy*, Philip Frank (1957) makes a distinction between two types of induction—induction by intuition and induction by enumeration—and reports about conflict between Mill and Whewell over their roles in the origins of scientific theories (Frank 1957, pp. 316-22). While this distinction may not be as sharp as Frank suggests, I will use it to characterize different ways teachers organize student empirical exploration in geometry.

Induction by Enumeration and Discovery Problems

According to Frank (1957, p. 316):

> [Induction by enumeration describes the collection of] a series of observed events in which we recognize that some sequences of events repeat themselves again and again . . . In this context, we mean by "law of induction" the assertion that after such uniformities of sequences have been observed through many repetitions without exception, or with few exceptions, this uniformity will go on forever, provided that the conditions in the surroundings are not changed.

This is Beta's view, that his table "suggested" the conjecture, yet it does not explain how one knows which aspects of experience to count and put in one's table. Frank goes on to identify this method for arriving at general conclusions with superficial views of the scientific method (Lakatos's teacher's myth of induction) and suggests that Mill believed that this kind of induction is used to find new theories (Frank 1957, pp. 318-19).



Task:
To explore the relationship among the interior angles in different types of triangles.

Procedure:
- Construct a triangle.
- Measure each angle.
- Draw the triangles and record the angle measurements on the chart below.
- Repeat this procedure on five other triangles.
- On the following page, state conjectures about your findings.

Triangle Drawings	∠ABC	∠BCA	∠CAB
1.			
2.			

Investigation: The Midsegment of a Trapezoid

The midsegment in a trapezoid connects the midpoints of the legs. In this activity, you'll investigate relationships between the midsegment and the bases of a trapezoid.

Sketch

Step 1: Construct $\overleftrightarrow{AB}$ and C not on $\overleftrightarrow{AB}$.
Step 2. Construct a line through C, parallel to $\overline{AB}$.
Step 3: Construct CD on this line and construct $\overline{AC}$ and $\overline{DB}$.
Step 4: Hide the line. $ACDB$ is a trapezoid.
Step 5: Construct E and F, the midpoints of $\overline{AC}$ and $\overline{BD}$.
Step 6: Construct $\overline{EF}$. $\overline{EF}$ is the midsegment of the trapezoid.

Investigate

Measure CD, EF, and AB. Do you see a relationship? Drag various parts of the trapezoid and watch the the lengths of the midsegment $\overline{EF}$ and the bases $\overline{AB}$ and $\overline{CD}$. Use Calculate to check your conjecture about the relationship of these lengths. Is there anything special about the direction of $\overline{EF}$ compared to $\overline{AB}$ and $\overline{CD}$? Try measuring slopes. Or construct a line parallel to $\overline{AB}$ through E. What can you say about that line and $\overline{EF}$? Does this relationship hold for all trapezoids?

Conjecture

Write your conjectures below.

Figure 3. "Discovery" problems (Yerushalmy and Houde 1987; Bennett 1992)

This description of the way people create general laws, or students create conjectures, is evident in the practice of many teachers and the suggestions of many curriculum materials (Carey et al. 1986). Geometry teachers using geometry construction software who hold this belief give their students problems like the one described earlier which, in addition to specifying a construction to be made, also suggest a set of measurements to be taken (see Figure 3).

"Discovery" problems of this type are typically designed toward a specific content goal. The goal is to bring students to "discover" an important relationship or theorem which the teacher has in mind and which is central to the course content (Serra [1993] takes this approach). For example, in Figure 3, in the problem on the left, students are asked to discover that the sum of the interior angles of a triangle is 180 degrees, and, in the problem on the right, they are expected to conjecture that the midsegment of a trapezoid is half as long as the sum of its bases.

The rationale for such focused exploration, as opposed to teacher telling, flows from a theory of discovery learning; students are more likely to remember/learn/transfer ideas which they discover themselves than those introduced to them in a lecture. For this reason, it is important to have students explore and discover the central ideas of the course, the ones which the teacher most wants the students to know.

Teachers using such tasks have Beta's linear view of the development of conjectures (the one criticized by Lakatos's teacher). First, students collect the data which the teacher or problem request, and then a conjecture is developed from analysis of patterns in the data. In such an approach, teachers ask students to go to the computer lab, create the construction, and collect the requested measurements. Frequently, they will then tell their students to take these data home "to make their conjectures." In this view, lab time is not a time to think about the data; lab time must be used to collect as much data as possible, and frequently there isn't "enough time" during this period for students to make their conjectures. Many teachers who work in tracked situations with students of low achievement feel that they must work this way with their students, though they might work with other students differently. They speak of their students as "needing structure."

Induction by Intuition and Guided Inquiry Problems

By contrast, in Frank's (1957, p. 317) description of induction by intuition:

> [People find general laws] by what we may call "intuition" or "imagination," or perhaps just "guessing" [Lakatos's teacher's "deductive guessing"?], and test the results of such intuitions by comparing the results with actual sense observations . . . This procedure leads us also from the observation of single facts to the statement of general laws because there is no guessing at a general law before a certain number of individual facts have been observed.[14]

This view of the creation of general laws is manifest in geometry teachers' pedagogy when they use more open-ended, "guided inquiry" problems which can lead to a wide range of conjectures. Teachers guide student inquiry by choosing constructions which have pedagogically relevant relationships. Students may still wonder why they are exploring a particular construction at a particular time. For example, Figure 4 is a problem that might be given after students have worked with the concept of similarity of triangles.

As in the description of "messing around," teachers using a problem like this one do not have an investment in the "discovery" of a particular conjecture; they would like students to use the ideas studied in class to explore the assigned construction. In such an approach, students occasionally raise ideas that are unfamiliar to the teacher or had not occurred to the teacher. For example, here are some of the ideas that a group of students developed in response to the problem in Figure 4 (Education Development Center 1987):

- The triangle whose vertices are the reflected points is similar to the triangle whose vertices are the "feet" of the altitudes.
- The sides of these two triangles are parallel.
- The sides of these two triangles are in a 2:1 ratio.
- The altitudes of the original triangle are the angle bisectors of the triangle whose vertices are the "feet" of the altitudes.

[14]Notice that, in Frank's description of induction by intuition, there is an examination of individual facts before the creation of a guess. Here is where the distinction between the two types of induction can become fuzzy. A skeptic might argue that it almost seems as if the difference boils down to the number of cases examined before making a guess.

- If the altitudes are extended, they also bisect the angles of the triangle whose vertices are the reflected points.

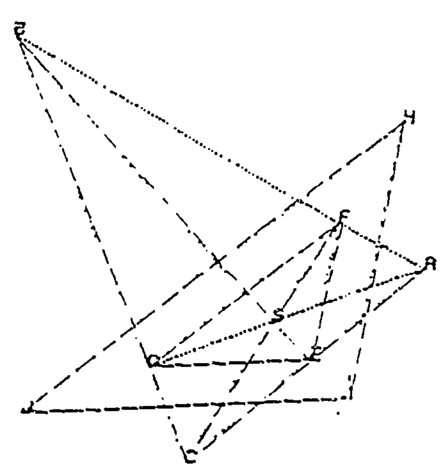
Students in other classes that used this problem explored relationships between the type of original triangle (acute, obtuse, right, . . .) and the types and locations of the triangles created by connecting the reflected points. Others noticed that the reflected points and the vertices of the original triangle lie on the same circle. One group also developed formulas relating the measures of the angles in the original triangle and those of the angles in the reflected triangle (Yerushalmy and Houde 1987).

Task: To explore the figure formed by reflecting the intersection point of the altitudes in each side of a triangle and connecting the three image points.

Procedure:

- Construct an acute ΔABC.
- Draw the three altitudes.
- Label G as their point of intersection.
- Reflect point G in each of the three sides of ΔABC producing points H, I, J.
- Draw ΔDEF and ΔHIJ.
- State your conjectures about the relationships among the points, elements, and triangles.
- Repeat the procedure for other types of triangles.

= *Drawings & Data*



= *Conjectures*

Figure 4. A "guided inquiry" problem (Yerushalmy and Houde 1987)

In geometry, guided inquiry problems tend to be about complex constructions which can be decomposed into many subfigures (more than those in discovery problems). The problem may suggest a few areas for exploration or measurement, but it assumes that students must take responsibility to decide which questions to explore and which measurements to choose to test the hypotheses they have developed based on their intuitions (from prior experience and knowledge developed in the class) about the figures. Some of these hypotheses will be expected by the teacher, others may not be.

A rationale for exploration of this sort is to have students use previously developed ideas as intellectual tools to create new ideas. The implicit theory of learning is that, by using the central ideas of the course to develop other ideas, the students will come to know these central ideas thoroughly. However, this theory of learning doesn't specify how these central ideas are learned. They may indeed be told, but they are learned or understood through use. Their importance and centrality become clear by virtue of their utility.

Teachers working with such problems suggest that their students go into the lab, create an instance of a construction, examine the resulting diagram, use all their accumulated knowledge to make hypotheses about that single case (or maybe a few others), test their hypotheses, and then ask themselves, "What if the figures were not the same? What if it was slightly different? Would these conclusions still hold?" as a spur to further empirical work. Finally, students are asked to develop written conjectures that outline the results which they have tested empirically and think warrant deductive proof.

Yet, these steps are not deemed to follow in a linear order. As ideas are explored, revised, or discarded, the student may move back and forth among each of these steps. They may even turn to proving deductively as a way to spur further questions for empirical study. To help their students learn to be proficient explorers and locate their progress among these steps, teachers using these problems distinguish between *hypotheses* which are ideas that have not yet been tested empirically, *observations* which are conclusive remarks about specific cases, *conjectures* for those general ideas which have withstood empirical testing but have not been proven deductively, and, finally, *theorems* for those which have been proven deductively.

A Critique of the Use of Discovery Problems

It is often very hard to have critical dialogue about differences in teaching. On the one hand, important differences are sometimes treated as purely stylistic or personal; on the other hand, there are many instances in which teaching is evaluated as good or bad in light of a small number of actions taken out of context and their match or mismatch with a reformer's vision. Yet, as mathematics teachers, following the *Standards*' emphasis on student exploration, begin to engage students in empirical exploration, it is important for them to develop a language with which to examine critically the ways exploration is being used as a pedagogical strategy. Such an examination may help inform changes in teaching practice adopted as a result of the reform efforts.

My initial misgivings about discovery problems stemmed from my observations of students' difficulties making conjectures. Teachers and students in the classes I described were having difficulties creating shared meaning for the term "conjecture." The task was a different one from those to which students and teachers were accustomed; yet the structure that teachers wanted to provide seemed only to exacerbate the problem. These misgivings deepened as I visited with other teachers around the country who were beginning to use geometry construction programs. It seemed that the first ideas people had for student explorations were usually discovery problems. While I appreciate the desire to help students be successful—a potential embedded in the structure of discovery problems—I prefer other types of problems. I present below two sets of reasons for this preference: one set is pragmatically related to students' difficulties making conjectures, while the other set focuses on the kind of teaching teachers seek to create.

The first set of reasons is structured around explanations for the phenomenon of students having difficulties making the conjectures, explanations which argue that this phenomenon is not surprising. In regard to students' difficulties making conjectures, discovery problems are reminiscent of "funneling"—"procedures which seem to begin with great openness, but then narrow the student's options until the 'desired' response is virtually guaranteed" (R. Davis 1992, p. 342-43)—and a "clozed questioning" style, where the teacher begins a sentence, pauses to have a student fill in a word, and then completes the thought as if the exchange had not occurred (Pimm 1987). In analyzing the sacrifices involved with the cloze questioning style, Pimm (1987, p. 53-54) notes the difficulties created because "the answer can usually only be a single word which fits in with the grammatical structure already specified by the 'floating' part of the teacher's utterance." There seem to be similar difficulties with discovery problems.

A related psychological argument, made by Inhelder (Karmiloff-Smith and Inhelder 1977, p. 305) in an article titled "If You Want to Get Ahead, Get a Theory," is that, in order to make thoughtful progress, learners need to consider goal and means simultaneously: "Only when goal and means are considered simultaneously do pauses *precede*[15] action."[16] In the discovery problems, this is not the case. The teacher indicates the means by telling students how to make the construction and which data to collect without indicating why these data are important or useful. After all, indicating why these data are useful or important may "give away" the central result, which is to be discovered. As a result, students may have no idea why they are collecting the requested data; there may not be a question guiding their exploration. (Yerushalmy et al. [1990] indicates that this criticism of discovery problems has some validity, especially with students who have been less successful in mathematics.) In contrast, in the guided inquiry problems, the construction is chosen for students, yet students then must decide which questions they would like to explore about this construction and figure out how to explore these problems.

A third explanation for why students have difficulty making conjectures with "simple" discovery problems is based on views of how knowledge is created. Those who theorize about the generation of knowledge simply no longer believe the common sense notions which underlie the view that knowledge is created by what Frank (1957, p. 303) terms "induction by enumeration." Much greater weight is now given to the categories which observers bring to their explorations. Thus, it does not make sense to have students "blindly" collect a lot of data to examine subsequently for patterns.

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[15]Author Inhelder's italics.

[16]Similar arguments appear in Kuhn et al. (1988).

The second set of reasons for being critical of discovery problems involve messages such problems send about the nature of mathematics and the roles of teachers and students in classrooms. Reform documents embrace a dynamic view of mathematical practice. In such a view, once a problem has been developed, isolated, or suggested to them, mathematicians must decide which aspects of the problem they wish to attend, which data to collect.[17] As Polya indicates, they must decide which aspects of the question are crucial, or of interest, and which are not. The explorations then undertaken are not linear ones directed towards a single goal (even though the title of one of his books, *How to Solve It*, and its four-stage heuristic might give a different impression). These explorations frequently end up in unexpected places or entail the creation of new mathematics. Thus, when "trying to find the solution, we may repeatedly change our point of view, our way of looking at the problem. We have to shift our position again and again. Our conception of the problem is likely to be rather incomplete when we start the work" (Polya 1945, p. 5). Finally, once results have been developed, they must be communicated to others who may be skeptical and who have not shared in one's exploration of examples. In such situations, deductive proofs are an established means in mathematical communities for communicating ideas and convincing one's peers of the validity of one's results.

The rhetoric of reform efforts emphasizes having students do this kind of mathematics, yet an approach based on induction by enumeration leads to a pedagogy which seems to contradict the view of mathematical practice embraced in reform documents. In these problems, there is only one answer of importance.

A final reason for being critical of discovery problems is their relationship to efforts to change the roles of teachers and students in classrooms. If such problems degenerate into an activity where students are guessing what is on the teacher's mind, the teacher is still the ultimate authority and arbiter of the correctness of student's ideas; the teacher's ideas are the most important ideas in the classroom.

By way of contrast, the guided inquiry type of problem supports a teaching style which alters the roles of students and teachers. It emphasizes the importance of students'

[17] A series of interesting examples are given in Gleick (1987).

ideas, allows students some choice, and changes the teacher's role from herder to that of facilitator, coach, and community builder. In geometry classes, when students have developed a wide-ranging set of conjectures, class discussions can become a forum for sharing these ideas which express the individuality of students. Some of the ideas will not be found in the text. Thus, the class (and even the teacher) may genuinely not know if the conjecture is true or false, though, based on their empirical work, they should have good reason to believe that it is true. The class as a whole can make progress through the diverse exploration of individuals. All of this is in stark contrast to students asked to explore "discovery" problems whose important results are found in the text and have been proven year after year.

Conclusions

> *To a mathematician, who is active in research, mathematics may appear sometimes as a guessing game: you have to guess a mathematical theorem before you prove it.* (Polya 1954, p. 158)

I believe that it is high time that secondary and tertiary mathematics education afford a larger role for plausible reasoning and empirical exploration. Yet, as we begin to do so, it is important that our first steps be carefully thought through. This paper distinguished between two views of induction and raised concerns about exploratory activities created in accordance with one such view. These activities, which may be carried out in mathematics classes under the banner of current reforms, can leave little room for students to explore and discover anything other than the single idea on the teacher's mind. Rather than help define new roles for teachers and students in classrooms, such activities may simply help perpetuate the view sometimes held by students that the purpose of school activities is for students merely to articulate to the teacher what the teacher already has in mind.

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