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ABSTRACT

This document presents a South Carolina statewide consensus of what children are expected to know and be able to do in mathematics and the changes necessary in the education system to support what teachers and students do every day in the classroom. Chapter titles are: (1) "A Vision for Change"; (2) "Learning and Teaching Mathematics"; (3) "K-12 Mathematics Curriculum"; (4) "Instructional Materials"; (5) "Assessment"; (6) "Professional Development of Teachers of Mathematics"; and (7) "Essential Support Systems," including administrators, school boards, parents, business and industry, elected officials, and the media. Appendices contain the K-12 mathematics curriculum standards by content strands and the South Carolina Chamber of Commerce Education study. (Contains 43 references.) (MKR)

ED 383 554

SOUTH CAROLINA

Mathematics

FRAMEWORK

SE056288

South Carolina Mathematics Framework

Developed by the
South Carolina Mathematics
Curriculum Framework Writing Team

Adopted by the
South Carolina State Board of Education
November 1993

Barbara S. Nielsen, Ed.D.
State Superintendent of Education
South Carolina Department of Education
Columbia, South Carolina

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Acknowledgments

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Foreword

Dear South Carolinians:

On November 10, 1993 the State Board of Education adopted the first three frameworks in Foreign Languages, Mathematics, and the Visual and Performing Arts to guide policy and practice throughout the state's education system. The learning standards outlined in the South Carolina Mathematics Framework are the result of over a year's discussion in which thousands of South Carolinians took part. After the dedicated teachers, higher education faculty members, and community members had written, reviewed and come to consensus about what we want students to know and be able to do in Mathematics, we proudly presented this framework to the State Board of Education for their adoption. Whether you are a student, parent, school staff, an administrator, a local business person, or a concerned community member, you can feel very excited about the accomplishments which this framework represents.

Now that we have a framework in place, the real work begins for the State Board of Education, the State Department of Education and all South Carolinians. That work is to carefully review the entire system of education against the recommendations in this framework and to propose and support changes in that system that can translate the South Carolina Mathematics framework into classroom practice. We believe that all students can learn at high levels. The three frameworks already adopted and those under development in Science, English Language Arts, Health and Safety, Social Studies, and Physical Education will serve as the guides to enable the system of education in our great state to deliver on that belief. We salute all of you for your involvement and dedication to that goal.

Sincerely,

Barbara Stock Nielsen, Ed.D.
State Superintendent of Education

Sincerely,

Samuel M. Greer, Ed.D.
Chairman, State Board of Education

Preface

The *South Carolina Mathematics Framework* presents a statewide consensus of what we expect children to know and be able to do in mathematics and the changes necessary in the education system to support what teachers and students do every day in the classroom. The framework challenges all of us to provide mathematics programs with greater focus on understanding essential mathematical ideas; more emphasis on investigation, analysis, and interpretation of real-life problems; and the realization that talking and writing about important mathematical concepts is as valuable as computational skills.

This framework is not a program or curriculum guide, but is intended to be used by policymakers, instructional leaders, teachers and communities as a broad instructional design for continuous improvement of the education system. The framework can serve as a common reference point to ensure that all components of the education system work together and reinforce the same vision of instructional excellence for **all** students in our classrooms.

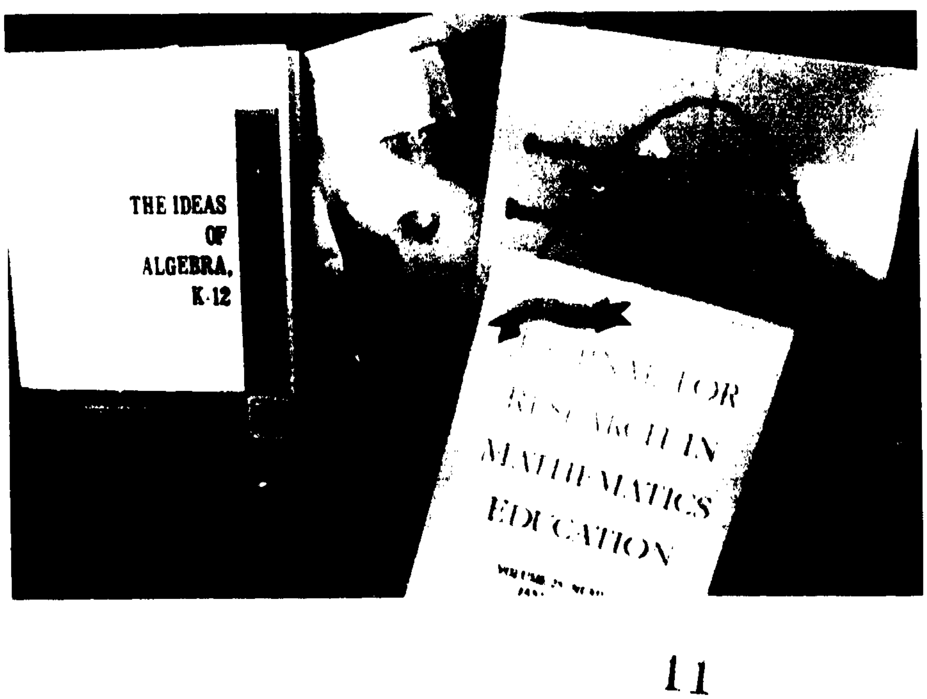
The planning for this framework, which was shaped over two years, began with the appointment of a writing team of teachers representing all grade levels, administrators, and post-secondary faculty who have either written, taught, or lectured in the discipline area. This team made fundamental decisions regarding the basic tenets for mathematics, student performance standards, how students learn and different ways to teach, instructional materials, and what parts of the system must change to support this vision. The team drew from the expertise and reports provided by the South Carolina Curriculum Congress and existing national and state documents including the *Education Study: "What Work Requires of Schools"* from the South Carolina Chamber of Commerce. The initial draft framework was distributed to districts, schools, county libraries, members of the business community, parents, and colleges and universities for extensive public review and comment. Final revisions to the document were made based on the results of the field review process and the framework was submitted to the State Board of Education for adoption.

Following State Board adoption, the frameworks will guide the State Department of Education and others responsible for the quality of mathematics education in South Carolina to pursue the policy and program changes advocated in the framework. No policy or program changes are automatically in effect as the result of the adoption of this framework. Many of the types of changes recommended in this framework will require formal approval of regulatory or statutory changes or are a matter of local authority requiring action or approval of school districts or local school boards. We urge all who have a stake in South Carolina education to use it in shaping their policies, programs, planning, budgets, and personnel decisions.

The changes outlined in the framework will take time. All the instructional goals cannot be met overnight, within a few months, or even a few years. These changes will require thoughtful discussion and the necessary support in place, such as the provision of professional development and instructional materials that support the type of instruction program that schools and communities want for their children.

The planning document *Using Curriculum Frameworks for Systemic Reform* provides a State Department of Education response to support this framework and represents a point of departure for statewide discussion and joint planning. Policy changes at the state level may include instructional materials selection, assessment, school accreditation, teacher certification/recertification, and professional development.

This framework appropriately and accurately describes the mathematics programs that should be established in our schools. We know what is required. Now we must move forward, each responding at our own pace, but all moving in concert to make the promise of this framework a reality for all students.



Prologue

What Is a Mathematics Framework?

This Mathematics Framework presents the essential components necessary for improving mathematics education in South Carolina. It is a visionary document designed to provide the guidance needed to ensure that *every* student in South Carolina has the opportunity to receive the best possible education in mathematics.

This framework communicates the core understandings in mathematics that every student is expected to learn. *It does not contain the specific, detailed curriculum that is actually taught in each school. Rather, it sets out broad curricular themes, topics, and objectives in multi-year blocks.* It communicates the spirit, not the specifics of the mathematics curriculum. It communicates the core understandings, attitudes, and a way of thinking about mathematics that are essential for effective mathematics instruction. However, it is left to districts, schools, and classroom teachers, and parents to determine how these curriculum standards can best be attained for the students.

This particular Mathematics Framework includes the following key components:

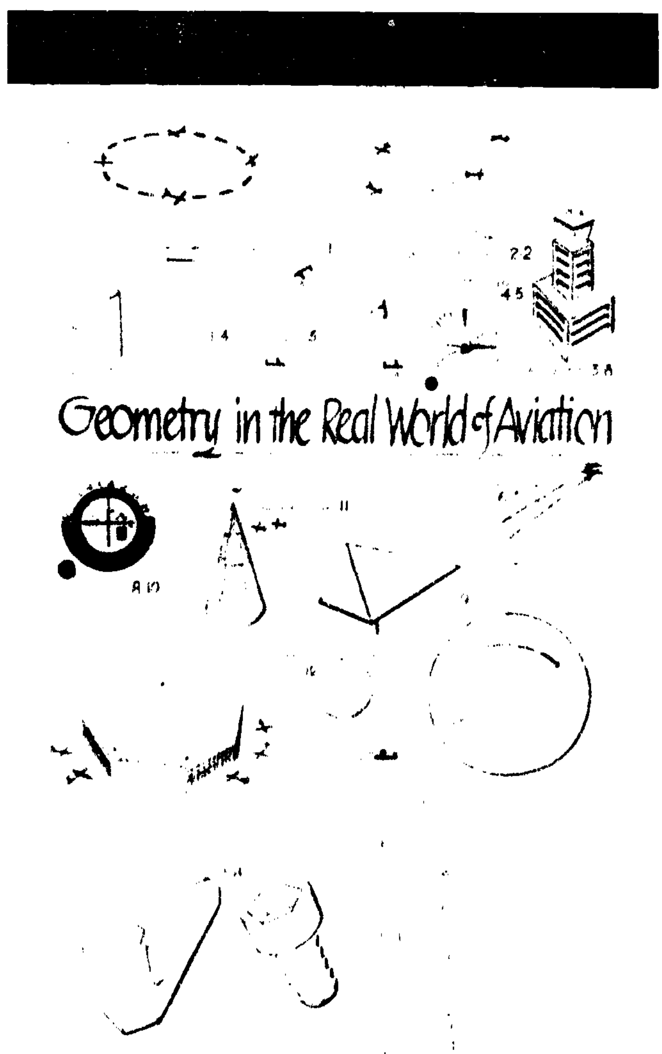
- the vision for mathematics education in South Carolina;
- how students learn mathematics and how teachers can teach mathematics effectively;
- the core curriculum in mathematics for grades K-12, with sample activities and problems that can be used to implement this framework;
- criteria for the selection of instructional materials;
- assessment of teaching and learning of mathematics;
- the professional development of teachers of mathematics; and
- essential support systems for translating the curriculum framework into classroom, school, and district practices.

This framework communicates clear expectations for all South Carolina students and programs. It is the basis from which state and local educators can obtain guidance and support in providing the best human and material resources possible for each student and school.

With a common set of content standards and a shared vision, this framework guides the development of state policies and programs concerning

- the adoption of instructional materials,
- the preparation of teachers and administrators,
- professional certification requirements,
- professional development opportunities, and
- student assessment.

The unity achieved through this framework provides the opportunity to create a strong system of high quality mathematics education throughout the state of South Carolina.



1: *A Vision For Change*

A Rationale for Change

If South Carolina students are to function effectively as adults in today's society, they must know and be able to use mathematics in both their personal lives and their professional lives. When they leave school today, they must be mathematically literate. To ensure that they have the opportunity to develop the mathematical literacy they need, our present system of mathematics education must be strengthened and improved. The reform of that system is the goal of this Mathematics Framework.

Mathematics in Everyday Living

An educated electorate is the foundation for a sound democracy. Our citizens must have the mathematical knowledge and the analytical skills necessary to make informed decisions about our political, social, and physical environments. Interpreting masses of data that appear in the media, understanding the financial costs of government actions and policies, or judging the quality and effectiveness of health care options require us to know and understand more mathematics than in the past.

Educated consumers are important for a healthy economy. Managing personal finances effectively requires mathematical knowledge to make informed decisions about spending, borrowing, saving, and investing. Mathematical illiteracy in financial matters can seriously affect the quality of our everyday living.

Mathematics is a universal part of human culture. This legacy should be part of the general education of everyone. Abstract concepts from arithmetic, geometry, algebra, statistics, and analysis have been developed over several thousand years by many races and cultures. These ideas need to be learned by each generation in order to pass this mathematical legacy on to future generations.

Society's Perception of Mathematics

Society has many misconceptions about mathematics and its role in our world. Mathematics is incorrectly viewed as a collection of rigid rules and procedures which are unrelated to each other and which are to be learned without understanding. Consequently, mathematics is perceived by many to be difficult and demanding and is considered to be a subject in which it is socially acceptable to do poorly. Society underestimates the pervasive role of mathematics in both the world of work and the world of everyday living. Many people consider school mathematics to be irrelevant, unnecessary, and unrelated to the mathematics they encounter in their professional and personal lives. These false perceptions and unfortunate attitudes about mathematics have a significant, negative impact on mathematics education. *These perceptions must be changed.*

Mathematics in the World of Work

Our economy has changed from an industrial base to an information base, with technology now playing a dominant role. Calculators are everywhere in the workplace, and computers are being used in ways undreamed of several decades ago. The fastest growing kinds of jobs require more analytical skills, not more mechanical skills. These changes are affecting not only what mathematics is important in the world of work but also how mathematics is used and applied there.

The kinds of mathematics needed in today's jobs are different from the kinds of mathematics needed in yesterday's jobs. For example, a technician in the semiconductor industry needs to understand algebra and have access to a computer to solve this problem:

> **The Semiconductor Industry:** You are using a Type J thermocouple to measure the temperature of a VLSI chip under a thermal stress test. You record a voltage of 0.025 volt. Use the calibration formula shown below to compute the temperature indicated by the thermocouple.
>
> $$T = -0.48868252 + 19{,}873.14503V - 218{,}614.5353V^2 + 11{,}569{,}199.78V^3 - 264{,}917{,}531.4V^4 + 2{,}018{,}441.314V^5$$
>
> where T is the temperature of the thermocouple in C°, and V is the thermocouple voltage. (Reprinted with permission from *Applied Mathematics*, Unit 14 Teacher's Guide, 1988, p. 51)

An engineer in the aerospace industry needs to have a good understanding of probability to set up and solve this problem:

The Aerospace Industry: An aerospace consulting company is working on the design of a spacecraft system composed of three main subsystems, A, B, and C. The reliability, or probability of success, of each subsystem after three periods of operation is displayed in the following table:

	1 day	**3.3 mos.**	**8.5 mos.**
A	0.9997	0.8985	0.6910
B	1.0000	0.9386	0.7265
C	0.9961	0.9960	0.9959

These reliabilities have been rounded to four significant digits. The 1.0000 in the first column means that the likelihood of the failure of subsystem B during the first day of operation is so remote that more than four significant digits are needed to indicate it.

(a) Consider the case of the series system shown in *Figure 1.1*. If any one (or more) of the subsystems A, B, or C fails, the entire system will fail. If P_s is the total probability of success of this system, find P_s for each of the three time periods.

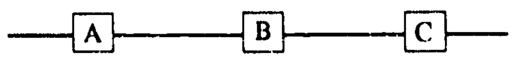


Figure 1.1

(b) The system shown in *Figure 1.2* will succeed if B succeeds and at least one of A or C succeeds. Find the probability of success for this system for the 3.3-month time period.

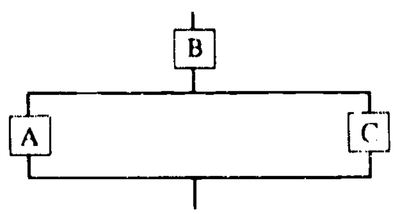


Figure 1.2

(Reprinted with permission from Kastner, *Space Mathematics*, 1985, pp. 75-76)

As the mathematics needed in the workplace is changing, our economic survival as a global power is being threatened. It is essential that we become more competitive internationally. The present levels of mathematics achievement in our schools, however, are not sufficient to sustain our nation's leadership in a global society that is information based and technology driven. A work force that is well educated in mathematics is necessary for our survival. Our students need to learn the mathematics that technology in their jobs will require. At the same time, their mathematics education must be broad enough to qualify them for tomorrow's jobs as well as today's jobs. Our students must be prepared to become both learners and users of mathematics during the rest of their lives.

Changing demographics are affecting the quality of our work force. The number of graduates of U.S. citizens in mathematics at both graduate and undergraduate levels has been declining. American universities and colleges grant more doctorates in mathematics to foreign students each year than to U.S. citizens. Furthermore, predictions indicate that today's minorities will become tomorrow's majorities. But our present system of mathematics education filters out many minorities and women from the study of mathematics, reducing our pool of workers with good mathematics training. Having the majority of our population mathematically illiterate is the waste of a valuable natural resource that we cannot afford. Equal opportunity *for everyone* in a high quality mathematics education is an economic necessity.

Business and industry have a vital stake in the mathematics education of our students. The South Carolina business community recently voiced its expectations for the graduates of our school system in a survey of members of the South Carolina Chamber of Commerce. This survey was designed to identify those skills and competencies high school graduates in South Carolina need in order to be successful in the workplace. Many of the workplace skills and competencies that emerged from the survey are related to the learning of mathematics; for example:

- Thinking skills: being able to identify and weigh all options and choose the best alternative; recognize problems and develop plans of action to address them; conceptualize and process information.
- Information: being able to interpret and communicate information to others; organize information in a way that suits their needs; use computers to process information.

- Interpersonal: being able to be a good team player; appreciate and work well with men and women from diverse backgrounds; teach others new skills.
- Basic skills: being able to communicate thoughts, ideas, information, and messages in writing; perform basic mathematics and apply it to everyday situations in the workplace; organize ideas effectively and communicate orally; locate, understand, and interpret written information in a wide array of documents, graphs, etc.
- Technology: being able to select and use appropriate procedures, tools, or equipment, including computers software, and related technology.

The complete list of all 37 skills and competencies appears in Appendix B.

The Current State of Mathematics Education

Our present system of mathematics education is not producing the level of mathematical literacy that our society needs. According to data collected by the National Assessment of Educational Progress during the 1985-86 school year, only about 51% of 17-year-olds can adequately handle decimals, fractions, percents, and elementary algebra. Only about 6% know enough about algebra and geometry for advanced study in mathematics (Lindquist, 1989, pp. 118-19). On the other hand, the average Japanese high school student (50th percentile) knows more mathematics than the average gifted and talented American high school student (*The Underachieving Curriculum*, 1987, p. vii). On South Carolina's Basic Skills Assessment Program Mathematics Exit Exam administered to tenth-graders in the spring of 1991:

- only 53.2% could determine the area of a triangle, using customary (English) units;
- only 47.6% could multiply units of length with conversion; and
- only 45.4% could solve a problem involving multiple operations, time, and extraneous information.

(State Level Basic Skills Assessment Program Reports - Spring 1991, Volume III)

Although statistics such as these are discouraging, the records of mathematics achievement in many other countries indicate that most of our students should be able to learn much more mathematics than our society commonly expects them to learn.

Our mathematics curriculum is dated. It fails to integrate mathematics with other disciplines and does not adequately reflect recent advances in mathematics and the impact of computers and calculators on mathematics. Our method of instruction does not incorporate our new understandings of how students learn mathematics, and our methods of assessment are too narrow and do not measure accurately the breadth and depth of the teaching and learning of mathematics. Describing our present state of mathematics education, John Dossey says:

> We have drifted into a curriculum by default, a curriculum of minimum expectations that resists the changes needed to keep pace with the demands of preparing students for contemporary life. (*Everybody Counts*, 1989, p. 74)

Research findings document that our system of mathematics education has serious problems. Our students can perform basic skills fairly well, but they do not do well on thinking and reasoning. Our mathematics textbooks fail to give adequate attention to major ideas or to pose challenging problems. Many teachers teach only for factual knowledge, not for understanding, and avoid involving students in thought-provoking work and activities. Our system of mathematics education thus provides little intellectually stimulating work and tends to produce students who are not capable of intellectual work (Kennedy, 1991, p. 660). Our teachers are highly likely to teach the way they themselves were taught.

> We are caught in a vicious circle of mediocre practice modeled after mediocre practice, of trivialized knowledge begetting trivialized knowledge. Unless we find a way out of this circle, we will continue re-creating generations of teachers who re-create generations of students who are not prepared for the technological society we are becoming. (Kennedy, 1991, p. 662)

There is, unquestionably, an urgent need to restructure our system of mathematics education. In short, we must *completely redesign* school mathematics – what mathematics is taught, how it is taught, and how it is assessed.

Conditions are currently ripe for restructuring the system of mathematics education in South Carolina. Many teachers are aware of this need and are preparing themselves for change by participating in workshops, by taking university courses, by attending professional mathematics education meetings at local, state, and national levels, and by taking advantage of staff development opportunities in their schools and districts. The reform of mathematics education in South Carolina already has a strong base to build upon.

The Vision for Mathematics Education

Every student in South Carolina must learn the mathematics necessary to experience a successful life. South Carolina schools advocate mathematical literacy as a lifelong necessity for an informed citizenry. Mathematical literacy enables an individual to use exploration, conjecture, logical reasoning, and a variety of mathematical techniques to solve problems effectively. South Carolina students must also learn to value mathematics and become confident in their use of mathematics. The combination of all of these components define "mathematical power."

Students must leave school with a solid foundation in mathematics that will enable them to use reasoning in order to improve the decisions that affect their lives and to become more productive members of society. To this end, schools must focus on mathematics as a means of communication and as a tool for the discovery and exploration of ideas. Mathematics instruction must emphasize problem solving and the interrelatedness of mathematical ideas rather than a series of isolated skills to be mastered independently. Therefore, a wide variety of mathematical materials, resources, and experiences, including the use of technology, must be available to enable students to develop their full potential and to become mathematically powerful.

Mathematics equity for all students, regardless of their race, gender, ethnic background, or previous success in mathematics, requires excellence for all and high expectations for all. Accordingly, all students in South Carolina must experience a common core of relevant mathematics. They must learn mathematics through an active, constructive approach that emphasizes understanding mathematics. They must be assessed in a manner that is consistent with what mathematics is being taught and how that mathematics is being taught. To ensure that all students have ample opportunity to learn this common core of mathematics, they must study mathematics every year they are in school.

This vision of high quality mathematics education for every student in South Carolina must be embraced by the entire community. Students, teachers, administrators, parents, other citizens, public policy makers, and the business community must unite as partners in order to make this vision become a reality.

Mathematical Power

The ultimate goal of mathematics education in South Carolina schools is the development of broad-based *mathematical power* for all students. Mathematical power is the ability to use mathematics effectively to solve real-world problems. An important factor in the acquisition of mathematical power is the development of personal self-confidence in, and appreciation for, mathematics. Mathematically powerful students can think, they can communicate, they can draw on mathematical ideas, and they can use mathematical tools and techniques. When students develop mathematical power during their school years, they will have the mathematical knowledge and skills to pursue the profession or vocation of their choice and to undertake further study of subjects that require proficiency in mathematics.

Learning Mathematics

In recent years, cognitive scientists and psychologists have been studying how children learn mathematics. Their research supports the work of Piaget and others from earlier decades:

> Rather than being passive absorbers of knowledge, children actively create their own understanding of the world. In fact, by the time they come to school, they have already developed a rich body of knowledge about the world around them — including well-developed, informal systems of mathematics (*Mathematics Framework for California Public Schools*, August 1991, p. 28).

Learning, then, occurs as children actively assimilate new information and experiences and construct their own meanings (NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 2). Accordingly, a fundamental, unifying principle for how students should learn mathematics is that they be *actively involved* in learning mathematics. As learners, students need to construct mathematical knowledge for themselves and to make sense of their own experiences. They should re-invent knowledge for themselves and actively interpret mathematical aspects of the world around them. To quote Piaget, "To understand is to invent" (*Reshaping School Mathematics*, 1990, p. 29).

All students should be actively involved in the learning process. They should learn mathematics under conditions like the following:

• **Students should learn mathematics with understanding.** Mathematics should be a discipline that helps them to make sense of things, not a discipline that is arbitrary and devoid of meaning.

• **Students should learn mathematics in familiar, realistic contexts having connections to other mathematics or to other disciplines.** The problems, examples, and activities that they study should exploit the connections between mathematics and the world outside the classroom. Students are especially responsive to situations that connect their personal lives to classroom activities in mathematics.

• **Students should learn mathematics in a sequence of activities proceeding from concrete** (actual objects) **to semi-concrete** (pictures) **to abstract** (symbols). To develop understanding at the concrete level, students at all levels need to use appropriate manipulative materials. Elementary school students, for example, can use base ten blocks to develop an understanding of face value and place value in numeration, while secondary school students can use algebra tiles to learn to multiply two binomials or to factor a trinomial.

• **Students should use appropriate calculators or computers in learning new mathematics as well as in doing mathematics.** Calculators or computers help students learn new concepts in mathematics by helping them look for patterns, explore and conjecture, or represent situations graphically. They help students do mathematics when they are used to calculate, to graph, or to simulate processes that are too tedious or time-consuming to do by other means. Calculators or computers can allow students to get past barriers like inadequate computational facility and thus permit all students to have access to good, challenging mathematical problems.

• **Students should talk and write about the mathematics they are learning as a means of strengthening their understanding of mathematics.** They must learn to communicate mathematically. "As they communicate their ideas, they learn to clarify, refine, and consolidate their thinking" (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 6).

• **Students should work in groups of various sizes, including small groups,** where they pool their knowledge and understanding of mathematics and capitalize on the strengths of others. Cooperative learning groups generate interest in mathematics, foster individual accountability, promote higher achievement, develop social skills, and involve all students actively in learning.

• **Students should take responsibility for their own learning and understanding of mathematics.** They must learn to question, to probe, to create, and to decide what to do. As they take responsibility for their own learning and understanding, they begin to develop mathematical power.

• **Students can learn mathematics more effectively if practice is distributed over an extended period of time.** They need to return to the same concept periodically in the same or different contexts in order to deepen and broaden their understanding of it.

• **Students can study topics in mathematics out of their traditional sequence,** a view supported by current research from cognitive science. Mastery of mathematics presents a complex picture. "Contrary to much present practice, it is generally most effective to engage students in meaningful, complex activities focusing on conceptual issues rather than to establish all building blocks at one level before going on to the next level" (*Reshaping School Mathematics*, 1990, p. 30).

How should students learn mathematics? The following verbs clearly indicate the spirit in which students should learn mathematics: explore, justify, clarify, create, represent, convince, construct, discuss, validate, investigate, describe, predict, simulate, verify, interpret, estimate, explain, model, classify, organize, gener[illegible]ize, reflect, analyze, apply, evaluate, prove, translate.

Teaching Mathematics

If students are expected to learn mathematics by constructing, understanding, and interpreting through active involvement, then teachers should teach mathematics with this *active, constructive* view of learning in mind.

> In reality, no one can *teach* mathematics. [illegible] who can stimulate students to *learn* mathematics. [illegible] search offers compelling evidence that student[illegible] well only when they *construct* their own mathematical [illegible] To understand what they learn, they must [illegible] that permeate the mathematics curriculum [illegible] "transform," "solve," "apply," "prove," [illegible] most readily when students work in [illegible] make presentations, and in other ways [illegible] ing (*Everybody Counts*, 1989, pp. [illegible])

No single method of instruction, no single kind of learning activity, can develop all of the different aspects of mathematical power that teachers want students to acquire. *What is needed is a variety of instructional activities,* including opportunities for

- appropriate project work;
- group and indiv[illegible]dual assignments;
- discussion between teacher and students and among students;
- practice on mathematical methods; and

- exposition by the teacher (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 10).

The following lesson and unit, each developed by a South Carolina teacher, illustrate how mathematics *could* be taught today.

Lesson:
Is Seeing Really Believing?

The teacher begins the class by posing the question, "Is seeing really believing?" The teacher then introduces the lesson with examples and questions so that the students gain the information necessary for comparing and ordering fractions. "I have a snack that is 9/16 of an ounce. Do you think it has more in it than the snack with 1/2 of an ounce because of its bigger numbers?" Students are working cooperatively in groups of four. Four different snack products are passed out to each group. By using their rulers, the students discover which product package is the largest or smallest in size. The recorder writes down their findings. Then conjectures are made about the amount of each product's contents.

After reading the labels and finding their fractional weights, the students make comparisons. They convert the fractional weights into decimals with the use of their calculators. The groups are using fraction bars and pattern blocks as they observe relationships and make discoveries. During this time the teacher is a facilitator and is going around observing each group and posing thought-provoking questions. Using labels with the different fractional weights on them, the students form a number line while they justify their findings.

The students use the nutritional information on the back of the packages to formulate real-life situational problems to be discussed and answered. "If I were concerned about my weight, what else besides the caloric value might I look for? What is the relationship between the calories and the fat content?"

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The students are continuously using their reasoning skills to estimate the amount in the bags. They take a poll of the class of their favorite snack product and graph their findings.

At the end of the final day they reflect on all they have done and write about their experience in their math logs.

Diane G. Boyd
Kingstree Elementary School, Grade 5
Kingstree, South Carolina

Unit:
Building a Scale Model of Moultrie Middle School

I. Purpose of the Unit: This unit is designed to be the culmination of a year's study of mathematics. While its primary emphasis is geometry, it is constructed to integrate computation, spatial sense, art, business, and the language arts.

II. Objectives of the Unit:

A. To sketch all elevations of the school building.
B. To estimate and measure the dimensions of all sides of the school building.
C. To determine the most cost effective materials and method for construction of the model.
D. To foster inter- and intra-group cooperation in creating a single model.
E. To formulate a persuasive presentation to explain the construction of the model.

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F. To encourage communication in and about mathematics.
G. To promote an appreciation of the utility and beauty of mathematics.
H. To construct a scale model of the school.

III. Activities:

A. The Planning Phase (5-7 class periods)

1. Discussion: In this initial phase students discuss in their groups and/or as a class the following questions:

- What cooperative skills are necessary for the group to work together?
- What steps are necessary to create a scale model?
- Why are scale models useful? What are some properties of scale models?
- What materials could be used to construct this model?
- Should customary (English) or metric units be used to measure the building? What are the benefits of each? What are the drawbacks? Which unit? What scale should be used?

2. Sketching and estimating: Each student sketches the side of the building assigned to his or her group and estimates the dimensions of the sides and openings in the unit of measure decided on by the class. The group then chooses a representative sketch of its work to submit.

3. Determining the height of the school: Students use similar triangles to determine the height of their assigned side of the school.

4. Drawing the school to scale: Each group draws its assigned side of the school to the scale determined by the class.

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B. Cost Analysis Phase (2-3 class periods)

1. Calculation of the surface area of the exterior of the building: Each group determines the number of square units of material needed to construct its side of the model. Collecting data from the other groups, they determine the total surface area of the building. The class then verifies these figures through comparison and discussion of each group's work.

2. Determination of the most cost effective materials to use to build the model: Students determine the most economical materials and supplies needed to complete their model of the school. The choice of materials is not only based on the cost but also on the conformity of materials to specifications agreed upon by the class.

C. Construction Phase (3-4 class periods)

1. Building the model: Each group constructs its assigned elevation of the school building. Then a team made up of one member from each group assembles these components into the model of the building.

2. Creating the scenery: While the construction team assembles the model, another team made up of one member of each group creates a mounting board and scenery for the model.

3. Planning the presentation: A third team made up of the final member of each group plans, writes, and prepares the necessary materials for the presentation of the model to the panel of judges.

4. The presentation: After a rehearsal in front of the class to gain suggestions and input, the members of the "sales team" make their presentation to the judges.

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IV. Classroom Organization: Cooperative learning groups are used for the major portion of this unit. Whole class and small group lecture and discussion are utilized to teach the mathematics skills necessary to complete each task, to process information, and to teach group process skills.

A. *Cooperative learning groups are teacher selected to insure heterogeneous grouping.* Students are rank-ordered based on prior performance in class, and the list is then divided into thirds. One high, one middle, and one low achieving student comprise each group. Group members are randomly assigned jobs appropriate to the task being undertaken (ex. Quality Control Officer, Draftsman, Surveyor, etc.). Jobs are well-defined to ensure equal participation among group members.

B. *Whole class interaction is used* to brainstorm ideas, to teach the skills necessary for the daily lesson, and to reach consensus on materials and procedures for developing the model.

C. *Small groups are used* to work on the various components of the final presentation.

V. Assessment Procedures: Students are assessed in this unit using a variety of techniques.

A. *Learning Log:* Each day a student enters at least a paragraph into his or her learning log to answer the question posed in the lesson or to summarize the learning for the day. Each entry in the log is scored using this rubric:

3 -- exceptional performance
2 -- acceptable entry
1 -- inadequate entry lacking substance
0 -- no entry

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B. *Project Book:* The sketches, calculations, and notes in this notebook are scored using the same rubric.

C. *Group Performance:* Students are assessed on a class observation sheet. Both individual and group performance is noted.

D. *Product Assessment:* Two mathematics classes are given the task of acting as competing companies to create a model of the school building. Upon completion of the model, a select group of students from each class presents its model and the mathematics behind it to a panel of three judges. Judges base their selection on accuracy and appearance of the model and on the accuracy and eloquence of the presentation.

Christine W. Pateracki
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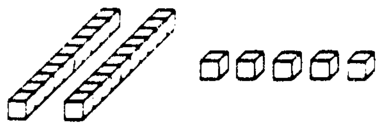
As the above lesson and unit illustrate, good mathematics teaching requires a variety of teaching practices.

Use a variety of activities that address different learning styles: auditory, visual, kinesthetic, convergent, divergent.

Structure learning activities to proceed from concrete (actual objects) **to semi-concrete** (pictures) **to abstract** (symbols). Take special care to bridge the transitions from one level to another so that students understand the connections. For example, students may use base ten blocks to learn numeration. At the concrete level they represent twenty-five by using two longs and five units. At the semi-concrete level they represent twenty-five by drawing pictures of two longs and five units. To bridge the transition to the abstract level and the numeral 25, students can draw pictures of the blocks as well as write the mathematical symbols to make the connection between the levels.



2 tens and 5 ones = 25.

Make effective use of appropriate manipulative materials at all grade levels with all students. Manipulative materials are essential for learning activities at the concrete level. Research on using manipulative materials shows that lessons using manipulative materials have a higher probability of producing greater mathematics achievement than do lessons in which such materials are not used. Concrete experiences with manipulative materials provide the means to abstract more complex ideas. The use of effective manipulative materials make lessons in mathematics come alive.

Use current technology in teaching mathematics. Calculators and computers are important aids to teaching and learning. Technology, used appropriately, can open the doors for all students to the learning of mathematics.

Elementary school students can use four-function calculators to develop their understanding of the multiplication algorithm for whole numbers with problems like the following:

$$\begin{array}{r} \square\ \square\ \square \\ \times\ \ \ \ \ \square \\ \hline \end{array}$$

Place the digits 4, 6, 7, 9 in the boxes to make the smallest possible product and to make the largest possible product. Use a calculator to help do this.

Middle school students can use computers and geometry-oriented software to study the properties of geometric shapes such as regular polygons, equilateral triangles, squares, regular pentagons, and regular hexagons.

Secondary school students can use graphing calculators to study the effects of the constants a, b, and c on the behavior of the function $f(x) = ax^2 + bx + c$.

Students should use calculators and computers throughout their school work. More important, they must learn when to and when not to use them.

> Contrary to the fears of many, the availability of calculators and computers has expanded students' capability of performing calculations. There is no evidence to suggest that the availability of calculators makes students dependent on them for simple calculations. Students should be able to decide when they need to calculate and whether they require an exact or approximate answer. They should be able to select and use the most appropriate tool. Students should have a balanced approach to calculation, be able to choose appropriate procedures, find answers, and judge the validity of those answers (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 8).

Teach mathematics for understanding. Develop mathematical reasoning and higher order thinking skills. Create a classroom climate where mathematics makes sense to students and where mathematical truth is logical. Learning to understand and to make clear, logical mathematical arguments is a goal of good mathematics instruction at all grade levels.

> Making conjectures, gathering evidence, and building an argument to support such notions are fundamental to doing mathematics. In fact, a demonstration of good reasoning should be rewarded even more than students' ability to find correct answers (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 6).

Develop concepts in context. Use situations that are familiar to students or that interest them. Use real-life applications in problem-solving to develop new ideas in mathematics.

For example, the concept of division of whole numbers can be introduced by posing a problem like the following:

> *The twenty-eight children in Mr. O'Keefe's class are going to the Riverbanks Zoo. If four children can ride in a car, how many cars are needed to take the class to the zoo?*

The difference between area and perimeter of geometric figures can be explored by posing a problem like the following:

> *Arrange sixteen card tables into different kinds of rectangular banquet tables. If one person sits at a side of a card table, what is the largest number of people that can be seated at a banquet table? The smallest number?*

Make connections with the mathematics being taught to other mathematics or to other disciplines. Students must come to view mathematics as an integrated whole rather than a set of isolated, unrelated topics and to recognize the usefulness and relevance of mathematics both in school and out of school.

[illegible]

Develop communication skills in mathematics – both oral and written. Students should learn to use the language of mathematics, both its symbols and its special vocabulary. They must have opportunities to read, to write, and to talk about the mathematics they are learning. When students talk and write frequently about mathematics, they are required to reflect on what they know and to organize and clarify their own thoughts. They learn to communicate effectively their ideas, understandings, and results with others.

The following activity is a good example of a problem that incorporates writing, problem-solving, technology, and number sense:

> *Use a calculator to find three different numbers whose product is 7429. How many different answers can you find? Write a paragraph explaining what you did, why you did it, and how well it worked* (Reprinted with permission from *Reshaping School Mathematics,* 1990, p. 19).

Use cooperative learning groups. Students need to learn to work collaboratively as well as independently. Collaboration is an essential part of social life; it is indispensable in the workplace; and most real mathematical work generally involves collaboration. When students confront mathematics in small cooperative learning groups, they compare alternative approaches, test different conjectures, share insights, express their own thoughts, state their own arguments, and listen carefully to others. Cooperative learning experiences in mathematics foster improved attitudes toward mathematics and build confidence in the ability to do mathematics.

Integrate assessment with teaching. Use assessment as a teaching tool. Methods of assessment must extend beyond traditional teacher-constructed tests and standardized tests to determine what students know and how they think. Some kinds of assessment, embedded in instruction, can monitor students' understanding as it evolves over time. For example, students can

> be encouraged to keep a mathematics journal. These journals can contain goals, discoveries, thoughts, and observations, as well as descriptions of activities. Journals not only allow students to chart their progress in understanding but also act as a focus for discussion between student and teacher, thereby fostering communication about mathematics itself. (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 197).

Reading students' mathematics journals enables the teacher to assess informally the progress that students are making in learning mathematics.

To summarize, good mathematics teaching at any grade level

- uses a variety of activities over time that address different learning styles;
- structures learning activities to proceed from concrete to semi-concrete to abstract;
- makes effective use of appropriate manipulative materials at all grade levels;
- uses current technology;
- teaches mathematics for understanding;
- develops concepts in context;
- makes connections with the mathematics being taught to other mathematics or to other disciplines;
- develops communication skills in mathematics – both oral and written;
- uses groups of various sizes; and
- integrates assessment with teaching.

An effective teacher in today's mathematics classroom plays many roles. No longer is the teacher simply a dispenser of knowledge.

> The most useful metaphor for describing the modern teacher is that of an intellectual coach. At various times this will require that the teacher be
>
> - A *role model* who demonstrates not just multiple paths to a solution but also the false starts and higher-order thinking skills that lead to the solutions of problems.
> - A *consultant* who helps individuals, small groups, or the whole class to decide if their work is keeping 'on track' and making reasonable progress.
> - A *moderator* who poses questions to consider but leaves much of the decision making to the class.
> - An *interlocutor* who supports students during class presentations, encouraging them to reflect on their activities and to explore mathematics on their own.
> - A *questioner* who challenges students to make sure that what they are doing is reasonable and purposeful and who ensures that students can defend their conclusions (*Counting on You*, 1991, pp. 13-14.).

Creating an Effective Learning Environment

In order to create a learning environment that will effectively support teaching and learning as envisioned by this Framework, careful attention must be given to the physical, supportive, and enabling environments of the mathematics classroom.

Physical Environment

The classroom should be supplied with a wide variety of teaching materials. In teaching mathematics, no single textbook can suffice for all instructional needs. Teachers need collections of supplementary materials to serve as resources. Such materials also help teachers meet the needs of all students, from those who need remediation to those who need enrichment. The classroom should be equipped with appropriate sets of manipulative materials and tools. All students should have calculators appropriate to their grade level, and each classroom should have at least one computer and software available at all times for both teacher demonstrations and student use. The school should have additional computers available for individual, small-group, and whole-class use as well as audio-visual equipment and materials - available from a well-stocked library media center.

The classroom itself should be bright, cheerful, well-lighted, and attractive. It should be furnished with either tables and chairs or flat-top desks to accommodate groups of different sizes, manipulatives, calculators, and printed materials. The room should be designed for the flexible use of furniture so that it can be arranged for different kinds of instruction, including cooperative learning groups, laboratory activities, learning centers, and project work. Ample space or room for movement is essential. The classroom needs to have convenient storage space for manipulative materials and supplies, for supplementary printed materials, and for calculators and computers.

Time is another critical factor in creating an effective learning environment. Teachers need more time to enable them to plan and develop the stimulating lessons that this framework envisions. Such lessons do not come ready-made in today's textbooks. Students need to have adequate time, free from interruptions, to learn from these lessons. Some lessons, for example, may be long-range activities or projects that may take several days or weeks to complete. Changing the present time structure of today's school may be necessary in order to provide the time needed for teachers to teach and students to learn.

To teach the mathematics of this framework, optimal student/teacher ratios, effective faculty configurations, and appropriate instructional time frames must be created. High student/teacher ratios make the use of active, constructive learning – so critical to the teaching of mathematics – impossible. Large classes supervised by one teacher cannot support the vision of mathematics education promoted by this framework. Furthermore, learning activities in which students construct mathematical ideas often require a time frame of more than 40 to 50 minutes. Consequently, it is imperative that scheduling be guided by the needs of students and what they are to learn rather than by arbitrary, inflexible time frames. Finally, schools must be cautious not to be locked into the traditional faculty configuration of one teacher responsible for one class of students. Alternative configurations are possible. For example, an experienced teacher working as the leader of other teachers with interns and/or paraprofessional educators *can* effectively teach large groups of students. This configuration further encourages a career-ladder approach for the teaching profession that rewards dedicated, accomplished teachers.

Supportive Environment

Creating a supportive environment is critical for the teaching and learning of mathematics. A supportive mathematics classroom atmosphere has characteristics like the following:

- Students feel free to take risks and feel free to make mistakes and to learn from their mistakes.
- The teacher believes that *all* students *can* learn mathematics. The teacher builds confidence in *all* students that they can learn mathematics.
- The teacher creates opportunities for all students to experience success.
- Students develop a positive disposition to do mathematics. They approach mathematics with persistence, confidence, self-reliance, flexibility, curiosity, inventiveness, and enthusiasm.

Enabling Environment

An enabling environment is one in which teachers stimulate the learning of mathematics by exhibiting characteristics of good teaching. They

- encourage all students to explore, conjecture, and invent for themselves;
- have students verbalize and write about their mathematical ideas and understandings;
- ask mathematical questions having more than one right answer and pose problems having more than one method of solution;
- encourage students to take risks in their mathematical reasoning and problem solving;
- set high expectations for *all* students;
- encourage students to validate and support their ideas with sound mathematical arguments;
- allow students to work together to make sense of mathematics;
- help students rely on themselves to determine whether something is mathematically correct; and
- expect students to accept responsibility for their own learning.

3: *K-12 Mathematics Curriculum*

All South Carolina students will acquire mathematical power through the study of a core curriculum of significant mathematics during grades K-12. The mathematics of this core curriculum must be sufficiently broad and deep so that all students will have the background needed for employment in the workplace or for further study in mathematics and related subjects. Any differentiation within the curriculum in terms of depth and breadth of treatment should be based on the needs, abilities, and backgrounds of students. All students must have the opportunity to study the fundamental, unifying ideas of mathematics as a discipline. These ideas are organized in the core curriculum into the following six content strands:

- **number and numeration systems,**
- **numerical and algebraic concepts and operations,**
- **patterns, relationships, and functions,**
- **geometry and spatial sense,**
- **measurement, and**
- **probability and statistics.**

These strands broaden the scope of what has been the traditional school mathematics curriculum. In this broadened curriculum it is unacceptable for elementary and middle school mathematics to focus primarily on arithmetic or for high school mathematics to concentrate exclusively on algebra and geometry. Appropriate mathematics from *each* of the six strands must appear at each grade level. The strands themselves are not disjointed collections of ideas, since many mathematical concepts and principles belong to several different strands. The strands thus interweave mathematical ideas throughout the curriculum. Solving a real mathematical problem, for example, rarely involves ideas from just one strand; rather, ideas from several strands must be integrated in order to create an appropriate solution to the problem.

The mathematical topics included in each content strand are stated as *standards*. "A standard is a statement that can be used to judge the quality of a mathematics curriculum or methods of evaluation. Thus, standards are statements about what is valued" (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 2). Stating mathematical topics as standards in this framework has been done to ensure quality, to indicate goals, and to promote change.

All students must encounter mathematics through the following four process strands:

- **problem solving,**
- **communication,**
- **reasoning, and**
- **connections.**

These process strands unify the mathematics curriculum, serving as vehicles for study of the content strands. Appropriate activities, tasks, and projects in mathematics should require students to use combinations of the process strands in their work. Students should be

- solving problems in realistic and meaningful contexts,
- communicating with each other about what they are doing,
- using reasoning to explain and justify their work, and
- making connections in their study to other aspects of mathematics and to other disciplines.

Thus, students should use problem solving, communication, reasoning, and connections to *do mathematics* as they study geometry, algebra, statistics, probability, number systems, numeration, and patterns and functions.

It is important to remember that this Mathematics Framework is *not* the actual curriculum taught in each classroom. Rather, it consists of broad content and process strands in grade-level divisions that serve as guidelines for each school district as it develops its own specific, detailed mathematics curriculum. No matter how a school district chooses to develop its own curriculum and to organize its students into classes, it is crucial that *all* students experience the full range of topics included in the strands of this core curriculum. The phrase "all students" includes

- students who have been denied access in any way to educational opportunities as well as those who have not;
- students ... of all ethnic origins;
- students who are female as well as those who are male; and
- students who have not been successful in school and in mathematics as well as those who have been successful

(NCTM *Professional Standards for Teaching Mathematics*, 1992, p. 4).

It is also essential that the instructional practices and materials discussed in Chapters 2 and 4 be an integral part of the mathematical experiences of all students.

The core curriculum is designed to provide a common body of significant mathematics accessible to all students. It can be modified in a variety of ways to meet the needs, interests, and backgrounds of individual students or groups of students to permit them to progress as far into mathematics as their achievement allows. Students, for example, who are gifted in mathematics, can master the topics of the core curriculum at a greater depth and at a faster pace. Such students would then be eligible to take advanced placement courses or college-level courses in mathematics when they are in high school. All students, however, regardless of whether they are gifted in mathematics or not, should study the topics of the core curriculum. This broadened curriculum is intended to set high expectations for all students. Districts are cautioned against early labeling of students and rigid tracking systems. Benefits can result from alternative patterns of grouping to those used traditionally. South Carolina should continue to explore such patterns where feasible. Decisions should be made by local school districts and communities, and if changes are to be made, they should take place only after considerable discussion has been given to proper planning, professional development, parent involvement, appropriate instructional materials, and curriculum objectives.

This framework endorses the NCTM *Curriculum and Evaluation Standards for School Mathematics* and the NCTM *Professional Standards for Teaching Mathematics*. Both of these documents have been a source of ideas and inspiration in the development of the mathematics curriculum for this framework.

In the sections that follow the mathematics curriculum is described in terms of the grade-level divisions K-3, 3-6, 6-9, and 9-12. The mathematics standards for each of the six content strands are listed and discussed. Sample problems and activities are included to illustrate each strand at each grade-level division. These are representative but not inclusive of the kinds of problems and activities appropriate for implementing this framework, since they use the four process strands as well as the instructional practices and materials advocated in Chapters 2 and 4.

The complete set of K-12 mathematics curriculum standards by content strands is found in Appendix A. Under each content strand are listed all of the curriculum standards for that strand as they appear in the grade-level divisions K-3, 3-6, 6-9, and 9-12, in order to show how the mathematical topics of that strand are developed from grades K through 12.

Mathematics Curriculum for Grades K-3

Young children enter school with a natural curiosity and enthusiasm for exploring patterns, relationships, and quantities. Through everyday experiences they have developed their own unique, creative problem-solving strategies. The strands and sample activities for grades K-3 support young children as they continue to explore mathematical concepts through relevant, interesting investigations using concrete materials. Furthermore, the strands support students as they construct, or invent, mathematics for themselves. This active approach to learning helps students develop an intuitive understanding of mathematics that makes sense to them.

Strand: Number and Numeration Systems – Grades K-3

In grades K-3, number sense serves as a foundation for the development of basic number concepts, including ordering and comparing numbers. Students should explore quantities and the relationships among quantities so that their understanding of place value will develop naturally from the idea of grouping. In all situations number symbols should be linked to real-world concrete materials. The focus of instruction should be based on a constructive approach to learning.

Students will participate in problem-solving activities through group and individual investigations so that they can

- establish a strong sense of number by exploring concepts such as counting, grouping, place value (other bases as well as base ten), and estimating;
- develop concepts of fractions, mixed numbers, and decimals;
- use models to relate fractions to decimals and to find equivalent fractions;
- communicate number relationships by exploring the comparing and ordering of whole numbers, fractions, mixed numbers, and decimals; and
- relate the use and understanding of numeration systems to their world.

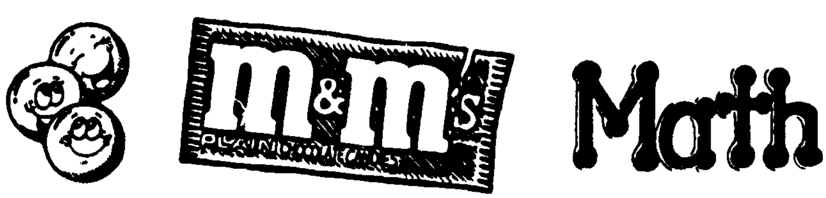
Activity:
MATH WITH "M&M'S" ® CANDIES

Overview: In this activity students will estimate the number of "M&M's" candies in a bag and complete activities of addition and subtraction. They will be asked to use letters to represent the colors of candies and perform operations of addition and subtraction or state relationships.

Materials: One bag of plain "M&M's" candies per cooperative group of 3 or 4 students and one set of student worksheets per group.

Directions: Assign the students to cooperative groups of 3 or 4 each, and pass out one candy bag and one set of worksheets for each group. Before a group opens its bag, the students should guess how many candies are in the bag. Then, they open the bag and count the candies. Afterwards they sort the candies on a worksheet having six circles with colors printed inside – red, orange, green, yellow, light brown, and dark brown.

Before each group completes the addition, subtraction, and comparison activities, be certain that the students know what the symbols on the worksheets mean; e.g., LB represents light brown colored candies. Allow students to manipulate and count candies as they solve the activities together.



1. Do not open your bag yet. Guess how many of the candies are in your bag. ______________

2. Open your bag, and count the candies. How many candies are in the bag? ______________

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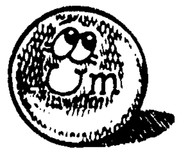
3. How far off was your guess? ________________

4. Now put your candies into sets by color.



green = G	red = R
orange = O	yellow = Y
light brown = LB	dark brown = DB

5. Write the number of candies in each set.

set G ____	set R ____	set LB ____
set O ____	set Y ____	set DB ____

6. Using > or < or =, show the relationship between these sets.

G ____ R	Y ____ DB	Y ____ LB
O ____ G	DB ____ LB	G ____ Y
G ____ LB	Y ____ R	G ____ DB

7. Do these problems:

LB + DB = ____	O + G = ____	O + LB = ____
R + O = ____	R + Y = ____	G + LB = ____
R + G = ____	Y + LB = ____	

8. Put 15 candies in a pile in front of you. Use them to do these problems.
   - How many piles of four can you make? ________
     How many are left over? ________
   - How many piles of seven can you make? ________
     How many are left over? ________
   - How many piles of five can you make? ________
     How many are left over? ________
   - How many piles of two can you make? ________
     How many are left over? ________

9. Put two of the candies in your mouth.
   How many are left? ________

10. Eat four more.

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Discussion: Have the class discuss questions like:

- Do all of the candy bags used in this activity have the same number of candies? How can you explain this?
- Why do you think that there were more dark brown candies in the bag?
- What combination of two colors produces the greatest amount of candies?
- How does the number of dark brown candies compare with the number of all the other candies together?

(Adapted with permission from AIMS Education Foundation, *Primarily Bears*, Book 1, 1987, pp. 65-70)

Strand: Numerical and Algebraic Concepts and Operations – Grades K-3

In grades K-3, students explore numerical concepts using manipulative materials to develop an understanding of the four basic operations, to discover and learn the basic facts, and to construct their own algorithms. They use manipulative materials to investigate the idea of number sentences and relationships among operations in specific situations. By emphasizing what actions are represented by simple mathematical operations, how these actions relate to one another, and what operations are appropriate in particular situations, students understand when to use an operation as opposed to simply performing the procedure to do the operation. By providing students the opportunity to describe, illustrate, estimate, and write about their experiences with concrete materials, students form the foundation for the generalizations that will emerge in algebra.

Students will participate in problem-solving activities through group and individual investigations so that they can

- use concrete models to develop an understanding of the concepts of addition, subtraction, multiplication, and division with whole numbers;
- investigate, model, and compare different strategies for constructing basic arithmetic facts with whole numbers;

- use models to allow students to construct their own algorithms for addition, subtraction, multiplication, and division of whole numbers;
- model, explain, and develop reasonable proficiency in adding, subtracting, and multiplying whole numbers and evaluating the reasonableness of results;
- compare and contrast different computational strategies for solving a specific problem;
- use mental computation, estimation, and calculators to predict results and evaluate reasonableness of results;
- use concrete models to explore operations on common and decimal fractions; and
- use whole numbers, common and decimal fractions, variables, equations, and inequalities to describe problem situations.

Activity: THINGS THAT COME IN GROUPS

Overview: By grouping objects in real-world contexts, children learn to link the idea of multiplication to the world around them.

After figuring out how many chopsticks are needed for everyone in the class, the children brainstorm other things that come in groups of 2. Then they collaborate in small groups to identify things that occur in sets of 3s, 4s, 5s, and so on, up to 12s. The groups' findings are compiled into class lists that can be used later for solving problems and investigating multiples.

Materials: A 12-by-18-inch sheet of newsprint for each group of four. Eleven sheets of 9-by-12-inch drawing paper or a large sheet of chart paper for the class lists.

Directions: Use chopsticks as an example to introduce this exploration. First make sure children know that two chopsticks are required for eating. Then pose a problem for class discussion: How many chopsticks are needed for four people? Hear from all volunteers, asking them to explain how they arrived at their answers.

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Pose another problem: How many chopsticks are needed if everyone in the class eats together? Ask the class to discuss and solve this problem in small groups. Then have individuals tell their answers, again asking that they explain their reasoning. Record on the chalkboard the methods they report, modeling how to use mathematical notation to represent their ideas.

Have the class brainstorm other things that come in 2s. List their suggestions. It's common for children to think of examples from their bodies – eyes, ears, hands, feet, thumbs, and so forth. If the students are limited by a particular category, offer a few suggestions to broaden their thinking – wheels on a bicycle, wings on a bird, slices of bread in a sandwich.

Present the problem of making lists for other numbers. Have pupils work in small groups and think of objects that come in 3s, 4s, 5s, and so on, up to 12s. Give each group a sheet of 12-by-18-inch newsprint on which to organize their lists.

Post a sheet of 9-by-12-inch drawing paper for each list, or one large sheet of chart paper for recording all the lists.

Have groups take turns reading an item from their charts. As an item is read, children from the other groups guess the list on which it belongs. Record each item on the correct list. Should uncertainty or a dispute arise about some items, start a separate "research" list and resolve these questions at a later time. Continue having groups report until all their findings have been offered.

Encourage students to suggest other items that could be added to the lists. You may choose to give them the homework assignment of asking their parents to help them think of additional items.

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You may want to have children write about their solutions to the chopstick problem. This gives them a chance to reflect on their thinking. It also gives you additional information about the reasoning of individual students. The writing need not be done immediately following the class discussion; waiting a day or so is fine.

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Strand: Patterns, Relationships, and Functions – Grades K-3

Students in grades K-3 initially explore patterns, relationships, and functions by observing, analyzing, and describing attributes such as color, size, position, rhythm, design, and quantity in their environment (e.g., classroom, stories, numerical sequences, music, art, and nature). Using this information, students extend or translate existing patterns from one medium to another. Furthermore, they create their own original patterns using a variety of media, including concrete objects, sound, movement, numbers, and pictures. The process of analysis and synthesis of patterns provides the opportunity for exploring mathematical relationships. Furthermore, it affords students the chance to form generalizations by shifting their thinking from the "actual manipulation of numbers to the general relationship between numbers" (*A National Statement on Mathematics for Australian Schools*, 1990, p. 190).

Students will participate in problem-solving activities through group and individual investigations so that they can

- recognize, describe, extend, and create a wide variety of patterns;
- represent, discuss, and describe mathematical relationships;
- use calculators to create and explore patterns;
- make generalizations based on observed patterns and relationships;

- explore the use of variables, equations, and inequalities to express relationships; and
- connect patterns, relationships, and functions with other aspects of mathematics and with other disciplines.

Activity:
EXPLORING PATTERNS ON A HUNDREDS CHART

Get ready. The purposes of these activities are to familiarize students with a hundreds chart, to identify some of the patterns found in this arrangement of numbers, and to practice counting patterns.

For the first activity, you will need a transparency of a hundreds chart or a large chart that all the children can see.
For the second part of the lesson, you will need a copy of the hundreds chart cut apart into puzzle pieces for each group of students (pairs or small groups). In the final activity, each student needs a copy of a hundreds chart, twenty to thirty counters (for example, paper squares, cubes, or bingo chips), and crayons.

Get going. Display the hundreds chart on the overhead projector (or give each student a copy) and ask the students to tell you about it. As they state their observations, record the information on the board. You may wish to have the students color some of the patterns they identify to help others see what is being described orally.

What do you notice about the numbers in the last column? Where are all the numbers with a 3 in the ones place? Where are all the 0s? How many 0s are there?

After the class has talked about the hundreds chart and has identified various patterns, divide the students into pairs or small groups and give each group a hundreds chart that has been cut apart. Challenge the teams to reassemble their charts without looking at a completed chart.

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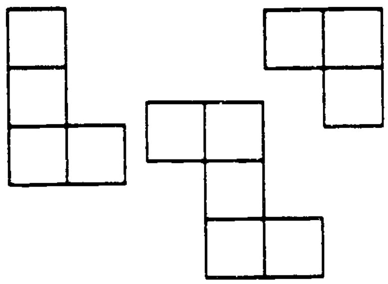
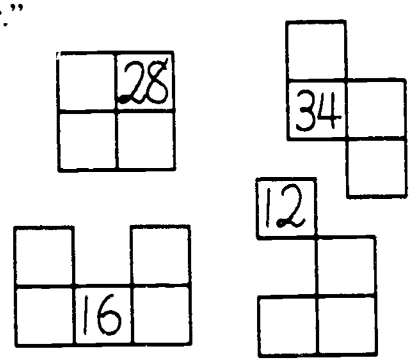
Keep going. Using the hundreds chart on the overhead projector and shapes like those below that you have cut out from plain paper, cover different numbers and ask the students to tell what is hidden. Notice that you can turn the paper cutouts in different ways as you cover parts of the hundreds chart.



How did you know what was covered up? Is there another way to figure out what I have hidden?

You could also furnish shapes like those below with one number given and guide the class in filling in the missing numbers. This challenging activity focuses on place value and counting concepts such as "10 more," "10 fewer," "1 more," and "1 fewer."



Continue using the hundreds chart throughout the year by focusing on skip counting patterns. Have the students cover their hundreds charts with counters or paper markers as they count by 2s, 5s, 10s, 3s, 4s, and so on. Let them record the patterns by coloring different hundreds charts and create a

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display of the counting patterns. Ask the students to go beyond what they have colored and can see.

If we extend the 1 to 100 chart, can you predict how each counting pattern will look? When we count by 5s, will we color 143?

(Reprinted with permission from Burton and others, *Second-Grade Book*, 1992, p. 4)

Strand: Geometry and Spatial Sense – Grades K-3

Students in grades K-3 should explore geometric concepts informally. Instruction should emphasize the interrelatedness of geometric concepts rather than isolated topics or skills. It should promote geometry as a way of perceiving the environment, not just as a set of shapes to be identified. Students should have many opportunities to construct and explore geometry in two and three dimensions, to develop their sense of space and relationships in space, and to generate and solve problems that involve geometry and its applications. "During the early years of schooling the emphasis should be on exploration, both free and structured, of the children's local environment and the objects within it" (*A National Statement on Mathematics for Australian Schools, 1990, p. 81*).

> Children need to investigate, experiment, and explore with everyday objects and other physical materials. Exercises that ask children to visualize, draw, [construct], and compare shapes in various positions will help develop their spatial sense. Although a facility with the language of geometry is important, it should not be the focus of the geometry program but rather grow naturally from exploration and experience (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 48).

> Children should develop spatial language in much the same way as they learn to talk about various animals and objects — by hearing it used appropriately by others (their teachers and peers) and being encouraged to use progressively more sophisticated language in describing their

... (A *National Statement on Mathematics for Australian Schools*, 1990, p. 82).

Students will participate in problem-solving activities through group and individual investigations so that they can

- describe, model, and draw two-dimensional geometric shapes to develop spatial sense;
- describe and model three-dimensional geometric shapes to develop spatial sense;
- identify, classify, and compare geometric shapes according to attributes;
- investigate and predict the results of transformations of geometric shapes, including slides, flips and turns;
- investigate and predict the results of combining and partitioning geometric shapes;
- explore informally tessellations, symmetry, congruence, similarity, scale, perspective, angles, and networks;
- connect geometry to related concepts in measurement and number; and
- identify and appreciate geometry in the world around them, including applications in science, art, and architecture.

Activity:
PROBLEM SOLVING WITH GEOBOARDS

Get ready. The purposes of this activity are to have children distinguish similarities and differences among geometric figures (visual discrimination) and to promote eye-hand coordination.

Give the children 5 x 5 geoboards and rubber bands. If you do not have enough geoboards, children can work in pairs.

Get going. Use the following activities to teach the meaning of the "boundary" of a plane figure. Note that the boundary will be used in later grades to determine perimeter. Look for different answers to each problem. Children should begin to realize

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that not all problems have just one correct answer and some problems may have no answer.

Have the children make a triangle on a 5 x 5 geoboard.

How many pegs does the rubber band touch? Make three triangles so that the rubber band touches three pegs, four pegs, and five pegs. Can you make a triangle so that the rubber band touches more pegs?

Is it possible to make a triangle that touches only two pegs? Is it possible to make a triangle that has the same number of pegs on two sides? On three sides? Have the children discuss why three is the fewest number of pegs the rubber band can touch when a triangle is made.

Make a square with the rubber band touching four pegs and a square with the rubber band touching eight pegs. What other squares can you make? Count the number of pegs on each side. Discuss why each side must have the same number of pegs. Establish a pattern with the number of pegs on the boundary of a square: 4, 8, 12, ...

Explore rectangles the same way. Help the children to see that the number of pegs may not be the same for all four sides but will be the same for opposite sides.

Once the children have grasped the idea of pegs on the boundary, illustrate what is meant by "pegs inside." Point out that although the pegs touching the rubber band are technically "inside," for this problem we will count only those pegs that are not touching the rubber band.

Ask the children to make a triangle with two pegs inside, then with three pegs inside. *What is the greatest number of pegs that can be inside a triangle on a geoboard?*

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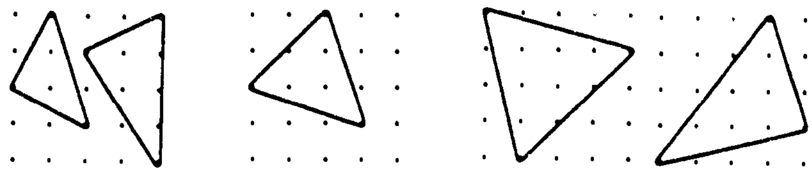
Answers



2 pegs inside **3 pegs inside** **6 pegs inside**

Repeat these problems using other figures, such as four-sided figures, and six-sided figures. *Is it possible to get more pegs inside a four-sided or a five-sided figure?* Note that the answers will vary according to the figures the children make.

Keep going. Vary the characteristics of the figures as you present more problems – the number of sides, the number of pegs inside, and the number of pegs on the boundary. For example,

Make a triangle with one peg inside
and four pegs touching the rubber band.

As you repeat the directions for these problems, the children will see that there are three things to remember: the number of sides, the number of pegs inside the figures, and the number of pegs in the boundary. Initially it may be helpful to display the numbers in a chart.

Number of sides	Number of pegs inside	Number of pegs on the boundary
3	1	4
3	2	3
4	0	4

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Let the children suggest other problems. Some problems may be impossible – for example, a triangle touching five pegs with two pegs inside. Exploring such possibilities gives children experience with problems that have no solution.

(Reprinted with permission from Del Grande and others, *Geometry and Spatial Sense*, 1993, pp. 12-13)

Strand: Measurement – Grades K-3

Students' understanding of measurement should begin with practical projects involving non-standard units and move toward the exploration of standard systems of measurement. Initially, students compare objects directly without the use of measuring tools. Children learn the importance of measuring tools by trying to compare objects that are far apart and cannot be moved closer together. Students learn that in order to determine *how much* larger or heavier one object is than another they can measure with units rather than compare objects directly. They gain an understanding of units by using a variety of standard and nonstandard measuring tools. Broad experiences with measuring help students decide what units of measure are most appropriate for certain situations and what degree of accuracy is needed in various situations. Students learn about measurement while they are using measurement to learn about the world.

Students will participate in problem-solving activities through group and individual investigations so that they can

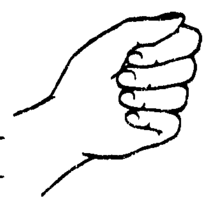
- explore the concepts of length, capacity, weight (mass), perimeter, area, time, temperature, and angle;
- classify angles as acute, right, or obtuse;
- explore, discuss, and use nonstandard and standard (customary [English] and metric) systems of measurement;
- use tools to compare units of measure within a given system;
- make and use estimates of measurement;
- make and use measurement in problems and everyday situations; and
- connect measurement to other aspects of mathematics and to other disciplines.

Activity:
HOW BIG IS YOUR HEART?

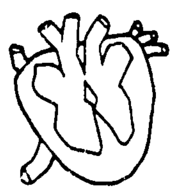
Objective: To measure the length, width, and circumference of each student's fist and relate these dimensions to the size of the heart.

Directions: Give each student a copy of the activity sheet. Read the introductory statement, and explain what the students are to do. To measure the circumference of the heart, a piece of string about 40 cm long should be wrapped around the fist including the thumb, and then that distance should be measured with a ruler. Students should record their actual measurements and their predicted measurements on their activity sheets. They should also make models of their hearts, using their fists as guidelines.

Your heart is a machine. It pumps blood to all parts of your body. The bigger the person, the bigger the heart. Your heart is about the size of your fist. Your heart and fist will grow at about the same rate.



1. Make a fist. Use string and a ruler to find
   - its length: ____________________
   - its width: ____________________
   - the distance around your fist: __________



2. On the basis of the measurements you made, about how long do you think your heart is?

   ____

3. Find the distance around the fist of
   - your teacher: ____________________
   - an older friend: ____________________
   - a parent: ____________________

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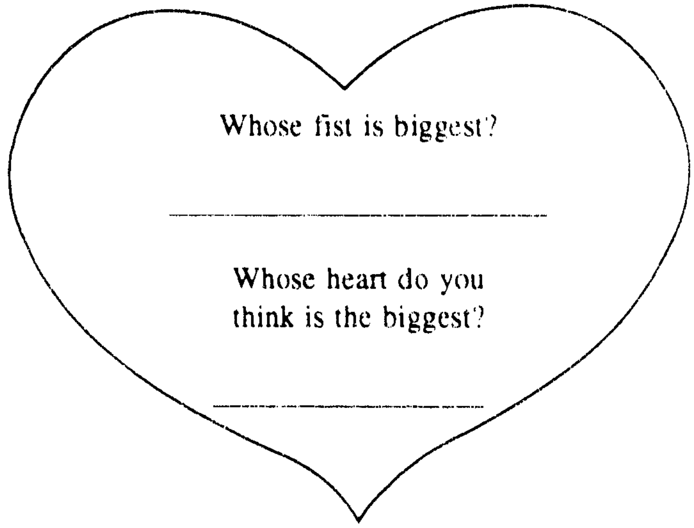


4. How long do you think your heart will be when you are 16 years old? ____________

5. Using your fist as a model, make a paper model of your heart. Write the length, width, and distance around the model of your heart.

(Reprinted with permission from Passarello and Fennell, *Arithmetic Teacher*, 1992, pp. 32, 34)

Strand: Probability and Statistics – Grades K-3

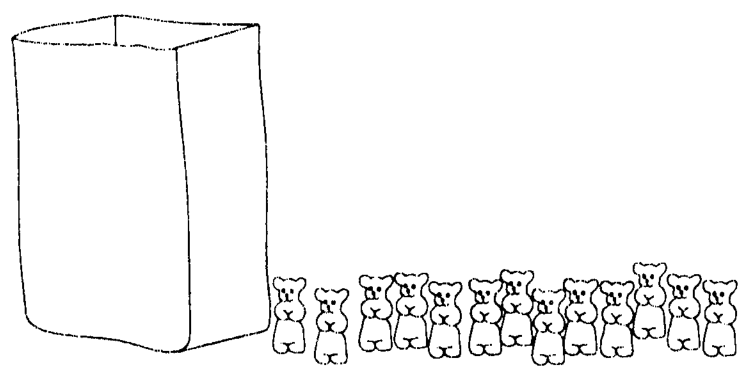
Children should learn about data collection and analysis while they use data collection and analysis to learn about the world. Students learn that they can generate and answer questions about the world around them by collecting, organizing, describing, displaying, and interpreting data. The questions investigated by young children should come from their natural interests. Children need regular opportunities to gather information and use that information to make decisions and solve problems. They should create their own graphic displays of information and interpret the results. This interpretation should include inferences regarding probability. Furthermore, students should have informal opportunities to predict outcomes based on previous experiences.

Students will participate in problem-solving activities through group and individual investigations so that they can

- explore concepts of the likelihood of events, including impossible, not likely, equally likely, more likely, and certain events;
- generate questions, collect data, organize and display information, and interpret findings;
- identify and appreciate examples of probability and statistics in the world around them; and
- connect probability and statistics to other aspects of mathematics and other disciplines.

Activity: BEARS IN THE BAG

The following game relies on student involvement to demonstrate some fundamental properties of probability.



Give each student a bag and a set of twenty teddy bear counters, ten blue and ten red. Throughout the exercise, students will put different combinations of red and blue counters in their bags, using no more than a total of ten counters at a time. At the teacher's signal, they will each pull out one counter. Record the number of blue counters and the number of red counters for the entire class.

For example, begin by having students put all ten blue counters in their bags. Tell them they will pull one teddy bear from the bag, but first have them predict the color of the teddy

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bear that they will remove. Students remove one teddy bear from the bag and record the fact that every student removed a blue bear.

Next, have students put seven blue teddy bears and three red teddy bears in the bag. Again, have them predict what color bear they will get when they pull one counter from their bags. Record these guesses. Then have students remove one bear from their bag and record how many students pulled out a blue bear and how many pulled out a red bear. Repeat the exercise two or three times. Compare the results with the predictions.

Try again using five red and five blue bears, seven red and three blue bears, and ten red bears. Repeat the exercise several times with each combination.

Students should discuss how their predictions compare with the results. Extend the activity by asking students to imagine what would happen if there were 10 red bears and 10 blue bears, 15 red bears and 5 blue bears, etc.

(Adapted with permission from AIMS Education Foundation, *Primarily Bears*, Book 1, 1987, pp. 48-51.)

Mathematics Curriculum for Grades 3-6

In grades 3-6, students need to continue to explore mathematics through active, hands-on investigations. They should continue to construct their own understanding of mathematics by relating their mathematical experiences outside the classroom to their mathematical experiences inside the classroom. They thus explore important connections between mathematics and other disciplines as well as connections within mathematics.

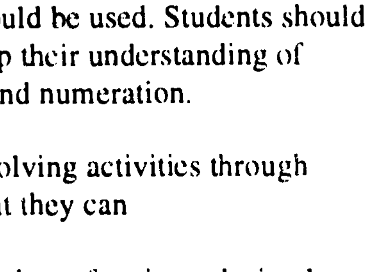
As students in grades 3-6 mature, more formal approaches to mathematics become a part of their daily learning experiences. Nevertheless, they continue to need many opportunities to experience hands-on, concrete activities to ensure that they discover and understand fundamental concepts of mathematics.

Strand: Number and Numeration Systems – Gra les 3-6

In grades 3-6, students continue to develop their understanding of number and numeration and to explore number quantities and their relationships. They build on their knowledge of whole numbers to develop an understanding of fractions and decimals and to extend their thinking and exploration of the relationships of these numerical concepts. Through these experiences, students come to recognize that numbers have multiple representations and learn to express fractions, ratios, decimals, and percents in a variety of meaningful ways. With this broader base, students expand their understanding of number and the idea of place value to work with larger and smaller numbers than previously encountered. In all aspects of these investigations, concrete materials and representational models should be used. Students should use calculators in creative ways to develop their understanding of numbers, relationships among numbers, and numeration.

Students will participate in problem-solving activities through group and individual investigations so that they can

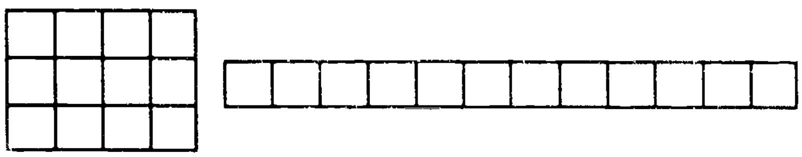
- develop number sense for whole numbers, fractions, decimals, integers, and percents;
- develop and use order relations for whole numbers, fractions, decimals, and integers;
- use concrete models to explore ratios and proportions;
- use concrete models to explore primes, factors, and multiples;
- extend their understanding of the relationships among whole numb , fractions, decimals, integers, and percents;
- connect number and numeration systems with other aspects of mathematics and with other disciplines; and
- relate the use and understanding of numeration systems to their world.



Activity: NUMBER SQUARES AND RECTANGLES

Using paper or tile squares to represent numbers helps students develop an understanding of factors, products, prime numbers, composite numbers, and square numbers. This activity can also provide the foundation for understanding fractions, ratio, and proportion.

Show students how to represent whole numbers with square tiles. For example, the number 12 could be a rectangle 3 tiles by 4 tiles or 1 tile by 12 tiles.



Students must know that a rectangle is a four-sided figure with four right angles. Show them several examples of rectangular numbers made with the square tiles. Then have students create the rectangle with the largest perimeter possible with the number of square tiles that they have. Ask them how they can determine the number their rectangles represent.

Next, select a number such as 18 and ask students to make as many rectangles as possible with 18 squares. Have them label each side of their rectangle according to the number of squares. Then have them look at what they have done and ask what observations they can make. For instance, a single line of squares is counting by ones, whereas a double line is counting by 2s. A rectangle of three squares by six squares is like adding 6 and 6 and 6; it can also represent a multiplication equation, 3 x 6 = 18.

Ask students to investigate the relationship between the rectangular dimensions of various representations of a number

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and the factors of a number. What does a prime number look like as a rectangular number (all on one line because it has only two factors, itself and 1)? How could they represent a composite number? A square number?

This activity can be extended by using graph paper. Outline a rectangular number on the graph paper and shade in a certain part to represent fractions graphically.

Strand: Numerical and Algebraic Concepts and Operations – Grades 3-6

In grades 3-6, students extend their understanding of whole number operations to fractions and decimals and use models and patterns to construct and understand algorithms for operations on whole numbers, fractions, and decimals. Students develop confidence in their ability to use algebraic ideas. They use manipulative models to investigate relationships of equality and inequality. Patterns are explored and extended to provide students with opportunities to develop mathematical and open sentences. Students construct geometric formulas through the use of concrete models, connecting the models to the appropriate algebraic language. By first solving algebraic equations informally and then by formal procedures, students become familiar with the language and symbols of algebra. As they confront an unfamiliar problem, they use mathematical thinking and reasoning to translate the problem into mathematical language and then to solve it.

Students will participate in problem-solving activities through group and individual investigations so that they can

- understand and explain how the basic arithmetic operations relate to each other;
- extend their understanding of whole number operations to fractions and decimals;
- use models, patterns, and relationships to construct and analyze

algorithms for operations on whole numbers, fractions, and decimals;
- model, explain, and develop reasonable proficiency in operations on whole numbers, fractions, and decimals;
- gain confidence in thinking and communicating algebraically;
- solve real-world and mathematical problem situations using algebraic concepts including variables and open sentences;
- use mental computation, estimation, and calculators to predict results and evaluate reasonableness of results;
- understand the concepts of variables, expressions, equations, and inequalities; and
- use models to explore operations on integers.

Activity: THREE BEAN SALADS

Why: This activity provides practice working with ratios and proportions. It includes some fairly difficult algebra problems that can be solved easily by trial and error using the beans.

Tools: Dry red beans, lima beans, and black-eyed peas; paper plates or paper cups to hold small portions of beans.

How: All three types of beans go into each salad in this activity. Students should be encouraged to guess and adjust as they work, using the beans to solve the problems. For each salad they must determine how many of each of the three types of beans are needed.

Each salad contains Red beans, Lima beans, and Black-eyed peas.

1

This salad contains:
2 Lima beans
Twice as many Red beans as Lima beans
10 beans in all

2

This salad contains:
4 Red beans
1/2 as many Black-eyed peas as Red beans
10 beans in all

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3

Lima beans make up 1/2 of this salad:
The salad has exactly 2 Red beans
The number of Lima beans is double the number of Red beans

4

This salad contains:
The same number of Red beans as Lima beans
3 more Black-eyes than Red beans
A total of 18 beans

5

This salad contains 12 beans
1/2 of the beans are Red
Lima beans make up 1/4 of the salad

6

This salad contains at least 12 beans
It has one more Lima bean than Red beans
It has one more Red bean than Black-eyes

7

This salad contains:
3 times as many Red beans as Black-eyes
One more Lima bean than Red beans
8 beans in all

8

This salad contains:
An equal number of Red beans and Black-eyes
5 more Lima beans than Red beans
No more than 20 beans

Make up a different salad.
Write instructions for someone else to make your salad.

Strand: Patterns, Relationships, and Functions – Grades 3-6

In grades 3-6, work with patterns continues to be informal and provides a transition to symbolic representation. Students should be encouraged to observe and describe many kinds of patterns in the world around them – for example: in architecture, tree leaves, pine cones, pineapple spirals, and street grids on maps. By observing the number of patterns that emerge from a variety of situations, students explore and understand the underlying structure of the number system and become familiar with commonly encountered number sequences such as odd numbers, multiples, and squares. Using technology, especially calculators, will further their understanding of patterns by allowing them to explore patterns in depth and to use larger numbers. When students create graphs, tables, mathematical expressions, equations, or verbal descriptions to represent the patterns they observe, they discover how different representations reveal different interpretations of a situation.

Students will participate in problem-solving activities through group and individual investigations so that they can

- use concrete models and calculators to create and explore patterns;
- explore, recognize, describe, extend, analyze, and create a wide variety of patterns;
- represent, discuss, and describe functional relationships with tables, one- and two-dimensional graphs, and rules;
- analyze and predict functional relationships and make generalizations based on observed patterns;
- explore the use of variables, equations, and inequalities to express relationships; and
- connect patterns, relationships, and functions with other aspects of mathematics and with other disciplines.

Activity: WHAT'S MY RULE?

Get ready. The purpose of this activity is to help students recognize number patterns and test their conjectures. Students solve number riddles by guessing the rules that apply to sets of three numbers. The focus is on describing a pattern both orally and in writing.

Get going. Explain to the class that the goal of the activity is to guess a rule that you have made up. The rule generates number

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triples. Think of a rule and write on the chalkboard three numbers related by the rule. For example, if your rule is "the third number is the product of the first two," write this sequence of numbers: 3 5 15. The students give three numbers that they think satisfy the rule. Respond yes or no and keep track of their guesses and your responses on a chart as shown below.

Attempt	Guess	Response
1	1 2 3	Yes
2	12 11 10	No
3	4 5 6	Yes
4	10 11 12	Yes

Rule: Each number is 1 more than the previous number.

Attempt	Guess	Response
1	70 21 7	Yes
2	7 77 17	No
3	3 6 9	No
4	35 42 56	Yes

Rule: All numbers are multiples of 7.

The class continues to name triples and to receive your feedback. When the students think they are ready to guess the rule, give them several triples. They must indicate whether or not the numbers satisfy the rule and describe the rule.

Keep going. Have small groups of students play the game. Let individual students invent their own rules and be rule makers for further class games at the chalkboard. Join the game and see if you can guess the students' rule. Students are delighted when a teacher plays along. Some possibilities for rules follow:

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1. The third number is the sum of the first two:

4	7	11
8	2	10

2. The first two numbers are even and the third is always odd:

4	6	7
8	2	5

3. The second number is greater than the sum of the first and third numbers:

2	9	6
10	30	14

(Reprinted with permission from Burton and others, *Fourth-Grade Book*, 1992, pp. 4-5)

Strand: Geometry and Spatial Sense – Grades 3-6

Students in grades 3-6 develop their spatial sense by examining and comparing objects from different perspectives. When given opportunities to notice how shapes change when viewed from different angles and how different shapes relate to other shapes, students begin to understand the mathematical relationships involved in geometry. By handling, thinking about, drawing, and comparing shapes, children become more familiar with geometric concepts and terminology. Their use of geometric ideas and language, as well as their ability to make simple deductions about geometric properties, develops out of their natural interest in the world around them when activities focus on representing, describing, and analyzing their own environment.

Students will participate in problem-solving activities through group and individual investigations so that they can

- construct two- and three-dimensional geometric figures with concrete materials;
- identify, describe, classify, and compare two-and three-dimensional geometric shapes, figures, and models according to their attributes;

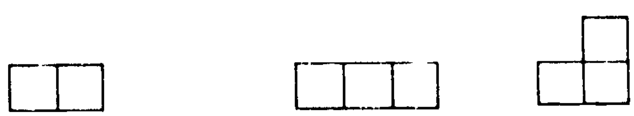
- develop spatial sense by thinking about and representing geometric figures;
- investigate and predict the results of transformations of shapes, figures, and models, including slides, flips, and turns and combinations of slides, flips, and turns;
- investigate and predict the results of combining and partitioning shapes, figures, and models;
- explore tessellations, symmetry, congruence, similarity, scale, perspective, angles, and networks;
- represent and solve problems using geometric models;
- understand and apply geometric relationships;
- develop an appreciation for geometry as a means of describing the physical world; and
- connect geometry and spatial sense to other aspects of mathematics and to other disciplines.

Activity:
WHAT ARE PENTOMINOES?

Objectives: This problem-solving activity helps students to develop their skills of spatial perception. It gives them an opportunity to experience informally geometric concepts such as congruence, flips, and turns. The students, working in cooperative groups, will discover and make all possible geometric shapes called pentominoes.

Materials: Each student needs 5 color tiles, a pair of scissors, and a sheet of one-inch squared paper. Each cooperative group will be given a 6 x 10 puzzle grid sheet made from one-inch squared paper *after* the group has discovered all of the different possible pentominoes.

Introduction: Discuss how to form a domino with 2 color tiles (there is only one way) and how to form a tromino with 3 color tiles (there are only two ways).



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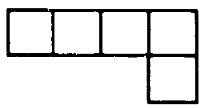
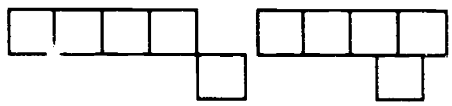
Then discuss how to form a pentomino with 5 tiles, stressing that at least one full side of a tile must touch one full side of another tile.

A pentomino



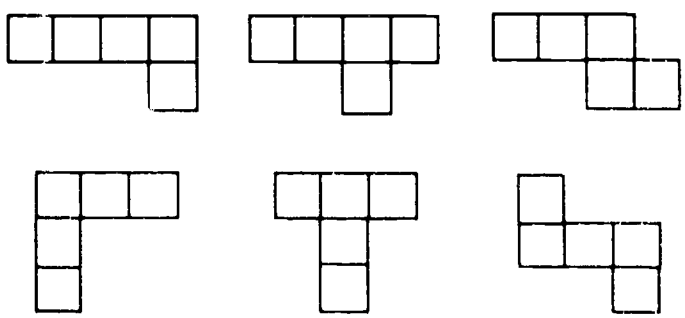
Not a pentomino



Activity: Have each group to find all the different pentominoes that are possible. Students should make each pentomino with color tiles, draw it on squared paper, and cut it out. If a pentomino can be flipped or turned to fit on top of another pentomino, it is not a different pentomino. The groups should find all the different possible pentominoes and should explain how they know they have found them all.

When a group has made a complete set of pentominoes, give the group a puzzle grid sheet with the challenge to fit all of the pieces together to make a 6 x 10 rectangle with no holes in it and with no overlapping pieces.

Closure: Show the 12 different possible pentominoes.



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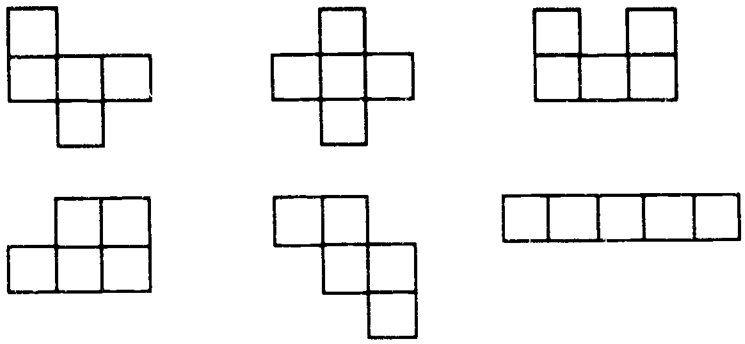
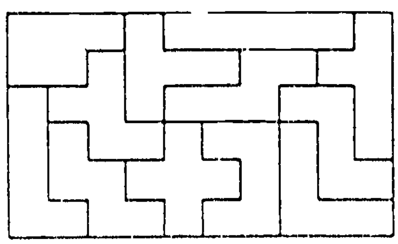


Discuss a systematic way of finding all 12 of them:

Maximum Number of Squares in a Row	Number of Different Such Pentominoes
5	1
4	2
3	8
2	1

Then, show a solution to the challenging puzzle, which has over 1,000 solutions:



6 x 10

Point out that the 6 x 10 puzzle grid sheet has an area of 60 squares, and since each pentomino has an area of 5 squares, there must be 60÷5=12 pentominoes.

(Adapted with permission from AIMS Education Foundation, *Hardhatting in Geo-World*, 1986, pp. 60-64)

Strand: Measurement – Grades 3-6

> In grades 3-6, measurement is of central importance to the curriculum, because of its power to help students see that mathematics is useful in everyday life and to help them develop many mathematical concepts and skills (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 51).

Students decide the most appropriate tool for a particular task and determine the most useful way to measure an object. As students acquire the ability to use appropriate measuring tools in the classroom, they should extend these skills to applications such as architecture, art, science, commercial design, sports, cooking, and map reading. Students' abilities to use estimation strategies develop naturally through their experiences in measuring the world around them.

Students will participate in problem-solving activities through group and individual investigations so that they can

- understand the concepts and attributes of length, capacity, weight (mass), perimeter, area, volume, time, temperature, and angle measure;
- understand the structure and use of nonstandard and standard (customary [English] and metric) systems of measurement;
- estimate, construct, and use measurement for description and comparison;
- select and use appropriate tools and units to measure to the degree of accuracy required in a particular situation;
- use concrete and graphic models to discover formulas for finding perimeter and area of common two-dimensional shapes;
- use measurements and formulas to solve real-world and mathematical problems; and
- connect measurement to other aspects of mathematics and to other disciplines.

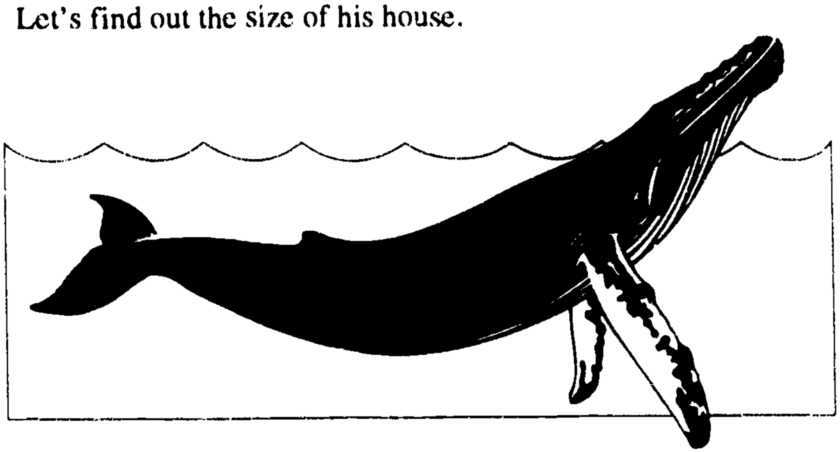
Activity: GETTING THE FACTS

Literature: Clement, Rod. *Counting on Frank.* Milwaukee, WI: Gareth Stevens Children's Books, 1991. (The narrator of this book likes to collect facts with the help of his dog Frank. Each two-page spread of the book includes a different fact involving such mathematical topics as counting, size comparison, or ratio.)

Objective: Students solve problems involving estimation of volume.

Directions: Read the entire story aloud. Tell students that they will need to know the size of the average humpback whale, and discuss where this information can be found. Divide the class into small, cooperative groups, giving each group a worksheet to complete. Because the data the students have collected will probably include the whale's length and width, they will have to make inferences about the dimensions of a box for the whale. They should justify these dimensions. They should also describe the processes they used to find their answers to the questions on the worksheet.

One of the facts shared in the book *Counting on Frank* is that only ten humpback whales would fit in the narrator's house. Let's find out the size of his house.



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Getting the Facts

How big is the average humpback whale? ________________

Using the Facts

If you placed one whale inside a box, about how big would the box need to be?

Length ____________________

Width ____________________

Height ____________________

Describe how you determined the box dimensions. __________

About how much space would ten boxes fill? ______________

About how big is the boy's house? ____________________

About how many humpback whales would fit in your school? ____________________

Write down how you arrived at your answer. ______________

(Reprinted with permission from Hopkins, *Arithmetic Teacher*, 1993, pp. 513, 517.)

Strand: Probability and Statistics – Grades 3-6

Students in grades 3-6 continue to investigate the world around them by gathering and interpreting data. At this level, students' emerging interest in music trends, movies, fashion, sports, and other consumer issues can serve as a natural source of questions to explore. Once data have been gathered and analyzed, students communicate their results to classmates, discuss their reasoning processes, and evaluate the conclusions reached by others. Students gain an understanding of the connection between statistics and probability by working with data that show the frequency or likelihood of an event and using this information to make predictions. Organizing data into charts and graphs helps students understand how they can present their findings visually and read graphic representations of data.

Students will participate in problem-solving activities through group and individual investigations so that they can

- model situations by devising and carrying out experiments or simulations to determine probability;
- extend their understanding of probability and statistics by systematically collecting, organizing, discussing, and describing data, using technology whenever appropriate;
- select and use a variety of representations for displaying data;
- construct, read and interpret tables, graphs, and charts; and
- make and justify predictions based on collected data or experiments, using technology whenever appropriate.

Activity:
THE AGE OF CENTS

Students will explore the minting dates of pennies in circulation and predict from a small sample to a nationwide population with a measurable degree of accuracy.

Have students collect 50-100 pennies from a variety of sources. Divide the class into groups of 4-5 students. Have each student predict what the total collection of coins will be like in such things as the

- distribution of minting dates,
- number minted in 1980, 1981, or any other year,
- probability of staying in circulation 10 or 20 years, and
- probability of being minted prior to 1970, etc.

After each has made the prediction based on its sample of pennies, have them check the prediction against the facts.

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In order to gather the facts, students must

- tabulate and graph the information from their sample,
- predict from their sample,
- check their predictions against the larger sample,
- predict from the larger sample to the population, and
- tabulate and graph the classroom sample.

In what ways can the information obtained from their combined samples be made more reliable with respect to the entire population? What is the median minting date of pennies in circulation? What percent of the total number of pennies in circulation were minted before 1965? In what year was the largest number of coins in circulation?

Students will write their findings in a report.

Extension: Find out from the U. S. Treasury how many coins were minted in given years and how the number to be minted is determined.

(Adapted with permission from AIMS Education Foundation, *Math Plus Science: A Solution*, 1987, p. 33)

Mathematics Curriculum for Grades 6-9

Students in grades 6-9 also need to be actively engaged, both physically and intellectually, in activities that connect their prior experiences and knowledge to new situations. The use of manipulative materials, calculators, and computer technology are essential tools to help them make these connections.

Since students in this transition period are making decisions that will affect their future, they should understand and appreciate the importance of mathematics in that future. All students in grades 6-9 must thus be provided with a full, rich mathematics curriculum that will open the doors to success as they continue their education and later as they enter the workplace.

Strand: Number and Numeration Systems – Grades 6-9

In grades 6-9, students broaden their experience with rational and irrational numbers as a foundation for manipulating algebraic expressions. Students need to explore very large and very small numbers represented in decimal form, in exponential form, and in scientific notation; to develop an understanding of negative numbers and irrational numbers; and to continue to compare fractions, decimals, and percents and their alternate representations. Students should feel at ease with integers and realize that their operations and properties are a natural extension of the whole number system. Teachers should reinforce number sense by exposing students to number lines, graphs, area models, and representations of numbers that appear on calculators and computers. Students should be encouraged to use estimation, to do work mentally, and to find alternate solutions to the same problem. They should be exposed to grouping schemes other than base ten.

Students will participate in problem-solving activities through group and individual investigations so that they can

- extend their development of number sense to include all real numbers;
- develop and use order relations for real numbers;
- understand, represent, and use real numbers in a variety of equivalent forms (integers, fractions, decimals, percents, exponentials, and scientific notation) in a variety of real-world and mathematical problem situations;
- understand and apply ratios, proportions, and percents in a wide variety of situations;
- develop and apply number theory concepts (primes, composites, factors, and multiples) in a variety of real-world and mathematical problem situations; and
- connect number and numeration systems with other aspects of mathematics and with other disciplines.

Activity:
McDONALD'S CLAIM

You and a friend read in the newspaper that 7% of all Americans eat at McDonald's each day. Your friend says, "That's impossible!"

You know that there are approximately 250,000,000 Americans and approximately 9,000 McDonald's restaurants in the U.S. You think the claim is reasonable.

Show your mathematical work and write a paragraph or two that explains your reasoning.

(Reprinted with permission from Dixon and others, *Performance Task Sampler*, Item Number: D14)

Strand: Numerical and Algebraic Concepts and Operations – Grades 6-9

In grades 6-9, students are making the transitions from number to variable and from arithmetic to algebra. They are learning to think proportionally. Students draw upon their experiences with patterns to explore the properties of algebraic relations. They should investigate patterns leading to formulas from home and work contexts, from other disciplines such as science, as well as from other strands and understand them as examples of functional relationships. Informal exploration should emphasize physical models, data, graphs, and other mathematical representations, leading students to sharpen their skills in recognizing patterns and regularities in mathematics. As students develop confidence in using algebra to represent and solve problems in informal situations, they extend these skills to more abstract and symbolic representations.

Students will participate in problem-solving activities through group and individual investigations so that they can

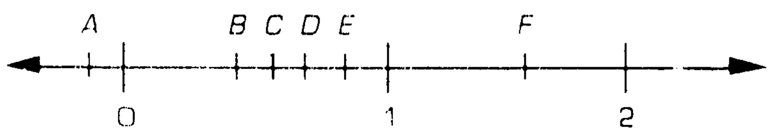
- use models, patterns, and relationships to construct, explain, and analyze algorithms for operations on integers and explain how the operations relate to each other;

- develop reasonable proficiency in operations on integers and rational numbers;
- develop, analyze, and explain techniques for estimation;
- develop, analyze, and explain procedures for solving problems involving proportions;
- select and use appropriate methods for computing from among mental arithmetic, paper-and-pencil, calculator, or computer methods;
- use mental computation, estimation, and calculators to solve problems, predict results, and evaluate reasonableness of results;
- understand the concepts of variables, expressions, equations, and inequalities and gain confidence in thinking and communicating algebraically;
- represent situations and number patterns with models, tables, graphs, verbal rules, and equations and make connections among these representations;
- analyze tables and graphs to identify properties and relationships;
- solve linear equations using concrete, informal, and formal methods;
- investigate inequalities and non-linear equations informally; and
- apply algebraic methods to solve a variety of real-world and mathematical problems.

Activity:
OPERATIONS ON FRACTIONS

To the Teacher: This activity is useful for clarifying students' understanding of the effects of certain operations on fractions and for helping to develop number sense.

Display the number line and ask students questions such as those indicated here and others that come to mind. Encourage students to justify their answers by explaining their reasoning.



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1. If the fractions represented by the points D and E are multiplied, what point on the number line best represents the product?

2. If the fractions represented by the points C and D are multiplied, what point on the number line best represents the product?

3. If the fractions represented by the points B and F are multiplied, what point on the number line best represents the product?

4. Suppose 20 is multiplied by the number represented by E on the number line. Estimate the product.

5. Suppose 20 is divided by the number represented by E on the number line. Estimate the quotient.

(Reprinted with permission from Reys and others, *Developing Number Sense in the Middle Grades*, 1991, p. 34)

Strand: Patterns, Relationships, and Functions – Grades 6-9

The study of patterns in grades 6-9 builds on students' experiences in grades 3-6, with a shift in emphasis to an exploration of functions. Students continue to explore patterns in the real world and begin to build mathematical models to make predictions about real-world situations. By observing and describing functions in informal ways, students learn that functions are composed of variables that have a dynamic relationship (i.e., a change in one variable results in a change in the other). The study of functions expands to include learning related concepts such as domain and range, using algebraic expressions to describe functions, and solving problems that involve functional relationships. Explorations of these concepts should be enhanced by the use of calculators and computers.

Students will participate in problem-solving activities through group and individual investigations so that they can

- use technology along with concrete, numerical, and abstract models to explore, describe, analyze, extend, and create a wide variety of patterns;
- represent, discuss, and describe functional relationships with tables, graphs, and rules;
- analyze and predict functional relationships and make generalizations based on observed patterns;
- use models and technology to analyze functional relationships to explain how a change in one quantity results in a change in another quantity;
- use variables, equations, and inequalities to express functional relationships;
- make, test, and utilize generalizations about given information as a means of solving real-world and mathematical problems; and
- connect patterns, relationships, and functions with other aspects of mathematics and with other disciplines.

Activity:
PATTERNS IN THE POWERS CHART

Problem: If 2^{100} is expanded, what is the digit in the units place?

Launch: Be sure students understand the question. Expand 2^{10}. What is the digit in the units place? [2^{10} = 1,024, so 4 is the digit in the units place.]

Explore: Allow students time to explore the problem. If students first try using a calculator, they may end up with scientific notation. Suggest looking at smaller exponents to find a pattern.

Summarize: The units digit repeats in a cycle of four: 2, 4, 8, 6, 2, 4, 8, 6, ... Students need to observe where each number occurs in a cycle: 2 occurs for the exponents 1, 5, 9, ..., or for the exponents that have a remainder of 1 when divided by 4 (or 1 more than a multiple of 4). Similarly, 4 occurs for the exponents 2, 6, 10, ..., or for the exponents that have a remainder of 2 when divided by 4; 8 will occur for the exponents that have

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a remainder of 3 when divided by 4; and 6 will occur for the exponents that have a remainder of 0 (or multiples of 4) when divided by 4. Thus, 2^{100} ends in 6, 2^{93} ends in 2, and 2^{51} ends in 8.

The original question can be extended for powers of all whole numbers 1 through 10. Let students investigate these powers. An amazing pattern emerges. The longest cycle for the units digit is four, and only 2, 3, 7, and 8 have a cycle of length four. A cycle of length one occurs for 1, 5, 6, and 10. A cycle of length two occurs for 4 and 9.

If this pattern is put into a table, other patterns can be observed. Some patterns to investigate are the endings for square numbers, cube numbers, or those that yield the same number. For example, could 292 be a square? [No, a square number does not end with a 2.] What are the endings for numbers that are a fourth power? [0, 1, 5, or 6] Another example is $2^4 = 4^2 = 16$. Why? Are there others? This can lead to the generalization that $(2^n)^m = 2^{nm}$. Other investigations can lead to a discovery of the other exponential rules. Does $2^2 \cdot 3^2 = 6^2$? [Yes] Why?

1	2	3	4	5	6	Units digit of the powers
1	1	1	1	1	1	1
2	4	8	16	32	64	2,4,8,6
3	9	27	81	243	729	3,9,7,1
4	16	64	256	1,024	4,096	4,6
5	25	125	625	3,125	15,625	5
6	36	216	1,296	7,776	46,656	6
7	49	343	2,401	16,807	117,649	7,9,3,1
8	64	512	4,096	32,768	262,144	8,4,2,6
9	81	729	6,561	59,059	531,441	9,1
10	100	1,000	10,000	100,000	1,000,000	0

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The last question provides a good opportunity to increase students' estimation skills. In the first example, 823,543 < 10^7 and the only numbers whose powers will have a 3 in the units place must end in a 7 or a 3. So the number is either 3 or 7, but closer to 7. Use a calculator to check that 7^7 = 823,543. Similar reasoning will yield 11^6 = 1,771,561 and 13^5 = 371,293.

Students could investigate the largest power of 2 that can be displayed completely in the calculator window. The answer will depend on the calculator. Most calculators will have an eight-place display. Thus 2^{26} = 67,108,864 is the largest power of 2 that can be displayed. If 2^{27} is entered, either the error symbol occurs or the display 1.3421773 08 occurs. The latter means 1.3421773 x 10^8. This is scientific notation, which would be an appropriate topic to investigate during a discussion of exponents.

Here are some other questions for student investigation:

- What is the digit in the units place for 17^{100}? 24^{50}? 31^{10}? These numbers will behave like 7, 4, and 1, respectively. So 17^{100} ends in 1, 24^{50} ends in 6, and 31^{10} ends in 1.

- How many zeros are there in 10^7? 10^{10}? 10^{100}?

- Which is larger: 6^{10} or 7^{10}? 8^{10} or 10^{10}? 6^9 or 9^6? Why?

- Try to find the missing numbers with one guess; check with a calculator.

$?^7$ = 823,543 $?^6$ = 1,771,561 $?^5$ = 371,293

(Reprinted with permission from Phillips and others, *Patterns and Functions*, 1991, pp. 12-14)

Strand: Geometry and Spatial Sense – Grades 6-9

Through the process of solving problems, students in grades 6-9 use their more advanced reasoning abilities to gain further understanding of geometric concepts and procedures. Students observe, analyze, and construct two- and three-dimensional models in order to explore relationships among shapes and their component parts. Students also use models and transformations to describe, quantify, and represent the real world. Models provide a foundation for conceptualizing geometric relationships, making inferences about geometric properties, and solving geometry problems. When teachers invite students to discuss their conclusions and challenge students to view problems from different perspectives, students' confidence in geometric concepts and language is strengthened.

Students will participate in problem-solving activities through group and individual investigations so that they can

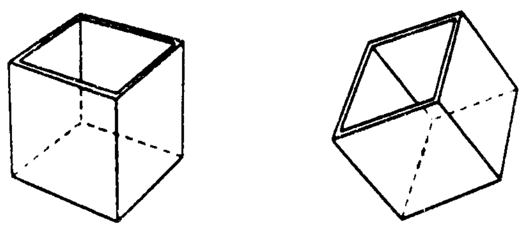
- model, identify, describe, classify, and compare two- and three-dimensional geometric figures;
- use technology whenever appropriate to explore concepts and applications of geometry;
- develop spatial sense by thinking about, constructing, and drawing two- and three-dimensional geometric figures;
- investigate and predict the results of combining, partitioning, and changing shapes, figures, and models;
- investigate the results of transformations, including translations, reflections, rotations, and glide reflections, to reinforce concepts such as congruence, similarity, parallelism, perpendicularity, and symmetry;
- apply coordinate geometry to locate positions in two and three dimensions;
- represent and apply geometric properties and relationships to solve real-world and mathematical problems; and
- connect geometry and spatial sense to the physical world, to other aspects of mathematics, and to other disciplines.

Activity: SHAPES AND WATER IN A CUBE

This learning activity should be done by groups of four or five students.

1. Each group should have a liter box containing some colored water (use food coloring); the approximate depth of the water should be 4 centimeters. A container with additional water should be available for use if necessary.

2. Each group should discuss what is meant by a "cross section," in particular, a plane polygon cross section made by the top surface of the water in the liter box. For example, when the liter box sits flat on the table, the polygon cross section formed by the top surface of the water is a square. If the box is tilted slightly, the shape of the polygon cross section changes.



3. Challenge: Each group should position, if possible, the liter box so that the polygon cross section made by the top surface of the water has the following shapes:

- A square
- A rectangle that is not a square
- A parallelogram that is not a rectangle
- A trapezoid
- An isosceles triangle
- An equilateral triangle
- A pentagon
- A hexagon
- An octagon
- Other (student choices)

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Is it possible to create all the figures above with water in the liter box?

If not, explain why particular shapes are impossible to create with water in a liter box.

(Reprinted with permission from Geddes and others. *Geometry in the Middle Grades*, 1992, p. 61)

Strand: Measurement – Grades 6-9

In grades 6-9, measurement activities focus on the transition from concrete to abstract mathematics. Such activities should be related to active exploration of the real world with careful selection and use of appropriate tools in measuring objects. However, students' ability to use these tools should be extended so that they can apply these skills to new situations. As students move through grades 6-9, there will be a shift in focus on using concepts and skills as students learn more efficient procedures and, ultimately, formulas for finding measurements

Students will participate in problem-solving activities through group and individual investigations so that they can

- extend their understanding of the concepts and processes of length, capacity, weight (mass), perimeter, area, volume, time, temperature, and angle measure;
- estimate, construct, and use measurements to describe and compare phenomena;
- use suitable methods of approximations to find areas and volumes of irregular shapes;
- understand the structure and use of nonstandard and standard (customary and metric) systems of measurement;
- select and use appropriate tools and units to measure to the degree of accuracy required in a particular situation;
- develop the concepts of rates and other derived and indirect measurements;
- use concrete and graphic models to discover formulas for finding perimeter, area, and volume of common two- and three-dimensional shapes;

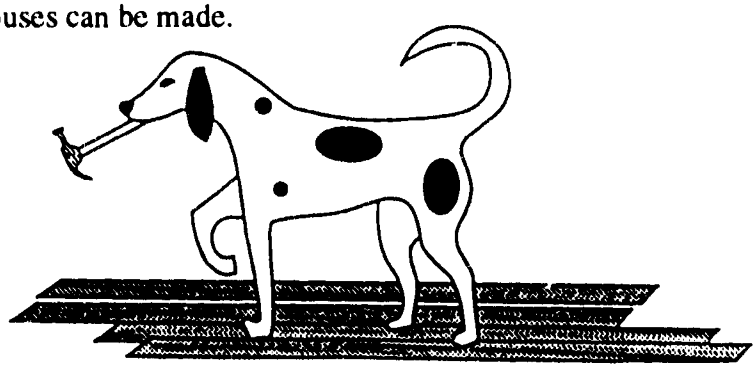
- use measurements and formulas to solve real-world and mathematical problems; and
- connect measurement to other aspects of mathematics and to other disciplines.

Activity:
CANINE CARPENTRY

This activity makes practical use of students' measurement skills and provides practice in working in two and three dimensions.

Give students a piece of plywood measuring 150 cm x 300 cm. Tell them that they must construct a dog house from this piece of wood. They should try to make the house as large as possible (by volume) from the available materials.

Students can begin by making a scale drawing to show how the parts of the dog house will be cut from the plywood and recording the measurements on the scale drawing. Students can draw sketches to show what the finished kennel will look like and record the measurements on these sketches. Encourage students to investigate how many different kinds of shapes of dog houses can be made.



This activity can be extended by actually constructing the "winning" dog house (largest interior volume) in the classroom or in a practical arts class. Have students volunteer to contact local builders to donate the plywood.

(Adapted with permission from National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 118)

Strand: Probability and Statistics – Grades 6-9

Students in grades 6-9 build on their knowledge of probability and statistics to further investigate real world applications of these topics. Middle school students have a keen sense of fairness but often misinterpret data, and their misconceptions lead to invalid conclusions. Through experimentation and simulation they can learn to make more reasonable predictions and verify their results. Sample space, experimental and theoretical probability, graphical representation of data, and the use of data to make inferences and to formulate and evaluate arguments are emphasized at this level.

Students will participate in problem-solving activities through group and individual investigations so that they can

- model situations by carrying out experiments or simulations to determine probabilities, using technology whenever appropriate;
- model situations by constructing a sample space to determine probabilities, using technology whenever appropriate;
- make inferences and convincing arguments based on an analysis of theoretical or experimental probability;
- collect, organize, analyze, describe, and make predictions with data, using technology whenever appropriate;
- construct, read, and interpret tables, graphs, charts, and other forms of displayed data;
- evaluate arguments that are based on data analysis;
- develop an appreciation for the pervasive use and misuse of probability and statistical analysis in the everyday world; and
- connect probability and statistics with other aspects of mathematics and with other disciplines.

Activity:
CHIPS

The story setting for this activity is that a toy manufacturer is considering marketing several new two-person games. The manufacturer would like to choose only fair games to market. The class is challenged to help the manufacturer make correct decisions about the games.

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The games are analyzed both experimentally and theoretically to determine fairness. This allows students to begin to see the effect of small samples of data versus larger samples and that theoretical probabilities reflect what happens in the long run (if the experiment is performed over and over). Coins and chips with each side lettered provide an easy introduction to experimentation-simulation. The theoretical analyses are easy since the number of outcomes is small. Making lists and simple tree diagrams help students learn a systematic way to determine the theoretical probabilities.

Finally, if a game is unfair, students are asked to show how to make it fair in two different ways – either by changing the points (payoff) or by changing the rules to change the probabilities.

Game One

Materials: Two chips – one with the letter X on both sides and one with an X on one side and a Y on the other side.

Rules: Flip both chips at once.

Score: Player I gets a point if the chips match. Player II gets a point if the chips do not match.

Play and Record:

	Player I Match	Player II No Match	
			P (Match) = _____
Tally			
			P (No Match) = _____
Total	__________ +	__________ =	__________

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Game Two

Materials: Three chips:
1 chip with an A side and a B side
1 chip with an A side and a C side
1 chip with a B side and a C side

Rules: Flip all three chips at once.

Score: Player I gets a point if there is a match. Player II gets a point if there is no match.

Play and Record:

	Player I Match	Player II No Match
Tally		
Total	________ +	________ = ________

P (Match) = _____

P (No Match) = _____

Adapted with permission from *Teaching Statistics and Probability* (1981 Yearbook); chapter 8, "Fair Games, Unfair Games," by George W. Bright, John G. Harvey, and Margariete Montague Wheeler; copyright 1981 by the National Council of Teachers of Mathematics.

Mathematics Curriculum for Grades 9-12

In grades 9-12, all South Carolina high school students will study a common core curriculum in mathematics. This Framework specifies the standards within each content strand that are essential for all students graduating from South Carolina high schools. Within each content strand are some topics that are appropriate for students intending to take advanced mathematics or AP calculus. No specific approach to implementing the core curriculum at the high school level is differentiated or enriched curriculum," are outlined in the NCTM book *A Core Curriculum: Making Mathematics Count for Everyone* (Meiring, 1991, pp. 117-141).

Like students in grades K-8, high school students should be actively engaged in a variety of different kinds of activities as they learn mathematics. Group work, individual explorations, emerging technology, and concrete models are helpful to students in grades 9-12 as they investigate and grow to understand mathematics, as they learn to make and create prove or disprove for conjectures, and as they develop and use a flexible collection of strategies for solving problems both within and outside of mathematics.

Strand: Number and Numeration Systems – Grades 9-12

In grades 9-12 students should come to understand and appreciate number systems with their operations as coherent structures rather than as isolated facts and rules. They should gain a thorough understanding of real and complex numbers, explore the fundamental properties of number systems, and develop conjectures and proofs of the properties of number systems.

The level of computational proficiency suggested in the strand number and numeration systems for grades K-8 is assumed for all students in grades 9-12. Although arithmetical computations are not a direct object of study in grades 9-12, number and operation sense, estimation skills, and the ability to judge reasonableness of results are strengthened in the context of applications and problem solving.

Students will participate in problem-solving activities through group and individual investigations so that they can

- develop the hierarchy of the real number system and compare and contrast the various subsystems of the real number system with regard to their structural characteristics;
- connect number and number systems with other aspects of mathematics and with other disciplines;
- use concrete models to explore fundamental properties of number systems;
- develop conjectures and proofs of properties of number systems;

and so that, in addition, students intending to take advanced mathematics can

- develop and understand the complex number system;
- develop an understanding of the concept of infinity;
- investigate limiting processes by examining infinite sequences and series; and
- connect the complex number system with other aspects of mathematics and with other disciplines.

Activity: KEEPING IN TOUCH

This activity demonstrates how one can take a simple consumer application mathematics problem and raise the complexity of the problem from pre-algebra to calculus.

Problem: Suppose your grandparents invested for your college education \$8,000 when you were eight years old at an interest rate of 8% compounded annually. What amount of money will be in the investment when you reach the age of 18?

Level 1: With a calculator students can compute the amount of money in the account each year by successively applying this relationship: the amount at the end of the year equals the amount at the beginning of the year plus the rate times the amount at the beginning of the year.

year 1: amount at the end of year = \$8,000 + .08 (8,000) = \$8,640

year 2: amount at the end of year = \$8,640 + .08 (8,640) = \$9,331.20

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.
.

year 10:

Level 2: Students at this level can use patterns to discover a general formula for finding the amount of money in an account compounded annually for a given number of years. This can be

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accomplished by allowing students to work in groups and assigning each group a different problem. Each group will report on its finding. The conclusion, or finding, will be used to generate the general formula $A = P(1+r)^t$.

Example:

year 1: amount at the end of year = \$8,000 + .08 (8,000) = 8000 (1.08)

year 2: amount at the end of year = \$8,000 (1.08) + .08 [8,000 (1.08)] = 8,000 (1.08) (1.08) = 8,000 $(1.08)^2$

year 3:

.

.

.

year 10:

... 8,000 $(1.08)^{10}$

Note: the result may be compared with a compound interest table.

Level 3:

(a) Further generalization of the formula found in Level 2 allows one to explore problems in which the annual interest is compounded semi-annually, quarterly, daily and so forth: $A = P(1 + r/m)^{mt}$.

(b) Students at this level should explore the derivation of the rule of 72 by solving for time in the formula in Level 2 using natural logarithms.

(continued)

(continued)

Level 4: After completing Level 3 students can solve for a given variable when values of the other three variables are known.

Level 5: Using calculus, students can derive the formula $A = Pe^{rt}$ for interest compounded continuously.

Various other extensions are possible. The formula $A = P(1+r)^t$ can be used in biology (exponential growth and decay), probability (estimating population), economics (prices), investments, and many others.

(Adapted with permission from National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics*, 1989, pp. 132-134.)

Strand: Numerical and Algebraic Concepts and Operations – Grades 9-12

Algebra is a fundamental language for communication in most of mathematics. It provides a means of operating with concepts at an abstract level and then applying them, and it permits generalizations and insights beyond the original context.

Students will participate in problem solving activities through group and individual investigations so that they can

- represent situations that involve variable quantities with expressions, equations, inequalities, and matrices;
- use tables and graphs as tools to interpret expressions, equations, and inequalities, using technology whenever appropriate;
- develop, construct, and evaluate formulas to solve a variety of real-world and mathematical problems;
- develop an understanding of and facility in manipulating algebraic expressions, performing elementary operations on matrices, and solving equations and inequalities;
- recognize the worth, importance, and power of the mathematics of abstraction and symbolism;

and so that, in addition, students intending to take advanced mathematics can

- demonstrate facility with operations on the complex number system;
- use matrices to solve linear systems, using technology whenever appropriate;
- demonstrate technical facility with algebraic transformations, including techniques based on the theory of equations; and
- represent problem situations using discrete structures such as finite graphs, matrices, sequences, and recurrence relations.

Activity:
WRITING AND EVALUATING VARIABLE EXPRESSIONS

Objectives:
- to write and interpret an algebraic expression in terms of a given situation
- to develop an intuitive notion for the nature of a variable and the relationships among variable expressions.

(continued)

(continued)

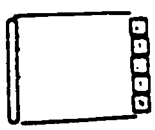
Materials: pipe cleaners, tongue depressors, unit square tiles.

Directions: Using pipe cleaners of length a, tongue depressors of length b, and tiles of unit length, form the following figures and state the lengths of all sides.

1. a triangle of perimeter $6a$
2. a rectangle of perimeter $2a + 6$
3. a regular hexagon of perimeter 12
4. a square of perimeter $8 + 4b$
5. a triangle of perimeter $a + 2b + 1$
6. a pentagon of perimeter $4b + 11$ with no two sides of equal length [Answers will vary.]

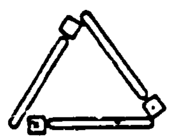
Write an expression for the perimeters of these figures:

7.



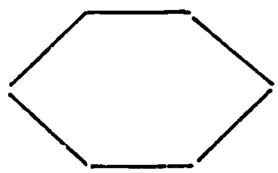
[$2a + b + 5$]

8.



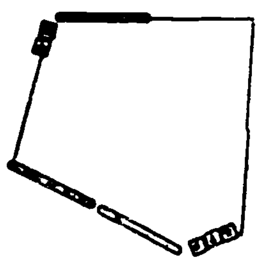
[$3(b + 1)$]

9



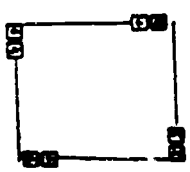
[$6a$]

10.



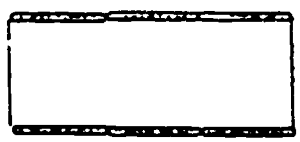
[$4a + 3b + 5$]

11.



[$4(a + 2)$]

12.



[$2a + 6b$ or $2(a + 3b)$]

(continued)

(continued)

Assessment Matters: When the instructional emphasis is on concept building through situations reflecting real-world questions and activities, the assessment should be of a similar nature. Open-ended, holistically scored questions, interviews, observation of group work, testing with the use of physical models like those used in instruction, and student self-assessment are appropriate approaches.

Teaching Matters: In this activity, variables represent specific, but unknown, lengths. The meaning of the algebraic symbols and expressions arises through applying the concept of perimeter with concrete materials (e.g., 6a means a + a + a + a + a + a rather than 6 + a). Students practice combining like terms and learn informally that different-appearing expressions may be equivalent (e.g., different expressions for the perimeters will arise in exercises 8, 11, and 12). The physical constraints of the figures force students to associate the terms of some expressions differently from what they first suppose. (In exercise 5, a triangle cannot be formed with lengths a, 2b, 1 but can be formed with lengths a + 1, b, b or a, b, b + 1). To extend this activity, ask students to evaluate perimeter expressions for specific values of variables.

(Reprinted with permission from Meiring and others, *A Core Curriculum: Making Mathematics Count for Everyone*, 1992, pp. 37-38)

Strand: Patterns, Relationships, and Functions – Grades 9-12

A central theme of mathematics is the study of patterns, relationships, and functions. This study requires students to recognize, investigate, describe, and generalize patterns and to build mathematical models to predict real-world phenomena that exhibit the observed patterns. The study of functions in grades 9-12 builds on students' experiences investigating patterns and relationships in grades K-8. The concept of function is a key unifying idea in mathematics relating arithmetic, algebra, and geometry as well as real-world applications.

The ability to develop and analyze algorithms is one aspect of this strand which is related to discrete mathematics.

Critical to the study of patterns, relationships, and functions is the use of technology. Calculators with their input-output process model the function concept. Graphing calculators and computers are ideal tools for investigating patterns and relationships as well as translating among various representations.

Students will participate in problem solving activities through group and individual investigations so that they can

- understand the logic of algebraic procedures;
- model real-world phenomena with a variety of functions, using technology whenever appropriate;
- represent and analyze algorithms and relationships using tables, verbal rules, equations, and graphs, using technology whenever appropriate;
- translate among tabular, symbolic, and graphical representations of functions, using technology whenever appropriate;
- recognize that a variety of problem situations can be modeled by the same type of function;
- explore calculus concepts informally from both a graphical and numerical perspective, using technology whenever appropriate;

and so that, in addition, students intending to take advanced mathematics can

- analyze the effects of parameter changes on the graph of a function, using technology whenever appropriate;
- understand the general properties, behaviors, and graphs of classes of functions, including polynomial, rational, radical, exponential, logarithmic, and trigonometric functions, using technology whenever appropriate; and
- understand operations on classes of functions, using technology whenever appropriate.

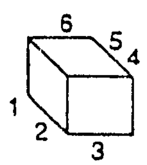
Activity:
MAKING BANQUET TABLES

This activity is appropriate as a ninth grade activity; it can be extended using different shapes to higher grade levels.

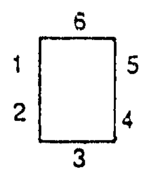
A problem is posed to entice students to examine a pattern of numbers related to how many people can be seated around a table. Using reasoning skills, students can determine the function which yields the number of people that can be seated using the number of tables as input. This activity illustrates an application of mathematics to the restaurant or banquet industry and connects the geometry of the table configurations to the algebra of the function.

Materials: At least five Cuisenaire Rods of length 2 for each group of students.

Introduce the class to the problem: The Cuisenaire Rod below represents a table that can seat 6 people:



Top View:



How can you arrange two tables (rods) in order to seat as many people as possible? Each arrangement must form a rectangle.

Answer: Two arrangements, one seats 8 and the other 10.

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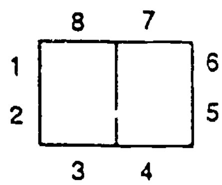
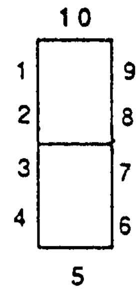




Students should work in groups of three to six to answer the following questions.

What is the greatest number of people that can be seated using three tables? four tables? five tables?

Describe how the tables should be configured to seat the maximum number of people. Describe how they should be configured to seat the fewest people.

Can you predict the answer for ten tables?

Answer:

# of tables	# seated
1	6
2	8, 10
3	10, 14
4	12, 18

Place the tables so that the short ends of the rectangle are together to seat the maximum number of people. Place tables so that the long ends of the rectangle are together to seat the fewest people.

10	24, 42

(continued)

(continued)

Ask student groups to summarize results. Lead discussion around the following question. Use table from group answers to generate hypothesis for formula.

Can you write a formula that takes the number of tables and gives the greatest number of people that can be seated?

Answer: If n is the number of tables, $4n + 2$ is the maximum number of people that can be seated.

Challenge: Prove the formula is true for all natural numbers n. (*Hint:* mathematical induction)

(Reprinted with permission of author, Ed Dickey. This and other examples may be found in *The Algebra Report, Video Series Lesson 1*, 1993.)

Strand: Geometry and Spatial Sense – Grades 9-12

Students in grades 9-12 should deepen their understanding of shapes and their properties. They should continue to develop their ability to visualize and make conjectures about two- and three-dimensional objects. Students should have opportunities to work with geometric figures, concepts, and models using a discovery approach to allow them to construct the skills fundamental to everyday life and necessary for many careers. Physical models, computer-based activities, and other real-world objects should be used to provide a base for the development of students' geometric intuition so that they can draw on these experiences in their work with abstract ideas. "Only when students have a sufficient grounding in the ideas of geometry should they be introduced to the formal deductive aspects of the subject" (Serra, 1989, p. 2).

One of the most important connections in all of mathematics is that between geometry and algebra. The interplay between geometry and algebra strengthens students' abilities to formulate and analyze problems from situations both within and outside mathematics. Although students will at times work separately in synthetic, coordinate, and transformational geometry, they should have as many opportunities as possible to compare, contrast, and translate among these systems.

Trigonometry has its origins in the study of triangles, and many real-world problems, such as those from navigation and surveying, require the solution of triangles. Trigonometric and circular functions are mathematical models for real-world phenomena, such as temperature changes, sound waves, and tide variations. Although students should explore data from such periodic phenomena, students studying advanced mathematics should identify and analyze the corresponding trigonometric models. Graphing utilities can and should be used to significantly enhance and facilitate the teaching of trigonometry. This technology will provide more class time to develop understanding and address realistic applications (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, pp. 157-163).

This strand is not to be considered as defining a single course in geometry. Instead it describes a great variety of topics and approaches that should be integrated throughout the curriculum in different courses at different grade levels 9-12.

Students will participate in problem-solving activities through group and individual investigations so that they can

- represent real-world and mathematical problem situations with geometric models and apply geometric properties related to those models;
- use technology whenever appropriate to explore concepts and applications of geometry,
- classify figures in terms of congruence and similarity and apply those relationships;
- deduce properties of and relationships between figures from given assumptions;
- translate between synthetic and coordinate representations;
- deduce properties of figures using transformations;
- deduce properties of figures using coordinate systems;
- analyze properties of Euclidean transformations and relate translations to vectors;
- apply trigonometry to problem situations involving triangles;
- explore periodic real-world phenomena using the sine and cosine functions;

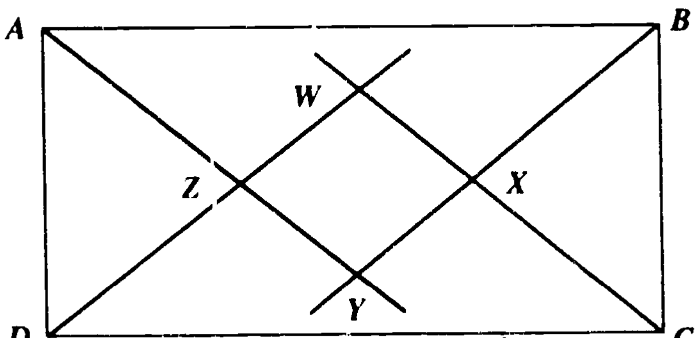
and so that, in addition, students intending to take advanced mathematics can

- develop an understanding of an axiomatic system through investigating and comparing various geometries;

- deduce properties of figures using vectors, using technology whenever appropriate;
- apply transformations, coordinates, and vectors in problem-solving situations, using technology whenever appropriate;
- understand the connection between trigonometric and circular functions;
- use circular functions to model periodic real-world phenomena, using technology whenever appropriate;
- apply general graphing techniques to trigonometric functions, using technology whenever appropriate;
- solve trigonometric equations and verify geometric identities, using technology whenever appropriate; and
- understand the connections between trigonometric functions and polar coordinates, complex numbers, and series, using technology whenever appropriate.

Activity:
Quadrilateral Investigation

1. If you were to draw the bisectors of the angles of a square, they would meet in a single point because they are the diagonals. However, draw a rectangle and construct the bisectors of the four interior angles using a Mira, compass, or paper folding, as described below. These four lines intersect in four points, *W, X, Y*, and *Z*, as shown below. List as many properties of the quadrilateral determined by these four points as you can. Compare your list with that of a partner. With a partner, justify each property in your combined lists.



(figure not drawn to scale)

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In Exercises 1 and 2, the instructions call for you to construct bisectors of the angles of a quadrilateral. You may do this by using a compass, a Mira, or paper folding. To use a Mira, construct the required quadrilateral, place the Mira so that one side of a vertex angle reflects onto the other side, and draw the reflecting line. In the figure above, the lines *AY*, *DW*, *BY*, and *CW* are the reflecting lines. To use paper folding, cut out the required quadrilateral. Make a fold through a vertex, creasing the paper so that the two sides of the angle fall on each other. The lines identified by the creases are the reflecting lines.

2. Carry out the procedure described above, using any of the three construction methods you wish, for each of the quadrilaterals identified below. Make a list of the properties of the new quadrilateral determined by the four points of intersection:

a. A rhombus
b. A parallelogram
c. A kite
d. An isosceles trapezoid
e. A trapezoid
f. A general quadrilateral, that is, a quadrilateral with no special characteristics

3. Examine the lists of properties you generated in parts a-f of Exercise 2. Are there any properties of the quadrilaterals in each of the above cases constructed that are common to *each*? If so, write your conjecture in if-then form and write a justification for it.

(Adapted from Coxford and others, *Geometry from Multiple Perspectives*, 1991, p. 31)

Strand: Measurement – Grades 9-12

Students in grades 9-12 should increase their facility in choosing, making, and interpreting measurements, both direct and indirect. They should be able to

[illegible]

These skills can be developed through the study of other aspects of mathematics or other disciplines. Students should also learn to make indirect measurements using rates, trigonometric ratios, and similarity relationships.

Students should consider the problem of approximating areas and volumes of irregular shapes and develop strategies for doing so. They can approximate areas using transparencies of square grids, and they can approximate volumes using water displacement. Computer software is available for approximating the area under a curve by partitioning it into rectangles and for approximating the volume of a solid of revolution by partitioning it into cylindrical disks, washers, or shells.

This strand is also a place to develop the conceptual underpinnings of the calculus, especially the derivative of a function as an instantaneous rate of change and the definite integral as the limit of a Riemann sum.

[illegible]

The definite integral can be developed as the limit of an approximating sum for finding the areas of plane geometric regions, the volumes of solids of revolution, and the lengths of arcs of plane curves.

Students will participate in problem solving activities through group and individual investigations so that they can

- estimate, construct, and use measurement for description and comparison;
- choose appropriate techniques, units, and tools to measure quantities;
- choose appropriate techniques for approximating the perimeter, area, or volume of irregular geometric figures or models;
- convert measurement units within a system to solve problems that involve various units, using technology whenever appropriate;
- apply the relationships between precision, accuracy, and tolerance of measurements, using technology whenever appropriate;
- use rates, similarity relationships, and trigonometric ratios to solve problems involving indirect measurements in two or three dimensions, using technology whenever appropriate;
- connect measurement with other aspects of mathematics and with other disciplines;

and so that, in addition, students intending to take advanced mathematics can

- understand the conceptual foundations of limits, infinite sequences and series, the area under a curve, the rate of change, and the slope of a tangent line, and their applications to other disciplines, using technology whenever appropriate.

Activity:
BOATS AND AMBULANCES

You are the captain of a ship, and one of your passengers has been injured. Your ship is 30 miles from a point that is 60 miles downshore from a hospital. You must order an ambulance to meet your ship at any point along a road that runs parallel to the shoreline. You would like to meet the ambulance at a point that will get your passenger to the hospital in the shortest possible time. Suppose that your boat travels at a rate of 20 mph and the ambulance will average 50 mph.

In the questions that follow, round all approximate answers to two decimal places.

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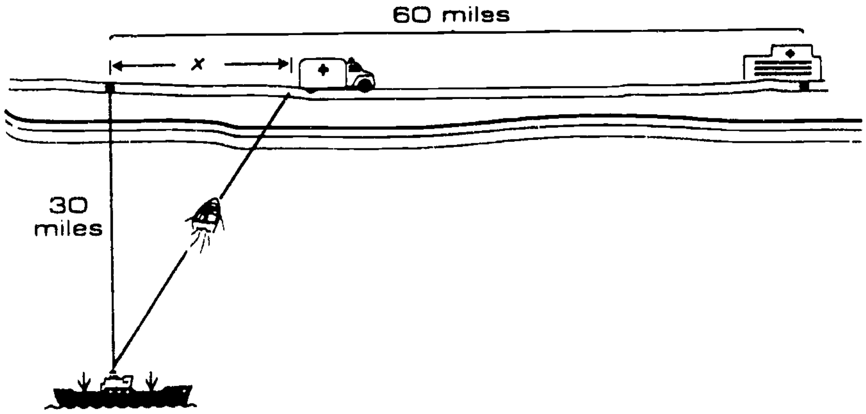

1. In the figure, x represents the distance downshore to the point where the ambulance is to meet the boat. Suppose the boat sails directly to shore. Here $x = 0$ miles.

a. How far must the boat travel? ________
b. How long will the boat take? ________
c. How far must the ambulance travel to the hospital? ________
d. How long will the ambulance take? ________
e. How long will it take to get the passenger to the hospital? ________

2. Suppose that $x = 10$ miles.

a. Determine the distance the boat must travel. ________
b. How long will the boat take? ________
c. How far must the ambulance travel to the hospital? ________
d. How long will the ambulance take? ________
e. How long will it take to get the passenger to the hospital? ________

(continued)

(continued)

3. Record your answers to the previous questions in the chart below and complete the chart by doing calculations similar to those you did in question 2.

X	Boat distance	Boat time	Ambulance distance	Ambulance time	Total time
0					
10					
20					
30					
40					
50					
60					

4. Which value of x results in the shortest time for the trip? ________

5. Try to shorten the total time required for the trip by using other values of x. Record the total time and the value of x for any better times you find. Compare your results with those of your classmates.

Teaching Notes: The total amount of time for the trip is the sum of the boat's time and the ambulance's time. Since time is equal to distance divided by rate, the total time is given by the function

$$f(x) = \frac{\sqrt{x^2 + 30^2}}{20} + \frac{60 - x}{50}.$$

A traditional approach to this problem uses calculus and consists of differentiating the time function f and finding the zeros of its derivative f′.

(continued)

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High school students can solve this problem without calculus by using calculator or computer technology. They can use an ordinary calculator or an inexpensive computer with a simple spreadsheet to solve the problem. They can also use a graphing calculator to draw the graph of the time function f and to locate the y–coordinate of the lowest point on the graph.

(Reprinted with permission from Froelich and others, *Connecting Mathematics*, 1991, pp. 40-43 and 54-55)

Strand: Probability and Statistics – Grades 9-12

Students in grades 9-12 should learn how to make sense of real-world data and to interpret that data. They should develop strong intuitive understandings of probability concepts by estimating probabilities associated with real-world data.

All 9-12 students should participate in hands-on, activity-based studies of real-world data in the context of *exploratory* data analysis and probability. Students need to explore, to discuss, and to pose hard questions about data that they find interesting and relevant to their lives. They should understand that statistical analysis is the careful and qualified *interpretation of patterns and associations* found in data. The emphasis in this strand should be on making sense of data, on estimating probabilities by working with real-world data, and on using expected values for decision making.

This strand should not be viewed as advocating a course in statistics and probability. Instead it describes topics and approaches that can be effectively integrated into the curriculum in different places at different grade levels 9-12. Instructional methods should have students – either as individuals or in small groups – exploring data sets, conjecturing, estimating, and refining conclusions under the guidance of their teacher. Students should make free use of calculators, especially graphing calculators that include easy-to-use, built-in software for curve fitting, enumeration, and elementary probability calculations. Computers can be used to handle large data sets effectively. Such technology is not for use just as an aid to computation but also for creative investigation.

Students will participate in problem-solving activities through group and individual investigations so that they can

- understand the relationships between theoretical and experimental probability and between probability and odds;
- use experimental or theoretical probability to represent and solve problems involving uncertainty, using technology whenever appropriate;
- use simulations to estimate probabilities from real-world situations, using technology whenever appropriate;
- understand the concept of random variable;
- create and interpret discrete probability distributions, using technology whenever appropriate;
- construct and draw inferences from charts, tables, and graphs that summarize data from real-world situations, using technology whenever appropriate;
- use curve fitting to predict from data, using technology whenever appropriate;
- describe in general terms the normal curve and use its properties to answer questions about sets of data that are assumed to be normally distributed;
- understand and apply measures of central tendency, variability, and correlation and apply the effects of data transformations on measures of central tendency and variability, using technology whenever appropriate;
- design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the results, using technology whenever appropriate;

and so that, in addition, students intending to take advanced mathematics can

- apply the concept of random variable to generate and interpret probability distributions including binomial, uniform, normal, and chi square;
- transform data to aid in interpretation and prediction, using technology whenever appropriate; and
- test hypotheses statistically, using technology whenever appropriate.

Activity: MONTY'S DILEMMA

On a popular TV game show, contestants are given a chance to choose a valuable prize behind one of three closed doors. Two of the doors hide gag prizes. Contestants choose a door, which remains closed while the host opens one of the two doors and reveals a gag prize. The contestants are then given the option of sticking with their original choice or switching to the other unopened door.

If you were a contestant, which of the following strategies would you adopt: stick with the original choice of doors, or switch to the other unopened door?

1. Students should first be asked to write which strategy they would choose and why.

2. Then, students should be divided into small groups to conduct simulation experiments to model the decision process. A spinner divided into three equal areas A, B, and C can be used to simulate each of the two strategies. Alternatively, students can use the random number generator on their graphing calculators. Each group should simulate each strategy many times, say 100-200 times, and record the relative frequency of winning. This will help them to better understand the two strategies and provide experimental estimates of the theoretical probabilities of winning the prize under each one. Suppose the prize is behind door A. The spinner reflects the initial choice of doors.

The stick strategy:

(a) Suppose that the spinner lands on Door B. The host opens door C. You stick to door B. Do you win or lose?

(b) Suppose that the spinner lands on door C. The host opens door B. You stick to door C. Do you win or lose?

(c) Suppose that the spinner lands on door A. The host shows you either door B or door C. You stick with door A. Do you win or lose?

(continued)

(continued)

The switch strategy:

(a) Suppose that the spinner lands on door B. The host opens door C. You switch to the unopened door. Do you win or lose?

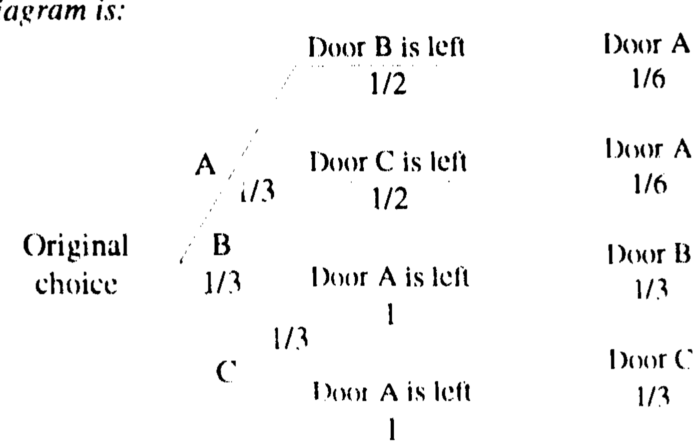
(b) Suppose that the spinner lands on door C. The host opens door B. You switch. Do you win or lose?

(c) Suppose that the spinner lands on door A. The host shows you door B or C. You switch. Do you win or lose?

Students with graphing calculators can use their random number generators to generate the numbers 0, 1, 2 randomly instead of using a spinner with equal areas A, B, C. After each group conducts many simulations of each strategy, it should reformulate its choice of strategies in writing again with supporting reasons.

3. The class can then run an interactive computer simulation of the game, say 100,000 times. (A program known as Monty's Dilemma, written in Apple II BASIC is available by sending a blank disk and a stamped return envelope to Dr. Thomas P. Dick, Mathematics Department, Oregon State University, Corvallis, Oregon 97331.)

4. More advanced students should then conduct a careful analysis of theoretical probabilities associated with each strategy by constructing tree diagrams. *For the stick strategy, the tree diagram is:*



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After constructing the tree diagrams, the class should prepare a table that compares the chances of winning under each strategy.

5. Finally, as an extension to the problem, students may consider a third strategy, the *flip strategy*: flip a coin; stick with the original door if it shows Heads; switch if it shows Tails.

As before, they should first conduct an experimental simulation of the strategy, then a much larger computer simulation, and finally develop the theoretical probabilities using a tree diagram. All three strategies should be compared.

(Excerpted from *Mathematics Teacher*, for more information, see the April 1991 issue.
Reprinted with permission from Shaughnessy and Dick, *Mathematics Teacher*, 1991, pp. 252-56)

4: Instructional Materials

The selection of instructional materials must be consistent with the goals, objectives, and guidelines of this Mathematics Framework (See Chapters 1-3). Depending upon the quantity of materials available for review, the South Carolina Department of Education should appoint from one to three committees to review and select programs for approval at the state level. For example, the state might appoint one committee for textbooks, another for computer software, and others for the remaining categories of materials. In 1992, the state of South Carolina enacted legislation to broaden the definition of instructional materials to any resource that assists in the instructional process, including but not limited to textbooks. Materials for mathematics instruction include

- textbooks and ancillary materials,
- other printed materials,
- manipulative materials,
- computer software,
- video materials (tape and video disks),
- distance education programs,
- teacher training programs,
- programs that may result from new technologies, and
- student assessment materials.

Local school districts should appoint their own committees to make selections from the state-approved list. The purpose of the state list is to ensure that ma' rials purchased with state funds are of the highest quality as defined by this framework. Local districts, however, should have the option to justify and select other materials purchased with state funds in lieu of the materials on the state list.

Instructional Materials Criteria

Instructional materials provide a foundation for a mathematics program, since they determine to a great extent the mathematics that students encounter. They influence what and how teachers teach and what and how students learn. Good materials can significantly improve students' attitudes toward and achievement in mathematics. The instructional materials criteria must be consistent with the goals and curriculum described in this framework. They should also be consistent with the goals of the NCTM *Curriculum and Evaluation Standards for School Mathematics.*

The South Carolina Department of Education and local school districts should develop evaluation processes to be used to select materials. Criteria used in developing the processes should include the following categories:

- Mathematical Content,
- Organization and Structure,
- Student Experiences,
- Teaching Strategies, and
- Assessment.

Mathematical Content

The mathematical content of the program should reflect the Strands outlined in this framework as well as the NCTM *Curriculum and Evaluation Standards for School Mathematics.* The curriculum should take into account the major goals of improving students' abilities to solve problems, to reason mathematically, and to communicate with mathematics. It should be comprehensive in addressing the elements of the core curriculum at appropriate instructional levels. It should provide for the natural and logical development of mathematical topics across grades. It should make connections among topics within the discipline of mathematics and between mathematics and the real world.

- Problem solving is built into the program at all levels through problem situations that are sufficiently simple to be manageable but sufficiently complex to provide a challenge. The problem situations should be adaptable to individual, small-group, or large-group instructional settings.

- Mathematics as communication is an important part of the program. Students should have many opportunities to communicate mathematical language and ideas. They should have opportunities to explain, conjecture, and defend their ideas orally and in writing.

- Mathematics as reasoning is built into the program at all levels. Students should have opportunities to explain and justify their thinking in keeping with their maturity level. At the high school level students should be asked to use formal methods of proof where appropriate.

- Mathematical connections are made throughout the program through instructional activities that interrelate concepts, procedures, and intellectual processes. Connections are made within the discipline of mathematics and between mathematics and the real world.

- The program is comprehensive and addresses standards in the curriculum identified in this Framework at each level: K-3, 3-6, 6-9, and 9-12. The presentation should be appropriate for the grade level for which it is intended.

Organization and Structure of the Core Curriculum

The program must be appropriate for all students and should be organized into cohesive units, multiday investigations, and worthwhile tasks. The purpose of the activities should be clearly defined. The units, investigations, and tasks must be of sufficient breadth and depth for students to develop ever increasing levels of understandings of mathematical concepts. They should give students opportunities to apply the mathematics they know to the discovery or investigation of new ideas in mathematics. The activities should include the appropriate use of technology.

- Units are organized around major mathematical ideas and are of sufficient duration for students to develop a broad understanding of mathematics.

- Many lessons are more than one day in length with more than one mathematics objective.

- Students work on worthwhile tasks which invite them to experiment with a variety of strategies and results.

- The programs incorporate the use of calculators and computers as tools for students to solve problems. The program should be designed with the expectations that calculators are available to students and that students have access to computers.

- Differentiation for students studying advanced mathematics can be made in terms of breadth and depth of treatment and the nature of applications.

Student Experiences

The program should emphasize active learning on the part of students. It should consistently include activities that call for the investigation and exploration of ideas, problem solving, conjecturing, and verification of results. It should include "friendly" activities showing students that problem-solving includes making false starts, evaluating solutions, and starting over again if necessary. It should encourage students to explore concepts at concrete, semi-concrete, and abstract levels in all grades.

- Materials encourage students to explore and conjecture in a risk-free environment, even allowing them to make and find errors.

- Materials engage students in mathematical discourse as they participate in concrete, semi-concrete, and abstract activities.

- Materials allow students to use manipulative materials to model mathematical situations and to use technology to analyze data, calculate numerical results, and solve problems.

- Materials encourage students to determine whether an exact answer or an approximate answer is appropriate for solving a problem. They are also expected to choose the appropriate computational procedure, whether paper-and-pencil, mental calculation, or calculator.

Teaching Strategies

The program should provide appropriate support for teachers as they implement the teaching methods recommended in this framework. It should include strategies for the active involvement of students in their own learning. It should include ideas on how teachers can teach the content in a variety of grouping patterns within the classroom. It should include suggestions on the appropriate use of calculators, computers, and other technology to enhance instruction.

- The materials provide suggestions that assist teachers to help students meet the major goals of learning to value mathematics, becoming confident in their own ability to learn mathematics, becoming mathematical problem-solvers, learning to communicate mathematically, and learning to reason mathematically.

- The materials assist teachers in meeting the instructional needs of all students.

- The materials provide suggestions for teachers on how to use time, physical space, and manipulative materials in ways to facilitate learning and how to teach students to work in groups.

- The materials provide suggestions on methods of observing and listening to students and on alternative methods of assessing student progress.

- The materials provide suggestions for how parents can be involved in the program.

Student Assessment

The student assessment materials in the program provide teachers with information about what students know and how they feel about math. The assessment must be aligned with the standards set forth in this framework. It should include multiple means of assessment integrated across the curriculum. Assessment tasks should be broad in scope and evaluate the extent to which students have internalized concepts and can apply them to new situations. Assessment activities should make appropriate use of technology.

- Assessment is integrated into the curriculum. Assessment activities are similar to learning activities and help teachers determine the extent to which students have made sense of information and whether they can apply it to problem-solving situations.

- Multiple means of assessment are used, including observations, oral and written work, student demonstrations, and group learning activities. The use of technology is built into assessment activities.

- All aspects of mathematical knowledge are assessed, including conceptual understandings and procedural knowledge.

Process for the Selection of Instructional Materials

The South Carolina Department of Education and local school districts should develop appropriate and clearly defined processes for the selection of instructional materials. All materials should receive a thorough evaluation through both analytical and holistic methods. Both methods involve making judgments. In analytic evaluation, the evaluator makes judgments about many independent aspects of the program; in holistic evaluation, the evaluator makes a general judgment about the entire program. The evaluation processes and any evaluation instruments should be developed by groups with expertise in mathematics education and with knowledge of the NCTM *Curriculum and Evaluation Standards for School Mathematics* and this framework. Appropriate and adequate training sessions on the implementation of the process must be provided for committee members who evaluate the instructional materials.

Establishing Review and Selection Committees

At both the state and local levels the review and selection committees should consist of professionally active classroom teachers, mathematics specialists, and administrators who are familiar with this Framework and the NCTM *Curriculum and Evaluation Standards for School Mathematics.* Teachers should be chosen from regular classroom teachers, teachers of gifted students, and teachers of students with special needs. Committees should also include a parent and a representative from the business community. While it is critical that committee members have expertise in mathematics, the committee itself should be balanced in terms of gender, grade-level representation, geographic location, and racial, ethnic, and socioeconomic backgrounds.

Training for the Committees

At both the state and district levels staff development should be provided early for committee members. During the staff development process committee members should receive a thorough review of South Carolina ethics legislation and the implications of that legislation on the work of the committee. Staff development should include a discussion of issues in mathematics education, the goals and learning standards of the programs for which the material is being selected, curricular and evaluation practices, selection criteria, and use of the evaluation process. Reliability and consistency in the selection process can be promoted in the following ways:

- Choose professionally active teachers and coordinators as committee members.
- Establish a common base of agreement on what constitutes a quality mathematics program.
- Select committee members with specific areas of expertise.
- Train committee members in the use of the evaluation instrument.
- Provide adequate time for members to review materials.
- Provide adequate time for members to share findings and ratings.
- Encourage members to justify ratings orally and in writing and to include examples to support their evaluation.

Identifying, Reviewing, and Evaluating Materials

The South Carolina Department of Education and local districts should seek out a wide variety of instructional materials, both nonprofit and commercial, for review and evaluation. Publishers and authors of materials should be provided opportunities to present their materials to the committee, pointing out features that might be overlooked by the reviewers. Guidelines for the presentations should be clearly defined by the South Carolina Department of Education or local school districts with each presenter given a proportionate amount of time relative to the amount of material presented. Sufficient time for the committees to study materials after the presentations is important.

Materials selected at the district or school level should go through a thorough review process. Teachers and parents should be involved in the selection of all instructional materials.

Committee meetings should be collaborative, with each member contributing to the process. Materials should be evaluated within grade levels and across grade levels. Different programs should be evaluated by the "shared comparison" method in which similar parts are laid out side-by-side to see how they compare. After individuals have shared their findings and opinions, the committee should reach consensus on recommendations.

Public Input and Challenging Process

Teachers who are not on the committees, parents, and the general public should be informed that a selection process is taking place and should be given an opportunity to review the materials being considered for selection. Materials should be placed in a central location at the South Carolina Department of Education, the local school district office, or a local school for this purpose. Individuals should, upon request, have an opportunity to provide input to the committee. In accord with state regulations, the South Carolina Department of Education or local school district should develop guidelines for non-committee members to make presentations to the committee. They should also develop a routine procedure to handle complaints about recommended materials after the final selections have been made.

Materials Selection

After committee members have thoroughly reviewed and evaluated the materials and ratings of the materials have been made by the committee, selections should be made for the materials to be placed on the state or local school district list for adoption. If it is determined that no mate als in a particular category meet state or local guidelines, the committee should recommend that purchase of materials in that category be delayed.

The instructional materials recommended by the committees should be adopted unless irregularities in the selection process re documented. Cost of materials should not be the only factor, since financial savings alone do not justify the selection of less desirable materials.

(*Note:* the 2/24/92 draft document *Guidelines for Selecting Instructional Materials for Mathematics* by the Association of State Supervisors of Mathematics and the National Council of Supervisors of Mathematics is the source for many of the ideas in this chapter.)

Principles and Goals for Mathematics Assessment

The successful reform of mathematics education in South Carolina requires not only that expectations, curriculum, and teaching be revitalized, but also that an assessment process documenting the results be redefined. The purpose of assessment is straightforward: to provide information to a variety of constituents in order that they may make informed decisions. For example, assessment may be designed for instructional decision-making or it may designed for accountability. Assessment information must be provided to

- **students**, who need to make decisions about their learning of mathematics;
- **teachers**, who must make careful decisions about how they can best help students develop their mathematical abilities;
- **parents**, who need to participate in and support decisions regarding their children's mathematics program;
- **school administrators**, who must make decisions concerning the effectiveness of the mathematics programs in their schools;
- **school boards**, whose decisions set policies that direct positive change in their local schools;
- **public policy makers**, who must make decisions about the best use of resources to develop and maintain mathematics programs of the highest quality; and
- **the public**, which makes decisions about the effectiveness of their mathematics education systems and the people who are responsible for those systems.

Assessment Principles

Since each of the above groups needs information from assessment to make its own special kinds of decisions, assessment must be based upon a set of guiding principles that provide a coherent rationale at all levels. This framework embraces three guiding principles, abridged from those adopted by the National Summit on Mathematics Assessment (*For Good Measure*, 1991).

- **Assessment must improve learning and teaching.** Whether with classroom assessment or external assessment, the results of assessment must inform and enhance the process of learning and teaching rather than narrow or restrict it.

• **Assessment must promote the development of the talents of all students.** The impact of assessment on any group of students must be determined before implementing a new assessment program, particularly if the assessment program will have high stakes for individuals or groups. To be fair to all, assessment must be sensitive to cultural, racial, and gender differences. There must be multiple ways to demonstrate mathematics achievement; no decision on a student's opportunity to learn mathematics or a student's placement in mathematics should be based on a single assessment.

• **The content of assessment must be derived from the consensus of the discipline.** Mathematics assessment must reflect the best judgment of the professional community of mathematics educators and mathematicians. The quality of assessment is defined by how well it measures the mathematical knowledge, skills, and processes defined by this framework.

Assessment Goal

This framework sets forth the following goal for the reform and revitalization of mathematics assessment in South Carolina.

South Carolina will institute a new, improved assessment system that will exert a positive influence on mathematics education.

A requirement for any realistic attainment of this assessment goal is that South Carolina eliminate the inappropriate use of test results for decisions regarding fund allocation, school deregulation, and teacher and school incentive programs. Furthermore, test results should never be the sole criterion for making student placement decisions.

Four objectives that support the above goal are as follows:

(1) **All assessments will be aligned with the mathematical knowledge, skills, and processes expected of all students in South Carolina.** When assessment is closely aligned with the curriculum, meaningful inferences can be made, and the curriculum itself becomes the standard against which an assessment is made.

(2) **Assessment practices will promote the development of mathematical power for all students in South Carolina.** Good assessment must provide information about what students know

and can do in using mathematics in meaningful ways. It should also provide information about students' self-confidence in mathematics, disposition for mathematics, and ability to communicate using mathematics.

(3) A variety of effective assessment methods will be used to evaluate learning standards of mathematics education in South Carolina. The methods should be substantially *performance-based* and *criterion-referenced.* They should encourage and allow the unrestricted use of the normal tools of mathematics teaching and learning -- calculators, geometric models, measuring devices, and other manipulative materials. The development of these methods must involve teachers, mathematics educators, mathematics assessment specialists, and professional test developers.

(4) South Carolina's citizens will be better informed about mathematics assessment, assessment practices, and the uses of assessment. To use assessment information wisely, all constituents of mathematics education within South Carolina must understand the various purposes of assessment and their meanings. They must understand that

- different information needs may require different assessments and that one assessment cannot serve all needs;
- multiple sources of information lead to better informed decisions (i.e., standardized tests, continuous assessment systems, teacher judgement);
- traditional tests of achievement are only incomplete measures of overall knowledge and capabilities; and
- potential unfairness to an individual or a specific group of people can result if decisions are based on assessment of performance without assessment of equity of opportunity to learn.

(*For Good Measure*, 1991, p. 16)

Classroom Assessment Alternatives

To be genuinely meaningful, assessments must measure what we want students to know and be able to do. They must gauge actual student performance in doing mathematics *(performance-based assessment)*. Students must do mathematics individually and in groups - reasoning, communicating, and solving problems in realistic situations *(authentic assessment)*.

When the performances and products of students' work are examined in detail, they provide a wealth of information about strengths and weaknesses that simple test scores can never reveal. Such information is precisely what teachers want in order to make the best decisions to help students learn; it can substantially enhance the teaching and learning process. And it is precisely what others need in order to make careful, informed judgments about the system of mathematics education.

While multiple-choice tests can be used effectively in some situations, an assessment task like the following one tests higher order thinking skills as well as computational skills. It provides opportunities for making and defending conjectures, for communicating with mathematics, and for connecting mathematics with other disciplines, social studies in this instance.

THE BUDGET DEBATE

The Facts: In 1980 the education budget of a certain community was \$30 million out of a total budget of \$500 million. In 1981 the education budget of the same community was \$35 million out of a total budget of \$605 million. The inflation rate for that one year period was 10%.

The Tasks: 1. Use the facts to discuss that the education budget increased from 1980 to 1981.

2. Use the facts to discuss that the education budget declined from 1980 to 1981.

(Adapted and reprinted with permission from *For Good Measure*, 1991, p. 7)

There has been an explosion of new thinking about assessment in mathematics learning over the last decade, and it is certainly beyond the scope of this framework to summarize it. In addition to such national documents as *Everybody Counts*, the NCTM *Curriculum and Evaluation Standards for School Mathematics*, *Measuring Up*, and *For Good Measure*, the following are excellent resources on new and alternative forms of assessment in mathematics, each one containing important references

- *Educational Leadership* 46 (April 1989), an issue devoted to assessment.

- Jean Kerr Stenmark, editor. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions.* Reston, VA: National Council of Teachers of Mathematics, 1991.
- *Educational Leadership* 49 (May 1992), an issue devoted entirely to performance-based assessment.
- Norman L. Webb and Arthur F. Coxford, editors. *Assessment in the Mathematics Classroom*, 1993 Yearbook. Reston, VA: National Council of Teachers of Mathematics, 1993.

A variety of assessment methods are gaining wide acceptance for their ability to help analyze student performance in mathematics. Though each method has its own unique features, these methods all work together in a complementary manner to provide different views of students' abilities and skills:

- **student work and responses to open-ended questions, problems, and tasks:** Open-ended questions, problems, and tasks have more than a single successful response.
- **projects and investigations:** Projects and investigations are more than short tasks and involve longer and perhaps ongoing work, often extending beyond the conclusion of a unit of instruction.
- **mathematics portfolios:** Portfolios are a showcase of student work, where a variety of projects, writings, assignments, and pieces can be collected. Progress in mathematics and toward understanding mathematics can be seen comprehensively and over time.
- **writing in mathematics:** Writings can be in the form of responses to teacher prompts, such as, "describe how you went about solving the problem ... ," or "how would you explain to your friend that ... ," or in the form of student logs maintained over a period of time.
- **demonstrations, discussions, and presentations:** Students give explanations, using appropriate models, manipulative materials, or technology.
- **observations, interviews, and conferences.** A teacher observes and interviews students at work on individual or group projects and holds conferences about the results with students and other teachers.

As South Carolina reforms its current standardized testing program, better assessments must be designed and implemented at district, school, and classroom levels.

Toward New Assessments

There is a strong national movement for the revision of all mathematics assessments. As a part of that movement the South Carolina Governor's Task Force for Educational Accountability undertook a comprehensive 18-month review and examination of South Carolina's current testing programs as well as the national testing and assessment movement. In November 1991, it issued 16 recommendations for testing and assessment in South Carolina at the public policy level. In October 1991, the report of the South Carolina Department of Education's Excellence Team for Testing and Performance Assessment also made specific recommendations with regard to testing and assessment in South Carolina.

The recommendations in both reports clearly delineate a dual purpose assessment system whose primary purpose is improvement in student learning and which differentiates between testing for accountability and for instructional support. They call for the system to be substantially performance-based with expectations that move well beyond basic skills and incorporate process and thinking skills. Assessment for instructional purposes should be a classroom, school, and district responsibility. The state's role would be one of providing assistance, as necessary, to develop local assessment instruments in terms of resources, examples, processes or ideas; ensuring that local expectations are consistent with state and national standards; and determining that standards are being met.

Assessment for accountability should have a two-pronged approach:

- a *norm-referenced test*, to provide a comparison of our students' performances to those of students in other states and nations, and
- a *criterion-referenced performance assessment*, to determine how well our students are performing against agreed-upon standards.

In the remainder of this section, the best ideas from these two reports have been combined into several specific recommendations.

Relative to a **norm-referenced test**, this framework recommends the following:

(1) South Carolina should use the *best available* such test, chosen by a panel of expert reviewers.

(2) The test should be given in two grade levels, but not before grade 4.

(3) Random matrix sampling by school and by mathematics content strands should be used. There are advantages of sampling when assessing students

(a) If the sampling is representative of gender, race, socio-economic status, ethnic background, and geographic location, it will provide aggregate scores that are as reliable and valid as the aggregate scores obtained through census sampling.

(b) Sampling, including matrix sampling, reduces the amount of time needed for testing. With sampling, only a subset of the population of interest is assessed; with matrix sampling, only a subset of the test items is used. When the amount of time spent on testing is reduced, schools have more time for instruction.

(c) Costs associated with testing are also reduced with sampling procedures. These costs include test booklets, test administration materials, scoring and reporting costs, and wages for test administrators and monitors.

With proper design, such sampling is extremely cost effective and will enable a highly reliable inference to be made about how South Carolina students compare to students from other regions of the country.

(4) Results should be analyzed to provide a state aggregate only. To report norm-referenced results to smaller entities (districts or schools) distorts the limited significance of the results and only serves to sharpen the focus on testing instead of learning.

Relative to **criterion-referenced performance assessments**, this framework recommends the following:

(1) Assessment should take place at the *beginning* of the school year in grades 4 and 8. These are the most appropriate places to assess educational system effectiveness. Assessment at the beginning of the year will provide an accurate profile of the *retained* learning that has occurred through grades 3 and 7.

(2) Every student in grades 4 and 8 will participate in the assessment. A report on each individual student's performance on the assessment will be sent to the student's parents or guardians by the school district.

(3) The assessment instruments should be provided by the state, designed by a task force of teachers of mathematics and test developers from across the state. They should be rich in mathematics content, performance-based, and should use realistic situations as much as possible.

(4) All assessments should encourage and allow free access to, and unrestricted use of, the normal tools of mathematics teaching and learning – calculators, geometric models, measuring devices, and other manipulative materials.

(5) Grading of the assessments should be by a task force of mathematics "readers," much as the national advanced placement calculus examinations are graded. Carefully designed scoring rubrics should be used.

(6) Results of the assessments should be reported to provide an overall state profile only. Results should be reported to each school and each school district to improve teaching and learning.

(7) Instructional and assessment support documents should be available.

Our final recommendations concern the exit examination in mathematics, which a student must successfully complete before earning a high school diploma. The current exam should be replaced with an examination that reflects the expectations of the new curriculum outlined in this framework (Chapter 3), beginning with the inclusion of open-ended problems and allowing the use of the normal tools of mathematics. Students may elect to take the examination as early as grade 10. The examination should be incrementally improved every two years so that it accurately assesses the mathematical literacy expected of all students. The examination and its incremental improvements should be designed and constructed by a task force of South Carolina's most knowledgeable teachers of mathematics, test developers, and business members. Sample copies of model exit examinations should be widely circulated so that students will be well informed of the expectations.

6: Professional Development of Teachers of Mathematics

A Call for Restructuring

A review of the mathematics training of pre-service and in-service teachers clearly indicates a need to restructure their education.

> ... instructional methods that are widely used in undergraduate programs foster a model of teaching - blackboard lectures, template exercises, isolated study, narrow tests - that is (inappropriate) for elementary and secondary school teachers (*Moving Beyond Myths*, 1991, p. 28).

Furthermore, continuing education courses for teachers of mathematics frequently fail to meet their professional needs, both in mathematics content and in pedagogy. And all too often other avenues of professional development are not integrated into a comprehensive program that has unity, purpose, and utility.

Students at all levels are currently taught mathematics primarily through the lecture method. This instructional strategy has probably dominated the educational experience of prospective teachers throughout their years in school. It is not an exaggeration to suggest that by the time they graduate, they have spent approximately 17,000 hours listening to lectures. It is not surprising, then, that the instructional style adopted by most teachers is one dominated by lectures.

If students "are to view mathematics as a practical, useful subject, they must understand that it can be applied to a wide variety of real world problems and phenomena" (NCTM *Curriculum and Evaluation Standards for School Mathematics*, 1989, p. 18). Unfortunately, current mathematics instruction often fails to do this. Connections between topics in mathematics, between mathematics and other disciplines, and between mathematics and the real world are not made.

Technology has had an enormous impact on mathematics education. Because of technology, particularly computers and calculators, some mathematics is no longer relevant, some has become more important, and some that is obscure without technology is now accessible. The pre-service and continuing education of teachers of mathematics must include courses in which appropriate and effective uses of technology are modeled by the instructor.

Many early childhood and elementary pre-service and in-service teachers express a negative attitude toward mathematics. They did not enjoy their undergraduate mathematics courses and do not believe that these courses are relevant to teaching elementary children.

> [illegible] elementary teachers take only one course in mathematics, [illegible] with [illegible] and leaving it with relief. Such [illegible] many elementary [illegible] totally unprepared to inspire [illegible] with confidence in their own mathematical abilities [illegible] 1989 [illegible]

A further problem is that early childhood and elementary teachers are often criticized for having an insufficient background in mathematics content. Most of these teachers were required in their pre-service programs to take 6-9 semester hours of mathematics content and 3 semester hours of methods. However, some teachers may have had as little as one 3-semester-hour course. Furthermore, few graduate-level courses in mathematics content are available to early childhood or elementary teachers.

South Carolina does not currently require, n r does it offer, a specialized middle school mathematics certification. Some of the mathematics teachers at the middle school level are certified at the elementary level while others are certified at the secondary level.

Middle school mathematics teachers must have a depth and breadth of understanding in mathematics considerably beyond that required at the elementary school level. They need to be able to make mathematics alive and exciting, to present a broad, integrated view of mathematics, and to effect the transition from elementary school mathematics to high school mathematics. Furthermore, they must understand the unique characteristics of the middle school student. Neither elementary nor secondary teacher education programs meet these special needs.

While secondary teachers are generally well prepared in mathematical content (at least 24 semester hours), the impact of "17,000 hours of lecture" is evidenced with them as well. The instructional strategy of choice of most secondary teachers, as with elementary, is the lecture method. Teachers need to design lessons in which their students have opportunities to construct knowledge, work with peers, or take advantage of appropriate technology. Furthermore, as with elementary teachers, secondary teachers' knowledge of applications of mathematics needs to be strengthened and broadened.

Besides undergraduate and graduate level coursework, the continuing education of teachers of mathematics takes place through school district in-service programs, workshops, seminars, professional conferences, reading journals and books, and a host of formal and informal gatherings or conversations. Many of these opportunities are limited to a single session without adequate follow-up support. The opportunity to reflect with the aid of the mentor on progress made during and after implementation is often missing. Consequently, even promising ideas may be forgotten or abandoned after initial attempts at implementation. A further problem is that teachers often have little input into the topics chosen for professional development.

Finally, teachers rarely have the time necessary to plan, to discuss teaching practices and activities with one another, to form discussion and study groups to explore new ideas, to meet with support groups when trying out a new technique or methodology, to collaborate, to share ideas, and to solve problems. Teaching should not be a profession where people are expected to perform miracles in isolation.

A Plan of Action

In order for such large scale restructuring to occur, *every* level of school mathematics must re-examine its role in the mathematics education process.

Colleges and Universities

Colleges and universities must examine the extent to which their undergraduate and graduate mathematics and mathematics education courses place subject matter in a context that is meaningful to teachers and the extent to which their courses model the variety of teaching practices and assessment strategies outlined in this framework. For example, a course for elementary teachers on probability and statistics would be useful if taught in a manner that is linked to elementary school instruction and placed in a context meaningful to the elementary student; a purely theoretical course would serve little useful purpose.

If teachers are expected to engage students in active learning, allowing students to construct their own understandings, they must experience this type of learning in their own education. Mathematics courses that teachers take at colleges and universities must model the instructional techniques, assessment strategies, and technologies that they are expected to use. (See Chapter 2.) Such courses should not be dominated by the lecture method of instruction.

> To believe that one can teach mathematics successfully by lectures, one must believe what most mathematicians know to be untrue — that mathematics can be learned by watching someone else do it correctly (*Moving Beyond Myths*, 1991, p. 24).

Furthermore, these courses must address the applications of mathematics so that teachers are able to make mathematics meaningful to their students.

> It is rare to find mathematics courses taken by prospective teachers that pay equal attention to strong mathematical content, innovative curricular materials, and awareness of what research reveals about how students learn mathematics. Unless college and university mathematicians model through their own teaching effective strategies that engage students in their own learning, school teachers will continue to present mathematics as a dry subject to be learned by imitation and memorization (*Moving Beyond Myths*, 1991, pp. 28-29).

In addition, colleges and universities must promote school/college collaboration in the development of courses to meet the continuing education needs of in-service teachers. Their departments of mathematics and education must collaborate to provide the simultaneous study of mathematics and mathematics pedagogy. Colleges and universities must also examine the extent to which they provide "multiple perspectives on ... the influence of students' linguistic, ethnic, racial, and socioeconomic backgrounds and gender on learning mathematics" (NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 144). Finally, colleges and universities must examine the extent to which they encourage participation in professional organizations by students and professors alike.

The South Carolina State Board of Education

The South Carolina State Board of Education must restructure its certification standards to include three different certifications: early childhood (K-4 or K-3), middle school (5-8 or 4-8), and secondary school (9-12).

The NCTM *Professional Standards for Teaching Mathematics* (1991, p. 136) recommends that the following topics be included in coursework for elementary teachers: number systems and number sense; geometry; measurement; probability and statistics; functions and use of variables.

> For teachers of grades K-4, a sufficient understanding of mathematical topics ... cannot be attained with less than nine semester hours of coursework in content mathematics. These mathematics courses assume as a prerequisite three years of mathematics for college-intending students or an equivalent preparation (NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 139)

The Mathematical Association of America's Committee on the Mathematical Education of Teachers also recommends a minimum of nine semester-hours in content mathematics for elementary school teachers (Leitzel, 1991, p. 11). South Carolina *elementary* level teachers should be required to take *at least 12 semester hours of courses that integrate mathematics methods and content* in a sequence designed especially for them. These courses should build confidence in teachers' mathematical ability, foster a disposition to do mathematics, and offer the appropriate content.

Middle school certification must require, in addition, the study of content, applications, and instructional methods in geometry and measurement, probability and statistics, and number theory and patterns (similar to the content preparation of elementary mathematics specialists). Furthermore, it must include the study of the unique characteristics and needs of the middle school student. *A minimum of 18 semester hours of coursework integrating mathematics methods and content* must be required for South Carolina *middle school* certification "These mathematics courses assume as a prerequisite four years of mathematics for college-intending students or an equivalent preparation" (NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 139). This recommendation is supported by the Mathematical Association of America's Committee on Mathematical Education of Teachers, which recommends at least 15 semester-hours of content mathematics for middle school teachers (Leitzel, 1991, p. 17).

Elementary and middle school teachers need to be supported in their schools by a mathematics specialist. The mathematics specialist should be a school-based professional whose full-time or part-time responsibility is serving as a resource to other teachers in the school.

South Carolina mathematics teachers in grades 9-12 must have *the equivalent of a major in mathematics*. NCTM *Professional Standards for Teaching Mathematics* (1991, pp. 138-139) recommends study in the following areas of mathematics: number systems; number theory; abstract algebra and linear algebra; geometry; statistics and probability; calculus and analysis; discrete mathematics. Experiences showing applications of mathematics to other disciplines and the real world are vital. An emphasis on problem solving and the history of mathematics is essential in all courses. Furthermore, the departments of mathematics and education must collaborate to provide the simultaneous study of mathematics and mathematics pedagogy. Four years of mathematics for college-intending students or an equivalent preparation is a prerequisite for all coursework for teachers at this level (NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 139).

School Districts

School districts must improve their programs for in-service education and professional development. To help do this, they should maintain a position for a mathematics supervisor or coordinator. In-service education programs should cover topics suggested by teachers and be planned so that the topics are revisited over a period of time. Resources such as equipment, printed materials, and consultants must support the planned activities. It is especially important that school districts provide extensive in-service education on this framework.

Professional development programs should support the concept of the developmental nature of teaching; that is, a teacher is constantly in the process of "becoming a teacher." Beginning teachers need the opportunity to learn from more experienced teachers during the first year of their careers through mentoring programs, induction year programs, or support groups of peers.

Professional development programs require that school district administrators commit the necessary resources, equipment, time, and funding to ensure successful implementation as well as participate in these programs themselves. For example, because of the use of technology in the workplace and its availability as a resource in mathematics teaching and learning, professional development programs must provide teachers with appropriate equipment and the training to use it effectively in the classroom. As a part of professional development, principals should provide release time for teachers to form collegial links with other faculty and to participate in professional meetings.

Professional Development Programs

Professional development programs for teachers of mathematics should be integrated, comprehensive experiences having unity, purpose, and utility. Single-session workshops, seminars, or presentations can be used to gather information from teachers or to disseminate general information to teachers (for example, about new legislation, a new framework, or planning for the future). But such isolated sessions are not appropriate for the development of new ideas or for their application and transfer to the classroom; they should *not* be the main focus of a comprehensive professional development program.

Professional development programs of high quality for teachers of mathematics should have characteristics like the following

- Experiences in the program are based on identified teacher, school, or district needs.
- Teachers are an integral part of the planning process for the program.
- The program is consistent with this framework, with current national documents such as the NCTM Standards, and with current exemplary practices in mathematics education.
- Mathematics content and pedagogy are integrated in the experiences of the program.
- Instructional materials are provided.
- The program uses good quality presenters whose knowledge and experience inspire, involve, and instruct teachers. The most effective teachers of teachers can be other teachers.
- The program is intensive; it provides long-term, in-depth, sustained activities that include a variety of strategies to help teachers apply what they are learning.
- The program focuses on one or two key topics over a period of weeks or months and includes an academic year follow-through.
- Teachers must have opportunities to collaborate, to network, to discuss issues, to share ideas, and to solve problems as they learn and follow through with the program.
- Instruction in the program must be modeled; that is, teachers must have experiences doing what they will ask their students to do.
- Teachers should have opportunities to observe new practices in action; for example, through visitation, demonstration lessons, or videotaped lessons.
- Administrative support of the program is essential – before, during, and after.
- Teachers should understand that in mathematics education there is no one right answer or no one best way – that the best practices are constantly emerging from real classrooms.

Teachers

"Teachers of mathematics should take an active role in their own professional development by accepting responsibility for

- experimenting thoughtfully with alternative approaches and strategies in the classroom;
- reflecting on learning and teaching individually and with colleagues;
- participating in workshops, courses, [professional meetings and conferences], and other educational opportunities specific to mathematics;
- participating actively in the professional community of mathematics educators;
- reading and discussing ideas presented in professional publications;
- discussing with colleagues issues in mathematics and mathematics teaching and learning;
- participating in proposing, designing, and evaluating programs for professional development specific to mathematics; and
- participating in school, community, and political efforts to effect positive change in mathematics education.

Schools and school districts must support and encourage teachers in accepting these responsibilities."
(NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 168)

Real change in mathematics education requires action by the *entire* community. Schools themselves are not the sole source of all the current problems in mathematics education and cannot be the sole source for improvement. The support and goodwill of the broader community is critical in creating the proper environment in which the teaching and learning of mathematics can flourish. Administrators and school boards, parents and guardians, business and industry, elected officials, and the media all have responsibilities to support change in mathematics education and to ensure the success of this Mathematics Framework. Significant, lasting improvement is simply not possible without their help.

Administrators and School Boards

Administrators at all levels as well as school boards have responsibilities to

- be knowledgeable of this framework and the NCTM Standards documents;
- support and encourage teachers of mathematics to participate in the professional development of their choice and to implement peer coaching endeavors;
- provide instructional leadership by
  - using the expertise of a district-level mathematics supervisor or coordinator,
  - creating a climate that allows for innovation and experimentation,
  - collaborating closely with teachers as they begin to implement this framework and the NCTM Standards documents,
  - having knowledge of current research on curricular, instructional, and assessment innovations,
  - ensuring equity in scheduling, and
  - providing sufficient resources (e.g., equipment, time, finances) to meet or exceed the resources recommended in the NCTM Standards documents;
- evaluate each teacher by using information gathered from a

variety of sources such as the teachers' goals and plans, the students' accomplishments, repeated classroom observations, and national professional standards (*Counting on You*, 1991, p. 22);
- regulate and control class size as a means for effective instruction;
- establish outreach activities with parents, guardians, leaders in business and industry, and other community members to build support for quality mathematics programs; and
- report mathematical successes of students and teachers to the media.

Administrators and school boards can take specific actions to fulfill these responsibilities. They can

- recognize mathematics honor students and those students who have made significant improvement in their mathematics achievement (e.g., at school board meetings, at the students' individual schools, through special luncheons, breakfasts, or assemblies or through awards of T-shirts or special privileges);
- value mathematics honor students with the same degree of enthusiasm as members of the football team;
- oversee funding so that all schools within a district have equitable funding and the equipment needed for mathematics instruction;
- attend a mathematics conference with their teachers;
- be more visible in their schools;
- plan teachers' in-service and workdays with meaningful activities (e.g., programs within schools to meet specific needs, programs requested by teachers, or peer-group meetings);
- arrange interaction among teachers of mathematics from elementary, middle, and high schools;
- sponsor family and community programs like the Family Math Program and make them district-wide events;
- give recognition to volunteers in the schools;
- substitute for classroom teachers on occasion to keep abreast of education needs, concerns, and reality;
- arrange sabbaticals for teachers;
- provide teachers with time for peer group meetings within their schools (e.g., use of teacher aides, creative scheduling);
- provide rewards (other than release to administrative posts) for outstanding teachers of mathematics;

- provide incentives for teachers outside the field of mathematics to upgrade their mathematical knowledge, particularly elementary school teachers and those responsible for a student's entire curriculum; and
- provide encouragement and compensation for working with mathematics clubs and competitions.

Parents and Guardians

Parents and guardians with children in school have special responsibilities to

- create a home environment that fosters high expectations, daily attendance at school, participation in extracurricular activities every year, a common-sense approach to mathematics as necessary and essential, and a positive attitude towards mathematics;
- participate and support educational endeavors through
  - volunteering their time (e.g., chaperoning field trips or making materials),
  - volunteering their expertise (e.g., tutoring, discussing career opportunities, or explaining the need for mathematics in the work place),
  - school-based councils (e.g., PTA, PTO, PTSA, or School Improvement Committees),
  - studying research and training related to changes in curriculum, instruction, assessment, and technology,
  - politically advocating sound educational reforms and tax levies,
  - endorsing the professional development of teachers,
  - supporting this framework and the NCTM Standards documents,
  - encouraging innovative activities in mathematics,
  - seeking information about their school's mathematics program, and
  - supporting the study of mathematics every year through graduation; and
- demand equity in mathematics education for under-represented groups such as females, minorities, and the handicapped.

Parents and guardians can fulfill their responsibilities in a variety of ways. They can

- attend their district school board meetings regularly;
- have family celebrations to recognize mathematics achievement;
- participate fully in local school events and programs;
- set a good example of lifelong learning by reading, attending classes, watching educational programs, using computers, and playing strategy or problem-solving games;
- use community resources such as libraries or museums and attend cultural events;
- cooperate and communicate with teachers and be aware of their child's progress in mathematics;
- take responsibility for their child's education in mathematics;
- promote a positive disposition toward mathematics by pointing out the many ways mathematics is used on a daily basis; and
- provide the appropriate study environment, discipline, and guidance in the home.

Business and industry have responsibilities to

- promote mathematical endeavors by
  - sponsoring and displaying mathematical activities for students in stores and restaurants,
  - providing release time for employees and parents to visit or assist in classrooms, and
  - encouraging parental participation on school-based councils;
- provide resources and personnel for field trips, scholarships or incentives for teachers, homework hotlines, career days, mathematics fairs, adopt-a-school or adopt-a-student programs, technology training, co-op programs, and participating in state and local coalitions;
- arrange for teachers to shadow business/industry roles and arrange for teacher/student teams to connect and apply technology, mathematics, and science; and
- articulate business expectations and assist
  - schools in curriculum development (real-world applications),

improving teaching strategies (cooperative learning and communication skills), and developing appropriate assessments, and
- school board members in hiring mathematically qualified teachers who are committed to implementing this framework and the NCTM Standards documents.

Business and industry can be responsible corporate citizens in their support of mathematics education. Specifically, they can

- allow employees time to share their expertise with students and serve as a resource to the schools;
- allow release time for employees to participate in school programs;
- network with schools to create realistic problem-solving activities involving mathematics;
- help provide better career counseling at elementary, middle, and secondary school levels to reinforce the need for mathematics in a wide range of careers;
- provide opportunities for educational field studies in business and industry that emphasize the need for mathematics;
- offer gift certificates to movie theaters, fast food restaurants, and department stores to students who show outstanding mathematical ability or improvement;
- donate useful equipment, either new or used, to schools; and
- continue the practice of offering reduced automobile insurance rates for students with above average GPAs (particularly in mathematics).

Elected Officials

Elected officials have responsibilities to

- support decisions made by the mathematics education professional community that set directions for mathematics curriculum, instruction, evaluation, and school practice; and
- provide resources and funding for, and assistance in, developing and implementing high-quality school mathematics programs that reach all students, as envisioned in this Framework and the NCTM Standards documents.

Elected officials can lend their support to mathematics education through the prestige of their positions in government. They can

- be more visible at school board meetings and at local school events honoring mathematics achievement;
- send letters of congratulations to students who excel or show great improvement in mathematics;
- mail students articles from newspapers or other publications which feature those students;
- attend mathematics classes at all levels to gain a better understanding of problems and needs;
- act as a resource for teachers of mathematics; and
- volunteer their time to assist or tutor in mathematics classes.

The Media

The various forms of the media have respon bilities to

- [report to] the public [on] the process of change in mathematics and science educatio ;
- inform the public about the new standards for curriculum, teaching, and assessment in mathematics;
- help counteract stereotypes that hinder women and minorities from achieving their full potential in science and mathematics education;
- report on mathematical success stories of students and teachers; and
- promote a positive image of mathematics and its importance to the economic future of the United States.

(*Counting on You*, 1991, p. 23)

Specifically, the media can

- provide more extensive coverage of mathematics honor students;
- create sections in its publications for outstanding mathematics achievement and success stories similar to what is done for sports; and
- be more responsible in providing a better balanced coverage of mathematics education by emphasizing its positive aspects.

(*Note*: the *Mathematics Summit Action Plan* of the Kentucky Department of Education, Frankfort, KY, is a source for some of the ideas in this chapter.)

Epilogue

This Mathematics Framework outlines the key components that describe necessary conditions for improving mathematics education in South Carolina – a philosophical base from which educators can make informed curricular decisions. It is in harmony with national documents that call for change – publications such as the Mathematical Sciences Education Board's *Everybody Counts* and the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* and *Professional Standards for Teaching Mathematics*. These documents are visionary and provide guidance and direction for mathematics education for the 1990s. Leaders in mathematics education agree that the realization of the goals presented in these documents will take a long time. In the January 1992 issue of the *ASCD Curriculum Update*, past president of NCTM Shirley Frye stated, "This major change in mathematics will probably take us a decade" (p. 6). In concurrence, past president Iris Carl stated that it will be the end of the decade before even those people who have concentrated on implementing the Standards can be said to have done so. This framework is not an end in itself but rather a means to an end, a work in progress. Therefore, it appears likely that when South Carolina begins work on revisions to this framework, there will still be unrealized goals. However, the progress made will be recognizable and significant:

- Each school district will have utilized this framework to provide the local support and leadership necessary to transform its local mathematics curriculum into one of which this framework is the core.
- A variety of new instructional materials developed both locally and nationally will be available to assist teaching and learning as described in this framework.
- Assessment instruments and techniques reflective of this framework will be commonly used at local, state, and national levels.
- All professional development models and opportunities available for teachers will be effective and will include the tenets of this framework.
- A vast infusion of technological tools will have occurred in mathematics programs at all levels.

- An increased articulation of mathematics content and teaching methodology from elementary school through higher education will be evident.
- Revised teacher certification and teacher preparation programs will reflect the recommendations of this framework.
- A readiness by mathematics educators to revise and expand the vision of mathematics education for South Carolina will be apparent.

Statements from the *Professional Standards for Teaching Mathematics* (1991) can be used to describe our efforts as we seek to implement this framework.

In addition, *Everybody Counts* (1989) provides a powerful summary of what needs to happen.

Good teaching is neither a set of activities, a program to be purchased, nor a textbook from which to teach. Good teaching is an art, and part of that art is being a watchful participant in the classroom. Among other things, this includes planning with (not simply for) students, capitalizing on their learning potential inherent in every class, and learning the best ways to teach from the students themselves.

In order to be effective teachers of mathematics, teachers themselves must begin thinking like mathematicians and must help students view the world through the eyes of mathematicians. Good mathematics classrooms are not quiet places where students do their own work. These classes should be filled with the joyous noise of question posing and discovery. These classes should have teachers as learners. Good mathematics classrooms are ever changing and do not necessarily look the same from one year to the next but rather reflect what teachers continue to discover about learning and the needs and interests of their students.

There is no single best method of teaching mathematics. Teachers, with the help of their students, must find their own ways. While this framework may be helpful in answering questions about current trends in teaching mathematics, it should also raise important questions that individual teachers themselves must attempt to answer:

How can I make mathematics more meaningful to the lives of my students?
How can I get my students to teach and learn from each other?
How can I help my students to pose good mathematical questions and empower them with the ability to seek answers to some of those questions?
How can I enable my students to see mathematics as a tool for learning?
How can I help my students and colleagues see the role of mathematics in other curricular areas?

Teachers who read this document will be left with important questions of their own about the teaching and learning of mathematics. The answers to these questions lie within the art of teaching.

Restructuring mathematics education is a formidable task. This framework presents us with a challenge. The opportunity for change is upon us, and we are aware that meaningful change takes time. An old saying states, "Where there is a will, there is a way." The way has been charted for the next several years by this framework. The next framework will point the way thereafter. Through our collective wills, we must now implement this framework so that *all* students will develop mathematical power and reap the benefits.

K-12 Mathematics Curriculum Standards by Content Strands

Strand: Number and Numeration Systems

Grades K-3

Students will participate in problem-solving activities through group and individual investigations so that they can

- establish a strong sense of number by exploring concepts such as counting, grouping, place value (other bases as well as base ten), and estimating;
- develop concepts of fractions, mixed numbers, and decimals;
- use models to relate fractions to decimals and to find equivalent fractions;
- communicate number relationships by exploring the comparing and ordering of numbers, fractions, mixed numbers, and decimals; and
- relate the use and understanding of numeration systems to their world.

Grades 3-6

Students will participate in problem-solving activities through group and individual investigations so that they can

- develop number sense for whole numbers, fractions, decimals, integers, and percents;
- develop and use order relations for whole numbers, fractions, decimals, and integers;
- use concrete models to explore ratios and proportions;
- use concrete models to explore primes, factors, and multiples;
- extend their understanding of the relationships among whole numbers, fractions, decimals, integers, and percents;
- connect number and numeration systems with other aspects of mathematics and with other disciplines; and
- relate the use and understanding of numeration systems to their world.

Grades 6-9

Students will participate in problem-solving activities through group and individual investigations so that they can

extend their development of number sense to include all real numbers;
develop and use order relations for real numbers;
understand, represent, and use real numbers in a variety of equivalent forms (integers, fractions, decimals, percents, exponentials, and scientific notation) in a variety of real-world and mathematical problem situations;
understand and apply ratios, proportions, and percents in a wide variety of situations;
develop and apply number theory concepts (primes, composites, factors, and multiples) in a variety of real-world and mathematical problem situations; and
connect number and numeration systems with other aspects of mathematics and with other disciplines.

Grades 9-12

Students will participate in problem-solving activities through group and individual investigations so that they can

develop the hierarchy of the real number system and compare and contrast the various subsystems of the real number system with regard to their structural characteristics;
connect number and number systems with other aspects of mathematics and with other disciplines;
use concrete models to explore fundamental properties of number systems;
develop conjectures and proofs of properties of number systems;

and so that, in addition, students intending to take advanced mathematics can

develop and understand the complex number system;
develop an understanding of the concept of infinity;
investigate limiting processes by examining infinite sequences and series; and

- connect the complex number system with other aspects of mathematics and with other disciplines.

Strand: Numerical and Algebraic Concepts and Operations

Grades K-3

Students will participate in problem-solving activities through group and individual investigations so that they can

- use concrete models to develop an understanding of the concepts of addition, subtraction, multiplication, and division;
- investigate, model, and compare different strategies for constructing basic arithmetic facts with whole numbers;
- use models to allow students to construct their own algorithms for addition, subtraction, multiplication, and division of whole numbers;
- model, explain, and develop reasonable proficiency in adding, subtracting, and multiplying whole numbers and evaluating the reasonableness of results;
- compare and contrast different computational strategies for solving a specific problem;
- use mental computation, estimation, and calculators to predict results and evaluate reasonableness of results;
- use concrete models to explore operations on common and decimal fractions; and
- use whole numbers, common and decimal fractions, variables, equations, and inequalities to describe problem situations.

Grades 3-6

Students will participate in problem-solving activities through group and individual investigations so that they can

- understand and explain how the basic arithmetic operations relate to each other;
- extend their understanding of whole number operations to fractions and decimals;
- use models, patterns, and relationships to construct and analyze algorithms for operations on whole numbers, fractions, and decimals;

model, explain, and develop reasonable proficiency in operations on whole numbers, fractions, and decimals;
gain confidence in thinking and communicating algebraically;
solve real-world and mathematical problem situations using algebraic concepts including variables and open sentences;
use mental computation, estimation, and calculators to predict results and evaluate reasonableness of results;
understand the concepts of variables, expressions, equations, and inequalities; and
use models to explore operations on integers.

Grades 6-9

Students will participate in problem-solving activities through group and individual investigations so that they can

use models, patterns, and relationships to construct, explain, and analyze algorithms for operations on integers and explain how the operations relate to each other;
develop reasonable proficiency in operations on integers and rational numbers;
develop, analyze, and explain techniques for estimation;
develop, analyze, and explain procedures for solving problems involving proportions;
select and use appropriate methods for computing from among mental arithmetic, paper-and-pencil, calculator, or computer methods;
use mental computation, estimation, and calculators to solve problems, predict results, and evaluate reasonableness of results;
understand the concepts of variables, expressions, equations, and inequalities and gain confidence in thinking and communicating algebraically;
represent situations and number patterns with models, tables, graphs, verbal rules, and equations and make connections among these representations;
analyze tables and graphs to identify properties and relationships;
solve linear equations using concrete, informal, and formal methods;
investigate inequalities and non-linear equations informally; and
apply algebraic methods to solve a variety of real-world and mathematical problems.

Grades 9-12

Students will participate in problem-solving activities through group and individual investigations so that they can

- represent situations that involve variable quantities with expressions, equations, inequalities, and matrices;
- use tables and graphs as tools to interpret expressions, equations, and inequalities, using technology whenever appropriate;
- develop, construct, and evaluate formulas to solve a variety of real-world and mathematical problems;
- develop an understanding of and facility in manipulating algebraic expressions, performing elementary operations on matrices, and solving equations and inequalities;
- recognize the worth, importance, and power of the mathematics of abstraction and symbolism;

and so that, in addition, students intending to take advanced mathematics can

- demonstrate facility with operations on the complex number system;
- use matrices to solve linear systems, using technology whenever appropriate;
- demonstrate technical facility with algebraic transformations, including techniques based on the theory of equations; and
- represent problem situations using discrete structures such as finite graphs, matrices, sequences, and recurrence relations.

Strand: Patterns, Relationships, and Functions

Grades K-3

Students will participate in problem-solving activities through group and individual investigations so that they can

- recognize, describe, extend, and create a wide variety of patterns;
- represent, discuss, and describe mathematical relationships;
- use calculators to create and explore patterns;
- make generalizations based on observed patterns and relationships;

explore the use of variables, equations, and inequalities to express relationships; and
connect patterns, relationships, and functions with other aspects of mathematics and with other disciplines.

Grades 3-6

Students will participate in problem-solving activities through group and individual investigations so that they can

use concrete models and calculators to create and explore patterns;
explore, recognize, describe extend, analyze, and create a wide variety of patterns;
represent, discuss, and describe functional relationships with tables, one- and two-dimensional graphs, and rules;
analyze and predict functional relationships and make generalizations based on observed patterns;
explore the use of variables, equations, and inequalities to express relationships; and
connect patterns, relationships, and functions with other aspects of mathematics and with other disciplines.

Grades 6-9

Students will participate in problem-solving activities through group and individual investigations so that they can

use technology along with concrete, numerical, and abstract models to explore, describe, analyze, extend, and create a wide variety of patterns;
represent, discuss, and describe functional relationships with tables, graphs, and rules;
analyze and predict functional relationships and make generalizations based on observed patterns;
use models and technology to analyze functional relationships to explain how a change in one quantity results in a change in another quantity;
use variables, equations, and inequalities to express functional relationships;
make, test, and utilize generalizations about given information as a means of solving real-world and mathematical problems; and

- connect patterns, relationships, and functions with other aspects of mathematics and with other disciplines.

Grades 9-12

Students will participate in problem-solving activities through group and individual investigations so that they can

- understand the logic of algebraic procedures;
- model real-world phenomena with a variety of functions, using technology whenever appropriate;
- represent and analyze algorithms and relationships using tables, verbal rules, equations, and graphs, using technology whenever appropriate;
- translate among tabular, symbolic, and graphical representations of functions, using technology whenever appropriate;
- recognize that a variety of problem situations can be modeled by the same type of function;
- explore calculus concepts informally from both a graphical and numerical perspective, using technology whenever appropriate;

and so that, in addition, students intending to take advanced mathematics can

- analyze the effects of parameter changes on the graph of a function, using technology whenever appropriate;
- understand the general properties, behaviors, and graphs of classes of functions, including polynomial, rational, radical, exponential, logarithmic, and trigonometric functions, using technology whenever appropriate; and
- understand operations on classes of functions, using technology whenever appropriate.

Strand: Geometry and Spatial Sense

Grades K-3

Students will participate in problem-solving activities through group and individual investigations so that they can

- describe, model, and draw two-dimensional geometric shapes to develop spatial sense;

- describe and model three-dimensional geometric shapes to develop spatial sense;
- identify, classify, and compare geometric shapes according to attributes;
- investigate and predict the results of transformations of geometric shapes, including slides, flips, and turns;
- investigate and predict the results of combining and partitioning geometric shapes;
- explore informally tessellations, symmetry, congruence, similarity, scale, perspective, angles, and networks;
- connect geometry to related concepts in measurement and number; and
- identify and appreciate geometry in the world around them, including applications in science, art, and architecture.

Grades 3-6

Students will participate in problem-solving activities through group and individual investigations so that they can

- construct two- and three-dimensional geometric figures with concrete materials;
- identify, describe, classify, and compare two- and three-dimensional geometric shapes, figures, and models according to their attributes;
- develop spatial sense by thinking about and representing geomet ric figures;
- investigate and predict the results of transformations of shapes, figures, and models, including slides, flips, and turns and combinations of slides, flips, and turns;
- investigate and predict the results of combining and partitioning shapes, figures, and models;
- explore tessellations, symmetry, congruence, similarity, scale, perspective, angles, and networks;
- represent and solve problems using geometric models;
- understand and apply geometric relationships;
- develop an appreciation for geometry as a means of describing the physical world; and
- connect geometry and spatial sense to other aspects of mathematics and to other disciplines.

Grades 6-9

Students will participate in problem-solving activities through group and individual investigations so that they can

- model, identify, describe, classify, and compare two-and three-dimensional geometric figures;
- use technology whenever appropriate to explore concepts and applications of geometry;
- develop spatial sense by thinking about, constructing, and draw ing two- and three-dimensional geometric figures;
- investigate and predict the results of combining, partitioning, and changing shapes, figures, and models;
- investigate the results of transformations, including translations, reflections, rotations, and glide reflections, to reinforce concepts such as congruence, similarity, parallelism, perpendicularity, and symmetry;
- apply coordinate geometry to locate positions in two and three dimensions;
- represent and apply geometric properties and relationships to solve real-world and mathematical problems; and
- connect geometry and spatial sense to the physical world, to other aspects of mathematics, and to other disciplines.

Grades 9-12

Students will participate in problem-solving activities through group and individual investigations so that they can

- represent real-world and mathematical problem situations with geometric models and apply geometric properties related to those models;
- use technology whenever appropriate to explore concepts and applications of geometry;
- classify figures in terms of congruence and similarity and apply those relationships;
- deduce properties of and relationships between figures from given assumptions;
- translate between synthetic and coordinate representations;
- deduce properties of figures using transformations;
- deduce properties of figures using coordinate systems;

- analyze properties of Euclidean transformations and relate translations to vectors;
- apply trigonometry to problem situations involving triangles;
- explore periodic real-world phenomena using the sine and cosine functions;

and so that, in addition, students intending to take advanced mathematics can

- develop an understanding of an axiomatic system through investigating and comparing various geometries;
- deduce properties of figures using vectors, using technology whenever appropriate;
- apply transformations, coordinates, and vectors in problem-solving situations, using technology whenever appropriate;
- understand the connection between trigonometric and circular functions;
- use circular functions to model periodic real-world phenomena, using technology whenever appropriate;
- apply general graphing techniques to trigonometric functions, using technology whenever appropriate;
- solve trigonometric equations and verify geometric identities, using technology whenever appropriate; and
- understand the connections between trigonometric functions and polar coordinates, complex numbers, and series, using technology whenever appropriate.

Strand: Measurement

Grades K-3

Students will participate in problem-solving activities through group and individual investigations so that they can

- explore the concepts of length, capacity, weight (mass), perimeter, area, time, temperature, and angle;
- classify angles as acute, right, or obtuse;
- explore, discuss, and use nonstandard and standard (customary and metric) systems of measurement;
- use tools to compare units of measure within a given system;
- make and use estimates of measurement;

- make and use measurement in problems and everyday situations; and
- connect measurement to other aspects of mathematics and to other disciplines.

Grades 3-6

Students will participate in problem-solving activities through group and individual investigations so that they can

- understand the concepts and attributes of length, capacity, weight (mass), perimeter, area, volume, time, temperature, and angle measure;
- understand the structure and use of nonstandard and standard (customary and metric) systems of measurement;
- estimate, construct, and use measurement for description and comparison;
- select and use appropriate tools and units to measure to the degree of accuracy required in a particular situation;
- use concrete and graphic models to discover formulas for finding perimeter and area of common two-dimensional shapes;
- use measurements and formulas to solve real-world and mathematical problems; and
- connect measurement to other aspects of mathematics and to other disciplines.

Grades 6-9

Students will participate in problem-solving activities through group and individual investigations so that they can

- extend thei nderstanding of the concepts and processes of length, capacity, weight (mass), perimeter, area, volume, time, temperature, and angle measure;
- estimate, construct, and use measurements to describe and compare phenomena;
- use suitable methods of approximations to find areas and volumes of irregular shapes;
- understand the structure and use of nonstandard and standard (customary and metric) system of measurement;
- select and use appropriate tools and units to measure to the degree of accuracy required in a particular situation;

- develop the concepts of rates and other derived and indirect measurements;
- use concrete and graphic models to discover formulas for finding perimeter, area, and volume of common two- and three-dimensional shapes;
- use measurements and formulas to solve real-world and mathematical problems; and
- connect measurement to other aspects of mathematics and to other disciplines.

Grades 9-12

Students will participate in problem-solving activities through group and individual investigations so that they can

- estimate, construct, and use measurement for description and comparison;
- choose appropriate techniques, units, and tools to measure quantities;
- choose appropriate techniques for approximating the perimeter, area, or volume of irregular geometric figures or models;
- convert measurement units within a system to solve problems that involve various units, using technology whenever appropriate;
- apply the relationships between precision, accuracy, and tolerance of measurements, using technology whenever appropriate;
- use rates, similarity relationships, and trigonometric ratios to solve problems involving indirect measurements in two or three dimensions, using technology whenever appropriate;
- connect measurement with other aspects of mathematics and with other disciplines;

and so that, in addition, students intending to take advanced mathematics can

- understand the conceptual foundations of limits, infinite sequences and series, the area under a curve, the rate of change, and the slope of a tangent line, and their applications to other disciplines, using technology whenever appropriate.

Strand: Probability and Statistics

Grades K-3

Students will participate in problem-solving activities through group and individual investigations so that they can

- explore concepts of the likelihood of events, including impossible, not likely, equally likely, more likely, and certain events;
- generate questions, collect data, organize and display information, and interpret findings;
- identify and appreciate examples of probability and statistics in the world around them; and
- connect probability and statistics to other aspects of mathematics and to other disciplines.

Grades 3-6

Students will participate in problem-solving activities through group and individual investigations so that they can

- model situations by devising and carrying out experiments or simulations to determine probability;
- extend their understanding of probability and statistics by systematically collecting, organizing, discussing, and describing data, using technology whenever appropriate;
- select and use a variety of representations for displaying data;
- construct, read, and interpret tables, graphs, and charts; and
- make and justify predictions based on collected data or experiments, using technology whenever appropriate.

Grades 6-9

Students will participate in problem-solving activities through group and individual investigations so that they can

- model situations by carrying out experiments or simulations to determine probabilities, using technology whenever appropriate;
- model situations by constructing a sample space to determine probabilities, using technology whenever appropriate;

- make inferences and convincing arguments based on an analysis of theoretical or experimental probability;
- collect, organize, analyze, describe, and make predictions with data, using technology whenever appropriate;
- construct, read, and interpret tables, graphs, charts, and other forms of displayed data;
- evaluate arguments that are based on data analysis;
- develop an appreciation for the pervasive use and misuse of probability and statistical analysis in the real world; and
- connect probability and statistics with other aspects of mathematics and with other disciplines.

Grades 9-12

Students will participate in problem-solving activities through group and individual investigations so that they can

- understand the relationships between theoretical and experimental probability and between probability and odds;
- use experimental or theoretical probability to represent and solve problems involving uncertainty, using technology whenever appropriate;
- use simulations to estimate probabilities from real-world situations, using technology whenever appropriate;
- understand the concept of random variable;
- create and interpret discrete probability distributions, using technology whenever appropriate;
- construct and draw inferences from charts, tables, and graphs that summarize data from real-world situations, using technology whenever appropriate;
- use curve fitting to predict from data, using technology whenever appropriate;
- describe in general terms the normal curve and use its properties to answer questions about sets of data that are assumed to be normally distributed;
- understand and apply measures of central tendency, variability, and correlation and apply the effects of data transformations on measures of central tendency and variability, using technology whenever appropriate;
- design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the results, using technology whenever appropriate;

and so that, in addition, students intending to take advanced mathematics can

- apply the concept of random variable to generate and interpret probability distributions including binomial, uniform, normal, and chi square;
- transform data to aid in interpretation and prediction, using technology whenever appropriate; and
- test hypotheses statistically, using technology whenever appropriate.

S.C. Chamber of Commerce Education Study

In 1992, the Sou'h Carolina Chamber of Commerce conducted a survey of member businesses "to identify what skills and competencies public school graduates in South Carolina need to have in order to be successful in the workplace." According to the Chamber's final report, this survey is "the first time that the South Carolina business community has collectively tried to voice its expectations for graduates of the public school system, grades K through 12." Following are the 37 workplace skills and competencies included in the survey, ranked in descending order from highest priority to lowest priority. (24 of the 37 skills and competencies were ranked as "high" or "very high" in priority.)

Personal Qualities -- Someone who displays responsibility, self-esteem, sociability, self-management, and integrity and honesty.

1. Be honest and ethical in all their dealings
2. Believe in themselves and maintain a positive outlook
3. Try hard and persevere until they achieve their goal
4. Assess themselves accurately, set personal goals, monitor progress, and exhibit self-control
5. Be understanding, friendly, and polite in group settings

Thinking Skills – Someone who thinks creatively, makes decisions, solves problems, conceptualizes, knows how to learn and how to reason.

1. Identify and weigh all options and choose the best alternative
2. Learn new skills
3. Recognize problems and develop plans of action to address them
4. Recognize relationships between people, ideas, or objects and use this information to solve a problem
5. Generate new ideas
6. Conceptualize and process information

Information – Someone who acquires and uses information.

1. Look for information they need
2. Interpret and communicate information to others
3. Organize information in a way that suits their needs
4. Use computers to process information

Interpersonal – Someone who works well with others.

1. Work to satisfy customer expectations
2. Be a good team player
3. Develop leadership skills and not be afraid to take the initiative
4. Appreciate and work well with men and women from diverse backgrounds
5. Teach others new skills
6. Know something of the art of negotiation

Basic Skills – Someone who reads, writes, listens, speaks, and performs math at a level that allows him or her to do their job well.

1. Communicate thoughts, ideas, information, and messages in writing
2. Perform basic math and apply it to everyday situations in the workplace
3. Organize ideas effectively and communicate orally
4. Listen and respond well to the words and non-verbal cues of others
5. Locate, understand, and interpret written information in a wide array of documents, graphs, etc.
6. Speak or understand a second language

Resources – someone who organizes, plans, and allocates resources.

1. Manage time wisely; prepare and follow schedules
2. Manage people in a way that maximizes their motivation and performance
3. Allocate and use materials or space efficiently
4. Use or prepare budgets, make forecasts, keep records, and make adjustments to meet objectives

Technology – someone who works well with a variety of technologies.

1. Select and use appropriate procedures, tools, or equipment – including computers, software, and related technology
2. Understand the overall intent and proper procedures for setup and operation of equipment
3. Maintain and troubleshoot equipment

Systems – Someone who appreciates and understands how social, organizational, and technological systems work.

1. Understand how social, organizational, and technological systems work and operate effectively within them
2. Distinguish trends, predict consequences, diagnose performance, and correct malfunctions
3. Improve existing systems or design new ones

Appendix C: *References*

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