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ABSTRACT

This investigation is part of a combined research/curriculum development project in which children's learning is being examined in the context of developing and testing instructional units on 3-D geometry in grades 3, 4, and 5. There are two components to the article. First, the strategies and cognitive constructions that students utilize to conceptualize and enumerate the cubes in 3-D arrays are described. Second, the change in thinking of students as they are involved in instructional tasks that help them develop more sophisticated thinking about enumerating cubes in 3-D arrays are examined. (Author/MKR)

Enumerating Cubes in 3-D Arrays: Students' Strategies and Instructional Progress

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ENUMERATING CUBES IN 3-D ARRAYS: STUDENTS' STRATEGIES AND INSTRUCTIONAL PROGRESS

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This investigation is part of a combined research/curriculum development project in which children's learning is being examined in the context of developing and testing instructional units on 3-D geometry at grades 3, 4, and 5. There are two components to the article. First, we describe the strategies and cognitive constructions students utilize to conceptualize and enumerate the cubes in 3-D arrays. Second, we examine the change in thinking of students as they are involved in instructional tasks that have been utilized to help students develop more sophisticated thinking about enumerating cubes in 3-D arrays.

Enumeration Strategies and Cognitive Constructions: An Overview

In previous research, we have described in detail students' strategies and difficulties in enumerating 3-D rectangular cube arrays (Battista & Clements, in press). Our theory suggests that students' initial conception of a 3-D rectangular array of cubes is as an uncoordinated set of faces of the prism formed. These are the students who count all or a subset of exterior cube faces. Eventually, as students become capable of coordinating orthogonal views of the array, and as they reflect on experiences with counting or building cube configurations, this conception is perturbed. They see the array as space filling and strive to restructure it as such. Those who complete a global restructuring of the array conceptualize the set of cubes organized into layers. Those in transition, whose restructuring is local rather than global, conceptualize the set of cubes as space-filling, attempting to count all cubes in the interior and exterior, but do not consistently organize the cubes into layers. They have not yet formed an integrated conception of the whole array that globally coordinates its dimensions. Indeed, our data supports this hypothesized sequence of conceptions. From 3rd to 5th grade, we saw that students made a definite move from seeing a 3-D cube array as an uncoordinated medley of faces toward seeing it in terms of layers. We also saw a significant number of students in transition, with these students exhibiting a wide range of sophistication in their structuring of such arrays.

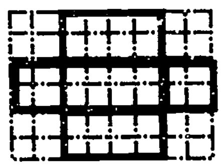
Our research suggests that *spatial structuring* is a fundamental notion in understanding students' strategies for enumerating 3-D cube arrays. We define spatial structuring as the mental act of constructing an organization or form for an object or set of objects. We found that in the process of determining the number of cubes in an array, students' spatial structuring of the array determined their enumeration of it; sometimes their spatial structuring supported a correct enumeration, sometimes it inhibited it.

The Evolution of Students' Thinking during Instruction

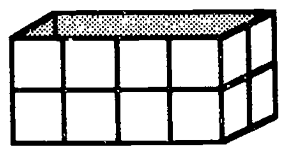
A fifth grade class was divided into pairs of students, each working on an activity sheet consisting of problems in which students were to predict how many cubes would fit in a box, then check their answer by making the box out of grid paper and filling it with cubes. The teacher circulated about the room, listening to students' conversations and asking questions. The first researcher observed and recorded the work of one pair of students, N and P, throughout the instructional unit. We will trace the course of these students' construction of a viable structuring and enumeration scheme for 3-D cube arrays.

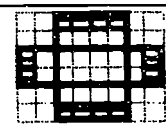
Box A

Pattern Picture



Box Picture





For Box A, N counted the 12 outer squares on the 4 side flaps, then multiplied by 2: "There's 2 little squares going up on each side, so you times them."

P counted the 12 visible cube faces showing on the box picture, then doubled that for the hidden lateral prism faces. So both students agreed on 24 as the prediction. After putting 4 rows of 4 cubes into the paper box, the boys exclaimed:

N&P: We're wrong. It's 4 sets of 4 = 16.

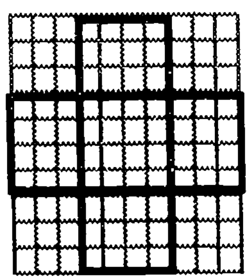
N: What are we doing wrong? [question directed at himself and his partner]

P: I know; we counted these twice [pointing to the column of 2 cubes on the right front corner of the box picture].

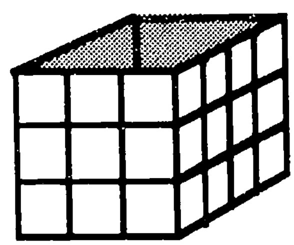
The boys then examined the box they constructed and concluded that they should have subtracted 8 for the 2 double-counted cubes at each of the 4 vertical edges (which would have given them a correct answer). So their reflection on the discrepancy between the actual and predicted answers caused them to discover their double counting.

Box B

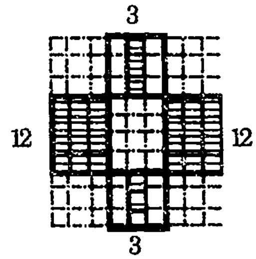
Pattern Picture



Box Picture



For Box B, P counted 21 visible cube faces on the box picture, then doubled it for the hidden lateral prism faces. He then subtracted 8 for the double counting (not taking into account that this box is 3 high, not 2, like Box A). He predicted 42 - 8 = 34.



N added 12 and 12 for the right and left side flaps on the pattern, then 3 and 3 for the front and back flaps, explaining that the outer columns of 3 on the front and back flaps were counted when he counted the right and left flaps. His prediction was 30.

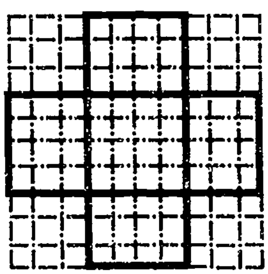
Both N and P accommodated their structuring and enumeration schemes in attempting to deal with the double-counting error. P compensated for the error by subtracting double-counted cubes. N tried not to double count.

After the boys constructed the box, filled it with cubes, and discovered that their answer was incorrect, P tried to figure out why their predictions were wrong: "If there's 21 here and 21 there, there's still some left in the middle. We missed 2 in the middle."

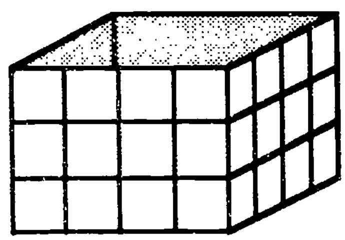
In this episode, the boys discovered yet another shortcoming of their original counting strategy—it ignored cubes in the middle. But as they attempted to compensate for this error, they focused on numerical differences, rather than carefully analyzing the spatial structure of the cube arrays. P concluded that they missed 2 cubes in the middle because 2 was the difference between his prediction of 34 and the actual answer of 36. The boys used a similar line of reasoning in making their prediction for Box B. They subtracted 8 for double counting because they needed to subtract 8 to make the prediction for Box A correct. However, although neither P nor N had yet developed a structuring of 3-D arrays that lead to correct enumeration of cubes, they were abstracting important aspects of the spatial organization of the cube arrays that would help them make the needed restructuring.

Box C

Pattern Picture



Box Picture

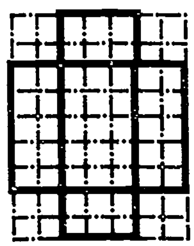


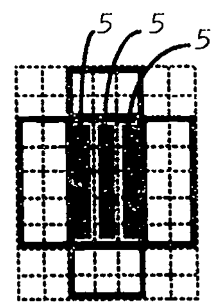
For Box C, N and P counted 24 visible cubes on the box picture then multiplied by 2 to get 48. They subtracted 12 for double counting the vertical edge cubes, getting a total of 36. But they decided that Box C was bigger than Box B, so they tried another analysis. This time they counted 21 outside faces (not double counting cubes on the right front vertical edge), times 2 for the hidden lateral faces. They then added 2 for the middle cubes (which is how many cubes they concluded they missed in the middle of Box B) to get a total of 44. They filled the box and found it contained 48 cubes.

The next day, N and P began class by trying to figure out what they did wrong with their prediction for Box C. They reviewed their method and concluded that they didn't add enough cubes for the middle—they needed 6 instead of 2. But they derived this conclusion by comparing their predicted amount, 44, to the actual number, 48, not by analyzing the spatial structure of the cube arrays. Analyzing only numbers can easily lead one astray in spatial situations. At this point, N and P's numerical reasoning was not properly linked to the spatial structure of the arrays.

Box D

Pattern Picture

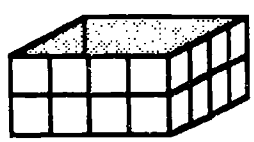




For Box D, N said there would be 30 cubes: "5 + 5 + 5 for the columns in the bottom, times 2 because there are 2 up." The boys cut out the pattern, filled it with cubes, and determined that it had 30 cubes. This excited them because it was the first time their prediction was correct. N explained his procedure to P, and P said he understood it: "You find how many are on the bottom, then you count how many you go up; 5 by 3 by 2 up. Add 15 and 15 and get an answer of 30."

When the observer asked N how he developed this strategy, N said that he generated the idea while looking at Box D, then tested it (silently) on Box C and found that he got the correct answer. N had been staring off into space for a while, clearly thinking about the problem. It seemed that he knew there was a better way to solve the problem, that he was reflecting on and analyzing the situation. On this problem, the boys seemed to abandon the counting of exterior cubes to find another structuring, possibly because of the shortcomings they were finding with their previous methods.

Box E



At this point, the boys' method for enumerating cubes was confined to examining box patterns, so problem 5 presented some difficulty for them. P counted 16 around the bottom and 16 around the top. But N replied, "Wait, that's not right because you counted these 2 twice [at the right front vertical edge]." P agreed, so they decided the count for the bottom layer should be 14. P said there were two horizontal layers, and predicted 32 - 8 = 24; taking 8 away because of the double counting on the 4 edges. But N said, "We don't know there's only two rows in this [meaning horizontal layers]. I think there might be 3." N predicted 28, which he arrived at by counting 14 on the front and right sides (not double counting the corner cubes), then multiplying 14 times 2, saying to P: "Maybe you should only take 4 away [so their predictions would be equal]."

After the boys correctly made the pattern, the observer asked them if they wanted to stick with their predictions, now that they could see the pattern. P said it was 16×2 = 32, + 16 = 48. N said it was just 32. But they decided to stick with their original predictions.

The boys seemed to be confusing parts of their old and new strategies. For instance, when only the box picture was available, it's possible that N used his count of the front and right sides in the same way he used his count of the bottom of a pattern. He didn't seem to be able to visualize what the bottom would look like. Even though looking at their pattern seemed to enable the boys to conceptualize the array in terms of layers, they didn't change their original predictions, seemingly unable to decide which of the two strategies was appropriate. However, when the boys put the cubes in the box and found that 32 fit, N said, "It <u>is</u> 32," as if coming to some realization.

Box F The bottom of the box is 4 units by 5 units. The box is three units high.

For Box F, the boys were unwilling to make a prediction until after they had made the pattern. N counted all the bottom squares in the pattern one by one—once for each of the 3 layers—counting 1-20 the first time, 21-40 the second, and 41-60 the third.

P: You counted 3 times, no 4.

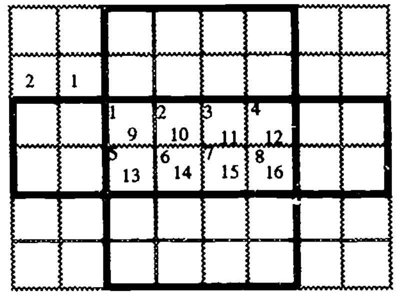
N: Why 4, it's 3 up? [with assurance]

The boys predicted 60 cubes, seeming quite confident in their prediction. They built the box and filled it with rows of 5 cubes, then counted the cubes by fives to 60. However, they didn't seem relieved that they were correct. Instead, they expected that their answer would be correct. Later, the boys read aloud the procedures they had written for determining the number of cubes in a box:

N: You count how many are on the bottom. Then you add how many go up.

P: You multiply to find the bottom. Then you multiply by how many high.

To test P's understanding of his procedure, the observer asked him how many cubes would be in a box that was 3×2 on the bottom and 5 high. He drew a 3×2 rectangle on graph paper, then correctly drew the four sides: "3×2 on the bottom, 6. 6 × 5 = 30."



Both boys seemed to have come to an understanding of a layering approach. They struggled, but they found viable methods to solve the "How many cubes?" problem and seemed pleased with themselves for doing so.

On the third day of the unit, N showed the observer an alternate way of finding the number of cubes in a box. He had described and illustrated his method in his journal. "There are two up, so you have to count two for each on the bottom." N demonstrated by counting by ones from 1 to 8 as he pointed to the 8 squares on the bottom, then counting from 9 to 16 as he pointed to each of these squares again.

Finally, the observer asked the boys how many cubes would be in a box that had the same bottom as Box A but was 3 cubes high.

P: 8 times 3 = 24.

N: Yeah, 8, 16, 24. I'm not too good at my multiplication facts.

Analysis

Throughout this account, N and P were trying to develop a theory of how to make correct predictions. The discrepancies between what they predicted and what they actually found caused them to reflect on their prediction strategies and their structuring of the cube arrays. At first, their enumeration strategies were based on more primitive, spatial structurings of 3-D arrays—seeing them in terms of the faces of the prism formed. The boys seemed to focus more on numerical strategies than a deep analysis of the spatial organization of the cubes. However, because their initial spatial structuring led to incorrect predictions, the boys refocused their attention on the structure of the cube arrays, which led to a restructuring of their mental models of the arrays. In fact, during their work on Box D, N and P seemed to develop a layer structuring of the array, a structuring that they verified and refined on subsequent problems.

The gains for N and P were typical of those achieved by students in this instructional unit. For instance, in one class of 47 fifth-graders, of the 31 students who did not use layering strategies on all the pretest items, 16 were doing so on the posttest, 9 increased their use of layering strategies, 4 did not increase, and 2 decreased. So, 81% increased their use of layering strategies. And 5 out of 6 of the students who did not increase used layering strategies on a box item similar to those on the student sheet discussed above. Forty-three of the students got this item correct; 2 of the 4 students who missed the item made computational mistakes.

Conclusions

Consistent with constructivist accounts of the learning process, two of the essential components of learning for N and P were reflection and cognitive conflict. Reflection and cognitive conflict were promoted by focusing students on *predicting* the number of cubes of 3-D arrays. Errors in predictions—which the boys themselves discovered—caused cognitive conflicts, or perturbations in the boys' current mental models for arrays. The boys attempted to resolve these conflicts by reflecting on the strategies they were using, all the while examining and restructuring their mental models of the arrays. In fact, the boys moved from an incorrect conception of the arrays, to a period confusion in which they vacillated between different conceptions, to a viable conception that resolved their confusion.

The account of N and P's work illustrates the constructivist claim that, like scientists, students are theory builders. They build conceptual structures to interpret the world around them. Cognitive restructuring is engendered when students' current knowledge fails to account for certain happenings, or results in "obstacles, contradictions, or surprises. The difference between the scientist and the student is that the student interacts with a teacher, who can guide his or her construction of knowledge as the student attempts to complete instructional activities" (Cobb, 1988). This guidance is often covert; in the present situation, the guidance came through the sequence of tasks, not by telling N and P problem solutions.

References

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Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist, 23*, 87-103.