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ABSTRACT

This study was designed to determine students' abilities and difficulties in articulating the relationship between function and derivative. High-school students were presented 15 problems during two 75-minute interviews in which they were asked to construct functions experimentally in three different contexts: motion, fluids, and number-change. In each of the environments the students utilized tools, ranging from computer software to manipulative materials, that enabled them to generate functions. Analysis of the interviews indicated that students had the tendency to assume resemblances between the behavior or appearance of the function and its derivative in all three contexts. Three types of cues that activated resemblances in the predicted functions were identified: syntactic, semantic, and linguistic. Approaches that students used to move from the derivative to the function and vice-versa are described. A case study of one 17-minute episode of a student working with an airflow device describes the evolution of one such approach. The report concluded that the construction of such an approach was complex and that tools to explore the mathematical ideas helped frame the interviewer/student discourse through which the approach was developed. (MDH)

Students' Tendency to Assume Resemblances Between a Function and its Derivative

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Students' Tendency to Assume Resemblances Between a Function and its Derivative

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Contents

Preface

TERC is a nonprofit education research and development organization founded in 1965 and committed to improving science and mathematics learning and teaching. Our work includes research from both cognitive and sociocultural perspectives, creation of curriculum, technology innovation, and teacher development. Through our research we strive to increase knowledge of how students and teachers construct their understanding of science and mathematics.

Much of the thinking and questioning that informs TERC research is eventually integrated in the curricula and technologies we create and in the development work we engage in collaboratively with teachers. Traditionally, TERC staff present their research at conferences and report their studies in journals. By launching the TERC Working Papers series, we hope to expand our reach to the community of researchers and educators engaged in similar endeavors.

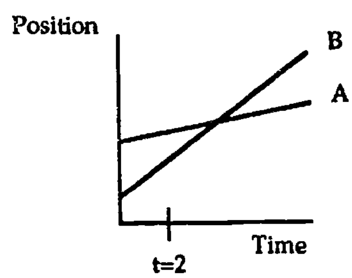
The TERC Working Papers series consists of completed research, both published and unpublished, and work-in-progress in the learning and teaching of science and mathematics. We are introducing the series with four papers and will add papers at regular intervals.

Introduction and Methodology

The relationship between a function and its derivative is a central theme of calculus. As such, it is closely connected to some concepts that are normally included in formal calculus courses, such as limits, continuity, and convergence. Although mathematically complex, the function/derivative relationship is embedded in many contexts of daily life, which allows us to construct intuitions from an early age. These intuitions allow students who have never taken calculus to think about, predict, and discuss situations involving functions and their derivatives.

Many studies have described students' abilities and conceptual difficulties in dealing with such problems (Clement, 1985; McDermott, Rosenquist, & van Zee, 1987; Monk, 1990). Particularly relevant for this paper is the so-called "height/slope misconception" in which the student seemingly confuses the slope of a line with its height in a Cartesian graph. Figure 1.1 shows a typical problem.

Figure 1.1



Question: Which car is going faster at t=2?

Student's answer: A

The correct answer to the problem is B, since the slopes of the lines represent the velocities. Some authors (Clement, 1985; Janvier, 1978; McDermott et al., 1987) distinguish two possible sources for this mistake: a representational one (e.g., "higher" in a Cartesian graph means "more," or the slope of a line is not meaningful to the student) and a conceptual one (position and velocity are not adequately distinguished). Another relevant distinction, elaborated by Monk (1990), is between point and across-time analyses of a function. The problem above can be seen as an across-time problem, in the sense that it might require the student to visualize the variation of position over time for each car when t is close to 2. Monk hypothesizes that students might instead be biased toward a point analysis, according to which the answer ought to be a comparison of point-values on the two curves corresponding to t=2.

This paper is part of a study of students' abilities and difficulties in articulating the relationship between function and derivative. However, the problems we worked with were posed in a different context from those described in the literature. We provided students with one of three contexts in which they could construct functions experimentally: motion, fluids, and numerical integration.

In each of the three environments students had a set of tools that enabled them to generate functions and thereby explore their conjectures about the shape of a function. For both motion and air flow, students could produce curves on a computer screen by manipulating a physical object monitored by a sensor. For motion, students worked with a small car and a motion detector to generate curves of position and velocity versus time. For air flow, students controlled a bellows to generate changes in air flow and in the volume of air accumulated in a bag.

For numerical integration, we developed a software environment in which a function is generated by accumulating the numerical values of another function. The functions can be displayed in either tables or graphs. The following shows a simple case, in which B is a constant function and A (which is accumulating the values of B) is a linear one. In Figure 1.2, the independent variable, called X, simply counts the number of accumulations.

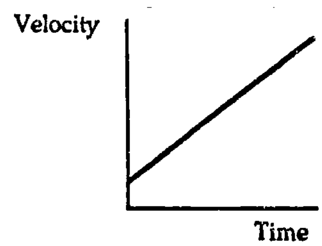
Figure 1.2

X	A	B	
0	8	3	(Initial values)

X	A	B
0	8	3
1	11	3
2	14	3
..	..	..

We presented parallel problems in each environment. The example in Figure 1.3 was one of the problems presented in the context of motion.

Figure 1.3



Question: If this is a graph of the velocity of a car versus time, what would be the corresponding graph of position versus time?

The same problem, in the context of air flow, was worded as:

Question: If this is a graph of the flow rate of air versus time, what would be the corresponding graph of volume of air versus time?

Finally, in the context of numerical integration:

Question: If this is a graph of B versus X, what would be the corresponding graph of A versus X?

All the students in our study were high school students who had not taken calculus courses. We carried out the teaching interviews with individual students, videotaping the session with two cameras, one for the computer screen and another for the overall action. The results analyzed here are based on interviews with six students, two using each environment.

The sequence included 15 different problems, presented in two teaching interviews of 75 minutes each. Usually the interviewer posed a problem, the student formulated a certain prediction by drawing it on a pad, the corresponding functions were generated experimentally by the student (frequently necessitating several trials), and the results were discussed. While we began by following the interview format, the session often took unexpected turns as the interviewer tried to follow the student's thinking.

The current analysis of the teaching interviews focuses on what we call learning episodes; that is, episodes during which the student changed her view or adopted a different way of thinking. Our analysis strives to investigate the critical elements in that transition and why it took place (Nemirovsky & Rubin, 1991; Rubin & Nemirovsky, 1991).

Three broad assumptions are important for our analysis:

1. Every normal human being, from early stages in life, has some intuitive knowledge about the relationship between function and derivative. Whether this

is manifested as knowledge about position and velocity, level and flow, or number and number-change, we construct complex bodies of knowledge that enable us to make sense of situations involving change (Piaget, 1970; Piaget, Grize, Szeminska, & Vinh, 1977).

2. The relationship between function and derivative is one of those notions that always remains open to further elaboration, with new and unresolved issues involving the fundamental nature of space, time, and number.

   These two assumptions help us dismiss simplistic explanations for students' performance, such as "the student does not distinguish position and velocity" or "the student has finally grasped the concept of position and velocity." In fact, we believe that all humans have some notion of the relationship between position and velocity (or other function-derivative pairs), but that none has a complete understanding.

3. Students' performance in solving problems involving the function/derivative relationship is strongly affected by contextual parameters. The same problem, from the point of view of its mathematical structure, may elicit very different responses if it is posed as a position/velocity problem or as a volume/flow rate problem. Different representational systems make some aspects of the situation more salient than others, and access to particular measurement devices has a strong influence on students' thinking (Monk, 1990). This is an important aspect of our study, and we plan to develop a detailed analysis accounting for some of these contextual factors. As an example, we have noted that students tend to think of "position of the car" as "distance travelled," not as a "relative position." This perception makes it difficult for them to accept functions involving decreasing positions or negative velocities. On the other hand, there is no equivalent intuition to "distance travelled" in the context of air flow, so negative air flow is a more acceptable concept to students.

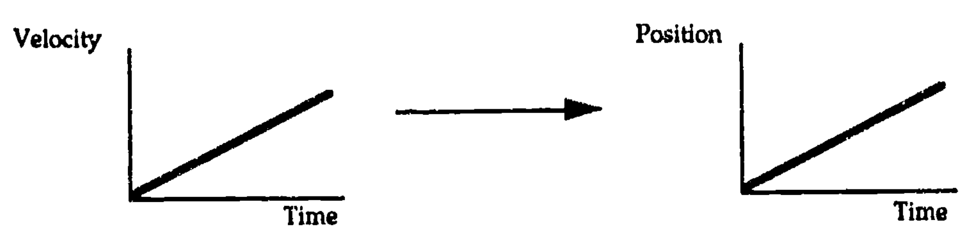
In the next section we examine a particular tendency that appeared repeatedly in all three contexts: students' tendency to assume resemblances between the behavior or appearance of a function and its derivative. We describe how this tendency was manifested by students solving different problems. Many learning episodes show how the student overcomes some of the assumed resemblances by constructing an alternative frame for understanding the relationship between a function and its derivative. To illustrate this, we provide a case analysis of a learning episode with one student working with air flow.

Assumptions of Resemblance and the Variational Approaches

From a perceptual point of view, the graph of a function may be characterized by several attributes, such as being increasing or decreasing, having straight or curved contours, and crossing or not crossing the horizontal axis. *Assumptions of resemblance* are premises that the graphs of two different functions will be perceived as having common attributes. Such a pair of graphs does not need to share all perceptual attributes, only those on which the characterization of resemblance is based at the moment. For example, an assumption of resemblance with respect to straight and curved contours leaves open the possibility that they do not match on the attribute of increasing/decreasing. Assuming that two graphs are both increasing does not determine whether each is a straight line or a curve.

An assumption of resemblance reflects an informal taxonomy of graphs based on the attributes that participate in the resemblance. Thus, these spontaneous and implicit taxonomies can indicate what graphical attributes the student considers relevant at the moment. For example, a student making the prediction described in Figure 1.4 might be expressing his conviction that the position curve has to be increasing, regardless of its shape. The fact that he draws a straight line may not show his belief that it *has* to be a straight line: it happens that a straight line is for him the simplest way to indicate "going up."

Figure 1.4



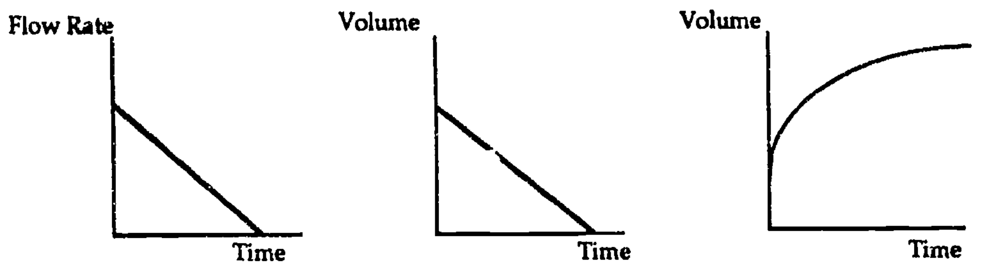
Assumptions of resemblance are implicit, but often easily elicited. Asking a student about the assumption is often enough to bring out his belief, and sometimes even enough to make him question it. For example, a student who consistently assumes that straight lines on velocity graphs generate straight lines on position graphs may find the following question disturbing: Are you saying that the position graph has to be a straight line because the velocity graph is a straight line? Sometimes explicitly stating or questioning an assumption leads a student to wonder if there is a real basis for it.

Types of Resemblances

Assumptions of resemblance show up in a variety of ways. The following examples of students using incorrect resemblances to solve problems occurred when students were predicting the graph of a function from that of its derivative (Figure 1.5 to Figure 1.11).

1. Simple replication (the predicted graph is identical to the original graph) (Figure 1.5)

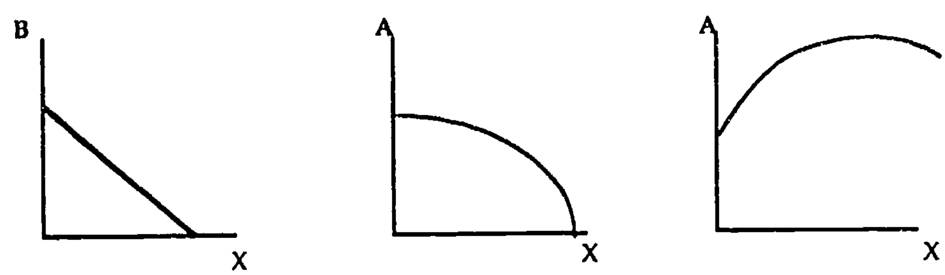
Figure 1.5



Original source graph *Student's prediction* *Possible correct graph*

2. Same direction of change (e.g., increasing derivatives correspond to increasing functions, and decreasing derivatives correspond to decreasing functions) (Figure 1.6)

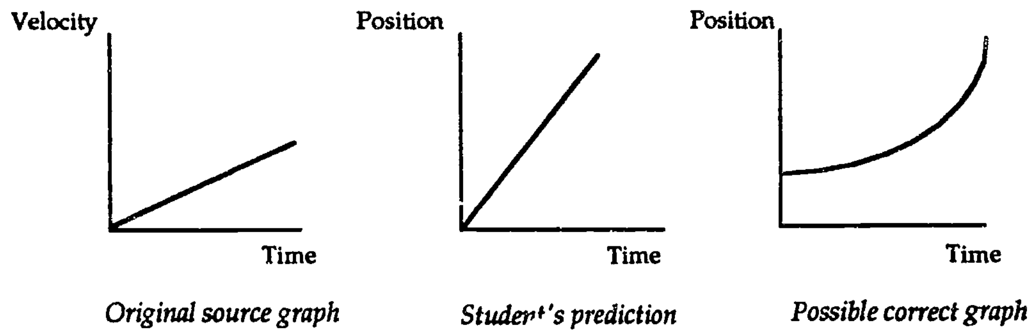
Figure 1.6



Original source graph *Student's prediction* *Possible correct graph*

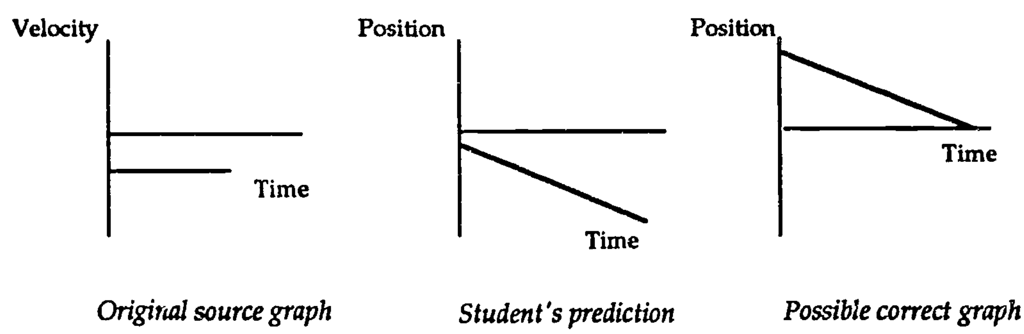
3. Same shape (e.g., straight lines correspond to straight lines) (Figure 1.7)

Figure 1.7



4. Same sign (graphs above the x-axis generate graphs above the x-axis and vice versa) (Figure 1.8)

Figure 1.8

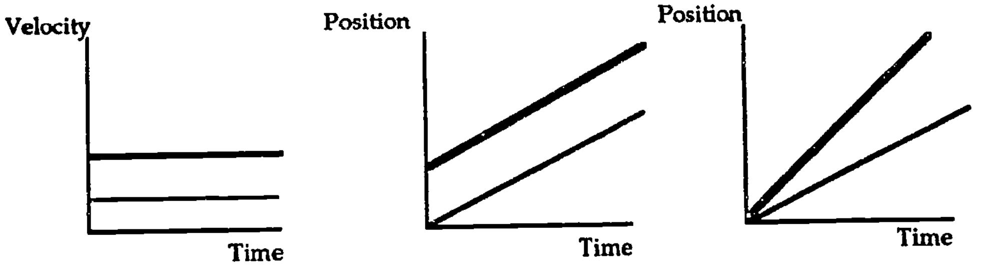


(Note that in our experimental set up, positions are always positive.)

5. Same geometrical transformation (Figures 1.9, 1.10, and 1.11)

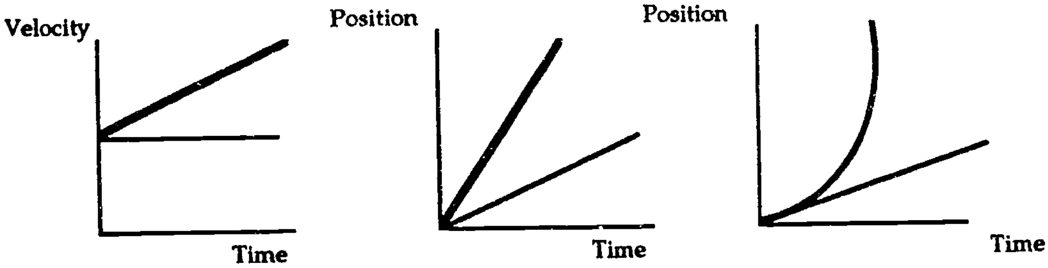
   Students using resemblance also have a method for inferring from two velocity graphs and one position graph what the second position graph should be. In these cases a change in the graph of the velocity is thought of as producing a similar change in the graph of position. In the following examples the thin line on both graphs had already been verified by the student and she was predicting the shape of position graph corresponding to the velocity function marked with a thick line.

Figure 1.9: Translation



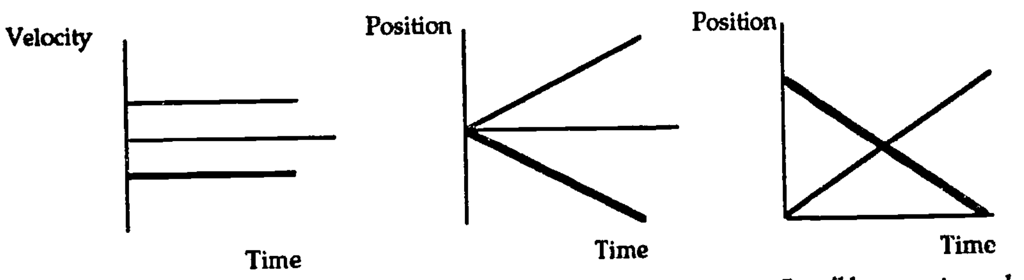
Original source graph *Student's prediction* *Possible correct graph*

Figure 1.10: Rotation

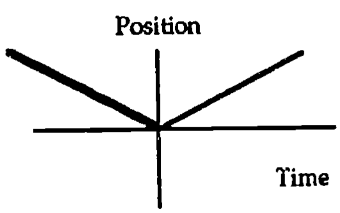


Original source graph *Student's prediction* *Possible correct graph*

Figure 1.11: Reflection



Original source graph *Student's prediction* *Possible correct graph*



Also Student's prediction

(One might argue that the student's first solution in Figure 1.11 is correct because she assumed that the initial positions of the two cars were the same. However, our experimental apparatus does not allow for negative positions since the position of the sensor is 0, and it measures in only one direction. It was also clear from the accompanying conversation that the student based her prediction on preserving symmetry, an assumption that led her to consider the second solution, even though it involved negative times.)

In all these cases the change in the velocity graph is seen by the student as a geometric transformation: parallel translation (Figure 1.9), rotation (Figure 1.10), or reflection (Figure 1.11), and the prediction for the corresponding position graph is accomplished by applying the same geometric transformation to the first position graph.

Cues for Resemblances

Resemblances give students tools for making sense of a complex situation. Students probably do not adopt resemblances because they have solid reasons to believe the tools are appropriate, but rather because the tools enable them to organize and solve a bewildering domain of problems. The choice to use resemblance is not made blindly. We believe this choice emerges from an interplay between expectations and cues. Students' expectations that the graphs of position and velocity (e.g., volume and flow rate) for the motion of the same object (the level of the same container) will show similarities derive from the perceived close connection between these quantities. They both describe the behavior of the same object over the same time period, so students have a general, probably tacit, expectation that the graphs will be strongly related. These expectations are validated, challenged, or even raised by the students' recognition of cues. Particular cues prompt the student to use specific resemblances. We have identified several types of cues, deriving from the situation or the student's background knowledge, that activate certain resemblances and support their use as appropriate or plausible. We distinguish three types of cues: syntactic, semantic, and linguistic.

Syntactic cues

Syntactic cues are distinguished by the fact that they are based on graphical features, unrelated to the student's knowledge of motion or air flow. These cues are compelling to students not so much because they shed light on resemblances but because they support a strong feeling of making sense of the problem. Their simplicity and replicability have an inexorable attraction. For example, assuming that geometric transformations are preserved through the function-derivative translation (Figures 1.9, 1.10, and 1.11) seems to make solving such problems relatively simple: find another, simpler situation where the position graph is known, determine the geometric transformation, and apply it to the original position graph. This procedure is so simple and "reliable" that it does not seem strange that students who recognize its potential power will strive to maintain it, even after some contradictions arise in particular cases.

Semantic cues

Semantic cues elicit students' ideas that the function and its derivative behave similarly on the basis of real-world knowledge. Since their reactions are overgeneralizations of experiences in which a function and its derivative actually do vary in the same way, we call this strategy "isomorphic variation." It has appeared quite frequently in our interviews. The student is aware that the function and the derivative are different entities, but assumes that they change in a similar way. Isomorphic variation (the assumption that a function and its derivative vary together) may be rooted in some powerful intuitions derived from common experiences in daily life. For example, the common experience that going faster implies traveling further sometimes results in the overgeneralization that velocity and position always move in the same direction, either both increasing or both decreasing.

We have repeatedly observed how students who are able to predict position from velocity in the problem described in Figure 1.12 have difficulty with the problem described in Figure 1.13 because the derivative is decreasing while the function is increasing. In the second example, students' assumption that functions and derivatives increase and decrease together leads them down the wrong path.

Figure 1.12

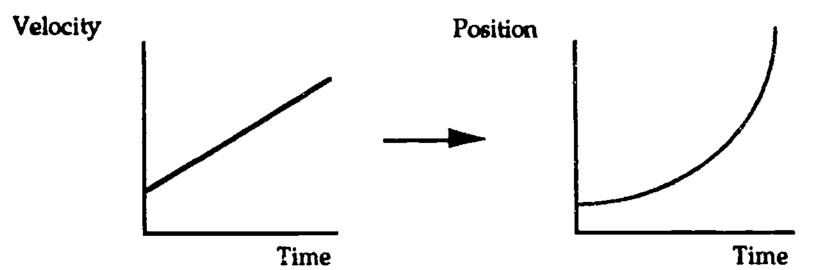
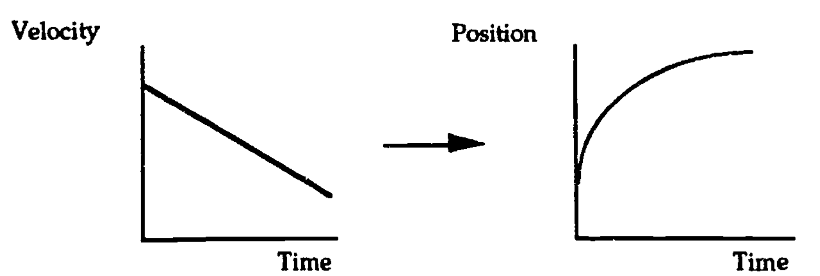


Figure 1.13



In using isomorphic variation the student significantly constrains the range of possible graphs, but still leaves room for many possibilities. For example, given a graph for a function of velocity, as described in Figure 1.14, any of the (incorrect) predictions for the graph of position shown in Figure 1.15 could be generated using isomorphic variation (i.e., more is more and less is less).

Figure 1.14

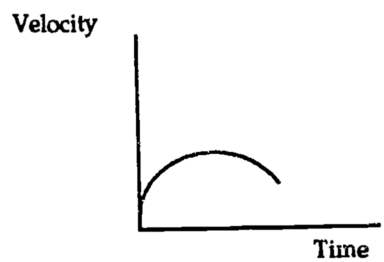
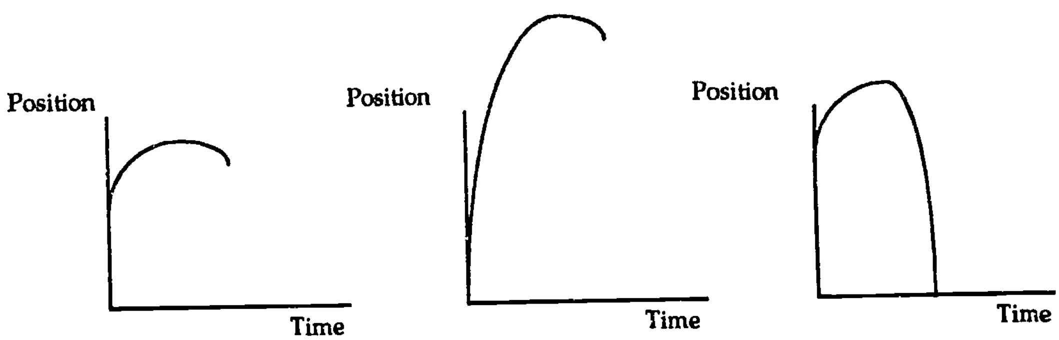


Figure 1.15



Each of these position graphs increases as the velocity graph increases and decreases as it decreases, but other graphical elements such as initial value, amount of increase, and amount of decrease may not match.

Linguistic cues

Linguistic cues are ambiguities of language that support resemblances between a function and its derivative or a function and its indefinite integral. We will analyze two examples: the uses of more/less and up/down.

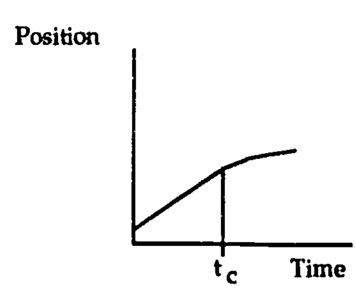
There is a common tendency to think that more velocity (flow rate) means more distance (volume) and that less of the first means less of the latter, as if they always change in the same direction. In part this may have to do with an ambiguous use of the words "more" and "less." We often say, "the more flow rate, the more volume" and we may similarly say that "the less flow rate, the less volume," meaning that less volume would result from a smaller flow rate than would accumulate with a greater flow rate. However, the same sentence can also be interpreted (incorrectly in this case) as meaning that a decreasing flow rate implies a decreasing volume.

The ambiguity is caused by the fact that we can use the comparative words "more" and "less" in comparing two parallel events or two successive values of a single event. For example, let us imagine two cars, A and B. It is true that less velocity for B than for A implies less distance travelled for B than for A (all other things being equal). But if we consider a single car A, it is not true that less velocity now than earlier implies that we have travelled less distance now than earlier.

However, a student may not be aware of this distinction when he says, "the faster the car, the more distance," and he may thus confuse change over time with the comparison of two parallel events. This confusion may lead to an incorrect assumption that decreasing velocity implies decreasing distance. Students who are aware of the difference sometimes use the expressions "more and more" or "less and less" to render those two meanings less ambiguous, since they make more explicit the reference to the process of a single event over time with the repeated adjective.

Another example of language ambiguities that may function to support the use of resemblances is the words "up" and "down." On the one hand, "up" and "down" are used in phrases to indicate increase and decrease respectively, such as in "speeding up" or "slowing down." On the other hand, "up" and "down" denote opposite spatial directions that can be used in describing curves on a graph or directions in the air flow apparatus. This ambiguity may underlie the assumption of certain resemblances. For example, a student who is sketching a curve of position versus time for a car that goes at a constant speed and, at time t_c, begins decreasing its speed may produce the description "at time t_c the car begins to slow down," eliciting the intuition that, after t_c, the curve should go down (see Figure 1.16), as would be the case in a graph of velocity versus time.

Figure 1.16



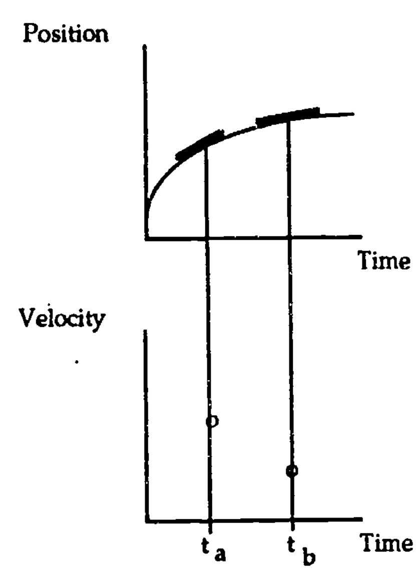
Using resemblance leads to a particular approach to solving problems of translation between a function and its derivative. It enables the student to make sense of the situation and to formulate a prediction for the solution. As we have shown in the examples, using resemblance often leads to mistaken predictions, but it is still a compelling and functional technique for many students.

Students often cling to this approach in the face of contradictory evidence because they lack a coherent alternative approach to figuring out a function from its derivative or vice versa. Through the learning episodes students often begin to make use of alternative approaches that focus on how one function (the derivative) describes the local variation in the other. Because they incorporate variation, we call such approaches *variational approaches* to understanding the relationship between a function and its derivative.

Variational Appproaches

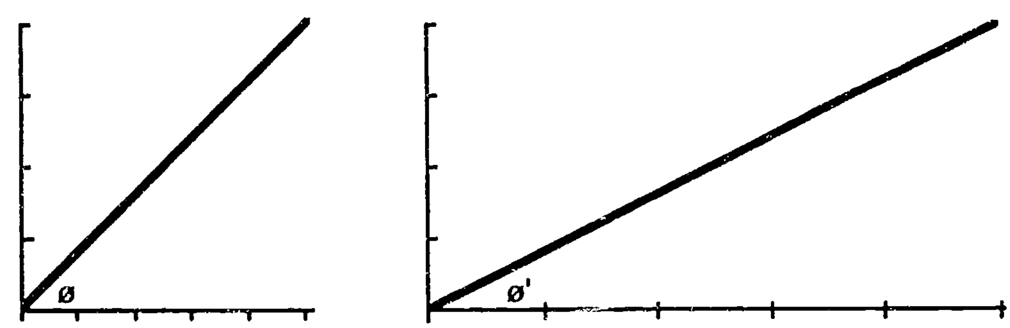
We will distinguish several related variational approaches by the mathematical notions the student uses in describing the local variation of a function. One of these is an approach based on the student's perception of the "steepness" of a graph. Steepness is a graphical-perceptual feature. Looking at steepness lets students make qualitative comparisons among points on the same curve. For example, variational reasoning based on steepness would produce the prediction, for Figure 1.17, that the velocity at t=a must be greater than the velocity at t=b because the curve is steeper at a than at b.

Figure 1.17



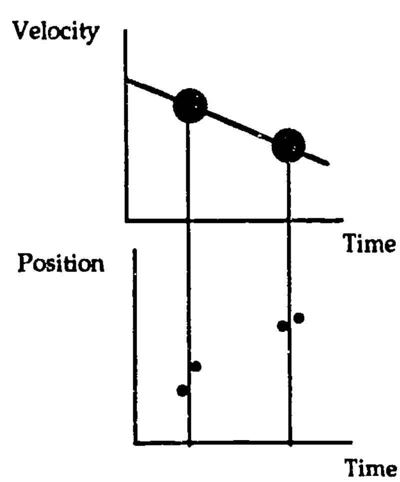
A different mathematical entity that can support a variational approach is the *slope* of a curve. The slope of a curve is the rate of change in y with respect to the change in x. Because a change in the length of the units of either variable in a graph affects the graphical steepness, a given slope may show up graphically as a curve that is more or less steep. Similarly, a certain steepness on a graph may correspond to infinitely many slopes. Figure 1.18 is an example of two lines with the same slope but different steepness ($\o \neq \o'$).

Figure 1.18



Students use a third variational approach when they are trying to predict a graph of a function from one of its derivative. This approach is based on what we call "local qualitative accumulation." A student using this approach recognizes that a positive velocity determines comparatively how much local distance a car gains over a short corresponding time. In the graph in Figure 1.19, a student would be able to say that the two points on the bottom left are further apart than those on the right.

Figure 1.19



There are other examples of variational approaches (such as the one based on the area under a curve), and each one of them involves specific ways of framing the translation between function and derivative or back. A variational approach involves two key pieces of knowledge that are not part of a resemblance approach:

1. How to construct a global shape out of many instances of local variation. A resemblance approach regards the entire graph as one entity, in the sense that what really counts are the global attributes of its shape (e.g., increasing, curved). A student without enough experience with the mathematical entities that enable one to describe the local variation of a function, such as slope or area under the curve, will be particularly prone to use resemblances.

2. How to focus on the *relationship* between the function and its derivative, rather than on each one of them separately. A resemblance approach makes it difficult for the student to think simultaneously about the function and its derivative; the student tends to focus on one or the other, except for the projection of the resemblances themselves. Even if the problem is focused on the relationship between the two functions, the student will not be able to think productively about the relationship. Often this gives the impression that the student does not even distinguish between position and velocity, or between volume and flow rate, because she tends to center, indiscriminately from a naive observer's point of view, on only one of them at a time. In fact, she is able to describe each individually; but she does not have access to conceptual tools to focus on the relationship between them.

The construction of a variational approach is a complex process, involving the coordination of many different pieces of knowledge. It takes place over a long period of time and includes both leaps forward and steps backwards. Often a student seems to have constructed elements of a variational approach but in solving a more complex problem regresses to using non-variational techniques, such as those based on resemblance.

Students move toward a variational approach in many different ways. Often a student will revise an earlier prediction that had been based on resemblance and clearly articulate why the resemblance is not appropriate. However, this same resemblance strategy may reappear later. It is clear that students' learning, at least in our teaching interviews, is not a progressive sequence of "getting" (or "not getting") one idea after another. Some students begin the second session repeating exactly the same predictions that they had made and revised during the first. However, because their former experience is within reach, students usually reconsider quickly and reconstruct ideas that were initially difficult for them to develop.

We do not consider the use of resemblances a matter of "confusion" in the sense that students cannot discriminate between volume and flow rate. There are several reasons why using resemblances seems like a promising approach for many students, such as the following:

1. Often it works! That is, in many cases a function actually does resemble its derivative (e.g., uniform velocity motion produces velocity and position curves that are both straight lines; positive accelerated motion may produce velocity and position curves that are increasing); the student may overgeneralize from these cases.

2. In using graphical resemblances the student gets a feeling of mathematical power by making sense of the problem through constraining possibilities and recognizing patterns.

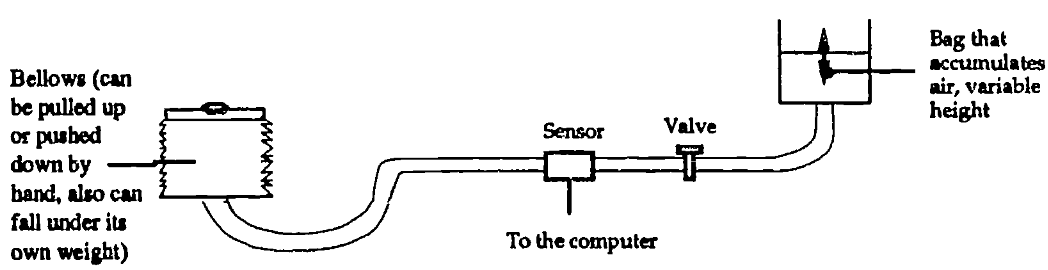
3. Ambiguities of language may elicit and support notions of resemblance.

4. The student may not have access to a variational approach as an alternative way to think about the problems. When the learning situation or the student's previous knowledge do not help to induce such knowledge, the student resorts to more graphical-oriented thinking, assuming partial resemblances between a function and its derivative.

Case Study: From Resemblances to the Beginning of a Variational Approach

To investigate these analyses further we chose a 17-minute learning episode of a student working with an airflow device, during which he began to acquire a variational approach. The basic structure of the apparatus used during the interviews for experiments with air flow is described in Figure 1.20.

Figure 1.20



The flow of air can be controlled in two ways: either by pushing and pulling the bellows with the valve open or by closing and opening the valve while the bellows is released and moving down under its own weight. The valve is a more accurate way of controlling the flow but the limitation is that it only controls air coming into the bag, not air leaving the bag. This is because the top of the bellows is heavy, and unless it is held up by the student, it drops, pushing air into the bag, as regulated by the valve.

We have divided the episode into six segments. The student, whom we call Dan, was an 11th grader, considered a good math student, had taken algebra, and was currently taking physics, but had not taken calculus. Through the episode we focus on how Dan manifested his use of resemblance between the functions of flow rate and volume, and how he began to move toward a variational approach. We view the learning episode as a joint interviewer-student construction of knowledge and mutual understanding. We trace how both the student's and interviewer's notions of the problem differed at several points, eliciting the need for them to negotiate its definition.

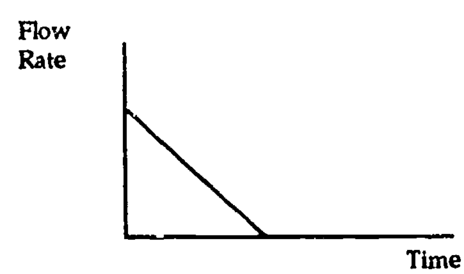
This episode took place during the second session, in which Dan worked with flow rate. We had worked before with problems of constant as well as linearly increasing flow rate.

Segment 1

Description

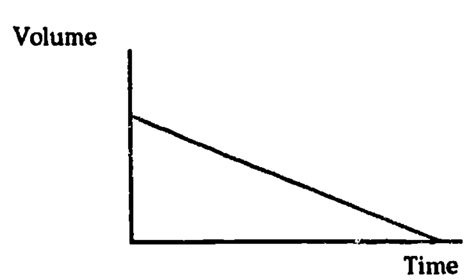
The learning episode begins when the interviewer asks about the case shown in Figure 2.1 below:

Figure 2.1



Dan sketches his prediction for volume (Figure 2.2).

Figure 2.2



He draws a volume line with a different slope but starting at the same point on the y-axis, which he indicates by labeling the point as "8" on the volume graph, then going back to the flow rate graph and indicating its intersection with the y axis as "8" as well.

The interviewer suggests that Dan try to produce the graph of flow rate experimentally. Dan tries to decide whether it is possible to perform the experiment with the valve instead of with the bellows. With this apparatus, only problems of inflow (positive flow rate) can be controlled with the valve because the weight of the bellows will push air into the bag. But in drawing his volume prediction, Dan has indicated that he thinks that this is a problem of outflow:

> Dan: First we have to get the flow rate like that [pointing to the flow rate graph], that means I should just pull this [the bellows] up the whole time. [so that air goes out of the bag and volume diminishes] Is there a way to do it [with the valve]? . . . Can't be . . . you know, there's no way to do that.

Actually, Figure 2.1 shows an inflow (positive flow rate), but Dan's prediction in Figure 2.2 portrays an outflow (volume decreasing). The interviewer, trying to bring these two conceptions of the problem together, calls Dan's attention to the fact that the flow rate is positive, but Dan does not note the discrepancy:

> Interviewer: Now, the flow rate is positive.
>
> Dan: It starts at positive, and then it goes down to zero. [pointing to the height of the bag which shows volume]
>
> Interviewer: The flow rate?
>
> Dan: Yes.

As he starts to measure flow rate, Dan sets the bag half full of air, with an initial volume of 12.5 liters. When the measurements for flow rate begin to appear on the screen, he notices that they start from zero:

> Dan: Oh. It had, it has to start at 12.5, it can't start at zero.
>
> Interviewer: But this is flow rate.
>
> Dan: All righty. I get confused . . . Oh. This can't work, then.

Analysis

Dan constructs his prediction by assuming several resemblances between the given curve of flow rate and his predicted graph of volume: same initial point, both straight lines, decreasing, and positive. The slope, however, is different. Dan does not assume that both graphs have to "diminish" (as he says) in the same way.

After he generates his prediction of the volume graph, Dan's behavior suggests a particular appropriation of the problem: He focuses on his predicted graph of volume, as if the graph of flow rate is irrelevant, and strives to create, with the apparatus, his predicted graph. That is, what was originally posed as a prediction from a certain function of flow rate versus time to a function of volume versus time, is transformed by Dan to the task of producing his predicted function of volume versus time—without questioning whether his translation was correct. Other pieces of evidence for this interpretation of the problem are Dan's judgment that this is a problem of outflow, or decreasing volume; his expectation that the initial value on his generated graph must correspond to the initial volume he set in the program; and his gesturing at the top of the bag (which is a measure of volume) when describing the graph of flow rate he is generating.

We think that this tendency to focus exclusively on either the function or the derivative, except at the time during which the graph characteristics are transferred, is typical of a resemblance approach. Dan knows that volume and flow rate are different measures (his prediction for volume does not have the same slope as the posed flow rate), but he has difficulty focusing his thinking on the relationship between them. Rather, he focuses on one of them (volume), perceiving the other as a piece of the background, and, as a consequence, transforming his conception of the problem. He perceives the mismatch between his focusing on volume and the measurements of flow rate as a state of confusion ("I get confused").

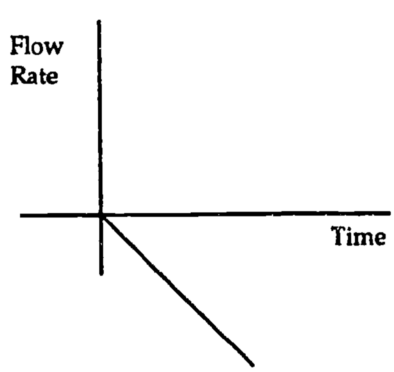
Segment 2

Description

Dan looks at the function of flow rate shown in Figure 2.1. Considering it along with his prediction of volume (Figure 2.2), he recognizes that Figure 2.1 is inconsistent with his prediction of volume. He then goes on to construct a graph of flow rate that would be consistent with his prediction (see Figure 3.1).

> Interviewer: It [the flow rate] cannot be like that [Figure 2.1]?
>
> Dan: No, 'cause then [with Figure 2.1] you can't get the volume to go, diminish, like this [Figure 2.2]. This [Figure 2.1] can't be like that. This would be different. Yeah, it has to be, I think it has to be like this [preparing the axis for a new graph]. Let me think. It has to be like this [Figure 3.1]. To start here, flow rate. Flow rate. This [flow rate] has, it has to start here at zero and to get the, to get that [Figure 2.2] it has to start here [at zero] and I believe has to just diminish:

Figure 3.1



The interviewer returns to the problem of generating the graph in Figure 2.1 (ignoring, for the moment, Dan's volume graph):

Interviewer: Well, just to get this [Figure 2.1].

Dan: Oh, to get that?

Interviewer: Yes.

Dan: You can't get that.

Interviewer: You cannot?

Dan: Can never get that to show up.

Interviewer: Mm hm.

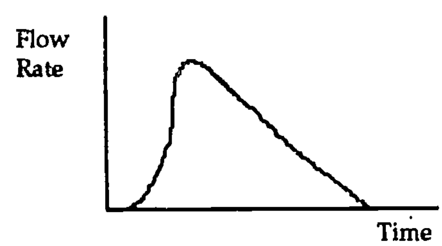
Dan: Doesn't, the flow rate always starts at zero. It never starts at eight.

Interviewer: Okay.

Dan: Unless there's some way to do it that I don't know about.

Dan is now considering another reason for the impossibility of the graph of flow rate (Figure 2.1): the flow rate must start at zero. In order to make the problem acceptable for Dan, the interviewer produces a "new version" for the function of flow rate versus time (Figure 3.2):

Figure 3.2



Interviewer: Let me, let me add here a new version. It's zero and it goes up, and then it's decreasing:

Dan: I can do that, yes.

Analysis

In this segment Dan's perception of the problem to be solved changes radically two different times. First he changes his idea about which graph is given and which is predicted, predicting a function of flow rate versus time from his constructed function of volume versus time (Figure 2.2). He concludes that the graph in Figure 2.1 is impossible, because it is incompatible with a decreasing volume and generates a new flow rate graph (Figure 3.1) that he believes is consistent with his graph of volume. In constructing the new graph from his volume graph (Figure 2.2), he assumes fewer resemblances than when he generated the volume graph. Let us compare both constructions:

Figure 2.1 —> Figure 2.2

Both: straight lines, decreasing, positive, same initial value

Differ in: slope

Figure 2.2 —> Figure 3.1

Both: straight lines, decreasing

Differ in: slope, *sign*, *initial value*

Two resemblances—sign and initial value—were assumed in the first but not in the second prediction. These two new dissimilarities reflect two pieces of knowledge that Dan uses: a decreasing volume must correspond to a negative flow rate (dissimilarity in sign), and the flow rate must start at zero, but the volume does not (dissimilarity in initial value). Both pieces of knowledge were elicited by his interaction with the apparatus. We see this as an example of how assumptions of resemblance are adjusted by the student according to his knowledge of the situation. In the absence of a reason to the contrary, the resemblances are simply assumed, but as new pieces of knowledge come into play, some resemblances are rejected so that the new pieces of knowledge can be expressed in the graphs.

A second change in Dan's perception of the problem takes place when the interviewer asks about the production of the original graph of flow rate (Figure 2.1). Dan argues that the graph is impossible because flow rate must start at zero. This is not true because it is possible to start the measurements at any time (i.e., when the flow rate is not zero). However, students often expect that a graph must represent the "whole story," which begins when air is first physically moved. In this case, the interviewer redefines the problem without challenging this consideration, by posing a new version (Figure 3.2). Here we see yet another step in the joint construction by interviewer and student of the definition of the "problem" to be solved.

Segment 3

Description

The interviewer asks Dan to formulate a new prediction of volume versus time given the new graph of flow rate versus time (Figure 3.2). (See Figure 4.1.)

> Dan: [as he is drawing Figure 4.1]. Well I, if it's like that [Figure 3.2], it had to, let me think. The flow rate would go up, so that means the volume would also go up, and then it, it would drop. Let me think. I'm trying to think if this [the decreasing part of Figure 4.1] would go very much below, if this would go below zero or not. Let me see. If the flow rate went up, volume increases,

then it starts to decrease. Well, the only thing I'm trying to think of is if this [the decreasing part of Figure 4.1], as soon as I stop the increase will it automatically just drop right down until it would be below here (pointing to the steeper decrease) and then it contin... or whether it just—see, see the . . . go like this (pointing to the no-so-steep decrease) . . . I'm not sure which one . . . I'm not sure if it will drop right away and then come down or would just gradually.

Figure 4.1

Dan prepares himself to measure. He rehearses his actions several times with the apparatus before he actually measures. He looks at the graph of flow rate shown in Figure 3.2 but interprets it as how the height of the bag (a measure of volume) should change. Therefore, what he actually rehearses is how to produce a function of *volume* like the *flow rate* curve in Figure 3.2 (or, equivalently, like the volume curve in Figure 4.1). The interviewer intervenes, reminding Dan that flow rate should be positive:

Dan: Yeah, so I'll have to, I'll have to go like this [adding air to the bag] and then bring it back down [taking air out of the bag].

Interviewer: Uh huh.

Dan: I'm thinking, I'm pretty sure that's what I have to do.

Interviewer: But the flow rate wouldn't be never, shouldn't be negative, there [pointing to Figure 3.2] always positive.

Dan: Always positive? You're asking me?

Interviewer: Well, it is positive in the sense that it is always —

Dan: [looking at Figure 3.2] Above zero.

Interviewer: Above zero, right.

Dan: Uh, I think you're right, I think it might have to always be above zero. I'm not sure though. If we go like this (moving the bellows so air comes out of the bag), no, it would go below zero, 'cause I pull the air in [at the beginning] and then I'd be doing the reverse. So it should go below zero [when I do the experiment]. Oh, I'll try it.

Analysis

In this segment Dan repeats the process described in Segment 1, first constructing a prediction of volume versus time by assuming resemblances, then focusing on the predicted function of volume. We believe that in assuming resemblances Dan uses mainly semantic and linguistic clues. His use of semantic cues—namely, the intuition that isomorphic variation holds for volume and flow rate—is manifested by his reliance on the apparatus to interpret the graph. He uses the apparatus to rehearse and to decide what should happen. His language also strongly suggests that linguistic cues are playing a part in his reliance on resemblance. He says: "The flow rate would go up, so that means the volume would also go up, and then it . . . would drop," and "If the flow rate went up, volume increases, then it starts to decrease." In both expressions his reference to the quantity that is decreasing is ambiguous. "It would drop" and "it starts to decrease" each contain an "it" that can ambiguously refer to either flow rate or volume.

Dan is in conflict. He is now juxtaposing two attributes that are incompatible. On the one hand he expects the air to come out of the bag (since, in the graph, the volume decreases after the initial rise). On the other hand he knows that the positive sign of flow rate implies that air goes into the bag (and that the volume has to increase). Despite his awareness of the contradiction, he is not sure which way of understanding the situation is appropriate, so he decides to experiment. This reflects his reconstruction of the problem: Dan will try to produce the volume graph (Figure 4.1), and is curious to see what will happen with the function of flow rate.

Segment 4

Description

Dan produces four measurements with the apparatus, but he is unable to produce the graph he wants. After the fourth trial he says, "Can't get it through." While performing the experiment he looks at the top of the bag. In his attempts the flow rate goes up initially and then drops below the X axis as he makes the bag go up (by pushing the bellows down) and then down (by pulling the bellows up). Once the graph is below the X axis Dan tries to produce a decreasing line. The graph in Figure 5.1 is the result of his fourth attempt.

Dan complains that the graph drops abruptly after the initial rise instead of going "nice and slowly." He expects it to drop slowly because he is slowly and carefully changing the direction of the bellows.

The interviewer tries again to re-orient Dan toward the problem of getting the graph in Figure 3.2 by reminding him that flow rate is never negative in that graph:

> Interviewer: How could you do something that here [after the initial rise in Figure 5.1] goes down slowly without becoming negative?
>
> Dan: Without becoming negative?

Interviewer: Right. Just that goes down slowly.

Dan: Um, let me think. If I bring it [the bag] I have to bring it down to zero maybe, go up, and then go down, down, down like that maybe—no, but, as soon as I pull down [the top of the bag] it [the flow rate] goes down to zero, so it has to, I can't, I don't think we can get that [Figure 3.2] then.

Dan concludes that the graph of flow rate (in Figure 3.2) cannot be produced.

Figure 5.1

Analysis

During the measurements Dan tries to get the volume to go up and then down as in Figure 4.1. He cannot make sense of the abrupt drop shown on the computer screen. He expects the computer graph to reflect the motion of the top of the bag, and, indeed, the bag does not drop abruptly.

The interviewer's question about going down "without becoming negative" prompts Dan to use another piece of knowledge: air coming out of the bag corresponds to negative flow rate. Dan juxtaposes the two features: decreasing volume (his assumption from a graph of decreasing flow rate) and positive flow rate shown in Figure 3.2, and states correctly that they are incompatible (i.e., a positive flow rate always implies an increasing volume). He is now convinced of the impossibility and feels there is no need to try a new experiment. Dan is not aware at this point that he has changed the terms of the problem by focusing on his predicted volume graph. This segment shows two characteristics typical of a resemblance approach:

1. extracting graphical features from one function and projecting them onto the other in an attempt to come up with a matched set of graphs; and
2. focusing exclusively on the function or the derivative alone; whichever is chosen as a focus is taken as the "given," regardless of how the problem was initially stated.

Segment 5

Description

After Dan states his belief that it is not possible to get a decreasing flow rate without becoming negative, while implicitly he assumes that the volume has to decrease, the situation appears to be a dead end. The interviewer says:

> Interviewer: Um, let me try.
>
> Dan: You want to try?

The interviewer generates a graph of flow rate decreasing but remaining positive by starting with the valve open and gradually closing it. The top of the bag always moves up.

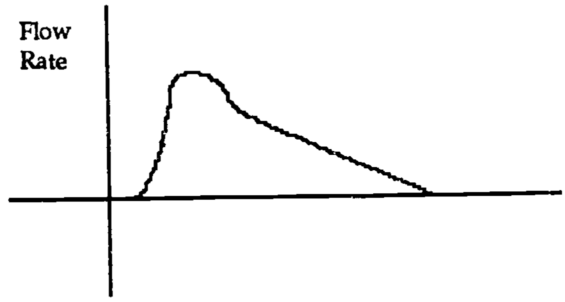
> Dan: Playing tricks on me again. I know what I told you, maybe there should have been a way to do that that I didn't know. [laughs]
>
> Interviewer: Let's do it without the valve.
>
> Dan: I want to see that!

Now the interviewer generates another graph by pushing down the bellows at a decreasing rate, the equivalent of his previous action with the valve. Dan looks carefully at the three elements: the interviewer's hand pushing down the bellows, the computer screen, and the bag accumulating the air.

> Dan: Well, it's just the amount of increase is less and less. I see... Yeah, I see. So what do I do, just pull it [the top of the bag] down, let some air go up and then we just let it go up slowly, is that what you did? Yeah.... That's, that's different, I didn't think of that.
>
> Interviewer: Does it make sense?
>
> Dan: Yeah, now it does... It's tricky, though, tricky to figure that out.
>
> ...
>
> Dan: I didn't think that maybe if we just did this [push down the bellows] and then let the flow rate just go nice and slow, that it would decrease from what it was before. I should have thought of that, but I didn't.

After the interviewer generates the function of flow rate Dan attempts to do it too. He finds that it is not a trivial effort to generate a smooth curve. The main problem is that to generate the graph in Figure 3.2, one must move smoothly between the increasing and decreasing segments of the curve. Dan, however, changes the rate of change abruptly and generates a sharp peak instead of a rounded curve. He tries four times before, finally, he gets an acceptable graph (like Figure 6.1) by using the valve to control the air flow. His difficulty generating the point on the graph that reflects a change in the rate of change foreshadows problems that arise in the next segment.

Figure 6.1



Analysis

At the end of Segment 4, Dan expressed his conviction that the graphs in Figure 3.2 and 4.1 are incompatible. Since he was focusing on his prediction (Figure 4.1), he concluded that the graph in Figure 3.2 was impossible. The interviewer's showing Dan how to generate that graph in Segment 5 is a landmark for Dan's view of the situation. On the one hand it elicits a cognitive conflict: a belief that something is impossible is challenged by the the factual evidence that it is possible, requiring Dan to revise his way of thinking. But more important, watching the apparatus provides Dan with clues about how his understanding of the situation might be improved, such as:

- The top of the bag continued "moving up" as the graph "moved down."
- Air was flowing into the bag both before and after the peak of the graph.

Both aspects challenge Dan's assumptions of isomorphic variation and suggest that the difference between the graph before and after the peak of Figure 3.2 is a difference between modes of inflow, rather than between inflow and outflow. In sum, Dan's insight is not just a product of cognitive conflict; it is also the outcome of the presence and role of the apparatus. Dan benefits both from being able to experiment with a tool that mediates his thinking about the situation and from watching the interviewer use it to generate graphs.

In describing how the graph of flow rate (Figure 3.2) is generated, Dan changes his language. Examples of this change in his descriptive language are shown below.

From Segment 3:

1. "If the flow rate went up, volume increases, then it starts to decrease."
2. "The flow rate would go up, so that means the volume would also go up."

From Segment 5:

3. "The amount of increase [of volume] is less and less."
4. "It [the flow rate] would decrease from what it was before."

Expressions (1) and (2) reveal the assumption of isomorphic variation and suggest that volume and flow rate are conceptualized as two parallel variations rather than one reflecting the way the other one changes over time. Figures 6.2 and 6.3 may help to clarify this point.

Figure 6.2

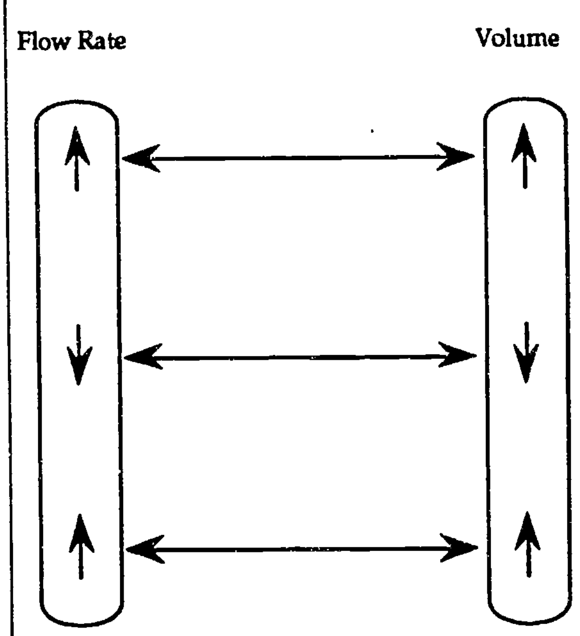
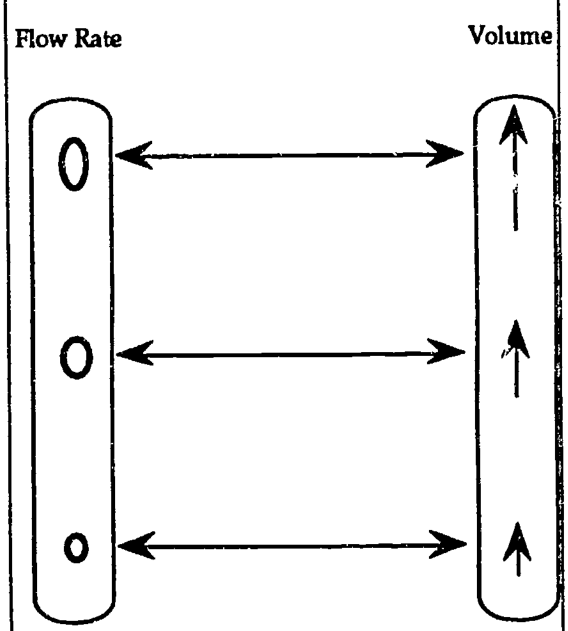


Figure 6.3



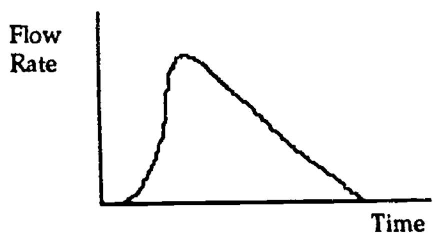
In Figure 6.2 volume is thought of as varying according to the variation of flow rate (more flow rate means more volume and less flow rate means less volume). Both vary in the same way; more of one implies more of the other, and less implies less. In Figure 6.3, however, flow rate describes the local variation of volume as time passes. A key step in constructing the framework depicted in Figure 6.3 is focusing on how volume changes locally over time and recognizing flow rate as the descriptor for this local change. While expressions (1) and (2) are consistent with the approach shown in Figure 6.2, expressions (3) and (4) are suggestive of the latter: "the amount of increase (of volume) is less and less" (recognition of the local variation depicted by the arrows on the right of Figure 6.3) and "it (the flow rate) would decrease from what it was before" (recognition of the local variation depicted by the shrinking circles on the left of Figure 6.3).

This conceptual shift is the beginning of Dan's construction of a variational approach. We believe that the role of the apparatus in this process is to support the joint interviewer-student construction of narratives, understood as descriptions of sequences of events that seem more or less consistent with the observational facts and their own expectations. To say "the amount of increase (of volume) is less and less" is a way to tell the story of the apparatus' behavior without assuming isomorphic variation: volume "increases" and at the same time flow rate is "less and less."

Segment 6

Description

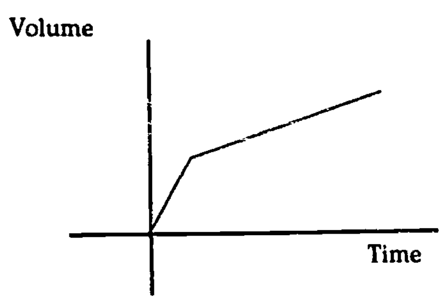
Figure 7.1



Interviewer: Now [given the graph in Figure 7.1] we will look at volume, what do you expect to see?

Dan: [while drawing the graph in Figure 7.2] Always increasing, but at the, at the beginning it will increase. At the beginning . . . I believe that, the volume will go up, and then, and then it will just sort of like go up slower and slower and slower. I'm pretty sure that's what it will do.

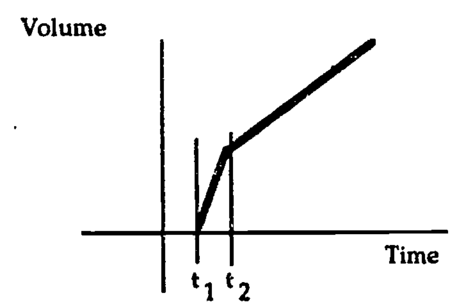
Figure 7.2



Dan divides his prediction into two parts: the first (when flow rate increases) and the second (when flow rate decreases). For the first he anticipates a sudden increase of volume, whereas for the latter he anticipates a slow increase. For the portion of decreasing flow rate he no longer holds the assumption of resemblance that volume will increase. On the other hand, his intuition is that the initial rise of flow rate will correspond to a similar sudden rise of volume.

The interviewer asks whether the second part of Dan's prediction for the volume graph, corresponding to the decreasing flow rate, is a straight line. In order to understand Dan's answer let us identify two salient times in his prediction: t_1 and t_2, shown in Figure 7.3.

Figure 7.3



> Dan: It will, this [after t_2] will not be as, will not, its incline will not be as much as [between t_1 and t_2]. It will reach, it will reach this point [t_2] here and then it will start to even off sort of. It might not, it might be a straight line, but I, I don't think it will. I think it will be just wavy, you know . . . It will always increase, but it won't increase as, as the amount of time did here [between t_1 and t_2] . . . Say we did, say right here [between t_1 and t_2] it was, I don't know, say it went up . . . 2 liters. Right here [after t_2], for a same interval as between whatever this is [the time interval between t_1 and t_2] it won't be 4 liters, it will be 2 and a little bit, and then it will be a little bit more, a little bit more. It won't, it won't be as much. The amount of air going in will not be as much as it did in this piece [between t_1 and t_2]. And very much less the following seconds after.

Finally Dan asks the computer to display the curve of volume versus time and gets a curve similar to the one shown in Figure 7.4.

Dan expresses disappointment at the fact that the curve does not show what for him is the most important aspect: a fast increase followed by a slow increase.

> Dan: Nah. I don't know, it didn't really do that. Didn't have that amount of volume.

Figure 7.4

Dan decides to look at the number values of flow rate and volume. In examining them he comes to the conclusion that the initial increase of volume did not appear because "the flow rate was not so great."

> Dan: [Initially I expected that] it would go up more, but it didn't. Maybe because that other line was not the amount of ... flow rate was not so great that it, it would make this [a bump] and then make it, make it go like that. Like make it go up and then even off like I thought it would. So, probably had, not having the great amount of flow rate at the beginning, it just sort of went up, and up, and up, but it, it, I think it, basically it did what I thought it was going to do. That it just sort of evened off here.

Dan solves the discrepancy between his prediction for the initial increase of flow rate and the observed measurements by using the ambiguity inherent in any experimental process. In a sense, he says, "What I expected is there, but you can't really see it."

Analysis

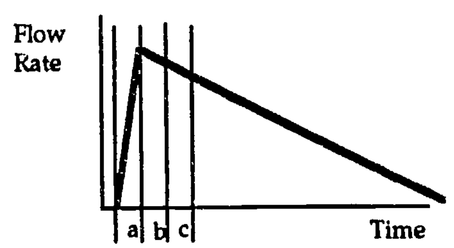
In this segment Dan begins to articulate an interval-based analysis for the variation of volume. In justifying his prediction for the volume graph that corresponds to the flow rate graph, he uses the increase in volume in the interval between t_1 and t_2 as a standard and compares it to the increase in volume in a similar interval after t_2. He goes on to describe how the volume continues to change as more time intervals pass: "It (the volume) will be two and a little bit, and then it will be a little bit more, a little bit more," implying that each time the "little bit more" will be smaller and smaller. This interval-based analysis of a change over time is a central phase in the construction of a variational approach.

While Dan expresses clear and precise verbal descriptions for the change of volume as a decreasing increase (e.g., "sort of like go up slower and slower and slower"), the mapping between this notion and graphical shapes is unstable for him. He has an intuition that the graph should not be a straight line, but he has difficulty figuring out what else it could be (e.g., "it might be a straight line, but I, I don't think it will. I think it will be just wavy, you know"). Thus, while he is articulating a verbal description of a decreasing increase, he has not yet attached a corresponding knowledge of graphs to his descriptions. Dan draws straight lines in his graphical predictions not because he knows that they should be straight lines, but because straight lines are sufficient for showing what, for him, are the most relevant features of the graph, namely, whether the function increases or decreases and how quickly, relatively, different functions or pieces of a function change.

Dan spends some time trying to explain how the volume graph behaves differently before and after the "peak" of flow rate. Before Segment 4 Dan explained this difference by assuming a resemblance: more flow rate means more volume (before the peak) and less flow rate means less volume (after the peak). In other words, before the peak he saw the situation as one of inflow, and after the peak it became for him a situation of outflow. In the current segment Dan no longer assumes this resemblance. He recognizes that there is inflow during the entire graph, but he must still account for the difference in the graph before and after the peak. Dan comes up with a new solution: before the peak the volume increases very fast, and after the peak it increases slowly. In constructing this solution, Dan assumes a different resemblance that focuses on "abruptness": An abrupt increase in flow rate means an abrupt increase in volume (before the peak) and a slow decrease of flow rate means a slow increase in volume (after the peak). This new resemblance links the speed at which volume changes with the speed at which flow rate changes. Thus, it is a different resemblance, since it deals with rate of change, rather than just increase and decrease.

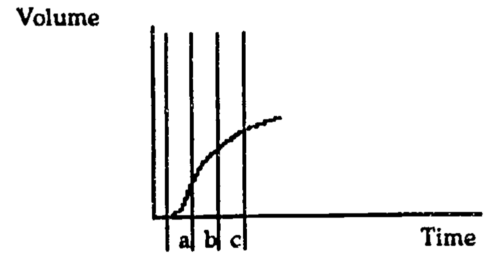
A graph like Figure 7.5, which echoes a peak in the volume graph, seems plausible; we have, in fact, seen similar predictions made by several people who know a fair amount about calculus. Why is it false? Consider this graph (Figure 7.5), an idealized version of the one in Figure 3.2.

Figure 7.5



Assume that intervals *a*, *b*, and *c* are equal. The average flow rate during the interval *a* is less than the average flow rate during either of the intervals *b* or *c*. Therefore, the amount of increase of volume during interval *a* is less than the increase of volume during either *b* or *c*. The graph in Figure 7.6 could be a curve of volume versus time corresponding to the one in Figure 7.5.

Figure 7.6



The peak of Figure 7.5 corresponds to an inflection point in Figure 7.6. The curve Dan produced with the apparatus did not show the pattern of Figure 7.6 because the increase of volume corresponding to interval *a* was too small to be noticeable as an inflection point in the experimental curve. (In fact, Dan used a similar argument about an insufficient increase in volume to explain why his experimental curve did not show his anticipated sudden initial rise of volume!)

In describing the curve he expects to see, Dan is able to use a variational approach for the slow decrease of flow rate, but for the sudden increase of flow rate he assumes a resemblance between the speed at which flow rate and volume change. One element that may have played a role in his clinging to resemblance is that a sudden increase is more likely to be perceived as a change between two close times, rather than as a sequence of changes. In this sense it is harder to develop an interval-based analysis for a sudden change. In other words, an interval-based analysis requires a concept of air flow changing over time, rather than "jumping" suddenly from one value to the next. The unit that Dan uses to split the time into intervals is the entire duration of the sharp increase of flow rate, eliminating the possibility of looking at the sharp increase as a sequence of intervals.

Conclusion

This paper has analyzed how high school students tend to solve problems of prediction between a function and its derivative by assuming partial resemblances between them. We observed this tendency in different physical contexts, such as in motion (translation between position and velocity), in fluids (translation between flow rate and volume), and in number-change (translation between a list of numbers and the list of its accumulated values or its differences). These assumptions of resemblance lead to a particular approach to problems of prediction between a function and its derivative, characterized by forcing a match of global features of the two graphs (e.g., increasing/decreasing, sign) and by focusing on one of them (function or derivative) rather than on their relationship.

The use of resemblances is not the result of the student's inability to distinguish between the function and derivative. Several perceptual and cognitive aspects of situations of change support the plausibility of such resemblances. We described three such aspects: semantic cues, syntactic cues, and linguistic cues. We compared resemblance approaches with variational approaches, which are based on the analysis of the local variation of a function and understanding how it is described by its derivative. A variational approach focuses on the relationship between a function and its derivative rather than on the global properties of each graph. We distinguished several variational approaches according to the mathematical entity that is used to describe this relationship (e.g., steepness, slope, local accumulation).

The construction of a variational approach is a complex process that we attempted to illustrate through the analysis of a 17-minute learning episode. This episode was built around a co-construction, by student and interviewer, of some insights that support a variational approach, such as an interval-based analysis for the variation of a function and the development of a connected story that recognizes that change over time in volume is described by its derivative, flow rate. But this development is not a monolithic insight. Even at the end of the learning episode, the student used a new resemblance to account for some troublesome aspects of his experience. The case study also supports the importance of our technique of using physical contexts to provide students with tools to explore mathematical ideas from a variety of directions, and gives us insight into how these tools help frame the interviewer/student discourse through which learning occurs.

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