DOCUMENT RESUME

ED 355 096 SE 053 270

AUTHOR	Bodenhausen, Judith
TITLE	Using Cognitive Research to Turn a High School 'Remedial' Mathematics Program Inside-Out: A Teacher's Perspective.
PUB DATE	'pr 92
NOTE	25p.; Paper presented at the Annual Meeting of the American Educational Research Association (San Francisco, CA, April 21, 1992).
PUB TYPE	Reports - Research/Tec 1 (143) -- Speeches/Conference Pap. (150)
EDRS PRICE	MF01/PC01 Plus Postage.
DESCRIPTORS	Attitude Change; *Cognitive Development; Cognitive Style; Computation; Context Effect; High Schools; *Mathematics Achievement; Mathematics Education; *Mathematics Instruction; *Problem Solving; *Remedial Mathematics; Secondary School Mathematics; Student Attitudes; Student Behavior; *Teaching Methods; Thinking Skills
IDENTIFIERS	Mathematics Education Research

ABSTRACT

The "Thinking Mathematics" approach to teaching is guided by two ideas: (1) that learning requires knowledge upon which new problems and situations are interpreted and (2) that mathematics skills are applied in context. This paper reports research that examined whether teaching mathematics students in high school remedial classes can be improved by applying the Thinking Mathematics approach. The teacher as researcher technique was employed in which the researcher examined how the research applied to teaching her own ninth-grade remedial classes in an urban high school with a multi-ethnic, multi-cultural student body over a 3-year period of time. In addition, observations of and discussions with colleagues teaching similar classes applying cognitive research were employed. Students scores on pre- and post-tests based on the district's proficiency exam were compared to those of students in similar classes and the students of the pre-study year. Two aspects of the teaching approach were described. First, warm-ups designed to help students develop number sense and skills in counting, estimation, proportional reasoning, mental mathematics, and properties were employed to begin each day. Secondly, situational problems on which primary instruction was based are presented and discussed. Results indicated that the percentage of students in the research classes passing the proficiency exam was 3 times that of students in comparison classes (16% to 48%). Student self-esteem concerning their mathematical ability rose significantly. student attendance was significantly better, and discipline referrals were significantly fewer. While recognizing flaws in the research design, the findings support the applicability of cognitive research to high school remedial mathematics classes. (Contains 52 references.) (MDH)

Using Cognitive Research to Turn a High School 'Remedial' Mathematics Program Inside-Out: A Teacher's Perspective

by

Judith Bodenhausen
University of California, Berkeley
Berkeley High School

Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA, April 20-24, 1992

"All students are capable of what were once considered higher order skills deemed appropriate for only the brightest. All students can solve problems and think critically given appropriate environments. We reject the notion that students should be labeled and categorized for instruction according to a strictly hierarchical view of knowledge That view has served to relegate many students to receiving instruction in only the simplest forms of knowledge — which has been delivered as isolated pieces to be learned by rote — with no thought of ever involving these children in the rich, exciting web of mathematical connections or in real problem solving" (Bodenhausen, Denhart, et al., in press, p. 1).

These words from the introduction to *Thinking Mathematics, Volume 2: Extensions*, reflect a philosophy which is unusual when applied to the elementary school students about whom it is intended. Its application to high school students is almost unheard of. At that level the notion of a hierarchy of mathematical concepts reigns supreme. Throughout the United States, the vast majority of secondary teachers have never even considered that it might be possible for a student who can't even divide (or add fractions or understand percents or...) to do significant, thought-provoking mathematics problems.

In the opinion of most secondary teachers, such students are not even capable of doing mathematics.— they are hopelessly mired in arithmetic. When asked why some high school students fail to learn basic arithmetic skills, a dozen applicants for high school teaching positions, ranging from beginning teachers to individuals with a decade of experience, each avoided answering the question directly but implied poor teaching in the past was usually responsible. When asked how they would teach high school students whose math skills were still below the 'proficiency'[1] level, without exception the interviewees replied that they would break skills down into small components, drill each skill thoroughly so that transitory understanding might become permanent, and progress to a subsequent skill only after a student had mastered the previous one.

---

[1]In California, students must pass a district-determined math 'proficiency' exam which is supposed to be a test of eighth grade skills. In practice, few districts require eighth grade skills for students to pass the test — perhaps to score 100%, but perhaps even that requires a lower level of skills.

Yet, there is no evidence that this 'conventional wisdom' approach works. In fact, circumstantial evidence exists that it does not. The number of students who never advance beyond remedial classes and the frustration of their teachers point in that direction. Furthermore, some research implies the same thing. Silver (1988) found some indication that remedial students add fractions by adding numerators and denominators because their conceptualization of the process is one of adding *pieces* (3 out of 4 pieces combined with 2 of 3 pieces gives 5 of 7 pieces). Resnick (1987) and VanLehn (1986) found that student malrules are intelligent constructions normally based on the student's knowledge at the time of the malrule formation. Resnick also notes that "buggy subtraction algorithms"[2] (and, by implication, other buggy algorithms) are deeply rooted and unlikely "to respond to superficial changes in instructional practices" (1987, p.45).

Few high school teachers, whether novice or veteran, have, however, had an opportunity to encounter this research or any other which might lead to a different philosophy for teaching students in remedial classes. In the first place, little work has been done at this level. While the California Department of Education has developed a course, Math A, in order to provide students unprepared for algebra with an opportunity to do meaningful mathematics, most districts maintain another course for students with low arithmetic proficiency.[3] Researchers have similarly paid little attention to the mathematics learning of students performing at this level. The literature which has begun to develop on the teaching and learning of high school mathematics focuses on grade-level math (Schoenfeld, 1988; Hall, Kibler, Wenger, & Truxaw, 1989; Resnick,

---

[2] Incorrect procedures that students use with regularity.

[3] While the Mathematics Framework recommends that all students unprepared for algebra be assigned to Math A, the researcher's district like many others has continued to offer a basic math class. Even where the school is trying to decrease or eliminate tracking, the attitude is often that assigning the weakest students to Math A dooms that class to failure. It should also be noted that, at one time, the Framework also called for a Math B class to be offered (as a follow-up to Math A). Many schools still offer this although the state no longer considers Math A+Math B to be equal to one year of algebra.

Cauzinille-Marmeche & Mathieu, 1987; Lesgold, Putnam, Resnick, & Sterrett, 1987; Mamona, in press).

In the second place, high school teachers have little contact with research findings which might be applicable to the teaching of students in remedial classes. To most high school teachers,

the substantial literature produced by the movement toward a cognitive understanding of mathematical learning seems to have no relevance for them. The topics for which this literature is well developed, including counting (Gelman & Gallistel, 1986; Fuson, in press) and additive structures (Carpenter & Fennema, 1988; Fuson, in press; Resnick, Bill & Lesgold, in press), for those familiar with them , clearly inform elementary school teaching. However, on the surface, those topics themselves seem to exclude high school teachers.

In areas such as number sense (Resnick, 1983; Sowder, in press; Sowder & Schappelle, 1989), multiplicative structures (Lampert, 1986, 1988, 1990, in press; Nesher, in press; Schwartz, 1988, Vergnaud, 1988), and general learning (Cobb, Wood, & Yackel, in press; Greeno, 1987; Hiebert, 1984, 1986; Kaplan, Yamamoto & Ginsburg, 1989; Nicely, 1986; Resnick, 1989, 1987, 1986, 1983; Resnick & Omanson, 1987), the literature is more formative in nature, but nevertheless guides the instruction of those familiar with it. Again, however, it seems on the surface to contain little which would transfer to the secondary classroom. The mathematical training received by secondary math teachers[4] discusses algebra, geometry, and trigonometry, not number sense and multiplicative structures.

A third segment of this literature does not have the problem of surface irrelevance. It would be conceded by many secondary teachers to be germane to their teaching even though it has also focused on young children. Throughout, it challenges

---

[4]To the extent that it exists at all.

assumptions that have traditionally guided much of the K-12 mathematics teaching in this country — namely that mathematics learning is hierarchical and sequential, that procedural learning is more important than conceptual,[5] and that the layers of the hierarchy involving word problems are more difficult than those which solely involve computation (Resnick, Bill, Lesgold & Leer, in press; Nesher, in press; Resnick & Klopfer, 1989; Hiebert & Lefevre in Hiebert, 1986). However, this literature, too, is largely unavailable to secondary teachers.

The lack of focus on the remedial high school classroom is understandable. Such classes are unlikely to be the focus of a school's internal restructuring efforts. Furthermore, the remedial high school classroom is a difficult venue for the researcher. The students, who would probably not be in the class if their experiences with math had been positive, are unlikely to be cooperative in providing either valid baseline data or insights into their learning process (especially for a stranger). The teacher is reasonably likely to be teaching out of field. Even if a math teacher, his secondary training and interests are unlikely to have prepared him to teach the elementary school subject matter of the class. In either case, he is not likely to welcome a scrutinizing outsider into his classroom.

The Research Question

This research examines the question of whether the problem of lack of guidance in meeting the needs of students in high school remedial classes can be solved by applying other research to teaching these students. In particular, it examines the applicability of cognitive research on the mathematical learning of young children and unschooled adults (Schliemann & Magalhaes, 1990; Saxe, 1988; Carraher, Carraher, & Schliemann, 1985) to the teaching of such students. It also explores the applicability of an approach to elementary school teaching (based on that body of research) that is developed in

---

[5]This has sometimes been reversed, notably in the New Math.

Thinking Mathematics (Bodenhausen, Denhart, et al., in press 1991) to teaching in high school remedial classrooms.

The Methodology

Problems associated with the inaccessibility of remedial classrooms were avoided by using the teacher-as-researcher technique employed by Lampert (1986, 1988, 1990, 1991, in press), and Ball (1990) as the primary research methodology. In addition, observations of and discussions with colleagues teaching similar classes and also trying to apply cognitive research[6] were employed as a secondary methodology.

In each of the previous five semesters,[7] the teacher-researcher investigated whether and how this research was applicable to her own remedial classes. As a teacher at an urban high school with a multi-ethnic, multi-cultural student body, she teaches a cross section of 'at risk' students in her remedial classes. Most live in poverty, many come from single-parent homes (or no parent homes), many do not speak English well (or at all), most have never been successful in school, some are abused, some must work to help support their families, some cannot read, some receive special education services.

The research involved keeping a log and then reflecting on the process, on student progress in developing understanding and procedure acquisition, and on differences between the way in which this group of students functioned as compared to how previous groups of similar students had done so. It also involved designing lessons as a part of this reflective process, recording the logic behind decisions, and recording the outcome of the lessons.

[6]Such colleagues were not always available. Over the period of the study, the teacher-researcher has always been assigned at least one remedial class because of her participation in the LRDC-AFT collaboration (Bickel & Hattrrup, in press, 1992, 1991a, 1991b, 1990, 1989; Hojnacki & Grover, 1992; Leinhardt & Grover, 1990). However, her colleagues are assigned such classes irregularly. The interests and teaching styles of these colleagues vary.

[7] The research has continued during the current semester.

Some analysis of quantitative data was also used in the research although that data was definitely confounded. At the beginning of the semester, each student was given a 'diagnostic'[8] test based on the district's graduation proficiency exam (8th grade level) which tests students' skills in basic computation, 'problem solving,' and applications.[9] In November (or April), each student took the actual proficiency test. At the end of the semester each student was given an exam similar to the diagnostic test.[10] The students' 'actual' test scores were compared to those of other students in similar classes and to those of the researcher's students the pre-study year. Their end-of-semester scores were compared to those of the researcher's students the pre-study year. The students were also compared to the pre-study year's students on the more subjective dimension of teacher assessment of mathematical learning and to both comparison groups as far as attendance and discipline referrals were concerned.

Due to the variation from class to class in student and teacher characteristics, neither the students in similarly constituted classes taught in a traditional manner nor the researcher's students of the pre-study year formed a true control group. However, each did provide a comparison group of students taught under the assumption that in learning mathematics a student progresses through a hierarchy of skills.

Instruction

The classes have presented students with a radical departure from the computational algorithm and key-word oriented remedial classes they have experienced

---

[8]The test was used both to inform the teacher as to the level of a student's skills and to determine whether a student was appropriately placed in the class. Most students who scored 60% or more were moved to a higher level class.

[9]Unfortunately, the district test changed twice during the course of this study, making comparison of scores rather subjective. Except for the first year, the exam was primarily a portion of a standardized test.making it a particularly poor measure for the purposes of this research. Finally, this past year, the district changed the rules for LEP/NEP students, prohibiting them from receiving translation help or extra time on the actual test.

[10]The students present for each of the tests were not necessarily the same ones. There is a great deal of transiency among such students.

for several years. The only thing remaining unchanged has been the emphasis on catching up as much as possible over the course of the semester. Shaped by the *Thinking Mathematics* approach to teaching, the instruction has also been guided by two additional notions: "Learning requires knowledge...[which] cannot be given directly to students [and] before knowledge becomes truly generative --...*[i.e.]* can be used to interpret new situations, to solve problems, to think and reason, and to learn -- students must elaborate and question what they are told, examine the new information in relation to other information, and build new knowledge structures" (Resnick & Klopfer, 1989, p. 5). Secondly, just as relatively unskilled Brazilians develop considerable mathematical skill within the context of their work (Carraher, Carraher, & Schliemann, 1985; Saxe, 1988; Schliemann, 1990; Schliemann & Magalhaes, 1990), students in remedial classes have likewise done so in their out-of-school life; the teacher's role is to help them make connections between what they know out of school and what they need to do in school.

Instructional goals have included efforts to maximize students' opportunities to

- build mathematical knowledge based on their own knowledge, both intuitive and schooled;
- fill in the gaps in their understanding of place value and number relationships;
- construct computational procedures that they could both understand and use successfully;
- focus on relatively complex problems which were not amenable to simplistic strategies such as the key-word.

There has been no expectation that students learn the 'standard' algorithm or even that they master any algorithm at all. An instructional style which provides students with numerous opportunities to discuss their ideas and solutions with their classmates and specifically validates multiple methods of solving problems has helped them understand that they are free to use any (correct) procedure. Whenever possible, corrections of students' misconceptions have been drawn from the students themselves.

In the past, math for these students has generally meant individual completion of ditto after ditto containing a single type of computation or a single type of word problem. The goal has generally been to memorize the algorithms sufficiently well that they could be recalled for the Proficiency Test. Rarely were insight and understanding even implied student goals. Estimation was used only when it could be reduced to an algorithm. Calculators were often provided whenever students were doing word problems but were not used in any other context so that computation and getting answers to practical problems became divorced in students' minds.

In all of the classes included in the study, including both those of the teacher-researcher and those of colleagues, students learned that they were expected to develop the abilities to understand and do mathematics and to use their math to do other things that require math skills. Math, in these classes, became more than doing a bunch of problems neatly on a piece of paper. It became understanding the data given; deciding on the correct procedure; carrying out the computation, graphing, etc. necessary to the problem; interpreting the results; and communicating those results to others. Furthermore, hopefully, math became something that was not boring -- it became interesting and even fun.

Specific Instructional Episodes

Warm-ups

Each day begins with a 'warm-up' designed to help students develop number sense.[11] They are grounded in counting, proportional reasoning, estimation, mental math and properties.

Counting: Young children develop understanding of the decimal number system and ability to solve problems in the additive structures by counting (Gelman and Gallistel, 1986; Fuson, in press). Many high school students in remedial classes have

[11] This is the case not only in remedial classes taught by teachers interested in applying cognitive research but in Math A classes taught by the same teachers.

not developed their counting-based skills beyond using versions of counting on[12] and counting back[13] to solve additive structures problems and computations. These students' sense of number rarely extends beyond the decades and perhaps the hundreds. It is uncommon for them to understand that neighboring digits in a number differ by a factor of 10; it is rare for them to be able to read a decimal fraction without using the word. "point" or to have any understanding of the meaning of a decimal point.

Counting based warm-ups work to remediate this lack of understanding. Stressing, for example, the difference between tenths and hundredths, they usually work best when done orally (although some can be adapted for use as a silent warm-up):

What's this number? (.32)
What's two tenths more?
What's two hundredths more?
What's two tenths more?
What's two hundredths more?

Or they can stress, for example, the structure of the decimal system of numeration:

What's this number? (.3)
What's two tenths more?
What's two tenths more?
What's two tenths more?
What's two tenths more?

The last question is a difficult one early in the semester. Most students can figure out that the answer is "eleven tenths." They cannot, however, figure out how to write the number. When they understand that it is written 1.1, that "one whole[14] and eleven tenths" is written 2.1, etc. they have crossed an important bridge.

12They count the ones up to the next decade and then "carry;" they then treat the tens as if they were ones and repeat the process."

13Here, they may not be able to reverse the process to "borrow" correctly. When this is the case and they are sufficiently motivated to get the correct answer, they use counting back in its simple form, utilizing hash marks if their fingers won't suffice. They may also use the hash marks to solve the problem by counting on.

14In Spanish, the number, 1.1, is read "*un entero y un desimo*" (one whole and one tenth); I have adopted that practice in English during warm-ups.

Proportional reasoning: In the opinion of Vergnaud (1988)and Lampert (in press), and contrary to what most secondary teachers believe,[15] multiplicative structures are those grounded in proportional reasoning. In their opinion, an understanding of this relationship is necessary to an understanding of the multiplication and division processes. Researchers studying unschooled Brazilians have found that many of them use proportional reasoning in doing their 'street math.' My students in remedial classes also often have developed their own proportional reasoning strategies which they use in their own 'street math.' For example, if they know that two pencils cost \$.25, they do not need to skip count to know that a dozen pencils cost \$1.50. However, they do not know that there is a relationship between this reasoning strategy and multiplication or division. Warm-ups in this area help them make this connection. For example:

> Let's develop a table telling us how much different numbers of pencils cost if one pencil costs 7¢.

number of pencils	1	2	3	4	5	6	7	8	9	10
cost	7	14	21	28	35	42	49	56	63	70

> What else do you notice?" "Can you extend the table but leave gaps?" If you spent \$7.00 on pencils, how many did you buy?" If you spent \$7.20 on pencils, how many did you buy? Explain how you could use this same strategy to answer these question if pencils cost 2@25¢.

Estimation: Most students in remedial classes have no idea that estimation has mathematical legitimacy. None have any understanding of the range of estimation techniques discussed by Sowder (in press) and Sowder and Schappelle (1989). Yet, according to these researchers, estimation skills must grow out of understanding. Thus estimation warm-ups not only involve having students estimate, but alsc their discussing different ways for arriving at the estimate, "good" and "not-so-good" estimates, and values of estimation. For example:

---

[15]Most define multiplication only as repeated addition and division only as repeated subtraction.

You just crossed the border at Tijuana. Now all of the distance signs are in kilometers. How many miles away are the following cities? Mexicali-190 km, Ensenada-85 km, Mazatlan-1850 km, Guadalajara-2365 km, Mexico City-3050 km.

How did you decide on your answer? Did anyone else have a different way? Still another way? What are the advantages of each?

Depending on the students,[16] this and other estimation problems can evolve into situational problems for the class. This topic lends itself, beyond the warm-up level, to discussion of multiple methods for problem solution, to the relationship between fractions and decimals, and to comparing magnitudes. It also permits students to get into a discussion — to learn that math is to talk about.

Mental math: Mental math techniques often differ from those students are taught to use in school (Bodenhausen, Denhart, et al., in press, 1991). Students in remedial class may have a reasonable grasp of these techniques, but seem not to connect them with the math they do at school; in fact they often believe that they may **not** use them at school. Warm-ups in this area are, of necessity, oral and are designed to help students make those connections and to sharpen their mental math skills. For example:

82-7=? 102-7=? 93-9=? 103-9=?
candy bars are 3 for 79¢; how much do 6 cost?

Property exploration: This is another area in which student intuitive knowledge often seems not lead to number sense for this group of students. Warm-ups are designed to help students make the connection. They are also designed to help students learn that (scientific) calculators are tools to understanding (Bodenhausen, Denhart, et al., in press, 1991) rather than merely devices to produce answers. For example:

Using your calculator, do the following computations. With your partner, decide what you have learned: 8(23+37), 8(23)+8(37), 15(91)+15(109), 15(91+109).

[16]One major difference between teaching students in remedial classes and students in grade level classes at the high school level is that in the former, the topic under consideration must engage the students. In the latter, the teacher can expect the students to be engaged, no matter what the topic.

Luisa bought 7 single cassettes and 4 packages of 5 cassettes each. How many cassettes did she buy? Calculate the answer to the following problem: 7 + 4 (5). Did you get the same answer or a different answer? Why?

Situational problems

Following the recommendations of the *Thinking Mathematics* approach, the primary instruction in each class is normally based on a situational problem. Designed so that students need to discuss it before beginning to work on it, the problem is usually amenable to more than one solution technique, requires several steps to solve, and is designed to interweave conceptual and procedural learning (Hiebert, 1984; J. Hiebert (Ed.), 1986). Problems are also designed to counteract a problem described by Schliemann (1990). She observed that, "once students had learned a set of rules, they did not want to engage in activities designed to create understanding of those rules" (Bodenhausen, Denhart, et al., in press, p.59).

As they tackle the problem, students are expected to explain their thinking both to each other and to the teacher. It is never enough for them merely to get the correct answer. Rather than practicing a lot on a skill shown them by their teacher, they develop their own solution procedures (with the assistance of the teacher, when necessary) and then justify them. Homework is more like that assigned to grade-level students in that it reinforces the lesson, but also requires them to think.[17]

Presented below are four sample situational problems together with the rationale behind key decisions. These problems are based on situations the students could reasonably expect to encounter and they use the students' names in order to strengthen those connections.[18] These problems vary in difficulty across the range used in a typical class.

---

[17]It is dissimilar in length and in that the thinking required is controlled so that students who can't figure something out don't quit from discouragement.

[18]Although in high school, students soon demand to see their own names in the problems. Name usage can thus become a low-level incentive.

Roberto is doing the shopping for the evening meal on his way home from school. The shopping list includes 4 cans of tomatoes, 2 large packages of spaghetti, 1 small jar of salsa, 2 loaves of sourdough bread, 1 jar of sliced mushrooms, 1 bell pepper. These are the prices at Safeway: tomatoes - 2 cans for 97¢; spaghetti - 89¢ a package; salsa - two jars for $1.69; bread - $1.89 a loaf; mushrooms - $1.45 a jar; bell pepper - 3 for $1.19. How much was Roberto's total? He brought his own bag (a 5¢ discount) and had two 35¢ coupons. How much did Roberto have to pay?

Students are required first to estimate Roberto's total. Then, in doing the problem,[19] they must use proportional reasoning in both multiplication and division situations, decide how to handle remainders, add a series of numbers which might be added more easily using an addition-subtraction procedure, and finally subtract. Students are expected to solve the problem in pairs or groups.[20] Two or three pairs are asked to put their solutions and any drawings they used to help find the solution on the board for the class to discuss.[21] The *Thinking Mathematics* approach encourages teachers to use manipulatives whenever possible.[22]

Ms B's class is going to an A's game. Because they have a special price for tickets, some younger brothers and sisters and parents are going too. Tickets are $3 for children, $5.75 for high school students, and $ 7.50 for adults. Fourteen children, 31 high school students, and 15 adults are going. BART to the Coliseum costs $1.30 each way. How much will the trip cost?

This problem involves some of the skills used in the previous problem but also requires multidigit multiplication and adding whole numbers and decimals (albeit in a monetary context). Again, students are encouraged to use drawings to help them develop their

---

[19]This problem can be cut down for students who have gotten into high school with extremely limited skills. Then, at a subsequent point of the semester, these same students can be given the original problem. They appreciate seeing how they have caught up.

[20]Groups of four are often difficult to use in these classes. Pairs can, however, usually be used successfully. Depending on the problem, pairs can be combined to form groups of four.

[21]At first, students often refuse to explain their work while at the board; they will, however, participate in discussions while at their seats.

[22]High school students in remedial classes often refuse to use manipulatives until they learn to trust the teacher. Drawings can, however, serve as manipulatives which these students will use.

solutions. They are also encouraged to explore the use of the distributive property in doing the multiplication. For example, 31 x \$5.75 = 30 x \$5.75 + 1 x \$5.75 = 30 x \$5 + 30 x.75 + 1 x \$5.75 = \$15 +\$22.50 + \$5.75 = \$43.25 — calculations which can be understood by most students who have difficulty multiplying 31 x \$5.75.

We need to paint this room. Paint costs \$13.95 a gallon and a gallon of paint covers about 400 square feet. The ceiling is 11' 4". How much will it cost?

This problem is a complex one which requires several days to solve. Students must plan out a strategy which begins with measuring the room, windows, doorways, and builetin boards (and perhaps with learning to read a ruler or yardstick and adding mixed numbers using a ruler as a manipulative in order to do this). In order to find the area in square feet, they must decide how to handle the measurements, to some extent by trying different options. They must either divide by 400 or use proportional reasoning strategies and they must decide how to handle the remainder. Finally, they must decide how to multiple by \$13.95. This problem can be followed by a similar (but smaller in scope)[23] problem using metric measures.

The above problem is an excellent example of one which requires students to do real mathematics, to think, and to do computations which can be done in any of several ways. It is sufficiently complex that it cannot be done by an algorithmic procedure. It is also a problem which requires inventive strategies since few of these students are willing even to try dividing by a 3-digit number (even one which ends in 00) and even fewer have any idea how to multiply a 4-digit decimal by a mixed number.[24] It is one

[23]If the scope is not cut down, the problem becomes boring. This group of students gets turned off quickly if it becomes bored — and it's difficult to turn it back on.
[24]The question might be raised about letting students do the computation with a calculator. While there certainly are arguments for doing so, the students in these classes must do there computation by hand because the school district, in its infinite wisdom, has chosen to use a proficiency test which does not permit the use of a calculator. It is also true that calculator use inhibits the development of innovative strategies and thus of thinking.

for which the logical role for the teacher is that of coach and for which the role of lecturer would be a difficult one.

Ms B. calculates grades in this class as follows: Warm-ups-20%, quizzes-15%, homework-20%, coming to class on time and with materials-5%, classwork-40%. Jaqueline has the following scores: warm-ups (48, 49, 48, 47, 50, 43-300 possible); quizzes (20/30, 35/50, 23/40, 43/50); homework (19/24); on-time with materials (10, 10, 10, 8, 10, 9 - 60 possible); classwork (85,92,87,94,89,91-600 possible). What is Jaqueline's grade this marking period? What is your grade this marking period?

This problem is not only one which relates to the students' daily lives, it is one which is likely to engage them. In order to do the problem, they have to interweave their conceptual and their procedural learning. They must devise a method of organizing their data and they have to learn what percent means and to differentiate between fractions and ratios. They have to learn how to weight calculations and to make sure of their referents when they add, but they nevertheless have a range of solution methodologies from which to craft their own.

When the class works on these and other situational problems, students are no longer *burros*[25] who aren't even entitled to a book.[26] They are students who are doing real mathematics. They are no longer children who have to practice second and third grade math, they are students who must think in order to do their work. Over time, they gain confidence that this new identity of theirs is the real one.[27] Their grade reflects not how well they do on tests that they are afraid of but how hard they work on tasks that

---

[25]In Spanish slang, this term means someone who is slow mentally and incapable academically. It's become a term that others also use.

[26]While they don't have a commercially printed book, they do have one that they create. They also have a portfolio that they keep. They must keep both in a binder, keep them organized, and keep them up to date (study skill that are also part of the class curriculum).

[27]Some may question the reality of this and other statements. While some students do not succeed in this type of a class, the proportion who fail is significantly lower than it is in 'traditional' remedial classes. It should also be remembered that there exists a gradualness to these events — they occur over time.

are challenging but not so challenging that they seem impossible.[28] They are acquiring "not only the fundamentals of a discipline, but also the ability to apply those fundamentals, and — critically — a belief in their own capabilities as learners and thinkers" (Resnick, Bill, Lesgold, & Leer, in press, p. 137).

Other Elements of a Cognitive *Thinking Mathematics* Approach

Four additional elements differentiate the instruction of teachers trying to use a cognitive, *Thinking Mathematics* approach to teaching their remedial classes. The first is that these teachers are trying to develop students' "trust in their own knowledge" (Resnick, Bill, Lesgold, & Leer, in press). Alluded to above, this development of students' 'trust in themselves' is crucial to the success of the approach for high school students in remedial classes. It is very difficult to do something you do not believe you can do. Students who do not trust in their ability to do mathematics cannot do math in school because they believe they cannot do math in school. Yet many of them can do math out of school (although they may not realize that they are doing math). From the beginning of the warm-up to the conclusion of the situational problem solution(s), developing students' trust in their own mathematical abilities must be a primary goal.

The second element is recording. Students in high school remedial classes are often caught in the nether-world of being without the skills they need in order to succeed in school but being at a grade level at which their teachers expect them to have those skills. A primary example of a skill which is often missing is recording. Students without this skill listen to things in class and perhaps think about what is happening, but they make no records. They have no record of their own ideas and abilities; they have no record of things they will need later on. Students who record their work in a *Thinking Mathematics* class link their own work and that of their classmates to formal

[28]The degree to which problems are challenging varies with the time of the semester and with the students in the class. As with homework, they cannot be so challenging that they defeat the student. These students don't bounce back easily from defeat.

mathematics. They begin to progress beyond the 'finger' mathematics[29] to which they have been limited to symbol mathematics which can be used to solve most of the problems which they can imagine themselves encountering. Recording also transfers to other classes; students who can record can take notes. It should be noted that this skill must be specifically taught — usually through a system of rewards.

For want of a better name, the third element can be called spiralling.(although for many secondary teachers that word conjures up images of 'drill and kill' textbooks).[30] It is meant to imply an approach which treats a topic and returns to it again and again, expecting mastery to develop over time rather than at first presentation. The returns to the topic may be presented in different ways so that students with different conceptualizations can also master it. A further element of both the spiralling approach and of *Thinking Mathematics* is that the acceptance of a variety of solutions, some less sophisticated than others, permits students whose skills are at different levels to work with each other on the same problem.

Finally, this approach to teaching provides a 'bridge to higher mathematics.' Students in remedial classes want the prestige that accompanies taking algebra — even though they have no concept of what that means. Because the *Thinking Mathematics* approach is real mathematics, students can legitimately be told that they are studying algebra concepts and that in their algebra class, they will link these concepts together. Furthermore, students who have investigated the distributive property, order of operations, triplet relationships,[31] mental math problem solving, referents, etc. will

---

[29]Math in which all computation can be done using fingers, counting, and skip-counting. It is not necessarily bad, but it can be limiting.
[30]In particular, one series of textbooks which is sold on the basis of its spiralling, focuses on rote learning.
[31]6+3=9 implies that 9-6=3 and that 9-3=6.

have an easier time with these concepts in an algebra class than will students who have never seen them before.[32]

Results

The percentage of students in these classes passing the proficiency exam (16% to 48%), while still low, was twice that of students in comparison classes — both those which were taught by other teachers during the research period and those taught by the researcher prior to the research period. The improvement over previous scores on the part of students who did not pass was significantly greater than was that of the comparison students.[33] Most importantly, students' concept and skill retention over the course of the semester, as measured both by exam and by teacher assessment, were significantly greater than they had been in any of the teachers' pre-research period classes. Student self-estimation of their own mathematical capability[34] rose significantly. Their attitudes toward math and their attitudes toward school changed for the better. Furthermore, attendance was significantly better (an average of 55% more of assigned students were present each day and the number of chronic attendance problems was 70% less) and discipline referrals significantly fewer for students in study classes than for those in the comparison classes.[35]

Discussion

While this research has some definite flaws — specifically a possible Hawthorne effect which is common to much teacher research — its findings support the notion of the applicability of cognitive research on young children's mathematical learning to the high

[32]In many, if not most, of the countries which out perform the United States on international comparisons, the curriculum is designed to introduce children to algebra concepts at an early grade. While things have improved in this country in recent years, the vast majority of algebra concepts are first encountered by American students in their algebra class.
[33]Because of the changes in the test, this figure is only valid in comparing fall and spring scores.
[34]This was measured by paragraphs written in their notebooks.
[35]This may not be due only to *Thinking Mathematics*.

school remedial classroom. While using the instructional approach set forth in *Thinking Mathematics* is not a panacea, this research points to a method of improving a very difficult type of teaching. It also points to a method of successfully reaching "at risk" students.[36] Furthermore, it points to a change that, because it is curriculum based, has the potential of being usable by large numbers of teachers, not just those who have some specific type of personality characteristics or those who have access to some specific training program.

---

[36]It needs to be stressed that this "reaching" is more gradual in a high school remedial class than it is in either a "regular" high school class or in an elementary class. Student patterns of being turned off need to be overcome.

References

Ball, D. The role of subject matter knowledge in teaching elementary school mathematics. Paper presented at the annual meeting of the American Educational Research Association, Boston.

Bickel, W. E., & Hattrrup, R. A. (in press). A case study of institutional collaboration to enhance knowledge use: Restructuring practitioner-researcher dialogue in education. *Knowledge in Society: The International Journal of Knowledge Transfer.*

Bickel, W. E., & Hattrrup, R. A. (1992). Lessons learned: The design of practitioner-researcher collaborations. Paper to be presented at the annual meeting of the American Educational Research Association, San Francisco, CA.

Bickel, W. E., & Hattrrup, R. A. (1991a). Paths to Professionalism. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.

Bickel, W. E., & Hattrrup, R. A. (1991b). *Disseminating new knowledge about mathematics instruction: A collaboration between the American Federation of Teachers and the Learning Research and Development Center, Annual Report.* National Science Foundation, Washington, DC.

Bickel, W. E., & Hattrrup, R. A. (1990). *Disseminating new knowledge about mathematics instruction: A collaboration between the American Federation of Teachers and the Learning Research and Development Center, Annual Report.* National Science Foundation, Washington, DC.

Bickel, W. E., & Hattrrup, R. A. (1989). *Disseminating new knowledge about mathematics instruction: A collaboration between the American Federation of Teachers and the Learning Research and Development Center, September 1987-April 1989, Interim Report.* National Science Foundation, Washington, DC.

Bodenhausen, J. A. (1991). Extrapolating from what we know about how young children learn mathematics to the high school 'catch-up' classroom. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.

Bodenhausen, J. A., Denhart, N., Gill, A., Kaduce, M., Miller, M., Grover, B., Resnick, L. B., Leinhardt, G., & Billups, L. (1991). *Thinking Mathematics, Volume 2: Extensions.* Washington, DC: AFT

Bodenhausen, J. A., Denhart, N., Gill, A., Kaduce, M., Miller, M., Grover, B., Hojnacki, S., Resnick, L. B., Leinhardt, G., Bill., V., Rauth, M., & Billups, L. (1991). *Thinking Mathematics, Volume 1: Foundations.* Washington, DC: AFT

Carpenter, T. P. & Fennema, E. (1988). Research and cognitively guided instruction. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating Research on Teaching and Learning Mathematics* (pp.2-17). Madison, WI: Wisconsin Center for Education Research.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the Streets and in Schools. *British Journal of Developmental Psychology*, 3, 21-29.

Cobb, P., T. Wood, & Yackel, E. (in press). Classrooms as learning environments for teachers and researchers. *Journal for Research in Mathematics Education.*
Fuson, K. C. (in press). Conceptual structures for multiunit numbers: Implications for learning and teaching addition, subtraction, and place value. *Cognition and Instruction.*

Fuson, K. C. (in press). Research on learning and teaching addition and subtraction of whole numbers. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), *Analysis of Arithmetic for Mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Gelman, R. & Gallistel, C. R. (1986). *The Child's Understanding of Number* (rev. ed.). Cambridge, MA: Harvard University Press.

Greeno, J. (1987) Instructional representations based on research about understanding. In A. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education* (pp. 61-88). Hillsdale, NJ: Erlbaum.

Hall, R., Kibler, D., Wenger, E. & Truxaw, C. (1989). Exploring the episodic structure of algebra story problem solving.*Cognition and Instruction*, 6(3), 223-283.

Hiebert, J. (1984). Children's mathematics learning: The struggle to link form and understanding. *Elementary School Journal*, 84, 497-513.

Hiebert, J. (Ed.) (1986). *Conceptual and Procedural Knowledge: The Case of Mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Hojnacki, S. & Grover, B. (1992) Thinking Mathematics: What's in it for the students? Paper to be presented at the annual meeting of the American Educational Research Association, San Francisco, CA..

Kaplan, R. G., Yamamoto, T. & Ginsburg, H. P. (1989). Teaching Mathematics Concepts. In Resnick, L. B. & Klopfer, L. E. (Eds.), *Toward the Thinking Curriculum: Current Cognitive Research* (pp. 59-82). Alexandria, VA: Association for Supervision and Curriculum Development.

Lampert, M. (in press). Teaching and learning long division for understanding in school. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), *Analysis of Arithmetic for Mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Lampert, M. (1991) Looking at Restructuring from Within a Restructured Role. *Phi Delta Kappan*, 72(9), 670-674.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27(1), 29-63.

Lampert, M. (1988) Connecting mathematical teaching and learning. In E. Fennema, Carpenter, T. and Lamon S.(Eds.), *Integrating research on teaching and learning mathematics* (pp. 132-167). Madison, WI: Wisconsin Center for Educational Research.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction*, 3(4), 305-342.

Leinhardt, G. & Grover, B. (1990). Transforming translations: A case of collaboration. Paper presented at the annual meeting of the American Educational Research Association, Boston, MA.

Leinhardt, G. Putnam, R., & Hattrup, R. A., (Eds.) (1992). *Analysis of arithmetic for matherratics teaching.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Lesgold, S. B., Putnam, R. T., Resnick, L. B.,& Sterrett, S. G. (1987). Referents and understanding of algebraic transformations. Paper presented at the annual meeting of the American Educational Research Association, Washington, D. C.

Mamona, J. C.(in press) Sequences and series - sequences and functions: Students' confusions. *International Journal of Mathematical Education in Science and Technology.*

Nesher, P.(in press). Solving multiplication word problems. In G. Leinhardt, Putnam, R., & Hattrup, R. (Eds.), *Analysis of Arithmetic for Mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Nicely, R. F. (1986). Synthesizing research-based instructional strategies for effective teaching. Mathematics Teaching and Learning Yearbook (pp. 30-39). University Park, PA: Pennsylvania Council of Teachers of Mathematics.

Resnick, L. B. (1989) Developing Mathematical Knowledge. *American Psychologist,* 44(2), 162-169.

Resnick, L. B. (1987). Constructing Knowledge in School. In L.S. Liben (Ed.), *Development and Learning: Conflict or Congruence?* (pp. 19-50). Hillsdale, NJ: Lawrence Erlbaum Associates.

Resnick, L. B. (1986) The development of mathematical intuition. In M. Perlmutter (Ed.), Perspectives on Intellectual Development: The Minnesota Symposia on Child Psychology, (19) (pp. 159-194). Hillsdale, NJ: Lawrence Erlbaum Associates.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. Ginsburg, (Ed.), The Development of Mathematical Thinking (pp. 109-151). Orlando, FL: Academic Press.

Resnick, L. B., Bill, V., & Lesgold, S. (in press). Developing thinking abilities in arithmetic class. In A. Demetriou, M. Shayer, & A. Efkides, (Eds.), *The Modern Theories of Cognitive Development Go to School.* London: Routledge.

Resnick, L. B., Bill, V., & Lesgold, S. & Leer, M. (in press). Thinking in Arithmetic Class. In B. Means & M. S. Knapp, (Eds.), *Teaching Advanced Skills to Educationally Disadvantaged Students.* San Francisco: Jossey-Bass.

Resnick, L. B. & Klopfer, L. E. (1989). Toward the thinking curriculum: An overview. In Resnick, L. B. & L. E. Klopfer (Eds.), *Toward the Thinking Curriculum: Current Cognitive Research* (pp. 1-19). Alexandria, VA: Association for Supervision and Curriculum Development.

Resnick, L. B. , Cauzinille-Marmeche, E., & Mathieu, J. (1987). Understanding Algebra. In J. A. Sloboda & D. Rogers (Eds.) *Cognitive Processes in Mathematics* (pp. 169-203) Oxford: Clarendon.

Resnick, L. B. & Omanson, S. (1987). Learning to understand arithmetic. In R. Glaser (Ed.) *Advances in Instructional Psychology*: vol. 3 (pp. 41-95). Hillsdale, NJ: Lawrence Erlbaum Associates.

Saxe, G. B. (1988). Candy Selling and Math Learning. *Educational Researcher*, 17(6), 14-21.

Schliemann, A. D. (1990). Seminar at the Learning Research and Development Center, University of Pittsburgh, Pittsburgh, PA. June 23.

Schliemann, A. D. & Magalhaes, V. (1990). Proportional Reasoning: From Shopping to Kitchens, Laboratories, and Hopefully, Schools. Paper presented at the Fourteenth International Conference for the Psychology of Mathematics Education, Oaxtepec, Mexico, July.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. Educational Psychologist, 23 (2), 145-166.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. J. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 41-52), Reston VA: National Council of Teachers of Mathematics.

Silver, E. (1988). Seminar at the Learning Research and Development Center, University of Pittsburgh, Pittsburgh, PA. June 30.

Sowder, J. C. (in press). Making sense of numbers in school mathematics. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), *Analysis of Arithmetic for Mathematics.* Hillsdale, NJ: Erlbaum.

Sowder, J. T. & Schappelle, B. P. (1989). *Establishing Foundations for Research on Number Sense and Related Topics: Report of a Conference*. San Diego, CA: Center for Research in Mathematics and Science Education, San Diego State University.

VanLehn, K. (1986). Arithmetic Procedures Are Induced From Examples. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp.133-180). Hillsdale, NJ: Lawrence Erlbaum Associates.

Tsu, R. (1991). Eliminating tracking in a high school. Presentation at the California Mathematics Council annual Conference at Asilomar, Dec. 6.

Vergnaud, G. (1988). Multiplicative Structures. In J. Hiebert & M. J. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 141-161), Reston VA: National Council of Teachers of Mathematics.