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ABSTRACT

In 1986 the United States (U.S.)-Japan Seminar on Mathematical Problem Solving convened to compare the state of problem solving in the classroom and in research in the two countries. The data and results given in this paper are the results of research conducted in the United States in response to the 1986 seminar. The U.S. and Japanese research groups decided to collect problem solving data at the 4th, 6th, 8th, and 11th grade school levels. Problems were collected from a pool of problems submitted by both the United States and Japan. U.S. subjects were from at least two 4th-, 6th-, 8th-, and 11th-grade classes from large rural, small urban, and large urban schools in areas around Carbondale and Champaign/Urbana, Illinois; Pittsburgh, Pennsylvania; Gainesville, Florida; and Athens, Georgia. In addition to the problems, student questionnaires collected data on students' attitudes towards problem solving, their comparison of the problems to textbook problems, and their reactions to the problems. A teacher questionnaire collected information about the schools, teachers' views of their classes, their reactions to the problems, and their perceptions of how students worked on problems. These questionnaires are included in the appendix. Five research reports include the results for each of five problems developed for varying grade levels, as follows: (1) the "Marble Arrangement Problem" for grade 4 (researched by E. Silver, S. Leung, and J. Cai); (2) the "Matchsticks Problem" for U.S. students in grades 4 and 6 (M. Hart and K. Travers); (3) the "Marble Pattern Problem" for U.S. students in grades 6, 8, and 11 (K. Fouche and M. Kantowski); (4) the "Arithmogons Problems" for U.S. students in grades 8 and 11 (J. Becker and A. Owens); and (5) the "Areas of Squares Problem" for U.S. students in grade 11 (J. Wilson). (Contains 39 references, as well as references with chapters.) (MDH)

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Report of U.S.–Japan Cross-National Research on Students Problem Solving Behaviors

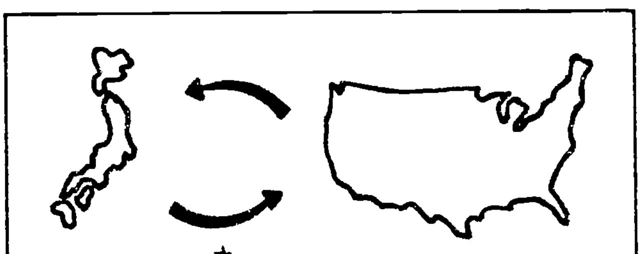
Edited by

Jerry P. Becker

Department of Curriculum and Instruction

Southern Illinois University at Carbondale

Carbondale, IL



SE053384

REPORT OF U.S.-JAPAN CROSS-NATIONAL RESEARCH ON STUDENTS' PROBLEM SOLVING BEHAVIORS

Edited by

Jerry P. Becker
Department of Curriculum and Instruction
Southern Illinois University at Carbondale
Carbondale, Illinois USA

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U. S. RESEARCHERS

Jerry P. Becker (Coordinator)	Southern Illinois University at Carbondale
Mary Grace Kantowski	University of Florida
Edward A. Silver	University of Pittsburgh
Kenneth J. Travers	University of Illinois at Urbana
James W. Wilson	University of Georgia

PREFACE

This is the U.S. report of the U.S.-Japan Cross-National Research on Students' Problem Solving Behaviors. The Japanese counterpart reports were edited by Professor Tatsuro Miwa (Report of the Japan-U.S. Collaborative Research on Mathematical Problem Solving (1991) and Teaching of Mathematical Problem Solving in Japan and the U.S. (1992)) and submitted to the Japanese Society For the Promotion of Science in Tokyo. Both are in Japanese.

Together, the reports mark the importance placed on problem solving in both the U.S. and Japan in school mathematics in the decades of the 1980's and 90's. Further, the reports mark the continuing evolution of cross-national communication, exchange and collaboration in mathematics education between the U.S. and Japanese mathematics education research communities. This is a collaboration that has great potential for improving the teaching of mathematics in both countries and to expanding research-based knowledge in the areas of teaching, curriculum and evaluation. The advancement of knowledge in this cross-national context can be promoted by research and by a growing variety of theoretical perspectives and research methodologies. Such research will almost certainly lead to improvement in mathematics education in both countries.

This research has been reported on programs of professional and research meetings regionally, nationally and internationally. The research has also led to translation of an important Japanese book, edited by Professor Shigeru Shimada, on the Open-end Approach in Arithmetic and Mathematics - a New Proposal Toward Teaching Improvement. The book was translated into English by Professors Shigeo Yoshikawa and Shigeru Shimada, edited by Jerry P. Becker and will be published by the National Council of Teachers of Mathematics in 1993. This represents another important dimension in cross-national research, for researchers are brought into contact with written reports of work done on both sides to which they may not otherwise have access.

The U.S. researchers express appreciation to all who contributed to making this study possible and to finalizing this report. To our Japanese colleagues, we express sincere appreciation for their cooperative attitude in all aspects of the research, including the many kindnesses shown to the U.S. group during its sojourns in Japan. To the National Science Foundation goes appreciation for the funding which made the research possible. We are particularly grateful for the diligent work of Joan Griffin, Lois Cornett and Karen Stotlar in typing the reports and to Ming Wang who provided software expertise. There are still others who contributed to the project in one way or another, and to them we express our thanks.

Jerry P. Becker

BACKGROUND TO THE RESEARCH

The data and results given in this report are part of the project on U.S.-Japan Cross-national Research on Students' Problem Solving Behaviors. The research has its origin in the U.S.- Japan Seminar on Mathematical Problem Solving held at the East-West Center in Honolulu, July 14-18, 1986 (Becker and Miwa, 1987).* At that seminar nine U.S. and ten Japanese mathematics educators met to examine the present state of problem solving, explore classroom practices in problem solving, and, in general, to compare the situations in both countries relating to various aspects of problem solving in the classrooms and research (Becker and Miwa, 1987, p. viii).

The last afternoon of the seminar dealt with future communication, exchange of materials and planning cross-national collaborative research. Subsequently, research proposals were submitted, on both sides, requesting funding to support research: in the U.S. to the Division of International Programs of the National Science Foundation (NSF) and in Japan to the Japan Society For the Promotion of Science (JSPS), under the U.S.-Japan Cooperative Science Program. A separate proposal was submitted to the Research in Teaching and Learning Program in the National Science Foundation. The proposals were funded and the research effort commenced with a meeting of the U.S. and Japanese groups at the University of Tsukuba in Fall 1988.** At that time, the U.S. group also made visits to Japanese classrooms and observed numerous problem solving lessons preliminary to conducting the research (Becker, Silver, Kantowski, Travers, and Wilson, 1990). These visits and the related discussions set the stage for the research which was further broadened and deepened by a visit to the U.S. in the Fall 1989 by the Japanese group, which made similar classroom visits followed by further discussions and planning.

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* The Seminar was supported by the U.S. National Science Foundation (Grant INT-8514988) and the Japan Society For the Promotion of Science.

** Members of the groups were: U.S.: Jerry P. Becker (Coordinator), Edward A. Silver, Mary Grace Kantowski, Kenneth J. Travers, and James W. Wilson; Japan: Tatsuro Miwa (Coordinator), Shigeru Shimada, Toshio Sawada, Tadao Ishida, Yoshihiko Hashimoto, Nobuhiko Nohda, Yoshishige Sugiyama, Eizo Nagasaki, Toshiakira Fujii, Shigeo Yoshikawa, Hanako Senuma, Junichi Ishida, Toshiko Kaji, Katsuhiko Shimizu, and Minoru Yoshida.

PROCEDURES AND METHODOLOGY IN THE RESEARCH

Grade Levels for Data Collection

The U.S. and Japanese groups, hereafter referred to as the group, made decisions to collect problem solving data at the 4th, 6th, 8th and 11th grade school levels. Problems were selected and administered as follows: one problem at the 4th grade only; one problem at both the 4th and 6th grades; one problem at the 6th, 8th and 11th grades (with a variation at grade 11 in the U.S.); one problem at both the 8th and 11th grades (two problems at the U.S. eleventh grade - a variation of one 8th grade problem); and one problem at the 11th grade only. The problems were selected from a pool to which both sides contributed and are included in the Appendix to this report.

Subjects

Subjects were at least two classes of 4th, 6th, 8th, and 11th grade students in areas around Carbondale (IL), Champaign/Urbana (IL), Pittsburgh (PA), Gainesville (FL), and Athens (GA). For all grade levels, students were attending school in large rural, small urban or large urban school districts. Schools were purposely selected to provide this mix, although the selection of schools and classes within a school was not made in a random manner. The descriptive nature of the study provides information which helps to document results pertaining to performance of U.S. students on certain kinds of problem solving behaviors as well as to provide some contrasts between subjects in these two U.S. and Japanese samples on these behaviors.

Questionnaires

In addition to the problems, student questionnaires were developed to gather information about students "liking" and "good at" math, their comparison of the problems to textbook problems, and their reactions to each of the problems in the research. A teacher questionnaire was also developed to collect information about the schools, teachers' views of their classes, their reactions to the problems and their perceptions of how seriously students worked on the problems. In addition, a set of instructions was developed for use by proctors when the problem booklets were administered. All are included in the Appendix to this report.

Tryout of Research Materials

Problem booklets and questionnaires were developed into preliminary form during the winter, 1988-89 following the Fall, 1988 meeting of the group in Japan. They were "tried out" in the Spring 1989 in classrooms in the Carbondale, IL area. The results were tabulated, reported and discussed at the group's second meeting in Japan in Fall 1989 (Becker, 1989). Subsequently, the materials were revised and finalized for data collection, which occurred at about the same point in each country's school year during 1989-90. (Note: The Japanese school year begins in early

April, and the U.S. in late August.)

Data Collection

U.S. data for each problem in the study were collected by the five U.S. researchers (Jerry Becker, Kenneth Travers, Edward Silver, Mary Grace Kantowski and James Wilson) in their respective centers in areas around Carbondale (IL), Champaign/Urbana (IL), Pittsburgh (PA), Gainesville (FL), and Athens (GA). In the formal data collection phase, subjects were given fifteen minutes to work on each of two problems (three at the 11th grade for the U.S. - time for the third problem was ten minutes) and were asked to write down all their work and to "line out" rather than erase writing. Further, proctors were directed if and when subjects asked questions, to respond by saying "I leave it to your judgment" or "Please judge for yourself." In general, students worked on the problems, asking no questions. Each problem was read aloud by the proctor before subjects began, subjects were asked to read the problem themselves, and were stopped promptly after fifteen minutes on each of the two problems (and after ten minutes on the third at the 11th grade for the U.S.). Subjects filled out the questionnaire during the last five minutes of the class period and teachers filled out their questionnaire while the problems were being administered. Total time elapsed was forty-five minutes for 4th, 6th and 8th grades, and fifty-five for 11th grade, the approximate length of class periods in the schools.

Analysis of the Data

Each researcher analyzed data for one problem which were collected at the four grade levels at each of the centers:

Name of problem	Grade(s)	Researcher
Marble Arrangement	4	Edward Silver
Matchsticks	4,6	Kenneth Travers
Marble Pattern	6,8,11	Mary Grace Kantowski
Arithmogons	8,11	Jerry Becker
Area of Squares	11	James Wilson

Results for the U.S. sample are given in this report, for each problem - the principal authors are Silver, Travers, Kantowski, Becker and Wilson, respectively. Each prepared his/her report which follows. Only very minor editing has been done to provide some consistency. Results for the Japanese sample are reported in Miwa (1992). Becker and Silver include contrasts of results for the U.S. and Japanese samples in their reports. The Comments on Results For This U.S. Sample and Conclusion were written by Jerry Becker.

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RESEARCH REPORTS

THE MARBLE ARRANGEMENT PROBLEM: RESULTS OF AN ANALYSIS OF U.S. STUDENTS' SOLUTIONS AND A COMPARISON WITH JAPANESE STUDENTS *

Edward A. Silver
Shukkwan S. Leung
Jinfa Cai

University of Pittsburgh

This report presents the analyses of the U.S. results on the *marble arrangement problem* and a comparison with the results from a sample of Japanese students who solved the same problem (Nagasaki & Yoshikawa, 1989; Nagasaki, 1990).

Method

Subjects

A total of 151 students (83 boys and 68 girls) from four U.S. locations participated in the study during Fall 1989. Most of these students (142) were fourth graders; the remaining students (19) were fifth graders from a combined 4th/5th grade class. All students attended classes that were judged to be of average ability, with the exception of the combined 4th/5th grade class which included gifted students.

Task and administration

Each student was given a workbook in which the marble arrangement problem (see Figure 1) appeared as the first of two problems. For the marble arrangement problem, students were instructed to determine the number of marbles in a given arrangement in as many different ways as they could. Nine copies of the marble arrangement, each with a separate solution space, were provided after the presentation of the problem and the instructions. The workbooks also included an attitude survey in the form of a

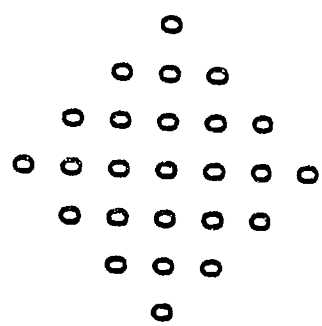
* The authors gratefully acknowledge Adam Deutsch and Jerry P. Becker for their helpful comments on earlier drafts of this report and Patricia Ann Kenney for her extensive editorial assistance in preparing the final version of this report.

questionnaire which students completed after working the two problems. Students were given 15 minutes to work on the problem and 5 minutes to complete the questionnaire. A copy of the relevant portions of the student workbook appears in the Appendix to this report.

Figure 1

<u>The Marble Arrangement Problem</u>

How many marbles are there in the picture below?



<u>FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN</u>. Write your ways of finding the answer and write your answer.

Coding Method

The coding method used for the U.S. sample was influenced by the desire to conduct appropriate comparisons to the results from the Japanese sample. For this reason the coding system developed by Nagasaki and Yoshikawa (1989) was used for U.S. student responses, and was applied at two different levels. Each workbook, hereafter referred to as a *script*, received a code based on correctness of the answers provided in all responses. Also, each individual response within a script received two separate codes, one based on the solution strategy used to solve the problem and another code based on the mode of explanation used to justify the given solution.

Script Codes. All of the responses within a script were examined for correctness of the answer (i.e., "25 " or an equivalent mathematical expression such as "5 x 5"). If at least one answer within a script was correct, the script was coded as *at-least-partially correct* (PC). If all answers were correct, the script was further coded as *completely correct* (CC). Since CC scripts meet the criterion of at-least-one-correct answer, CC scripts are a subset of PC scripts.

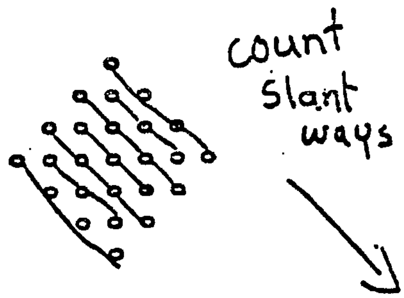
Solution strategy. Three categories of solution strategies identical to those identified by Nagasaki and Yoshikawa (1989) were used for the responses to the marble arrangement problem: *enumeration*, *find-a-structure*, and *change-the-structure*. Figure 2 contains examples of student responses for each of these categories. In order to be coded as *enumeration*, a student's response had to show some evidence of a counting procedure such as counting one-by-one, counting in a specific direction, or counting by drawing a continuous line. Responses coded as *find-a-structure* gave some evidence of the student having used grouping. This strategy could involve placing the same number of marbles in each group or forming groups based on some other convenient arrangement such as rows, columns, diagonals, or a combination of any of these. The final category, *change-the-structure*, involved a restructuring of the given arrangement of marbles based on a displacement of marbles by drawing arrows to show the marbles' new positions or adding on (and later subtracting off) additional marbles to facilitate the calculation process.

Figure 2

Examples of Solution Strategies in Each Category

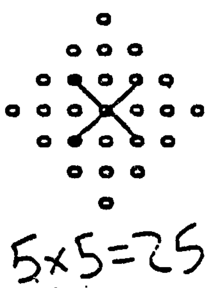
Enumeration

Enumeration

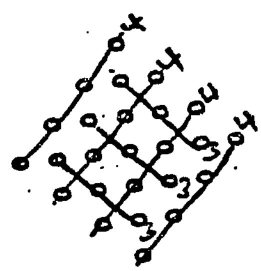


Enumeration

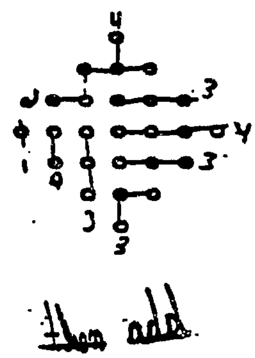
Find-a-structure



Find-a-structure

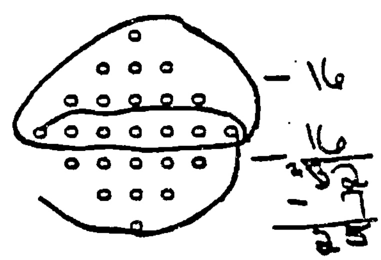


Find-a-structure



Change-the-structure

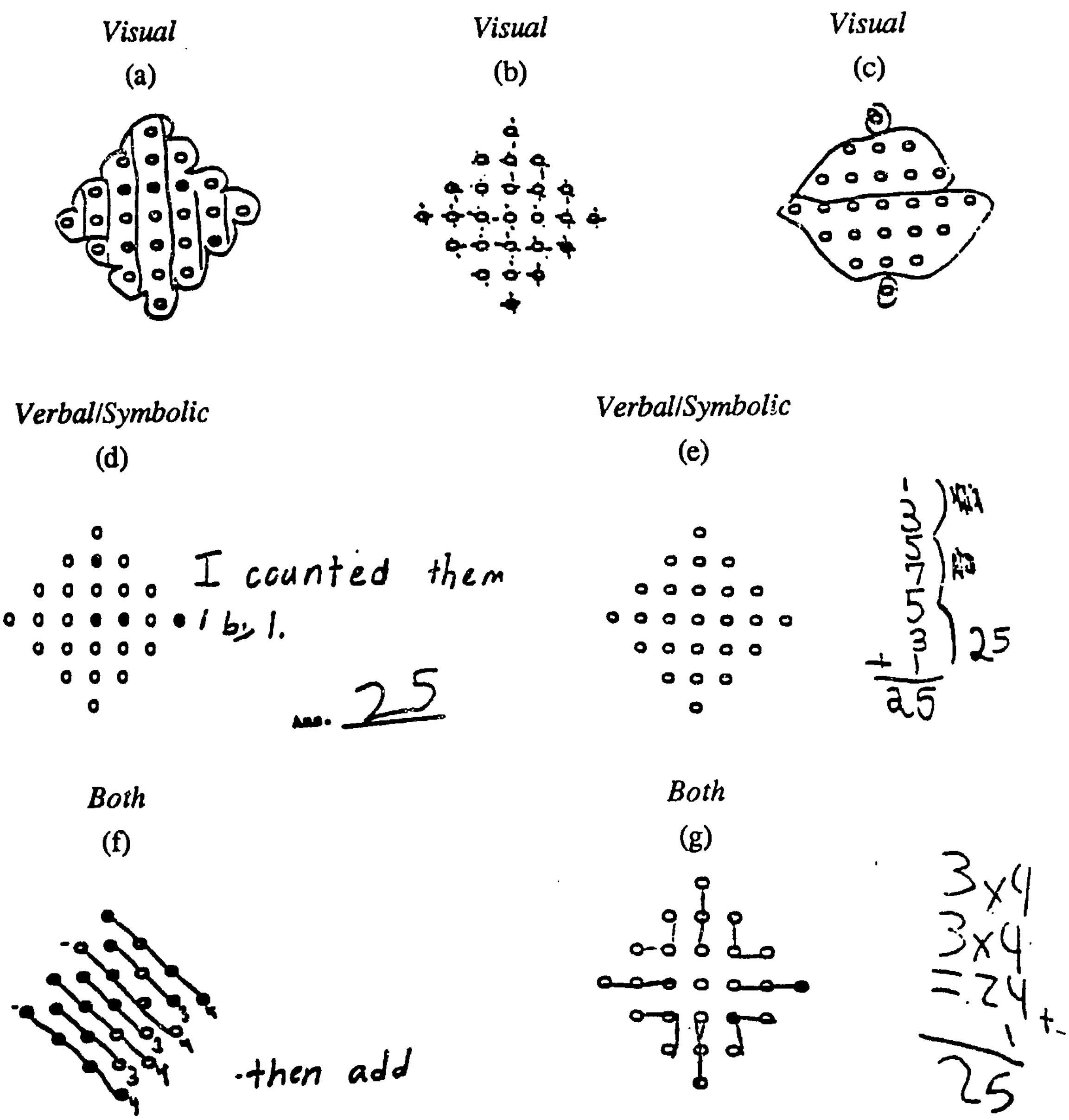
Change-the-structure



Mode of explanation. Five main categories were used to describe the manner in which the response was justified by the student: *visual*, *verbal/symbolic*, *both*, *neither*, and *inconsistent*. The first three categories were those used by the Japanese researchers (Nagasaki & Yoshikawa, 1989); the last two categories were added to make the coding of the U.S. sample more complete. Figure 3 contains examples of responses from each of the five categories.

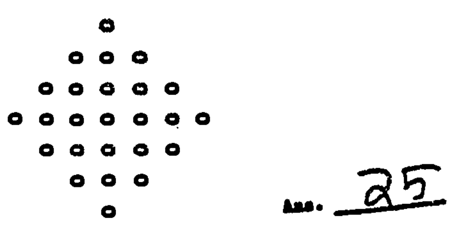
Figure 3

Examples of Modes of Explanation in Each Category



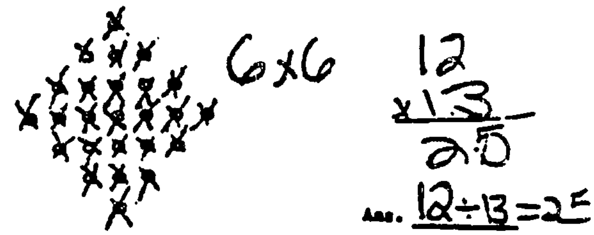
Neither

(h)



Inconsistent

(i)

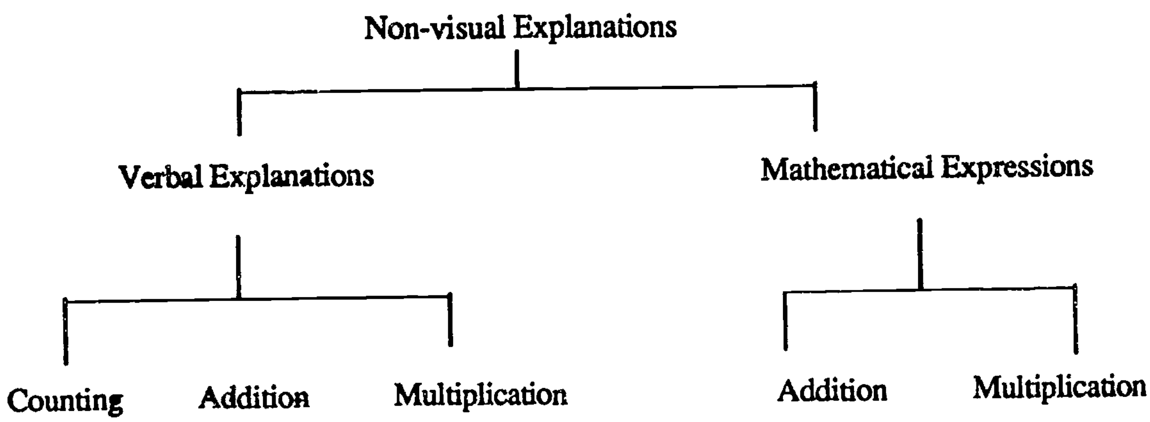


A response was coded as *visual* if the student marked the figure given in the problem and provided no words or mathematical expressions as justification. A response consisting of an unmarked figure accompanied by words and/or mathematical expressions that explained the answer was coded as *verbal/symbolic.* If the student marked the figure and wrote either words or a mathematical expression as justification, the mode of explanation was recorded as *both* because it contained both visual and verbal/symbolic features. If an answer appeared without any attempt at justification, the response was coded as *neither.* A response in which a verbal/symbolic explanation was internally inconsistent or in which such an explanation did not match the markings on the figure was coded as *inconsistent.*

All responses classified as *verbal/symbolic* (including those classified as *both*) were further categorized in a manner similar to that employed by Nagasaki (1990). Figure 4 contains a schematic diagram of the categorization scheme for these non-visual explanations. In addition to distinguishing responses involving the use of *verbal explanations* from those involving *mathematical expressions,* this process also identified the mathematical process involved (i.e., *counting, addition, multiplication* in the verbal explanation category; *addition* and *multiplication* in the mathematical expression category). For example, the *verbal/symbolic* response labeled "d" in Figure 3 was further coded as *verbal explanation - counting*; the *both* response labeled "g" in Figure 3 was further coded as *mathematical expression - multiplication.* Since responses could involve features of more than one coding category, the following rules were used: When responses involved both multiplication and addition, they were coded as *multiplication* (as in example "g" in Figure 3), when responses contained both a mathematical expression and a verbal explanation (as in "Multiply 5 x 5 = 25" in figure 2), the response was coded as *mathematical expression.*

Figure 4

Categories of Non-visual Explanations



Inter-rater Reliability In coding the responses from the U.S. sample, one rater first coded for solution strategy all responses contained in the 151 scripts. A second rater then randomly selected 20% of all scripts and independently coded all of the correct solutions, and the inter-rater reliability coefficient from this coding exercise was computed (Kappa coefficient = .94). The same two raters completed a similar coding exercise for the five mode of explanation categories. There was virtually unanimous agreement from this exercise, and the few disagreements were resolved through discussion.

Results

The results section is comprised of two parts. The first part contains the analysis of responses from the U.S. sample of students; the second part focuses on a comparison of the U.S. results with those from the Japanese sample of students.

Results for the U.S. Sample

This section contains analyses of four components of the results from the U.S. sample: responses, solution strategies, modes of explanation, and questionnaire. In order to facilitate comparisons, methods of analysis corresponding to those used by Japanese researchers (Nagasaki & Yoshikawa, 1989; Nagasaki, 1990) were used whenever possible.

Responses

In the U.S. sample there was a total of 151 scripts, which contained a total of 1083 responses to the marble arrangement problem. There were 142 PC scripts (representing 94% of all scripts), and within these scripts 90% of the responses were correct. Of the PC scripts, 99 scripts were also CC scripts (representing 66% of all scripts). In the group of 99 CC scripts, there was a total of 740 solutions (344 from boys and 396 from girls).

Table 1 shows the distribution of the frequency counts on the number of responses per script. Although there were a few blank scripts, no student gave exactly one response. This suggests that the students were willing and able to find additional solutions to the problem after they had found a first solution. In fact, 86% of the students gave 5 or more responses. The mean number of responses was 7.2 and the modal number of responses was 9, which corresponds to the number of solution spaces provided. The two students who provided a tenth response used the figure that accompanied the instructions.

Table 1
Distribution of the Frequency of Responses per Script by Gender

Number of Responses per Script

	0	1	2	3	4	5	6	7	8	9	10
Boys (n=83)	1	0	1	5	8	9	9	11	5	33	1
Girls (n=68)	2	0	2	0	2	3	10	7	4	37	1
All subjects (N=151)	3	0	3	5	10	12	19	18	9	70	2

There were several gender-related differences evident in the students' responses. Compared to the boys in the sample, the girls gave significantly more responses (girls: 7.7, boys: 7.0, $t = 2.33$; $p < .05$) and significantly more correct responses (girls: 90%, boys: 83%, $z = 3.88$; $p < .001$). Moreover, the percentage of CC scripts for girls was significantly higher than that for boys (girls: 75%, boys: 58%, $z = 2.21$; $p < .05$). No other gender differences related to response frequency were noted.

Solution Strategies

Strategy use. Frequency of strategy use was examined in two ways: overall frequency by response ($N = 1083$) and frequency by student ($N = 151$). Overall, 28% of the responses showed evidence of *enumeration*, 71% *find-a-structure*, and 1% *change-the-structure*. Using the student as the unit of analysis, the results of this counting showed that 58% of the students used the *enumeration* strategy at least once, 88% used *find-a-structure* at least once, and 4% used *change-the-structure* at least once. Therefore, students used *enumeration* fairly frequently, but not consistently; whereas, they used *find-a-structure* frequently and consistently. The contrast in frequency of use of *find-a-structure* and *change-the-structure* suggests that most students were able to detect or impose a structure in the marble arrangement problem, but only a small percentage of students were able or willing to change the structure of the original problem.

Strategy shifts. This analysis was done by comparing the strategy used in the first response at solving the problem with those used in subsequent attempts. It was thought that, since the answer to the problem was not yet known, the first response was different from all other responses. In their analysis of results from the Japanese students, Nagasaki and Yoshikawa (1989) investigated shifts in strategy use by comparing the first response to the fifth response. For the purposes of comparison, a similar analysis was done on the responses of the U.S. students; moreover, strategy shifts between the first and the second responses were also investigated for the U.S. sample.

For the first response, about 50% of the students used *enumeration* and about 50% used *find-a-structure*; only one student began with *change-the-structure*. Table 2 contains the data on strategy shifts from the first to the second response and from the first to the fifth response. The overall tendency was that once students selected a solution strategy, they retained that strategy in subsequent attempts to solve the problem. If students changed strategies, the most likely shift was from *enumeration* to *find-a-structure*.

Table 2

Percent of Students Exhibiting Shift in Solution Strategies

	Response Occasion	
	1st to 2nd	1st to 5th
No change	60%	53%
Enumeration to *Find-a-structure*	25%	23%
Find-a-structure to *Enumeration*	5%	3%
Enumeration to *Change-a-structure*	1%	--
Change-a-structure to *Enumeration*	1%	--
Missing	9%	21%

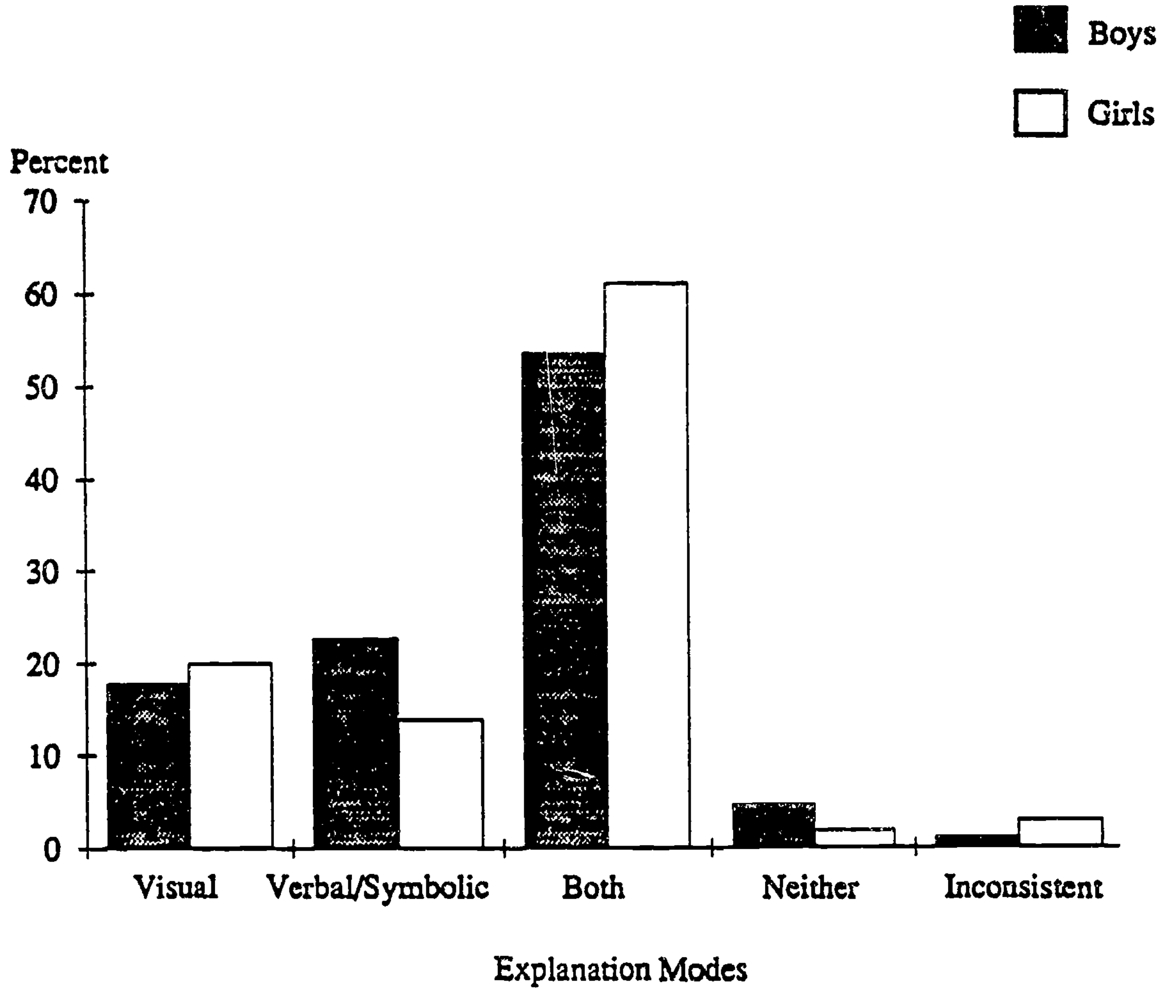
Modes of Explanation

Use of Explanation Modes. Using the set of 1083 responses, an examination of the distribution of explanation modes showed that 19% of the explanations were categorized as *visual*, 19% as *verbal/symbolic*, 57% as *both*, and the remaining 5% as *neither* or *inconsistent*. This distribution pattern was similar for responses within the set of PC script and the set of CC scripts.

Boys and girls differed in their relative frequency in using explanation modes, as can be seen in Figure 5. For example, a significantly larger percentage of boys' responses (23%) than girls' responses (16%) involved *verbal/symbolic* explanations in the justification of the answers (Wilcoxon-Mann-Whitney test, $z = 2.53$; $p < .05$).

Figure 5

<u>Distribution of Explanation Modes by Gender</u>



Although *both* was the mode of explanation used in a majority of the <u>responses</u>, it was not clear that a majority of <u>students</u> used this as the dominant mode of explanation; that is, the mode with the highest proportion of use. Therefore, a further analysis was conducted, in which the focus was on the proportion of each explanation mode in each student's script. For each student's script, the proportion of explanations in the *visual*, *verbal/symbolic*, and *both* categories was computed. The results of this analysis showed that 57% of the students used *both* as the dominant mode of explanation. Thus, the mode of explanation *both*, involving both *visual* and *verbal/symbolic* aspects, was not only used in the majority of responses but also was used by a majority of the students.

The Non-Visual Modes of Explanation. An analysis done by Nagasaki (1990) on the non-visual responses (i.e., those involving verbal explanations and/or mathematical expressions) to the marble arrangement problem provided the basis for a similar analysis on the U.S. results. In addition to including responses that had been categorized as *verbal/symbolic*, responses categorized as *both* were also included because they contained non-visual information. These categories of responses were further classified as *verbal explanations* and *mathematical expressions* along with their appropriate subcategories (cf. Figure 4). Due to the significantly larger percentage of *verbal/symbolic* explanation modes in boys' responses than in girls' (cf. Figure 5), the results were further analyzed for gender differences.

The analysis was done on a total of 849 responses that were coded *verbal/symbolic* or *both*. Of these responses, 440 were given by boys and 409 were given by girls. Table 3 contains the distribution of the responses by gender within the various subcategories. The percentage of *verbal explanations* for boys (64%) was lower than the percentage for girls (80%). Therefore, although the percentage of explanations involving the *verbal/symbolic* mode previously appeared to be larger for the boys than the girls, girls produced a larger percentage of verbal expressions when the category of *both* was included in the analysis. The fact that boys produced a large proportion of mathematical expressions (36% as opposed to 19% for girls) probably explained their higher proportion in the *verbal/symbolic* category.

The data in Table 3 also show that there were more verbal explanations (72%) than mathematical expressions (28%) used in students' explanations of the problem. Also, the responses showed evidence that more students used *addition* to solve the problem (62%), than used either *counting* (26%) or *multiplication* (12%).

Table 3

Percent Distribution of Categories of Non-visual Explanations by Gender

	Verbal Explanations			Mathematical Expressions	
	Counting	Addition	Multiplication	Addition	Multiplication
Boys' responses (n=440)	25%	38%	1%	23%	13%
Girls' responses (n=409)	27%	52%	1%	10%	9%
All responses (N=849)	26%	45%	1%	17%	11%

Questionnaire Responses and Their Relationship to Problem-Solving Success

The student workbook included a questionnaire consisting of seven questions designed to investigate students' thinking and attitudes about mathematics in general and about the marble arrangement problem and one other non-routine problem (cf. Appendix). Table 4 presents the percentage of students giving the indicated responses to questions concerned with the marble arrangement problem.

Table 4

Distribution of Indicated Responses on Student Questionnaire

Indicated Response	% of Students
Like math	70%
Good at math	57%
Problem is interesting	69%
Problem is easy	52%
Different from textbook problems	61%
Like more than textbook problems	52%
Have seen similar problems before	61%

The relationship between students' interest in this problem and their success in solving it was examined. Students who responded that they found the marble problem interesting had a mean of 7.1 correct responses, which was significantly better than the performance of the other students, who had a mean of 5.6 correct responses ($t = 1.82$; $p < .05$, one tailed). The relationship between success and familiarity with problems of this type was also examined. Students who reported that they had previously seen similar problems were compared to those who indicated that they had not. The mean number of correct responses given by students who indicated familiarity ($M = 6.9$) was significantly higher than the mean for the other students ($M = 5.9$) ($t = 2.03$; $p < .025$, one-tailed).

Comparison of U.S. and Japanese Results

This section focuses on the comparison of results obtained from students in the U.S. sample with those obtained from the Japanese sample. The Japanese sample included a total of 206 fourth-grade students (102 boys and 104 girls) from six schools -- five public schools and one national school. The areas of comparison were responses, solution strategies, and mode of explanation. In general, the comparisons were made by referring to the results reported by Nagasaki & Yoshikawa (1989). However, in a few instances, comparisons were made to results reported in a subsequent analysis by Nagasaki (1990), in which he used a more restricted sample by excluding the national school students, who were judged to be of higher-than-average ability.

Responses

Table 5 shows the distribution of scripts by gender by national sample. In the Japanese sample there was no separate analyses on PC scripts (i.e., *at-least-partially correct*) due to the large number of CC scripts (i.e., *completely correct*). Students in the Japanese sample produced significantly more CC scripts than students in the U.S. sample (U.S.: 66%, Japan: 96%, $z = 7.46$; $p < .001$). It is interesting to note, however, that the percentage of PC scripts from the U.S. sample was about the same as the percentage of CC scripts from the Japanese sample. The percentage of girls' CC scripts in the U.S. sample was significantly higher than the percentage of boys' CC scripts; however, no gender-related difference was found in the Japanese sample.

Table 5

Percent Distribution of Scripts by Gender by National Sample

	Boys	Girls	All students
U.S.			
CC Scripts	48 (58%)[a]	51 (75%)	99 (66%)
PC Scripts	77 (93%)	65 (96%)	142 (94%)
All scripts	83 (100%)	68 (100%)	151 (100%)
Japan			
CC Scripts	97 (95%)	100 (96%)	197 (96%)
PC Scripts	NA	NA	NA
All scripts	102 (100%)	104 (100%)	206 (100%)

a Percents in parentheses show the proportion of PC or CC scripts by gender (or total) by national sample (e.g., 48 CC scripts is 58% of the 83 scripts produced by boys in the U.S. sample.)

Although there were significantly more CC scripts produced by students in the Japanese sample, U.S. students produced a significantly great number of solutions per CC script (U.S.: 7.5, Japan: 5.8, $t = 6.68$; $p < .001$). In fact U.S. students produced more overall responses per script than Japanese students; the modal number of responses was 9 for the U.S. sample and 6 for the Japanese sample. These differences may be related to the number of solution spaces provided in the student workbook (i.e., 9 for the U.S. students and 6 for the Japanese students). Although U.S. students gave more responses than the Japanese students, Japanese students were more likely to persevere beyond the limitations imposed by the workbook. For example, despite the fact that there were only 6 solution spaces given in their workbook, 6% of the Japanese students produced 10 to 15 responses. In contrast only 2% of the U.S. students gave a tenth response and no one produced more than 10 responses. The range of the number of responses produced by Japanese students was 1 to 15, in contrast to the range of 1 to 10 for the U.S. students. Although there were some differences in response frequency, it is important to note an overall similarity: over 80% of the students in both national samples provided 5 or more responses.

Solution Strategy

The distribution of solution strategies in the U.S. sample was similar to the distribution in the Japanese sample. In both national samples, about 60% of the students used *enumeration* at least once, 90% used *find-a-structure* at least once, and less than 5% used *change-the-structure* at least once.

Regarding solution shifts between the first response and subsequent responses, the patterns were quite similar in the samples. Most students in both national samples continued to use the type of strategy they used on the first response occasion. About 33% of U.S. students and about 50% of Japanese students used a different strategy on the fifth response than on the first response, and the change was more likely to be from *enumeration* to one of the other two "structure" strategies.

Mode of Explanation

The comparison of the distribution of explanation modes was based only on the group of CC scripts. Figure 6 shows the distribution of explanation modes by national sample. In both samples, most responses (about 60%) were categorized as *both* (i.e., both *visual* and *verbal/symbolic* modes). The percentage of *verbal/symbolic* explanations was higher for Japanese students' responses, whereas the percentage of *visual* explanations was higher for U.S. students' responses. In the U.S. sample, about 20% of the responses involved *visual* explanations and 20% involved *verbal/symbolic* explanations, whereas in the Japanese sample less than 5% involved *visual* explanations but 36% involved *verbal/symbolic* explanations.

Figure 6

Distribution of Explanation Modes by National Sample

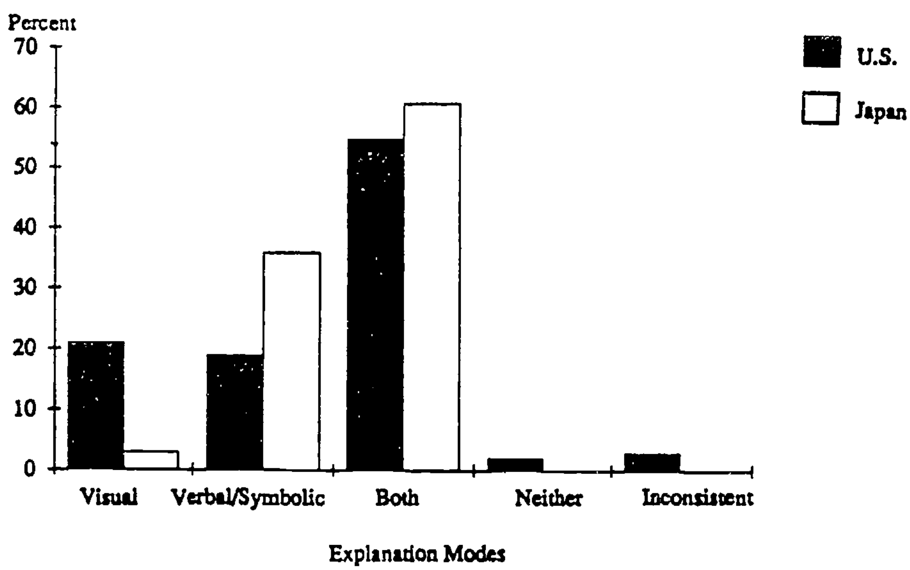


Table 6 can be used to compare the results on *verbal explanations* versus *mathematical expressions*, as given in the second report on the Japan sample (Nagasaki, 1990). Responses from the Japanese sample tended to involve *mathematical expressions* (59%), while responses from the U.S. sample more frequently involved *verbal explanations* (72%). Moreover, Japanese responses tended to involve explanations related to multiplication (55%) while U.S. responses tended to involve explanations related to addition (62%).

Table 6

Percent Distribution of Categories of Non-visual Explanations by National Sample

	Verbal Explanations			Mathematical Expressions	
	Counting	Addition	Multiplication	Addition	Multiplication
U.S. (n=849)	26%	45%	1%	17%	11%
Japan (n=930)	22%	1%	18%	22%	37%

Note: n=number of non-visual responses involving explanations

Discussion

The marble arrangement problem was administered as part of a U.S. - Japan collaborative study on nonroutine problems. For students in the U. S. sample, the problem was thought to be nonroutine in at least two ways. First, the structure of the problem-solving activity, in which one answers a single problem a number of times, was thought to be somewhat novel for U.S. students. Moreover, the problem-solving task called for students to provide explanations of their solution methods or justifications of their answers, which was also thought to be novel for U. S. students. Some reports of instructional activity in Japanese mathematics classrooms (e.g., Becker, Silver, Kantowski, Travers & Wilson, 1990; Stigler, Lee & Stevenson, 1987) have suggested that students in that country are often given opportunities to solve problems in more than one way and to present different solutions to the same problem. In contrast, reports of activity in U. S. mathematics classrooms (e.g., Fey, 1981; Silver, Lindquist, Carpenter, Brown, Kouba & Swafford, 1988) rarely suggest such a picture.

In light of the expectations of task novelty for U. S. students, it is somewhat surprising that about 60% of the students reported having previously seen a problem similar to the marble arrangement problem. Assuming veridical responses and a lack of sampling bias, the large number of students responding that they had seen such a problem may reflect the fact that U. S. mathematics teachers at the fourth grade level make more frequent use of such problems than was originally assumed, or that many U. S. students are exposed to such nonroutine problems in settings other than the mathematics classroom, or that the students were basing their response on surface features of the task (i.e., counting the objects in a figural display) rather than the structure and demands associated with the task. Because the Japanese students' questionnaire did not include a question about familiarity with similar tasks, it is impossible to make a direct comparison of task familiarity between the students in the national samples. However, there was another question that provides some indication of the relative familiarity of this type of task to Japanese and U.S. students. Students in both national samples were asked whether the marble arrangement problem was similar to problems that appear in their textbooks. On this question, 42% of the students in the Japanese sample reported that this problem was different from problems in their textbook (Nagasaki, 1990); in contrast, 61% of the U.S. students reported that the problem was different. These data suggest that the marble arrangement task may have been more familiar to the Japanese students than to the students in the U.S. sample.

Although it is impossible to reach a definitive conclusion regarding task familiarity and its impact on students' performance from the data obtained in this study, and although neither sample was systematically chosen to be nationally representative, it is nevertheless interesting to examine some of the most salient findings. In many ways, students in both national samples behaved quite similarly with respect to the marble arrangement problem. Although there were differences favoring the Japanese students in the number of *completely correct* (CC) scripts, there was virtually no difference when the percent of Japanese CC scripts was compared with the percent of U. S. PC (*at-least-partially correct*) scripts. The occurrence of U. S. scripts in which some but not all answers were correct may be a direct result of the U. S. students' relative unfamiliarity with tasks in which one is asked to answer the same question many times. Students may have assumed that the marble arrangement was different at least some of the time and treated each occurrence as a new problem rather than an occasion to display a new method of solving a problem whose solution was known. Moreover, the fact that the Japanese students were more likely to persevere beyond the limitations imposed by the workbook than the U. S. students, by drawing additional figures in order to produce solutions after filling all the given answer spaces, may reveal their increased comfort and familiarity with this kind of task.

Nevertheless, the majority of students in both national samples were constrained by the presentation format of the tasks and produced exactly the same number of solutions as there were answer spaces available in the workbook.

In most studies involving a comparison of the mathematical proficiency of Japanese and American children (e.g., Robitaille & Garden, 1990; Stevenson, Lee & Stigler, 1986), Japanese children far outperform their American counterparts. Thus, it is noteworthy that the U. S. and Japanese students in this study exhibited many quite similar behaviors. For example, in both samples, over 80% of the students produced 5 or more solutions. Thus, the students in both countries were able to solve the problem and produce multiple solutions and explanations of their solutions. This was especially remarkable for the U. S. students, since, as noted above, such problem-solving behavior is not regularly evoked in typical U. S. mathematics classrooms.

The analysis of solution strategies also revealed some interesting similarities between the students in the two national samples. In particular, the solutions produced by the U. S. students were easily analyzed using a coding scheme developed by Japanese researchers to code responses from students in that country. Moreover, the frequency and patterns of strategy use across response occasions were almost identical in the two national samples. For example, in both countries, about 90% of the students used the *find-a-structure* strategy at least once, about 60% used *enumeration*, and less than 5% used the *change-the-structure* strategy. The findings on strategy use suggest that the students in both national samples were comfortable with counting and grouping the objects in the figural display, but they were less comfortable moving the objects to create a new display.

Examination of the findings on students' mode of explanation suggests another similarity and some important differences between the two national samples. Students in both samples used the same kinds of explanations, although there was differential frequency of use in some categories. A major similarity was the finding that about 60% of the responses from students in both samples involved explanations that had both visual and verbal/symbolic features. This finding is reminiscent of results obtained by Ben-Chaim, Lappan and Houang (1989), in a study of eighth-grade children's ability to describe in writing to another person a three-dimensional block display. Like the children in that study, it would appear that children in both national samples in this study occupy a middle position rather than either extreme on the hypothesized verbalizer-visualizer continuum (Richardson, 1977).

Despite the general preference in both countries for a mixed mode of explanation, a substantial number of responses involved "purer" forms of explanation. Within this group of responses, there were some differences between the national samples. The proportion of visual explanations in the U. S. responses was about four times greater than in the Japanese responses. On the other hand, the proportion of verbal/symbolic explanations in the set of Japanese responses was nearly twice that found in the U. S. sample.

One of the most important differences between the two national samples was the level of mathematical sophistication evident in the students' explanations. Although students in both national samples were likely to provide explanations that combined the use of visual and verbal/symbolic features, Japanese students produced a much higher proportion of responses involving mathematical expressions than did their U.S. counterparts, who tended to favor explanations involving verbal statements. Moreover, Japanese students produced a higher proportion of mathematical explanations that involved multiplication than U.S. students, who were more likely to use explanations involving addition. The tendency to use mathematical expressions rather than verbal statements, and the tendency to use multiplication rather than addition are both indications of the increased mathematical sophistication of the Japanese students' responses when compared to those provided by the U.S. students.

Another major difference in the findings for the two national samples is the detection of significant gender differences in the U. S. sample, but not in the Japanese sample. The observation that U. S. and Japanese students exhibited similar solution strategies and response tendencies provides some good news for American educators. Nevertheless, the findings of differential performance for U. S. boys and girls, and the lower levels of mathematical sophistication evident in the U. S. students' responses suggest that much work remains to be done in order to assist **all** U. S. students to achieve the recently promulgated national goal of reaching "world class standards" of mathematical proficiency.

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THE MATCHSTICKS PROBLEM: RESULTS OF AN ANALYSIS OF U.S. STUDENTS' SOLUTIONS

Mitchell E. Hart

Kenneth J. Travers

University of Illinois at Urbana/Champaign

Introduction

This report presents analyses of the U.S. results on the matchsticks problem. Each student was given a workbook in which the matchsticks problem (see Figure 1) appeared as the second of two problems at the fourth and sixth grade levels. A copy of the relevant portions of the student workbook appears in the Appendix to this report.

Problem II

Squares are made by using matchsticks as shown in the picture below.

When the number of squares is eight, how many matchsticks are used?

DO NOT ERASE ANYTHING YOU WRITE DOWN; JUST DRAW A LINE THROUGH ANYTHING YOU FEEL IS IN ERROR.

(1) Write a way of solution and the answer to the problem above.

Ans. ____________________

(2) Now make up your own problems like the one above and write them down. Make as many problems as you can. <u>You do not need to find the answers to your problems</u>.

(3) Choose the one problem you think is best from those you wrote down above, and write the <u>number</u> of the problem in the space: ____________________________

Write the reason or reasons you think it is best.

Figure 1

The following aspects are considered in the analyses of data:

1. Rate of correct answer
2. Methods of solution used to solve the problem
   A. Breakdown of problem
   B. Use of drawings
3. Problems made up by students
   A. Type of problem
   B. Comparison to matchsticks problem
      i. Object asked for
      ii. Use of overlap
      iii. Increased dimensions
   C. Use of illustrations
4. The problem chosen as best
5. Responses to the questionnaire.

The results in each section are examined with respect to grade level, sex, and correctness of response to the problem. The analysis is also carried out with respect to the geographic location of the students as well as the relation between method of solution and the questionnaire responses concerning preferences for problem types and other subjective data.

1. Correct solution

Grade

The average rate of correct response was 37% in the fourth grade (N = 84), 58% in the fifth grade (N = 19), and 52% in the sixth grade (N = 105). The fifth grade class, while not part of the design, was included because it was a part of a combined fourth and fifth grade gifted class in Champaign, Illinois. The mean score of the fifth graders was the highest of the three grade groups.

Geographic Location

The Florida students answered correctly most often at both the fourth and sixth grade levels. One curious result is that fourth grade students in Florida answered correctly more often than their sixth grade counterparts. In fact, the fourth grade class from Florida was within one percentage point of the class with the highest rate of correct response.

Sex

In Carbondale and Champaign, the boys' rate of correct solution was more than ten percentage points higher than that of the girls for each location at each grade level. In Florida the fourth grade boys had a slightly higher rate of correct response than did the girls. In the sixth grade, the girls had a slightly higher score.

2. Method of solution

A. Problem analysis

Three ways of solving the matchsticks problem were identified: (i) repetition of squares, or groups of three matchsticks; (ii) draw a picture and/or count, without noticeably; (iii) miscellaneous or other.

Grade

The fourth grade students were most likely to use drawing/counting (70%) and least likely to figure using repetition of squares (7%). The sixth graders were more likely to use drawing/counting (46%) than repetition of squares (30%). The fifth graders were most likely to use repetition of squares (42%) and least likely to use drawing/counting (37%) (again, recall that this was a small, special group of advanced fifth grade students). Uses of "other" strategies were relatively constant from grade to grade, ranging only from 21% to 24% among grade levels.

Sex

In the fourth grade, there was little difference between the methods used by boys and girls. Both boys and girls used drawing/counting most often and few students of either sex used the repetition of squares approach. In the sixth grade, the girls were more likely to use repetition of squares (37%) than were the boys (22%). The boys, on the other hand, were slightly more likely to use drawing/counting (49% to 43%) and "other" strategies (29% to 20%) than were the girls.

Correct Response

At both the fourth and sixth grade levels, the students who answered correctly were more likely to have used counting and less likely to have used repetition of squares than those who answered incorrectly. In both categories, the majority of students used drawing/counting, except for those sixth grade students who answered incorrectly. Of this group, 44% used repetition of squares and 26% drawing/counting. The majority of boys in this group used repetition of squares (53%). Of the girls, more used drawing/counting, but only 35% did so. Regardless of grade, sex, or correct answer, over 60% from all other groups used the drawing/counting approach.

The sixth grade students were more likely than the fourth graders to attack the problem using the repeating pattern approach. But students who used drawing/counting were much more likely to figure the answer correctly than those who used repetition of squares at both grade levels, regardless of sex.

Geographic Location

There was little variation between locations in the proportions of students at a given grade level using each method.

B. Use of drawings

Grade

At all grade levels, students were more likely than not to use a drawing in solving the problem. Seventy percent of the fourth graders used a drawing, 79% did so in 5th grade, and 71% did so in 6th grade. These results are remarkably consistent, especially considering the low number of 5th grade students examined. Due to the low number, data from the 5th grade will not be discussed further in this subsection.

Sex

In the fourth grade, 78% of the girls used a drawing, but only 65% of boys did so. In the sixth grade, boys were more likely to do so, but there was very little difference (73% for the boys, 70% for the girls).

Correct Response

The majority of students in both grades and of both sexes used drawings, whether they got the problem correct or not. However the students who used a drawing were more likely to answer correctly (42% in 4th grade, 63% in 6th grade) than were those who did not use a drawing (24% in 4th grade, 27% in 6th grade).

Location

At the fourth grade level, the students from Florida were more likely to use a drawing (86%) than were those from Carbondale (66%). In Champaign, 80% of the 4th grade students used a drawing, but there were only five such students tested. At the 6th grade level, there was little between location variation in the use of drawings.

3. Problems made up by students.

In this section, only the first question created by each student is examined. As mentioned in the analysis of the Japanese student's responses, the first problem is most likely to reflect the student's initial impression of the given problem.

A. Type of problem

The problems made up by the students were broken down into four types.

(i) Problem similar to the given problem. (That is, a repeating pattern is described and a number of parts must be determined given a number of repetitions of the pattern).

(ii) Basic arithmetical problem.

(iii) Simple counting or measuring.

(iv) Other problems.

Grade Level

The sixth grade students were more likely to create problems similar to the given problem (55%) than were the fourth grade students (25%). The fifth grade students were the most likely to create similar problems (58%) as well as counting and measuring problems (21%). However, due to the low number of fifth students tested, these data will not be discussed further in this subsection.

Fifty-eight percent of the 4th grade students created problems that fit into the "other" category. Many of these problems were unintelligible, so it is possible that some students meant to create problems of other sorts. For this reason, comparisons between grades of problems in categories besides "similar" and "other" problems (that is, categories in which a few more problems would mean a significant difference) will not be made.

Sex

At the fourth grade level, the biggest gender difference is that the boys created more problems in the "other category (65%) than did the girls (50%). In the 6th grade, the boys made more problems similar to the given problems (62%) than did the girls (52%), while the girls made more arithmetical problems (17%) than did the boys (9%).

Correctness

In the 4th grade, students who answered the given problem correctly were more likely to create problems similar to the given problem (35%) than were those who answered incorrectly (22%). These numbers varied little between boys and girls.

In the 6th grade, the percentage of students creating problems similar to the given problem was similar for students who had answered the given problem correctly (58%) and those who answered incorrectly (54%). Among those who answered incorrectly, there was very little between gender difference in creating problems similar to the one presented. Among those who answered correctly, however, 68% of the boys and only 50% of the girls created similar problems.

Location

In the 4th grade, there was a large variation between locations in types of problems created. 48% of students in Florida created problems similar to the given problem. Only 17% of Carbondale students and 20% of the Champaign students did so (but again, there were only five such students in the Champaign group). In the 6th grade, there was much less variation between the groups in terms of the types of problems created.

B. Comparison to matchsticks problem

In this subsection, only those students who created problems categorized as similar to the given problem are considered. These are examined to see how they differ from or resemble the given problem according to three criteria:

i. Object asked for

The vast majority of students who created problems similar to the given problem also asked how many sides (or objects such as matchsticks) would be required to complete a certain number of unit figures (like squares in the given problem). This was what the given problem required as well.

The number of students who asked for something else in the problem was too small to analyze in terms of differences between sexes, grade levels or correctness of response. The most common alternatives were to ask the reverse question (that is to give the number of sides and to ask for the resulting number of unit figures) or to ask for the number of corners.

ii. Use of overlap

The questions categorized as similar to the given question were examined in terms of the use of the condition of overlap.

Questions were categorized as retaining the condition of overlap found in the given problem, changing the condition of overlap, eliminating the condition of overlap or unclear on the condition of overlap.

Grade

The uses of overlap were almost identical for 4th and 6th grade students. Thirty three per cent of the fourth graders and 34% of the sixth graders retained the condition of overlap. Ten per cent of students at each grade level changed the condition. 48% of 4th graders and 44% of 6th graders eliminated the condition.

Sex

Given the small number of stude ·s in this category, the small differences between boys and girls were not considered to be significant.

Correctness

Correctness of response seemed to be related to the use of overlap. Students who had answered the given problem correctly were more likely to retain the condition of overlap (37%) than were those who had answered incorrectly (25%) and this difference was fairly consistent across sexes and grade levels.

At the fourth grade, students who changed the condition of overlap had all answered the given problem correctly, but this finding did not apply to other grade levels.

Students who had answered the given problem correctly were less likely to eliminate the condition of overlap (37%) than were those who had answered incorrectly (55%).

These findings are not suprising because recognizing the condition of overlap was one key to successfully solving the matchsticks problem. It could therefore be expected that students who solved the problem correctly would be more likely to include a condition of overlap when asked to create similar problems.

iii. Increased dimensions

The given problem involved matchsticks being used to form a row of squares. These have been categorized as single-dimensional. Some students created problems in which a two-dimensional array of squares or a three-dimensional pattern was formed. These have been called multi-dimensional. Another category, called "special forms," includes questions in which unit figures form a pyramid, a circle, or a set of concentric circles. The last category consists of those questions in which the pattern of unit figures is irregular.

The vast majority of students who created problems similar to the given problem created single-dimensional problems. This was true across grade levels, sexes, and correctness of responses to the given question. None of the fifth grade boys who answered incorrectly created single-dimensional problems, but there were only three such students. The number of students who asked questions in the other categories was too small to analyze in these terms.

C. Use of illustrations

The vast majority of students used an illustration in the problems they created. This was a consistent result across categories of grade level, sex, correctness of response and geographic location.

4. The problem chosen as best

The reasons given for choosing one problem as best have been organized into five categories: (i) It was hardest, (ii) It was easiest, (iii) Because of the particular content of the problem (i.e., because it has fractions), (iv) Because of the value of the problem (e.g. because it is different, educational, or can be solved in many ways, (v) Other responses. Some examples of other reasons for choosing a problem are that it is "fun," "neat," or "best" as well as blank responses.

Except for the "other" category, the only reason frequently given for choosing a problem as best was that it was the hardest. Students were more likely to choose the hardest problem than the easiest problem regardless of grade level, sex, or correctness of response to the given problem.

Boys, students who answered incorrectly, and 4th graders were slightly less likely to choose the hardest problem as best than were students who didn't fit into these categories (or who fit into fewer of them). The differences were not large, but they were fairly consistent.

The fourth grade students were more likely than older students to choose reasons which fit into the "other" category.

5. Responses to the questionnaire

Note: Due to the small number of fifth graders, only fourth and sixth grade students are considered in this section when comparing students at different grade levels.

Do you like mathematics?

The sixth graders were less likely to express a liking for mathematics (50%) than were the fourth graders (62%). This increased tendency to "like mathematics less" from fourth to sixth grades was greater for the boys (44% to 65%) than for the girls (53% to 58%). The sixth graders were more likely than the fourth graders to say they were neutral. Few at either grade said they disliked math.

At the 4th grade level, students who has solved the given problem using repetition of squares were more likely to say they like mathematics (83%) than were those who had used drawing/counting (59%) or "other" methods (63%). The differences at the 6th grade level were much smaller.

Are you good at mathematics?

The sixth graders were less likely to say they were good at mathematics (31%) than were the fourth graders (49%). The sixth graders were more likely to say they were neutral. Few in either grade said they were not good at mathematics.

Those students who had answered the given problem correctly were more likely to say they were good at mathematics (49%) than were those who had answered incorrectly (37%). This difference was more marked for boys than for girls at both the fourth and sixth grade levels. There was little difference between boys and girls when taken as a whole at either grade.

Do you think today's problems are interesting?

In almost any combination of grade, sex, and correctness of response, slightly over 50% of the students reported that they found the problem interesting. The only category which was noticeably different from others in its responses was the fourth grade girls who had answered the given problem incorrectly. Interestingly, these students were much more likely to find the problem interesting (71%) and much less likely to report "neutral" than were other students.

At the fourth grade level, the students who had solved the given problem using repetition of squares were more likley to find the problem interesting (83%) than were those who had used drawing/counting (56%) or "other" methods (47%). At the sixth grade level, the group differences were much smaller.

Do you think today's problems are easy?

At both the fourth and sixth grade levels, the girls were more likely to say the problem was easy (50% in 4th and 55% in 6th) than were the boys (38% in 4th and 44% in 6th).

The differences with respect to combinations of grade, sex, and correctness of response do not seem to follow a pattern.

Are today's problems the same as the problems in your mathematics textbook?

The fourth graders were more likely to say the problems were different (52%) than were the sixth graders (34%). The sixth graders were more likely to state that they could not say. Very few students thought the problems were the same as in their textbooks.

In comparison to the problems in your mathematics textbook, did you like today's problems more, the same, or less?

In almost any combination of grade, sex, and correctness of response to the given problem, slightly over 50% of the students liked this problem more. There was little difference in responses between students in these categories.

Have you seen problems like this before?

The sixth grade students were more likely to say yes (75%) than were the fourth graders (49%). Of those in fourth grade, the girls were more likely to say yes (64%) than were the boys (38%).

Finally, a note about the relation between methods of solution to the given problem and questionnaire results. Only the first four questions were included in this analysis since these seemed the most relevant. Of these questions, whether or not a student used a drawing to solve the given problem did not appear to be related to the survey results. As noted above, the students who had solved the given problem using repetition of squares were more likely to like mathematics and to think the problem was interesting than were those who used other methods.

THE MARBLE PATTERN PROBLEM: RESULTS OF AN ANALYSIS OF U.S. STUDENTS' SOLUTIONS

Katheryn K. Fouche
Mary Grace Kantowski

University of Florida

This report presents the analysis of U.S. results on the *marble pattern problem.* A copy of the relevant portions of the student booklet appears in the Appendix to this report.

Method

Subjects

A total of 791 students in grades 6, 8, and 11 participated in the study. The distribution by gender in each grade is given in Table 1.

Table 1

Number of Students Involved in the Survey by Grade and Gender

Grade	(total)	Gender	Number
6	(179)	female	93
		male	86
8	(368)	female	189
		male	179
11	(244)	female	124
		male	120

Task

A workbook containing a selection of nonroutine problems and an attitude survey in the form of a questionnaire was given to each student. The marble pattern problem, as illustrated in Figures 1-3, appeared as the first of two problems for the grade 6 and 8 students and as the last of three problems for the grade 11 students. Students were asked to complete the attitude survey after solving the problems.

In Part 1 of the problem, students were asked to determine the number of marbles in the fourth place using as many different solution methods as possible. Space was available for seven solution methods. In Part 2 students were asked to show one method of solution to find the number of marbles in the sixteenth place. Part 3 on the grade 6 and grade 8 tests required a formula for finding the number of marbles in the one hundredth place while the eleventh grade students were asked to find a formula for the number of marbles in the nth place.

Directions: Read the question carefully and follow directions.

<u>You should write down all your work</u>. Do not erase anything you write down, just draw a line through it rather than erase it.

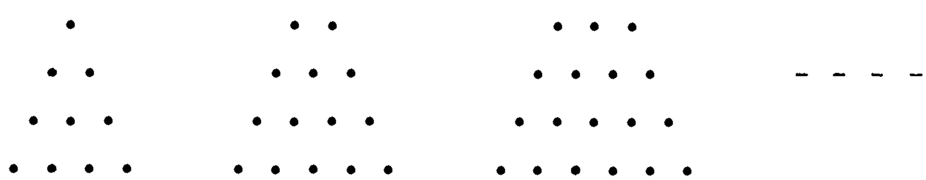
There are two questions and you will have 15 minutes for each question.

Do not turn the page until the teacher tells you to.

Problem I

Marbles are arranged as follows:

first second third fourth



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Way of solution 1)

Ans. ________

(2) How many marbles are there in the sixteenth place? Show your way of solution and your answer.

Ans. ________

Figure 1. Parts 1 and 2 of the marble pattern problem.

(3) Try to find a formula for finding the number of marbles in the hundredth place.

Figure 2. Part 3 of the marble pattern problem for students in grades 6 and 8.

(3) Try to find a formula for finding the number of marbles in the nth place.

Figure 3. Part 3 of the marble pattern problem for students in grade 11.

Coding Method

Prior to coding the student solutions, several possible methods of solutions were identified. Below is a brief description of each solution method identified.

Methods of Solution for Part 1 (Finding the number of marbles in the 4th place)

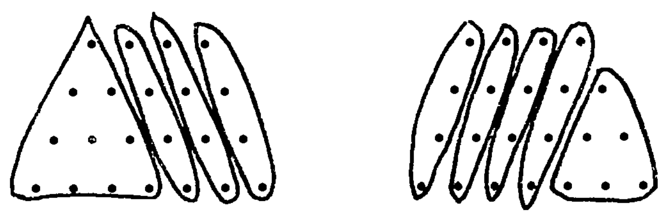
Examples of student solutions using these methods are found in Appendix A.

1. Enumeration: Drawing a representation of the number of marbles in the fourth place and counting the marbles.
2. Pattern:
   a. Table: Indicating with a table or by noting, using successive differences, that each stage increased by 4.
   b. Adding 1 to each row and finding the sum: Indicating the addition of one marble to each row either with a picture or in writing.
   c. Net gain: Indicating that for each stage the top row was removed from the previous stage and a new bottom row was added. (The net gain is the difference between the number of marbles removed from the top row and the number of marbles added to the bottom row.)
3. Addition of four consecutive integers: Noting that the number of marbles is the sum of four consecutive integers (4 + 5 + 6 + 7).
4. Grouping:
   a. 10 + 3(4): Grouping marbles by adding "diagonals" consisting of four marbles to

the original structure containing 10 marbles. Figure 4 illustrates this method of grouping.

b. 4(4) + 6: Grouping marbles by adding an additional six marbles to the "diagonals" consisting of four marbles. See Figure 4 for an illustration.

c. Other grouping: Some variation of the groupings mentioned above.



10 + 3(4)   4(4) + 6

Figure 4: Grouping examples.

5. "Other": If five or fewer students employed a method of solution, the method was coded as "other". A formula solution was coded as "other" for students in grades 6 and 8 on Part 1.

Methods of Solution for Part 2 (Finding the number of marbles in the 16th place)

1. Enumeration: Drawing a picture representation of the number of marbles in the sixteenth place and counting the marbles.
2. Pattern: Completing a table of the number of marbles to the 16th place.
3. Addition of four consecutive integers: Noting that the number of marbles is the sum of four consecutive integers (16 + 17 + 18 + 19).
4. Grouping:
   a. 10 + 15(4): Grouping marbles by adding "diagonals" consisting of four marbles to the original structure containing 10 marbles.
   b. 16(4) + 6: Grouping marbles by adding an additional six marbles to "diagonals" consisting of four marbles.
   c. 22 + 4(12): Grouping marbles by adding "diagonals" consisting of four marbles to the solution for Part 1.
5. "Other": If five or fewer students employed a method of solution, the metl od was coded as "other". Using a formula solution was coded as "other" for students in grades 6 and 8 on Part 2.

Methods of Solution for Grades 6 and 8 Part 3 (Finding the number of marbles in the 100th place)

1. Addition of four consecutive integers: Noting that the number of marbles is the sum of four consecutive integers (100 + 101 + 102 + 103).
2. Grouping:
   a. 10 + 4(99): Grouping marbles by adding "diagonals" consisting of four marbles to the original structure containing 10 marbles.
   b. 4(100) + 6: Grouping marbles by adding an additional six marbles to "diagonals" consisting of four marbles.
   c. 22 + 4(96): Grouping marbles by adding "diagonals" consisting of four marbles to the solution for Part 1.
3. Application of a formula.

Methods of Solution for Grade 11, Part 3 (Finding the number of marbles in the nth place)

1. n + (n + 1) + (n + 2) + (n + 3): Noting that the number of marbles is the sum of four consecutive integers.
2. 10 + 4(n - 1): Grouping marbles by adding "diagonals" consisting of four marbles to the original structure containing 10 marbles.
3. 4(n) + 6: Grouping marbles by adding an additional six marbles to "diagonals" consisting of four marbles.
4. Other grouping: Some variation of the groupings mentioned above.

Each solution method for each student was coded as one of the identified methods of solution; as an irrelevant method of solution if computations that did not relate to the problem were shown; as an incorrect method; or as "didn't understand" if the student gave evidence that he/she was misled by the question. Each method used to solve the problem was evaluated independently.

Results

The analysis of the data for the American sample will be followed by a discussion of the results of the Japanese study as reported by Junichi Ishida of the Yokohama National University. Comparisons between the results from Japan and the United States conclude this section. The first discussion includes the following:

1. The percentages of students who found the correct solution.
2. The mean number of different methods of solution for Part 1 for those students who found at least one correct method of solution.
3. The distribution of the categories of metnods of solution for each part.

4. The frequency with which successful methods of solution were repeated in successive parts of the problem.
5. The sophistication of methods of solution employed by students, and the determination of whether the level of sophistication increased as students progressed through the process of searching for more solution methods.

Correct Solutions

Approximately eighty-two percent of 6th grade, ninety-three percent of 8th grade, and ninety-six percent of 11th grade students were able to find the correct solution for the first part using at least one method. The results are shown in Table 2. One possible explanation for the low percentage of correct methods of solution among the sixth grade students was a misunderstanding of the wording of the question. Part 1 read, "How many marbles *are there* in the fourth place?". Many students responded that there were no marbles in the fourth place. A few went on to point out that "There are only lines in the fourth place but no marbles". Had the question been worded to give some suggestion of a pattern of stages, more students might have followed a correct solution path.

Percentages of correct solutions for Parts 2 and 3 were low at all grade levels. Examination of the papers revealed that many students did not attempt Parts 2 or 3, possibly because they used the entire 15 minutes allotted to find multiple solution methods for Part 1 of the problem.

Table 2

Percent of Students at Each Grade Level Finding the Correct Solution.

	Grade 6	Grade 8	Grade 11
Question 1	82.1	92.7	96.0
Question 2	25.7	52.5	68.0
Question 3	17.3	29.9	40.1

Different Methods of Solution for Part 1

In comparing the number of different methods of solution, students were given credit for each different solution method. For the students who gave at least one correct method of solution for Part 1 there were no grade or gender-related differences in the mean number of different methods of solution. The means are listed in Table 3 by grade and gender and in Table 4 by gender only.

Table 3

Mean Number of Different Methods of Solution for Part 1 by Grade and Gender

Grade	Sex	Mean	Std. Dev.
6	female	1.91	.97
	male	1.95	.97
8	female	2.10	.95
	male	2.02	.91
11	female	1.93	.92
	male	2.06	.95

Table 4

Mean Number of Different Methods of Solutions for Part 1 by Gender

Gender	Mean	Std. Dev.
Female	2.00	.95
Male	2.02	.93

Distribution of the Categories of Correct Methods of Solution for Part 1

As illustrated in Table 5, the most frequently chosen methods for finding the number of marbles in the fourth place were finding a pattern, enumeration and adding one marble to each row. The few sixth and eighth grade students who used a formula were included in the count for "other". The percentages shown in Tables 5, 6, and 7 are based on the number of students who found at least one correct method of solution for the part under consideration.

Table 5

Percentage Categories of Correct Methods of Solution for Part 1 by Grade

Part 1	Grade 6	Grade 8	Grade 11
Enumeration	38.1	33.1	45.1
Pattern			
a. Table	46.3	63.3	54.0
b. Adding 1 to each row and finding the sum	44.9	51.3	24.1
c. Net Gain	1.4	3.8	3.8
Addition	30.6	28.2	35.4
Grouping			
10 + 3(4)	13.6	11.7	9.3
4(4) + 6	8.8	9.1	11.8
other grouping	3.4	2.3	1.7
Other	5.4	3.2	5.5
Formula	N/A	N/A	12.0

In Part 2 students were asked to show one method of solution for finding the number of marbles in the sixteenth place. When the student used two different methods, credit was given for each. As Table 6 indicates, the most frequently occurring methods were addition, finding a pattern and grouping. The majority of sixth grade students favored addition. In the other two grades the three methods were distributed fairly evenly with pattern finding slightly favored.

Table 6
Percentage Categories of Correct Methods of Solution for Part 2 by Grade

Part 2	Grade 6	Grade 8	Grade 11
Enumeration	6.5	8.2	1.8
Pattern	15.2	35.1	36.3
Addition	52.2	28.9	33.9
Grouping			
16(4) + 6	6.5	9.8	11.9
10 + 15(4)	6.5	10.3	11.9
22 + 4(12)	10.9	10.3	6.0
Other	17.4	5.7	8.3
Formula	N/A	N/A	11.9

In Part 3 students in grades 6 and 8 were to find the number of marbles in the one hundredth place. The results are presented in Table 7. Addition was the most frequently chosen method by sixth grade students while addition and grouping were about evenly distributed among eighth grade students..

Table 7

Percentage Categories of Correct Methods of Solution for Part 3 for Grades 6 and 8

Part 3	Grade 6	Grade 8
Addition	61.3	45.5
Grouping		
4(100) + 6	6.5	14.5
10 + 4(99)	16.1	8.5
22 + 4(96)	12.9	8.2
other grouping	0	6.4
Formula	6.5	10.9

Table 8 presents the results of the solution of Part 3 of the problem by the eleventh grade students. They were asked to find a formula for the number of marbles in the *n*th place. Eleventh grade students most often generalized one of the grouping methods to obtain $4(n) + 6$ or $10 + 4(n - 1)$.

Table 8

Percentage Categories of Correct Methods of Solution for Part 3 for Grade 11

Part 3	Grade 11
4(n) + 6	45.5
10 + 4(n - 1)	27.3
n + (n + 1) + (n + 2) + (n + 3)	17.2
other	11.1

Repetition of Methods of Solution for Each Successive Problem

Once a student successfully employed a method of solution, did that student choose the same method to answer the next part? Of the 405 students who found a correct solution for both Parts 1 and 2, a total of 294 (72.6%) repeated one of their methods from Part 1 to solve Part 2. Of the sixth grade students who repeated a method of solution, 48.7% used addition for both parts whereas 20.5% found some type of pattern. Only 15.4% chose grouping for both solutions. In the eighth grade, of the students who repeated a method of solution, 52.2% found a pattern while 19.8% and 18.0% used addition and grouping respectively. Of the eleventh grade students who used a method of solution from Parts 1 for Part 2, 46.7% found a pattern, 28.9% used addition, while 17.8% used grouping. The results are shown in Table 9. Examination of student papers indicated that a solution that had not been used before was usually a generalization of one used previously.

Table 9

Percent of Students Who Repeated a Method of Solution from Part 1 to 2 by Grade

Repeated Method	Grade 6	Grade 8	Grade 11
Enumeration	7.7	7.0	.1
Pattern	20.5	55.2	46.7
Addition	48.7	19.8	28.9
Grouping	15.4	18.0	17.8
Other	7.7	0	2.2
Formula	N/A	N/A	3.7

Of the 141 sixth and eighth grade students who found the correct solution for Parts 2 and 3, 72.2% employed the same method on both parts. The sixth grade students tended to more often repeat the addition method (54.8%) while the solution methods of eighth grade students were about evenly split between grouping and addition (34.5% and 32.7%, respectively). Results for grades 6 and 8 are shown in Table 10.

Table 10

Percent of Students Who Repeated Methods of Solution from Part 2 to 3 by Grade

Repeated Method	Grade 6	Grade 8
Addition	54.8	32.7
Grouping	25.8	34.5

Sophistication of Responses

For each student the first and second solution methods employed in Part 1 of the problem were identified in addition to the number of methods of solution. This was followed by a determination of whether, for at least one method after the first, the student chose a more sophisticated method. One justification for encouraging multiple methods of solution might be rooted in results indicating that in their search for additional solution methods, students' methods tend to become more sophisticated. The seven identified methods were ranked 1 - 7 according to the level of understanding represented and their generalizability.

Sophistication of methods of solution (from lowest to highest)

1. Enumeration
2. Table
3. Adding 1 to each row or net gain
4. Addition
5. Grouping
6. Other
7. Formula (11th grade only)

The results for each identified category are presented in Table 11. Of the students who solved Part 1, 45.5% progressed to a more sophisticated method than a method of solution used earlier in Part 1. That percentage increased to 52.4% when the sample was limited to only those students who began with one of the methods identified as lower level.

Table 11

Percent of Students in Each Level of Sophistication Category by grade *

Sophistication	6	8	11	Total
Began with lower level** solution and did not increase in sophistication	43.9	45.7	38.8	43.2
Began with lower level** solution but progressed to a higher level	43.2	44.2	42.7	43.6
Began with a higher level*** and did not increase in sophistication	9.5	9.4	15.1	11.3
Began with a higher level*** but moved to an even higher level	3.4	.6	3.0	1.9

* Percent is of the students who found at least one method of solution on Part 1.

** The first 3 methods, enumeration, pattern, and adding 1 to each row and finding the sum were considered lower level methods.

*** Higher level methods were identified as addition, grouping, other, or formula (formula for 11th grade only.)

Questionnaire Responses

A seven-question survey concerning students' attitudes about mathematics and about each of the nonroutine problems was administered immediately following the completion of the problem booklet. (Figure 5.) The student questionnaire examined attitudes about mathematics, as well as attitudes about their mathematical ability in general and about the marble pattern problem in particular.

Questionnaire to Students

1. Do you like Math?
   1. like math 2. neutral 3. dislike math
2. Are you good at math?
   1. good at math 2. neutral 3. not good at math
3. Do you think that today's problems are interesting?
   1. interesting 2. neutral 3. not interesting
4. Do you think that today's problems are easy?
   1. easy 2. average 3. difficult
5. Are today's problems the same as the problems in your math textbook?
   1. the same as 2. can't say 3. different from
6. In comparison to problems in your math textbook, did you like today's problems?
   1. more 2. the same as 3. less
7. Have you seen problems like this before?
   1. yes 2. no

Figure 5. Attitude questionnaire.

The relationship between each of the seven attitude variables and the number of different methods of solution for Part 1 was investigated. The results for each grade level are shown in Table 12. The results indicated a relationship between a positive response on the attitude survey and a greater number of different methods of solution for Part 1 on six of the seven variables. The number of different solution methods was not related to the student's response to question 1 (Do you like math?).

Table 12

Attitude Responses by Percent of Each Grade

Questions	Grade	Positive	Neutral	Negative
1. Do you like math?	6	60.1	33.1	6.7
	8	50.5	40.4	9.0
	11	54.5	36.4	9.1
2. Are you good at math?	6	42.7	51.7	5.6
	8	34.5	57.5	7.9
	11	46.7	43.4	9.9
3. Do you think today's problems are interesting?	6	46.1	36.5	17.4
	8	41.1	40.2	18.8
	11	62.7	27.3	10.0
4. Do you think today's problems are easy?	6	30.7	50.3	19
	8	40.2	44.5	15.3
	11	52.7	36.1	11.2
5. Are today's problems the same as the problems in your math textbook?	6	5.6	37.2	57.2
	8	15.9	44.1	40.0
	11	22.5	29.6	47.9
6. In comparison to problems in your math textbook, did you like today's problem?	6	40.2	31.0	28.7
	8	31.3	34.3	34.3
	11	45.2	38.9	15.9
7. Have your seen problems like this before?	6	66.9		33.1
	8	73.8	N/A	26.2
	11	77.0		23.0

Concerning gender differences, the only attitude variable that indicated a relationship was response 2 (Are you good at math?). Males more often than females thought that they were good in mathematics even though the results on performance show no significant gender differences. During the examination of the students' work, it was noted that girls offered more written explanations than the boys. Many girls wrote as if in conversation with the grader. Generally the boys included only the necessary computations with fewer explanations. Table 13 presents the responses by gender.

Table 13

Percentage of Responses in Each Category of the Questionnaire by Gender

Questions	Gender	Positive	Neutral	Negative
1. Do you like math?	Female	52.1	37.7	10.2
	Male	55.9	37.3	6.8
2. Are you good at math?	Female	35.2	53.8	10.9
	Male	45.3	49.7	5.0
3. Do you think today's problems are interesting?	Female	49.0	35.9	15.1
	Male	48.7	34.8	16.5
4. Do you think today's problems are easy?	Female	41.3	42.1	16.7
	Male	42.4	44.5	13.1
5. Are today's problems the same as the problems in your math textbook?	Female	16.9	39.7	43.4
	Male	14.2	36.5	49.3
6. In comparison to problems in your math textbook, did you like today's problem?	Female	36.8	35.1	28.1
	Male	38.4	34.9	26.7
7. Have your seen problems like this before?	Female	72.9	N/A	27.1
	Male	73.5		26.5

U. S. - Japan Comparison

Some observable comparisons between the Japanese and American results were noted but not subjected to statistical tests. Students from both countries employed the same methods of solution for each part of the problem. Overall the sixth and eighth grade Japanese students found methods of solution that were identified as higher level more often than did their American counterparts. In Japan, the eighth grade students were asked to find a formula for the number of marbles in the nth place, while in the U.S. only the eleventh grade students were asked to generalize with a formula. Eleventh grade students in Japan were not administrated this problem. In Japan the eighth grade students used the same methods [4(n) + 6; 4(n - 1) + 10; n + (n + 1) + (n + 2) + (n + 3)] to determine their formula for the nth place as did the eleventh grade students in the U.S.

In the area of repetition students in each country tended to use one of the methods of solution from a previous part to answer subsequent parts. There is also a parallel in the kinds of mistakes made by both groups of students. Students in the U.S. often calculated the number of marbles in the sixteenth place by multiplying the number in the fourth place by four; [i.e. 4(22)]. Common errors for Part 2 in both countries were often in the form [4(16) + 10] or 4(16). For the number of marbles in the one hundredth place, students from both countries frequently made similar mistakes to those mentioned for the sixteenth place; [i. e. 4(100) + 10 or 4(100)].

The difference between the way students in the U.S. and students in Japan solve problems seems not to be in the methods chosen or the number of methods, but in the age at which students are able to employ the methods. In Japan 33.0% of the students in the eighth grade were able to find a formula to represent the number of marbles in the nth place. Eleventh grade American students performed only slightly better with 39.7% finding a formula. The additional number of school days per year in Japan which places Japanese eighth grade students a full school year ahead of their American counterparts and the fact that students in Japan study algebra at an earlier age could have given the Japanese students an advantage in this portion of the study.

Conclusion

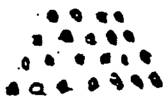
Although this report only covers the analysis of one problem, the richness of these data and the potential for additional implications for mathematics education highlights the value of the collaborative study. In addition to the statistical results, the creativity that was apparent in many of the student papers was very encouraging. For students who do not often fare well when their achievement scores are compared internationally, American students were often impressive with the ingenious ways in which they found solutions to the problem. The U.S.-Japan Collaborative Research on Problem Solving should serve as the ground work for future international research projects in the area of nonroutine problem solving.

Appendix A

Examples of Students' Solutions

Part 1 Methods of Solution:

Enumeration:

 - count how many

Ans. 22

Pattern:

Table:

First = 10 balls
Second = 14 balls
Third = 18 balls
Fourth = ?

$$\begin{array}{r} 18 \\ 4 \\ \hline 22 \end{array}$$

Ans. 22 balls

Adding 1 to each row and finding the sum:

Take the third arrangement and re-add the last row therefor ?

 = 22

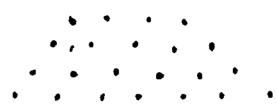
Ans. 22

60

Part 1 Methods of Solution: *Continued*

Net gain:

take top row off, add another row at bottom containing marbles from top row plus 4 new marbles



Ans. 22

Addition of four consecutive integers:

1 + 2 + 3 + 4 = 10
2 + 3 + 4 + 5 = 14
3 + 4 + 5 + 6 = 18
4 + 5 + 6 + 7 = 22

Ans. 22

Grouping:

10 + 3(4):

10
+4
+4
+4
22 marbles

Ans. 22 marbles

Part 1 Methods of Solution: *Continued*

4(4) + 6:

4321 44321 444321 4444321

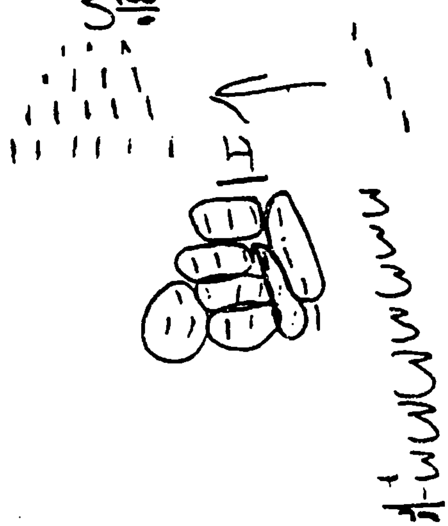
22

Ans. 22

Other grouping:

Find Groups of a number (3) when counting. add 4

3rd



Ans. 22

62

Part 1 Methods of Solution: *Continued*

"Other"

odd numbers starting with 5 (like 5, 7, 9, 11, ...)
X's 2

1st place	2nd	3rd	4th
$\boxed{5} \times 2 = 10$	$\boxed{7} \times 2 = 14$	$\boxed{9} \times 2 = 18$	$\boxed{11} \times 2 = 22$

Ans. 22 marbles

Find a formula

$(\boxed{10} + (\triangle 3 \times 4) = \text{Answer}$
22

many
$\triangle$ = How times you want to find. (3)

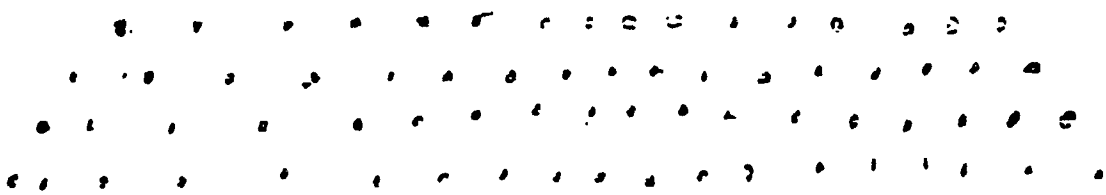
$\square$ = The number of marbles you started out with.

Ans. 22

63

Part 2 Methods of Solution:

Enumeration:



Ans. 40

Pattern:

5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
26	30	34	38	42	46	50	54	58	62	66	70	74	78	82	86

Ans. 70

Addition of four consecutive integers:

16 ← 4 rows, start with 16 marbles in first row
17
18
+ 19
70

1 more for each row

or 16 × 4 = 64, + 6 = 70.

Ans. 70

Grouping:

10 + 15(4):

$10 + [4 \cdot (16-1)] = 70$

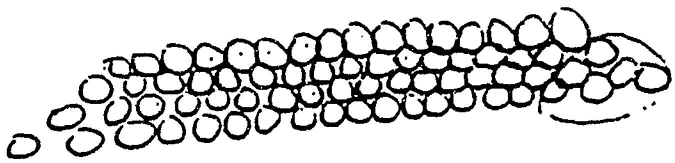
16 − 1 = 15; 15 × 4 = 60; 60 + 10 = 70

Ans. 70

64

Part 2 Methods of Solution: *Continued*

16(4) + 6:



$$\begin{array}{r} 16 \\ \times\ 4 \\ \hline 64 \\ +\ 6 \\ \hline 70 \end{array}$$

16 rows of 4 marbles

Ans. 70

---

22 + 4(12):

$$22 + \underbrace{4+4+4+4+4+4+4+4+4+4+4+4}_{12} = 70$$

$$\begin{array}{r} 12 \\ \times\ 4 \\ \hline 48 \end{array} + 22$$

Ans. 70

"Other"

$$\begin{array}{r} 16 \\ 17 \\ +18 \\ \hline 51 \\ +19 \\ \hline 70 \end{array}$$

take the # of marbles in the 15th place on the top row (15 marbles), add another row at botto[m] containing original 15 marbles plus 4 extra. Then coun[t] # of marbles in the 15th place, rows 2+3+4 + add 19 to this #

1st row in 15th place plus 4

Ans. 70

Part 3 Methods of Solution for Grades 6 and 8:

Addition of four consecutive integers:

one hundredth

line 1 = A
" 2 = B
" 3 = C
" 4 = D

A + 1 = B
B + 1 = C
C + 1 = D

A 100
B 101
C 102
D 103

Grouping:

10 + 99(4):

$$10 + [4 \cdot (100 - 1)] =$$

100(4) + 6:

$$\cancel{100(4) + 3 + 2 + 1}$$

$$100(4) + 6$$

Part 3 Methods of Solution for Grades 6 and 8: *Continued*

22 + 4(96):

Same as (2)

384
22
406

∴ spots awy from fourth place

96 # added each time
× 4
384

Formula would be

spots way from 4th place times number added each time ~~added~~ the result added to number of the 4th

---

Application of a Formula:

$x+(x+1)+(x+2)+(x+3)$

$X+(x+1)+(x+2)+(x+3)$

67

Part 3 Methods of Solution for Grade 11:

n + (n + 1) + (n + 2) + (n + 3):

# of marbles = (n) + (n+1) + (n+2) + (n+3)

10 + 4(n - 1):

$a_n = a_1 + d(n-1)$

$a_n = 10 + d(n-1)$

$a_n = 10 + 4(n-1)$

4(n) + 6:

n x 4 +6

Take your # in the nth

multiply n by 4 + add 6

4(N) + 6 = # marbles

Other grouping:

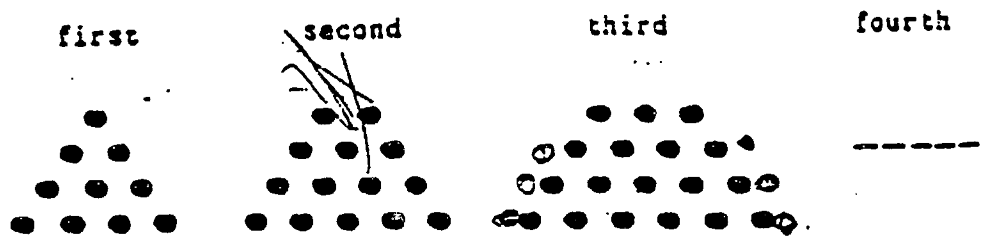
$n^{th} = 4^{th} + 4(n - 4^{th})$

$n^{th} = 22 + 4(n - 4)$

Examples of Incorrect
Solution Methods

Problem I

Marbles are arranged as follows:



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Way of solution 1)

as they went on they keep taking
a row away so u took a row away and
added the three rows

Ans. 19

(Way of solution 2)

u first took a row away and added
it to the bottom till they turned
up the way they usually do
and added

Ans. 19

Continue on the next page.

(Way of solution 3)

first i took away 1 off the first one two off the second one 3 off the 3 one and added 1,2, and 3 together and got 6 and ~~[illegible]~~ got a fraction 19/6 = and reduced to lowest terms

Smile

Ans. 3 1/6

(Way of solution 4)

I would predict and try how i got 19 and turn in to a improper fraction 19/6 then divide 3/6 = but put as the - and get - reduce

Ans. 16/6 = 2 4/6 = 2 2/3

If you need more space, write on the back of this page.

(2) How many marbles are there in the sixteenth place? Show your way of solution and your answer.

I don't understand besides i think you muliply the dots times 16.

Ans. 140

(3) Try to find a formula for finding the number of marbles in the hundredth place.

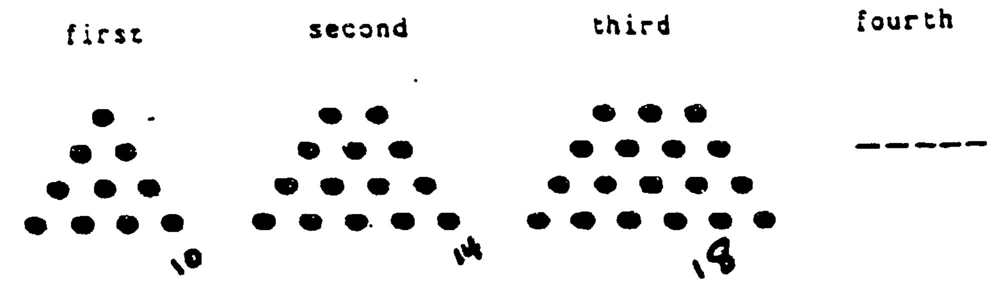
Don't understand

71

Stop working when the teacher says "STOP."

Problem I

Marbles are arranged as follows:



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place? none

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Way of solution 1) look at it there are no marbles.

Ans. none

(Way of solution 2)

see if it looks like the other ones.

72

Ans. none

Continue on the next page.

(Way of solution 3) See if the design in the fouth space looks the same as the others.

Ans. none

(Way of solution 4)

In the fourth space there are nothing but lines. Those are not marbles.

Ans. none

If you need more space, write on the back of this page.

---

(2) How many marbles are there in the sixteenth place? Show your way of solution and your answer.

1 10	11 50
2. 14	12 54
3- 18	13 58
4 · 22	14 62
5. 26	15 64
6. 30	16. 68
7 34	
8 38	
9 42	
10. 46	

Ans. 68

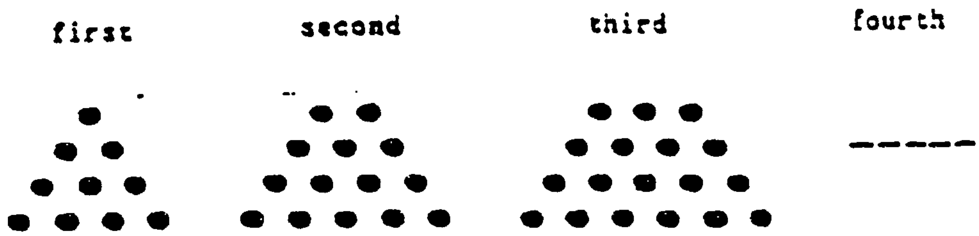
---

(3) Try to find a formula for finding the number of marbles in the hundredth place. Each time you go up a number add four.

Stop working when the teacher says "STOP."

Problem I

Marbles are arranged as follows:



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place? none

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Way of solution 1)
There is only 3 places.

Ans. ________

(Way of solution 2) So if there is only 3 places there can not be any on the fourth place.

Ans. ________

74

Continue on the next page.

(Way of solution 3) unless you take some marbles from the other places.

Ans. none

(Way of solution 4) you can take the marbles from the second third + just + even them up.

Ans. none

If you need more space, write on the back of this page.

---

(2) How many marbles are there in the sixteenth place? Show your way of solution and your answer.

There is none. you would first have to take marbles from the otherplaces + even them up.

Ans. none

---

(3) Try to find a formula for finding the number of marbles in the hundredth place. you can't all of them until you do find a formula.

75

Stop working when the teacher says "STOP."

(Way of solution 5) There is no other way I can figure this out.

Ans. ________

(Way of solution 6) unless if I take the marbles from the 4 + 6th place when I took it from the other places before.

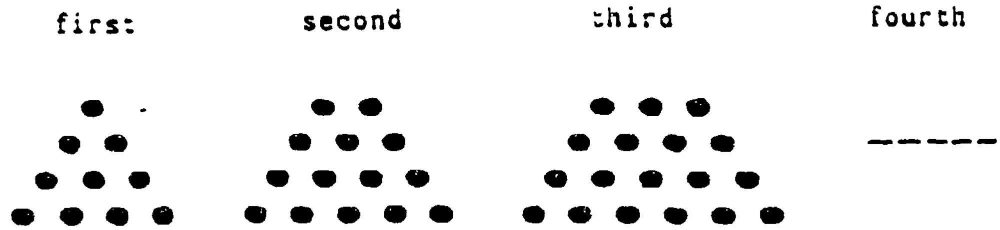
Ans. ________

(Way of solution 7) I can't think of another way.

Ans. ________

Problem I

Marbles are arrang_d as follows:

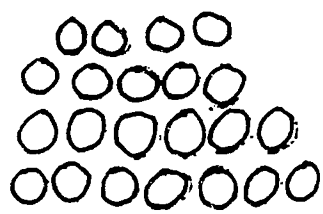


Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

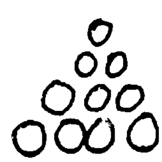
FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Way of solution 1)



Ans. 5

(Way of solution 2)

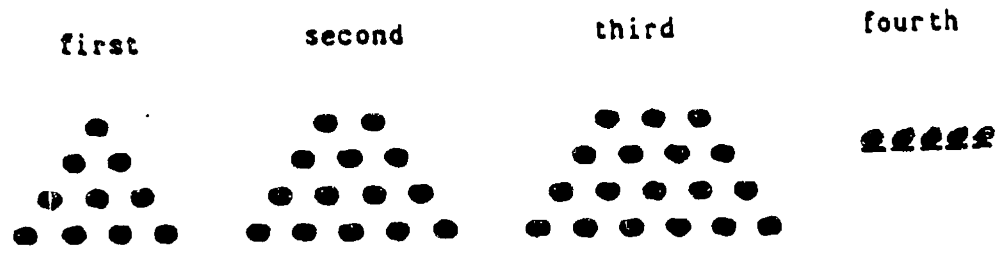


Ans. 2

Continue on the next page.

Problem I

Marbles are arranged as follows:



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Method 1)

Look at the fourth space there are five lines about the size of five marbles.

Ans. 5

(Method 2)

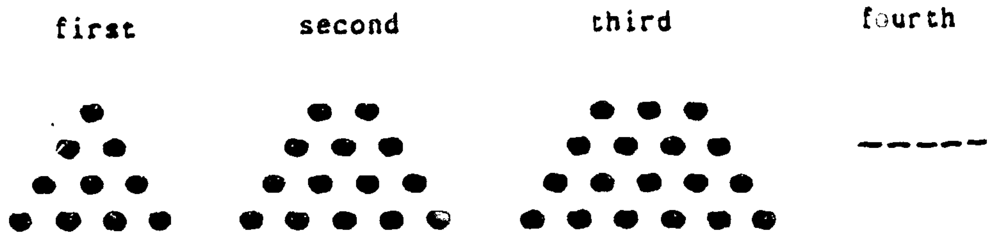
circle in the lines to get your answer.

Ans. 5

Continue on the next page.

Problem I

Marbles are arranged as follows:



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Method 1) no marbles

____ = 0

Ans. 0

(Method 2)

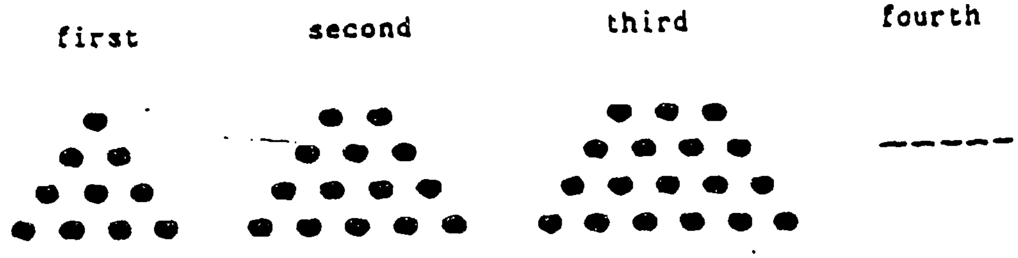
40 marbles all together answer 40

Ans. 40

Continue on the next page.

Problem I

Marbles are arranged as follows:

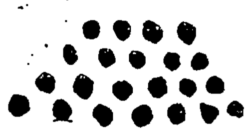


Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

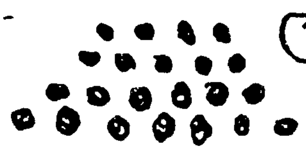
(Method 1)



① I counted first, second and third places and every timit it was ~~more~~ 4 more so I added four onto the third place and that was the answer to the fourth place.

Ans. 22

(Method 2)



② There were 5 Blanks under the word forth. So I measured it to the third place and those blanks represented the marbles in the second row, Every row you go down 2 more is added.

Ans. 22

Continue on the next page.

THE ARITHMOGONS PROBLEMS: RESULTS OF AN ANALYSIS OF U.S. STUDENTS' SOLUTIONS AND A COMPARISON WITH JAPANESE STUDENTS

Jerry P. Becker
Southern Illinois University at Carbondale

Allison Owens
St. Louis, MO

This report presents analyses of U.S. results on the *Arithmogons Problems*, and a comparison with the results from a sample of Japanese students who solved the same problems. A copy of the relevant portions of the student booklet appears in the Appendix to this report.

The *Arithmogons problem* and its variation was the second of two problems administered during one class period at the eighth and eleventh grade levels.. Subjects proceeded to the Arithmogons problem immediately after the proctor stopped work on the first problem in the booklet. Subjects filled out the questionnaire during the last five minutes of the class period and teachers filled out their questionnaire while the problems were being administered. Total time elapsed was forty- five minutes for 8th grade and fifty-five for U.S. 11th grade, the usual length of class periods in the schools.

The results for the U.S. sample are reported here for one problem and a variation, the Arithmogons problem (McIntosh and Quadling, 1975). It was administered at the 8th and 11th grade levels in both countries. The results for the Japanese sample are reported in Senuma and Nohda (1989) and Miwa (1991). Some are reported for the purpose of some contrasting comparisons in a later section. The descriptive nature of the study provides information which helps to document results pertaining to performance of U.S. students on certain problem solving behaviors as well as to provide a contrast between U.S. and Japanese students on these behaviors (cf., Bradburn and Gilford, 1990).

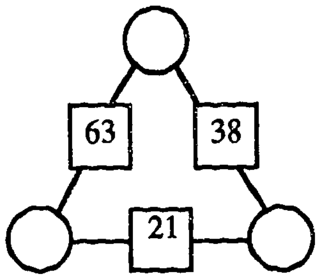
THE PROBLEMS

The Arithmogon problem (Problem I) and its variation (Problem II) are shown below. Subjects were provided with six (6) different work spaces following the problem statement in which they could write their different ways of (approaches to) solving problem I. For problem II, subjects were given space as indicated in the figure which follows.

Problem I

Given a three-sided arithmogon as in the figure below. We put three numbers in the three □ -- the number in each □ must equal the sum of the numbers in the two ○ on either side.

Find the numbers for ○ at each corner. The numbers in ○ may be negative numbers.



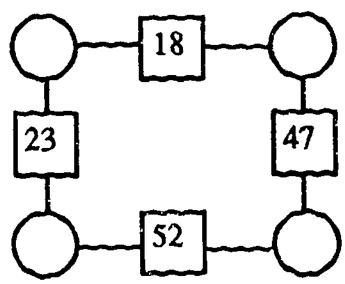
Do not erase anything you write down, just draw a line through anything you feel is in error.

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN.

Problem II

Now change to a square (four-sided) arithmogon as in the figure below. The number in each □ must equal the sum of the numbers in the two ○ on either side.

Try to find the numbers for ○ at each corner.

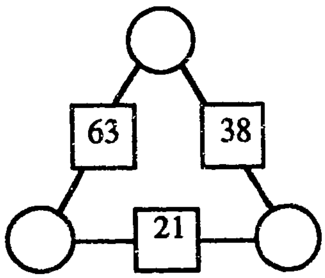


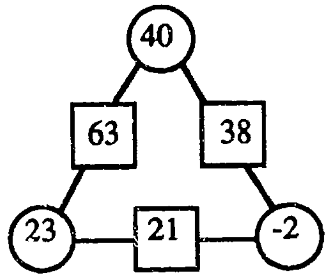
If you need more space, write on the back of this page.

Ways (Approaches) of Solving the Problems

It was anticipated by Japanese and U.S. researchers that students would exhibit from none to all of the following ways of solving the Problem I and possibly others:

Problem I: Unique Solution





(1) Random Trial and Error
Here subjects might guess a number for the top ◯ and, by subtraction and moving counterclockwise, see if they would end up with the same number in the top ◯ . Alternately, subjects might (a) work clockwise or (b) work both clockwise and counterclockwise starting with a guess in the top ◯ , to see if they end up in both directions with 21 at the bottom.

(2) Systematic Trial and Error
Here subjects might reason that the numbers in the top ◯ and lower left ◯ must add to 63. After picking a pair adding to 63, work around counterclockwise or clockwise, using subtraction, to see if they end up with the same number in the top ◯ . If not, pick a different pair and proceed similarly.

(3) One Equation in One Unknown
Let x represent the number in the top ◯ . Then the lower left ◯ is 63 – x and the lower right ◯ is 38 – x. The two must add to 21; so

$$(63 - x) + (38 - x) = 21$$

(4) System of Two Equations in Two Unknowns
Here subjects might let x represent the number in the top ◯ and y the number in the lower right ◯ . Then x + y = 38 and 63 – x = 21 – y; so

$$x + y = 38$$
$$x - y = 42$$

(5) Three Equations in Three Unknowns
Here subjects might let x represent the number in the top ◯ , y the number in the lower right ◯ , and z the number in the lower left ◯ ; so

$$x + y = 38$$
$$x + z = 63$$
$$y + z = 21$$

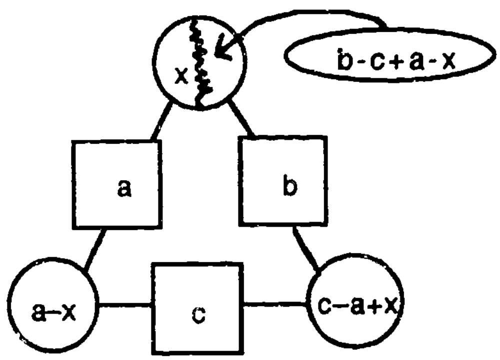
(6) By Adding 63, 38, 21 (Seeing a structure)

$$63 + 38 + 21 = 122$$
$$122 + 2 = 61$$
$$61 - 63 = -2$$
or
$$61 - 38 = 23$$
or
$$61 - 21 = 40$$

(7) Difference of the two smallest □ 's (Seeing a structure) □

a. Find the difference of the numbers in the two smallest □ 's.
b. Subtract the difference from the number in the largest □ .
c. Divide the second difference by 2, which is one of the numbers in the ○ 's.
d. Add this number to the first difference to get the number for the next ○ .
e. Determine the number for the third ○ .

(8) General Solution (Changing perspective and solving a "bigger" problem first)
Let **x** represent the number in the top ○ and let the numbers in the □ 's be represented by **a**, **b**, and **c**. Then, work counterclockwise.

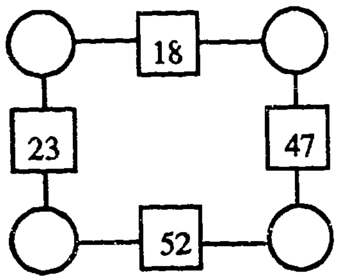


then, $x = b - c + a - x$.

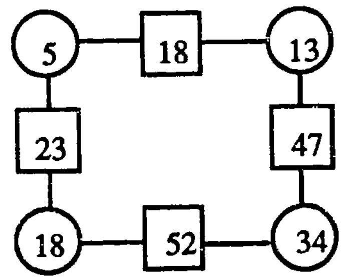
so, $x = \dfrac{a + b - c}{2} = \dfrac{63 + 38 - 21}{2} = 40$.

so, 40, 23, and –2 are the solution.

Problem II:



One non-unique Solution



It was anticipated that students would exhibit one or more of the following approaches to solving problem II.

(1) Trial and Error

Let the top left ○ be 5 (or any integer). Then the lower left ○ is 18; then the lower right ○ is 34; then the upper right ○ is 13; and 5 + 13 = 18.

Note: Will subjects recognize that starting with any number in any ○ will lead to a solution, and that there is more than one (infinitely) many solutions?

(2) Four Equations in Four Unknowns

Let x, y, z, w represent the numbers in the four ○ 's. Then

$$x + y = 23$$
$$y + z = 52$$
$$z + w = 47$$
$$x + w = 18$$

(3) Two Equations in Two Unknowns

Let x represent the number in the upper left ○ and y the number in the lower right ○ .

Then, $18 - x = 47 - y$

$23 - x = 52 - y$

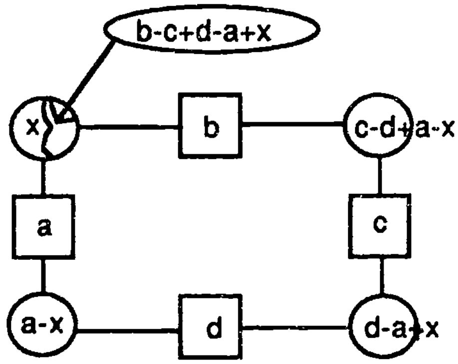
So, $x - y = -29$

$x - y = -29$

Therefore, there are infinitely many solutions.

(4) Addition of Pairs of Numbers in Opposite $\square$'s.

Will subjects see that 23 + 47 = 52 + 18 and, therefore, there are infinitely many solutions, or reason as follows?



So, $x = b - c + d - a + x$

So, $a + c = b + d$ (condition for a solution to exist)

SUBJECTS

There were 368 (178 male and 190 female) eighth-grade students in mathematics classes in schools in the areas around Carbondale (IL), Champaign/Urbana (IL), Pittsburgh (PA), Gainesville (FL), and Athens (GA). There were 246 (124 male and 122 female) eleventh-grade students in the same areas except for Champaign/Urbana (IL). In general, for both grade levels students were attending school in large rural, small urban or large urban school districts. Schools were purposely selected to provide this mix, although the selection of schools and classes within a school was not made in a random manner.

RESULTS FOR THE U.S. SAMPLE

All classes of students for both grade levels were reported by their teachers to be either "regular" or "above average" classes: at eighth grade 29% "regular" and 71% "above average"; at eleventh grade 40% "regular" and 60% "above average." At both grade levels, teachers also reported that their students accepted the problems, liked them, found them challenging (in three classes difficult), in a few cases wanted more time, and represented their best effort (with one class as an exception). Teachers further commented that they liked the problems themselves, thought they were "thinking" problems, and that there is a need for more such problems in the curriculum. In particular, teachers commented that problems with several or many ways to solve them are needed in the curriculum and, further, that this was the first such experience their students had with such problems. Some teachers reported that their students wanted to discuss the problems afterwards. A few teachers reported that their students had seen problems like these before but, when further questioned, they meant problems in which a pattern(s) could be used or that students were periodically assigned non-routine problems in which "process" was emphasized. There is no evidence that subjects had seen these problems before.

Table 1 shows the number of male and female eighth and eleventh grade subjects for each of the five centers. Note that the numbers of male and female students are about the same for both grade levels.

Table 1

Number of Eighth and Eleventh Grade Male and Female Subjects in the Five Centers, and Totals (Percents)

Grade Level / Location	Eighth Grade N = 368		Eleventh Grade N = 246		Total N = 614	
	Male	Female	Male	Female	Male	Female
Center 1	28	23	23	25	51	48
Center 2*	15	20	--	--	15	20
Center 3	22	30	7	26	29	56
Center 4	25	19	26	24	51	43
Center 5	88	98	68	47	156	145
Totals (Percent)	178 (48%)	190 (52%)	124 (50%)	122 (50%)	302 (49%)	312 (51%)

*No data were collected at this Center for grade eleven.

Table 2 shows the distribution of male and female Correct and Incorrect Solutions and No Attempts for Problem I along with percentages at each grade level in the five centers where data were collected. Note that, for eighth-grade subjects, nearly twice as many male than female subjects got Problem I correct though, for eleventh-grade subjects, there is no difference. For both eighth and eleventh-grade subjects, male subjects also got fewer incorrect solutions. Table 3 shows the percents for each center of Correct, Incorrect, and No Attempts to Problem I for eighth and eleventh-grade subjects. Table 4 shows that, at the eighth-grade level, 57 of 368 subjects (15%) got a correct solution to Problem I, 293 (80%) got an incorrect solution, and 18 (5%) made no attempt at the solution. In contrast, for eleventh-grade subjects, 113 of 246 subjects (46%) got a correct solution and significantly fewer eleventh-grade than eighth-grade subjects got an incorrect solution. Similarly, the number of non-attempts decreased from eighth to eleventh grade. In terms of getting a correct solution, eleventh-grade subjects did much better than eighth-grade subjects, as would be expected. Taking both groups together, 170 of 614 subjects (28%) got a correct solution.

Table 2

PROBLEM I

Number of Correct and Incorrect Solutions and No Attempts by Males/Females to Problem I for Eighth and Eleventh Grade Subjects for Each Center, and Totals (Percents)

Grade Level / Location	Eighth Grader N = 368						Eleventh Grade N = 246						Total Eighth and Eleventh Grades N = 614					
	Correct		Incorrect		No Attempt		Correct		Incorrect		No Attempt		Correct		Incorrect		No Attempt	
	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female
Center 1	10	3	18	20	2	0	13	11	10	14	0	0	23	14	26	34	2	0
Center 2	5	2	9	18	1	0	--	--	--	--	--	--	5	2	9	18	1	0
Center 3	4	5	18	21	0	4	5	11	2	12	0	3	9	16	20	33	0	7
Center 4	9	1	14	18	2	0	15	10	11	14	0	0	24	11	25	32	2	0
Center 5	9	9	74	85	5	4	28	20	39	25	1	2	37	29	113	110	6	6
Total	37	20	131	162	10	8	61	52	62	65	1	5	98	72	193	227	11	13
(Percent)	(10%)	(6%)	(35%)	(44%)	(3%)	(2%)	(25%)	(21%)	(25%)	(26%)	(1%)	(2%)	(16%)	(12%)	(31%)	(37%)	(2%)	(2%)

Table 3

PROBLEM I

Percent of Correct and Incorrect Solutions and No Attempts to Problem I for Each Center for Eighth and Eleventh Grade

Grade Level / Location	Eighth Grade			Eleventh Grade		
	Correct	Incorrect	No Attempt	Correct	Incorrect	No Attempt
Center 1	26%	71%	3%	50%	50%	0%
Center 2*	20%	77%	3%	--	--	--
Center 3	17%	75%	8%	48%	42%	10%
Center 4	23%	73%	4%	50%	50%	0%
Center 5	10%	85%	5%	42%	56%	2%

*No data were collected at this Center for grade eleven.

Table 4

PROBLEM I

Total Number and Percent of Correct and Incorrect Solutions and No Attempts to Problem I for Eighth and Eleventh Grade Subjects in All Centers, and Totals

Grade Level / Solution	Eighth Grade N = 368		Eleventh Grade N = 246		Total N = 614	
	Number	(Percent)	Number	(Percent)	Number	(Percent)
Correct	57	(15)	113	(46)	170	(28)
Incorrect	293	(80)	127	(51)	420	(68)
No Attempt	18	(5)	6	(3)	24	(4)
Totals	368	(100)	246	(100)	614	(100)

*No data were collected at this Center for grade eleven.

Tables 5 and 6 provide results for male and female Correct and Incorrect Solutions and Non-Attempts on Problem II for eighth and eleventh-grade subjects at the five centers. Note from Table

5 that the number of non-attempts is much higher at Center 3 for eighth-grade subjects than for any of the others. Perhaps the proctor did not remind subjects to work on Problem II as well as Problem I in the time limit. Note also that the success rate for eighth-grade subjects in Center 2 is considerably higher than for all the others. At Center 3, eleventh-grade subjects have a much higher success rate than for all the others, and also a lower percent of incorrect answers. Table 7 shows the numbers and percents of Correct and Incorrect solutions and No Attempts, in aggregate, for each grade level. Note that 95 of 368 eighth-grade subjects (26%) got a correct solution, and 135 of 246 eleventh-grade subjects (55%) got a correct solution. Again, eleventh-grade subjects were much more successful than eighth-grade subjects and had a lower rate of both incorrect solutions and non-attempts, as expected.

Table 5

PROBLEM II

Percent of Correct and Incorrect Solutions and No Attempts to Problem II for Each Center for Eighth and Eleventh Grades

Grade Level / Location	Eighth Grade			Eleventh Grade		
	Correct	Incorrect	No Attempt	Correct	Incorrect	No Attempt
Center 1	28%	28%	44%	58%	27%	15%
Center 2*	51%	15%	34%	--	--	--
Center 3	21%	16%	63%	73%	6%	21%
Center 4	30%	25%	45%	48%	24%	28%
Center 5	21%	42%	37%	51%	18%	31%

*No data were collected at this Center for grade eleven.

Table 6

PROBLEM II

Number of Correct and Incorrect Solutions and No Attempts by Males/Females to Problem II for Each Center for Eighth and Eleventh Grade Subjects, and Totals (Percents)

Grade Level / Location	Eighth Grade N = 368						Eleventh Grade N = 246						Total Eighth and Eleventh Grades N = 614					
	Correct		Incorrect		No Attempt		Correct		Incorrect		No Attempt		Correct		Incorrect		No Attempt	
	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female	Male	Female
Center 1	8	6	8	6	12	11	16	12	5	8	2	5	24	18	13	14	14	16
Center 2*	7	11	1	4	7	5	--	--	--	--	--	--	7	11	1	4	7	5
Center 3	6	5	2	6	14	19	6	18	0	2	1	6	12	23	2	8	15	25
Center 4	10	3	5	6	10	10	16	8	4	8	6	8	26	11	9	14	16	18
Center 5	23	16	30	48	35	34	31	28	11	9	25	11	54	44	41	57	60	46
Total	54	41	46	70	78	79	69	66	20	27	34	30	123	107	66	97	112	109
(Percent)	(15%)	(11%)	(13%)	(19%)	(21%)	(21%)	(28%)	(27%)	(8%)	(11%)	(14%)	(12%)	(20%)	(17%)	(11%)	(16%)	(18%)	(18%)

*No data were collected at this Center for grade eleven.

Table 7
PROBLEM II

Number and Percent of Correct and Incorrect and No Attempts to Problem II for Eighth and Eleventh Grade Subjects, and Totals

Grade Level / Solution	Eighth Grade N = 368 Number	(Percent)	Eleventh Grade N = 246 Number	(Percent)	Total N = 614 Number	(Percent)
Correct	95	(26)	135	(55)	230	(37)
Incorrect	116	(32)	47	(19)	163	(27)
No Attempt	157	(42)	64	(26)	221	(36)
Totals	368	(100)	246	(100)	614	(100)

Tables 8 (eighth grade) and 9 (eleventh grade) give the distribution of responses for the student questionnaires. In Table 8, we see that eighth-grade subjects show a very strong tendency towards "liking math," feel strongly they are "good at math," and strongly found the problem "interesting." The vast majority found the problem "difficult" and felt it was different from typical textbook problems. More reported that they "liked the problem less" than textbook problems and, about evenly, said they had seen the problem before, which appears inconsistent with their success on the problem. In Table 9, we see similar results for eleventh-grade subjects for "liking math" and "good at math." They more strongly found the problem "interesting" and less "difficult" than eighth-grade subjects, which is not surprising. They similarly strongly reported the problem to be "different from typical textbook problems" and "liked it" more than eighth-grade subjects. They more strongly reported that they had "seen the problem before" than eighth-grade subjects, which again appears inconsistent with their performance. These results are interesting in that subjects at both grade levels report that they "like math" and "are good" at it, the problem "interesting," and had "seen it" before, feelings that seem inconsistent with their performance.

Table 8

EIGHTH GRADE: STUDENT QUESTIONNAIRE

Tabulation of Responses to Items on the Student Questionnaire (N = 368)

Items / Location	Like Math			Good at Math			Problem Interesting			Problem Easy			Same as Textbook Problem			Comparison to Textbook Problem			Seen Problem Before	
	L	N	D	G	N	NG	I	N	NI	E	A	D	S	CS	D	LM	LS	LL	Y	N
Center 1	30	18	3	23	28	0	28	15	8	2	10	39	1	22	28	13	16	22	25	26
Center 2*	20	13	1	8	25	0	25	5	4	0	10	24	4	25	5	14	13	7	27	7
Center 3	21	25	6	13	33	6	32	9	11	1	9	42	0	18	34	19	10	23	26	26
Center 4*	13	22	8	10	30	3	19	11	13	2	7	34	0	22	21	14	7	22	26	17
Center 5*	101	69	15	70	97	18	98	38	49	6	32	147	9	63	113	49	36	100	83	102
Totals	185	147	33	124	213	27	202	78	85	11	68	286	14	150	201	109	82	174	187	178
Percent	(51%)	(40%)	(9%)	(34%)	(59%)	(7%)	(56%)	(21%)	(23%)	(3%)	(19%)	(78%)	(4%)	(41%)	(55%)	(30%)	(22%)	(48%)	(51%)	(49%)

*One questionnaire not filled out

Legend:
- L, N, D: Like Math, Neutral, Dislike Math
- G, N, NG: Good at Math, Neutral, Not Good at Math
- I, N, NI: Interesting, Neutral, Not Interesting
- E, A, D: Easy, Average, Difficult
- S, CS, D: The Same As, Can't Say, Different From
- LM, LS, LL: Like More, Like Same As, Like Less
- Y, N: Yes, No

Table 9

ELEVENTH GRADE: STUDENT QUESTIONNAIRE

Tabulation of Responses to Items on the Student Questionnaire (N = 246)

Items / Location	Like Math			Good at Math			Problem Interesting			Problem Easy			Same as Textbook Problem			Comparison to Textbook Problem			Seen Problem Before	
	L	N	D	G	N	NG	I	N	NI	E	A	D	S	CS	D	LM	LS	LL	Y	N
Center 1	25	20	3	20	25	3	35	8	5	3	25	20	5	15	28	22	11	15	32	16
Center 2	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--
Center 3	16	9	8	8	19	6	22	8	3	6	18	9	2	13	18	15	10	8	26	7
Center 4	28	18	4	27	19	4	34	10	5	6	20	23	4	12	34	23	18	9	34	16
Center 5*	65	42	8	62	41	12	90	18	6	8	51	55	6	35	72	59	30	24	66	47
Totals	134	89	23	117	104	25	181	44	19	23	114	107	17	75	152	119	69	56	158	86
Percent	(54%)	(36%)	(10%)	(48%)	(42%)	(10%)	(74%)	(18%)	(8%)	(9%)	(47%)	(44%)	(7%)	(31%)	(62%)	(49%)	(28%)	(23%)	(65%)	(35%)

*One questionnaire not filled out; no response for several items (where total ≠ 116).

Legend:
- L, N, D: Like Math, Neutral, Dislike Math
- G, N, NG: Good at Math, Neutral, Not Good at Math
- I, N, NI: Interesting, Neutral, Not Interesting
- E, A, D: Easy, Average, Difficult
- S, CS, D: The Same As, Can't Say, Different From
- LM, LS, LL: Like More, Like Same As, Like Less
- Y, N: Yes, No

97

98

Table 10 shows that for Problem I, nearly all eighth and eleventh -grade subjects who got a correct solution used a Trial and Error approach. Only 21 of 246 eleventh-grade subjects (9%) used simultaneous equations with two or three variables and got a correct solution and 8 of 246 (3%) used these approaches but were not able to get a solution. Subjects were asked to "find the answer (solution) in as many different ways as you can" but, as Table 10 shows, the number of subjects at both grade levels who used more than one way (approach) is nearly negligible (3%). A large number of eighth-grade subjects (152/41%) did not understand the problem and, similarly, a significant number of eleventh-grade subjects (51/21%) did not understand the problem - even though the problem was carefully read to them. Even though a systematic Trial and Error approach was used by subjects at both grade levels, only infrequently did subjects choose to start with the number 21 which might provide an answer in a small number of combinations/trials in getting an answer (e.g., 3 and 18, 2 and 19, 1 and 20, 0 and 21, –1 and 22, –2 and 23). There were many eighth-grade subjects who showed evidence in their scripts of difficulty working with negative integers and even among eleventh-grade subjects there was such evidence - even though subjects were given a cue to negative numbers in the problem statement (see Appendix A for examples). We should comment that in the early "try out" of the problem, the numbers in the squares were 14, 12, 18 and a Trial and Error approach was very commonly used which was easy and led to a solution. Accordingly, the problem was revised to its present form with the view that it would be more challenging and that more ways of solving the problem would obtain. Our hopes were not realized.

Table 10

PROBLEM I

Ways of Solving Problem I Leading to Correct/Incorrect Solutions by Eighth and Eleventh Grade Subjects

Correct / Incorrect	Ways of Solving	Eighth Grade N = 368	Eleventh Grade N = 246	Eighth and Eleventh Grades (Total) N = 614
CORRECT	1	2	31	33
	2	41	48	89
	3	0	1	1
	4	0	0	0
	5	0	14	14
	6	0	1	1
	7	0	0	0
	8	0	0 *	0
	9	13**	16**	29**
	2 & 5	0	5	5
	2 & 9	0	1	1
	5 & 6	0	2	2
INCORRECT	1	136***	23	159
	2	6	35	41
	3	1	0	1
	4	0	0	0
	5	5	8	13
	6	0	0	0
	7	6	0	6
	8	0	0	0
	9	0	0	0
	Not Understand	152****	51	203
	No Attempt	6	10	16

* One subject showed some evidence of this approach.

** Subjects got correct solution but showed no work or work not discernible, didn't seem to know they had solved the problem, or used incorrect approach, but got correct solution (i.e., (63 + 38 + 21) + 3 = 40.6, so used 40).

*** 49 subjects showed clear evidence of not being able to work with negative integers.

**** 81 subjects were from one center.

Legend of Ways of Solving:

1: Random trial and error
2: Systematic trial and error
3: Linear equation with one variable
4: Simultaneous equations with two variables
5: Simultaneous equations with three variables
6: Adding 63 + 38 + 21 and dividing by 2
7: Difference of two smallest squares
8: General solution
9: Other way of solving

Table 11 shows that, for Problem II, only 96 of 368 eighth-grade (26%) and 130 of 246 eleventh-grade subjects (53%) got a correct solution, even though starting with any integer would lead to a solution. Many subjects at both grade levels did not try (157/43% of eighth and 71/29% of eleventh) or did not understand the problem (e.g., they added the numbers in the squares to get the numbers in the circles), and very few noticed that there was more than one solution. For both grade levels, only 10 of 614 (2%) noticed that there was more than one solution and only one (1) subject mentioned that all integers lead to a solution. In a few instances a subject showed, for example, three solutions but did not indicate there were infinitely many (see Appendix D). Many subjects spent most or nearly all their time on Problem I even though they were instructed to work on both problems in the fifteen-minute time limit.

An analysis of the success of subjects at both grade levels on <u>both problems</u> was also done. These results are shown in Tables 12 and 13.

Table 11

PROBLEM II

Ways of Solving Problem II Leading to Correct/Incorrect Solutions and Noticing More than One Solution by Eighth and Eleventh Grade Subjects

Correct / Incorrect	Ways of Solving	Eighth Grade N = 368	Eleventh Grade N = 246	Eighth and Eleventh Grades (Total) N = 614
CORRECT	1	95	118	213
	2	1	8	9
	3	0	4 *	4
INCORRECT	1 and Not Understand	115**	34**	149
	2	0	9	9
	3	0	2***	2
No Attempt		157****	71****	228
Noticed More than One Solution		4*****	6	10

* For example, subject averaged 23, 52, 47, 18, and then placed 35 in a circle and solved.

** Almost all did not understand; a few started with a number in a circle, but then made a computational error and got no solution.

*** One subject used an equation with one variable, but didn't solve the problem.

**** The number of No Attempts is probably due to a time factor; i.e., subjects spent all their time on Problem I.

***** Only one subject mentioned that all integers lead to a solution.

Legend of Ways of Solving:
1: Try a number
2: Simultaneous equations with four variables
3: Other ways of solving

Table 12

EIGHTH GRADE

Number and Percent of Eighth Grade Subjects Who Got Problems I and II Correct/Incorrect and Number and Percent Who Got No Solution or Made No Attempt on Problem II

Problem I Correct/Incorrect / Problem II Correct/Incorrect	Male N = 177 Number(Percent)	Female N = 191 Number(Percent)	Total N = 368 Number(Percent)
Problem I Correct/Problem II Correct	23 (13)	11 (6)	34 (9)
Problem I Correct/Problem II Incorrect	14 (8)	10 (5)	24 (7)
Problem I Incorrect/Problem II Incorrect	110* (62)	138* (72)	248* (67)
Problem I Incorrect/Problem II Correct	30 (17)	32 (17)	62 (17)
No Solution or No Attempt on Problem II**	79 (45)	78 (41)	157 (43)

* About 55% from Center 5

** A majority of subjects attempted Problem I.

Table 13

ELEVENTH GRADE

Number and Percent of Eleventh Grade Subjects Who Got Problems I and II Correct/Incorrect and Number and Percent Who Got No Solution or Made No Attempt on Problem II*

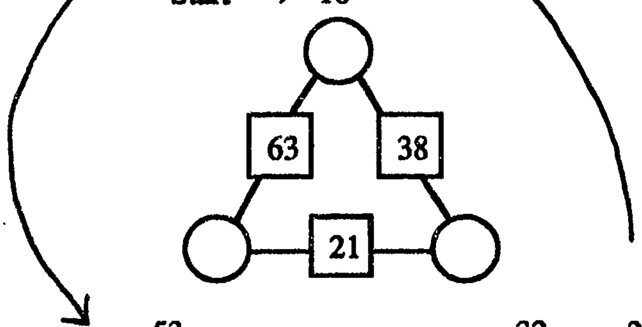
Problem I Correct/Incorrect / Problem II Correct/Incorrect	Male N = 122 Number(Percent)	Female N = 124 Number(Percent)	Total N = 246 Number(Percent)
Problem I Correct/Problem II Correct	50 (41)	34 (27)	84 (34)
Problem I Correct/Problem II Incorrect	14 (12)	15 (12)	29 (12)
Problem I Incorrect/Problem II Incorrect	38 (31)	44 (36)	82 (33)
Problem I Incorrect/Problem II Correct	20 (16)	31 (25)	51 (21)
No Solution or No Attempt on Problem II	32 (26)	30 (24)	62 (25)

* No data from Center 2

Table 12 shows that, for eighth-grade subjects, twice as many males (13%) as females (6%) got correct answers to both problems though the percentages are not large, as mentioned earlier. In about the same percentages, males and females got both problems incorrect and got no solution to or did not attempt Problem II. In about the same percentages, for both males and females, eighth-grade subjects got one problem correct and the other incorrect. Similar results were found for male and female subjects at the eleventh grade. We might wonder whether there would be a higher success rate on both problems if the order were reversed.

DISCUSSION OF RESULTS FOR THE U.S. SAMPLE

There are some comments that should be highlighted regarding the results for this sample of U.S. subjects. As mentioned, the sample consisted of at least two classes of eighth and eleventh-grade students in five different centers in the eastern half of the U.S. For Problem I, the success rate at the eighth-grade level is low (15%) and fairly consistent across the five centers. There is a very large rate of incorrect answers. Further, there is ample evidence in student scripts that subjects had difficulty understanding what they were to do (see Appendix A) and, when they did, Trial and Error was seemingly the only way students could approach the problem. Further, even when subjects made progress towards a solution using such an approach (i.e., two numbers in the circles correct), they could not make an adjustment and reason how starting with another number or pair of numbers could carry them to (or closer to) a solution. For example, if students were to pick a number for, say, the top circle and reason counterclockwise (or clockwise) as follows, it would lead to a solution.



40
30
20
Start → 10

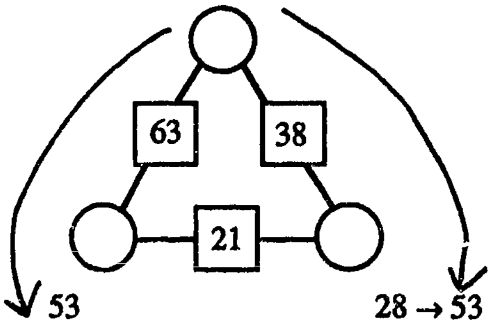
53   −32 → 21 − 53 = −32; 10 + (−32) = −22 ≠ 38; start larger, say 20.

43   −22 → 21 − 43 = −22; 20 + (−22) = −2 ≠ 38; start larger, say 30.

33   −12 → 21 − 33 = −12; 30 +(−12) = 18 ≠ 38; start larger, say 40.

23   −2 → 21 − 23 = −2; 40 + (−2) = 38 Done!

Nor could subjects reason both counterclockwise and clockwise to get a solution:



40
Start → 10

53   28 → 53 + 28 = 81 ≠ 21, but, 81 − 21 = 60; 60 ÷ 2 = 30; so try 10 + 30 = 40

23   −2 → 21 − 23 = −2 Done!

Similarly, eleventh-grade subjects had only moderate success in solving Problem I. Here, too, there was little evidence in subjects' scripts of reasoning as illustrated above to get a solution. Further, there was scarcely any evidence of reasoning, as shown earlier in (6), (7), and (8) of ways (approaches) to solving the problem (see Appendix E for example, nearly, of (8)), for either eighth or eleventh-grade subjects of seeing a structure in the problem. With respect to approach (6), if subjects could understand from the problem statement that the number in each square is the sum of the numbers in the circles on the two "ends," it is not a large conceptual leap to infer that, therefore, the sum of the numbers in the three squares is twice the sum of the numbers in the three circles on the "ends." Yet there is also precious little evidence of this way of thinking in subjects' scripts. More commonly, subjects found the average of the numbers in the three squares, which is incorrect (but works in this problem).

Now we turn to application of algebraic techniques in solving Problem I. It is well known that, in general, U.S. students do not begin study of algebra until their ninth year. Thus, perhaps we should not expect eighth-grade students to use algebra in solving the problem. Even so, however, it seems almost natural to think in terms of letting the three numbers in the circles be represented by, say, a, b, and c; then $a + b = 63$, $b + c = 21$, and $a + c = 38$. Yet, there was scarcely any evidence of such thinking in eighth-grade subjects' scripts, say nothing of solving the system of equations. Indeed, there was practically no algebraic thinking exhibited at all in students' scripts.

Eleventh-grade subjects, on the other hand, might reasonably be expected to use an algebraic approach though students were just a few weeks into their Algebra II course. While more eleventh-grade than eighth-grade subjects used a simultaneous equations approach, the occurrence was marked more by an absence than a presence. For both grade levels, as mentioned earlier, trial and error was the "approach of choice" or, we might comment, very nearly the only tool subjects seemingly had in their problem solving repertoire to use. Also, as mentioned earlier, there were too many subjects at both grade levels that indicated, by actually writing it or by the work shown, that they did not understand what they were to do. So, while it was expected when data were collected that subjects' scripts might be rich with data about subjects' thinking for Problem I, in fact there was little to analyze by way of different approaches or ways of thinking about the problem.

In the earlier tryout of problems and booklet format (Becker, 1989), the numbers in the squares for Problem I were 14, 12, and 18. In this case, as mentioned earlier, the Arithmogons problem was too easily solved by straightforward trial and error and there appeared to be little motivation to use any other approach. Accordingly, the problem was revised to use 63, 38, and 21 which, of course, introduced negative numbers into the picture. Subject scripts show clearly at the eighth-grade level, and to a lesser but still significant degree at the eleventh-grade level, that subjects had difficulty with computation involving negative numbers (see Appendix B). Though this finding is not completely surprising, it is, nonetheless, discouraging. It simply is not unreasonable to expect eighth and eleventh-grade students to be able to deal with operations on integers in a competent manner. Following Flanders (1987) findings, not only is there a great deal of repetition of content in the elementary/middle school mathematics curriculum, with much emphasis on computation, but the teaching and learning is not effective.

The results show clearly that subjects in this sample not only do not have algebraic techniques to apply in solving Problem I, but subjects' work also demonstrates that they could neither see a structure in the problem nor reason in any significant way, even when using a trial and error approach as a "starter" (for one exception, see Appendix E). We need to more carefully consider Usiskin's (1987) admonition to teach algebra in eighth grade and, at the same time, provide more learning experiences which help students to learn to think (i.e., appeal to their natural ways of thinking) and use algebraic techniques in a variety of problem situations.

A few further comments seem in order. When subjects used a trial and error approach, it was rare to see "nice numbers" such as 10 or 20 used as "starters" or as first "guesses." Does this indicate something about subjects' number sense (i.e., lack of it)? Also, student scripts were rarely written in an orderly manner, say nothing of neatness, which perhaps indicates something about expectations of students in classroom teaching. Some subject scripts were also incomprehensible when there was "a lot" of scratch work shown. While this may be regarded as a shortcoming of the "paper and pencil" approach to collecting data, perhaps it also indicates something about the work and thinking habits expected of students. While the results for eighth and eleventh-grade subjects are discouraging, there were, nevertheless some "points of light" in subject scripts. Appendix C shows some examples of these.

The results for Problem II present a somewhat different picture. Here a little more than one-fourth (26%) of eighth-grade and more than half (55%) of eleventh-grade subjects got at least one correct solution. The latter result is, perhaps, somewhat encouraging. But Problem II has the property that trying any number (barring computational errors) will lead to a solution. Combine this with the fact that subjects commonly used a trial and error approach and, perhaps, we have the explanation. But when subjects got one correct solution, they rarely tried another number and, so, did not get the solution (i.e., that there are infinitely many solutions). In one case, a subject got three correct solutions, but made no mention of more (see Appendix D).

Many subjects, at both grade levels, did not understand or attempt the problem. Perhaps they worked most of the fifteen minutes on Problem I and were not reminded by proctors to be sure to try Problem II also. We note that, for Problem I, a significant number of subjects who did not seem to understand were from one center (53%). But we need to ask, with respect to understanding the problems, whether it is asking too much of students? Can't we expect students to be able to read such a problem and at least begin and show some level of understanding? We think we can - it is not asking too much.

A few other overall observations should be made by way of summary. Teachers reported that students in their classes (with one class as an exception) gave their best effort - they took their task seriously. Moreover, the problems were read to students before they began work and, in the case of Problem I, they were given a cue that the numbers in the circles could be negative. Even so, the number of different approaches used by subjects, at each grade level in aggregate, was negligible. Moreover, given that only one approach was used, subjects far too commonly used only a trial and error approach. There was scarcely any mathematical sophistication reflected in the technique or approach used and there is virtually no evidence that students could "shift gears" or "switch" thinking or strategy and try another (other) approach(es). On the contrary, for the vast majority of subjects who tried the problem and whether successful or not, scripts reflected a stubborn determination to grind out a solution for Problem I by either random or systematic trial and error.

Finally, results of the questionnaire indicate that, for this sample, both eighth and eleventh-grade subjects "like" math (51%/54%), feel they are "good at" math (34%/48%), found problems "interesting" (56%/74%), felt the problem was "easy" or "average" (22%/56%) and had seen the problem before (51%/65%). Yet these results seem inconsistent with their performance. Perhaps we need to ask what is it that they "like" and feel "good at" (is it a conception of mathematics as computation and algebraic manipulation only, to which they have been exposed so much)? Further, what do subjects mean when they say the problem is "easy" or "average" and have "seen the problem before"? With respect to the latter, their teachers state clearly that the students have not seen these problems before in their classes. Perhaps the problem was seen in a different class or even outside of school. We cannot say for sure, but these are questions that could be pursued.

RESULTS FOR THE JAPANESE SAMPLE*

In Japan, at the eighth-grade level, there were 189 subjects (96 male (51%) and 93 female (49%)). There were 234 subjects (135 (58%) male and 99 (42%) female) at the eleventh-grade level. Subjects were from public lower secondary and upper secondary schools, except for 39 in one class of a school attached to a National University. In particular, subjects were not from private schools or schools attached to National Universities, except for the one class (Senuma and Nohda, 1989).

Table 14 shows overall results for Japanese eighth and eleventh-grade subjects on Problems I and II.

*Data in this section are taken from Senuma, 1991, pp. 97-113.

Table 14

JAPAN

PROBLEMS I AND II

Percent of Japanese Eighth and Eleventh Grade Subjects Who Got Correct, Incorrect, or No Answer for Problems I and II**

Grade / Level / Answer	Problem I		Problem II	
	Eighth Grade	Eleventh Grade	Eighth Grade	Eleventh Grade
Correct	39%	90%	1%*	1%*
Got One Answer	--	--	38%	24%
Incorrect	52%	8%	21%	55%
No Answer	9%	2%	40%	20%

* Correct answer means subjects indicated there were more than one or infinitely many solutions on Problem II.

** Taken from Senuma, 1991, p. 100 [Tables 2 and 3]

In Table 14 above we see that 39% and 90% of Japanese eighth and eleventh-grade subjects, respectively, got the correct answer for Problem I. Only one percent of eighth and eleventh-grade subjects got the correct answer for Problem II - here subjects indicated that there was more than one answer or infinitely many; however, 38% and 24% of eighth and eleventh-grade subjects, respectively, got exactly one answer (i.e., did not indicate there was more than one or infinitely many answers). From eighth to eleventh grade there is a dramatic increase in percent of subjects getting a correct answer for Problem I, a similarly dramatic decrease in percent who got an incorrect answer, and a decrease in percent of No Answers. It is surprising that the percent of incorrect answers increases significantly from eighth to eleventh grade for Problem II, which will be discussed later.

Table 15

JAPAN

Percent of Japanese Eighth and Eleventh Grade Subjects Who Got Problems I and II Correct/One Answer/Incorrect or No Answer*

Problem I / Problem II**	Eighth Grade	Eleventh Grade
Correct / Correct	1%	1%
Correct / One Answer	24%	21%
Correct / Incorrect	6%	53%
Incorrect / Correct	1%	0%
Incorrect / One Answer	12%	2%
Incorrect / Incorrect	14%	2%
No Answer for either Problem I or Problem II	42%	21%

* Taken from Senuma, 1991, p. 101 [Table 4]
** Correct answer means subjects indicated there was more than one or infinitely many solutions.

In Table 15 above we see that 1% of both eighth and eleventh-grade subjects got both problems correct; however, 24% and 21%, respectively, got a correct answer for Problem I and one answer for Problem II. Also, 6% and 53% of eighth and eleventh-grade subjects, respectively, got Problem I correct and Problem II incorrect. It appears that eleventh-grade subjects spent most of their time on Problem I. The percent of no answer for eleventh-grade subjects is half that of eighth-grade subjects, which is expected.

Table 16

JAPAN

PROBLEM I

Ways or Approaches Used by Japanese Eighth and Eleventh Grade Subjects in Solving Problem I, in Percents*

Way (Approach) of Solving / Grade Level	Eighth Grade	Eleventh Grade
1	19%	82%
2	5%	1%
3	1%	3%
4	2%	1%
5	4%	2%
6	40%	6%
7	21%	3%
No Answer	9%	3%

Legend of Ways of Solving:
1 Simultaneous linear equations with three variables
2 Simultaneous linear equations with two variables
3 Linear equation with one variable
4 Adding 63 + 38 + 21 and dividing by 2
5 Systematic substitution (trial and error)
6 Random Trial and Error
7 Other way of solving

*Taken from Senuma, 1991, p. 105 [Table 5]

Table 16 above shows the different ways eighth and eleventh-grade subjects solved Problem I. It is noteworthy that 19% of eighth-grade subjects used three simultaneous equations in three variables to solve the problem, and it is remarkable that 82% of eleventh-grade subjects used the same approach. We note also that the dramatic decrease in number of subjects who used a trial and error approach in grade eight (44%) to grade 11 (8%). Algebra is taught in grades 7 and 8 in Japanese schools and eleventh-year students have had considerable exposure to algebraic methods and would appear to have learned it well.

Table 17

JAPAN

PROBLEM II

Ways or Approaches Used by Japanese Eighth and Eleventh Grade Subjects in Solving Problem II, in Percents*

Way (Approach) of Solving	Eighth Grade	Eleventh Grade
1	9%	64%
2	42%	12%
3	10%	5%
No Answer	40%	20%

Legend of Ways of Solving: 1 Simultaneous linear equations with four variables
2 Random Trial and Error
3 Other way of solving

*Taken from Senuma, 1991, p. 106 [Table 6]

Table 17 shows that, for Problem II, Japanese subjects at both grade levels used simultaneous equations with four variables, with 64% of eleventh-grade subjects using the approach. Also, Japanese subjects used a trial and error approach but the number drops dramatically from eighth to eleventh grade. It is probable that many subjects spent a great deal of time on Problem I and did not try Problem II, which is discussed later. The vast majority of eighth and eleventh-grade subjects failed to give evidence that they knew there was more than one or infinitely many solutions.

Table 18

JAPAN

PROBLEM I

Percent of Japanese Eighth and Eleventh Grade Subjects Who Used Each Way (Approach) Correctly in Solving Problem I*

Way (Approach) of Solving	Eighth Grade	Eleventh Grade
1	54%	94%
2	47%	100%
3	50%	100%
4	100%	100%
5	79%	100%
6	52%	89%
7	6%	62%

Legend of Ways of Solving:
1 Simultaneous linear equations with three variables
2 Simultaneous linear equations with two variables
3 Linear equation with one variable
4 Adding 63 + 38 + 21 and dividing by 2
5 Systematic substitution (trial and error)
6 Random Trial and Error
7 Other way of solving

*Taken from Senuma, 1991, p. 107 [Table 7]

Table 18 shows that Japanese subjects at the eighth-grade level were able to get a correct solution more than half (with one exception) of the time no matter what approach they used. There was a very high success rate for eleventh-grade subjects no matter what approach was used. The results are different for Problem II, however, as seen in Table 19.

Table 19

JAPAN

PROBLEM II

Percent of Japanese Eighth and Eleventh Grade Subjects Who Used Each Way (Approach) Correctly in Solving Problem II*

Way (Approach) of Solving	Eighth Grade	Eleventh Grade
1	19%	21%
2	87%	89%
3	6%	28%

Legend of Ways of Solving:
1 Simultaneous linear equations with four variables
2 Random Trial and Error
3 Other way of solving

*Taken from Senuma, 1991, p. 108 [Table 8]

Japanese subjects who used trial and error were quite successful at both grade levels, but not nearly as successful when simultaneous equations with four variables were used.

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Table 20

JAPAN

EIGHTH AND ELEVENTH GRADES: STUDENT QUESTIONNAIRE

Percent Tabulation of Japanese Eighth and Eleventh Grade Subjects' Responses to Items on the Student Questionnaire*

Questions / Grade	Like Math			Good at Math			Problem Interesting			Problem Easy			Same as Textbook Problem			Comparison to Textbook Problem		
	L	N	D	G	N	NG	I	N	NI	E	A	D	S	CS	D	LM	LS	LL
Eighth Grade	24%	49%	27%	11%	46%	43%	19%	38%	43%	2%	19%	79%	12%	40%	48%	22%	34%	44%
Eleventh Grade	18%	53%	29%	7%	42%	51%	29%	39%	32%	13%	48%	39%	14%	42%	44%	30%	52%	18%

Legend:
- L, N, D: Like Math, Neutral, Dislike Math
- G, N, NG: Good at Math, Neutral, Not Good at Math
- I, N, NI: Interesting, Neutral, Not Interesting
- E, A, D: Easy, Average, Difficult
- S, CS, D: The Same As, Can't Say, Different From
- LM, LS, LL: Like More, Like Same As, Like Less

*Taken from Senuma, 1991, p. 111 [Table 12]

Table 20 gives the results for Japanese eighth and eleventh-grade subjects on the questionnaire. Overall, the results indicate that for eighth-grade subjects:

1. slightly more subjects dislike math than like it (27%/24%), and nearly half have neutral feelings (49%);
2. nearly four time as many subjects feel they are not good at math than good at math (43%/11%), and nearly half have neutral feelings (46%);
3. just under three times as many subjects found the problems not interesting than interesting (43%/19%), and 38% have neutral feelings;
4. 79% of subjects found the problems difficult compared to 2% who felt they were easy, and 19% thought the problems of average difficulty;
5. four times as many subjects thought the problems different from textbook problems than the same (48%/12%), with 40% who can't say;
6. twice as many subjects like the problems less than textbook problems (44%/22%), and 34% like them the same

For eleventh grade subjects:

1. nearly twice as many subjects dislike math than like it (29%/18%), while 53% have neutral feelings;
2. more than seven times as many subjects feel they are not good at math than good at it (51%/7%), and 42% have neutral feelings;
3. slightly more subjects found the problems not interesting than interesting (32%/29%), and 39% have neutral feelings;
4. three times as many subjects find the problems difficult than easy (39%/13%), and 48% feel they are average;
5. more than three times as many subjects feel the problems are different from textbook problems than the same (44%/14%), and 42% can't say;
6. more subjects like the problems more than textbooks problems than less than (30%/18%), and 52% like them the same.

Overall, subjects at both grade levels have a tendency to dislike math, strongly feel they are not good at math, find the problems not interesting, find the problems difficult, and strongly find the problems different from textbook problems. In comparison to textbook problems , eighth-grade subjects strongly tend to like the problems less, and eleventh-grade more. There is consistently (nearly half) neutral feelings for each question.

CONTRASTING THE RESULTS FOR THE U.S. AND JAPANESE SAMPLES

Tables 14-20 show results for the Japanese sample of eighth and eleventh-grade subjects. For Problem I, 39% of Japanese eighth-grade subject got the correct answer, in contrast to 15%

for U.S. subjects; and 90% of Japanese eleventh-grade subjects got Problem I correct, compared to 46% of U.S. subjects. Thus, more than twice as many Japanese eighth-grade subjects than U.S. subjects, and nearly twice as many Japanese eleventh-grade subjects than U.S. subjects got the correct answer to Problem I. The difference is quite large.

For Problem II, 38% of Japanese eighth-grade subjects compared to 26% of U.S. got one correct answer. One U.S. eighth-grade subject indicated there were infinitely many solutions compared to 1% of Japanese subjects. But, the results for eleventh-grade subjects are different: here, 24% of Japanese subjects got one correct answer compared to 55% of U.S. subjects, and 1% of Japanese eleventh-grade subjects indicated that there were infinitely many solutions compared to 0% for the U.S. A probable explanation for the reverse results on Problem II for eleventh-grade subjects is that since 15 minutes were allowed for subjects to do both problems and Japanese subjects far more commonly used simultaneous equations for the first problem (83%-Japan (Table 16) and 9% - U.S. (Table 10)), they therefore quite naturally used simultaneous variables for the second problem (which are not linearly independent and therefore do not have a unique solution) and ran short of time. Since 64% of Japanese subjects used the same approach for Problem II as Problem I, this time with four variables, in contrast to U.S. subjects who commonly used trial and error (and it is an approach that would work starting with any integer), Japanese subjects had a lower success rate than their U.S. counterparts.

If the above contrasting results are interesting and show big differences in overall performance between U.S. and Japanese subjects, then ways (approaches) of solving Problems I and II provide even more interesting and contrasting results. For Problem I, Table 16 shows that Japanese eighth and eleventh-grade subjects, in aggregate for each level, used all the approaches listed compared to U.S. subjects who very predominantly used only trial and error and, in particular, did not use some of the approaches at all. Though Japanese eighth-grade subjects use trial and error fairly commonly too (44%), they also use simultaneous equations in two and three variables (24%), in contrast to U.S. subjects (1%/all incorrect). The results for eleventh-grade subjects are even more striking, in contrast: 83% of Japanese subjects use simultaneous equations compared to 12% of U.S. subjects (8 of 28 U.S. subjects were incorrect (28%)); far fewer Japanese subjects use trial and error (8%) than U.S. subjects (58%) - 58 of 143 U.S. subjects were incorrect (41%). For the Japanese sample, eighth and eleventh-grade subjects begin study of algebra in grades seven and eight and have algebraic techniques to work with in contrast to U.S. subjects who commonly begin study in algebra in grade 9. Moreover, Table 18 shows that Japanese eighth and eleventh-grade subjects, in aggregate at each grade level, in contrast to U.S. subjects, exhibit a propensity to use all the various approaches to Problem I accurately, whether the approach(es) involve reasoning, computation (including negative numbers), or algebraic techniques. For this sample, Japanese subjects would appear to have acquired algebraic knowledge and an ability to apply it very effectively. Table 10 also shows that a large number of U.S. eighth-grade (158/43%) and eleventh-grade (61/25%), in contrast to Japanese subjects, did not understand what to do. U.S. subjects far more than Japanese also showed difficulty in working with negative numbers, though this difference was more prominent at the eighth-grade level. Overall, the evidence would seem to support Usiskin's (1987) and other mathematics educators' recommendation to teach algebra in eighth grade. Japanese mathematics education exhibits clearly that students can learn algebra, beginning in grade 7 - an existence proof!

Similar contrasting results emerge for Problem II. Tables 11 and 17 show that U.S. subjects (57%) use "Try a Number" very commonly along with Japanese subjects (42%) at the eighth-grade level; but, at the eleventh grade, Japanese subjects far more commonly (64%) than U.S. subjects (7% - used correctly or incorrectly) use simultaneous equations. Further, eleventh-grade Japanese subjects far less frequently than eighth-grade subjects use trial and error. While only 21% of Japanese eleventh-grade subjects used simultaneous equations correctly, this may be due to a time factor discussed above; i.e., given more time they may have been able to work on the system of four equations in four variables and see that, perhaps, there are infinitely many solutions.

Tables 12, 13, and 15, which show results for both Problems I and II, provide still further contrasts. Here 9% of U.S. eighth-grade subjects compared to 25% of Japanese subjects got both problems correct (i.e., one correct answer or indicated infinitely many solutions for Problem II). About the same percentages got Problem I correct and Problem II incorrect (7% - U.S., 6% - Japan) and Problem I incorrect and Problem II correct (17% - U.S. and 13% - Japan). But far more U.S. subjects (67%) got both problems incorrect than Japanese (14%). These results seem consistent with other overall results. For eleventh-grade subjects in both samples, more U.S. subjects (34%) than Japanese (22%) got both problems correct. Fewer U.S. subjects (12%) than Japanese (53%) got Problem I correct and Problem II incorrect, and more U.S. subjects (21%) than Japanese (2%) got Problem I incorrect and Problem II correct. Also, more U.S. subjects (33%) than Japanese (2%) got both problems incorrect. These results reflect earlier discussion of the reversal of results for the two samples from Problem I to Problem II, but are otherwise consistent with earlier results for the two samples. Moreover, for both grade levels, the number of No Attempts for Problem II are about the same (Tables 5 and 17): eighth grade: 40% - Japan and 42% - U.S., and eleventh grade: 20% - Japan and 26% - U.S.

There is a uniformity in performance among male/female subjects in this Japanese sample. In contrast, the results vary and differ among males/females in this U.S. sample, though not consistently. Perhaps this reflects clear expectations of performance among both male and female students in Japanese education (cf., Leestma et al, 1987).

Tables 8, 9, and 20 give the results for the questionnaires for the two samples. Taken together, they show that at the eighth-grade level:

1. (a) Far more U.S. subjects than Japanese express feelings of "liking math" (51% - U.S., 24% - Japan)
   (b) Far more Japanese subjects than U.S. express feelings of "disliking math" (9% - U.S., 27% - Japan)
   (c) More Japanese than U.S. subjects have "neutral" feelings (40% - U.S., 49% - Japan)
2. (a) Far more U.S. subjects than Japanese express feelings of being "good at math" (34% - U.S., 11% - Japan)
   (b) Far more Japanese subjects than U.S. express feelings of being "not good at math" (7% - U.S., 43% - Japan)
   (c) More U.S. than Japanese subjects have "neutral" feelings (59% - U.S., 46% - Japan)
3. (a) Far more U.S. subjects than Japanese found the problems "interesting" (56% - U.S., 19% - Japan)
   (b) Far more Japanese subjects than U.S. found the problems "not interesting" (23% - U.S., 43% - Japan)
   (c) More Japanese than U.S. subjects have "neutral" feelings (21% - U.S., 38% - Japan)
4. U.S. and Japanese subjects felt the problems were "easy," "average," and "difficult" in nearly the same percentages, with 78% (U.S.) and 79% (Japan) feeling they were "difficult."
5. U.S. and Japanese subjects felt the problems were "the same as," "can't say," and "different from" textbook problems in about the same percentages, with 55% (U.S.) and 48% (Japan) feeling they were "different from" textbook problems.
6. More U.S. subjects (30%) than Japanese subjects (22%) "liked the problem more" than textbook problems, fewer U.S. subjects (22%) than Japanese (34%) "liked the problems the same as" textbook problems, and slightly more U.S. (48%) than Japanese (44%) "liked the problems less."

At the eleventh grade level:

1. (a) Far more U.S. subjects than Japanese express feelings of "liking math" (54% - U.S., 18% - Japan)
   (b) Far more Japanese subjects than U.S. express feelings of "disliking math" (10% - U.S., 29% - Japan)
   (c) Far more Japanese subjects than U.S. have "neutral" feelings (36% - U.S., 53% - Japan)
2. (a) Far more U.S. subjects than Japanese feel they are "good at math"

(48% - U.S., 7% - Japan)

(b) Far more Japanese subjects than U.S. feel they are "not good" at math (10% - U.S., 51% - Japan)

(c) The same percent (42%) of U.S. and Japanese subjects have "neutral" feelings

3. (a) Far more U.S. than Japanese subjects thought the problems were "interesting" (74% - U.S., 29% - Japan)

   (b) Far more Japanese than U.S. subjects thought the problems were "not interesting" (8% - U.S., 32% - Japan)

   (c) Far more Japanese than U.S. subjects expressed "neutral" feelings (18% - U.S., 39% - Japan)

4. Slightly more Japanese than U.S. subjects though the problems were "easy," about the same percentages "average," and slightly more U.S. than Japanese thought the problems were "difficult" (44% - U.S., 39% - Japan).

5. Japanese more than U.S. subjects felt the problems were "the same as" and "can't say" in comparison to textbook problems, but far more U.S. subjects felt they were "different from" textbook problems (62% - U.S., 44% - Japan).

6. Far more U.S. than Japanese subjects "liked the problems more" than textbook problems, far fewer "liked them the same," and roughly the same percentages "liked them less."

The results above are consistent with findings of other researchers (e.g., McKnight, 1987; Becker et al., 1988; and Becker, 1992) in which Japanese students more than U.S. dislike math and feel they are not good at it; however, they perform better. Though their results are for first and fifth grade students, Stevenson, Lee and Stigler (1986) and Stigler, Lee, and Stevenson (1990) report that the status of Japanese students achievement remains high and relatively constant across grade levels and the relative status of U.S. students shows a striking decline. (p. 13)

There are several considerations deriving from these analyses that mathematics educators in both countries must address. Looking at the U.S. results in their own right, we need to be concerned about the overall performance of students at both grade levels on these problems. For the U.S. sample, students far too frequently show little ability to understand the problems, see any kind of structure in them, and reason about them. We should be able to reasonably expect that students be capable of reading and understanding the problems as well as learning more algebraic techniques earlier in the curriculum that can be applied in their solutions. Also students at these grade levels should not, in any marked degree, be having difficulty operating on negative numbers - this is simply unacceptable.

But there is more we can say. Subjects in the U.S. sample also show remarkably little mathematical sophistication in their solutions of the problems (i.e., use of mathematical expressions, equations, simultaneous equations and seeing some structure in the problems). As indicated earlier, however, trial and error predominates and students show little "fluency" in dealing with the problems, i.e., there is precious little evidence that students can think about the problems in different ways. As is well known, U.S. students believe that there is one and only one solution and way to get the solution to a problem, and we see evidence of this in these results. We need to provide students with more experiences so they can see otherwise. Further, students have a tendency to see math as easy and have a feeling that they are good at math, but this is entirely inconsistent with their performance. Where do these attitudes come from, and how are they formed? These questions need to be addressed. Finally, we need to address the problem of student work habits - there needs to be more order and neatness in their work. This is something that can and should be addressed by teachers, beginning in the early elementary grades.

From the Japanese perspective, for this Japanese sample, it is seen that students have rather marked knowledge of and more sophisticated algebraic techniques which they bring to problem solving situations. At the same time, they seem unable, at the eleventh-grade level, to switch from a sophisticated approach to a more naive one in transition from Problem I to Problem II (i.e., more flexibility was needed to cope with the situation). This is a source of concern to our Japanese counterparts. Further, Japanese students have a strong tendency to dislike math and express feelings of not being good in math. Exactly why this is the case is not known, but it is something Japanese colleagues are now addressing as a serious problem.

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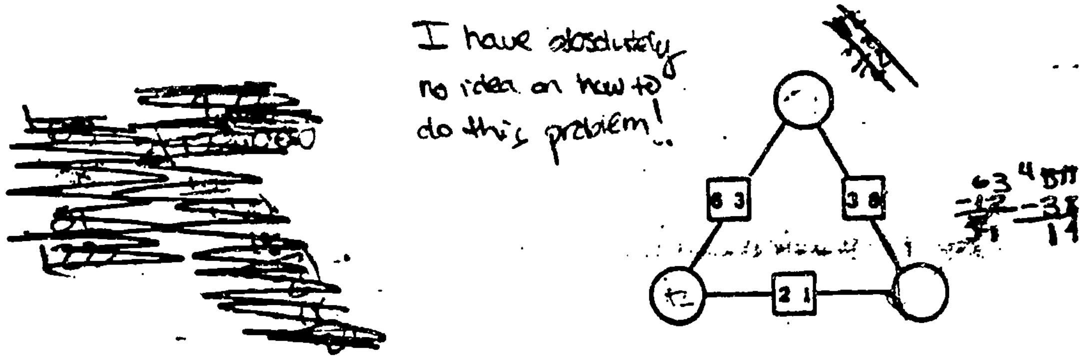
APPENDIX A

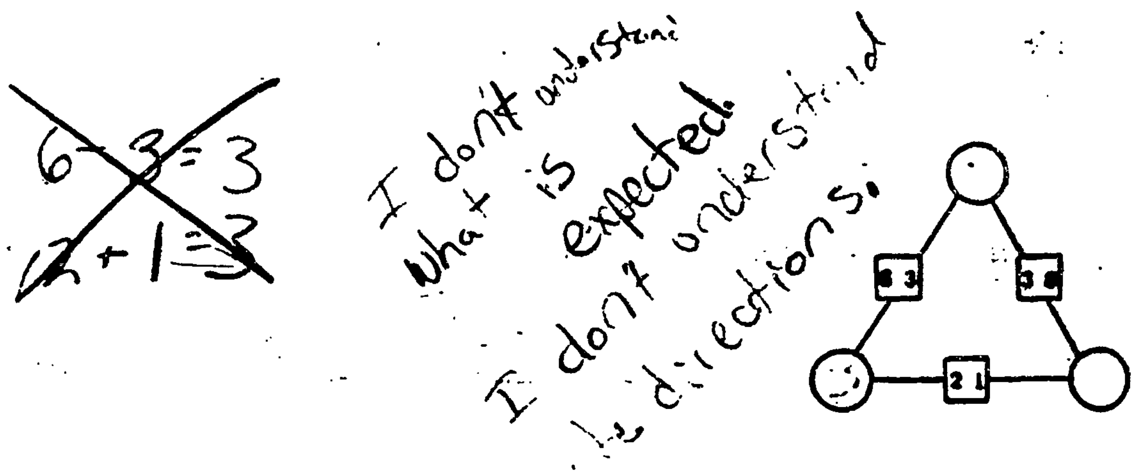
Examples of Subjects' Work Illustrating Not Understanding the Problem

1. Subjects write that they do not understand

a)

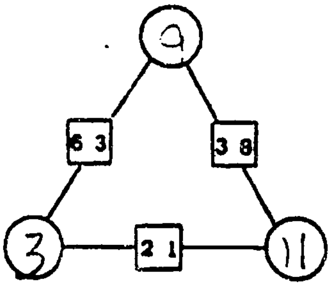


b)

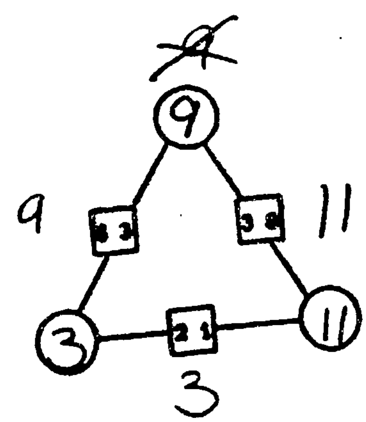


2. Adding the digits of numbers in the squares

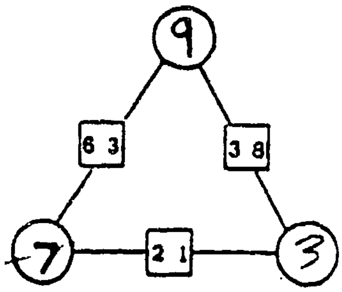
a)



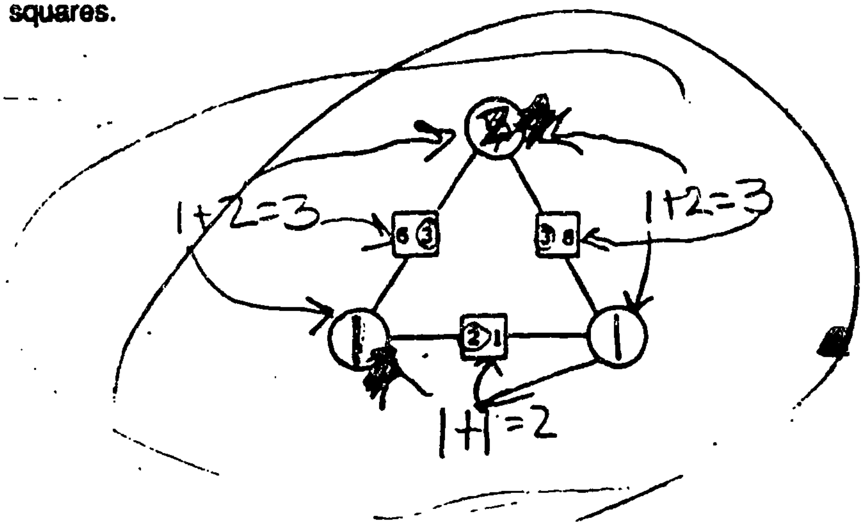
b)



3. Finding numbers for circles whose products are the numbers in the squares.

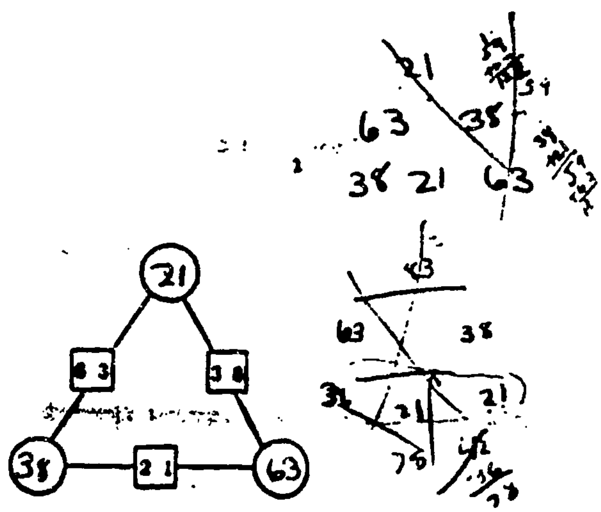


4. Determining numbers for circles so that their sums equal one of the digits in the numbers in the squares.



5. Putting the numbers in the square into the circles so that the sum of the numbers in the circles and squares will be equal.

X So there will end up being all 3 numbers added together will be together



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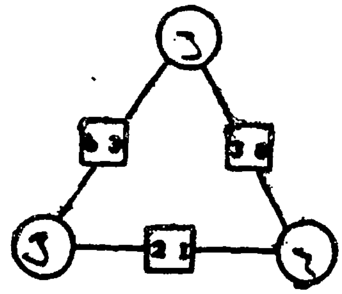
6. Using a different operation

$3 \div 63$

$63 \div 38$

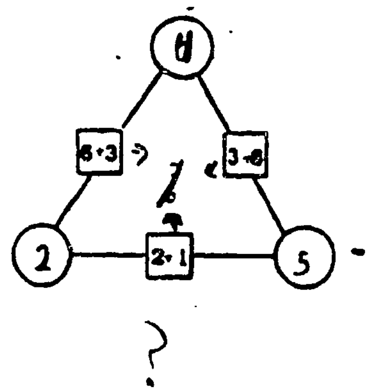
$63 \div 21$



7. Incomprehensible

a)

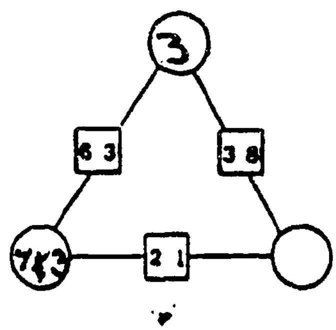
Guess



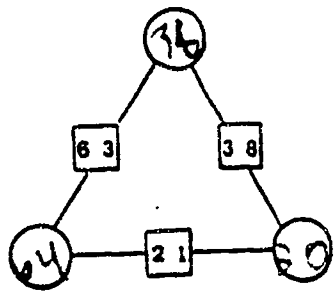
b)

$$\begin{array}{r} 21 \\ \times\ 3 \\ \hline 63 \end{array}$$

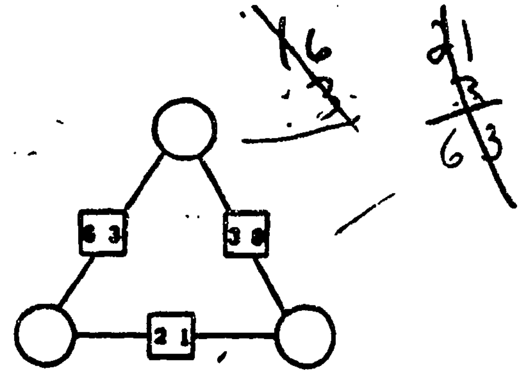
$$\begin{array}{r} 12.2 \\ 3\overline{)38} \\ 3 \\ \hline 08 \\ 6 \\ \hline 2 \end{array}$$



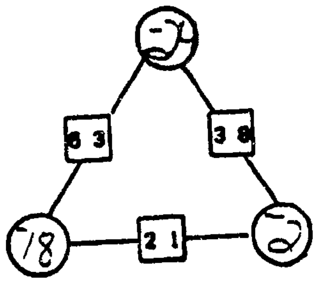
c)



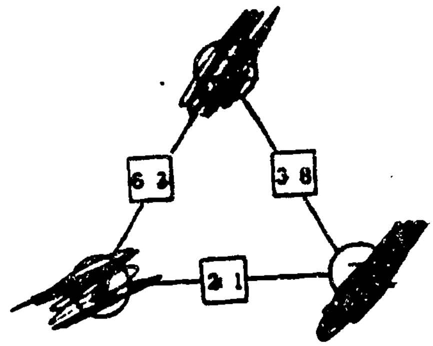
e)



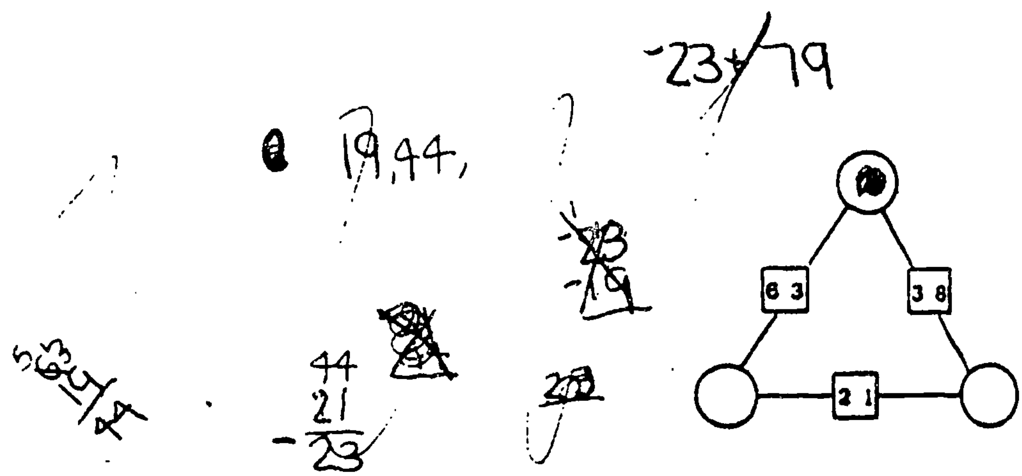
d)



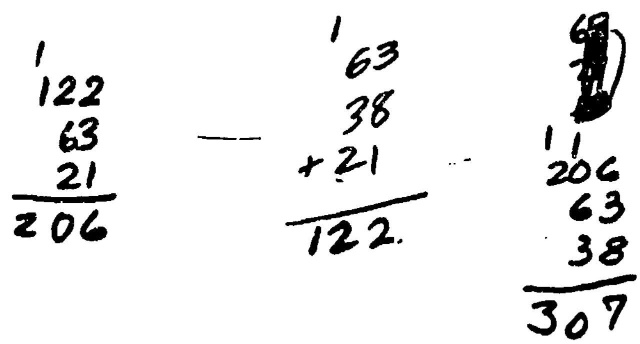
f)

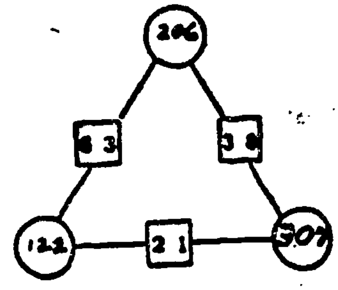


g)



8. Adding numbers in squares, with no apparent goal.

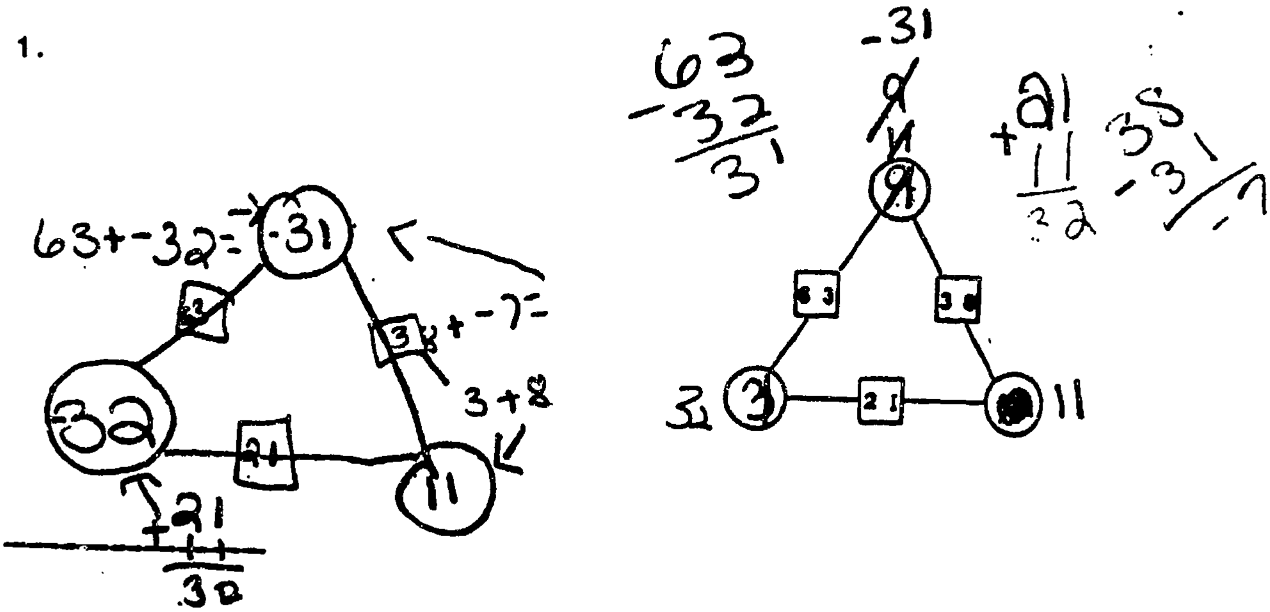




APPENDIX B

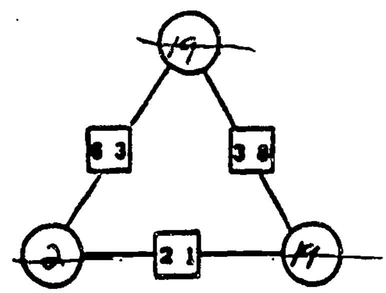
Examples of Subjects' Work Illustrating Difficulty With Negative Numbers

1.

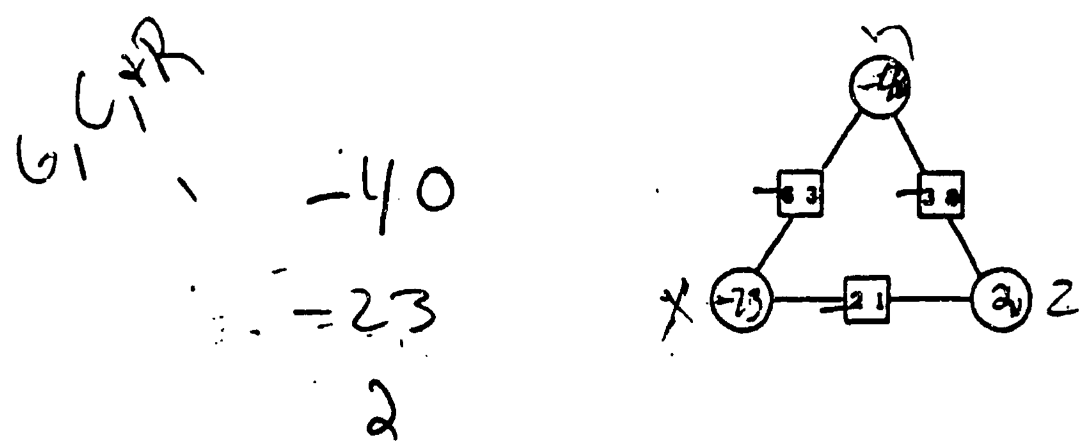


2.

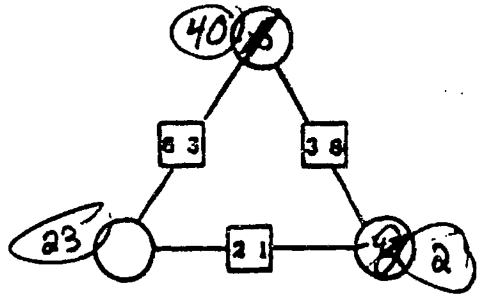
~~you can say that 63 is the sum of 17 and 19. 19 − 21 = 2. 63 − 2 = 61~~



3.

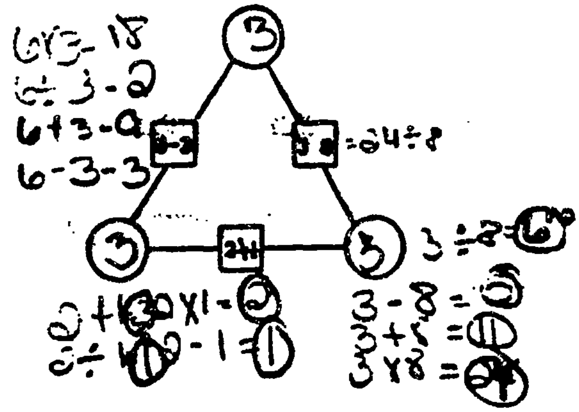


4. 40 +23 63   23 -2 21   40 -2 21



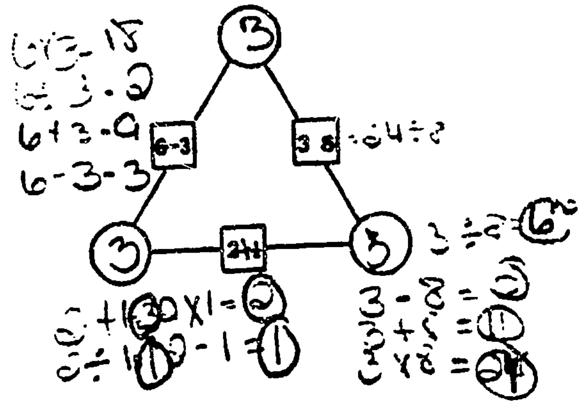
5.

-5 -4 -3 -2 -1 0 +1 +2 +3



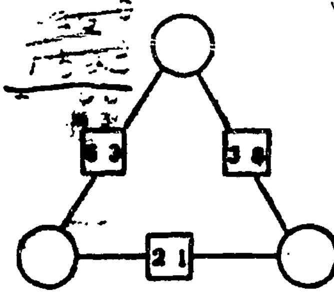
6.

-5 -4 -3 -2 -1 0 +1 +2 +3



7.

9 -11 2



3 -11 8

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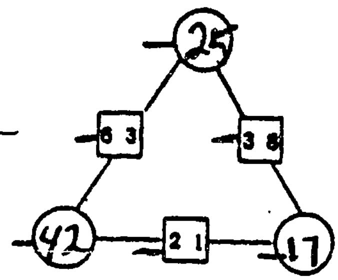
8. add negative

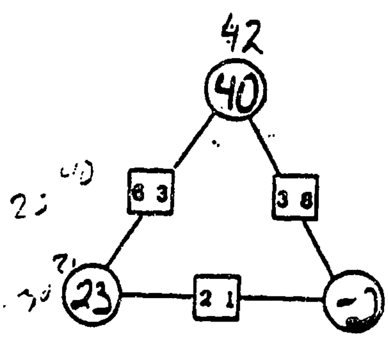


APPENDIX C

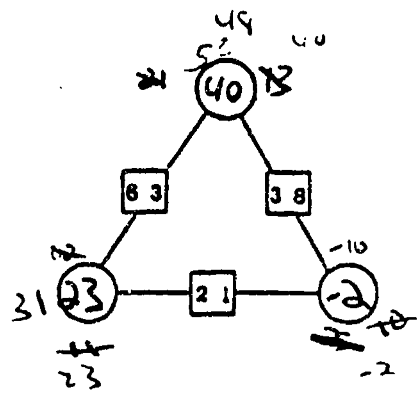
Examples of Correct Student Work on Problem I

1. Trial and Error

a)



b)

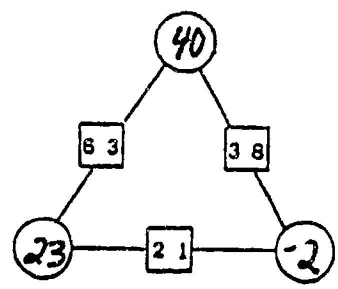


c)

$$x + y = 63$$
$$30 + 33 = 63$$

$$\begin{array}{r} 40 \\ 23 \\ \hline 63 \end{array} \qquad \begin{array}{r} 40 \\ + -2 \\ \hline 38 \end{array}$$

$$23 + -2 = 21$$



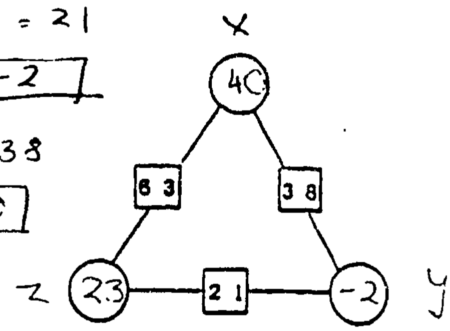
2. Simultaneous equations with three variables

a)

$$\begin{cases} x + y = 38 \Rightarrow y = 38 - x \\ y + z = 21 \\ x + z = 63 \Rightarrow x = 63 - z \end{cases}$$

$$38 - x + z = 21$$
$$\cancel{38 - 63} = z$$
$$38 - (63 - z) + z = 21$$
$$38 - 63 + z + z = 21$$
$$2z = 46 \quad \boxed{z = 23}$$

$$23 + y = 21$$
$$\boxed{y = -2}$$
$$-2 + x = 38$$
$$\boxed{x = 40}$$



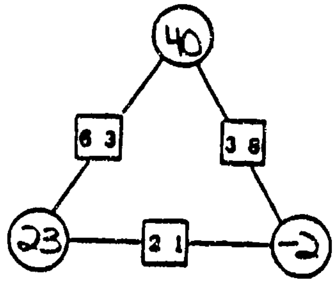
b)

$x+y=63$
$y+z=21$
$x+z=38$

$$\begin{array}{r} x+y=63 \\ -x-z=-38 \\ \hline y-z=25 \\ -y-z=-21 \\ \hline -2z=4 \\ z=-2 \\ x=40 \\ y= \end{array}$$

$x+z=38$
$x-2=38$
$x=40$

$y+z=21$
$y-2=21$
$y=23$



c)

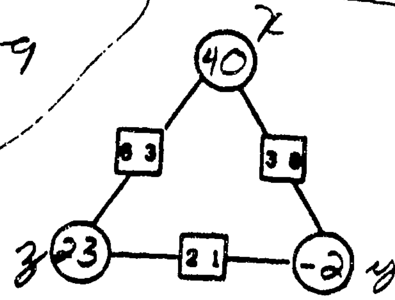
$x-2=38$
$x=40$

$40+z=63$
$z=23$

$x+y=38$
$y+z=21$
$x+z=63$

$2x+2y+2z=122$
$x+y+z=61$
$x+2y+z=59$
$-y \quad 0$

$x+2y+z=59$



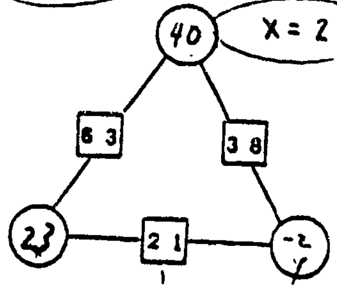
d)

$x+y=21$
$y-z=38$
$x-z=63$

$x=63-z$
~~$y-z=38$~~
$y=38-z$
$x+y=21$
$(63-z)+(38-z)=21$
$101-2z=21$
$2z=80$
$z=40$

$y+40=38$
$y=-2$

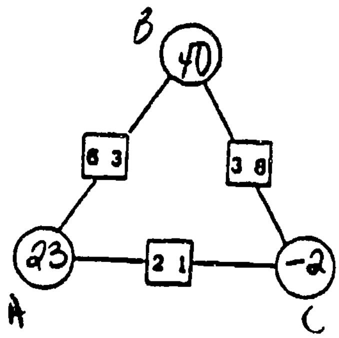
$x+y=21$
$x+-2=21$
$x=23$



e)

$A+B=63$
$A+C=21$
$B+C=38$

$63-A+21-A=38$
$84-2A=38$
$-2A=-46$ $A=23$

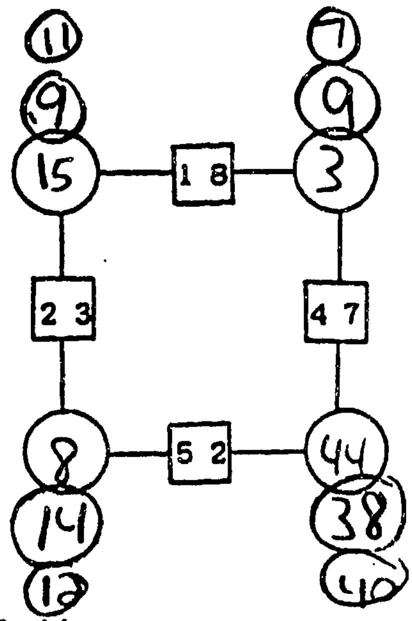


APPENDIX D

Three solutions, but no indication of more or infinitely many, for Problem II

(2) Now change to a square (four-sided) arithmogon as in the figure below. The number in each ☐ must equal the sum of the numbers in the two ◯ on either side.

Try to find the numbers for ◯ at each corner.



If you need more space, write on the back of this page.

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APPENDIX E

Example of Trial and Error and General Solution (almost) for Problem I and One Solution for Problem II [same subject]

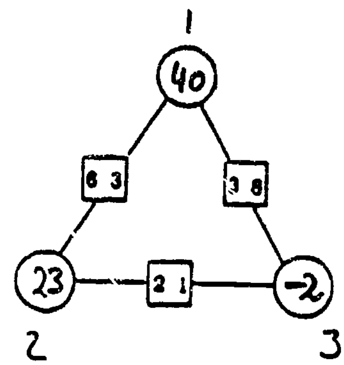
FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN.

(Way of solution 1)

Trying numbers:

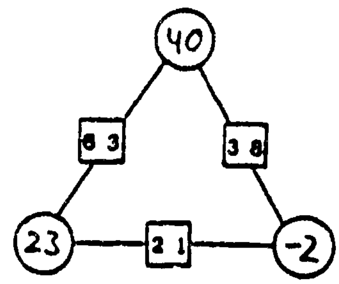
If ① = 40 than ② = 23
If ② = 23 than ③ = -2

I found them by accident



(Way of solution 2)

$63 + 2x = 21 + 38 = 59 \rightarrow x = -2$
$38 + 2x = 63 + 21 = 84 \rightarrow x = 23$
$21 + 2x = 63 + 38 = 101 \rightarrow x = 40$



I don't know why I took 2x I just found out that $\square_1 + 2x = \square_2 + \square_3$

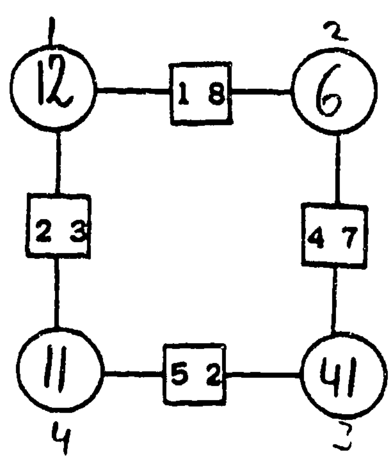
Continue on the next page.

(2) Now change to a square (four-sided) arithmogon as in the figure below. The number in each □ must equal the sum of the numbers in the two ○ on either side.

Try to find the numbers for ○ at each corner.

Found them by accident in first try.
Maybe I've got the feeling for it.

If ① = 12 than ② : 6
If ② = 6 than ③ = 41
If ③ = 41 than ④ = 11



THE AREAS OF SQUARES PROBLEM: RESULTS OF AN ANALYSIS OF STUDENTS' SOLUTIONS

James W. Wilson

University of Georgia

This report presents the analyses of U.S. results on the *areas of squares problem*. A copy of the relevant portions of the student booklet appears in the Appendix to this report. In the problem which follows, we are concerned with:

a. Students' performance on a non-standard problem amenable to Algebra II.
b. The strategies and approaches the students use.
c. Students' ability to generate similar problems.
d. The impressions students have about the problem.

The Problem

The "Areas of Squares" problem was given to 11th graders in a few Algebra II classes in Georgia, Florida, Illinois, and Pennsylvania. The problem statement is given in Figure 1. The language and format in Figure 1 is the same as that presented to the students. The format took

---

Pick a point P on the line segment AB and make squares: One side of one is AP and one side of the other is PB. Where should the point P be located to satisfy the condition that the sum of the areas of the two squares is a minimum?

Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) Write a way of solution and the answer to the above problem.

Ans. ____________________

Figure 1. The Areas of Squares Problem

approximately 2/3 of the left hand side of a double page. Then at the bottom of the same space, just below the answer blank, was the question in Figure 2 asking the students to make up problems like the one just presented. This was followed by 5 numbered spaces, one at the

(2) Now make up your own problems like the one above and write them down. Make as many problems as you can. <u>You do not need to find the answers to your problems</u>.

Figure 2. Follow-up Question for Areas of Squares Problem

bottom of the left-hand side of the double page and four on the right-hand side. At the bottom of space 5 was an instruction, "If you need more space, write on the back of this page." and three additional numbered spaces were provided on the back of the page. The third part of the question attempted to have the students pick the "best" problem from among the ones they had generated and to give a reason for their choice. (See Figure 3).

(3) Choose the problem you think is best from those you wrote down above and write the number in the space: __________

Write the reason or reasons you think it is best.

Figure 3. Choosing the Best

The students were given 15 minutes to complete the three parts to this problem. Each booklet contained three problems and was administered by a proctor. Near the end of the period the students were asked to respond to the questions in Figure 4 to assess how they perceived these problems.

1. Do you like Math?
2. Are you good at Math?
3. Do you think that today's problems are interesting?
4. Do you think that today's problems are easy?
5. Are today's problems the same as the problems in your math textbook?
6. In comparison to problems in your math textbook, did you like today's problems?
7. Have you seen problems like this before?

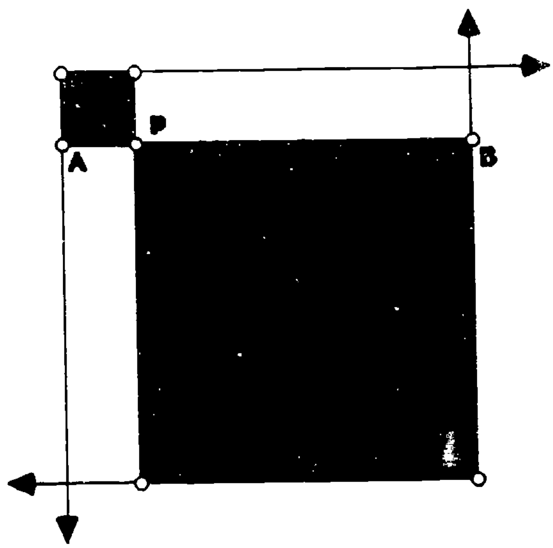
(Note: The students made separate responses to Questions 3 - 7 for each of the three problems.)

Figure 4. Questions to Assess Students Perceptions of the Problem

Some Ways to Solve

Our expectations were that this would be a difficult problem for U.S. students in Algebra II. The problem can be approached in several different ways and it was hoped that the types of attempts would be of as much interest as the complete solutions. It was judged to be a non-standard problem in each of the countries. This is a very nice problem as a teaching task, where in the context of instruction and student discussion, the many alternative approaches can be explored. In the development of this item for the survey, it was hoped to capture some of that flavor as an assessment item.

The problem is understandable within the context of Algebra II. It asks for the minimum of the sum of two squares as shown below. If the sides of the squares are extended in our sketch to form a square of length AB on each side, four regions are formed: the squares AP^2 and PB^2, and the two rectangles each AP by PB. Now the total of the four regions is always AP^2. Therefore the minimum sum of the squares $AP^2 + PB^2$ occurs when the two rectangles have maximum area. But a rectangle has maximum area when it is a square or when AP = PB.

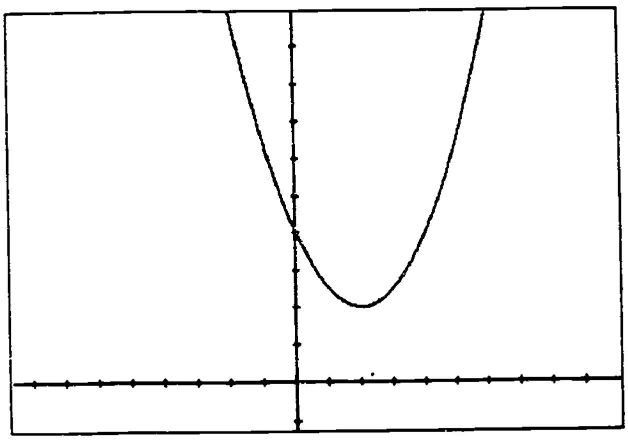


Variations on that approach include the following. Let $AB = x$ and $PB = y$. Then we want to minimize $x^2 + y^2$. By the Arithmetic Mean-Geometric Mean inequality,

$$x^2 + y^2 \geq 2xy, \text{ with equality iff } x = y.$$

Therefore the sum of the two squares is always greater than the combined areas of the two rectangles except when $x = y$. So the minimum area occurs when P is the midpoint.

Another approach is to formulate the area as a function of a single variable. Let $AP = x$ and $PB = AB - x$. The area $f(x) = x^2 + (AB - x)^2$. This simplifies to $f(x) = 2x^2 - 2(AB)x + AB^2$. This might be recognized as a parabola with the following graph where the vertex is at $(AB/2, AB/2)$.



On the other hand $f(x) = -2x(AB - x) + AB^2$. By the Arithmetic Mean-Geometric Mean Inequality,

$$f(x) \leq -2\left[\frac{x + AB - x}{2}\right]^2 + AB^2, \text{ with equality iff } x = AB - x$$

$$= -AB^2/2 + AB^2$$

$$= AB^2/2$$

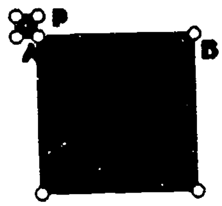
For the rare algebra II student that has had just enough cookbook calculus to take $f(x) = 2x^2 - 2(AB)x + AB^2$, find its derivative $f'(x) = 4x - 2(AB)$, and set equal to 0, the result is about it.

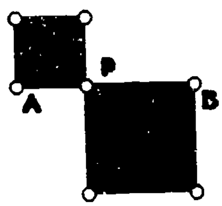
Another approach is to particularize the length AB and compute a sequence of values for $AP^2 + PB^2$ as P is placed along points on the line. Let AB – 10 and x = AP. Then the following table can be generated quickly.

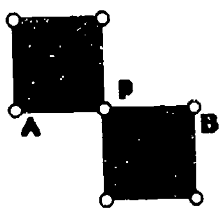
x	0	1	2	3	4	5	6	7	8	9	10
10 – x	10	9	8	7	6	5	4	3	2	1	0
sum	100	82	68	58	52	50	52	58	68	82	100

This provides good intuition that the desired location for P is at the midpoint of AB.

A variation is to draw a sequence of figures such as the following







Results

As a part of the Japan-U.S. Joint Seminar common survey, the problem was attempted by 247 U.S. Algebra II students. We had the advantage of seeing a preliminary analysis of data on this problem for Japanese 11th graders. Only 14% of the Japanese students gave correct answers that included complete solutions. In the Japanese scoring, inductive approaches such as calculating areas or developing intuition by drawing pictures were called "inappropriate reasoning, correct answer in the end."

We examined the problem booklet for one Algebra II class and developed the following scoring categories. Independent scorers showed close agreement on using these categories for that class and so the 247 booklets were scored with these categories.

A *The student produced a drawing or sketch that was a reasonable interpretation of the problem.*

B *The student produced some work in addition to or without a drawing to show some understanding of the problem.*

C *The student produced an argument, line of reasoning, or sequence of steps leading to a correct answer.*

D *The student produced the correct answer.*

E *The student explicitly said "I do not understand [the problem].*

The percentage of students in each of the categories is shown in Table 1. The data are given for each geographic region as well as for the total. An additional line in the table presents the data for 48 preservice secondary mathematics teachers.

Table 1

Types of Responses to the Areas of Squares Problem

	N	A	B	C	D	E
Athens	116	85.3	65.5	46.6	73.3	7.8
Gainesville	50	80.0	38.0	38.0	80.0	0.0
Carbondale	48	81.3	39.6	39.6	85.4	0.0
Pittsburgh	33	69.7	51.5	39.4	81.8	3.0
TOTAL	247	81.4	53.0	42.5	78.1	4.0
Prospective Sec. Math Teachers	48	56.3	68.8	72.9	75.0	0.0
Japanese 11th Grade					84.0	

N = Number of Students Tested

A = Made a reasonably correct drawing or sketch

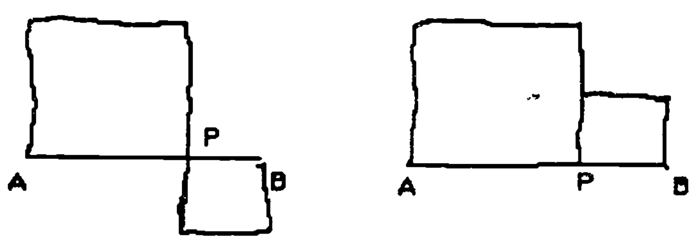
B = Something beyond the drawing to show some understanding of the problem

C = Reasoning or argument presented

D = Answer "In the middle," "at the midpoint," or equivalent

E = Explicitly write "I do not understand [the problem]"

Draw a figure. Over 80 percent of the students produced a drawing that was a reasonably correct interpretation of the problem statement. These seemed about equally divided between drawing the squares on opposite sides of AB vs. drawing them on the same side.



Many made only a drawing with P at th. midpoint and wrote a correct answer. It would seem that these students already had some intuition that P should be at the midpoint and made the drawing to fit that intuition. This is reasonable in that the symmetry of the situation would allow the midpoint as the only unique placement for P but no one made any explicit argument of the symmetry.

It is encouraging that 80 percent of the students could make a problem translation from the verbal mode to an iconic one. It is discouraging that so many of them reasoned only from their drawing.

The preservice teachers had only 56% of them making a drawing. In part this was probably due to many of them producing calculus solutions -- setting up a function, taking the derivative, setting the derivative equal to 0, and solving -- without making a drawing.

Evidence of understanding. This category was checked if the student made a verbal restatement of the problem, drew multiple drawings to show P could vary along AB, or did an incomplete argument before writing the correct answer. It was used to capture the cases where the student made some progress beyond a single sketch of the situation.

Reasoning or argument presented. The Japanese students had 14% who present a correct answer with mathematical reasoning. The corresponding figure for the U.S. students would be 0%. Of the 247 students, none produced an algebraic equation to represent the problem. One student wrote $x^2 + y^2 \leq ((x+y) / 2)^2$ but did not connect it to anything in his drawing (just as well since it is not true).

For U.S. Algebra II students about 42% presented reasoning or argument leading to a correct answer. In every case this reasoning was inductive. The most common was to particularize the length AB and to calculate a sequence of areas, usually accompanied by a sequence of drawings. For the Japanese student using "inappropriate reasoning," 26% calculated areas and 21% reasoned by drawing pictures.

Correct answer. The question asks where to locate P to minimize the two areas. In retrospect, if we wanted explanation and justification we should have asked. About 78% of the Algebra II students produced a correct answer (i.e., "at the midpoint of AB" or something equivalent). Many of them drew a single figure and wrote an answer. The Japanese students had 84% with a correct answer of which 70% used "inappropriate" reasoning.

"I do not understand." This category came about because it occurred a few times from our one class used in developing the scoring scheme. For the total it occurred only 10 times out of 247 and 9 of these 10 were in the Georgia classes. One student wrote a correct answer and the comment, "This problem is a perfect example of how I know the answer but have no idea how to go about getting it." One suspects the student is not unique in this respect among the 247 Algebra II students.

Making up problems. Students were asked in Part 2 of the question to make up problems like the one they had just attempted. In scoring these responses a count has been made of the number of problems written (Table 2) and a judgment of the quality made on each problem, using a scale from 0 (poor) to 5 (excellent). These quality ratings are summarized in Table 3. A "0" rating was given to problem statements that were incomprehensible for either context or substance, a "1" rating was used when a statement was comprehensible but did not present a problem, a "2" rating was used for incomplete but clearly on target questions, a "3" rating was used for essentially restating the given question or some part of it, a "4" rating was given for problems with some similarity to the given problem and which asked a question that might be attempted (e.g., constructing triangles on the segments), and "5" ratings were for questions showing substance, correctness, or insight. The results of this question are given in Table 3.

Table 2

Number of Students Writing Problems

	Number of problems written							
	0	1	2	3	4	5	6	7
11 Graders	37	75	63	42	17	11	2	0
Prospective Sec. Math Teachers	12	16	12	6	1	0	0	1

This question produced less than hoped. First, there were many, 37 11th graders and 12 preservice teachers, who made no response. Second, the quality of the responses was very low. Responses rated 0, 1, or 2 were best described as evidence of inability to respond to the question. Take away the large number of 3 ratings that are essentially copying of the original problem and there is little of use, even from the preservice teachers. Writing the problems was clearly a task beyond the experience of these students.

In part 3 of the problem the students were asked to select what they perceived to be their best problem. The ones who write 0 or 1 problems (112 of the 247 11th graders and 28 of the 48 preservice teachers) really had no response to give to this part of the problem. There were 113 11th graders who selected a be problem from their set and 19 of these were students generating only one. The nonresponses makes this data have little value. Table 4 shows that of those who responded the majority selected their first problem generated. The reasons given for selection are summarized in Table 5. Here the responses were placed in three general categories -- Most like the original, Only one thought of, or Most challenging. There were only 79 11th graders and 17 preservice teachers who gave a reason for their selection. It is interesting that the majority who gave a reason indicated the challenge for giving a problem the "best" award. Note also that even though there were 48 11th graders who indicated the "challenge" choice, there were not 48 of them who wrote challenging problems.

Table 3

Number of Students Writing Quality Problems

Index of Quality	First Problem	Second Problem	Third Problem	Fourth Problem	Fifth Problem
11th Graders					
0	90	59	26	15	5
1	38	32	22	10	3
2	23	12	9	2	2
3	47	20	8	2	3
4	11	11	7	1	0
5	0	1	0	0	0
Preservice Teachers					
0	10	1	1	0	0
1	4	0	0	0	0
2	4	1	1	0	1
3	11	4	4	3	1
4	5	0	0	0	0
5	2	2	2	0	0

Table 4

Selection of Best Problem

	First	Second	Third	Fourth	Fifth
11th Graders	55	31	19	7	1
Preservice Teachers	13	6	0	0	0

Table 5

Reason Given for Selection
of Best Problem

	Most Like the Original	Only One Thought of	Most Challenging
11th Graders	13	19	48
Preservice	4	5	8

The perceptions these students have of this problem are summarized in Table 6. The majority indicated they liked mathematics and considered themselves good at it. Yet, the majority of 11th graders found this problem

- not interesting
- hard
- more difficult that textbook problems
- liked less than textbook problems
- totally unfamiliar

The majority of preservice teachers, however, believed this problem to be the same as textbooks now used. Clearly, this problem was not a hit with these 11th graders.

Table 6

Students' Perception of Mathematics
and the Areas of Squares Problem

11 Graders

Yes	Neutral	No	
118	80	16	1. Do you like Math?
108	85	20	2. Are you good at Math?
56	76	81	3. Do you think that today's problems are interesting?
37	80	96	4. Do you think that today's problems are easy?
32	69	106	5. Are today's problems the same as the problems in your math textbook?
22	70	120	6. In comparison to problems in your math textbook, did you like today's problems?
0	90	122	7. Have you seen problems like this before?

Preservice Teachers

Yes	Neutral	No	
40	8	0	1. Do you like Math?
35	13	0	2. Are you good at Math?
20	20	7	3. Do you think that today's problems are interesting?
9	21	16	4. Do you think that today's problems are easy?
36	8	2	5. Are today's problems the same as the problems in your math textbook?
13	23	8	6. In comparison to problems in your math textbook, did you like today's problems?
0	35	11	7. Have you seen problems like this before?

Discussion

The performances of 11th grade algebra students and preservice secondary teachers could be reason for despair. It is inconceivable that out of 247 algebra students, not one would set up an algebraic expression in an attempt to solve the problem. Very few saw any need to present reasoning as part of their response.

Why?

Clearly, something more needs to be done to understand what produced these results. From personal experience I know that typical algebra II class such as those in this study can, in an <u>instructional</u> setting, work meaningfully on this problem and others like it. I also know this is not the usual fare for algebra II. Is our attempt to put this problem into a test format so far off the mark that it misleads students?

A few one-on-one interviews with students -- in no sense a representative group of students -- has suggested a few things that may have gone wrong on this. First, in U.S. classrooms any testing and problem solving is the same as answer getting. Students have examined this problem and said, using no algebra, that the answer (there is an "answer blank") is the midpoint of AB. When I ask "why?" they point out that the problem does not ask for "why?" but rather asks for an answer. I think this is a prevailing attitude among U.S. Students that may be much less prevalent among Japanese students. It is a question that needs to be examined and can not be answered by survey testing.

Second, testing for problem solving is more than moving performance situations from the classroom to the printed page. The poor results here may be as much inherent in the limitations of that process as they are to be seen as limitations in the students who were tested. Our enthusiasm for national performance standards testing ought to be tempered by the need to fully understand what is being measured -- deal with the issues of validity -- before the tests are widely administered.

The extensive use of visualization by these algebra II students goes against some of the prevailing folklore of mathematics education -- that students can not translate from a verbal description to a drawing and visualize implications from the drawing. One of the things that happened with this problem is that a large number of students did make a good sketch or drawing to visually interpret the problem and to solve it visually. At that point they could write the answer and seeing the need for algebra or giving reasons did not occur to them -- algebra and reasoning were not asked for. Here again we need more research to understand how to help students use visualizing abilities in a constructive way.

Another facet that needs better understanding is just what students believe mathematics to be. It oversimplifies the situation, but makes clear a distinction, to say that answer getting is the sum and substance of mathematics for U.S. students, while Japanese students at this level have a better conception of the role of argument and reasoning in mathematics. This is a byproduct of our extensive testing in U.S. classroom and the almost complete absence of examinations. One hypothesis is that the U.S. students approached this survey booklet as a test while the Japanese students approached it as an examination.

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COMMENTS ON RESULTS FOR THIS U.S. SAMPLE

As mentioned earlier, the focus of the U.S.-Japan Cross-national Research on Students' Problem Solving Behaviors was not on a comparison of overall or specific performance of U.S. and Japanese students with respect to, for example, rates of correct or incorrect answers. Though some of the results reported reflect such differences, it is noted again that while these two samples were selected with a "mix" of small and large schools, in particular, the samples were not selected in a random manner either with respect to schools or classes within schools. The study is descriptive in nature and analyses of the data show a number of results for the U.S. sample worthy of further consideration and study. Accordingly, the comments below derive from results reported earlier for the U.S. sample. Comments are presented for each problem separately.

<u>Marble Arrangements Problem (Grade 4)</u>:

A significant percent of U.S. subjects gave all correct answers, with the results for female subjects significantly greater than for male subjects. Perhaps we should pursue the question of why this should be the case. Further, subjects show clear evidence of switching strategies; however, the results are noteworthy for a lack of higher level thinking or mathematical sophistication. For example, subjects commonly use counting and less commonly multiplication, and there is a marked absence of a significant use of mathematical expressions in finding and representing answers.

The results show clearly that subjects can solve the problem in different ways when asked to do so; nevertheless, we need to ponder why this mathematical behavior is not more common and what we can do in classrooms to help students to develop this facility. As is well-known, students at all school levels express belief that there is one and only one answer to problems in mathematics and, also, only one correct way to find the answer.

<u>Matchsticks Problem (Grades 4 and 6)</u>:

As expected, 6th grade subjects show a higher success rate in finding the correct answer than 4th grade subjects. Unlike the Marble Arrangements Problem, male subjects more than female show a higher success rate in finding the correct answer. Here also, results show that subjects use an unsophisticated mathematical approach to finding the answer to the problem - Drawing and Counting is predominantly used by both 4th and 6th grade subjects even though we might reasonably expect them to be able to use a more sophisticated approach. Many subjects recognize that there is a shared side among the squares, but either they prefer to not use this characteristic or are unable to use it. We need to ponder why this is the case. We also need to consider the question why students do not have a more natural inclination to use more mathematically sophisticated ways of thinking about the problem.

Similar comments can be made with respect to students' formulation of problems similar to the given one. Here again 6th grade subjects perform better than 4th grade subjects, as expected, and male subjects have a higher success rate than female subjects among subjects who formulated problems structurally similar to the given one. But problems given by subjects tend, once again, to reflect unsophisticated mathematical approaches; as the reader may recall, most problems involve only simple extensions and seemingly too few subjects keep the shared-side characteristic of the Matchsticks problem in their problem formulations. As a matter of fact, 37% of subjects who solved the problem correctly used the shared-side characteristic; but 25% of those who were incorrect also used this characteristic of the problem - there is not a large difference between successful and unsuccessful subjects in this respect. The results, any way one views the situation, however, reflect the fact that the results are marked, again, by an absence of a higher level of sophistication in formulating problems. It may be that subjects have not had experience with this kind of mathematical activity or mathematical thinking. If so, perhaps this argues for providing these kinds of activities and experiences to students in the classroom.

Finally, it is noted that both male and female subjects prefer problems that are hard - both selected the problem they liked best consistently giving this characteristic as the reason for doing so. Perhaps we need to provide more challenging problems to students in our mathematics classrooms.

<u>Marble Pattern Problem (Grades 6, 8, and 11)</u>:

Subjects were asked to find the number of marbles in the 4-th, 16-th, and 100-th places at grades 6 and 8, and the number in the 4-th and 16-th places and the n-th term formula at grade 11. Overall for problem 1, 96% of 11th grade subjects got correct answers, followed by 93% at the 8th and 82% at the 6th grade. For problem 2, 68% of 11th grade subjects got correct answers, followed by 53% at the 8th and 26% at the 6th grade levels. The latter results are not encouraging. For problem 3, only 40% of 11th grade subjects got the n-th term formula correct, and 30% and 17% of 8th and 6th grade subjects, respectively, got correct answers for the number of marbles in the 100-th place. Again, none of these results are very encouraging. There were virtually no differences by gender.

What is more disturbing are the results for different approaches used and how often each was used for the three questions. Subjects at all grade levels tended to use counting for problem 1, even at the 11th grade level, and rarely used more sophisticated approaches at any grade level for any of the problems. Moreover, there is a strong tendency for subjects at all grade levels, who got correct answers, to use the same approach on all problems; also, there is a non-trivial number of subjects who used incomprehensible approaches for all problems at all grade levels.

These results are consistent with those for the Marble Arrangement and Matchsticks problems, and the results are consistent across gender and grade level.

Arithmogons Problem (Grades 8 and 11):

There is further consistency in results on the Arithmogons problem with results for the three previously reported problems for the U.S. sample. Here, again, overall performance leaves much to be desired. Overall success rates are not high for either problem, even given that trial of any number for Problem II would lead to at least one solution. Similarly, the mathematical sophistication of approaches to solving both problems is at a low level: Trial and Error is predominantly used by subjects at both the 8th and 11th grade levels, practically no use is made of simultaneous equations in attempting to solve the problem, and a large number of subjects did not understand or try the problems, even though the problems were read aloud by the proctors and subjects were asked to read the problem statements carefully before beginning.

Moreover, the number of subjects who used more than one approach, as requested, was negligible. Eighth grade subjects exhibited significant evidence that they could not work with negative numbers, and this was true, though somewhat less so, for 11th grade subjects as well. We can note that, to a large extent and understandably, 8th and 11th grade subjects do not have setting up and solutions of simultaneous equations in their problem solving repertoire; still, the problems could be solved using other forms of reasoning. Subjects were not only not able to apply other reasoning skills successfully, but there is scant evidence that they attempted to do so, though teachers consistently reported that students in their classes gave their best efforts. Further, numerous student scripts reflected a stubborn determination to grind out a solution to Problem I by random or systematic trial and error. Evidence abounds that subjects could seemingly not shift gears in thinking or strategy and try another (other) approach(es).

Perhaps we need to re-examine the extent to which Trial and Error (random or systematic), as a viable problem solving strategy, is taught in our mathematics classrooms. Further, following these results, we need to re-examine where, when and how operations on negative numbers are taught and the need for students to have more sophisticated approaches (i.e., algebraic and reasoning) in their problem solving repertoire. Note is made here of Usiskin's (1987) strong admonition that we can, should, and must teach algebra to eighth grade students and the excellent study of Flanders (1987) showing the huge amount of repetition of content in the U.S. elementary and junior high school curriculum (p. 22):

> Although the percentage of new content is actually quite low (49 percent or less) in all texts beyond third grade, the seventh- and eighth-grade texts have the least new material. It is in comparison with the 88 percent new content in algebra books that the 30 percent eighth-grade level seems shockingly small. . .

It is my opinion that we should ask more of students (and textbooks and teachers) in earlier grades.

Area of Squares Problem Grade 11):

The results for the U.S. sample on this problem present further evidence that subjects are seemingly unable to reason in any kind of a sophisticated manner in getting a solution to the problem. While it is encouraging that 80% of subjects could translate the problem statement to a figure or diagram, it is discouraging that such a large percent reasoned only from the drawing. Zero percent (0%) of subjects in the sample presented a solution with mathematical reasoning (Wilson, 1991, pp. 8-9). For example, of 247 subjects, none produced an algebraic equation to represent the problem statement. When reasoning or an argument was presented (42%), the reasoning was inductive in every case.

General Comment:

There is ample evidence that subjects in the U.S. sample express feelings that they "like math" and are "good at math." However, their performance on the problems, in general, is inconsistent with their expressed feelings. Moreover, the results found in this study are consistent with the findings of other researchers (cf., for example, McKnight et al., (1987) and Becker and Owens, 1991.

CONCLUSION

Probably the most important reason for participating in this kind of cross-national descriptive research is that it may contribute towards understanding and improving our own approaches to mathematics teaching. As such, it may be viewed as a complement to other studies in the U.S. (e.g., the NAEPs) as well as to international comparisions (e.g., the First and Second International Mathematics Studies (FIMS and SIMS)) (see Bradburn and Gilford, 1990).

Though we do not know from these research results exactly why the differences exist, for various cultural, societal, and other factors may play a role (see, for example, Becker, 1992), still, there is mounting evidence that what goes on in the Japanese mathematics classrooms is very different from the U.S. For example, Becker et al (1990) and Stigler and Stevenson (1991) write about observations made of classroom lessons in Japan. These lessons may be characterized as carefully crafted, organized, teacher managed and coherent lessons with a focus on one main idea. Further, drawing on students' thinking is part of the pedagogy along with a lot of teacher-student and student-student interaction. Very frequently, lessons begin with a carefully chosen problem situation and the teacher, far from being a dispenser of knowledge, acts as a <u>guide</u> in the lessons <u>using student input</u>, aware of how much time remains for the lesson and what the teacher wants accomplished by the end of the lesson. Classroom management is critically important. Even when the problem around which a lesson plan is constructed is more "mathematical" than it is real-life, the teacher knows the characteristics of the problem (e.g., knows that the problem lends itself to multiple approaches to its solution or to multiple correct answers) and can therefore draw upon students' solutions for discussion. However the lesson unfolds, the teacher ends the lesson with a summary of what the problem was to begin with, how various approaches contribute to its solution, and what was learned in the lesson. This kind of "lesson closure" is critically important. Various other characteristics of Japanese and Chinese classroom lessons in mathematics are discussed by Stigler and Stevenson (1991). They end by asking "Why not here?" and discuss changes necessary in the philosophy and structure of U.S. education in order to improve the effectiveness of our teachers in the classroom. The problem is not with the teachers, it is with the whole context of teaching in which we expect our teachers to perform (Stigler and Stevenson, 1991).

But there are still other considerations. The mathematics curriculum is carefully set by the Ministry of Education in Japan, in terms of overall objectives, content, and the teaching of the content. Teachers must, to a very large extent, teach the prescribed mathematics curriculum - textbooks used in classrooms must first be approved by the Ministry of Education. It would be valuable for U.S. mathematics educators to see and study what mathematical content is in the curriculum, how it is organized and how it is taught (philosophy, approaches, materials, and classroom management). This requires translation of these books and related materials for U.S. curriculum developers to study and see what might be useful and adapted to the U.S. The University of Chicago School Mathematics Project is doing this at present. Perhaps the work should be intensified and continued.

It would also be useful to continue research as carried out in the U.S.-Japan Cross-National Research on Students' Problem Solving Behaviors described here. While there are no absolute standards of achievement in mathematics education, in any country, comparative types of studies like the one reported here may have potential for improving instruction and learning in the U.S., setting attainable standards, and monitoring our success in achieving those standards. Further, instances are increasing in which having an American-only sample is not efficient for developing improvements in effective delivery of education. As Bradburn and Gilford (1990) comment, "the issue here is not whether an observed pattern is typical, but rather whether something that exists in another country, but not in the United States, would be useful here (p. 3).

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APPENDIX

1. Introduction

Thank you very much for your cooperation in administering this survey. The survey is a part of the U.S.-Japan Collaborative Research on Students' Strategies and Difficulties in Solving Mathematics Problems. It will be administered in both countries using the same problems. Results will be analyzed from a cross-cultural point of view.

In order to use the same methodology in both countries, you are requested to follow the directions below:

2. Contents of Booklet

Please make sure that you have enough booklets, each containing
(1) the directions (cover page).
(2) survey problems.
(3) Questionnaire for Students (at the end).
(4) Questionnaire for Teachers.

3. Outline of the Survey

The time schedule for administering the survey is as follows:
* five minutes to hand out booklets, have students fill in name, date, name of school and sex.
* fifteen minutes for Problem I.
* fifteen minutes for Problem II. [Note: For Grade 11 only, ten minutes for problem III]
* five minutes for Questionnaire to Students.

In total, you will need approximately forty-five minutes (fifty-five minutes for Grade 11) for the survey.

Note: While administering the survey, please do not answer any questions students ask concerning the problems or answers. You may say to a student "I leave it to your judgment" or "Judge for yourself" if questions are asked.

4. Directions for Administering the Problems - please follow the directions below:

(1) At the beginning, you may briefly explain the aim of the survey. Then ask the students to:
   a) Please relax and just try to do their best.
   b) Do not turn the first page until asked to.
   c) Stop working when asked to "STOP."
   d) Read each problem and follow directions carefully.
   e) Write down all work: Do not erase anything written down; draw a line through any work that might be in error.
   d) They have fifteen minutes for each problem (Grade 11 only: ten minutes for problem III).
(2) Distribute the booklets to the students and have them fill in information at top of cover page.
(3) Read directions on cover page.
(4) Say "begin" when every student has finished filling in information and you have read the directions.
(5) Fifteen minutes later, say "stop."
(6) Then for problem II, repeat the directions (4) and (5). Note: For Grade 11 only, say "stop" after 10 minutes for problem III.
(7) Have students fill out Questionnaire to Students.
(8) Collect all booklets when everybody is finished.

SPECIAL NOTE: If you observe anything out of the ordinary about a student working, write down the name, grade level, problem number, and your comment.

We greatly appreciate your cooperation in the U.S.-Japan Collaborative Research survey. Thank you very much indeed.

Questionnaire to Teacher

1. Name of Teacher: ______________________________

2. Name of School: ______________________________

3. Date: ______________, 1989

4. Time of day: From ____________ to ____________.

5. Describe the school in which you teach:

   School enrollment: ________ Number of teachers: ________

   Number of teachers who teach math? ________

6. Have you ever taught or used problems similar to these?

   _____ yes _____ no _____ can't say

   If yes, how?

7. In your view, how did students react to these problems?

8. In your view, do you think the students represented their best work?

   _____ yes _____ no _____ can't say

9. What is your reaction to problems like these?

10. Describe your class.

    ___ remedial ___ below average ___ regular ___ above average

4G1

4th Grade

Name ______________________ Date ______________

School ______________________ Sex: _____Boy _____Girl

Directions: Read each question carefully and follow directions.

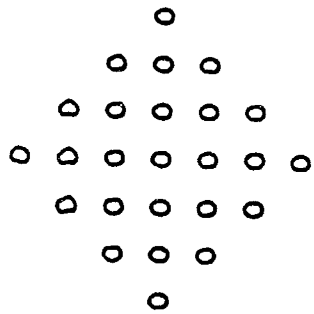
You should write down all your work. Do not erase anything you write down, just draw a line through it rather than erase it.

There are two questions and you will have 15 minutes for each question.

Do not turn the page until the teacher tells you to.

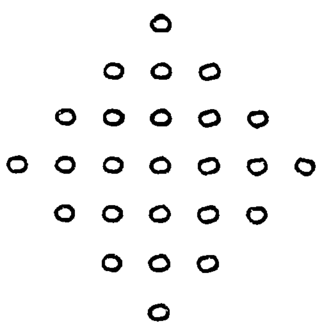
Problem I

How many marbles are there in the picture below?



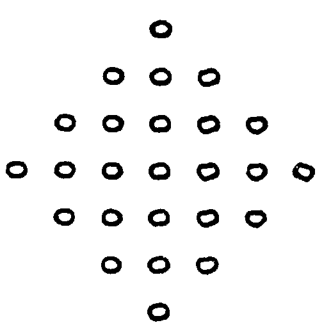
FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your ways of finding the answer and write your answer.

(Way 1)



Ans. ________

(Way 2)

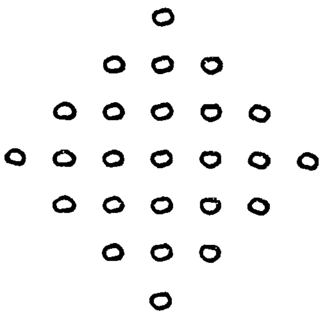


Ans. ________

Continue on the next page.

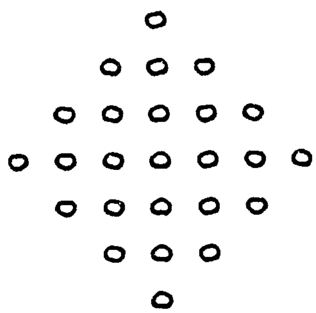
4G3

(Way 3)



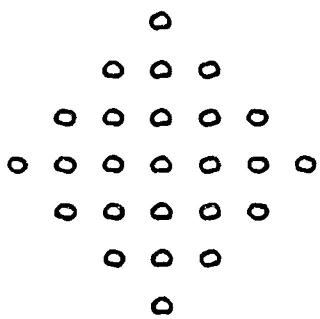
Ans. __________

(Way 4)



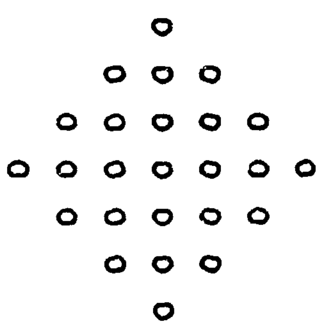
Ans. __________

(Way 5)



Ans. __________

(Way 6)



Ans. __________

If you need more space, turn the page.
Stop working when the teacher says, "Stop."

(Way 7)

```
      O
    O O O
  O O O O O
O O O O O O O
  O O O O O
    O O O
      O
```

Ans. ________

(Way 8)

```
      O
    O O O
  O O O O O
O O O O O O O
  O O O O O
    O O O
      O
```

Ans. ________

(Way 9)

```
      O
    O O O
  O O O O O
O O O O O O O
  O O O O O
    O O O
      O
```

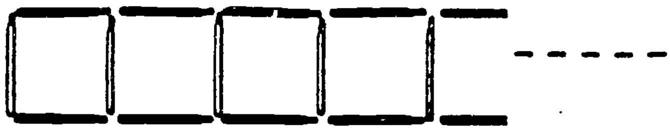
Ans. ________

Please do not turn this page until the teacher tells you to.

Problem II

Squares are made by using matchsticks as shown in the picture below.

When the number of squares is eight, how many matchsticks are used?



DO NOT ERASE ANYTHING YOU WRITE DOWN; JUST DRAW A LINE THROUGH ANYTHING YOU FEEL IS IN ERROR.

(1) Write a way of solution and the answer to the problem above.

Ans. ___________

(2) Now make up your own problems like the one above and write them down. Make as many problems as you can. <u>You do not need to find the answers to your problems</u>.

1

Continue on the next page.

2

3

4

5

If you need more space, turn the page.

---

(3) Choose the one problem you think is best from those you wrote down above, and write the <u>number</u> of the problem in the space: __________

Write the reason or reasons you think it is best.

6

7

8

Please do not turn this page until the teacher tells you to.

4G10

Questionnaire to Students

Grade 4: Name ______________________________

* Please mark your answers to the following questions by putting a ✓ mark in the space.

1. Do you like Math?

   _____ like math _____ neutral _____ dislike math

2. Are you good at math?

   _____ good at math _____ neutral _____ not good at math

3. Do you think that today's problems are interesting?

Problem I (Marble)	Problem II (Matchstick)
_____ interesting	_____ interesting
_____ neutral	_____ neutral
_____ not interesting	_____ not interesting

4. Do you think that today's problems are easy?

Problem I (Marble)	Problem II (Matchstick)
_____ easy	_____ easy
_____ average	_____ average
_____ difficult	_____ difficult

5. Are today's problems the same as the problems in your math textbook?

Problem I (Marble)	Problem II (Matchstick)
_____ the same as	_____ the same as
_____ can't say	_____ can't say
_____ different from	_____ different from

6. In comparison to problems in your math textbook, did you like today's problems?

Problem I (Marble)	Problem II (Matchstick)
____ more ____ the same as ____ less	____ more ____ the same as ____ less

7. Have you seen problems like this before?

Problem I (Marble)	Problem II (Matchstick)
____ yes ____ no	____ yes ____ no

6G1 6th Grade

Name ______________________________ Date ____________________

School ____________________________ Sex: _____Boy _____Girl

Directions: Read each question carefully and follow directions.

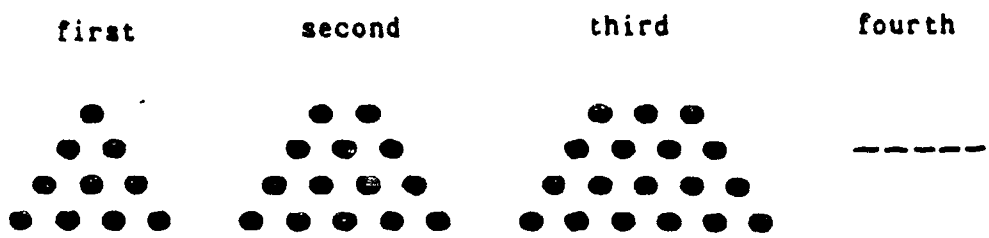
You should write down all your work. Do not erase anything you write down, just draw a line through it rather than erase it.

There are two questions and you will have 15 minutes for each question.

Do not turn the page until the teacher tells you to.

Problem I

Marbles are arranged as follows:



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Method 1)

Ans. __________

(Method 2)

Ans. __________

Continue on the next page.

(Method 3)

Ans. ________

(Method 4)

Ans. ________

If you need more space, write on the back of this page.

(2) How many marbles are there in the sixteenth place?

Show your way of solution and your answer.

Ans. ________

(3) Try to find a formula for finding the number of marbles in the one hundredth place.

Stop working when the teacher says, "Stop."

(Method 5)

Ans. ________

(Method 6)

Ans. ________

(Method 7)

Ans. ________

Please do not turn this page until the teacher tells you to.

Problem II

Squares are made using matchsticks as shown in the picture below.

When the number of squares is eight, how many matchsticks are used?

Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) Write your way of solution and the answer.

Ans. __________

(2) Now make up your own problems like the one above and write them down. Make up as many problems as you can. <u>You do not need to find the answers to your problems.</u>

1

Continue on the next page.

(2)

(3)

(4)

(5)

If you need more space, write on the back of this page.

(3) Choose the problem you think is the best from these you wrote down above, and write the number of the problem in this space: ________

Write down the reason or reasons you think it is the best.

Stop working when the teacher says, "STOP."

6

7

8

150

Please do not turn this page until the teacher tells you to.

Questionnaire to Students

Grade 6: Name ______________________________

* Please mark your answers to the following questions by making a ✓ mark in the space.

1. Do you like Math?

_____ like math _____ neutral _____ dislike math

2. Are you good at math?

_____ good at math _____ neutral _____ not good at math

3. Do you think that today's problems are interesting?

Problem I (Marble)	Problem II (Matchstick)
_____ interesting	_____ interesting
_____ neutral	_____ neutral
_____ not interesting	_____ not interesting

4. Do you think that today's problems are easy?

Problem I (Marble)	Problem II (Matchstick)
_____ easy	_____ easy
_____ average	_____ average
_____ difficult	_____ difficult

5. Are today's problems the same as the problems in your math textbook?

Problem I (Marble)	Problem II (Matchstick)
_____ the same as	_____ the same as
_____ can't say	_____ can't say
_____ different from	_____ different from

6. In comparison to problems in your math textbook, did you like today's problems?

Problem I (Marble)	Problem II (Matchstick)
_____ more	_____ more
_____ the same as	_____ the same as
_____ less	_____ less

7. Have you seen problems like this before?

Problem I (Marble)	Problem II (Matchstick)
_____ yes	_____ yes
_____ no	_____ no

8th Grade

Name ______________________________ Date ____________________

School ____________________________ Sex: _____Boy _____Girl

Directions: Read each question carefully and follow directions

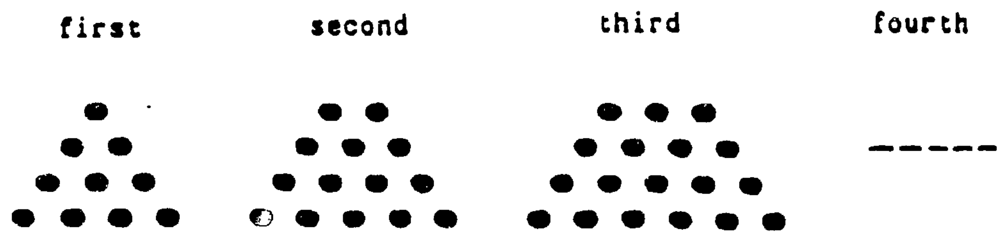
You should write down all your work. Do not erase anything you write down, just draw a line through it rather than erase it.

There are two questions and you will have 15 minutes for each question.

Do not turn the page until the teacher tells you to.

Problem I

Marbles are arranged as follows:



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Way of solution 1)

Ans. __________

(Way of solution 2)

Ans. __________

Continue on the next page.

(Way of solution 3)

Ans. __________

(Way of solution 4)

Ans. __________

If you need more space, write on the back of this page.

(2) How many marbles are there in the sixteenth place? Show your way of solution and your answer.

Ans. __________

(3) Try to find a formula for finding the number of marbles in the hundredth place.

Stop working when the teacher says "STOP."

(Way of solution 5)

Ans. ________

(Way of solution 6)

Ans. ________

(Way of solution 7)

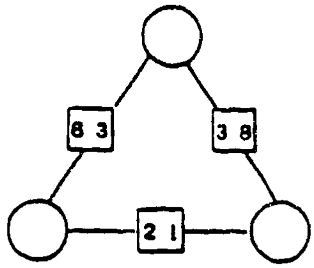
Ans. ________

Please do not turn this page until the teacher tells you to.

Problem II

Given a three-sided arithmogon as in the figure below. We put three numbers in the three □ - the number in each □ must equal the sum of the numbers in the two ○ on either side.

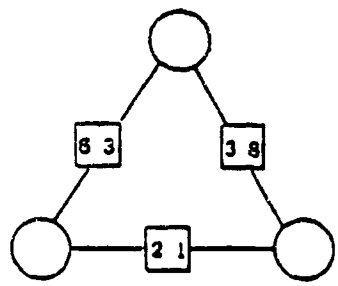
Find the numbers for ○ at each corner. The numbers in ○ may be negative numbers.



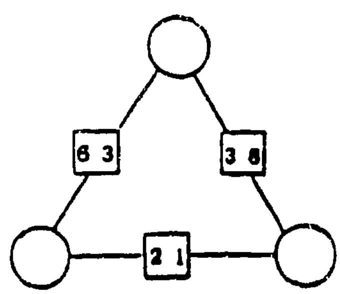
Do not erase anything you write down, just draw a line through anything you feel is in error.

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN.

(Way of solution 1)



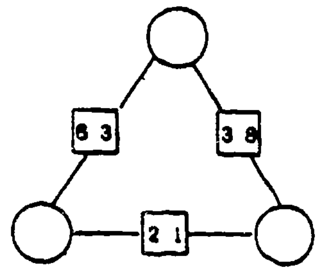
(Way of solution 2)



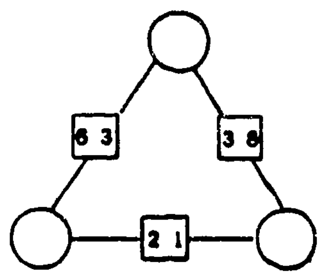
Continue on the next page.

8G7

(Way of solution 3)



(Way of solution 4)



If you need more space, write on the back of this page.

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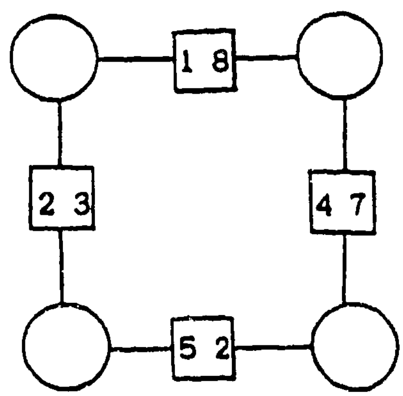
(2) Now change to a square (four-sided) arithmogon as in the figure below. The number in each □ must equal the sum of the numbers in the two ○ on either side.

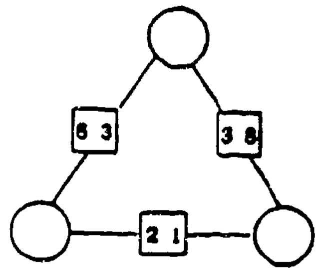
Try to find the numbers for ○ at each corner.



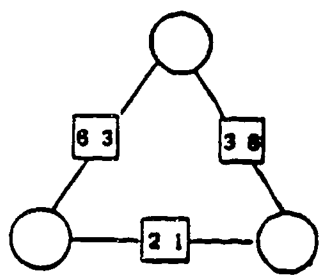
If you need more space, write on the back of this page.

8G8

(Way of solution 5)



(Way of solution 6)



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Please do not turn this page until the teacher tells you to.

Questionnaire to Students

Grade 8: Name ______________________________

* Please mark your answers to the following questions by making a ✓ mark in the space.

1. Do you like Math?

_____ like math _____ neutral _____ dislike math

2. Are you good at math?

_____ good at math _____ neutral _____ not good at math

3. Do you think that today's problems are interesting?

Problem I (Marble)	Problem II (Arithmogon)
_____ interesting	_____ interesting
_____ neutral	_____ neutral
_____ not interesting	_____ not interesting

4. Do you think that today's problems are easy?

Problem I (Marble)	Problem II (Arithmogon)
_____ easy	_____ easy
_____ average	_____ average
_____ difficult	_____ difficult

5. Are today's problems the same as the problems in your math textbook?

Problem I (Marble)	Problem II (Arithmogon)
_____ the same as	_____ the same as
_____ can't say	_____ can't say
_____ different from	_____ different from

6. In comparison to problems in your math textbook, did you like today's problems?

Problem I (Marble)	Problem II (Arithmogon)
_____ more _____ the same as _____ less	_____ more _____ the same as _____ less

7. Have you seen problems like this before?

Problem I (Marble)	Problem II (Arithmogon)
_____ yes _____ no	_____ yes _____ no

Name ______________________ Date ______________________

School ______________________ Sex: _____Boy _____Girl

Directions: Read each question carefully and follow directions

You should write down all your work. Do not erase anything you write down, just draw a line through it rather than erase it.

There are three problems and you will have 15 minutes for Problems I and II, and 10 minutes for Problem III.

Do not turn the page until the teacher tells you to.

Problem I

> Pick a point P on the line segment AB and make squares: one side of one is AP and one side of the other is PB. Where should the point P be located to satisfy the condition that the sum of the areas of the two squares is a minimum?

Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) Write a way of solution and the answer to the above problem.

Ans. ________________

(2) Now make up your own problems like the one above and write them down. Make as many problems as you can. <u>You do not need to find the answers to your problems</u>.

1

Continue on the next page.

2

3

4

5

If you need more space, write on the back of this page.

---

(3) Choose the problem you think is best from those you wrote down above and write the number of the problem in the space:__________

Write the reason or reasons you think it is best.

Stop working when the teacher says, "STOP."

6

7

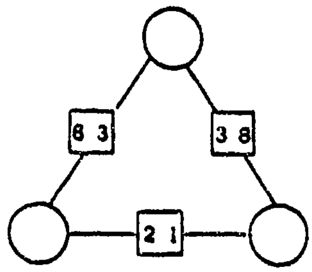
8

Please do not turn this page until the teacher tells you to.

Problem II

Given a three-sided arithmogon as in the figure below. We put three numbers in three □ - the number in each □ must equal the sum of the numbers in the two ○ on either side.

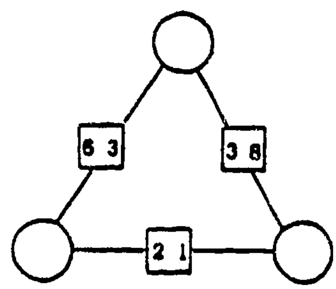
Find the numbers for ○ at each corner. The numbers for ○ may be negative numbers.



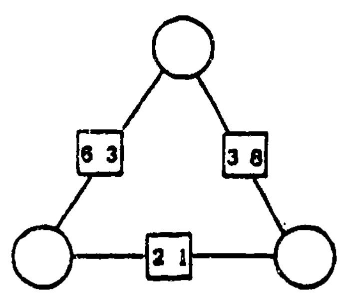
Do not erase anything you write down, just draw a line through anything you feel is in error.

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN.

(Way of solution 1)

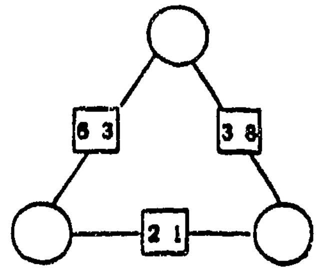


(Way of solution 2)

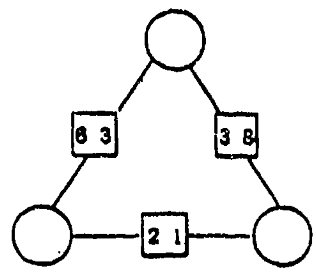


Continue on the next page.

(Way of solution 3)



(Way of solution 4)



If you need more space, write on the back of this page.

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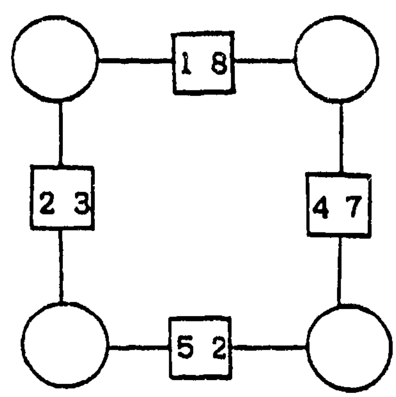
(2) Now change to a square (four-sided) arithmogon as in the figure below. The number in each □ must equal the sum of the numbers in the two ○ on either side.

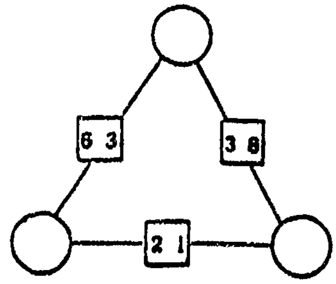
Try to find the numbers for ○ at each corner.



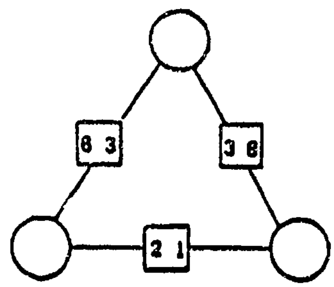
If you need more space, write on the back of this page.

11G8

(Way of solution 5)



(Way of solution 6)

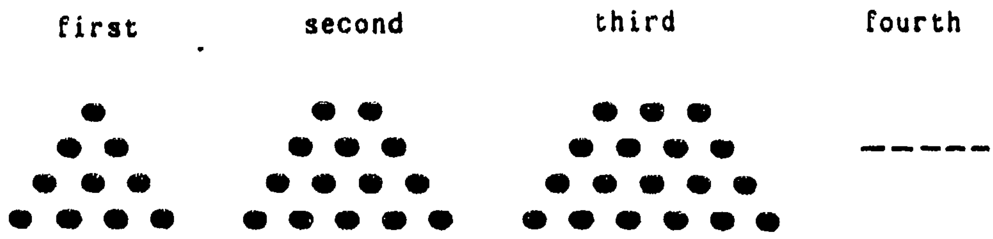


Please do not turn this page until the teacher tells you to.

11G10

Problem III

Marbles are arranged as follows.



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(Way of solution 1)

Ans. __________

(Way of solution.2)

Ans. __________

Continue on the next page.

(Way of solution 3)

Ans. ________

(Way of solution 4)

Ans. ________

If you need more space, write on the back of this page.

(2) How many marbles are there in the sixteenth place? Show your way of solution and your answer

Ans. ________

(3) Try to find a formula for finding the number of marbles in the n-th place.

If you need more space, write on the back of this page.

(Way of solution 5)

Ans. ________

(Way of solution 6)

Ans. ________

Please do not turn this page until the teacher tells you to.

Questionnaire to Students

Grade 11: Name ____

* Please mark your answers to the following questions by making a ✓ mark in the space.

1. Do you like Math?

_____ like math _____ neutral _____ dislike math

2. Are you good at math?

_____ good at math _____ neutral _____ not good at math

3. Do you think that today's problems are interesting?

Prob. I (Minim. Area)	Prob. II (Arithmogon)	Prob. III (Marble)
_____ interesting	_____ interesting	_____ interesting
_____ neutral	_____ neutral	_____ neutral
_____ not interesting	_____ not interesting	_____ not interesting

4. Do you think that today's problems are easy?

Prob. I (Minim. Area)	Prob. II (Arithmogon)	Prob. III (Marble)
_____ easy	_____ easy	_____ easy
_____ average	_____ average	_____ average
_____ difficult	_____ difficult	_____ difficult

5. Are today's problems the same as the problems in your math textbook?

Prob. I (Minim. Area)	Prob. II (Arithmogon)	Prob. III (Marble)
_____ the same as	_____ the same as	_____ the same as
_____ can't say	_____ can't say	_____ can't say
_____ different from	_____ different from	_____ different from

6. In comparison to problems in your math textbook, did you like today's problems?

Prob. I (Minim. Area)	Prob. II (Arithmogon)	Prob. III (Marble)
_____ more	_____ more	_____ more
_____ the same as	_____ the same as	_____ the same as
_____ less	_____ less	_____ less

7. Have you seen problems like this before?

Prob. I (Minim. Area)	Prob. II (Arithmogon)	Prob. III (Marble)
_____ yes	_____ yes	_____ yes
_____ no	_____ no	_____ no