DOCUMENT RESUME

ED 385 497 SP 036 118

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TITLE Learning about Teaching for Understanding through the Study of Tutoring.
PUB DATE Apr 95
NOTE 40p.; Paper presented at the Annual Meeting of the American Educational Research Association (San Francisco, CA, April 18-22, 1995).
PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS Fractions; *Individual Instruction; Intermediate Grades; *Knowledge Base for Teaching; Learning Strategies; Master Teachers; *Mathematics Instruction; Mathematics Teachers; *Reflective Teaching; Teaching Methods; *Tutoring; *Tutors
IDENTIFIERS *Teaching for Mastery

ABSTRACT

This study, part of a larger study focusing on studying a human tutor to design a computer simulation of expert tutoring (i.e., a self-improving intelligent tutoring system), conducted a detailed analysis of one expert tutor's (Nancy Mack) tutorial actions as she attempted to help students learn fractions with understanding. As Mack tutored students in two different research studies, she guided students through fraction topics by drawing on her understandings of her students' understanding and her content knowledge, focusing instruction on the development of students' conceptual knowledge, and using discourse to continually assess students' knowledge and assist them in learning. Additionally, unexpectedly and on her own initiative, Mack used the analytic framework employed to study her tutoring to deepen her own understanding of her teaching. The results of the study suggest that Mack's tutorial actions were guided by the interrelationships between her instructional goals, content knowledge, pedagogical content knowledge, and knowledge of students' thinking with respect to the content domain. (Contains 23 references.) (ND)

# LEARNING ABOUT TEACHING FOR UNDERSTANDING THROUGH THE STUDY OF TUTORING

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This paper was presented at the annual meeting of the American Educational Research Association, San Francisco, CA in April, 1995. The order of authorship on this paper is alphabetical; the authors view this paper as a collaborative effort to which they both contributed equally in important ways. This paper is based on portions of the first author's doctoral dissertation completed at the University of Wisconsin-Madison in 1993 under the direction of Jude Shavlik.

## Abstract

This study conducted a fine-grained analysis of one expert tutor's (Nancy Mack) tutorial actions as she attempted to help students learn fractions with understanding. As Mack tutored students in two different research studies, she guided students through fraction topics by drawing on her understanding of her students' understanding and her content knowledge, she focused instruction on the development of students' conceptual knowledge, and she utilized discourse to continually assess students' knowledge and assist them in learning. Additionally, unexpectedly and on her own initiative, Mack utilized the analytic framework employed to study her tutoring to deepen her own understanding of her teaching.

# Learning About Teaching for Understanding Through the Study of Tutoring

"Teaching and tutoring mathematics represent the basic core of mathematics education" (Davis, 1992a, p. 291). Although teaching and tutoring are central to mathematics education, many of the intricacies involved in teaching and tutoring mathematics have eluded researchers over the years (Fennema & Franke, 1992; Koehler & Grouws, 1992). Therefore, mathematics teaching and tutoring have primarily been described generally as "complex processes" rather than by detailed descriptions focusing on critical factors. A number of researchers concur that teaching and tutoring are complex processes and understanding their complexities will provide insights into ways to improve and communicate about teaching (Koehler & Grouws, 1992; Shavelson, Webb, Stasz, & McArthur, 1988). Ball (1993) further suggests that understanding the complexities involved in teaching and tutoring will provide insights into one of the most critical issues currently facing mathematics educators concerning how mathematics might be taught for understanding.

In recent years, researchers have begun to provide deep insights into the complexities involved in teaching and tutoring by closely examining teachers' and tutors' thought processes and knowledge in relation to their instructional actions and students' knowledge (Ball, 1993; Collins & Stevens, 1982; 1983; Fennema, Franke, Carpenter, & Carey, 1993; Lampert, 1986; 1990; Putnam, 1987). Results of these studies suggest that important relationships exist between a teacher's and tutor's instructional actions and his/her instructional goals, content knowledge, and knowledge of students' knowledge. However, the intricacies of the interrelationships that exist between these factors are still not yet clear.

Schoenfeld and colleagues (Schoenfeld, Gamoran, Kessel, Leonard, Or-Bach, & Arcavi, 1992) have begun to examine details of the interrelationships between various factors involved in the teaching/tutoring process by focusing on tutors as they tutor students in complex mathematical content domains. Based on their work with tutors, Schoenfeld et al. proposed a general theory of tutoring that focuses on describing interactions that occur between the tutor's goals, content knowledge, pedagogical content knowledge, affective factors, and tutorial actions during the tutoring process. Although Schoenfeld et al.'s theory helps make explicit many of the complexities involved in the tutoring process, their theory balances components of its structure and thus, is limited to describing interrelationships between "value-neutral" factors (Davis, 1992a).

Davis (1992a & b), as well as other researchers (Clark & Peterson, 1986; Fennema & Franke, 1992; Thompson, 1992), suggests that to gain deep insights into the complexities involved in the teaching/tutoring process, investigations of teaching and tutoring should focus not only on value-neutral factors such as those embedded in the theory proposed by Schoenfeld et al. (1992), they should also focus on the teacher's or tutor's philosophies underlying his/her instructional actions. As Davis (1992a) explains, investigations into philosophies underlying instructional actions are needed to help examine existing differences that may occur between teachers/tutors and their instructional actions. In light of current recommendations for mathematics education, Davis further suggests that it is particularly important at this time to examine the role of philosophies in the teaching/tutoring process as teachers and tutors struggle to interpret and act upon recommendations to teach mathematics for understanding.

How might the details of the interrelationships between a teacher's/tutor's philosophies and factors such as the teacher's/tutor's goals, content knowledge pedagogical content knowledge, and instructional actions be revealed? Davis (1992a) suggests that these interrelationships "can be revealed only by fine-grained analysis" because "the deep realities of teaching are often hidden in very fine-grained details" (p. 291). Anderson (1983), Klahr, Langley, and Neches (1987), and Van Lehn (1991) suggest that one way to conduct fine-grained analyses of teaching and tutoring is through the use of <u>production rules</u>. Production rules can be characterized generally as "a set of preconditions and associated actions; if a rule's preconditions are satisfied at a particular moment, then its actions may be taken " (Gutstein, 1993, p. 3). As a methodological framework for studying teaching and tutoring, production rules can be utilized to make explicit tacit knowledge and philosophies underlying each instructional action. Thus, production rules have the potential to effectively aid in providing detailed descriptions of the complexities involved in the teaching/tutoring process as teachers and tutors attempt to teach for understanding.

A growing number of studies suggest that rich descriptions of teaching and tutoring can emerge from investigations not only utilizing methodological techniques that focus closely on teachers and tutors and their instructional actions, but also by focusing on expert teachers and tutors as they attempt to teach mathematics for understanding (Ball, 1993; Fennema et al., 1993; Lampert, 1986; 1990; Pirie & Kieren, 1994 Schoenfeld et al., 1992). Therefore, the study reported in this paper sought deep insights into the complexities involved in the tutoring process by focusing on one expert tutor as she attempted to help students learn mathematics with understanding in a complex content domain. More specifically, this study focused on Nancy Mack as she tutored third-, fourth-, and sixth-grade students in two different studies focusing on students' learning of fractions with understanding (see Mack, 1990; in press). This study was part of a larger study focusing on studying a human tutor to design a computer simulation of expert tutoring (i.e., a self-improving intelligent tutoring system). The purpose of this paper is two-fold: (a) to conduct a fine-grained analysis of Nancy Mack's tutorial actions to investigate the interrelationships between her beliefs about learning and teaching for understanding, her instructional goals, content knowledge, and tutorial actions as she attempted to help students learn fractions with understanding, and (b) to describe how Mack, unexpectedly and on her own initiative, drew on the analytical framework (production rules) utilized to study her tutoring to analyze and deepen her own understanding of her teaching.

## Method

### Subject

Nancy Mack was selected as the subject of this study because she is considered to be an expert tutor in a complex mathematical content domain (fractions), and Mack's research involves her in directly tutoring students as she focuses on their learning of fractions (see Mack, 1990; in press for reports of Mack's research). Additionally, as Mack tutors, she attempts to help students learn fractions with understanding by building on their informal knowledge, which can be characterized generally as applied, real-life circumstantial knowledge constructed by the individual student that may be either correct or incorrect and can be drawn upon by the student in response to problems posed in the context of real-world situations familiar to him or her (Leinhardt, 1988). Although Mack's research presents fine-grained analyses of students learning fractions with understanding, her research does not directly address the complexities involved in her tutoring as she attempts to help students give meaning to fraction symbols and procedures. Therefore, this study focused on analyzing Mack's tutorial actions as she tutored students in two different research studies that were conducted two years apart, Mack (1990) and Mack (in press).

### General Characteristics of Procedures Utilized In Mack's Research Studies

Mack (1990) involved Mack in tutoring eight sixth-grade students 11 to 13 times over a six week period. Mack (in press) involved Mack in tutoring four third-grade and three fourth-grade students twice a week for three weeks. Similar procedures were utilized in both studies.

Each student was tutored individually. All tutoring sessions combined clinical interviews with instruction based on the guiding principles of viewing students' learning and teachers' instruction as problem solving and the student and teacher as cooperative problem solvers.

The majority of the problems were presented to the students verbally. Students were encouraged to think aloud as they solved problems. If students failed to think aloud, they were asked to explain what they had been thinking as they solved the problems. Concrete materials in the form of fraction circles and fraction strips were available for students to use, and their use was encouraged as long as students thought they were needed. Paper and pencil were also available.

The instructional content deviated from the topics covered on fractions in traditional textbook series in two important ways: (a) Students' informal knowledge about fractions provided the basis for instruction, and (b) students' informal conceptions of partitioning, solving fraction problems by separating units into parts and dealing with each part as though it represented a whole number, provided the basis for the development of concepts and procedures related to fractions.

The first tutoring session for each student focused on assessing the student's knowledge of a variety of fraction topics to determine what knowledge the student initially possessed both about fraction concepts and symbolic representations for fractions. Assessment was not limited to the first tutoring session but continued throughout all sessions. Each question a student was given during any tutoring session was regarded as an assessment task. In general, the tasks were based on four central ideas: (a) The more parts a unit is divided into, the smaller the parts become, (b) a fraction represented symbolically is a single number with a specific value rather than two independent whole numbers, (c) selected ideas of equivalence related to concrete and symbolic representations, and (d) addition and subtraction of fractions represented symbolically requires common denominators.

The specific tasks a student received were based on the student's responses to previous questions and on his or her choice of context for the problems (symbolic representations or real-world situations familiar to the student). The purpose of each task was to encourage the student to draw on informal knowledge related to specific fraction concepts, to relate fraction symbols and procedures to informal knowledge, and to draw on these relationships to construct additional relationships.

The tasks were used not only to assess the student's ability to relate informal knowledge and knowledge of fraction symbols and procedures, but also to provide directions for instruction during each tutoring session. In general, in situations where the student was unable to successfully solve a problem, the student was given a simpler problem. In situations where the student successfully solved a problem presented in the context of a real-world situation, the student was given a corresponding problem represented symbolically. If the student successfully solved a problem represented symbolically by relating fraction symbols and procedures to informal knowledge but the relationship appeared to be tenuous, he or she was given a similar task. If the student appeared to understand the relationship between fraction symbols, procedures, and informal knowledge, the student was given a problem that was closely related but more complex.

After each tutoring session, a lesson was planned for the student's next session based on the student's informal knowledge, the student's misconceptions related to fraction symbols and algorithmic procedures, the student's responses to problems presented in previous sessions, Mack's knowledge of how students think about fractions, and Mack's understanding of relationships between fraction concepts and procedures. Because the purpose of the tutoring sessions was to aid the student in drawing on informal knowledge to construct meaning for fraction symbols and procedures, the lessons were designed to be flexible with respect to the topics covered, the amount of time spent on a topic, whether the student was required to master a topic before moving on to another, and the sequence in which the student covered specific fraction topics.

Analyzing Mack's Tutoring

Mack provided the authors with 34 transcripts of her choice of one-to-one tutoring sessions between herself and four elementary-school aged students. Mack selected the transcripts from 131 transcripts generated by two research studies focusing on students' understanding of addition and subtraction of fractions, Mack (1990) and Mack (in press). Mack selected the transcripts to represent variability with respect to students' ability to make connections between fraction symbols, procedures, and informal knowledge. More specifically, Mack selected complete transcripts for one student from each of the two studies who readily made connections between fraction symbols, procedures, and informal knowledge and complete transcripts for one student from each study who struggled to make connections.

On her own initiative, Mack also provided the authors with a rational task analysis she constructed for addition and subtraction of fractions. Mack related that she constructed her rational task analysis from her own understanding of fractions and her research focusing on addition and subtraction of fractions. Additionally, Mack communicated that she did not consider her task analysis to be definitive, rather she considered it as a guide for possible sequencings of topics during tutoring.

One of the authors selected a sample of the transcripts and initially analyzed them using a form of protocol analysis focusing on production, or "tutorial", rules (Anderson, 1983; Klahr, Langley, & Neches, 1987; Van Lehn, 1991). This method of analysis involved focusing on each interaction in the transcript between Mack and a student and examining the interaction with respect to immediately preceding and following interactions to develop a rule-based formulation to describe the interaction. For example, one interaction involved the following actions. Mack wrote 1 - 4/5 on Ted's paper and asked him to solve the problem. Ted replied, "It's three-fifths". Mack then turned over Ted's paper and asked him to mentally solve a problem she posed verbally involving eating four-fifths of one whole pizza. This interaction involving first presenting a problem symbolically and then verbally in the context of a real-world situation can be described by the tutorial rule "If student unable to solve problem represented symbolically, then vary problem representation, go to real-world context".

After completing the initial analysis, one of the authors met with Mack to discuss the tutorial rules generated from the transcripts. At this time, this author and Mack also discussed two types of situations that did not readily generate tutorial rules. These situations involved instances where Mack shifted the focus of instruction to a different but related topic for a short time then returned to the original topic and instances where Mack could have taken alternative actions. Mack explained her actions in these situations by referring to her beliefs about learning and teaching, her instructional goals, her rational task analysis, and her knowledge about how students come to learn fractions with understanding.

After the discussion of the initial analysis, one of the authors analyzed the transcripts more deeply by focusing on each interaction in the transcripts with respect to Mack's instructional goals, her rational task analysis, what appeared to be the student's knowledge and needs, and prior and subsequent actions on both Mack's and the student's part. This analysis not only generated tutorial rules, but also enabled one of the authors to make inferences about Mack's beliefs about how students learn mathematics with understanding, ways instruction can be utilized to facilitate the development of understanding, and Mack's instructional goals.

During the second phase of the analysis, one of the authors met with Mack five times (2 hours each time) and had multiple one hour phone conversations with Mack to discuss the tutorial rules generated by the analysis of the transcripts. The discussions also focused on the author's inferences about Mack's beliefs and instructional goals and her validation of these inferences.

All discussions were audio taped and documented by field notes. Both Mack and three expert mathematics educators reviewed portions of the transcript analysis to check for agreement. When disagreements occurred, they were discussed, and if needed, the analysis was revised. Additionally, the tutorial rules were applied to samples of previously unanalyzed transcripts to determine if they accurately described interactions between Mack and the students. In all cases, the tutorial rules provided valid descriptions.

## Results

As Mack reflected on her interactions with students during the tutoring sessions and the analysis of these interactions, she communicated her beliefs about learning and teaching for understanding by relating that she viewed teaching and tutoring as problem solving processes that involved both students and teachers as cooperative problem solvers drawing on both students' prior conceptual knowledge and teachers' knowledge of the content and students' knowledge. Mack also stated that her goals for each tutoring session had been two-fold: (a) to comprehend how students learn addition and subtraction of fractions with conceptual understanding, and (b) to ensure that students learn as much as possible about addition and subtraction of fractions and concepts underlying these operations. These beliefs and goals were further suggested by the reports of Mack's studies where she states that the purpose of each study was to examine the development of students' understanding about fractions during instruction and she describes the tutoring sessions as clinical interviews combining assessment and instruction based on students' thinking.

Throughout all tutoring sessions, Mack's interactions with students suggested that Mack's tutorial actions were tied to her beliefs about learning and teaching for understanding and her instructional goals, as well as her content knowledge. The interrelationships between Mack's beliefs, goals, knowledge, and actions were suggested in the same way with respect to her interactions with each of the students involved. These interrelationships were suggested by four primary themes: (a) Mack guided students through fraction topics by drawing on her understanding of students' understanding and her content knowledge, (b) Mack focused instruction on the development of conceptual understanding of fraction symbols and procedures, (c) Mack utilized discourse to both continually assess students' knowledge and assist them in learning, and (d) unexpectedly, and on her own initiative, Mack utilized the analytical framework employed to study her tutoring to deepen her own understanding of her teaching. Although these four themes are not independent of one another, the results are organized into four sections based on these themes. Additionally, fractions written in words (e.g., three-fourths) denote fractions stated verbally, and fractions written in the form of a/b (e.g., 3/4) denote fractions represented symbolically.

## Guiding Students in Extending Knowledge Through Fraction Topics

All eight sixth-grade students in Mack (1990) and all seven third- and fourth-grade students in Mack (in press) came to instruction with informal knowledge related to adding and subtracting like fractions without regrouping (e.g., If you have three-eighths of a pizza and I give you two-eighths more or a pizza, how much pizza do you have?). One of Mack's instructional goals was to help students extend their knowledge to increasingly complex problem situations so that they could solve problems represented symbolically involving adding and subtracting unlike fractions with regrouping (e.g., 5 1/3 - 1 1/2) in meaningful ways. Mack attempted to guide students in extending their knowledge by drawing on both students' responses and her own understanding of the mathematical content. As Mack drew on these sources of knowledge, she guided students through fraction topics by presenting them with problems appropriate to what they understood, which frequently involved digressions to related topics and decomposing problems into one or more simpler problems. The results in this section focus on these actions.

Prior to the commencement of her tutoring, Mack constructed a rational task analysis for addition and subtraction of fractions. Mack related that she constructed the task analysis by drawing on her own mathematical understanding and her knowledge of students' thinking about fractions to identify critical mathematical ideas underlying addition and subtraction of fractions and possible connections between these ideas. Mack intended for her task analysis to serve as a guide for possible sequencings of fraction topics during instruction. Throughout the tutoring sessions, Mack's tutorial strategies were informed by her rational task analysis of the domain; however, her problem selection and each interaction she had with a student was ultimately determined by what a student did and did not know.

As Mack guided students in drawing on and extending their informal knowledge of fractions, she asked students to "think aloud" as they solved each problem or to explain their solution processes if they failed to think aloud. Students' explanations provided Mack with insights into what students did and did not understand about specific concepts, forms of representation, and algorithmic procedures. Mack consistently drew on these insights to give students problems appropriate to what they understood. For example, Mack asked Tony during his first tutoring session how fractions should be added when represented symbolically. Tony replied, "Across, you add the top numbers across and the bottom numbers across". Tony's response suggested to Mack that his knowledge of symbolic representations and algorithmic procedures for fractions was faulty. Mack's following tutorial action suggested that she drew on this insight in her attempt to determine what Tony did know about fractions by asking him to solve the following problem mentally, "If you have three-eighths of a pizza and I gave you two-eighths more of a pizza, how much pizza would you have?". Tony correctly responded, "Five-eighths" and then recorded the problem symbolically (3/8 + 2/8 = 5/8) and remarked, "I don't think that's right. . . I think this (the 8 in 5/8) just might be sixteen. I think this'd be five-sixteenths". Tony's response related to the real-world problem suggested to Mack that Tony possessed informal knowledge related to adding like fractions; however, Tony's response related to the symbolic representation once again suggested that his knowledge of algorithmic procedures was faulty and had a strong influence on his thinking. Mack drew on her knowledge of Tony's informal knowledge and faulty knowledge of algorithmic procedures in subsequent tutorial actions as she removed symbolic representations for four tutoring sessions and posed problems that drew on Tony's informal knowledge of fractions.

As Mack presented students with problems appropriate to what they understood and helped students extend their thinking to increasingly complex problem situations, she did not lead students linearly through a sequence of topics for fractions. Mack presented students with problems involving addition and subtraction concurrently, and she structured the sequencing of fraction topics around increasingly complex addition and subtraction problems that emerged from her rational task analysis of the domain. These addition and subtraction problems can be characterized generally by the following sequence of problems: add and subtract like fractions without regrouping, one minus a fraction less than one, one plus a fraction less than one, a whole number greater than one minus a fraction less than one, subtraction of like mixed numerals, addition of like mixed numerals, addition and subtraction of unlike fractions without regrouping, and addition and subtraction of unlike mixed numerals with regrouping. Mack did not often lead students from one type of addition or subtraction problem directly to the next. She frequently led students through digressions to related topics and led them back to simpler problems before helping students solve more complex problems in meaningful ways.

Mack led students through digressions to related topics to extend students' understanding of various concepts as they were encountered through addition and subtraction problems. More specifically, Mack's sequencing of fraction topics was based on addition and subtraction problems; however, when the problems involved concepts such as equivalent fractions, mixed numerals, and improper fractions, Mack led students away from the addition and subtraction problems for awhile and guided them in exploring the concept in some depth. After exploring the concept, Mack led students back to the addition or subtraction problem from which they had departed.

The following protocol, which was taken from Ted's third tutoring session, illustrates how Mack utilized digressions to extend students' understanding.

(This was Ted's first experience with problems involving subtracting a fraction from a whole number. Prior to this, Ted had solved problems involving adding and subtracting like fractions without regrouping.)

Mack: (Wrote 1 - 5/8 on Ted's paper). What's that equal to?

Ted: I don't know.

Mack: (Covered Ted's paper). How about if we think about our pizzas and you've got one whole pizza and you eat five-eighths of one pizza, how much pizza do you have left?

Ted: Three-eighths. It was five pieces of pizza that I had out of eight pieces. I have three left.

Mack: If you had a pizza and it's cut into six pieces and you get the whole pizza, how much of the pizza do you get?

Ted: Six-sixths.

Mack: What about if the pizza's cut into four pieces and you get all of them, how much would you get?

Ted: Four-fourths, 'cause there's four pieces of pizza and I had them all. . .Eight-eighths. . . Ten-tenths. . .

Mack: Can you write the different names for one whole pizza on your paper?

Ted: (Wrote 4/4, 5/5, 6/6, 8/8, and 10/10).

Mack: Let's go up here to this problem (1 - 5/8), can you think of how to solve that problem now? . . .

<u>Ted</u>: (long pause) I'm going to put eight-eighths for the one whole pizza, it's three-eighths. "Cause this (numerator) would be eight pieces up here and there was eight of that, there was eight in one whole pizza and I wanted eight there (referring to numerator) and eight there (referring to denominator). (Wrote 8/8 - 5/8 = 3/8).

Mack's focus on the problem one minus five-eighths at the beginning and end of the protocol suggested that addition and subtraction problems provided structure for her instruction with Ted, but the structure was flexible. Mack's digression for the problem one minus five-eighths suggested that she was drawing on both her knowledge of Ted's knowledge related to fractions equivalent to one and her knowledge of concepts underlying the problem to guide Ted in exploring concepts and symbolic representations related to fractions equivalent to one. Furthermore, Ted's writing of 8/8 - 5/8 = 3/8 suggested that Mack's digression to fractions equivalent to one helped him extend his informal knowledge to solve the problem represented symbolically in a meaningful way.

Mack led all students in digressions to related topics in a manner similar to the way she led Ted as she provided students with problems appropriate to what they understood and attempted to extend their understanding. Mack presented students with addition and subtraction problems and when new concepts were embedded in the problems, Mack led students in explorations of these concepts and then returned them to the problem from which they digressed. This action proved to be effective in helping all students extend their knowledge to give meaning to more complex problems and problems represented symbolically.

In addition to digressing to a related topic to help students extend their knowledge, Mack also led students back to simpler problems before helping them solve more complex problems in meaningful ways. This latter action occurred when Mack presented a student with a problem that was closely related but more complex than the problems he/she had previously solved and the student was unable to solve the new problem even when it was presented in the context of a real-world situation. When this occurred, Mack decomposed the new problem into one or more simpler problems that were similar to problems the student had previously solved.

The following protocol, which was taken from Laura's third tutoring session, illustrates how Mack decomposed complex problems into simpler ones students could solve.

(Prior to this time, Laura successfully compared unit fractions such as one-eighth and one-fourth by explaining that the more parts a whole is divided into, the smaller the parts become. This was Laura's first experience comparing fractions with like numerators other than one.)

Mack: You have five-eighths of a cup of white sugar and five-fourths of a cup of brown sugar, which type do you have more of?

Laura: Five-eighths. . . It sounds bigger because eight is bigger than four.

Mack: Are eighths bigger than fourths?

Laura: (Pause).

Mack: Is one-eighth bigger than one-fourth?

Laura: No. (Pause).

Mack: If you have a pizza and cut it into eight pieces and you cut another one into four pieces, and you get five-eighths. . .

Laura: (Interrupting). Oh, five-fourths is bigger because fourths are bigger than eighths and you have five of both of them.

Mack's questions "Are eighths bigger than fourths?" and "Is one-eighth bigger than one-fourth?" suggested that she was drawing on both her knowledge of Laura's knowledge related to comparing unit fractions and her own content knowledge related to unit fractions and non-unit fractions with like denominators to lead Laura back to a simpler problem and help her successfully compare five-eighths and five-fourths. Additionally, Laura's response "Oh, five-fourths is bigger. . . " suggested that Mack's actions of decomposing the problem into a simpler real-world problem aided her in successfully comparing the complex fractions.

Mack decomposed problems for all students in a manner similar to the way she decomposed problems for Laura when students were unable to solve problems that were new to them. Mack drew on both her knowledge of students' knowledge and her content knowledge to decompose problems and lead students back to explore these simpler problems before leading them to explore more complex problem situations. This tutorial action, as well as actions related to digressions and presenting students with problems appropriate to what they understood, resulted in Mack leading students non-linearly through a sequence of fraction topics and guiding students in solving increasingly complex problems in meaningful ways.

## Focus on Development of Conceptual Understanding

All eight sixth-grade students came to instruction with a rich store of informal knowledge related to addition and subtraction of fractions and prior knowledge of procedures for these operations. This latter type of knowledge was often disconnected from students' informal knowledge, faulty, and limited to rote knowledge of procedures. All seven third- and fourth-grade students came to instruction with informal knowledge related to addition and subtraction of fractions and no formal knowledge of algorithmic procedures for these operations. As Mack guided students in building on their informal knowledge, her tutorial actions were guided by her goal of helping students develop conceptual understanding of addition and subtraction of fractions. Therefore, Mack continually focused instruction on the development of students' conceptual understanding. Three common themes characterized the way in which Mack focused instruction for all students: (a) Mack promoted the development of students' conceptual knowledge prior to their procedural knowledge, (b) she introduced symbolic representations in relation to problems posed verbally in the context of real-world situations students could solve, and

(c) she guided students in extending ideas and concepts within and between representational systems. The results in this section focus on these three themes.

Although students came to instruction with different prior knowledge of fractions, Mack attempted to help all students develop conceptual knowledge of fractions prior to addressing procedural knowledge. Additionally, Mack made an effort to teach as little procedural knowledge as possible in the sense of telling students specific step-by-step ways to solve problems or encouraging students to use their prior knowledge of procedures to solve problems.

The manner in which Mack promoted the development of conceptual knowledge was the same for all students. Mack continually drew on students' existing conceptually rich knowledge and encouraged students to solve problems in ways that were meaningful to them. More specifically, Mack initially posed problems verbally in the context of real-world situations (e.g., If you have one-fourth of a pizza and I give you two-fourths more of a pizza, how much pizza do you have?), rather than symbolically (1/4 + 2/4 = ?). Mack encouraged students to solve these problems in their own ways by visualizing the problems and solving them mentally or by using manipulative materials, rather than by using pencil, paper, and previously learned procedures.

After students solved real-world problems in meaningful ways, Mack guided students in building on their conceptually rich knowledge to give meaning to fraction symbols, procedures, and increasingly complex problem situations. Realizing that students' prior rote knowledge of procedures frequently interferes with their efforts to build on their informal knowledge and that students frequently confound whole number and fraction concepts when symbolic representations are involved, Mack attempted to minimize the dominating influence of students' prior knowledge related to symbolic representations by introducing students to symbolic representations in relation to real-world problems they could solve. More specifically, after a student solved a real-world problem presented verbally and Mack thought the student could relate symbolic representations to his/her conceptual knowledge, Mack asked the student to write the number sentence for the problem. For example, after Mack asked Todd a real-world problem involving having two-fourths of a pizza and getting one-fourth more, she asked, "Can you write that problem on your paper?. . .Can you write the number sentence for that problem?". Although Todd initially wrote 2/4 + 1/4 = 3, Mack drew on Todd's conceptual knowledge by asking, "Do you have three whole pizzas (pointing to the 3)?" to help him write an appropriate number sentence. Thus, although students did not always generate appropriate number sentences when symbolic representations were introduced with respect to real-world problems, Mack utilized these student-generated representations to further the development of students' conceptual knowledge.

Mack also attempted to further the development of students' conceptual knowledge by guiding students in extending a single mathematical idea within and across different representational systems. This involved Mack in drawing on her content knowledge to present problems within the same representational system that were closely related from both a mathematical and cognitive perspective but differed in complexity (e.g., real-world problems involving 4 - 7/8 and 4 1/8 - 7/8). Guiding students in extending their knowledge across representational systems also required that Mack made a directed effort to minimize the influence of students' prior knowledge related to symbolic representations. Additionally, guiding students in extending their knowledge required that Mack and the students work together to stay focused on specific mathematical concepts.

The following two protocols illustrate how Mack guided students in extending their conceptual knowledge. The first protocol, which was taken from Teresa's fifth tutoring session, illustrates how Mack guided students in extending their knowledge across different representational systems. The second protocol, which was taken from Ted's fifth tutoring session, illustrates how Mack guided students in extending their knowledge within the same representational system.

<u>Teresa</u> - (Prior to this time, Teresa had solved problems involving subtracting a fraction from a whole number (e.g., real-world contexts and symbolic representations for 4-7/8), and she had just encountered a real-world problem involving four and one-eighth minus seven-eighths. At this point, Mack digressed to helping Teresa extend her conceptual knowledge related to fractions equivalent to one to mixed numerals and improper fractions. Additionally, Mack knew from Teresa's initial assessment that Teresa possessed faulty prior rote knowledge of procedures for converting mixed numerals and improper fractions.

<u>Mack</u>: Suppose I told you that you have three and one-eighth cookies, how many eighths is that? (Pause). If you have three and one-eighth cookies?

<u>Teresa</u>: Twenty five, wait? (Pause). Twenty five-eighths, because eight-eighths go into three, I mean eight-eighths go into one, and eight times three equals, wait eight times, wait eight times

<u>Mack</u>: (Interrupting). You have three and one-eighth cookies.

<u>Teresa</u>: (Wrote 8, 16, 24, 25 on her paper as she talked). And then you have, you have to do it eight, sixteen, twenty four, that's three plus another eighth, that's twenty five. . .

<u>Mack</u>: Suppose you have eleven-eighths cookies, how many cookies is that?

<u>Teresa</u>: Umm.

<u>Mack</u>: Do you have more than one whole cookie?

<u>Teresa</u>: Yea, . . . one-third more. . . I mean three-eighths more. . . (Wrote 1 3/8 on her paper as she talked). I meant to say one and three-eighths [cookies].

<u>Mack</u>: I want you to write three and five-eighths as an improper fraction.

<u>Teresa</u>: Twenty nine-eighths, eight goes into three, I mean eight-eighths goes into one, so it's eight, then sixteen, then another one is twenty four, plus five is twenty nine.

Mack: Now write fourteen-thirds as a mixed numeral.

Teresa: (Wrote 3/3 3/3 3/3 3/3 2/3). Four and two-thirds. I had to write it down or else I'd get it mixed up in my head.

Ted - (Prior to this time, Ted had solved problems involving subtracting like mixed numerals with regrouping (e.g., real-world problems and symbolic representations involving 8 2/6 - 4 5/6). As Ted solved these subtraction problems, Mack guided him in developing conceptual knowledge related to fractions equivalent to one and to one-half. At this time, Mack was attempting to guide Ted in extending his knowledge related to equivalent fractions to symbolic representations involving adding unlike fractions.)

Mack: Let's try something different. Let's try five-eighths plus one-half. . . If you had five-eighths of a pizza and I gave you one-half. . .

Ted: (Interrupting). Oh, a half a pizza. (Wrote 5/8 + 1/2 = 9/8). Nine-eighths. I took eight, a half of eight is four and four plus five equals nine, nine-eighths. . . one and one-eighth.

Mack: Let's try one-half plus one-third.

Ted: (Wrote 1/2 + 1/3). Two and a half-thirds. There's supposed to be, a half of three is one and a half, plus another one would be two and a half-thirds. . . I'm trying to make them the same size of a whole pizza, 'cause it's easy to tell how much I have then.

In both protocols, Mack drew on students' conceptual knowledge of equivalent fractions by referring to the quantities in the context of real-world situations. Mack's asking Teresa about quantities first with respect to cookies and then in a more mathematical sounding way suggested that Mack was attempting to guide Teresa in extending her conceptual knowledge related to fractions equivalent to one to symbolic representations involving mixed numerals and improper fractions. Additionally, Mack's lack of comment about Teresa's solution method suggested that she was attempting to guide Teresa in extending her knowledge across representational systems in a way that avoided Teresa's faulty prior rote knowledge related to converting mixed numerals and improper fractions. Mack's presenting Ted first with "five-eighths plus one-half" and then "one-half plus one-third" suggested that Mack drew on both Ted's response to the first problem and her own content knowledge related to fractions equivalent to one-half and adding unlike fractions to present Ted with a closely related but more complex problem within the same representational system.

Mack promoted the development of all students' conceptual knowledge in a manner similar to the ways in which she interacted with Teresa and Ted in the above protocols. Mack continually drew on her knowledge of both students' knowledge and the content as she posed problems verbally and asked students to solve increasingly complex problems in ways that were meaningful to them. At times students' solutions reflected traditional algorithms for fractions and at other times students developed alternative algorithms for solving problems. However students solved problems, Mack accepted students' solutions and focused on the students' underlying conceptual knowledge. Therefore, when Mack guided a student in extending his/her conceptual knowledge to solve a problem such as 2/6 + 3/6 and the student explained "two plus three equals five and then you just put down the six because they're both the same, they're both six", Mack asked questions such as "Why do you put down the six?" and "What does the denominator of a fraction mean?". Thus, Mack's focus remained on the development of students' conceptual knowledge throughout the tutoring sessions.

There were three instances in all of the protocols when Mack directly addressed procedural knowledge. One instance involved Ted's solution to 6 3/5 - 1 4/5. Ted solved the problem by changing both mixed numerals to improper fractions. Ted obtained an answer of 24/5 and asked if his answer was correct. Mack responded, "I solved it a different way. Do you want to see how I solved it?" . Ted responded affirmatively and Mack explained, "I know I couldn't take four-fifths from three-fifths and wrote 5 8/5 - 1 4/5. As she wrote the problem, Ted remarked, "Oh, you borrowed one!" even though "borrowing" had not been previously mentioned during Ted's tutoring. Ted then utilized Mack's method "because it's easier and I don't make as many mistakes".

The other two instances where procedural knowledge was addressed involved students asking how to find equivalent fractions. Mack told the students they could use multiplication, but she did not tell them specifically how they could use multiplication to find equivalent fractions. Whether Mack told students ways to solve problems or guided them in solving problems in their own meaningful ways, Mack's tutorial actions focused on her goal of developing student's conceptual understanding of fractions.

Use of Discourse

As Mack drew on students' informal knowledge and guided students in building on their knowledge to give meaning to fraction symbols and procedures, she utilized discourse to both continually assess students' thinking and assist them in learning. Mack's use of discourse was guided by her beliefs about students and teachers working together as cooperative problem solvers and her goals related to the development of students' understanding. Two common themes characterized the way discourse transpired between Mack and all students: (a) Mack encouraged students to talk about their solution processes with respect to both correct and incorrect answers, and (b) Mack "turned situations back" to students to engage them is discussions focusing on misconceptions and extending understanding. The results in this section focus on these two themes.

Throughout the tutoring sessions, Mack asked students to think aloud as they solved problems or to explain their solution processes if they failed to think aloud. Regardless of whether students obtained correct or incorrect answers, Mack encouraged students to talk. As students explained their solution processes, Mack probed the depths of students' understanding by engaging them in discussions about their solutions. More specifically, Mack asked students why they performed specific actions (e.g., Why did you change the three, in 3 - 5/8, to two and eight-eighths?) and why they did not take alternative paths during their solution processes (e.g., Why didn't you add the denominators when adding two-sixths and three-sixths?).

The following protocol, which was taken from Jane's second tutoring session, illustrates how Mack encouraged students to talk about their solutions. The protocol also illustrates how Mack questioned students about paths not taken.

<u>Mack</u>: (Wrote 1/4 and 1/10 on Jane's paper. Jane read each fraction correctly). What I want you to do is I want you to circle the fraction that's the biggest.

<u>Jane</u>: You mean which pieces are going to be bigger?

<u>Mack</u>: Yes.

<u>Jane</u>: (Circled 1/4). When it's cut into more pieces it gets smaller.

<u>Mack</u>: Why does it get smaller?

<u>Jane</u>: The more pieces there are, the smaller they have to be if they all fit into the same size thing.

<u>Mack</u>: Now when I asked you to circle which one was bigger, you asked me do I mean the pieces. What if I said which number is bigger, which one would you circle then?

<u>Jane</u>: One-tenth, because there's more pieces of pizza in the thing. . . Pieces and numbers aren't the same thing.

Although Jane successfully compared the two fractions and explained her solution in a logical way, Mack's question "Why does it get smaller?" suggested that Mack was not completely satisfied with Jane's explanation. Her question suggested that she needed more information to see the depth of Jane's understanding. Additionally, Mack's question "What if I said which number is bigger. . ." suggested that Mack drew on both Jane's questions about pieces and her own knowledge that students think of "pieces" as referring to fractions and "numbers" as referring to whole numbers to ask Jane about an alternative way to compare the fractions.

Mack engaged all students in discussions of their solution processes in a manner similar to that utilized in Jane's protocol above. As students explained their solution processes, Mack engaged them in discussions designed to probe the depths of their understanding. During these discussions, students talked at least as much as Mack talked, students talked about conceptual issues, and correct answers generated as much discussion as incorrect answers.

As Mack engaged students in discussions of their solution processes, she frequently "turned situations back" to students to assist them in extending their understanding. More specifically, rather than directly answering students' questions or telling students how to solve problems, Mack asked students questions that guided them in determining answers on their own. Therefore, when a student asked a question about a specific concept or procedure (e.g., What's two-fifths? I've never heard that fraction before.), Mack tried to have the student answer the question for him/herself by asking the student to think about a response to a previously solved problem or to think about the problem in a simpler form (e.g., What do you think two-fifths would look like? If we're talking about fifths?). When a student solved a problem by utilizing a faulty procedure, Mack asked the student to relate the mis-solved problem to a past correctly solved problem to encourage the student to reflect on the solution process. For example, after Matt solved a real-world problem involving three minus four-eighths and wrote an appropriate number sentence for the problem, Mack presented him with the problem 4 - 3/8. Matt responded, "I don't know how to do that". Mack attempted to help Matt solve the problem by asking, "Can you think of the problem you just solved to figure out how to solve this one?". Additionally, when a student generated inconsistent answers to the same problem in different representations, Mack counterpoised the student's responses against each other and asked the student to resolve the discrepancies between his/her answers.

The following protocol, which was taken from Aaron's first and second tutoring sessions, illustrates how Mack "turned situations back" to students by counterpoising students' inconsistent responses against one another.

First Tutoring Session

Mack: Now I want you to solve this problem (shows Aaron a piece of paper with 4 - 7/8 printed on it).

Aaron: (Writes 4 - 7/8 on his paper). Well, you change this (the 4) to four-fourths.

Mack: Why four-fourths?

Aaron: 'Cause you need a whole, so you have to have a fraction and that's that fraction, and then you have to reduce, or whatever that's called, that (the 4) times two, so you'll have eight-eighths. Eight-eighths minus seven, so it's one-eighth.

Mack: Now suppose I told you that you have four cookies and you eat seven-eighths of one cookie, how many cookies do you have left?

Aaron: You don't have any cookies left. You have an eighth of a cookie left.

Mack: If you have four cookies. . .

Aaron: (Interrupting). Oh! Four cookies!

Mack: . . . And you eat seven-eighths of one cookie, how many cookies do you have left?

Aaron: Seven-eighths of one cookie? Three and one-eighth.

Mack: Now how come you got three and one-eighth here (referring to what Aaron had just said) and you got one-eighth there (referring to paper)?

Aaron: (Pauses, looking over problem). I don't know. (Contemplates problem; repeats problem). Well, because on this you're talking about four cookies, and on this you're talking about one.

Second Tutoring Session

Mack: Last time we were working on the problem four minus seven-eighths.

Aaron: (Immediately writes 4/4 - 7/8). That's impossible! This (4/4) is smaller than that (7/8) in fraction form it is. This (4/4) actually equals one.

Mack: Suppose you have a board four feet long and you cut off a piece seven-eighths of a foot long to make a shelf. how much of the board do you have left? . . . Don't look at your problem [on paper]. . .

Aaron: (Draws a line for the board. . . marks off the board to show four feet). Oh, I know now, three and one-eighth feet.

Mack: Very good. Now you said the problem couldn't be worked. . . You couldn't figure that problem out last time.

Aaron: I thought four was the same as four-fourths, but it's really the same as three and four-fourths, three and eight-eighths.

Mack's questions about Aaron's responses to the cookie problem and the symbolic representation during the first tutoring session suggested that Mack was trying to make Aaron aware of and assume responsibility for resolving the inconsistencies between his answers. Mack's actions during the second session suggested that Mack drew on Aaron's previous responses related to the problems, her knowledge that students' informal knowledge is often disconnected from their knowledge of fraction symbols, and her knowledge of the dominating influence of students' knowledge of whole numbers as she guided Aaron in resolving the inconsistencies between his answers. Mack's comment "You couldn't figure that problem out last time" further suggested that Mack was not satisfied with Aaron deciding that the answers to both problems should be three and one-eighth, she wanted to know the thoughts behind Aaron's reasoning. Additionally, Mack's questions and actions during both tutoring sessions suggested that both Mack and Aaron were working together as cooperative problem solvers focusing on extending Aaron's understanding.

Mack turned situations back to all students in a manner similar to that illustrated in Aaron's protocol above. Whether students asked questions, mis-solved problems, or obtained inconsistent answers to the same problem in different representations, Mack did not tell students answers or ways to solve problems. Instead, Mack consistently drew on students' responses and her knowledge of how students' think about fractions to pose questions and work with students as cooperative problem solvers focused on extending students' understanding of fractions.

## Utilizing the Analytic Framework to Study Her Own Teaching

Mack's goal of learning as much as possible about the development of students' understanding of fractions guided her actions during the tutoring sessions. Although Mack gained deep insights into students' understanding through her tutoring, Mack's goal related to students' understanding continued to guide her actions after the conclusion of tutoring. Simply through her discussions with one of the authors about the tutorial rules generated from her transcripts, Mack learned the analytic framework employed to study her tutoring. Unexpectedly, and on her own initiative, Mack utilized this framework to deepen her own understanding of her teaching. The results in this section focus on Mack's report of how she utilized the analytic framework to gain insights into her teaching. The report is written in Mack's own words.

From my tutoring of sixth-grade students, I knew that students often come to instruction with informal knowledge of fractions that is disconnected from their knowledge of formal symbols and procedures. I also knew that I could help students draw on and extend their informal knowledge by asking them guiding questions, presenting them with the same problem in different representational systems (e.g., real-world contexts and symbolic representations), and presenting them with closely related but more complex problems. I drew on this knowledge as I guided the third- and fourth-grade students in drawing on and extending their informal knowledge of fractions. Each of these younger students however, encountered one or two instances (12 instances total) where they failed to make connections between their informal knowledge, fraction symbols, and procedures when everything suggested to me that these connections were possible and should have been easily made.

The following protocol, which was taken from Todd's second tutoring session, illustrates one of these situations where students failed to make connections I thought probable.

I: You have one whole sausage and pepperoni pizza, and you're going to eat two-fifths of that pizza. . .

Todd: (Interrupting). I would have three-fifths left, because if I had, you said I had two-fifths of a pizza and all that I would have left is three-fifths because five take away two is three.

I: Can you write that problem?

Todd: (Wrote 1 - 2/5 = 3/5 vertically). Five would be one whole pizza and the two is how much I ate.

I: (Wrote 1 - 2/3 on Todd's paper). . .

Todd: I would have one-third, because two take away one is one, so I just put the one up here (referring to numerator) so I could put the three down here (referring to denominator).

Todd's response to the real-world problem and his symbolic representation for 1 - 2/5 suggested that Todd understood he could think of the "one" whether represented symbolically or in a real-world context in terms of an appropriate fractional quantity.

Therefore, I was surprised when Todd appeared to think of the "1" in 1 - 2/3 as one-third rather than three-thirds. Todd's inability to move from solving 1 - 2/5 to 1 - 2/3 in a logical way left me puzzled, as did the other instances where students failed to make connections I thought probable. I kept wondering why students failed to make these connections.

My initial analysis of these perplexing situations revealed nothing more than students' failure to make connections was not due to the interference of prior rote knowledge of procedures for fractions. Therefore, to try to gain deeper insights into why students failed to make connections I thought probable, I decided to try analyzing these situations and all other interactions I had with students by using the framework that was currently being utilized to study my tutoring. My purpose in using this framework was not to generate tutorial rules per se, but to analyze each problem I posed and its related probing questions with respect to preceding and subsequent questions. More specifically, I examined each interaction I had with a student and described how the problem or question I posed differed from the preceding question from both a mathematical and cognitive perspective. I focused on describing these differences in terms of actions such as backing up to a simpler problem, continuing with the same type of problem in the same representational system, varying the representational system, varying the problem along one dimension (e.g., following the problem 4 - 7/8 with 4 1/8 - 7/8 varies along one dimension by changing the whole number to a mixed numeral), and digressing to a related concept.

The following example, which was taken from Matt's fifth tutoring session, illustrates how I analyzed my interactions with the third- and fourth-grade students. Parenthetical notes marked with a "*" describe how the question immediately preceding the notes varies from its preceding question.

<u>I</u>: Can you write four and two-thirds as an improper fraction?

<u>Matt</u>: (Wrote 14/3).

I: Can you write five minus two-thirds? What's that equal to?

* (vary 1-dimension mathematically - go from mixed numeral to whole number; apply concept)

Matt: (Very long pause). One-fifth, 'cause see I had three pieces and I took two away so I'd have one-fifth left.

I: (Turned over Matt's paper). Suppose you have five whole pepperoni pizzas and you eat two-thirds of one pizza, how much pepperoni pizza do you have left?

* (continue with the same problem but vary the form of representation - use real-world context; remove symbols)

Matt: Four pizzas and one piece of pizza. . . One-fourth, 'cause there's four pizzas and I had one piece left of the pizza that I ate before.

I: Okay, suppose this time you have one whole pepperoni pizza and you eat two-thirds of it, how much pepperoni pizza do you have left?

* (back up to a simpler problem; keep the same context)

Matt: One-third.

I: Now this time you have two whole pepperoni pizzas and you eat two-thirds of one pizza, how much do you have left?

* (vary 1-dimension cognitively and mathematically by going from one whole to multiple wholes; keep the same context)

I analyzed all transcripts from the third- and fourth-grade students in the manner illustrated above. From this analysis I validated that I was able to help student make connections when I presented them with problems that were closely related but more complex. More specifically, when I presented students with problems that varied along only one dimension from a mathematical or cognitive perspective, students were able to make connections. However, presenting students with problems that varied along two dimensions did not necessarily prevent them from making connections. The six times that I did present students with problems that varied along two dimensions, students made connections by drawing on their informal knowledge on their own or after I asked one or two guiding questions.

My analysis also provided insights into the instances where students failed to make connections I thought probable. I discovered that in all 12 situations, I had not guided students by encouraging them to write their own symbolic representations. I had been the one to write the representation (e.g., as seen in Todd's protocol above when I wrote 1 - 2/3 after Todd wrote 1 - 2/5). Thus, when students wrote their own representations, no matter how inappropriate they were, students were able to draw on their informal knowledge to give meaning to these representations and eventually recorded problems appropriately. However, when I wrote the symbolic representations, students' knowledge of formal symbols and procedures remained disconnected from their informal knowledge of fractions.

These results suggested that as I worked with students to further the development of their understanding of fractions, actions such as telling or showing students things they could construct for themselves were ineffective. These results further suggested that my actions needed to match the way in which I was attempting to promote the development of understanding. Therefore, as I encouraged students to draw on and extend their informal knowledge of fractions, it was critical that I guided students through questions and encouraged them in constructing all aspects of their knowledge, which included recording symbolic representations.

## Discussion

This study provides a fine-grained analysis of one expert tutor's tutorial actions as she attempted to teach fractions for understanding. The picture of tutoring that emerged from this analysis further illustrates the complexities involved in the tutoring process and helps advance a theory of tutoring. Schoenfeld et al. (1992) suggest that "skilled tutors. . . enter the tutoring interaction with a well-developed sense of what they would like to have transpire" (p. 316-7), but how interactions actually occur depends on a complex set of interrelationships between the tutor's goals and various aspects of the tutor's knowledge. The results of this study show that Mack approached each interaction with the idea of helping students build on their informal knowledge to give meaning to fraction symbols and procedures. The results also show that Mack drew on multiple aspects of her knowledge at the same time to determine appropriate directions for instruction. The knowledge Mack drew on and the manner in which she drew on this knowledge reflected major components of the theory of tutoring proposed by Schoenfeld et al. (1992). Thus, the results suggest that Mack's tutorial actions were guided by the interrelationships between her instructional goals, content knowledge, pedagogical content knowledge, and knowledge of students' thinking with respect to the content domain.

Mack's goals and knowledge did not operate in isolation of other factors as they guided her tutorial actions. They operated in conjunction with her beliefs about learning and teaching for understanding. While all three factors influenced the specifics of each interaction Mack had with a student, the influence of Mack's beliefs extended beyond specific interactions to the manner in which she approached the tutoring process as a whole. As Mack posed questions, probed students' thinking, and turned situations back to students, her actions suggested that she viewed learning and teaching for understanding as an interactive process involving both her and the students. Mack's actions further suggested that it was important to her that she work together with students in a specific way, as cooperative problems solvers focused on extending students' understanding. Thus, Mack's beliefs about learning and teaching for understanding provided a foundation for all her tutorial actions. These results support Davis' (1992b) proposition that teachers and tutors come to instruction with well-developed philosophies that underlie their instructional actions. The results also support Davis' proposition that theories of teaching and tutoring should address not only instructional goals and various aspects of teachers'/tutors' knowledge, but also teachers'/tutors philosophies if they are to provide deep insights into the complexities involved in the process of teaching/tutoring for understanding.

In addition to providing insights into the complexities involved in the process of tutoring, this study also helps advance theories related to the development of mathematical understanding. Pirie and Kieren (1994) characterize the development of mathematical understanding as a dynamic, leveled, but non-linear process of growth. Additionally, they propose that the development of understanding can be facilitated by instructional actions that match characteristics of the development of understanding. More specifically, Pirie and Kieren suggest that mathematical understanding can be promoted by instructional actions that cause students to reflect on and at times "fold back" to previous levels to extend existing levels of understanding. This study provides an existence proof of Pirie and Kieren's proposition related to facilitating the development of understanding. The results show that as Mack guided students in extending their knowledge, she led students non-linearly through a sequence of fraction topics by utilizing digressions and decompositions of problems. The results suggest that these actions were needed to effectively guide students in extending their knowledge. Presenting students with problems that were closely related but more complex was not sufficient for facilitating the development of their understanding. Thus, as Pirie and Kieren (1994) suggest, to facilitate the growth of mathematical understanding, instructional actions are needed that reflect critical characteristics of the development of individuals'understanding.

The analytic framework employed in this study demonstrated that "production rules" can provide an effective way to gain deep insights into the complexities involved in the process of tutoring. Additionally, Mack's actions related to utilizing production rules to deepen her own understanding of her teaching suggested that this analytic framework may be accessible to others engaged in the teaching/tutoring process to help them deepen their own pedagogical content knowledge. Accessibility of this analytic framework is particularly important at this time. A number of researchers (Ball, 1993; Fennema et al., 1993; Lampert, 1990) suggest that gaining insights into one's own teaching plays a critical role in struggling with issues related to teaching for understanding. Thus, production rules may prove valuable to teachers and tutors as they attempt to find effective ways to teach mathematics for understanding.

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