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ABSTRACT

The relationships among types of errors observed during probability problem solving were studied. Subjects were 50 graduate students in an introductory probability and statistics course. Errors were classified as text comprehension, conceptual, procedural, and arithmetic. Canonical correlation analysis was conducted on the frequencies of specific types of procedural and conceptual errors. Variables loading on the first canonical pair suggest that general ability is a factor in successful probability problem solving. Additive trees were fit to the correlation matrix for procedural and conceptual errors, and salient clusters identified on the additive tree provided a visual correspondence to the results of the canonical correlation analysis. Additive trees were also used to investigate the structure of the correlation matrix for errors in all four main categories. Results indicate that difficulties in text comprehension and poor arithmetic skills are responsible for a considerable proportion of observed errors in probability problem-solving. Greater emphasis should be placed on these prerequisite skills, as well as on fostering an appreciation of probability concepts themselves. Suggestions for teaching probability concepts are presented, including a model of normative performance that could be taught in an effort to build more efficient schemas for probability problem solving. Three tables and two figures are included. An appendix summarizes error codes. (Contains 23 references.) (Author/SLD)

Investigating the Relationship between Conceptual and Procedural Errors in the Domain of Probability Problem-solving

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Paper presented at the annual meeting of the Mid-South Educational Research Association
New Orleans, LA; November, 1993

Abstract

The purpose of the study was to investigate the relationships among types of errors observed during probability problem solving. Subjects consisted of 50 graduate students enrolled in a first course in probability and statistics at a large urban university. Errors were classified into four main categories: text comprehension, conceptual, procedural, and arithmetic errors. Canonical correlation analysis was conducted on the frequencies of specific types of procedural and conceptual errors. The first canonical correlation was significant ($p=.02$). Variables loading on the first canonical pair suggest that "general ability" is a factor in successful probability-problem solving. Next, additive trees were fit to the correlation matrix for procedural and conceptual errors, with an $r^2=.70$. Salient clusters identified on the additive tree provided a visual correspondence to the results of the canonical correlation analysis. Additive trees were also used to investigate the structure of the correlation matrix for errors in all four main categories ($r^2=.62$).

Results indicate that difficulties in text comprehension and poor arithmetic skills are responsible for a considerable proportion of observed errors in probability problem-solving. To work towards improved student learning in this domain, greater emphasis must be placed on developing these prerequisite skills, as well as on fostering an appreciation for the probability concepts themselves. Suggestions for teaching probability problem-solving are presented, including a model of normative performance which could be taught to students in an effort to build more efficient schemas for probability problem solving.

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INTRODUCTION

The National Council of Teachers of Mathematic's current vision of mathematics teaching and learning, as presented in Curriculum and Evaluation Standards for School Mathematics (1989), is a shift from a quantitative approach to education, which concentrates on getting the right answer or passing the class, to an approach that is more qualitative or process-oriented, where concentration is placed on the nature of students' understanding of the material. In fact, NCTM suggests that, "good reasoning should be rewarded even more than students' ability to find correct answers" (p. 6). This change in focus from quantitative outcomes of learning to more qualitative outcomes directs a great deal of attention on how people learn, as well as on the extent of student preconceptions and misconceptions in a subject domain (Wittrock, 1991). Certainly, NCTM's Standards suggest strategic changes in the mathematics curriculum which are applicable to all grade levels, including college instruction.

There has been a great deal of recent interest and research in the area of diagnostic teaching, which involves qualitative analysis of the kinds of errors students make and attempts to adapt instruction so as to eliminate such errors. Studies in domains such as algebra (Matz, 1982; Sleeman, 1982), physics (Chi, Feltovich and Glaser, 1981), geometry (Anderson, Boyle and Yost, 1985), computer programming (Anderson and Reiser, 1985), and subtraction (Brown and Burton, 1978; Brown and VanLehn, 1980; VanLehn, 1982) have all focused on the errors or misconceptions that students exhibit or acquire as they are learning a new skill. Additionally, extensive research documents the existence of biases in people's reasoning about probability and probabilistic events (Tversky and Kahneman, 1974, 1983; Konold, 1989), as well as some of the conceptual difficulties students have when learning elementary probability (Hansen, McCann, and Myers, 1985; Garfield and Ahlgren, 1988).

The assessment of what a student knows versus what the teacher would like the student to know is an important part of the learning process. However, the ability to create such a "model" of student knowledge (i.e., a representation of what the student knows and does not know) can be extremely difficult, particularly in complex domains. According to Payne (1988), the generation of a student model must be based upon "a theory of cognitive skill and its acquisition: a theory which determines how knowledge in the target domain is represented, at various levels of expertise; and how it changes from one level to another, under what conditions" (p.69).

A particular challenge to understanding the cognitive structure of student's knowledge is the determination of the relationship between different kinds of misconceptions in a domain. Modeling of a student requires the ability to determine both correct and incorrect concepts or procedures which the student holds. Most of the information about knowledge and misconceptions in a domain can be obtained through the use of verbal or written protocols of student or expert work, or through the use of diagnostic tests that can be designed to elicit predicted errors (Ginsburg, Kossan, Schwartz and Swanson, 1983; Brown and Burton, 1978). However, it is not enough to recognize only surface errors. An investigation of the relationships among misconceptions is critical if instruction is to benefit from knowledge of common, or uncommon, student errors.

In the case of probability problem-solving, an in-depth investigation of the learning process will be aided by an understanding of how misconceptions are related, as well as how they might influence or build upon one another. Such information would provide an opportunity for the design of better instructional strategies in order to help students improve their knowledge of probability and their ability at problem-solving in the domain of probability.

Misconceptions in probability problem-solving may be manifested as errors in comprehension of questions concerning probability, as errors due to misunderstanding of basic concepts, as errors resulting from misguided application or use of particular formulas or rules, or as errors in arithmetic and computation (O'Connell and Corter, 1992). In this paper, the results of two different statistical techniques, canonical correlation analysis and hierarchical clustering using additive trees, are presented in an attempt to determine the relationship among observed errors in the domain of probability problem-solving. Although the focus of this research was on observable errors, a thorough investigation of the relationships among observable errors, particularly among conceptual and procedural errors, provides valuable information as to how flawed conceptual knowledge may impede the correct application of probability formulas and/or rules, and suggests a starting point for the development of pedagogical strategies aimed at improving instruction of probability concepts and procedures.

METHOD

Subjects for this study were 50 graduate students enrolled in a first semester course in probability and statistics at a large urban university in New York City. A detailed coding scheme was used to classify observed errors in these student's written work into four broad categories: Text comprehension errors (T), conceptual errors (C), procedural errors (P), and arithmetic errors (A). A fifth code (X) was used to identify an incorrect solution for which the error could not be determined or classified. Only 5.7% of the total errors observed for this group of students were classified as X. The coding scheme is presented in Appendix A.

The coding scheme was developed during a comprehensive hand analysis of the written work of 180 students at three different universities in New York City, who attempted to solve a total of 93 different problems in probability (O'Connell, 1993). All students received the same basic curriculum, however, not all students solved the same problems. Students were instructed to show all their work during solution to a problem. Inter-rater reliability of the coding scheme was determined in terms of percent agreement for identification of errors from each category, with the following results: T (84%), C (93%), P (82%), and A (89%).

The 50 students chosen for this research constituted all students from one class in the original group of 180. These 50 students were assigned a total of 13 problems, consisting of 39 individual questions. The problems were given at the completion of the probability sequence of the introductory probability and statistics course. Each question was analyzed separately, and the type of error made on each question was recorded. One difficulty with this type of analysis is that when a problem contains several different questions, an error on one part may affect the solution to a subsequent part of the same problem. With this understanding, the following guidelines were adhered to during the error analysis.

1. If a student made an arithmetic error in one part of the problem which affected the solution to any of the remaining questions, the arithmetic error was coded only once. If, however, the student made an error in text comprehension or a procedural or conceptual error which affected the correct solution to subsequent questions, the error was coded each time it affected the solution. This approach is justified because such errors of understanding "carry over" from problem to problem in a manner that is vastly different from a simple calculational or arithmetic error.

2. Often, one student's solution process contained several different errors. All of the observed errors for a solution process were coded according to the coding scheme described above.

3. If a student attempted the problem in more than one way, only the first solution attempt was coded.

Table 1 describes the types of errors which were classified under each of the above four main categories, and provides the frequency of each error type observed in the sample of 50 students. Appendix A describes the specific errors which were included under each error type. For example, several different conceptual errors concerning mutually exclusive events constitute one type of conceptual error; each distinct error was coded individually and errors of the same class or type were combined. For each student, a frequency score on each error type was computed by summing over the frequencies of errors on all 39 problems. Three students made no errors on any of the problems. Hence, the relationships among different kinds of errors was assessed using the frequencies of error types for the remaining 47 students.

The relationships among procedural and conceptual errors were assessed using two techniques: (1) canonical correlation, and (2) hierarchical clustering using additive trees. Only error types with a frequency greater than two were used for these analyses. Correlations between these variables are reported in Table 2. All of the error types used in the analyses were inspected for redundancy to determine if two types of errors were consistently coded together as a pair in a

Table 1

Type and frequency of observed errors in probability problem-solving (n=50 students)

Text Comprehension Errors

Type	Label	Freq.	% of T
T1	Missassigning stated probability value	53	38.4
T2	Incorrect specification of goal (equality)	13	9.4
T3	Choosing pairs instead of triples/singles, etc.	0	0
T4	Misinterpretations of inequalities	16	11.6
T5	Selection with vs. without replacement	2	1.4
T6	Real world knowledge errors	1	0.7
T7	Incorrect model of experiment described in problem	42	30.4
T8	Interference from another (previous) problem	11	7.8
Total		138	(23.1)*

Conceptual Errors

Type	Label	Freq.	% of C
C1	Misconceptions: defn. of probability/sample space/n(S)	0	0
C2	Misconceptions: frequency vs. probability	2	1.8
C3	$p>1.0$	11	10
C4	$p<0$	0	0
C5	$P(S)\neq 1.0$	0	0
C6	formal language of probability	7	6.4
C7	Misconceptions: equally likely events	63	57.3
C8	Misconceptions: mutually exclusive events	17	15.5
C9	Misconceptions: independence	4	3.6
C10	Misconceptions: mutually exclusive vs. independence	5	4.5
C11	Misconceptions: complementary events	1	0.1
Total		110	(18.4)*

Procedural Errors

Type	Label	Freq.	% of P
P1	Procedural errors in determining sample/event space	9	3.3
P2	Incomplete/unfinished	19	7.0
P3	General use of formulas	11	4.1
P4	Procedural errors involving independence	96	35.4
P5	Procedural errors involving mutual exclusiveness	27	10.0
P6	Procedural errors involving sequential experiments	6	2.2
P7	Procedural errors involving use of tabled data	45	16.6
P8	Procedural errors involving conditional probability	34	12.5
P9	Procedural errors involving complementary events	11	4.1
P10	Inventing incorrect procedures or rules	13	4.8
Total		271	(45.4)*

Arithmetic Errors

Type	Label	Freq.	%
A	Totals: Arithmetic errors	54	(9.0)*

Unclassified Errors

Type	Label	Freq.	%
X	Totals: Unclassified errors	34	(5.7)*

* Percent of total errors (total=597)

Table 2

Correlations among all error types (frequency greater than two)

	T1	T2	T4	T7	T8	C3	C6	C7	C8	C9	C10
T1	1.000										
T2	.162	1.000									
T4	-.081	.410[b]	1.000								
T7	.017	-.140	-.094	1.000							
T8	.011	-.087	-.113	-.118	1.000						
C3	-.050	.164	.374[b]	-.067	-.105	1.000					
C6	.003	.302[a]	-.126	-.005	-.059	-.168	1.000				
C7	.051	.129	-.100	-.185	-.091	-.036	.064	1.000			
C8	-.004	.374[b]	.262	-.054	-.115	-.056	.325[a]	-.044	1.000		
C9	.164	-.017	.070	-.157	-.070	.158	-.078	.015	.163	1.000	
C10	.120	.303[a]	.018	-.106	.075	.067	-.053	-.048	-.103	.233	1.000
P1	.167	-.202	-.177	-.116	-.083	.147	.045	.137	-.069	.464[b]	-.074
P2	.238	.251	.208	-.224	-.114	.383[b]	.142	.208	.363[a]	.409[b]	-.102
P3	.271	.060	.135	-.210	-.064	.115	.097	.029	.343[a]	.306[a]	.147
P4	.104	-.042	-.055	.615[b]	-.052	-.150	.010	-.229	.239	-.034	-.114
P5	-.008	.443[b]	.204	.010	-.129	.229	.391[b]	.090	.491[b]	.275	.045
P6	-.029	.030	.186	.066	-.060	.050	-.067	-.110	.110	-.080	-.054
P7	.083	-.128	.281	.029	-.010	.230	-.085	-.087	-.007	.142	-.099
P8	.405[b]	-.015	-.121	-.046	-.108	.108	.029	-.043	-.058	.270	.227
P9	.296[b]	-.096	.147	-.203	-.115	.037	-.127	.060	.188	.172	-.103
P10	.287	.275	.033	-.243	.312[a]	-.003	.073	.144	.236	-.015	.125
A	.206	.037	.187	.183	-.061	.309[a]	-.060	-.188	.104	.450[b]	.338[a]

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	A
P1	1.000										
P2	.263	1.000									
P3	.147	.434[b]	1.000								
P4	.040	-.058	-.039	1.000							
P5	.188	.382[b]	.229	.023	1.000						
P6	-.094	-.023	-.119	-.154	-.019	1.000					
P7	-.021	.197	.088	.106	-.104	-.065	1.000				
P8	.155	.208	.144	.095	-.027	-.088	-.040	1.000			
P9	.161	.363[a]	.301[a]	-.147	.074	-.037	.057	.196	1.000		
P10	.113	.378[b]	.384[b]	-.070	.056	-.054	-.049	-.048	.166	1.000	
A	.276	.419[b]	.441[b]	-.187	.066	.004	.208	.218	.193	.193	1.000

[a] $p<0.05$

[b] $p<0.01$

solution containing multiple errors. No reciprocal redundancy was found, which means that for the case of multiple errors in a single solution, the errors involved were also observed for different students on different problems. Results of the analyses are presented and compared below. Relationships among types of errors in all four main categories (T, C, P, and A) were also assessed using additive trees.

RESULTS

Canonical Correlation Analysis

The relationships among specific types of conceptual and procedural errors was investigated using canonical correlation, which involves searching for the number of ways in which two sets of variables are related, and identifying the strength and nature of these relationships. Canonical correlation seeks to determine weighted linear combinations of each of two sets of variables which maximizes the correlation between these combinations. These linear combinations of variables, taken from each set, make up a pair of canonical variates, and the correlation between the pairs is the canonical correlation. Canonical correlation generally requires 10 cases for each variable used, unless the reliability is quite high. Although the reliability of the coding scheme was fairly strong, the analysis was used for descriptive purposes only; significance of the canonical correlations was noted but greater emphasis was placed on interpretability of the resulting pairs of canonical variates.

All ten types of procedural errors were included in the analysis. Six types of conceptual errors had a frequency greater than 2: C3, C6, C7, C8, C9, and C10. Inspection of the correlation matrix between conceptual and procedural errors after removal of the low frequency variables showed no correlations greater than .62, indicating that multicollinearity was not an issue. The determinant of the correlation matrix for the two sets of variables was low (.02), but for the conceptual errors alone the determinant was calculated as .71, and for the procedural errors, the determinant of the correlation matrix was .28. Bartlett's test of sphericity on the correlation matrix for conceptual and procedural errors together was significant ($p=.02$), suggesting that reliable canonical variates between the two sets could be expected.

The canonical correlation analysis was conducted with the ten procedural errors as dependent variables and the six conceptual errors as covariates. The overall test for the existence of any linear relationship between the two sets of variables was statistically significant (Wilk's $\Lambda = .106$, $F(60, 167.46) = 1.49$, $p=.02$). The first pair of canonical variates correlated .78; 61% of the variance in the variates from the first set (P) could be explained by variates in the second set (C). The second pair of canonical variates correlated .71, with 50% of the variance shared between variates of the two sets. The third canonical pair correlated .54, with 29% of the variance in the set 1 variates explained by the variates in set 2. In total, six canonical pairs were extracted, corresponding to the number of variables in the smaller set (C).

Based on Wilk's criteria, dimension reduction analysis indicated that only the first canonical correlation was significant (Wilk's $\Lambda = .106$, $F(60, 167.47) = 1.49$, $p=.02$). However, the first three canonical pairs were judged interpretable. Table 3 gives the canonical correlations, correlations of the variables from each set with the first three canonical variates, percent of variance explained in each set by their corresponding canonical variates, and redundancies (percent of variance in the other set explained by the canonical variates).

A cutoff correlation of .30 was used for interpreting the canonical pairs. A canonical variate that has a strong positive correlation with an error type can be taken to indicate that a high frequency of occurrence for that error type is important to the interpretation of the canonical pair. Similarly, a variate that has a strong negative correlation with a specific error type indicates that its absence, or low frequency of occurrence, is important to the interpretation of that pair.

For the first canonical variate pair, nearly all the loadings were negative, suggesting that this first pair could be interpreted as indicating "general ability" at probability problem-solving. Students with a low frequency of procedural errors would have a low score on the first procedural variate; low frequencies on the conceptual error types would also produce a low score on the first conceptual variate. Therefore, results suggest that general ability at understanding

Table 3

Canonical correlations, correlations of variables with canonical variates, percent variance explained and redundancies for procedural and conceptual error variables

	First Canonical Variate		Second Canonical Variate		Third Canonical Variate		
Procedural	corr.	% var.	corr.	% var.	corr.	% var.	totals
P1	-.458		.450		.012		
P2	-.833		-.173		-.145		
P3	-.433		-.326		.201		
P4	.067		-.341		-.273		
P5	-.631		-.477		.323		
P6	.030		-.215		-.252		
P7	-.252		.112		-.308		
P8	-.205		.174		.391		
P9	-.263		-.125		-.366		
P10	-.078		-.368		.204		
(same set) % var.		17.7%		9.3%		7.3%	34.3%
redundancy		10.7%		4.6%		2.1%	17.4%
Conceptual							
C3	-.564		.098		.045		
C6	-.288		-.249		.515		
C7	-.226		.156		.129		
C8	-.411		-.873		-.086		
C9	-.739		.241		.135		
C10	.105		-.067		.787		
(same set) % var.		11.9%		7.6%		4.4%	23.9%
redundancy		19.6%		15.3%		15.5%	50.4
Canonical Corr.	.779	60.7%	.705	49.7%	.536	28.7%	

concepts in probability problem solving is paired with general procedural ability in solving these kinds of problems.

Only one conceptual error, misconceptions involving mutually exclusive events, is strongly correlated with the second canonical variate, with a negative loading. Five procedural errors are strongly correlated with the second canonical variate, although only one, procedural errors in determining the sample or event space, has a positive loading. Conceptual errors with the strongest positive loadings on the second canonical variate include misconceptions of equally likely events and misconceptions involving independence. The combination of conceptual and procedural errors most important to the interpretation of this canonical pair indicates that those students harboring misconceptions about equally likely events and independence tend to exhibit procedural errors when determining the correct sample or event space for a problem. In particular, these types of conceptual errors could lead a student to neglect to include permutations of outcomes in a sample or event space. Additionally, such students may experience difficulty if a question involves determining the outcomes or probabilities of outcomes included in an event space when such an event is dependent on any given prior event or outcome in the sample space.

Nearly all the conceptual errors are positively correlated with the third canonical variate. Interpreting the variables with the strongest positive loadings on this third variate, we see that misconceptions involving the distinction between mutually exclusive and independent events and confusion about the formal language of probability are related to procedural errors involving mutually exclusive events and difficulty working with conditional probability. In particular, it was observed that students often interpret the word "and" as "addition", with the following sequence of procedures ensuing: P(A and B) ==> P(A + B) ==> P(A) + P(B), an inappropriate procedure for <u>any</u> two events, regardless of whether or not they are mutually exclusive or independent. This term, P(A and B) also forms the numerator of the formula for determining conditional probability.

Hierarchical Clustering using Additive Trees

Hierarchical clustering was used to discern a more natural structure to the set of conceptual and procedural errors. The regular Pearson correlation provided a measure of proximity between distinct error types. As with the canonical correlation analysis, only those error types with a frequency greater than two were included in the analysis. The ADDTREE/P program (Corter, 1982), based on Sattath and Tversky's (1977) algorithm for fitting additive trees, was used to investigate the hierarchical cluster structure of the proximity matrix.

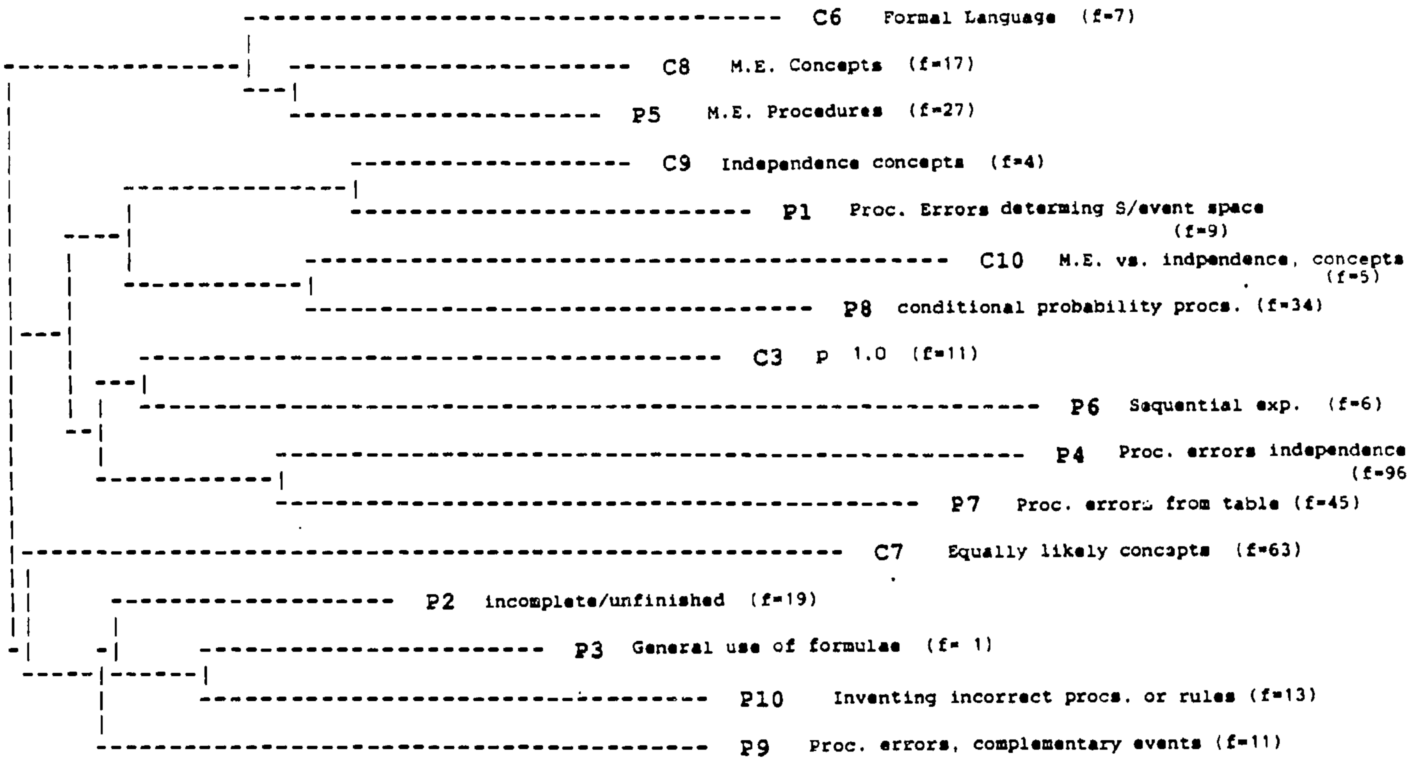
Additive trees provide a convenient visual representation of the relationships among a set of variables, in terms of their cluster hierarchy as well as for interpretation of the unique and common features of items in the tree. Unlike traditional hierarchical clustering schemes, which measure cluster distances in terms of average inter-cluster distances, or in terms of the furthest or nearest neighbor algorithms, additive trees are more faithful to the original distances among the items. In an additive tree, the distance between items in two different clusters is no longer represented as equal for all items in the two clusters; instead, the original distance relation is preserved.

According to Sattath and Tversky (1977), each arc on an additive tree "defines a cluster which consists of all the objects that follow from it. Thus, each arc can be interpreted as the features shared by all the objects in the cluster and by them alone. The length of the arc can thus be viewed as the weight of the respective features, or as a measure of the distinctiveness of the respective cluster" (p. 330). The additive tree analysis for the set of conceptual and procedural error types resulted in three identifiable clusters of procedural and conceptual errors (see Figure 1). A correlation of .83 was obtained between the estimated and the original proximities, explaining 70% of the variance.

The first cluster on the additive tree indicates that conceptual difficulties in working with the formal language of probability (which includes difficulty working with the algebra of sets versus the algebra of real numbers) are related to misconceptions and procedures involving mutually exclusive events. Due to the long arc emanating from this cluster, this is also one of the

Figure 1

Additive tree for conceptual and procedural error types



stress formula 1 = 0.0860
stress formula 2 = 0.4721
r(monotonic) squared=0.7771
r-squared (p.v.a.f.)=0.6988

more prominent relationships observed. As mentioned above during the discussion of the third canonical variate, some students tend to interpret the word "and" as implying "addition", which may lead them to apply the rule for determining the union of two mutually exclusive events when they have actually been asked to find the intersection of two events.

The second cluster identified on the additive tree includes several combinations of conceptual and procedural errors. Taken together, the items in this cluster seem to indicate a very general relationship between misconceptions of independence, conceptual difficulty in distinguishing between independent and mutually exclusive events, and procedural difficulty in solving probability problems which require some knowledge of independent versus non-independent events, such as conditional probability and working with data in table form.

The third cluster on the tree can be identified as procedural difficulty in working with formulas in general. The errors in this cluster include unfinished solution attempts, inventing procedures or rules to "fit" one's understanding of a problem, and difficulty working with formulas for complementary events, which are often confusing for students, particularly if determining the complement of an event is required as a first step towards solution.

One item which appears to stand alone in relation to the other clusters is the concept of equally likely events. This item, then, is relatively unique, although it is clustered together with difficulty working with formulas in general. The assumption that events are equally likely, whether justified or not, makes the computation involved in many probability problems easier. Therefore, students who have difficulty working with formulas in general may also feel comfortable relying on this assumption simply to reduce the complexity of the solution process.

Overall, the additive tree suggests the following clustering scheme for the relationship among conceptual and procedural errors: (1) concepts/procedures involving mutually exclusive events, (2) concepts/ procedures involving independent events and related formulas, and (3) general use of formulas.

To shed more light on the relationship between conceptual and procedural errors, a second additive tree was fit to the correlation matrix for error types in all four categories. The data for this analysis consisted of correlations between 5 text comprehension errors, the 10 procedural and 6 conceptual error types, and 1 class representing errors in arithmetic. The resulting additive tree is shown in Figure 2. Six clusters were identified, as labeled in the Figure. The correlation between the observed and estimated proximities is .79, accounting for 62% of the variance. The fit of this second additive tree is not as good as the fit of the additive tree to the matrix of conceptual and procedural errors alone, possibly due to low frequencies for some of the items. However, some useful information is obtained about the factors which may influence a student's tendency to exhibit certain conceptual and procedural errors.

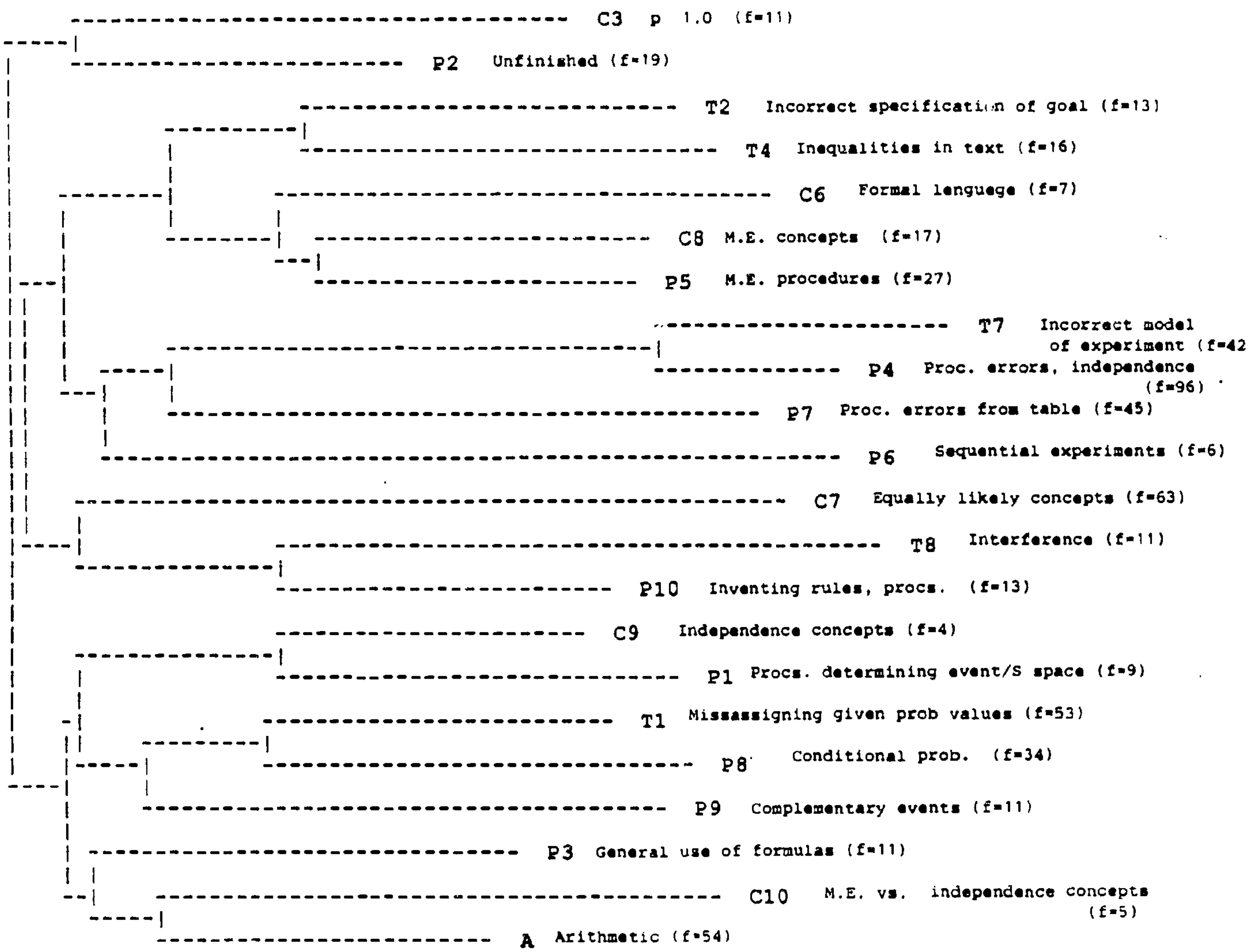
In particular, errors in arithmetic are clustered together with procedural errors in the general use of formulas (cluster 6), suggesting that poor arithmetic skills may be one reason why students are often unsuccessful at probability problem-solving. Difficulty in assigning a given probability value to the correct event forms a cluster (cluster 5) with procedural errors involving both conditional probability complementary probability. This is not surprising, since many students tend to interpret sentences such as "the probability is 1/20 that a female student will be taking a science course" as a conjunction (i.e., P(F and Science)=1/20) instead of a conditional probability (P(Science | F)=1/20). Similarly, if the text of a problem supplies complementary probabilities, such as the probability of an elevator not working, misinterpretation of the given information is likely to occur.

Cluster 4 indicates that interference with information given in a previous problem may be a factor in the tendency to invent procedures or rules, as well as in the tendency to assume that events are equally likely. This clustering suggests that the wording of traditional probability problems, as well as their contextual placement in a set of problems, may be confusing to some students.

The third cluster identified on the additive tree combines one text comprehension error and three procedural errors. Those students who have difficulty representing a situation as described in the text of a problem also have a tendency to exhibit procedural errors involving independence (i.e., applying the formula for the intersection of two independent events without

Figure 2

Additive tree for text comprehension, conceptual, procedural, and arithmetic error types



stress formula 1 = 0.0883
stress formula 2 = 0.5509
r(monotonic) squared=0.6965
r-squared (p.v.a.f.)=0.6169

verifying that the events are indeed independent), have difficulty working with tabled data, particularly in representing the given information in table form, and have difficulty working with sequential experiments, especially in setting up tree diagrams. Assisting students in the correct interpretation of text information may help to alleviate many of these procedural errors.

Cluster 2 contains two related text comprehension errors: incorrect specification of the goal of a probability problem and difficulty interpreting inequality statements. These two errors are combined with misconceptions about the formal language of probability and mutually exclusive events, and procedural errors concerning mutually exclusive events. Those students who exhibit a poor assessment of the quantity requested in a probability problem also tend to have difficulty with these particular concepts and procedures. The first cluster combines unfinished solution attempts with misconceptions concerning the validity of a probability value greater than one, suggesting that students who tend to accept probability values greater than one also have difficulty following a solution strategy through to completion.

DISCUSSION

This study was an investigation into the nature of observed errors in the domain of probability problem-solving. The error analysis was based on the written work of a sample of 50 graduate students enrolled in a first course in probability and statistics. Written protocols of students' problem-solving attempts are helpful in identifying associations among specific types of errors, and comprehensive analyses of these kinds of protocols yield valuable suggestions for the improvement of teaching and learning.

Results of this research have shown that specific relationships exist among conceptual, procedural, arithmetic and text comprehension errors in probability problem-solving. The techniques used to investigate the relationships among errors in each of these categories (canonical correlation analysis and additive trees) constitute a new approach to investigations of misconceptions in the domain of probability problem-solving. Results of the three analyses are similar in their identification of some of the relationships among the errors, but each provides a different perspective as to how these errors are related.

Probability problem-solving involves both an appreciation for probability concepts and an understanding of the terminology and procedures (equations, formulas, rules and their interrelationships) that are used to represent these concepts. From an educators perspective, a student's understanding of probability is recognized by the ability to work within this formal system of concepts and procedures. However, as this research has shown, the ability to correctly solve probability problems is often hindered by poor arithmetic skills and difficulty translating textual information into appropriate probability statements. Stressing normative learning of concepts and procedures is not necessarily enough to promote good problem-solving skills in this domain. Pre-requisite arithmetic skills and the ability to understand information presented in words, as well as in symbols, are crucial to students' development of appropriate cognitive models for probability problem-solving.

As a result of the inter-relationships among the errors, the additive tree fit to the correlation matrix for error types in all four main categories seems to provide the most useful information in terms of implications for teaching and learning probability. In particular, it was shown that poor arithmetic skills are related to difficulties in working with formulas in general. One suggestion for improving instruction, then, is to encourage a prerequisite course in arithmetic and basic algebra before students are allowed to enroll in a first course in probability and statistics. This is especially pertinent for graduate students who may not have had a math course for quite some time. A refresher course in basic mathematical concepts may also help to alleviate the difficulty which some students have in working with inequalities.

Due to the high proportion of errors which could be attributed to text comprehension difficulties (23%), especially regarding translation of probabilities given in the text of a problem and the identification of the goal of a probability problem, students should be given practice at reading and interpreting word problems in probability. Students need the ability to relate natural language to the language of probability. Since many probability problems require understanding of relational operators such as "less than", "at least", etc., students should also be given practice in representing these phrases in set notation. This should be done at the same time that specific procedural methods are being taught.

Incorrect interpretation of the information contained in a problem should not be a primary reason for a student's difficulties. Probability problem-solving involves reasoning about the question being asked; this is a skill teachers cannot assume as prerequisite knowledge in all of their students. In fact, instruction in probability problem-solving should also be approached as instruction in reasoning about probability.

In this research, several specific relationships between text comprehension errors, conceptual errors, and procedural errors were described. Some of the more interesting combinations are briefly re-examined here. In particular, difficulty in determining what a question is asking for (the "goal") was found to be related to difficulty working with sets (formal language of probability), misconceptions about mutually exclusive events and procedural errors involving mutually exclusive events and related formulas. Perhaps students who have trouble understanding the question being asked choose a convenient alternative: assume events are mutually exclusive, and "add" probabilities, whether appropriate or not.

Another interesting combination of conceptual, text, and procedural errors concerns the assumption of equally likely events, interference from information given in a previous problem, and the tendency to invent incorrect procedures or rules. Additionally, procedural errors in working with conditional probability was related to a tendency to inaccurately assign the probability values provided in the text of a problem to the appropriate event.

Due to the structure of the relationships identified in this research, instruction in probability problem-solving needs to address ability in three areas concurrently: text comprehension; an understanding of basic concepts, including set notation and the formal system used to express probability concepts; and the application and manipulation of specific formulas. Instruction should proceed with: (1) knowledge of the relationships among text, conceptual and procedural errors; (2) knowledge of which types of errors are frequently observed in student work; and (3) knowledge of the conceptual determinants of common as well as uncommon procedural errors, such as interpreting the word "and" as implying "addition". Procedural knowledge should be integrated with conceptual knowledge and the ability to accurately discern information contained in the text of a problem. The formulas typically taught in a first course in probability must be taught with greater emphasis on why certain formulas are appropriate and in what situations, as well as how to computationally execute these formulas.

Interpretation of many of the relationships uncovered in this study are consistent with current discussions about the nature of problem solving and the relationships between different kinds of knowledge. For example, Riley, Greeno and Heller (1983) describe three types of knowledge necessary for successful problem solving: a problem schema, for understanding a word problem; an action schema, for relating the representation of the problem to procedures; and strategic knowledge, for planning a solution. These researchers state that "conceptual knowledge can influence which actions get selected" (pg. 188).

It is believed that helping students develop a more efficient schema for solving probability problems would improve their performance on such problems. The following model for normative performance in probability problem-solving is offered as a guideline for how people might typically work towards a successful solution in probability problem-solving. This model provides a framework for the knowledge required and the steps typically involved in successful solution to many different types of probability problems. A description of each of the seven steps included in the model is given below, although the steps do not need to be followed in the particular order given.

1. Understand the given information.
In order to understand the information provided and develop an appropriate representation of the problem, as well as to interpret the given information as a mathematical or probabilistic expression, the student needs to have adequate knowledge of the natural language of probability, as well as an understanding of the concept of probability itself.

2. Identify what is being asked (the goal).

Identifying the goal statement involves the ability to translate the question being asked into a probability statement suitable for solution to the problem. The student needs to distinguish between the following possibilities: is the question asking for a numerical solution to a problem, for the verification of an assumption, for a particular quantity (i.e., complementary probability, joint or conditional probability), etc.? In addition, the student must know the meaning of "at least", "at most", "no more than" and other relational operators.

3. Develop notation for the given information and the goal statement.
This step requires successful completion of steps one and two above, as well as an understanding of the formal, symbolic language of probability in terms of events being described as sets of outcomes. The student must correctly develop a notation for expressing the given information and the relationship between the given information and the goal.

4. Identify the correct sample space for the problem.
The student needs to review possible assumptions or the state of events given in the problem, such as: are the events equally likely or not, are the events independent, are they mutually exclusive? This requires some real world knowledge about events (cards, coins, elevators that work independently, electrical components in series, people's opinions given independently, etc.). In addition, the student requires an understanding of when to assume that a concept (equally likely, mutually exclusive, independent) holds, and how to verify that it is true, if necessary.

5. Select a method of solution.
There may be many different methods of solution appropriate for any particular problem. Successful solution rests on choosing an appropriate method, such as the use of equations, tree diagrams, contingency tables, or Venn diagrams. Successful solution also depends on the ability to switch to a different method if the first does not offer a helpful path towards the desired goal. Recognizing problem types will facilitate the choice of a particular method of solution. Therefore, recall of a method of solution used for a similar problem has an important impact on problem solving ability.

6. Computing the solution.
The procedure chosen for determining the solution depends on both the problem involved and the solution method decided upon. Generally, these fall into four categories: using equations only, the use of tree diagrams, contingency tables, and/or Venn diagrams. Occasionally, problems may be solved with a combination of these four methods. Besides knowledge specific

to the application of each of these four strategies, computing a solution requires basic computational and arithmetical skill, as well as ability at manipulating specific formulas to solve for an unknown quantity.

7. Is the solution reasonable?

Evaluating the feasibility of a solution is one of the most important steps in successful probability problem-solving. It requires real world knowledge, and also an appreciation for the basic tenets of probability theory, i.e., that probability is never negative or larger than one. The solution found for any particular problem should "make sense" to the problem-solver.

To encourage the development of a problem-solving schema, this model for probability problem-solving could be taught to students as a form of "meta-instruction". Normative performance on a problem is highly dependent on the specific type of problem a student is requested to solve, because some problems are by nature more difficult than others, and may require qualitatively different kinds of knowledge. Familiarity with problem types, and how different types of problems might be solved using the above model, should also lead to improved problem-solving schemas, thereby enhancing performance in probability problem-solving.

Probability problem-solving is a difficult task, both to teach and to learn successfully. The major contribution of this research has been in the identification and description of specific associations among text comprehension errors, conceptual error and procedural errors in this domain. Future research based on the results of this study can take many interesting directions, and will help to refine our understanding of the process of probability problem-solving. Several suggestions for the improvement of teaching and learning in this area were presented. My hope is that the results of this study will further stimulate research in the area of probability problem-solving.

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APPENDIX A: SUMMARY OF ERROR CODES

Text Comprehension Errors

• T1 Assigning the probability value provided in the text of a problem to the wrong event. (f=53, 38%)

• T2 Incorrect specification of the goal (when expressed as an equality), i.e., answering a question that is different from the one requested. (f=13, 9%)

• T3 Choosing pairs of outcomes when question asks for only one selection, or vice versa, etc. (f=0)

• T4 Misinterpretations of statements which define sets of outcomes using verbal descriptions of inequalities. (Overall f=16, 12%)

  • • T4.1 Confusing "at least" with "greater than", "equal to", etc. (f=16, 12%)

  • • T4.2 Confusing the word "neither" with "both", "either", etc. (f=0)

• T5 Confusing drawing with replacement and drawing without replacement. (f=2, 1%)

• T6 Real world knowledge errors. (f=1, 1%)

• T7 Setting up an incorrect model of the experiment presented in the text of the problem. (f=42, 30%)

• T8 Interference from another problem, i.e., using information which was provided in a previous, distinct problem. (f=11, 8%)

Conceptual Errors

• C1 Misconceptions involving definitions of probability, sample space or outcomes in a sample space. (Overall f=0)

  • • C1.1 Confusing probability with the sample space. (f=0)

  • • C1.2 Confusing the sample space with the number of outcomes included in the sample space. (f=0)

  • • C1.3 Confusing the probability of an event with the description or list of outcomes making up that event. (f=0)

  • • C1.4 Confusing the probability of an event with the number of outcomes or the number of ways of selecting an outcome. (f=0)

• C2 Not making/recognizing a distinction between frequency of an outcome and probability. (f=2, 2%)

• C3 Accepting a probability value greater than 1.0 as valid. (f=11, 10%)

• C4 Accepting a negative probability value as valid. (f=0)

• C5 Accepting $P(S) \neq 1.0$ as valid where S represents the sample space. (f=0)

- C6 Confusion about the formal language of probability or the distinction between algebra of sets and algebra of real numbers. (f=7, 6%)
- C7 Misconceptions involving equally likely events. (Overall f=63, 57%)
  - •• C7.1 Incorrect definition of equally likely vs. non-equally likely events. (f=0)
  - •• C7.2 Assuming that all outcomes in the sample space are equally likely without appropriate justification. (f=63, 57%)
- C8 Misconceptions involving mutually exclusive events. (Overall f=17, 15%)
  - •• C8.1 Incorrect definition of mutually exclusive events, or the inability to distinguish between mutually exclusive vs. non-mutually exclusive events. (f=9, 8%)
  - •• C8.2 Believing that a single event can be mutually exclusive or not mutually exclusive. (f=6, 5%)
  - •• C8.3 Claiming events are mutually exclusive when the intersection of these events is either provided or observable from data in a table. (f=2, 2%)
  - •• C8.4 Not recognizing that the intersection of mutually exclusive events is null, so that the probability of their intersection is zero. (f=0)
- C9 Misconceptions involving independence: Incorrect definition of independence of events or inability to distinguish between independent vs. non-independent events. (f=4, 4%)
- C10 Confusion between mutually exclusive events and independent events. (f=5, 5%)
- C11 Incorrect definition of complementary events. (f=1, 1%)

Procedural Errors

- P1 Procedural Errors when determining sample space/event space. (Overall f=9, 3%)
  - •• P1.1 Lack of closure when describing sample/event space -- forgetting outcomes, missing specific permutations, repeating pairs or triples, etc. (f=1, 0.4%)
  - •• P1.2 Incorrect construction of the sample space as defined in the problem. (f=0)
  - •• P1.3 Incorrect construction of the event space as defined in the problem. (f=0)
  - •• P1.4 Omitting outcomes or including incorrect outcomes in an event space. (f=0)
  - •• P1.5 Miscounting or omitting one or more outcomes when determining the probability of an event, such as neglecting permutations of outcomes. (f=8, 3%)
  - •• P1.6 When determining the outcomes contained in the union of two events, common outcomes are repeated. (f=0)
- P2 Incomplete or unfinished solution attempts. (Overall f=19, 7%)
  - •• P2.1 Omitting a step in the solution process. (f=0)
  - •• P2.2 Unfinished strategies. (f=19, 7%)
- P3 Procedural errors involving general use of formulas. (Overall f=11, 4%)
  - •• P3.1 Substituting the wrong values into the correct formula. (f=5, 2%)

• • P3.2 Using outcomes in a procedural formula in place of probabilities. (f=0)

• • P3.3 Expressing a probability formula with frequencies of events in place of probabilities of these events. (f=0)

• • P3.4 Confusing formulas for union and intersection of events. (f=1, 0.4%)

• • P3.5 Circular modifications of a formula. (f=5, 2%)

• P4 Procedural errors involving independent events or formulas for independent events. (Overall f=96, 35%)

• • P4.1 Determining the probability of the intersection of two events by multiplying the probabilities of the simple events, without verifying independence. (f=59, 22%)

• • P4.2 Using the relationship P(A | B) = P(A) without checking or verifying if the two events are independent. (f=1, 0.4%)

• • P4.3 Incorrect procedures when investigating independence of events. (f=1, 0.4%)

• • P4.4 Using the multiplication rule for independent events to show that two events are not mutually exclusive. (f=0)

• • P4.5 Assuming independence to prove independence (i.e., applying the same multiplicative relationship on both sides of an equation to 'prove' independence). (f=4, 1.5%)

• • P4.6 Claiming that events are independent/not independent without mathematical verification, when data is in table form. (f=0)

• • P4.7 Claiming that events are independent/not independent without mathematical verification, when data is presented verbally in the text. (f=10, 4%)

• • P4.8 Incorrect formulas: 'or' (or union) of events implies multiplication of probabilities. (f=17, 6%)

• • P4.9 P(A|B) = P(A) * P(B). (f=4, 1.5%)

• P5 Procedural errors involving mutually exclusive events or formulas for mutually exclusive events. (Overall f=27, 10%)

• • P5.1 Incorrect formulas: 'and' (or intersection) of events implies addition of probabilities. (f=22, 8%)

• • P5.2 Determining the probability of the union of two events by summing the probabilities of the simple events, without verifying if the simple events are mutually exclusive. (f=5, 2%)

• P6 Procedural errors resulting from a sequential experiment. (Overall f=6, 2%)

• • P6.1 Each <u>trial</u> or <u>stage</u> in a sequential experiment is described with its own sample space. (f=4, 1.5%)

• • P6.2 Misreading the tree diagram when determining sample space or probabilities. (f=0)

• • P6.3 Believing that when constructing a tree diagram, the immediately prior event cannot be repeated, even when drawing with replacement. (f=0)

• • P6.4 Omitting branches, non-systematically, on a tree diagram. (f=0)

• • P6.5 In a sequential experiment, only one possible outcome is considered during the first stage of the experiment. (f=0)

• • P6.6 In a sequential experiment, each sequence of events is described as a separate sample space. (f=0)

• • P6.7 In a tree diagram, and when drawing more than once without replacement from the same population, the denominator after each selection remains the same, while the number of possible outcomes for the numerator decreases by one. (f=0)

• • P6.8 In a tree diagram, frequencies are used in place of probabilities to label branches at each step. (f=0)

• • P6.9 In a tree diagram, selection and/or probabilities are determined from an individual stage (trial) of the experiment, instead of from the entire experiment. (f=0)

• • P6.10 Using the conditional probabilities obtained on the last step of the sequence as if they were joint probabilities. (f=1, 0.4%)

• • P6.11 Writing the probabilities obtained at the end of the experiment as conditional probabilities instead of joint probabilities. (f=1, 0.4%)

• • P6.12 Incorrect construction of tree diagram or incorrect sequencing of events. (f=0)

• P7 Procedural errors involving data presented or compiled in a table. (Overall f=45, 17%)

• • P7.1 Incorrect determination of simple probabilities or frequencies when reading data from a table. (f=11, 4%)

• • P7.2 Incorrect determination of intersection of events or conditional probability when reading data from a table. (f=4, 1.5%)

• • P7.3 Ignoring the probabilities of simple events as provided in the problem, while using incorrect substitution of conditional probability as a joint probability to complete the cells of a table. (f=30, 11%)

• • P7.4 Incorrect determination of complementary probability when reading data from a table. (f=0)

• P8 Procedural errors concerning conditional probability. (Overall f=34, 13%)

• • P8.1 Using incorrect denominator when determining a conditional probability. (f=8, 3%)

• • P8.2 Using the formula for conditional probability when determining the union of two independent events. (f=0)

• • P8.3 $P(A \mid B) \Longrightarrow P(A \text{ and } B) * P(B)$ or $P(A \mid B) \Longrightarrow P(A \text{ and } B) * P(A)$. (f=1, 0.4%)

• • P8.4 The 'given' sign in a conditional probability statement implies division of the two probabilities. Form is $P(A \mid B) = \frac{P(A)}{P(B)}$. (f=8, 3%)

• • P8.5 Incorrect formulas when attempting to use Bayes Rule: Incorrect numerator or incorrect denominator. (f=6, 2%)

• • P8.6 $P(A) \Longrightarrow P(A \mid M) + P(A \mid F)$ or $P(A) \Longrightarrow P(M \mid A) + P(F \mid A)$. (f=6, 2%)

• • P8.7 $P(A \mid B) \Longrightarrow \frac{P(A)}{P(B \text{ and } A')}$. (f=2, 1%)

• • P8.8 $P(A \mid B) \Longrightarrow P(B \mid A)$. (f=0)

• • P8.9 $P(A \mid B) \Longrightarrow P(A) + P(B)$. (f=0)

• • P8.10 $P(A \text{ and } B) \Longrightarrow P(A \mid B) * P(A)$. (f=3, 1%)

• P9 Incorrect procedures involving complementary events. (Overall f=11, 4%)

• • P9.1 Using the complementary probability of an event as if it were the probability of that event. (f=0)

• • P9.2 The complement of an event is the other event specified in the problem. (f=0)

• • P9.3 $P(\text{not } A) \Longrightarrow \frac{P(A)}{P(A) + P(B)}$. (f=0)

• • P9.4 $P(A \text{ or not } B) \Longrightarrow P(A) \Longrightarrow 1\text{-}P(B)$. (f=0)

• • P9.5 Incorrect procedures involving conditional complementary events. (f=2, 1%)

• • P9.6 Not recognizing the importance of parentheses when describing a set, particularly involving complementary events or probabilities. (f=4, 1.5%)

• • P9.7 Neglecting to take the complement of an event when it is necessary as the final step in the solution. (f=1, 0.5%)

• • P9.8 Given four events, A B C D: $P(A') \Longrightarrow P(B) + P(C) + P(D) - P(A)$. (f=3, 1%)

• • P9.10 The union of two events is the complement of their intersection. (f=1, 0.4%)

• P10 Inventing incorrect procedures or rules. (Overall f=13, 5%)

• • P10.1 Using the number of sequential steps or selections as the frequency when determining probability. (f=2, 1%)

• • P10.2 When the probabilities of an event A, conditional on several other events B, are provided in the text of a problem, the sum of these probabilities is used to determine P(A). (f=0)

• • P10.3 Dividing the probabilities determined or given in the problem by the number of events or selections (an average probability). (f=7, 3%)

• • P10.4 When the set of mutually exclusive outcomes of S are given, multiply probabilities of the outcomes to find the probability of any event of S. (f=0)

• • P10.5 When events are not independent or not mutually exclusive, multiply probabilities. (f=0)

• • P10.6 When all but one of the probabilities of individual outcomes are provided, subtract the given probabilities from each other to calculate the missing probability. (f=0)

• • P10.7 P(A or B) ==> $P(A \cap B) - P(A \cup B)$. (f=0)

• • P10.8 P(A and B) ==> $\frac{P(B)}{P(A \text{ or } B)}$. (f=0)

• • P10.9 P(A and B) ==> P(A) - P(B). (f=1, 0.4%)

• • P10.10 P(A or B) or P(A and B) ==> $\frac{P(A \mid B)}{P(B)}$. (f=1, 0.5%)

• • P10.11 Adding and subtracting successive probabilities when determining the probability of the union of more than two events. (f=1, 0.4%)

• • P10.12 Choosing an answer from one of the probabilities provided in the problem. (f=1, 0.4%)

Arithmetic or Algebra Errors

- A1 Incorrect cancellation of similar terms in numerator and denominator. (f=1, 2%)
- A2 Transcription or copy errors. (f=10, 19%)
- A3 Arithmetic miscalculation. (f=38, 70%)
- A4 Arithmetic error in division: $\frac{a}{b}$ ==> b + a . (f=0)
- A5 Skipped a digit when writing down answer. (f=0)
- A6 Switching variable names in the middle of a solution, switching +/- signs in the middle of a solution. (f=1, 2%)
- A7 Writing probability as a percent without moving the decimal point correctly, or vice versa. (f=0)
- A8 Miscalculation when writing probability as a percent. (f=0, 0%)
- A9 Moving the decimal point. (f=3, 6%)
- A10 Given two values, setting a=b when in fact $a \neq b$. (f=1, 2%)
- A11 Dividing by 100 without moving the decimal point. (f=0)
- A12 Wrote = but multiplied, wrote + but subtracted, etc. (f=0)