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ABSTRACT

Participants at a conference on mathematics teaching discussed what it means to know and understand a subject and what teachers need to know, understand, and believe in order to teach a subject to elementary learners. The first section of the report covers the first day of the conference that was spent in a series of whole-group sessions. Participants made personal responses to three questions posed by the conference organizers: How might we consider research on learners' mathematics knowledge and thinking in the study of teaching? How might we use mathematical content analyses in the study of teaching? How might we consider research on teachers' knowledge and thinking in the study of mathematics teaching and learning? Following the statements, a summary is presented of the issues that came up in the dialogue between the participants. The second section presents the results of the small group sessions that focused on (1) students' mathematical thinking, (2) the conceptualization of mathematics teachers' beliefs, (3) learning from studying teacher change, and (4) facilitating shared roles and understanding between researchers and teachers. Issues for further thinking are raised and directions for future research are suggested. (MDH)

Elementary Subjects Center
Series No. 40

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PROCEEDINGS OF A CONFERENCE HELD AT
MICHIGAN STATE UNIVERSITY, OCTOBER 4-8, 1989

Penelope L. Peterson and
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Center for the
Learning and Teaching
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This conference was organized and sponsored collaboratively by the Center for the Learning and Teaching of Elementary Subjects at Michigan State University, East Lansing, and the National Center for Research in Mathematical Sciences Education at the University of Wisconsin-Madison. The Center for the Learning and Teaching of Elementary Subjects and the National Center for Research in Mathematical Sciences Education are funded primarily by the Office of Education Research and Improvement, U.S. Department of Education (Cooperative Agreement No. G0098C0226 and Grant Number G008720279, respectively). The opinions expressed in this report do not necessarily reflect the position, policy, or endorsement of the Office or Department.

Center for the Learning and Teaching of Elementary Subjects

The Center for the Learning and Teaching of Elementary Subjects was awarded to Michigan State University in 1987 after a nationwide competition. Funded by the Office of Educational Research and Improvement, U.S. Department of Education, the Elementary Subjects Center is a major project housed in the Institute for Research on Teaching (IRT). The program focuses on conceptual understanding, higher order thinking, and problem solving in elementary school teaching of mathematics, science, social studies, literature, and the arts. Center researchers are identifying exemplary curriculum, instruction, and evaluation practices in the teaching of these school subjects; studying these practices to build new hypotheses about how the effectiveness of elementary schools can be improved; testing these hypotheses through school-based research; and making specific recommendations for the improvement of school policies, instructional materials, assessment procedures, and teaching practices. Research questions include, What content should be taught when teaching these subjects for understanding and use of knowledge? How do teachers concentrate their teaching to use their limited resources best? and In what ways is good teaching subject matter- specific?

The work is designed to unfold in three phases, beginning with literature review and interview studies designed to elicit and synthesize the points of view of various stakeholders (representatives of the underlying academic disciplines, intellectual leaders and organizations concerned with curriculum and instruction in school subjects, classroom teachers, state- and district-level policymakers) concerning ideal curriculum, instruction, and evaluation practices in these five content areas at the elementary level. Phase II involves interview and observation methods designed to describe current practice, and in particular, best practice as observed in the classrooms of teachers believed to be outstanding. Phase II also involves analysis of curricula (both widely used curriculum series and distinctive curricula developed with special emphasis on conceptual understanding and higher order applications), as another approach to gathering information about current practices. In Phase III, models of ideal practice will be developed, based on what has been learned and synthesized from the first two phases, and will be tested through classroom intervention studies.

The findings of Center research are published by the IRT in the Elementary Subjects Center Series. Information about the Center is included in the IRT Communication Quarterly (a newsletter for practitioners) and in lists and catalogs of IRT publications. For more information, to receive a list or catalog, or to be placed on the IRT mailing list to receive the newsletter, please write to the Editor, Institute for Research on Teaching, 252 Erickson Hall, Michigan State University, East Lansing, Michigan 48824-1034.

Abstract

This is a report of the proceedings of a conference on "Mathematics Teaching and Learning: Research in Well-Defined Mathematical Domains" that was held from October 4-6, 1989, at Michigan State University. The conference was sponsored by two centers funded by the U.S. Department of Education's Office of Educational Research Improvement: The Center for the Learning and Teaching of Elementary Subjects at Michigan State University and the National Center for Research in Mathematical Sciences Education at University of Wisconsin-Madison. The Elementary Subjects Center focuses on the study of teaching for conceptual understanding, problem solving, and higher level learning in mathematics and other subjects. The Mathematical Sciences Education Center seeks to further the integration of research on teaching and research on learning. This conference provided an opportunity for researchers from around the nation to come together to discuss what it means to know and understand a subject and what teachers need to know, understand, and believe in order to teach a subject to elementary learners.

The first section of this report covers the first day of the conference which was spent in a series of whole-group sessions. This section consists of excerpts of the personal statements from participants made in response to three questions posed by the conference organizers: How might we consider research on learners' mathematics knowledge and thinking in the study of teaching? How might we use mathematical content analyses in the study of teaching? How might we consider research on teachers' knowledge and thinking in the study of mathematics teaching and learning? Following the statements, a brief summary is presented of the issues that came up in the dialogue between participants.

On the second day, small working groups met to develop ideas for a future research agenda based on their approaches to research on mathematics teaching and learning. The second section of this report summarizes some of the thinking, ideas, and discussion from these sessions which focused on (a) students' mathematical thinking, (b) the conceptualization of mathematics teachers' beliefs, (c) learning from studying teacher change, and (d) facilitating shared roles and understanding between researchers and teachers. Authors of the working group summaries also raise issues for further thinking and discussion and suggest needed directions for future research on the learning and teaching of mathematics.

LIST OF CONFERENCE ORGANIZERS AND PARTICIPANTS

Conference Organizers

Penelope L. Peterson is Co-director of the Institute for Research on Teaching and the Center for the Learning and Teaching of Elementary Subjects at Michigan State University where she is also a Professor of educational psychology and teacher education. Previously she was Sears-Bascom Professor of Education at the University of Wisconsin-Madison. Her current research focuses on children's mathematics knowledge, thinking, and learning and on teachers' knowledge, thinking, and beliefs. She is past Vice-President of the American Educational Research Association for Division C (Learning and Instruction) and the former editor of Review of Educational Research.

Elizabeth Fennema is Professor of Curriculum and Instruction with an emphasis on mathematics education at the University of Wisconsin-Madison. Her research and writing has been in two areas: gender issues in mathematics and teachers' use of knowledge about children's thinking. She is Co-director (with Thomas Carpenter) of the Cognitively Guided Instruction Project and Associate Director of the National Center for Research in Mathematical Sciences Education.

Thomas Carpenter is Professor of Curriculum and Instruction with an emphasis in mathematics education at the University of Wisconsin-Madison. Currently he is editor of the Journal of Research in Mathematics Education. He has written extensively about young children's thinking in addition and subtraction. His ongoing work involves studying teachers' use of knowledge about children's thinking. He is Co-director (with Elizabeth Fennema) of the Cognitively Guided Instruction Project and Associate Director of the National Center for Research in Mathematical Sciences Education.

Small Group Leaders

Ralph Putnam is a senior researcher with the Center for the Learning and Teaching of Elementary Subjects and Assistant Professor of educational psychology at Michigan State University. He received his Ph.D. from Stanford in 1985 and completed a two-year postdoctoral fellowship at University of Pittsburgh's Learning Research and Development Center. He combines perspectives from cognitive and instructional psychology with those from research on teaching to study the teaching and learning of mathematics in elementary classrooms.

Robert E. Orton is an associate professor in the Department of Curriculum and Instruction at the University of Minnesota, Minneapolis. His area is mathematics education. His area of research interest is mathematics teachers' knowledge and reasoning.

Karen C. Stoiber is an assistant professor in the Department of Psychology, Northern Illinois University, De Kalb. Her research specialization is in the area of teacher thinking, teacher beliefs, and knowledge construction.

Participants

Deborah Lowenberg Ball, National Center for Research on Teacher Education, Michigan State University
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Paul Cobb, Purdue University
Thomas Cooney, University of Georgia
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Linda Weisbeck, National Center for Research in Mathematical Sciences Education

Table of Contents

WHOLE-GROUP SESSION

OPENING STATEMENTS OF CONFERENCE ORGANIZERS

Elizabeth Fennema, Conference Co-Organizer

This conference is sponsored jointly by the National Center for Research in Mathematical Sciences Education at the University of Wisconsin-Madison (NCRMSE) and the Center for the Learning and Teaching of Elementary Subjects at Michigan State University. The portion of NCRMSE that is directed by Thomas Carpenter and me is focused on encouraging the integration of research on teaching and research on students' learning. Both are dynamic research areas that are producing real and useful knowledge about how learners think and about teachers and their instructional decisions. However, these two areas of research have been progressing almost independently. Thus, the purpose of the conference was to get researchers on teaching together with researchers on learners' thinking to explore the integration of the research in the two areas.

The current research on learning that is producing information useful in curriculum change has been done in well-defined content domains in mathematics. Researchers have taken small subsets of mathematics, examined and defined those subsets, and used the definitions to obtain insights into learners' thinking. Not only have we learned more about thinking by using these definitions, but the content definitions have also helped to focus the curriculum on important mathematics. However, this research has given us very little evidence about how this knowledge can or should be used to influence teaching and learning in the classroom.

Research on teaching, in contrast, has given us important insights into the teaching act, teachers' thoughts which influence instructional decisions, and the impact of knowledge and beliefs on the decisions. However, much of the research has been content neutral, if not totally devoid of the consideration of content. Thus, it seems reasonable that researchers on learning should begin to think and explore how knowledge of teaching can be of use in the classroom, and researchers on teaching should be considering mathematics that is important mathematics to learn.

Content analyses are not all the same. One of them, which has focused on addition and subtraction, has been based on the semantic analysis of a variety of problem types which enabled researchers to develop a categorization of how kids think about those problem types. Other kinds of content analysis have been different. The rational number analysis did not start from a semantic analysis, but by identifying some very basic ideas that are necessary in order to understand rational numbers, such as the idea of unit, and the part-whole relationship. People who have worked in other content domains such as multidigit algorithms, place value, or multiplication and division have done somewhat different things. Content analyses in subjects usually taught in the secondary school are just beginning.

It is not totally clear how content analyses will be of use in research on teaching, nor how the knowledge and methodologies developed by teaching researchers can be used as students' thinking is studied. However, if we wish to have an impact on teaching and learning mathematics in classrooms, the integration of these two research areas is essential. So in order to begin to consider this integration, we have organized this conference. The Mathematics Center has sponsored other conferences around specific content themes. However, this conference is being focused on the study of teachers, and in particular the study of teaching which considers students' thinking in well-defined content domains in mathematics.

Penelope Peterson, Conference Co-Organizer

The purpose of this conference is to try to conceptualize, think about, decide upon, and figure out the kinds of research that we think needs to be done on mathematics teaching and learning and the kinds of research that we want to do in the next three years. It's very important to think about this conference and the purpose of this conference within the context of reform of mathematics teaching and learning in our nation's schools.

This morning, I was reading the summary document entitled, "Setting a Research Agenda," (Sowder, 1989) from the Research Agenda Project which was conducted under the auspices of the National Council of Teachers of Mathematics (NCTM) and the National Science Foundation. The project had as its purpose the development of an agenda to guide research on the learning and teaching of mathematics. I know that many people in this room were involved in one or more of the conferences that took place as part of this project, and many of you have probably read the summary document. In rereading it myself this morning, I was struck particularly by some of the statements made on the page where the authors talk about research. The heading on the page is "Research," and I think that it is worth thinking about some of the points that are made here so I quote:

> Given that changes are needed, many individuals and groups will inevitably make claims about actions, programs, and policies that should be followed. Herb Larrabee (1945) argued that anyone who has surveyed the long history of claims about knowing is "struck by the discrepancy between the pretentiousness of most knowledge claims, and the small amount of evidence actually available with which to back them up" (p. 82). Researchers try to differ from this stereotype in that they admit their ignorance, spend considerable efforts gathering evidence, so that whatever information is acquired is reliable, and marshal the evidence into well-argued briefs to justify their assertions (p. 10).

By reliable knowledge we mean any claim to knowledge that is substantiated as trustworthy for some given purpose. The gathering of the evidence and the construction of the argument are the means by which researchers substantiate conjectures. This is an arduous and endless task that requires a substantial amount of training and effort; in the more complex cases it taxes the patience and ingenuity of the most gifted thinkers. Nor does it, once achieved, stay finished and complete; it has to be continually reachieved since what constitute both a reasonable argument and the given purposes continually change.

It also must be noted that most of the body of past and contemporary research addresses the past vision of school and literacy. For example much research on effective teaching has been based on predicting residualized-mean-gain scores on standardized tests. Such research does not support the pursuit of reliable knowledge about the new vision of mathematical literacy. New questions are now being posed with expectations of different outcomes. Our research must address questions using methods of gathering information in the light of the reform. We need to develop a comprehensive research plan and recruit a network of scholars to carry it out.

> Thus the role of research in the reform movement is to provide reliable knowledge about important aspects of the reform. . . . One key aspect of the strategy being posed is the creation of networks of scholars working collaboratively on common issues. (Sowder, 1989, p. 10.)

I think that's what we have here--a network of s[illegible] who are considering the important issues and the kinds of research that we need to do, and that the research community needs to do, within

the context of mathematics reform. What we hope to be able to achieve by the end of the next two days is some conceptualization of approaches, some listing of the kinds of questions we need to think about, and some descriptions of the kinds of research we need to be doing on the teaching and learning of mathematics.

Our purpose here has personal meaning for me because I have children who are attending public school. The problems and dilemmas of mathematics reform have become more and more clear to me this year as I have been having conversations with my eight-year-old son, Joshua, about mathematics. For example, when we were having such a conversation several weeks ago, Josh asked me what a decimal was. After we had conversed a bit about decimals, he queried me with, "Mommy, is .5 the same as .50? Does the zero make a difference?" And I replied "What do you think, Josh?" He countered with, "Mom, just tell me the answer!" And I responded, "But what do *you* think Josh?" Finally, in a exasperated tone, Josh admonished me with, "Mom, my teacher doesn't do this. *My teacher tells me* the answer!" My son's statement was like a stab in the back. His next statement was even harder to bear: "Just *tell me* the answer, Mom, *so I don't have to think about it anymore.*" I said to Josh, "That's exactly why I am not going to give you an answer so you *will* have to think about it some more!" I didn't give Josh an answer to his question then, and I still haven't.

Similarly, we are posing some questions here. We posed these questions to you in the letters that we sent prior to the conference. We want to know what you think about these questions and why you think that way. As we all know, there aren't any "right" answers to these questions. The questions we asked you to consider in preparation for this session are:

> How might we consider research on learners' mathematical knowledge and thinking in the study of teaching?
>
> How might we use mathematical content analyses in the study of teaching?
>
> How might we consider research on teacher's knowledge and thinking in the study of mathematics teaching and learning?

What we thought we would do is ask each person to respond to these questions. It is really important that everybody get a chance to say something. We need to come to understand each others' perspectives in order to develop some kind of shared understanding.

Thomas Carpenter

Maybe I can fill in a little bit of the history on why the title of this conference refers to a "well-defined content domain" in the case of mathematics. I guess part of my problem this morning is that as people have been talking about things they have been looking at in their research, some people have questioned whether mathematics is a well-defined content domain. In our title for this conference, we did not intend to convey that mathematics is a well-defined content domain. Mathematics isn't a well-defined domain. When we talked about a well-defined domain, we were refering to a subset of mathematics, that we thought might be studied in a reasonable way. For example, in the last 20 years, some researchers have studied the interplay between specific ideas in a particularly narrow domain of mathematics, such as addition and subtraction or rational numbers, and students' learning and thinking in those areas. That has turned out to be a very rich, important line of research. Basically, that research shows that kids come into any task knowing something before you introduce any symbolism or any labels. For example, Greeno talked about children having a kind of pseudo mathematical schema. Then you negotiate labels to give meaning to those things children already knew. If children don't know it, then whatever mathematical symbols you might give them are totally

meaningless. This is what happens in a lot of mathematics teaching--nothing is built on any prior knowledge of the student.

The whole notion of what constitutes a well-defined domain does not just involve content. Rather, it is content and the way in which kids, in the different circumstances with different kinds of problem situations, think about ideas in that mathematical domain. It does evolve. It doesn't happen by a mathematician simply sitting down and saying, "This is what addition and subtraction is," or "This is what multiplication and division are." That would be a very sterile representation of what we are talking about. What we are talking about is much more complex.

The question being raised in this meeting is, "Can we use some of that knowledge from research on students' mathematics learning in well-defined domains in actual classrooms and in classroom research?" I think several speakers put their fingers on important problems. Classrooms don't tend to be organized to teach mathematics the way researchers typically do it in laboratory settings. Teachers may not even teach fractions, or they may teach pieces in the domain of fractions without tying much of it to students' prior knowledge or to different meanings. They may teach multiplication as repeated addition or grouping, but they may not teach multiplication as a way of thinking about partitive problems. Teachers might not see that the same formal mathematical symbols will fit that problem situation. Partitive problems are unlikely to come up in most classrooms or to appear in most textbooks so there will be exploration of that part of the domain in most elementary mathematics classrooms. Or, you have things like the operational definition of a rational number. Why would anybody know that? Looking at most textbooks, you find almost no examples. Why would a teacher know that that's a nice example to deal with? Rational numbers were never covered in any of the courses teachers typically take.

So part of it is the domain itself--understanding the complexity of the mathematics--and part of it is understanding how kids' mathematical thinking evolves through that domain. When we talk about a "well-defined domain," we are talking about mathematics in the plural sense. There are lots of domains, lots of different pieces of mathematics, but only some are recently well-understood both from the mathematical point of view and from learning point of view. If we focus on our new understandings in those domains, can we use any of that information to make sense what is happening or what could be happening in mathematics classrooms? What we are saying is that we have found this knowledge in at least one domain to be useful to teachers--the domain of addition and subtraction of whole numbers with young children. It seems possible that such knowledge in other domains may also be useful to teachers.

STATEMENTS BY CONFERENCE PARTICIPANTS

[*Note from Penelope L. Peterson, editor*: As the day progressed, more and more dialogue occurred between conference participants. Participants who spoke later in the day sometimes built on or referred back to earlier statements made by participants. The following series of statements were excerpted in sequential order from an ongoing conversation that took place the first day between conference participants. They were taken from verbatim transcripts of the dialogue during the whole-group sessions. In editing the excerpts we kept the exact wording used by the participant, but we sometimes added a word or two if needed to complete a thought or a sentence. In the case of participants who made lengthy statements, we tried to capture what seemed to be the participant's main message. Following the statements of the conference participants, we present a brief summary of some issues that came up in the dialogue between participants that occurred between, among, in, and around these individual statements.]

Richard Lesh

The research that I have done in the last 10 years has to do more with how students learn than with how teachers teach. . . . It's pretty clear in mathematics education that there are three ways you could organize mathematical ideas. One is a mathematical structure--that is the way a mathematician organizes the body of knowledge. Then there is a cognitive structure which is the way kids think about it. And there is the instructional structure which is the way teachers or textbooks think about it. And those three things aren't the same--instructional structure, cognitive structure, and mathematical structure.

But my problem, in so far as I'm a practitioner as well as a researcher, is to try to bring those in sync with one another so that we at least make them mutually supportive. As a researcher, I think the way we should be telling people to teach ought to be consistent with the way kids learn. As a practitioner--mainly I'm a teacher trainer--I think we ought to be educating teachers to use techniques that we want them to use with kids. Let me illustrate the first one that deals with the relationship between teaching and learning. There are two ways that we have done that. One is called the teaching experiment; the other is a local conceptual development session.

In a teaching experiment we set up experiences for children that often go from 10 weeks to 16 weeks. We have focused on rational numbers and proportional reasoning especially because it is an area that is rich enough to be interesting, but simple enough to be manageable. What we were trying to do was to study the way ideas grow and to figure out the nature of mathematical ideas in kids' minds. If you are going to study the way ideas grow, you end up studying kids because those ideas grow in kids minds. But our real goal was to study the way ideas grow, and that's important to know to accentuate what we do that is different than what other people are doing. In many ways, what we have done grew out of Piagetian-like studies from 20 years ago, except the ideas we are studying are unlike those that psychologists were studying--they don't evolve naturally. In fact many of them don't evolve at all in most of the populations. It is very easy within the domain of rational numbers to find ideas that teachers don't understand as well as kids, and that normal people out in the street don't understand. If we are going to study the way those ideas grow, we have to create an artificial environment in which these will grow. The teaching experiment is a longitudinal developmental study of kids in a mathematics-rich environment.

Now we have a problem because if we are interested in the way ideas grow, we are looking at kids, but what we see grow is a function of the instruction the kids have and is also a function of the environment that you create. What we see develop depends partly on what conceptual instruction the kids bring to the task, but it depends partly on the environment we create. If we are going to describe the way ideas grow, our problem is: How do we discuss without interaction? We have to be explicit about the way we create the environment. To that extent, we are studying teaching. Now to

do that, we've actually looked at things two ways. If we want to see ideas grow, we look at kids and see them growing. We can see how kids' ideas go from concrete to abstract.

But another interesting thing is to go to teachers and investigate how teachers make ideas concrete. If what we are interested in is how ideas go from concrete to abstract, we find it just as interesting to look at kids going in one direction as it is to watch teachers go in the other direction. We've been trying to work out a way to use mathematical language, mathematical structures, and mathematical systems to describe the interaction of the environment with kids, and the environment includes teachers.

There is an important distinction to be made between studying teachers and studying teaching. Teachers are only a part of what is going on in those environments. Kids learn from a lot of sources--teachers are one of them, textbooks are another, their everyday experiences are another. When we are structuring the environment, we are interested in what teachers do, but we are also interested in what kids do. In sum, that's one of the lines of research we've been doing and that is how I see teaching and learning.

The other line of research is the local conceptual development session. I call them problem-solving sessions, but they are the 40-minute problem-solving session variety. They are realistic problem-solving episodes. They are the kind of things that might occur in the kids' everyday lives, but the distinctive feature is that these are 40-minute episodes in which the principal problem for kids is to find the way to think about the problem. Let me give you an example so you know what I am talking about.

One is an inflation problem we did where kids gathered newspapers from 10 years ago and a series of catalogues from 10 years ago from today. We gave them a bit of information about a person 10 years ago, for example, a teacher whose salary was a certain amount. Then we asked them what the teacher should be getting as a salary today based on the information in the series of catalogues and the newspapers. Now that's a problem that has too much information and at the same time not enough information. If you watch seventh graders, they have to go through modeling cycles to figure out the way to think about the problem. In the course of finding ways to think about the problem, the seventh graders actually invent some mathematical ideas, and so we see ideas evolve in 40 minutes.

One interesting thing is when we compare these findings with findings from the other kind of research that we've done over a 16-week period, we often find that the stages that the kids go through in one of the 40-minutes sessions are identical to the ones the kids went through in 16 weeks. For that matter, if you look at Piagetian literature and you look at proportional reasoning for that particular problem, we find that the stages and thinking about the problem are often identical to those that kids go through over years in conceptual development. Now that turns out to be an important thing for all kinds of reasons.

For example, when I consider the findings from the rational numbers project over 16 weeks, I can get depressed about what teachers understand and what kids understand. We found that teachers often didn't understand much more than their fourth graders when we asked them problems that probed deeper and broader understanding. It slaps you in the face what people *don't know* and what they *don't understand*. On the other hand, in the 40-minute problem-solving sessions, I think what turns out to be really obvious is what people *do understand*--that is, *average and lower ability kids invent mathematics all the time*. If teachers learn how they can fit into that, then I think you've got a whole different scene.

Deborah Lowenberg Ball

My starting point for the work that I do is that I am interested in how to change what goes on in school in mathematics, so that kids have opportunities to become engaged in serious mathematics so they can develop power to use and to learn mathematics. I am interested in how we can do that in schools in such a way so as to bring together the substance of mathematics with the goal of helping kids to reason and communicate about mathematics and to think mathematically. My basic assumption here (where I differ from Dick) is that teachers are key. My research, as I take you on a path in it, may sound "messy."

My research is really on teacher education and teacher learning in order to address my starting point which is how to change what goes on in classrooms. My second assumption, (where I think Dick and I also differ) is that I don't see what he laid out as the three structures of mathematics, learning and instruction as somehow separated, but rather as very much interwoven. What I am interested in is this:

From the point of view of what teachers do in classrooms, how could teaching work to enable the things to happen that I have described; that would bring together mathematics, instruction and learning; and would change the kinds of outcomes for kids in schools?

The easiest way for me to try to explain what I do is to talk about it in two parts. The first part has to do with my teaching a third-grade mathematics class. I am going to talk about that by referring to it almost as an intellectual practice. What I am attempting to do by teaching is to try to articulate images of the kind of teaching that brings together mathematics teaching and learning and enable kids to learn serious mathematics. There are two parts. First, I am working hard to create a picture of practice for myself that achieves these things. I am experimenting on a daily basis. In the context of teaching as a teacher, I am trying to track and follow what my students are learning in a way that any teacher would do except that I have more time and opportunity to do that. I work hard trying to think about creating this practice and looking carefully at what students are learning from that both individually and as a group. I am interested in how the group is changing as a mathematical community because that is part of my goal. I think about getting mathematics together with pedagogy, and also about how individuals within the community in my classroom are changing--how they are learning. So that's one piece of what I'm doing--I go out there everyday at 12:30, and I think hard about how to teach my students and how to bring them along, and I think about different parts of mathematics and mathematical thinking.

Based on what I am doing, I spend a lot of my time analyzing and reflecting on what teachers need to know and on what I have to think about--what I need to know about mathematics; how I think about mathematics; how I think about students; how I learn about both of these things; what I need to know about learning; what assumptions I make and how I think about them; how I reason about those things; how I reason about mathematics, about learning, about children; what kinds of things I pay attention to.

That kind of analysis and reflection gives you an idea of why I think of this work as an "intellectual laboratory." My analysis provides me with conceptual frames, for instance, for what kind of subject matter knowledge teachers really need in order to teach. This laboratory also provides me with questions and with genuine pedagogical problems and challenges that arise in a real classroom. (I don't attempt to make the claim that I am a regular elementary school teacher. I think of this as an opportunity to think seriously and hard about mathematics teaching.) These frames, problems, dilemmas, challenges, and questions that I distill out of paying attention to my own work then provide me with starting points and ways to organize the research that I do in the second part of my work which focuses on how it is that teachers can learn to teach mathematics.

There are two aspects to the way I think about how it is that teachers can learn to teach mathematics. One aspect focuses on research on learning to teach mathematics. The things that I learn from paying close attention to my own teaching of mathematics give me lenses and angles and frameworks for thinking about what the key questions are to pay attention to in studying people who are either beginning to teach or have been teaching and are attempting to change their practice. My own teaching gives me lenses for looking at teachers and for thinking about how they are changing or what they are learning. For that kind of work, I look at teachers with whom other teacher educators are working and at teachers who are teaching and learning from their own practice. I try to make some sense of how it is that people change and what kinds of things teachers learn as they evolve their own practice as teachers.

The second aspect of my interest in teacher education (and now this sound just like what I do in third grade only one level up) is to create alternative practices for preparing teachers, for helping teachers think about teaching, and for studying the impact on teachers in terms of changes in their own knowledge, skills, dispositions, and practices. Ultimately, I think the key question here involves a full cycle which is something that the individuals involved in the CGI [Cognitively Guided Instruction] Project have tried to build. That full cycle involves looking at changes in teachers and also at changes in their own students. That is not something that I currently try to do. I am interested in paying very close attention to what other teacher educators or other contexts for teacher learning provide as opportunities for teacher learning, and I explore what kind of changes then occur for teachers in those contexts. The teacher is the learner in this part of my research, and the teaching consists of the educational opportunities for learning that teachers have. The cycle for me then is to use my own teaching of mathematics as a way of getting clearer about what teachers might be trying to create.

Suzanne Wilson

I think we are making too many general statements about what mathematics is and what we know for sure about this and that. I think that all of these terrains are very contested. Maybe the people in this room agree about what "understanding" is, but I think the world at large has quite a diverse set of opinions what it might mean to "know mathematics" or to "understand mathematics."

I will use the field of history as an example because I don't know mathematics as well. When I think about teaching in an elementary or secondary history classroom, I have two very different images. One is that I could model the classroom after a meeting of the American Historical Association where a group of people who think very differently about what history is, how one does history, and what the purposes of history are argue with one another. That is fine, but those people come to the meeting with different sets of beliefs and understanding about the things they are talking about. That is one image of the classroom. Another image of the classroom is to take a particular stance and model my classroom after one image of history. You could take a subgroup, for instance, the women studies community where all the historians are studying the same content or a particular methodological group. I would pick that image of history based on my understanding of what children are doing, how children learn, what they are interested in, and what I think it is important to them to do. But in both of these cases, whatever choice I make (and I might make one choice one year, and one choice another year), I am acknowledging the fact that within the field of history, there is absolutely no agreement about what history is and about how one does history.

I agree with Deborah that teachers are integral in this because teachers are where the choice gets made about the environment. If teachers don't understand those things about the subject they are teaching, then they are unable to provide and set up the learning environment in which kids learn things. Although kids may learn things when the teacher isn't there, nevertheless, the teacher is the orchestrator of that environment.

When you were talking about mathematical structures, you said *the mathematical structure*. We have to acknowledge that when we study teaching, we study it in a content domain. How we understand subject matter is one of the things that is important to understand when we think about the teaching of a particular subject matter. We organize our thinking around research on teaching and learning of mathematics based on how we understand mathematics, and teachers do the same thing in classrooms. And so I have a question about whether there is *one* mathematical structure or whether there were other mathematical structures that other mathematicians would come up with.

Thomas Cooney

I think we all understand in some sense that mathematics isn't as fixed in the way that history isn't. On the other hand, [Bertrand] Russell said something like "mathematics is a subject in which nobody really knows what they're talking about," so in some sense, I think that mathematics is as arbitrary as history. But whether it is or isn't, I think we certainly want to convey to kids that there is a lot of arbitrariness in mathematics.

I like the notion of mathematical domain, but I think it's a little tricky to talk about "mathematical structures." There's something that's sort of dehumanizing about that word in the sense that the structure is "there" and then somehow you get the kids to acquire that structure. If you think about the advent of non-Euclidian geometry, there was an awful lot of humanization and human drama that characterized the rise and fall of different people struggling with that topic. Although I am probably not really disagreeing with Dick, there is a red light that goes on for me when I think about "mathematical structures" and about "trying to give kids mathematical structures."

When I talk to mathematicians at the University of Georgia, I am not so confident that I get some uniform idea of what constitutes mathematics. Quite frankly, we would be aghast at what they think mathematics is--some of them. In having a conversation with a really enlightened mathematician, by our standards, I said, "Henry, you understand there are some people teaching calculus and other subjects as if everybody in their class is a young Gauss [a famous 19th-century German mathematician]." He thought for a little bit and he said, "Unfortunately, they wouldn't know a young Gauss if they saw one." I think that really captures an awful lot. There are a lot of mathematicians who think we ought to help calculus students by sending them over to a computer so that they can acquire the rules they need to acquire in order to conceptually deal with calculus issues because the high schools failed miserably in teaching students the rules and procedures that they need to have. Another mathematician makes statements like, "My job is to get people to see mathematics come alive. If my lectures are carefully planned enough, I think I can do that." That's a direct quote. I just wanted to give you an idea of how mathematicians think about the world.

Your structures, Dick, are probably helpful, but I can remember listening to you talk at PME [the meeting of the group for the Psychology of Mathematics Education] about one of your more interesting studies where you had teachers solve problems. You monitored how those teachers solved those problems. Then you had teachers present those same problems to kids. I recall that what you found was that the way that the teachers helped the kids solve the problems was exactly how they solved the problem themselves. It seems to me that result has an awful lot to do with the cognitive structures of the teachers and how the teachers are trying to impose their cognitive structures on the kids' cognitive structures which I think provides you with the link. I realize that you are using the terms a little differently. I don't want to get into a game of just talking about kids' cognitive structures because I think that the teacher's cognitive structure is every bit as potent a force in influencing how the kids are going to think about mathematics.

Paul Cobb

I typically avoid talking about "mathematical content" because the term carries with it the metaphor as something to be put into the the container of the curriculum, but I'm happy to use the word if we might personally redefine it and talk about it as "taken to be shared or normative mathematical knowledge." Now that's obviously relative to a community. Those communities can be very small like three people on a research team. I think you're also talking about the problem of the social distributive nature of knowledge, and different people have different views. If we were to go around and talk about what "place value" means to us, in one sense we will understand it and agree, but the further we would go we would find incompatibilities. The view of "taken for granted normative activity" comes from the work of Wittgenstein, George Herbert Meade, and Alfred Schutz. Now if you think of content that way as opposed to this object or this structure, then analyzing content and analyzing normative conceptual interpretive activities has possibilities. I would argue that you'd have to have some thoughts about where you might want the kids to go or what the developmental possibilities as you interact with kids in the classroom.

I think my colleagues [Erna Yackel and Terry Wood] and I have a different perspective on looking at the relationship between learning and teaching. We are in the fifth year of an ongoing research and development project. What our current problem is right now is just to find a way of coping with the complexity of what's going on in the classroom. That's it. I'm not talking right now about how we induct teachers into our project, and I'm not talking about how we develop the instructional materials. I'm talking about just, "This is what happened. How do we account for that? How do we make sense out of it?" That's the only issue here.

As we've gone through this, we've come to the conclusion that we can't talk about learning without talking about teaching and vice versa. I want to try and explain why that is, and what I think the linkage between the two is. When we get a transcript, one thing we do is to conduct analysis of the social interaction, for example, when the kids work in groups and during whole-class discussions of their solutions and interpretations. When you take that perspective, you're looking at interaction patterns that emerge as the teacher and students interact together. Those are much easier to identify in the whole-class than in the small-group situation. You look at the nature of the discourse. You look at what the participants, the teacher and the students, take to be shared in order to communicate successfully. You look at situations where there are conflicts or at least differences in their interpretations and how those evolve and get resolved and so forth. What you're doing is you're taking an outsider's perspective--you're looking down on them, and you're trying to figure out how they coordinate and fit together their activities and how they negotiate a relatively smooth flow of classroom life. What we can also do is analyze exactly the same transcripts from a cognitive perspective. Here you're focusing not on, "What is the structure of the interaction?" but on, "What was the experience of the teacher in this process?" or "What was the experience of the student?"

I think there are two sorts of things to ask here. One is to think about the social level. What sorts of expectations do they have for each other's activities? What sorts of obligations do they accept for their own activity? I think there that research on both teachers' and students' beliefs becomes very important. That's very relevant to us. We're interested in the general nature of mathematical activity, the individual's own role, and their beliefs about others' role, and that applies to both teachers and students. I think you can also ask: "How does this particular individual interpret the task or the problem? What meaning does their own activity have for them? How do they interpret other people's activities with whom they're interacting?"

It's clear that research on learning (in the case of students) comes to the fore. One reason why we focused on arithmetic in the second grade is because that's where we felt the most progress was being made in terms of developing models of students' learning. Also, coming in there are what is known about teachers' knowledge of second-grade mathematics, their students' mathematical

thinking, and their students' developmental possibilities. What we hope to get out of that is an analysis of both teachers' and students' learning in the process of classroom social interaction. We are taking the view that the classroom is the primary site of teacher learning. (I think this also comes out of the Wisconsin CGI project.)

We see those two--student and teacher learning--as very much interrelated in that kids and teachers are constantly, continually orienting each other's activity. Suppose any one person in there learns, figures something out. That means they're going to act differently, and they're going to interpret other people's activity differently. In other words, that has repercussions for everybody else, and so it goes. It's not a linear thing starting with the teacher or the students, it's a very dialectical, self-organizing sort of view.

The point I'm trying to make is egocentric to the story--it seems to me that the place to integrate research on how we go about thinking about learning and how we go about thinking about teaching is by looking at classroom social interactions because, in the course of those events, teaching and teachers' learning and students' learning are occurring at the same time. I don't think it's just a matter of whether we could coordinate them; I think we should coordinate them. It seems to me that's the place to go. The other thing I would say (given that "constructivism" seems to be *the* thing right now) is that what we find very helpful in terms of the social perspective is social interactionism and ethnomethodology. I think that also gets you around the view of having a dualistic philosophy--where there's what the kid knows and there's mathematics out here--and how we get them into contact. It's a matter of negotiation where the teacher knows mathematics crudely, has a thought system, and society is negotiating meaning with the student.

Magdalene Lampert

I think Paul spoke eloquently about the fact that you can't address the notion of understanding unless all the people in the room have some sense of negotiated meaning and have some sense that they're talking about the same thing. Part of teaching mathematics for understanding is developing languages or representations that express both the important mathematical structures and the way kids would come at these in whatever grade level you're working on. An important aspect of teaching is building in several opportunities to check on whether you're talking about the same thing. I think that we have a lot to learn from the area of linguistics and sociolinguistics about the notion of how people come to be talking about the same thing in a setting. Communication is a biggie.

The second biggie in my research on teaching is how the teacher acts to build a culture of sense-making in the classroom. That also has to do with what roles people assume, what expectations people have for one another, how those expectations are communicated, teacher and students reasoning together about what all this means.

The third thing is: "How do you move around in the territory that you want kids to learn, in a way that represents both a response to their engagement and some sense of what's worth learning, and make a connection between those two?" In my view at that level, as a teacher, a lot of very hard issues are raised: How do you deal with diversity? How do you justify the fact that some kids learn one thing and other kids learn something else in a school setting where it's assumed that equity is an important issue? How do you assess what kids are learning, both on the fly, and in some way that you can represent, to their parents, to the principal, and to them, what they're learning? How do you pace yourself through a mathematical terrain in a way that is adjustable to the incredible diversity of what students bring, not only because students are themselves different from one another, but also because they're different on different days? (It depends on what went on at recess, who walked to school with whom that morning, or whether it was the night before that they saw the person that tutors them in computational facts, and so on.) All of that enters in when (as a teacher)

you walk around the classroom, watching kids do very interesting things with interesting problems and you think about, "What am I supposed to do next?"

I was thinking about the [mathematics as a] "contested terrain" issue, and I noticed that even Liz recognized the fact that people don't agree on what constitutes the domain of multiplicative structures. Within the mathematics education community people don't agree about either how to represent rational numbers or what's included in rational numbers. Now, as a person who thinks about these things from the perspective of the mathematical structures, I find that those questions face me in spades every day in the fifth grade because I am trying to teach fractions, decimals, division, multiplication, ratio, proportion. You give kids an interesting problem, like the Sears Roebuck Catalog problem, and all of those things emerge all at once, and you have to somehow construct a relationship between them in the conversation that you're having with the children. Whether you have a very sophisticated sense of what ratio and proportion entails, or a very primitive sense, you are constructing that meaning with children in the classroom. That is a place where I think we need to figure out what teachers can bring to that conversation that enriches it in the direction of mathematical structures as they are thought about in the mathematical discourse community, but we need to be careful about the fact that we not see that as a monolithic kind of thing.

An issue in mathematical discourse that occurs to me as I'm teaching fifth grade is: "What constitutes evidence for a mathematical assertion?" The question is there of providing an intuitive assertion that makes sense versus a deductive argument which is constantly going on in mathematics (and Paul alluded to). I have think about that every minute that I interact with my kids. Is what that child said enough to substantiate the answer that he gave? When I say "Why do you think that?" or "How do you know your answer is right?" I have to make a judgement about what I'll accept as an answer.

Once you open up mathematical education beyond learning what Paul called "school mathematics," that is "the record of knowledge," and you start thinking about constructing knowledge, and connecting that with the record of knowledge, then it becomes much more complicated to try to produce a coherent set of activities in the classroom.

Andee Rubin

I want to tell two stories, which I hope will connect to what Maggie has said and to what other people were saying although these are more concrete than they are philosophical. One is to say a bit about where I come from. I spent about seven years working with the Center for the Study of Reading in Champaign, Urbana, and, as part of that, I was developing a self-awareness curriculum for teaching writing in upper elementary schools. Although this may, in some ways, seem far from the teaching of mathematics in elementary schools, I see a lot of parallels that I think are important. I want to use that perspective from outside to look at the process of teacher change and at how classrooms change when an innovative way of thinking about a subject is introduced.

Some of the issues that I think are important arise out of looking at a whole series of classrooms that were in an experiment to use computers in teaching writing where the change was toward a lot of collaborative writing and the use of the computer, not just as an editing environment, but also as a way for students to send mail to one another and to plan their compositions as well as to write them. One thing we had talked about was having students choose their own topics and genre for particular pieces of writing which hardly ever happens in typical classrooms. Choice of topics and genres was so unfamiliar and uncomfortable for many teachers that they allowed it to happen only in a setting that we had constructed that was not like school. This was the electronic mail sending. Because this setting didn't count as school, teachers felt comfortable breaking the rules, but as soon as the focus came back to the pieces of writing that teachers were going to grade and put in students' portfolios, teachers didn't feel comfortable breaking the rules. Teachers themselves divided their

language of practice into "serious" and "not so serious." Teachers saw it as OK to experiment with the "not-so-serious," but they thought they had to conform to their old practice in the "serious" pieces of teaching language.

I have seen these things happen in mathematics classrooms as well. When someone is faced with a radical change, a lot of what happens is segmentation of practice and a lot of what happens is assimilation of practice to earlier practices, but with small changes that look like they're changes but may not be or are not full changes.

In terms of parallels, one of the most important factors in what kinds of changes teachers could make was whether teachers saw themselves as writers. In the same way, whether teachers see themselves as mathematicians is a big factor in how much they set up a context that allows teaching their students to think of themselves as mathematicians. Writing was useful in the sense that we set up an electronic network that allowed teachers to write to one another so what they were doing on this network actually was symbiotic to what they were doing in their classrooms. The setting up of the social environment gave teachers purpose and audience for their writing that allowed them to see that activity as purposeful. That helped teachers transfer that vision of writing as purposeful back to their students. Now it seems to me this has to do with mathematics because if you interviewed elementary school teachers about the purpose of mathematics in elementary school, I imagine you would get some very different and not terribly coherent answers. There is not, in fact, a community in which elementary school teachers participate where mathematical thinking is valued as a piece of that community. There is *not* a community in mathematics parallel to what we were able to set up for writing for elementary teachers that would allow them to have a reason to think mathematically in an audience and that would enable them to take that insight back to their classrooms.

Merlin Wittrock

I would like to address the issue that you raised--that very good question on what research is needed on the teaching of mathematics. One of the things we need to do research on in the teaching of mathematics is ways to teach for learning with understanding that build on the research that are in those three areas that you identified: the content analyses of mathematics, the research on teacher beliefs and knowledge of mathematics, and the research on student novices, both on learning as well on what they believe and understand about mathematics. If somehow or other, those things can be put together and research can be built on those things, then I think we are going to do something that I think really needs to be done.

I would like to start with an analogy. Imagine that we are back in the days when people believed in medicine that patients were dying on the operating table not because of germs infecting the wounds, but because of other reasons. Imagine that you had a theory that said there were all these strange little microscopic organisms floating around in the atmosphere, and it's those things that you can't see, but that are really there, that get into the wound and cause the patient to die.

Now it seems to me that that is somewhat like what students are facing when they try to learn some new idea in science or mathematics. Sometimes the ideas that we have to present to them as teachers seem as weird as that. Now imagine you were a teacher in that context. Would you simply stand up and tell the people, "Don't wipe your scalp, fellow, on your cuff, because that may get some of these little guys into the wound, and that's really what's going to cause the damage." Would you stand up and say that? You would probably be laughed out of the room if you tried that, wouldn't you? You would need something more than that.

If you follow that analogy further, you find that what you need in order to be effective as a teacher is some understanding of the belief structure and the knowledge structure of the people you are trying to teach. You need all of the elements that we have talked about here, or all the things that are

involved in the questions you posed to us in the beginning. You need all that kind of information to build on, and you still aren't home yet because it's not that easy to go about changing beliefs. You might think about: What did finally bring about changes in those understandings, in those beliefs? Was it done by teaching, or how did it happen? Did it happen as conceptual change? Did it happen gradually, or what?

I have been studying that kind of problem in mathematics and in science. Some things that we found and some things that we don't understand are the following: First, we have been studying how children see current flowing in simple DC circuits, and we find that there are basically just three or four different conceptions that kids have of how current flow.

One of them is that there are two different currents that go from the each side of the battery. When they bang together, that causes the light bulb to glow. These ideas really "blow you out of the water" when you first sit there and hear these little kids saying these things. But what really "does you in" is when you try to change children's beliefs. They are impervious to many kinds of teaching practices. You can't simply come in and tell them, "You know, you're wrong. This is it--the way it is in the book is right." Even if you give children demonstrations that show them it doesn't work, they rebut you by saying, "That's *your* demonstration. That's *your* battery. That's *your* light bulb. Teachers are sort of magic; they can make these things happen!" You say, "OK, take the stuff home with you, and do it there," so the children take it home. Then they say, "Yes, it works, but it is still your stuff. We don't believe it because if you really could get inside the wall and see what's going on inside the wall and in the light bulbs, you would see that we're really right!" So, it's not such a simple problem. At least with children of that age--sixth grade.

Years ago, I did several studies on teaching kinetic molecular theory to first and second graders and had a lot of success. A year later we gave them standard Piagetian tasks such as asking them, "Where does the water come from on the outside of the glass that has ice water on the inside?" Two thirds of the first graders who had gone through the program would say that the water condensed from the atmosphere. That's the language they would use. So, the first graders could adopt that kind of conceptual structure without too much "intellectual pain." But they did have difficulty where we thought they never would. For example, we would use words like "gas," which horribly confused them, because to those kids, "gas" was something that you put into the tank of an automobile. For us to say, "Gas as you know it is not gas. Gas is a liquid. What you're calling gas is a liquid. Gas is really gas." Later children said to me, "Look. This whole thing is just a big joke. There really aren't things like molecules. This is all a game, right?" The problems are really different from what one thinks they are.

I think one has to learn how to put the pieces together--in this case the pieces being the content analysis, the teacher's beliefs, the students' models, and what we know about learning with understanding. We need to put those pieces together and address the problems that involve teaching. We know something about those teaching issues, but we really don't know enough. We need to find out more and more about the complexities involved in teaching children. One of those complexities is going to revolve around the notion that we've got to help children to construct some of these answers. That doesn't mean we can't tell them, but it means that even if we do tell them, they still don't understand necessarily, until they are somehow led by us to find a way to make sense out of these things. It takes more than letting children take the materials home and manipulate them. We have got to find out what that something really is.

Audrey Jackson

One of the comments that Merl made was that teacher beliefs about students is one of the crucial issues to take a look at as we try and take a look forward. I had an opportunity to spend an entire semester with a teacher. She was tired of the way things were going--the pre- and the post-testing, and the kids being slotted into certain groups, and the kids weren't learning. She wanted to do

things differently, so I brought in a lot of different construction kinds of things, more drawing, more visual, more communication--all the things that we've been talking about. But she didn't feel comfortable in the domain of mathematics. She saw herself as a very, very strong language arts individual, so she felt very comfortable in that domain to make changes and to make alterations in the curriculum. She did not feel comfortable in mathematics in doing that. She would say, "The kids don't know their facts; they don't know their computations." When I would present it to her that she could address computation by another vehicle, by looking at geometry, by talking about measurement, or by solving problems, she would tell me, "The kids don't understand by looking at measurement, by doing construction." Now it's finally making sense to her, but I had to end up spending an entire semester with her for her to see that. I met with her last week, and now her ideas are just flowing. She feels far more comfortable doing it this year than she did second semester of last year. But there are tons of teachers out there who can't do that.

Teachers believe and accept what the mathematics community is telling them. They know that's the right thing to do, but they don't know how to do it. They really don't know how to do it. Even though textbooks now are saying, "Let's do this; let's do this." Textbooks don't assist teachers in getting at the "hows" of the teaching. We talk a lot about communication in mathematics and a lot about verbalizing in mathematics. Well, how do you communicate in mathematics? Do you talk about how you feel about the product of four and six? How do you get at these things in a math class? When I talk with teachers, I tell them, "You want your kids to discuss; you want them to talk about all these different strategies, all these different ways of finding the answers." That's a hard message for an elementary classroom teacher to grab a hold of.

Part of it is that teachers have the idea that a mathematics class has to be rigid. That conception is still out there. And from my perspective it goes beyond the teachers. It goes to the administration. In the 17 elementary schools that I service, I have found that, if the principal understands what needs to happen and what needs to be done, is flexible in his or her thinking about education, and is supportive of innovation and creativity, then change takes place. But if you happen to be in a school where the administrator feels that you're going to have to look at a standardized test score and if that individual does not know how to read the standardized test scores and make some sense out of them then there are a lot of problems there.

Recently, we were doing mental practice exercises in an inservice with teachers, and a teacher was adding 49 and 11. The teacher had never thought about making it either 50 and 11 or 10 and 49 and 1 more. And she said, "OK, I did that one. Now what do I do?" I can't go past that. There's a stumbling block that's there. . . . Teachers believe in the Standards (National Council of Teachers of Mathematics, 1989)--that message is out. But teachers are struggling with implementing the Standards, and the research community needs to take a look at ways of addressing teachers' struggles.

Susan Jo Russell

I want to ask two questions. One question is: How do we make occasions for teachers to grapple with content differently so they can recognize when students are grappling with content in some significant way? For example, I recently observed an interesting discussion in a fourth-grade mathematics class. The kids were planning how they were going to compare their heights in their class with children's heights in a first-grade class--how much taller is a fourth grader than a first grader? What were they going to do in order to answer that question? They hadn't even gathered any data yet; they were just thinking about the problem and what they were going to need to do. One kid said, "Well, after we've measured everybody's heights, we should find the one number that's maybe, in the middle, or maybe that all the other numbers are crowded around. Then we should do the same thing with the first graders, and then we should compare them." Now there he was inventing the whole idea of center--there it is! But you've got to notice something great is happening there, and grab hold of it.

What did the teacher do? The teacher accepted that as an interesting idea and went on to the next idea. She really didn't have the tools to recognize that that was a spectacular idea. This teacher is a teacher who said to us at the very beginning, "I'm really dissatisfied with mathematics." "Kids don't like to be inventors any more," is what she said, and yet much of what she did in her practice, in fact, discouraged kids from being inventors.

There was another incident in the same classroom in which kids were trying to describe a set of data about how long they could hold their breath. They looked at this gathered data under a number of conditions. Kids came up with all different answers, some of which we would consider reasonable, and some of which, we, in this room, would not consider reasonable. And, finally, after some discussion during which the kids were very interested, a student asked, "So, which is right?" The teacher, taken aback wanting to maintain this atmosphere of "everybody encouraged to participate" said, "Well, they're really all right. Nobody's really wrong." And you can understand what she was trying to do there. She was trying to do something very important there, but what happened was that the discussion immediately fell flat. The sense that the students had was, "Well, no matter what I say, it's kind of OK, but it's not really very important. What I say is just as important as what anyone else might say." So there was no challenge there, and there should have been.

For me this case raises the question of "How do we make better occasions for teachers to grapple with mathematical content?" In some clinical interviews that we have been doing with students and with their teachers about ideas about average and their ideas about data sets, we have found that teachers have many of the same misconceptions that students have.

During one very interesting interview I did with a woman who is a staff developer in mathematics for a K-5 school, the woman actually discovered what the structure of the mean is. She started out with an interesting misconception that we see frequently among kids. The problem was, "If you know the average allowance that a kid gets is a dollar fifty [$1.50], what might the data set have looked like that produced that average?" When children try to construct the data set with tiles or with a graph, we find that some children balance the amount of money on each side of the average. They are balancing the total amount of money, not looking at deviations. The staff developer did this, and she went through that wonderful state of knowing it wasn't quite right. Then she went on in the interview to test her own notion by simplifying the problem and using wonderful strategies. By the end, she realized that it was the differences from the mean that mattered, not the absolute total. The woman turned to me and said that she has never had the opportunity to do that, to delve into just one, small piece of mathematics--deeply in a way that got her past the surface understanding that she had picked up along the line earlier.

So there is a lot of work to get to happen with teachers. There are important policy questions and staff development questions related to "How do we get that to happen?"

My second question is, "What are the places that teachers pass through on the way from where they are to where it is that they (and we) might like them to be?" I want to read a quote from another teacher, Eric, who was a field test teacher for us in the data analysis curriculum work. Eric got very excited about this curriculum. It was very different from what he had done before. It allowed him to do more activity and have kids working in groups. The children liked it; they were collecting real data; the problems were problems that were of interest to them. Teachers like Eric see this as a vehicle they might use to change their mathematics teaching. This is what Eric, a third-grade teacher, said about that experience:

> I find that bringing in different methods and techniques, different ways of trying something, getting the kids involved on a personal level, is a much better teaching style than saying, "Get your books out, open to this page and

> we're going to have a lesson. And your homework is to give me those numbers back." And what I find is I make the room a little noisier, I make movement and motion happen more often, but I make the learning more concrete. And working with an urban population, the kids need those experiences, because it reinforces all of the numbers, and indeed when the numbers come up, the experience is much more success-oriented for them, because they've had the actual physical involvement.

Note the way Eric characterizes what he's doing differently. He is somebody who is in transition, trying to replace the old mathematics with some new mathematics. He uses all the words that I have heard lots of teachers use--activity, success, involvement, concrete, movement, cooperation, concrete, real--but none of these describes a change in mathematics, or in intellectual content. In fact, there was some change in mathematics content, on some occasions, in Eric's classroom, but more often not. I characterize this as a teacher's move from where he was, teaching traditional mathematics, to a place where he is beginning to tolerate physical mess. But he hasn't gotten to the place where he is going to tolerate intellectual mess, and that's much harder. I don't know how do you get from this change. This is a very significant change, and these teachers really feel that they are doing something differently; they are trying to give their kids a different experience; internally teachers feel like this is a significant change. But how do teachers get to that place where they can tolerate the intellectual mess? What are the leaps and plateaus? What are the increments, if there are increments? A teacher doesn't just go from here to there. It's a kind of wrenching experience.

Hilda Borko

I think that I come from a really different perspective than other people here, based on my background as a teacher educator and as an educational psychologist, and also based on the kind of project that we are doing. We are are not looking at students' learning of mathematics, and we are dealing very little with students' knowledge of mathematics or with an analysis of the content in mathematics. What we are very much interested in is teachers' knowledge, beliefs and thinking, and what we are doing is following teacher education students through their last years of teacher preparation, including their methods courses and their student teaching. We did that last year. Then this year we are following these individuals into their first year of teaching, and we are trying to understand the process that these teachers go through in learning to teach.

Our project, also funded by the National Science Foundation (NSF), is similar to Paul Cobb's NSF project in that we bring together the perspectives of psychology and anthropology. We are looking at learning to teach in context, and we think that we can't understand the process of learning to teach without understanding the kinds of experiences that people are having in multiple contexts--their teacher education program, their methods courses, and their other university courses, and in the public schools. Last year we spent a lot of time observing student teachers. We sat through entire mathematics methods courses, and we interviewed the mathematics methods instructors, before and after each course. We interviewed the students in the course about what they were learning. Then we followed these same individuals into their classrooms during student teaching. We watched them all year in their student teaching, and we talked to them about what they were trying to accomplish, what they thought they did, and how they thought they were doing. We also talked to the cooperating teachers with whom they were working and to the principals in the cooperating teachers' schools. We try to get at individuals' understanding of what learning to teach is, and what kinds of experiences teachers ought to have. We are trying to paint a picture of how a small number of people--eight people last year, and then four people this year--learn how to think like mathematics teachers, and how their knowledge of mathematics and their knowledge of mathematics teaching is changing over the two years, and how their experiences in the university and in public school settings influence the learning to teach process. So, our questions have most to do with their thinking and with what we are calling their "claims." What I bring to

this discussion then is a sense of how more and less experienced teachers think, not so much their knowledge, but their thinking--how they plan, how they come up with examples, how they come up with explanations.

We struggled a long time, with an incredible number of personnel, with the issue of what's knowledge, and what's beliefs? The more we struggled, the less satisfied we were with our answers. We struggled with the idea that it is how teachers hold them or that it is the kinds of evidence that will help prove or disprove the statement that makes it "knowledge" or "belief." That falls apart. We argued the issue among mathematicians, psychologists, and anthropologists, and every distinction that we tried to make between "knowledge" and "belief" fell apart. Finally, we decided that if our data help us understand knowledge and beliefs better, we will try to make some statements about differences between knowledge and beliefs. Right now we are looking at what we term teachers' "claims" about mathematics and about teaching, and we are tracing teachers' claims over two years. By using the term "claims," we are acknowledging that we do not see any clear-cut, neat distinctions between teachers' knowledge and teachers' beliefs. We are trying to understand the structure of the teacher's claim, and the teacher's thinking, and the teacher's action. Those are the three kinds of constructs that we are using to help us understand the process of change that student teachers go through in learning to teach.

Our present study is a descriptive study one. For example, we know that there is a clear gap between the message that the mathematics educator tries to get across in his or her methods course, and what student teachers end up doing in their own classrooms. There's a breakdown there, and we're trying to understand where that breakdown is. After this study, we would like to be able to talk about some interventions that will help teachers learn to teach mathematics in better ways than currently exist.

Peter Kloosterman

At Indiana University, one intervention that we are trying involves a significant change in the mathematics course requirements. We have thrown all the elementary teachers into a finite mathematics course that is also taken by five thousand other people at Indiana University. The hope was that being is such a course would build prospective elementary teachers' confidence and make them feel like they really know something about mathematics because they "got through" this course. Although the course involves a lot of problem solving, it is taught in a fairly routine manner, and I am not sure if that helps prospective elementary teachers understand what mathematics is all about.

Our project's major focus is on problem solving, and on problem solving in the narrow sense of solving word problems. Beginning last year and continuing for the next two years, we are working with one elementary school to try to make it a model site for teaching mathematical problem solving. It is a typical elementary school. We are spending a lot of time working with the teachers there, with the hope that when we send preservice teachers out there to be involved in field experiences, the student teachers are going to see something better than what they would normally see. They are actually going to see teachers who know something about mathematics and about how to teach it. Whether that will happen, we don't know. Although we have made a lot of progress in one year, we haven't changed everybody's mind. There are a lot of teachers who are still struggling with, "Here's the mathematical idea that I want to get across, and I don't have any idea how to do it."

We are interested in determining how effective this project is going to be. There are many parts of the project. We are developing new mathematics courses, we are developing new methods courses, and we are going to have actual products including course syllabi. We

are looking at the effects of the project on our population of preservice teachers as well as on the teachers and students in the one elementary school where we are working intensively.

One focus of our project is on teachers' beliefs about mathematics, about what it means to know and do mathematics, and about how mathematics is learned. I was laughing at Hilda's comment about not being able to tell the difference between knowledge and beliefs, because we too have gone around and around on that issue for quite some period of time without coming to any resolution on it. I think if you can change a teacher's conceptions of what mathematics is and what it means to teach it, then you have really done quite a bit. So, we are going to try and document whether and how we have had any success at doing that over several years.

One of the new projects for which we are seeking funding involves looking at what teachers do to transmit beliefs to kids. Obviously, there are explicit messages that the teacher transmits such as saying, "Story problems are a waste of time," and then the kids end up thinking that story problems are a waste of time. But there are also a lot of unintentional influences of teachers, the curriculum, parents, and textbooks. These all influence the conceptions that children end up having about mathematics, but we are particularly interested in what role the teacher plays.

Ralph Putnam

When I first heard about the Wisconsin Center's agenda of trying to make a closer link between research on teaching and research on learning, I thought, "Oh! That's just what I've been trying to do for a while." I feel like I have been at that for quite a while now, and still not quite making it. I feel like I am still searching for a paradigm. Coming from a research on teaching background, I was always left with this feeling in my mind: Am I getting at the heart of instruction. Am I getting at the content? Am I getting at what students are really learning? Nonetheless, the learning research always seems to miss something about the classroom context, and I have tried various ways of getting at that.

Currently, I am still trying to link research on teaching and research on learning in the work that I am now doing in the Center for the Learning and Teaching of Elementary Subjects. Penelope Peterson is involved with it, and Dick Prawat as well. We are doing case studies, and in these case studies, we are trying to attend simultaneously to what's happening with teachers and to what's happening with students in terms of learning. One way I like to think of it is trying to find the "best practice I can find" in a classroom that is close enough to East Lansing for me to study systematically over a period of time. Then I try to understand what it means for this teacher in this classroom to teach mathematics for understanding and what effects that mathematics teaching has on the students in that classroom.

Now I want to address the three issues of teacher knowledge and thinking, student learning, and the mathematics content. The first, teacher knowledge and thinking, is an easy piece for me because the teacher and her thinking was what led me to select this classroom. The teacher is the person I interact with a lot over the year, and it is very easy for me to think in terms of getting a handle on how she might be changing and on how she's thinking about mathematics.

When I start thinking about how to track students, it becomes much more difficult. I think, "Well, I want to really get into the content. I want to know how these kids are learning, say for example, fractions, or rational number concepts." But I hit this dilemma of not being able to hook into the research in that domain for a couple of reasons. Often it seems like questions that the researchers in these particular domains, such as rational numbers

and multiplicative instruction, are not always posing their questions in ways that seem to reflect what is going on in the classroom. Another problem is that although I care very much about how people learn particular mathematical concepts, I also am interested in the bigger picture, so I don't want to just look at learning of fractions, for instance. Also, I haven't found a way to manage to "drop into" the classroom so that I always am always there for the teacher's instruction on fractions. As taught in the classroom, topics don't cluster themselves necessarily around these kinds of domains. This problem is compounded by the fact that two teachers I am studying are using the CSMP (Comprehensive School Mathematics Program) curriculum (CEMREL, 1985) which is the ultimate spiral curriculum in the sense that from day to day, the topic varies. I am unable to go into the classroom knowing what topics will be taught that day. However, I can use general knowledge of the domain for the mathematical topic that the teacher happens to be teaching, and I can try to find out more about it later. It works out better if I am studying a teacher who happens to structure the content so that she says, "We're going to do fractions now for three weeks," and I can study that. I can interview the kids before, after and during this three-week period. That's not working for me now, because that's not the way the teacher structures it. So, I guess this is the biggest problem I face in trying to take very seriously the research on learning of particular topics within the context of doing a case study of a teacher's mathematics teaching.

The other driving question for me is: "How do teachers deal with the tension between attending to student's thinking and moving through the mathematics content?" A related issue is the tension between telling kids and having kids build on their own understanding and construct meaning. I am interested in how teachers deal with that tension between finding opportunities for students to construct meaning and, at the same time, moving through a particular set of mathematical ideas.

Richard Prawat

I teach educational psychology, and I am beginning to think more and more that I am contributing to the problem that prospective teachers have in putting together children's learning and the curriculum. As educational psychologists, we push this individual differences notion--that there are fixed individual differences. We teach that these are "hard and fast" categories for looking at students. In examining interview protocols of some elementary teachers that we have been interviewing in California, I have been surprised to see how often this sort of thinking figures in their talking about the teaching of math. These teachers talk frequently about "learning styles." They have this notion that there are auditory, visual, and kinesthetic learners. Many of these teachers think that you just try a lot of different approaches to teaching mathematics with the hope somehow that these different approaches will reach learners who learn differently. Recently, I was reading Harold Stevenson's (1989) study comparing American teachers with Chinese teachers. Stevenson had teachers rank order a number of different factors in terms of how important they considered those factors in their teaching. For American teachers, sensitivity to individual differences emerged as being of primary importance. What does that mean? It seems to me that that sort of attitude really works against teachers being able to think interactively about the child and the curriculum.

Earlier in this century Dewey talked about how educators need to abandon the notion of subject matter as something fixed. He said, "Cease thinking of the child's experience as something hard and fast." (This is where I think the individual differences notion comes in.) "See it as something fluid, embryonic, vital." If we take Dewey seriously, we realize that the child and curriculum are simply two limits which define a single process. How do we get teachers to begin to put these things back together? There is not much historical precedent for getting teachers to think that way. Rather, teachers' talk tends to drift

naturally towards either a subject-centered or a child-centered way of thinking. Those two frames seem to structure teachers' thinking. I think we really need to begin to look at what we, as researchers and professors, may be doing to contribute to that kind of mind-set. I think it's difficult to get teachers to break from that.

Perry Lanier

I want to suggest two explanations for the existence of the discrepancy that Paul Cobb noted between what we would like to see in mathematics classrooms versus what we typically see in mathematics classrooms. I have two explanations for that--knowledge constraints on the part of the teachers and contextual constraints. Today participants have said quite a bit about the knowledge constraints, but I am just as interested in the contextual constraints.

If I look at the complexity that Maggie laid out very clearly, then I realize that classrooms are complex places, and teachers need to be reflective. How do teachers look at this complexity? How do they make sense of it? Dick Lesh is going to let them have a learning community or observe what the kids are thinking on the computer. That does give teachers further insight. But then I think of Ralph's question about why he's not seeing more of this--we can identify a few teachers that are doing quite well. But why are the predominant number of them not? I think that isolationism is one factor and contextual constraints are another factor.

Jaime Escalante came to Flint in June, and he made the statement to an audience of educators that he really didn't like to leave school because he hated leaving his kids. He said that he really wasn't sure what his students were getting while he was gone. Every teacher in the room nodded in agreement with that. To me, that epitomizes two things--first, that the kids don't expect anything when their teacher is gone, and second, that the teacher doesn't expect anything to happen when he is gone. If you listen to what we have talked about here today from a research perspective, we have talked about doing staff development and then letting teachers go back into their isolated classrooms.

I think classrooms are going to be there, and there are going to be sets of kids there, but I am not convinced that there has to be one teacher there. I said to a group of teachers in a discussion session that in some professions, when the professionals go to conference, then clients aren't scheduled for those days. The teachers looked skeptical and said, "Yeah, yeah, yeah," so I agreed, "OK, you're right. Teachers will never get this." We won't have the kids stay home, but I think there are some analogs from our profession that we might look at so teachers don't have this isolationism. For example, Tom doesn't think about these being "his" students, and we don't think about Tom as being their "teacher," but rather, these students have "teachers." That there is a set of professionals working with Tom, and they have an opportunity to look at the same educational phenomena.

In considering the three questions posed for this conference, I would urge us not to neglect the context of learning and teaching mathematics. Some of the interventions that we try and that we study as researchers should involve having professionals work with other professionals. As professionals, let's not give teachers "the stuff" and then have them go into the classroom and try to do it by themselves. Let's go in with them, not just to study them, but as an ongoing, everyday part of the way that classrooms and schools are organized. I think that if we're going to push for teaching and learning mathematics for understanding, then some interventions need to be studied in this professional context, and I would urge that this become a part of the agenda of this group.

Karen Stoiber

In looking at teaching, I see conflicts among our theoretical notions, how they can be applied practically and what is meaningful for teachers. I have had some experiences in going and talking with teachers. Interestingly, when I ask teachers to provide for me some conceptualization of their beliefs or their roles, teachers do not use words such as "decision maker," "problem solver," or "mathematician." Those are not conceptions or images that teachers have of themselves, yet those are images that we, as researchers, have them and for their students. I am struggling with that discrepancy and wondering what we, as researchers, might do. One thing that we might do is to study teachers who do have those conceptions or images of themselves. However, in studying one such teacher, I have faced additional dilemmas.

I have been studying an expert teacher who is teaching fractions for three weeks to elementary children in a local public school. The teacher, Nancy, is also a fellow researcher and professor at my university. Nancy plans her instruction to build on her students' informal knowledge. She is teaching sessions with individual students and also teaching sessions to the whole class. I worry about what I will learn from studying an expert teacher like Nancy. How useful will that information be if it's not very realistic and if that's not how most teachers function? Also, in working with Nancy, I have faced many dilemmas in working out our relationship. For example, I wanted to have an opportunity to interview the students. Nancy was going to be doing some preassessment using measures of students' informal knowledge of fractions. I said that I would like to try some other kinds of measures that looked at more global kinds of problem solving. Nancy's response was, "Why? I'm going to be doing the assessment. That's really not your goal. How would that provide any additional information for what we're looking at?" I think that Nancy and I may have different conceptions of what constitutes authentic mathematical content. Also, in interviewing Nancy about her knowledge and beliefs, I fear that I may be intervening. I may be affecting how she is speaking. By asking her questions, I am affecting Nancy's thinking, and I can never get a sense of how she would be thinking if she were completely on her own and I was not there. These are just some problems that I think we need to consider as we try to collaborate as researchers and as we work together with teachers.

Robert Orton

I am trying to look at a conception of knowledge that is more normative. In that conception, I think it is pretty clear that knowledge is different from belief. If you "know" something, you have to be able to provide some kind of justification for it. We only call something "knowledge" if we can provide a reason or some sort of support for it. In considering what role research on teaching might have, I would argue that research can provide some sort of support or justification for teacher knowledge. One of the questions for this conference asked, "What role do content analyses play in helping us think about research on teaching?" It seems that we do not do a content analysis in the abstract; we do a content analysis for a certain purpose. If the purpose of the content analysis is to help us understand better how kids learn the content, then I think it is pretty clear what role that analysis plays in thinking about teacher knowledge--namely, that content analysis can provide some sort of justification or support for teacher knowledge. Dewey has this notion of a logical ordering of the subject matter that serves the purpose of advancing the subject matter versus a psychological organization of the subject matter, which is the concern of the teacher. It seems to me that content analyses provide that psychological orientation of the subject matter, which in turn, provide what might be called a normative justification for a conception of teachers' knowledge. We want to say a teacher knows something if he or she can provide some sort of argument that relates ultimately back to that psychological conception of the subject matter. That's the content business. It seems to me that the research on student knowledge and student beliefs might play an

analogous role. We might say that a teacher knows something, to the extent that what teachers do can be grounded ultimately in what we know about student learning. We want to say that what teachers do is "knowledge" if it can be traced to what we know about how kids learn something. It doesn't mean that it has to be shown empirically, but from a normative point of view, we want to ascribe the accolade "knowledge" to what teachers do, if what teachers do can be traced ultimately to what we know about students' learning. As to the final question about research on teaching and what role that can play, I see it again playing an analogous role. Ultimately, we want to provide some sort of intervention where we are helping teachers. It seems like the research on how teachers learn, again can provide a normative framework for what we do when we go out there and try to change things. If I had to sum it up, my basic idea is that we only want to call something "knowledge" if it can be supported. It seems that the research provides an ample base for constructing arguments that support action.

ISSUES THAT EMERGED AS THEMES IN THE DIALOGUE AMONG PARTICIPANTS

Penelope L. Peterson and Elizabeth Fennema

Several issues emerged as themes in the dialogue that occurred interspersed within, between, and among participants' statements. The first important issue was whether mathematics was indeed, a "well-defined" content domain. Suzanne Wilson raised this as a question in her opening statement. After Suzanne's statement, there ensued a discussion about what constitutes mathematical knowledge, whether or not there are mathematical structures, and to what extent knowledge is contested in the field of mathematics. Participants returned to this question over and over again during the day, and many participants addressed the question of mathematics as a "well-defined" content domain in their statements. Later in the day Tom Carpenter attempted to clarify what the conference organizers had meant by "well-defined content" by saying that when they had been referring to a subset of mathematics that they thought might be studied in a reasonable way. He gave the example of selected studies in which researchers have examined the interplay between specific ideas in a particularly narrow domain of mathematics, such as addition and subtraction or rational numbers, and students' learning and thinking in those areas. Belatedly, the conference organizers realized that dialogue might have been facilitated if they had provided the participants with a set of these studies as part of readings for the conference so at least participants would have had a concrete referent for studies in a "well-defined content domain." As it was, conference participants seemed to differ widely in their familiarity with these kinds of studies and their knowledge of the methodology and findings from these studies. Thus, many participants were unable to judge the usefulness of this line of learning research for studies of teachers and teaching.

Second, early on in the dialogue clear differences in perspectives emerged between participants depending on whether their foci of study was the learner or the teacher. Dick Lesh's clear foci on the learner and on the development of mathematical ideas were made evident in his opening statement. Deborah Ball's statement, following directly after Dick's, cast the two perspectives in sharp relief. Although Deborah cast the teacher and teaching as figure, and the mathematics and learners as ground, she acknowledged her need, as a teacher, to work continuously on making connections between these. Deborah questioned whether for Dick Lesh, the teacher is even in the picture. Although the conference organizers had intentionally invited participants to represent these two perspectives--researcher on mathematics teaching or researcher on mathematics learning--the organizers may have underestimated the extent of the differences that existed between these perspectives and among the participants at the conference. The differences became more and more evident throughout the day as the researchers on teaching continued to respond and critique some of the ideas that Dick Lesh introduced in his opening statement.

A third issue that became evident during the first day was a lack of common knowledge and shared understandings among participants in the group. This lack of shared understandings was evident particularly between researchers on learning and researchers on teaching, but also was present within each of these research communities as represented at the conference. As Magdalene Lampert noted in her statement to the group, "You can't address the notion of understanding unless all the people in the room have some sense of negotiated meaning and have some sense that they're talking about the same thing." Because of lack of shared understandings, most of the statements and dialogue on the first day seemed to reflect an attempt to make clear what individuals meant by certain terms like "mathematics," "content" "teaching," and "learning." Individuals' responses to the questions also served to make visible to others in the group their thinking and understanding of what is known about learning and teaching of mathematics and what needs to be known. These differences in understandings seemed to reflect different

epistemological views of the participants reminiscent of Sowder's (1989) summary of the research agenda-setting conferences held by the National Council of Teachers of Mathematics. Sowder distinguished five contemporary scholarly views of what mathematics is and how one comes to know mathematics:

According to the first view, mathematics is external to the knower, static, and bounded. Learning and teaching mathematics involves the acquisition of information. In the second view, mathematics is also external to the knower, but it is a growing unbounded discipline that changes over time. Learning and teaching mathematics focuses on how students acquire meaning for what is to be learned. The other three views involve some variation on "constructivist" ideas that knowledge is personal or social. According to the first of the constructivist perspectives, to know mathematics means to "do mathematics" by "abstracting, inventing, proving, and applying" (Sowder, 1989, p. 22). Sowder saw Romberg (1988) as representing this perspective. The second constructivist position assumes an epistemology of mathematical knowledge that is consistent with the contents of individual minds. The CGI approach of Carpenter, Fennema, and Peterson is consistent with this perspective as is the work of Lesh. Finally, the last constructivist perspective regards mathematical knowledge as the product of social and cultural processes. This latter perspective seems to best describe the views of Lampert (1990) and Ball (1990) and Cobb.

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SMALL-GROUP DISCUSSION SUMMARIES

WHAT ROLE SHOULD STUDENTS' THINKING PLAY IN INSTRUCTION?

Participants: Ralph Putnam, Richard Lesh,
Peter Kloosterman, James Reineke

Summary prepared by Ralph Putnam

The group's discussion revolved around a cluster of questions and issues concerning students' thinking about and learning of mathematics: What role should students' thinking play in instruction? How do you get teachers to attend to students' thinking? What should teachers know about students' thinking? How do teachers currently conceptualize student thinking and learning in mathematics and how might they conceptualize them? How should research on students' learning of particular mathematical topics inform research on mathematics teaching?

To summarize our discussion, I will describe some of the key issues and ideas that came up (not necessarily in the order we discussed them) and then present a brief list of some of the relevant research projects that group members described. The last section includes my thoughts on the role students' thinking should play in teaching mathematics for understanding.

Difficulties and Tensions in Attending to Students' Thinking as an Integral Part of Teaching

First, the group agreed that these issues about students' thinking and learning mathematics are important and difficult ones to tackle if we are to facilitate meaningful change in the way mathematics is taught in schools. As Peter Kloosterman pointed out, the idea of attending to students' thinking is foreign to most teachers, unless they've gone through something like the CGI program. How to get this idea in a form that is understandable and usable by teachers is clearly an issue we don't know much about.

Ralph Putnam pointed out three tensions or dilemmas that keep coming up for him in thinking about the role students' thinking does or should play in teaching mathematics. One problem is that, although virtually all research on students' learning (of mathematics and in other domains) highlights the importance of the learner's prior knowledge and active construction of knowledge, much of this research has been highly individualistic--focusing on the learning and knowledge of individual students. The images of good teaching that emanate from this research also seem to be too individualistic. One popular image is that of a teacher working with an individual child [illegible]tting an accurate sense of his or her current cognitive state, and then presenting the [illegible] with some activity or situation that helps him or her move just a bit further. If we [illegible] exactly what this particular child knows and exactly where we want this child [illegible]ove, then we can move the child to that new knowledge.[1] This individualistic model seems difficult to map onto the classroom where you don't have an individual child; you have a group, with interaction among students being a prominent feature.

A second concern is the tension in teaching between paying serious attention to students' thinking and attempts at sense making while still helping students learn particular mathematical ideas or skills. Even if you can convince teachers that they need to give students opportunities to express their thinking, some teachers provide the opportunities for

[1]One version of this view of ideal teaching is put forth by Glaser, R. (1984). Education and thinking: The role of knowledge. American Psychologist, 39, 93-104.

students to express their ideas, but then they don't go anywhere with the ideas; the sharing of ideas has become an end in itself. The teacher somehow has to lead the students or direct the discourse toward the more powerful mathematical ideas. Although this kind of teaching seems to be emphasized by the various documents of the current reform movement (e.g., NCTM Standards), we don't have enough ideas of what that teaching might look like in the classroom. We do ha important examples of teaching that seem to be successful in attending to students' think useful ways: Lampert, the teachers that Cobb and his colleagues have been working with to develop teaching based on a constructivist view of learning, and CGI teachers. But the mathematics education community needs to continue to work on characterizing good mathematics teaching in real classrooms if we hope to approach the vision being put forth in various reform documents.

The third concern is that to attend in powerful and flexible ways to students' thinking, the teacher has to know the domain--to be comfortable in the mathematical territory. And many teachers are not comfortable with the mathematical ideas they are expected to teach. Reform documents like the NCTM Curriculum Standards, in addition to asking teachers to deal with content that is new for them (e.g., probability and functions), are asking teachers to teach mathematics in ways in which they themselves have never experienced. It's a bit like trying to teach someone good writing style when you have never developed good writing skills yourself. To be able to respond in useful ways to students' mathematical thinking, teachers have to have engaged in that kind of thinking and developed powerful mathematical knowledge themselves.

Views on the Nature of Mathematics Knowledge and Learning

We all agreed that we would like to help teachers think about learners and attend to their thinking in more powerful ways. Lesh suggested that although we want to let teachers know what researchers have learned about how students' learn and know mathematics, there is not a single dominant view of mathematics learning within the mathematics education research community. Rather, there are a variety of perspectives on the nature of mathematical knowledge and how it is known and learned by individuals--general world views or global metaphors that shape our thinking about teaching and learning mathematics. We can think about these world views as being held both by researchers and by teachers and as being important in determining how these people approach learners. Lesh offered the following tentative list of such views of knowing and learning mathematics:

1. Collection of rules. First is the view that mathematical knowledge is best thought of as rules and facts that can be stated as behavioral objectives. Mathematics from this perspective is a collection of procedural rules, primarily for computation. This is the view of mathematics that seems to underlie much traditional mathematics curriculum and instruction and is held by many teachers.

2. Buggy algorithms. An alternative view that has been popular among some psychologists is the "buggy" algorithm perspective. Instead of a collection of rules, the mathematics knowledge of an individual is conceived of as a collection of computer programs. Some of these programs are faulty, or buggy. The instructional implication often drawn from this perspective is that the goal of instruction should be to analyze the buggy algorithms and correct them. Lesh pointed out that this is a band-aid approach in that it focuses on correcting specific erroneous procedures. Putnam pointed out that many of the basal mathematics textbooks now include notes to the teacher based on this sort of analysis of computational errors.

3. Problem solving processes. A third perspective is that mathematics has to do primarily with problem solving. Learning mathematics is more a matter of learning global problem solving processes than of learning specific mathematical content. This view is in many ways a return to the views of Pólya.

4. Modeling perspective. A final perspective is that the fundamental things individuals acquire as they learn mathematics are conceptual models or structural metaphors, much like the mental models students construct in physics or other scientific domains. These models are linked to real experiences and are used to reason about many things, but they are unique to mathematics--they provide a way of structuring experiences mathematically. Individuals gradually acquire these models as they reason about situations. From a curriculum standpoint, if one takes this view, it becomes important to identify the important conceptual models or structures that students should develop and study how kids go about constructing them.

Lesh argued that the trend in the research community is toward the modeling perspective, but that it has not been well articulated and does not seem to be represented at all in most teacher training programs. This view does not seem to underlie the NCTM Standards, although the Standards are not inconsistent with the modeling perspective. Lesh feels that the NCTM Standards are driven more by a problem-solving perspective (#3 above) and is concerned that unless the mathematics education research community gets more articulate about their view of learning, implementation of the Standards is likely to degenerate into discovery learning and generic problem-solving approaches, which have been tried before and failed.

Lesh argued that these world views about the nature of mathematics learning influence both researchers and teachers and that we need to be more explicit about what views teachers hold and what views are starting to prevail in the mathematics education community. Kloosterman pointed out that most teachers seem to hold the mathematics as rule or as buggy algorithms perspectives. There seems to be consensus in the mathematics education community that we need to help teachers move away from those perspectives, toward something that is more focused on understand and kids constructing knowledge (i.e., as expressed in the NCTM Standards), but there is much less agreement about the specifics.

Putnam pointed out that in addition to the mathematics content and problem-solving processes incorporated in these world views, there is another important aspect of mathematics that some scholars (e.g., Lampert, Ball, Romberg) are arguing has been underrepresented. That is the whole cluster of skills and dispositions entailed in mathematical argument, conjecture, and proof. These, too, are essential aspects of mathematics that we need to attend to in our teaching. Lesh commented that he regards these things as the social dimensions of understanding--that the need for them can be derived from the modeling view of mathematics learning.

We agreed as a group that teachers' beliefs or perspectives about the nature of mathematics knowledge and learning were important influences on their teaching, whether or not we ascribed to the particular set of models above. These world views about learners should play an important role in how teachers think about their work. And we can talk about how teachers' perspectives need to change to be more consistent with what we know about how kids learn mathematics. We agreed that these sorts of views are not easily changed, and that we need more research to help us understand the views of teachers and how they might change and develop.

Models or Beliefs of Teaching

In addition to views or beliefs about the nature of mathematics knowledge and how it is known by individual learners, teachers have beliefs and assumptions about teaching. They have, in short, world views or conceptual models of the teaching/learning process, which play an important role in shaping their instruction. Thus, for example, we agreed as a group that mathematics educators ascribing at least in a broad sense to the sort of modeling view that Lesh laid out might have quite different notions about how mathematics should be taught. We generated three possible alternative perspectives, thinking in terms of how different individuals might conceive of ideal teaching (without thinking that this was a systematic or exhaustive list):

1. Lesh said his view of good teaching was to carefully structure the environment that learners would work in and let them go to work within that environment. The teacher might not even be in the room for much of the class. Rather, the task would have been structured in ways that allowed students to interact with a situation that would lead to the construction (ultimately) of the desired powerful structural models.

2. In characterizing Magdalene Lampert's teaching (in absentia!), we pointed out that for her, group interaction is important, but with a much higher level of involvement by the teacher. For Lampert, the teacher plays a critical role in mediating and facilitating the sense that students are making during the instructional discourse.[2]

3. We characterized Les Steffe's (also in absentia!) ideal view of ideal teaching as a one-to-one interaction between teacher and student in which the teacher tries to understand the kinds of mathematics the student is using to deal with a task and gradually guiding the student to develop more powerful strategies and constructs.

The point of characterizing these different views of teaching was not to portray the views of particular individuals accurately but to demonstrate to ourselves that individuals who have some important commonalities in their views about the learning of mathematics may very well hold quite different views about teaching. We should not make immediate assumptions that a particular perspective on learning will automatically imply a particular way of teaching. And as researchers thinking about how to help teachers in their efforts to change, we need to attend to both. Just as research on the ways students think about particular mathematical situations has taught us a lot about the ways kids know and learn mathematics, research on the "structural metaphors" of teachers is needed to help us understand the teaching context and how it might be enhanced.

We spent some time talking about whether a teacher would hold a single well-developed model of teaching versus some collection of models and the process of changing from one model to another (i.e., how teachers change the beliefs or perspectives that shape their teaching). Lesh argued that students solving problems usually worked within a community of partly formed models, rather than having a single well-developed model for solving a problem. As they worked with problems, students gradually developed models that worked better for them; thus they developed more mathematically powerful models, without there being a particular "best" model in place. Lesh argued that the same should be

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[2]Lampert has written extensively on her own teaching, particularly the role of discourse and the teacher's role in shaping it. Our description in the group discussion was just a thumbnail sketch of her teaching, as is the subsequent description of Steffe.

true for teachers--that they would have a community of models of learners and teaching that they drew upon as they taught. Even though we can't make ultimate statements about which of these models are "best," we can make judgments about which models seem more powerful for helping kids learn mathematics.

A related concern that came up several times was a fear that documents like the NCTM Standards might promote a "naive discovery" view of learning in which it is thought that if students are simply actively engaged in mathematical activities, they will develop powerful mathematical knowledge. Kloosterman, after seeing many teachers use manipulative materials as if they had no relation to paper-and-pencil mathematics, wondered if this kind of discovery learning view was a necessary step on the way to a more sophisticated constructive view of mathematics teaching and learning.

Relevant Research Efforts

During the course of our conversation we each brought up research we had done or were doing that seemed somehow relevant to these issues of how teaching might be informed by attention to student thinking. I present here a brief description of some of these efforts.

Peter Kloosterman is collaborating on a project with Frank Lester and John LeBlanc that involves working intensely in an elementary school with about 20 teachers to help improve the teaching of problem solving. Within that context, Peter is focusing on children's views of what it means to learn mathematics and on teachers' views about teaching and learning mathematics. He is examining the ways different teachers are responding to the various efforts to get them thinking about and changing their mathematics teaching.

Dick Lesh mentioned a variety of research projects in the course of our discussion, including work at Northwestern in which teachers were helped to view the same mathematical content from a variety of perspectives (solving a problem with manipulatives, thinking about the same content in terms of how it might be taught, and actually teaching a lesson on the content), research on students' and teachers' knowledge of rational numbers with Merlyn Behr and Tom Post, and research on solving problems set up to be as much like problems in the "real world" as possible. Dick recently completed a paper on "on-the-job teacher training" in which he deals with some of the issues we discussed. At the Educational Testing Service, he is currently working on ways to measure "higher order thinking" in mathematics, on the belief that assessment and technology may be two of the most powerful levers we have at this point to effect meaningful change in the mathematics education system.

Ralph Putnam briefly described his dissertation research and subsequent efforts to understand better how teachers make decisions about what to teach--how they deal with the tension between attending to students' thinking and sensemaking and still cover particular mathematics content. The issues of teachers' beliefs about the nature of mathematics knowledge, learning, and teaching also play an important role in various case studies of mathematics teaching and learning he is conducting with the Center for the Learning and Teaching of Elementary Subjects.

STUDENTS' MATHEMATICAL THINKING: REFLECTIONS ON ITS ROLE IN PEDAGOGY

Ralph T. Putnam

These are some of my thoughts on the role students' mathematical thinking should play in teaching mathematics for understanding. I believe that finding ways for teachers to attend seriously to students' thinking is one critical aspect of helping them foster the sort of mathematical power or mathematical literacy being called for in the current reform movement. In this brief paper, I first offer a rationale for why attending to students' thinking seems so important. I then consider some proposals for and thoughts about how teachers might successfully incorporate more attention to thinking into their pedagogy.

Why Attend to Students' Thinking?

The argument that teachers need to attend to students' mathematical thinking more than they typically do can be justified along at least three lines. First is the voice of educators arguing that good teachers must engage seriously and intellectually with their students--that virtually by definition good teaching entails focusing on students' thinking, because that is what education is all about (Cohen, 1989; Dewey, 1938; Hawkins, 1974; Lampert, 1988). The second justification concerns the goals of mathematics teaching that are so much a part of the current rhetoric of reform. Reformers are trying to shift the goals of mathematics education away from fluency in isolated computational skills and toward mathematical understanding and the ability to solve diverse problems. As we have heard so often, traditional instruction focuses on students producing answers and teachers verifying their correctness. But reformers want instruction to focus on mathematical thinking, understanding, and problem solving. If we are to take these new goals[3] seriously, students' thinking must somehow become more public or visible in the classroom. For that, thinking becomes the object or goal of instruction, not correct performance on a set of computational exercises. The third justification for attending to students' thinking is grounded in psychological theories of learning. In part because it is from a psychological perspective that I study teaching and learning in classrooms, it is this psychological argument that I elaborate in this paper.

Virtually all current cognitive theories of learning posit that learners actively construct knowledge based on their existing cognitive structures (Putnam, Lampert, & Peterson, 1989; Resnick, 1985; Shuell, 1986). In contrast to earlier behaviorist theories, which posited that learning could be determined by carefully structuring the environment, these cognitive theories assume that students will construct understandings based on an interaction of what they already know and their actions in the environment. As Kilpatrick (1987) has pointed out, this constructive view of knowledge is one to which "most cognitive scientists outside the behaviorist tradition would readily give assent, and almost no mathematics educator alive and writing today claims to believe otherwise" (p. 7).

This does not mean that students cannot learn from having ideas or procedures presented to them--indeed, people often learn from lectures and from the traditional modes of "teaching as telling" that are so frequently criticized. But it does mean that a teacher cannot assume students will learn what was presented--no matter how carefully instruction is designed. Because learning is a matter of students incorporating

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[3]Actually these goals aren't so new--mathematics educators have been making similar arguments for a long time. But the goals (e.g., as put forth in the NCTM Curriculum Standards) are a clear departure from much of current classroom practice.

experience based on their current knowledge, learners will never construct knowledge that is somehow a perfect reflection of what was presented in a lesson or a textbook. As Donald Norman (1980) has argued, you cannot assume that students are learning what you think they are learning.

For teaching, this means that it is not enough to craft careful explanations of the ideas or procedures being taught.[4] Stopping there assumes that all learners will learn what they are expected to learn from the explanations. But because each learner brings somewhat idiosyncratic perspectives and knowledge to the learning setting, each will make somewhat different sense of the lesson. Thus, for teachers to know what their students are learning it becomes essential that they find ways to attend to how students are thinking and constructing knowledge.

This notion of attending to whether students are or are not learning is not a new one in teaching. "Checking for understanding" is an important component of the models of direct instruction or active teaching emerging from process-product research on teaching (Good, Grouws, & Ebmeier, 1983; Rosenshine & Stevens, 1986). And diagnosing the errors that students make in computational tasks has been emphasized by a number of mathematics educators (e.g., Ashlock, 1982). How recent cognitive views of learning differ from these perspectives is in a shift in emphasis to the thinking of students--their solution strategies and the ways in which they make sense--and not just to the products of their thinking--correct solutions to problems.

A critical tension for teaching that arises with the view of learning as constructive process is between students making personal (and possibly idiosyncratic) sense of mathematics and helping them learn mathematical ideas that are accepted by the broader community and an expected part of what students should learn (Cobb, 1988; Lampert, 1988; Putnam, Lampert, & Peterson, 1990). We want students to make sense of what they are learning but there are particular mathematical concepts, procedures, and ways of thinking we expect them to learn--a curriculum to cover. Teachers who seek to help students understand mathematics must deal with this tension. Researchers vary considerably in where they fall on the dimension of emphasizing students' personal meanings versus emphasizing the accepted mathematics (Putnam, Prawat, & Reineke, 1990). But wherever one falls along this dimension, attending to students' thinking and ways of making sense remains important. The radical constructivists (Steffe & Cobb, 1988; von Glasersfeld, 1989) emphasize the importance of attending carefully to the sense children are making of mathematical situations. But researchers who are much more explicit and direct about presenting particular models or ways of thinking to students also recognize that it is the child who ultimately constructs his or her understanding and that teachers must therefore attend carefully to how students' are making sense. Nesher (1989), for example, argues for being highly specific about the kinds of cognitive models children are to acquire and for developing highly structured instructional environments (microworlds or learning systems) to help students construct those understandings. But the teacher must provide ample opportunities for the child to work with the ideas to see whether he or she is, indeed, constructing the expected understandings; a teacher cannot simply present the models to children and expect that to automatically acquire the models in way they were intended.

In sum, current cognitive views of learning virtually mandate paying careful attention to students' thinking the knowledge they are constructing during mathematics instruction. The mandate holds across a wide variety of perspectives, all of which share the basic

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[4]I am not arguing here that teachers should neglect the quality of their explanations or the ways in which mathematical ideas are represented in lessons.

constructivist assumption that learning involves the learner's active constructing of meaning based on existing knowledge.

But traditional modes of teaching do not place a premium on careful attention to students' mathematical thinking. It is not clear how teaching might change to better reflect these constructivist assumptions about learning--how teachers can productively bring students' thinking more to the fore in their instruction.

An Individualistic Solution?

When psychologists (or other researchers taking a psychological perspective) have tried to answer the question of how teaching might better reflect a constructive view of learning they have brought with them a highly individualistic focus. The starting point for psychologists in thinking about learning has been the individual learner, not so much the learner in interaction with the teacher or the learner embedded in the social and cultural context of the classroom. Most of their research has entailed figuring out how individuals think about and learn to carry out various mathematical tasks. When psychologists have applied these ideas about learning to problems of teaching and instruction they have often maintained this focus on the individual, seeking to optimize instruction by tailoring it to the knowledge and characteristics of the individual learner. Thus, tutoring appears to be the ideal form of instruction--for a tutor can carefully diagnose the cognitive states of the learner and adjust subsequent instruction accordingly. This faith in the power of diagnosis between an individual teacher and student is typically presented as a starting assumption, rather than as a research question:

> One of the greatest talents of teachers is their ability to synthesize an accurate "picture," or model, of a student's misconceptions from the meager evidence inherent in his errors. A detailed model of a student's knowledge, including his misconceptions, is a prerequisite to successful remediation. (Brown & Burton, 1978, pp. 155-156)

> Although one-to-one tutoring has been regarded as the most effective method of teaching (Bloom, 1984), surprisingly little is understood about tutoring expertise. (McArthur, Stasz, & Zmuidzinas, 1990, p. 197)

Even when they have adopted much richer views of the nature of knowledge and thinking, psychologists have typically maintained their individualistic view of teaching. Glaser (1984), for example, after making a strong case for the importance of thinking being embedded in rich domain-specific knowledge, posited an pedagogical model. The model entailed figuring out the students' current schemata or ways of thinking, and then presenting to the student ideas or information intended to help the student question or modify his current understanding. Again, the image here is very much one of an individual teacher interacting with an individual student. The teacher engages in fairly extensive diagnosis of the learner's current state and then makes an instructional decision about what to do to push the student's thinking and understanding to a new level.

There are at least two big problems with these individualistic instructional approaches--the view of ideal teacher as tutor. The first problem is practical. Teachers in classrooms of 20 to 30 students rarely have the opportunity to have the sustained individual interactions with students that would be needed to realize a view of instruction built around teacher as tutor and student as tutee. And performing detailed diagnosis of the thinking of individual students is not consistent with the way most elementary school teachers do their work. Because they must deal with the learning of large numbers of students simultaneously,

teachers are likely not to strive for detailed diagnosis of the cognitive states of individual students (Putnam, 1987).

But these are practical problems. For if a individualistic instructional model is, indeed, the optimal way for students to learn, we can work hard to change the conditions of teaching to allow for more individual tutoring. Or we can work to develop computer-based intelligent tutoring systems (ITS) that may approximate the diagnostic capabilities of a good human tutor (and possibly surpass those of many teachers, ITS advocates would argue). It may be feasible for sophisticated computers to interact one-to-one with students and keep track of the detailed individual diagnoses required by such an approach[5] (McArthur et. al., 1990; Sleeman & Brown, 1982).

There is, however, a more fundamental problem with this tutoring view of instruction. It stems from an overly individualistic view of the learning process, ignoring social dimensions of learning. I would not want to see children taught primarily by tutors, in part because I consider developing the ability to interact collaboratively and supportively with other people as an important goal of schooling. I also believe that it is through interaction with others around important ideas that much significant learning--especially learning with understanding--takes place. This view is supported by the increasingly popular social constructivist views of learning, which hold that knowledge is jointly constructed through social interaction and resides in culture and groups, not solely in the minds of individuals (e.g., Bruner, 1990; Vygotsky, 1978). And many of psychologists who have previously posited individualistic views of learning and teaching are now arguing for a much more socially and culturally mediated perspective (e.g., Brown, Collins, & Duguid, 1989; Resnick, 1987).

I won't attempt here to unpack the recent views of learning as a social and cultural phenomenon--that is too big a task for this paper. Regarding the issue of teachers' attending to students' thinking, it is enough to say that cognitive research on learning has taught us a lot about how individuals learn and understand mathematics and provides a strong argument for the importance of attending carefully to students' mathematical thinking during instruction. But the cognitive perspective, at least in its highly individualistic form, has not provided viable models for how teachers might bring students' thinking to the fore.

Some Critical Features of Pedagogy

I have thus far given a one-sided argument (from the perspective of psychological theories of learning) about why it is important that teachers attend to students' mathematical thinking. And I have argued that a highly individualistic or tutoring-based approach is not an appropriate way to bring students' thinking into the picture. It is not so clear how teachers might or should bring students' thinking into a more central role. In concluding this paper, I offer what I see as critical features or issues that need to be considered in trying to make student thinking a more central part of pedagogy. In discussing these issues, I assume that there are multiple ways to teach mathematics well--that there is not one best way that teachers should make use of students' thinking in instruction.

Discourse. Having abundant opportunities for talking about mathematical ideas is important--students and teachers must engage in meaningful discourse around mathematical ideas. For it is primarily through language that we communicate our

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[5]Whether ITS programs that can do the sort of rich diagnosis and interaction required to foster student understanding of mathematics can actually be developed is still an open question.

thinking to one another, and thus discourse is a critical way for students to make their mathematical thinking more public. Students need to have opportunities to discuss solution strategies, to offer explanations and justifications--all ways to reveal their thinking. This emphasis on language should not be taken to exclude as part of the discourse the use of other means of representing mathematical ideas, such as manipulatives or visual representations. Such tools are important ways to communicate about mathematical ideas, but seem most powerful when used in conjunction with, not instead of, language. The importance of talking about mathematics is supported by the centrality of changing the discourse of teachers and students in successful attempts to forge new pedagogies that bring students' thinking to the fore (Ball, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb, Yackel, & Wood, 1988; Lampert, 1988). The role that discourse plays in these approaches--how teachers and students are expected to talk about mathematics--varies, but in all of them, language is a primary means of making students' thinking public and an object of instruction.

Teachers' knowledge of mathematics. But just talking about mathematics is not enough. Teachers have to be able to help students use the discourse to construct powerful and acceptable mathematics. Thus, to deal seriously with students' thinking, a teacher must have rich knowledge of the mathematics being taught. As David Hawkins (1974) argued,

> For such a teacher a limiting condition in mapping a child's thought into his own is, of course, the amplitude of his own grasp of those relationships in which the child is involved. His mathematical domain must be ample enough, or amplifiable enough, to match the range of a child's wonder and curiosity, his operational skills, his unexpected ways of gaining insight. (p. 112)

A teacher needs to know how a learners' ideas might connect with more established mathematical ideas. And just what mathematical ideas might come up can never be entirely predicted. Furthermore, if teachers are going to foster particular kinds of mathematical thinking and understanding in students, they must have developed that mathematical understanding themselves. For instruction in which teachers attend seriously to mathematical thinking is not something that can be specified in a textbook or activity that the teacher simply "delivers" or "presents" to students. Teachers need to have engaged in the kind of mathematical thinking they are trying to foster. Reform documents such as the NCTM Standards, in addition to asking teachers to deal with content that is new for them (e.g., probability and functions), are asking teachers to teach mathematics in ways in which they have never experienced themselves (Cohen & Ball, 1990).

Authority. Finally, closely related to the visibility of students' thinking is the issue of authority for truth or correctness in mathematics. Pervading the rhetoric of the current calls for reform and writing about teaching mathematics for understanding is the idea that we want students to make sense of mathematics. We want students to know mathematics in ways that makes them confident about their knowledge--believing that the solution to a problem or a mathematical idea is "true" or "correct" because it makes sense, not solely because it matches the answer provided by the teacher or the textbook. This kind of sense making entails thinking through and justifying ideas--if it is to become a goal of instruction, it seems essential that this thinking somehow be made visible.

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CONCEPTUALIZATION OF MATHEMATICS TEACHERS' BELIEFS

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Summary prepared by Robert K. Orton

The discussion focused on the issue of the "conceptualization of teachers' beliefs." What follows is a discussion of several points that the group felt to be worthy of consideration when trying to conceptualize and understand teachers' beliefs. Though an attempt has been made to preserve the spirit of the group, some additions have been made to this paper that were not strictly part of the discussion. For example, some comments on the relationship between "belief" and "knowledge" have been added at the beginning of the paper. Also, the last few pages of the paper include my thoughts regarding what I learned from the conference and what I think might be interesting lines of further inquiry in this domain of mathematics teachers' beliefs.

Understanding teachers' beliefs would appear to be central in understanding mathematics teaching. More specifically, it would seem that mathematics teaching will be better understood if more is learned about (a) teachers' beliefs about mathematics (e.g., whether mathematics is problem solving, symbol manipulation, or something else) and (b) teachers' beliefs about what students can do. Teachers' beliefs would seem to represent a critical research site.

Before any attempt is made to describe, measure, or evaluate teachers' beliefs, we must have a way to conceptualize what these beliefs are. The conceptualization should form a base for modeling teachers' beliefs, and perhaps resulting behaviors, so that related research can be programmatic to the extent possible. A long-term goal of this line of inquiry would be to more from conceptualizing to constructing a foundation for mathematics teacher education programs.

Three questions will be considered below to structure the discussion on teachers' beliefs. First, the question of what beliefs are of interest to study in conceptualizing teachers' beliefs will be considered. This will be referred to as the "what question." A second question pertains to how we come to understand these beliefs, both from a methodological and a theoretical point of view. This might be regarded as a "how question." Third, implications of knowing and learning about these beliefs will be considered, both for teachers and for researchers. This might be regarded as the "what does this mean?" issue. Thus, the topic of teachers' beliefs will be approached from conceptual ("What?"), methodological/theoretical ("How?"), and pragmatic ("What does it mean?") points of view.

Before considering, in turn, each of these three issues, a preliminary point needs to be made regarding the notions of "belief" and "knowledge." The approach taken below is to remain intentionally vague regarding these key notions. The role of undefined terms is well known in geometry and has also been described by thinkers as a central aspect of building theories in science. Quine (1953), for example, characterizes scientific theory as a "network" wherein concepts are implicitly defined by their relationship with a constellation of other terms (see also Waissman, 1951). The concepts of "belief" and "knowledge" will, in a like manner, be left undefined or primitive, allowing the conventional, everyday connotation of these terms to keep the discussion afloat.

Similarly, no attempt will be made to draw a sharp distinction between "belief" and "knowledge." This is not to claim that these two concepts are equivalent. It is rather to say

that common intuitions regarding these key ideas will guide the discussion, at least as long as this is possible. The approach is to keep these distinctions intuitive, sharpening them when necessary to communicate some meaningful questions that might be raised regarding teachers' beliefs.

The "What" Question

What teachers' beliefs are of interest to study? At least two approaches are possible in addressing this "what question." The first would be to generate a "laundry list" of possible questions. This would be a descriptive task and a necessary part of any conceptualization of teachers' beliefs. At the other extreme, one might begin with some theory or grand scheme for understanding teachers' beliefs and then use this theory to suggest what questions might be sensible to ask. This approach may ultimately prove more fruitful than the descriptive tack, but it presupposes some theory which can be used to guide the inquiry.

The approach taken below, in a spirit consistent with the dialogue that generated this report, is eclectic. Both a laundry list of possible answers to the "what question" and suggestions of ways in which a unifying theory might be used to conceptualize teachers' beliefs are discussed. The present section will put forward descriptive answers to the "what question." The answers will be organized around the three questions that were used to initially structure the conference. These questions focus on (a) learners' mathematical knowledge and thinking, (b) mathematical content analyses, and (c) teachers' knowledge and thinking.

It needs to be pointed out from the start, however, that the three areas of learner, content, and teacher are not discrete. Central themes, such as "belief in authority" or "belief in how learning takes place," may provide a more logical organization of answers to the "what question." This issue will be discussed in the second section below, where answers to the "how do we come to know question" will be considered. This second section will examine possible theoretical underpinnings for the study of teachers' beliefs.

Beliefs About Content

It would be of interest to know what teachers believe about mathematics. For example, How do teachers believe that mathematics is related to other aspects of life, such as volleyball or retail sales? How is mathematics related to other disciplines, such as chemistry or history? How is mathematics related to other branches of mathematics within itself? For example, How is arithmetic related to algebra, or algebra to geometry? How is mathematics related to different representations of the subject matter? For example, How is factoring polynomials related to rearranging squares and rectangular regions to form single rectangles? How is factoring related to the arrangement of desks within a classroom into a rectangular array?

It would be of interest to look at teachers' beliefs regarding connections between mathematics and other subjects (e.g., mathematics and history) and connections within mathematics (e.g., between factoring and rearranging tiles). A rationale for looking at these beliefs about relationships between mathematics and other things is found in the nature of comprehension. Namely, there exists a close relationship between comprehension and knowing relationships. Or, as Resnick puts it succinctly, "To know something is to know relationships" (Resnick, 1983, pp. 477-478).

Another set of beliefs of possible interest are related to the organization of mathematics. Most secondary mathematics teachers will have spent approximately 45 university quarter

hours or 30 semester hours in college mathematics classrooms. They will also have spent at least 12 years as students in elementary and high school mathematics classrooms. It would be of interest to investigate teachers' beliefs regarding the connection between these two "mathematics." Expressed as a question: What are teachers' beliefs regarding the relationship between collegiate level mathematics and school mathematics? How do teachers believe the subject matter of mathematics to be organized? For example, Is mathematics organized as a set of hierarchal clusters of different bodies of information? This would imply a relationship between college and school mathematics as one between "knowledge" and "prerequisites for knowledge." Or, Is mathematics more a way of thinking about particular types of problems? This would lead to a view of more continuity between college and school mathematics. Or, Is mathematics organized as a set of procedures and skills? This, again, would make school mathematics a preparation for "real" mathematics. How is the discipline called "mathematics" organized?

Beliefs About Students

In addition to beliefs about subject matter, several areas of teachers' beliefs about students and learners would seem worthwhile to investigate. The most general question would be: What do we believe about how students learn mathematics? Does a teacher believe in a transmission model of learning, a constructivist model, or something else? A related question pertains to teachers' conceptions of student abilities. What do teachers believe regarding the nature of student abilities? Are they inborn traits that students either have or do not have, or are they socially and environmentally determined characteristics that patience and the "right" instruction can further develop? What are teachers' beliefs about mathematical abilities?

These issues themselves (as opposed to teachers' beliefs regarding these issues) have been the topic of a vast amount of educational research. Also, teachers would have been exposed to some of this research in their preservice and inservice training. It would therefore be of interest to investigate teachers' views regarding the research on student learning. What do teachers believe about research on the learning of mathematics? How are teachers' views of learning related to the "academic views" on this topic?

Another line of questions within this area pertain to teachers' beliefs about who can learn mathematics. Such questions would have obvious overlaps with teacher views about human abilities. However, the questions would also address social concerns and would have equity implications. For example, Do teachers believe that all students can learn mathematics? If yes, do they really mean <u>all</u> students? For example, do they believe that brighter students can learn inquiry based mathematics and slower students skill based mathematics? Do teachers believe that boys can learn mathematics more easily or in a different way from girls? How do teachers tend to define mathematics for different student populations?

Beliefs About Teaching

The third facet of the "what" question is related to teachers' beliefs about the teacher and teaching. It might be of interest, for example, to investigate visions of a "good mathematics teacher." What do teachers believe that the ideal mathematics teacher "looks like"? Does the ideal teacher tell students the answers? Does the teacher help students to find answers themselves? How far should a teacher let a student "flounder" before jumping in with some assistance? Is the teacher a dispenser of information, a facilitator, a judge, a partner in an investigation, a representative of the public, or something else?

Another area that might be investigated pertains to teachers' views on the nature of an ideal mathematical task. What do teachers believe such an ideal task to be? Is a good task one wherein the students are all quietly doing calculations at their seats? Is a good task one where students work together in small groups to come up with a common solution to a problem, such as in Lesh's "camp problem" (Lesh & Zawojewski, 1988)? Is a good task one where students all talk about mathematics? What are the ingredients of an ideal task?

It would also be of interest to look at teachers' beliefs as to how an ideal task is related to other parts of the curriculum. Do teachers believe that a "good task" is one that develops a particular subskill that is related to a larger skill? Is a good task related to a development of the subject matter according to a learning hierarchy? For example, Does a good task focus on the skill of factoring polynomials when both the linear and constant terms are positive, which then leads to the case where one term is positive and the other is negative? Alternatively, Is a good task one that provides ideas as to how the task can be applied to some real world problem? For example, Is a good task one that the student would likely need every day as an engineer or a shopkeeper? What do teachers believe a good mathematical task to be?

Teachers' beliefs about a good mathematical task obviously is intertwined with the three areas of beliefs about students, content, and teaching. From the point of view of content, the issue translates into how teachers believe mathematics is organized. Do teachers believe mathematics is organized hierarchically, holistically, tied to applications, related to developmental levels, or some combination of these (these categories may not be discrete)? Is "good mathematics" discrete, molecular, or molar? Beliefs about a good mathematical task also ties into beliefs about student learning. Is a good task one in which students are working individually or cooperatively? Do students need to learn mathematics in a particular way? How is a mathematical task best structured so as to optimize student learning? Beliefs about a good task also ties into teachers' beliefs about their roles as teachers. Is a teacher ideally an orchestrator of tasks, a dispenser of information, a judge, a facilitator, or something else? What do teachers believe their roles to be in an ideal mathematical task?

The area of teachers' beliefs about a good mathematical task brings up the need to organize answers to the "what question" using a scheme of greater generality than content-teacher-student distinction. These categories can be thought of as initial, organizational heuristics, not as conceptually distinct domains. A "grand scheme" of teacher types, similar to the models of teaching described by Joyce and Weil (1980), would be useful in relating beliefs about teaching with beliefs about content and about students. Also useful for doing research in this area would be a normative conception of "good mathematics teaching." Ball (1988) describes a conception of "mathematical pedagogy," which is different from teaching that focuses either on just skill development or the learning of mathematical concepts. Ball's description of mathematics pedagogy might be used to suggest further questions for studying beliefs about teaching mathematics. These issues will be addressed more carefully in the next section on theoretical and methodological aspects of teachers' beliefs, the "how do we come to know" question.

The "How Do We Come to Know" Question

In addition to identifying those teachers' beliefs worthy of study, some thought needs to be given to how one might come to know about these beliefs. This might be regarded, in part, as a methodological question; a complete answer would require a review of the various techniques that have been used to study teachers' beliefs. However, the question is not just methodological. An answer would also require a theoretical framework or set of guiding assumptions which could be used to structure the study of teachers' beliefs. For example,

the belief cluster metaphor used by Green (1971), the developmental scheme of Perry (1970), the models of teaching by Joyce and Weil (1980), or the conception of mathematical pedagogy developed by Ball (1988) might all (either singly or in some eclectic combination) be used to study teachers' beliefs. These theoretical tools would help provide a rationale for using empirical methods (grid techniques, smallest space analysis, Likkert questionnaires, etc.) which are commonly used to study beliefs. Thus, the "how to we come to know" issue must be understood as raising both methodological and theoretical questions.

Several points can be made regarding possible answers to the "how" question. One of the most central, which would drive all research on teachers' beliefs, is the principle: Cognitions precede actions. Beliefs guide actions. This is in contrast to the behaviorist position, whose proponents held that beliefs were mere "operants," or shorthand statements about behaviors. The principle: "Cognitions precede actions," might be regarded as an affirmation that the study of beliefs is possible. For example, verbal reports of what a teacher is thinking or why he or she engaged in a certain action might be regarded legitimate data for inferring beliefs. Similarly, a teachers' actions in the classroom might be regarded as data for inferring teachers' beliefs.

The main point here is that beliefs can be inferred. An inferred belief is different from the mere "process of making an inference." In other words, there are things called beliefs, which are different from the inferences based on observations of teacher behavior or teacher verbal reports. The assumption that beliefs precede actions makes the study of teachers' beliefs compelling.

Another point that would assist in the study of beliefs might be called the "topographical principle." By this is meant that the landscape of beliefs is not flat. The topography of the belief landscape is pitted with hills and valleys, due to the fact that some beliefs are related to each other in different ways and some beliefs are more important or central than others. Green, for example, distinguishes logical, quasi-logical, psychological, and "cluster" relationships between beliefs (Green, 1971). Beliefs fall into what might be called metaphorically a "belief system" which contain "belief clusters" organized around "core belief" (Green, 1971, see Quine and Ullian, 1978). This metaphor might be useful in mapping out the belief landscape and identifying those beliefs which are more central than others.

Important for conceptualizing teachers' beliefs would be the identification of those belief clusters which are organized around some core beliefs. This "cluster and core" organization would likely result in a "ripple effect" when more central beliefs are changed or given up. In conceptualizing teachers' beliefs, it would be important to provide a better explication of this notion of the "centrality of beliefs" and the ways in which ripple effects work on other beliefs.

A scheme that might be useful for surveying the topography of beliefs is the developmental theory of Perry (1970). Perry hypothesizes (and corroborates, to some degree) that adolescents and adult reasoners pass through a dualistic stage, where they basically view the world as "black and white," to a multiplistic stage, where they recognize shades of gray but focus on discrete, multiple entities lacking an interconnection, to a relativistic stage, where the believer/ knower both recognizes the shades of gray and the interrelationship among these multiple perspectives. The usefulness of Perry's scheme in conceptualizing teachers' beliefs has been shown by Cooney and his students (see Cooney, 1988).

The "models of teaching" proposed by Joyce and Weil (1980) might provide another idea that would be useful for examining the topography of belief systems. Following the spirit of

Joyce and Weil, one might begin with certain "models of beliefs," such as those of a constructivist teacher, a "tell 'em" teacher, or a humanistic teacher. These models would then be used to guide what sorts of questions to be asked in conceptualizing teachers' beliefs. These models might also provide a means of describing which beliefs are more central than others and thereby suggesting possible ways to intervene and change teachers' beliefs. For example, identifying which beliefs are central in certain models might suggest implications such as, If you believe teaching is telling, then you are likely to believe authority lies in the textbook. In short, an economical construct label, such as "constructivist" or "humanist," would communicate meaning about teachers' beliefs in a way that a mere laundry list of questions might not.

Ball's conception of mathematical pedagogy might also prove useful in surveying the topography of teachers' beliefs (Ball, 1988). A normative framework of mathematics teaching, such as that described by Ball, would provide a sort of "advance organizer" or map for the researcher who begins to explore the mountains and valleys of the belief landscape. Equipped with a conception of mathematics pedagogy, a researcher would have some way of assessing which sorts of beliefs are of more interest to examine than others. For example, the conception of mathematics pedagogy would provide a foundation for justifying belief, which could then be used to explore the justifications that teachers might provide for their beliefs.

The topography of justification would be of interest to investigate because of the possible leverage points it might provide in attempting to change beliefs. Beliefs that are related to each other in some form of "justified relationship" can be more easily discussed. As Green (1971) argues, those who hold their beliefs on the basis of evidence or justification will likely engage in a rational discussion. "Mathematical pedagogy" would provide one possible entry point for such discussions.

In addition to the topographical principle, another important point in the study of teachers' beliefs would be a "multiple perspectives" principle. The study of teachers' beliefs would need to be guided by various sources of evidence. For example, both a sequence of classroom observations and a group of structured tasks might be used in tandem to study teachers' beliefs. Cooney (1985) describes a method used in his research, which includes an observation of behavior, a dialogue with teachers, and a card-sorting procedure that attempts to identify belief clusters. "Episodes" such as the following are used to get the dialogue started:

> Describe a particular anecdote during your teaching that held special meaning for you.
>
> If you could be another person (or famous person) when teaching, whom would you pick? Why?

Most teachers find it virtually impossible to do for their students all they had hoped to accomplish when they first entered the profession. There are all kinds of obstacles that get in the way. Imagine that you had three magic buttons to push that would enable you to eliminate these obstacles. Tell me which buttons you would press. What do you think the consequences would be of pushing each of them?

Data from these dialogues are then recorded on cards, sorted by teachers into piles representing belief clusters, and then described by propositions which attempt to capture the meaning of that belief cluster. As another example of the multiple perspective principle, teachers are observed in their classroom and then provided with an audio or video transcript of the classroom for their comments. With the transcript in hand, teachers are

asked to point to a particularly representative statement or highlight an incident that seems particularly important or significant. These data are then used to help describe a teacher's network of beliefs.

The topographical and multiple perspective principles give rise to another issue that needs to be addressed in the study of beliefs. Given that some beliefs are more central than others, and given that a multiplicity of sources need to be used for inferring beliefs, inconsistencies among beliefs are likely to arise. For example, a teacher might say that he or she believes in the importance of letting students construct their own mathematical knowledge and then teach in a way that involves telling the students the material that is in the textbook. What might be done when the data point to something contradictory?

Several approaches for dealing with these inconsistencies can be suggested. One is to look more carefully at the way in which the data are interpreted. For example, if a verbal report conflicts with an observation, it may be that the researcher just did not understand the verbal report. The teacher, for example, may have meant something different from "let students construct their own mathematical knowledge" than the researcher. Another possibility is that the beliefs that are inferred might belong to different belief clusters. Green (1971) describes how it is possible for a person to hold a series of incompatible beliefs. An important goal of education is to try to minimize the number of these discrete belief clusters.

A third way in which inconsistencies might be addressed is to note that certain beliefs are more central than others. A teacher might believe that students should construct their own mathematical knowledge, except when they are learning skills that will be tested next week on the district proficiency exam. The "constructivist" belief might be suspended in light of belief that an expository approach will maximize the chances that students will score well on the standardized test. The beliefs are not contradictory, but a more central belief may take over in a particular case. This might make it appear that an inconsistency exists.

The above example underscores another point that needs to be made regarding the study of teachers' beliefs. This is the importance of the teacher as a participant in the research process. Though the use of questionnaires and observations require a type of teacher participation, the study of teacher thinking requires that participants become involved in the data gathering in an even more active way. One method for doing so, mentioned above, is to work through interview or observational data with the assistance of the teacher. Given a transcript of a lesson or an interview, the teacher might be asked which points seem to be the most significant. Teachers might be asked, for example, to highlight those lines of an interview transcript that are particularly characteristic of what they believe. Similarly, card-sorting tasks in which teachers group various statements together into belief clusters require the teacher to become an active participant in the research. By having teachers reflect on the data in more than one sitting (e.g., during the initial collection and then at a second time), the multifaceted terrain of a belief structure is more apt to become apparent.

What Does This All Mean?

A third area to be considered in the study of mathematics teachers' beliefs is the issue of application. What are the implications of knowing and learning about teachers' beliefs? What does this all mean?

Perhaps the most obvious area of relevance pertains to teacher education. The findings of descriptive studies on teachers' beliefs would seem relevant for designing preservice and inservice teacher education programs which are more effective. Given the fact that

cognitions (and, in particular, beliefs) precede actions, then the more that we can understand regarding teachers' beliefs, the greater the chance we have of producing those types of actions that are in some sense "desirable." Knowing what is inside the "black box" will help in the direction of these behaviors which are ultimately controlled by the "black box." Thus, the products or findings of research on teachers' beliefs would seem to have implications for teacher education.

However, the products are not the only thing of possible benefit in this work with teachers' beliefs. The research also has process implications. For example, the process of asking teachers to highlight statements which are particularly relevant and to collect their beliefs into "clusters" can add not only to research knowledge but also teacher's self knowledge. In working with teachers on identifying and clarifying beliefs, information of interest to both researchers and participants would come to the surface. The questions used by Cooney in some of his work, such as, "If you could be an animal when teaching, what animal would you be?" have the potential to cause teachers to reflect on their beliefs in a way that a therapist might have a patient reflect on a particular incident. Benefits for both researchers and teachers can be imagined.

Several points of caution need to be added to temper these optimistic possibilities regarding the study of teachers' beliefs. The first is that a study of beliefs can, from a "process" point of view, lead to unpredictable consequences. The religions of old expended a great deal of effort to ward off heresy and make sure that its members were believing the "right" sorts of things. When members were confronted with the "wrong" beliefs or otherwise forced to examine some of their beliefs which had always been taught as the "truth," the results were sometimes unpredictable. It is not easy to give up a deeply seated but perhaps not very well justified belief, and if this belief is given up, then the consequences might be undesirable. For example, if the belief that is given up is a linchpin for other beliefs, then the emotional experience that surrounds this change of belief can become shattering.

Only the coolest of rational temperaments can stand an inquiry into their deepest beliefs and be able to walk away from the process of reflection unscathed. The rest of us are likely to become very uncomfortable and perhaps even unable to function if our beliefs are scrutinized in a way that makes them difficult to maintain. Though we may espouse faith in reason and argue that everything should be subject to the critical test and examination, most of us cannot engage in a discussion of deeply seated beliefs without some discomfort.

The issue here is whether teachers' knowledge of teachers' beliefs (or their "self-awareness") will help the teachers themselves. It is not clear from the start whether this is necessarily so. Some teachers might be "at risk" regarding the type of belief scrutiny that would become part of the process of research on teachers' beliefs. It is not clear, without further study, whether this "risk" is worth the taking. One would like to think, in a Popperian spirit, that good theories are those that must stick their necks out and that the belief systems of teachers are best if they are subject to the test. But teaching is a social institution, and it might be that researchers in teachers' beliefs should investigate the consequences of experiments on human subjects. In short, the unintended consequences of research on teachers' beliefs need to be mentioned and given careful consideration.

Laundry List Versus Grand Theory

Another point regarding possible consequences of a study of teachers' beliefs pertains to the difference between generating a "laundry list" versus trying to propose a "grand theory" that can be used to guide research. As mentioned above, the approach taken here is eclectic, including both some thoughts on a laundry list as well as some ideas on what possible grand theories, for example, those proposed by Green, Perry, and Joyce and Weil.

However, the consequences of research on teachers' beliefs are apt to be different, depending on what sort of approach one decides to adopt.

A laundry list has the benefit of being as descriptive as possible. One attempts to list areas of interest in the study of teachers' beliefs and to let the data "speak for themselves." There are at least two drawbacks to this approach. The first is that it is not really honest or possible. As it has been pointed out in the literature on theory formation, inquiry is always guided by a theory (see Popper, 1972). All observation is guided by some type or theoretical framework or paradigm (Kuhn, 1962; Quine, 1953).

A second drawback is that a more descriptive approach may have less of an impact on research. One of the greatest values of "theory" lies in the economy provided by a theoretical description of a phenomenon, as opposed to a mere description of traits associated with a phenomenon (Cronbach and Meehl, 1955). For example, making decisions about a student characterized as "anxious" is probably easier than making decisions about a student whose "personality profile shows high loadings on the aggression scale, middle loadings on the fear of strangers scale, low loadings on a tolerance for ambiguity, and high loadings on a fear of new situations scale." An economical description of teachers' beliefs, provided by a theory-based approach, would help to circumvent the information processing load of an lengthy description of various beliefs. It might be much easier, from a decision-making point of view and for suggesting further research hypotheses, to be able to characterize a teacher as "constructivist" or "tell 'em" versus trying to provide an exhaustive list of belief characteristics of that teacher. As Karmiloff-Smith and Inhelder put it succinctly in the title of one of their papers: "If You Want to Get Ahead, Get a Theory" (Karmiloff-Smith & Inhelder, 1975; see also Hiebert, 1984).

However, there are also some drawbacks to a more theory guided approach. Though great economy and heuristic power can be provided by an economical description of some phenomenon, there are some prerequisites to such a description. For one, a theory is needed. The ideas of Green, Perry, and Joyce and Weil have been mentioned in the previous discussion, but a more careful synthesis of these ideas would be needed before anything of the scale of a "theory of mathematics teachers' beliefs" could be proposed. Since the economy and power provided by a theoretical description requires work that may not yet exist, there is something to be said for remaining vague and eclectic at this point. The ideas suggested here might be regarded as "spadework" prior to a more thorough conceptualization of the domain.

Another point is that there are dangers associated with a more theoretical approach to the study of mathematics teachers' beliefs. In particular, a theoretical characterization of beliefs might be used for the wrong purposes. Students are often classified as having a "behavioral disorder," as "gifted," or as "mildly mentally retarded." Though these labels have all the power of suggesting hypotheses and facilitating the making of decisions alluded to above, they also have the danger (from a humanistic point of view) of putting people into straightjackets. The decisions made on the basis of these labels are not necessarily justified.

Applied to the area of teachers' beliefs: Characterizing certain teachers as "tell 'em" or "constructivist" might be a boon to theory creation and decision making but a bust to teachers. These labels, like those used so liberally in special education, might be used by those in authority to make decisions which are not sufficiently informed by the research or the complexities of teacher cognitions or are otherwise too simplistic. There would always be a danger of using belief descriptions to compartmentalize teachers' beliefs or to

otherwise do teachers injustice by discriminating against those whose do not hold the "right" beliefs. A theory of beliefs may not work in the best interests of teachers.

The issue here might be formulated as a question: Is it better to have the practical advantages of a theory, which can lead to hypotheses such as, If a teacher has this constellation of beliefs, then he or she is most likely to act in that particular way? Or is it better to stick to a laundry list approach and attempt, as far as this is possible, to describe teachers' beliefs in "low inference" terms? A possible middle ground would be to attempt to say something, both theoretical and empirical, about the relationships among various beliefs. The research discourse would be about relationships, not necessarily categories. This would be one step above a raw, descriptive characterization of the data on beliefs and one step below a tight, theoretical categorization of teachers' beliefs. This approach would also be in keeping with the eclectic spirit advocated above and would enable researchers to make statements about which beliefs are likely to strongly influence behavior. A systematic study of the relationships among the beliefs of mathematics teachers might lead to some fruitful intervention, both in preservice and inservice education, without casting teachers in molds that are hard to break.

Some Possible Extensions

One of the main deficiencies in the study of mathematics teachers' beliefs is the lack of an overarching framework. The above discussion has emphasized the eclectic nature of this field, and ambiguity is maybe necessary at this point in time. However, at least one theme might tie together some of the loose ends. This theme, which might be described, alternatively, as "teachers' beliefs in authority" or "teachers' beliefs in how learning takes place," was touched on at the MSU conference, but it was not discussed in detail.

Teachers' beliefs about authority and learning tie into several areas that would be of interest to investigate. One would be teachers' beliefs about the relationship between authority and mathematics. From where does the "truth" of mathematics derive? Does it come from the fact that famous mathematicians have "said so"? Is the mathematical process so mysterious and complicated that most human beings cannot comprehend the source of this truth? What do teachers' believe about the rules of mathematics? Are these like moral rules that we just "should not break"? Are they like rules of a game which we do not want to break if the game is to continue?

A related question pertains to teachers' beliefs about the relationship between mathematical knowledge and authority. Wherein does the teachers' mathematical authority lie? Is the teacher "in charge" merely out of a matter of convenience? Does the fact that he or she possesses a deeper understanding of the subject matter justify her role? How far does this authority extend into the classroom? Do teachers believe that their authority should guide them to "correct" students misconceptions and tell students what the truth "is"? Do they see their authority as enabling them to become, with their students, a fellow investigator in the discovery of new mathematics? How do teachers' conceive their role relative to the discipline of mathematics?

Another issue pertaining to authority is a teacher's conception of his or her relationship to peers and the rest of the community. What do teachers believe to be their charge from the school board or the principal? How much choice does the teacher feel that he or she has? Does the teacher believe that it is possible to depart from the textbook, or does the teacher believe that the curriculum is something mandated by peers and the community? Does the teacher believe that he or she has the authority to make professional decisions about curriculum and learning, or is this area something best left to the "experts"? What does the teacher believe to be his or her relationship to the rest of the educational establishment or to

the "knowledge base" of research on teaching? How does the teacher perceive his or her authority in relationship to the nation, community, principal, peers, parents, and so forth?

The work of Doyle (1983) might be used to help guide inquiry into the area of teachers' beliefs. Doyle argues that teachers' conceptions of academic work is apt to be very different from students' conceptions. Whereas a teacher might see a task as contributing to higher order thinking, a student will try to circumvent the necessity to think during that task and try to get teachers to just tell them the answers. The students want to play the traditional game of school. Students often resist teachers' attempts to change the rules and become a facilitator in inquiry rather than a dispenser of information. These different conceptions of academic work are likely related to differing conceptions of authority in the classroom.

Teachers' beliefs regarding authority would appear to be closely related to their beliefs regarding how learning takes place. At one extreme, the teacher might think that the teacher's authority in relationship to mathematics is really that of a learner, like the students. In this extreme case, one would have the teacher in virtually the same position as the student, and teacher and students alike would be in the classroom attempting to discover or create mathematical knowledge. At the other extreme, the teacher would be a master and judge of the subject matter, and the teacher's authority would derive from the fact that he or she "knows best." Here the teacher would be one of validating or certifying correct student answers to problems.

An important issue regarding teachers' beliefs in authority and learning is when and where the teacher believes that students should be left to struggle versus when they should be told what to do. This issue would be of interest to investigate because it is possible to make some statements, based on learning research, as to when the struggling should be permitted. Namely, if students have adequate conceptual knowledge to make some sense out of the struggle, then letting the students flounder might be sensible. Thus, it might be possible not only to describe but also to evaluate teachers' beliefs about when students should struggle.

Teachers' beliefs in authority and learning might be what Green refers to as "core beliefs" in a belief system (Green, 1971); that is, these beliefs would be central in describing the landscape of a belief structure. In addition, questions regarding teachers' beliefs in authority and learning bring to light the utility of a more general scheme for classifying models of teaching, such as that described by Joyce and Weil (1980). Depending on a teacher's views about his or her role in student learning, the teacher might be classified as a constructivist, a "tell 'em teacher," a humanist, or something else. These construct labels would, in turn, suggest other hypotheses and questions which might be proposed for empirical investigation.

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LEARNING FROM STUDYING TEACHER CHANGE

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At all grade levels and in all subject areas, teachers in the United States are being pressed to change their classroom practice in ways that will result in students leaving their classrooms with the knowledge, skills, dispositions, and habits of mind that will be needed to succeed in the world of the twenty-first century. Although reformers are not in complete agreement as to exactly what knowledge, skills, dispositions, and habits will be needed, reformers do agree that students are not being prepared adequately by typical classroom teaching as it now exists in practice. In the rhetoric of reform, one message comes through loud and clear: Current classroom practice is outmoded, and teachers need to change.

In the area of mathematics, reformers are in somewhat greater agreement than in other areas, such as social studies, for example. The Curriculum and Evaluation Standards for School Mathematics prepared by a working group of the National Council of Teachers of Mathematics (NCTM, 1989) represents a consensus of a cross section of mathematics educators, classroom teachers, supervisors, educational researchers, teacher educators, and university mathematicians. Buttressed by three subsequent documents--Everybody Counts (National Research Council, 1989), Reshaping School Mathematics (National Research Council, 1990), and Professional Standards for Teaching Mathematics (NCTM, 1991), the NCTM Standards lays out new goals, encompassed by a new vision of mathematical literacy, and provides a rationale for these new goals and vision.

In the area of mathematics, teachers are being asked to enact a vision of mathematics learning and teaching different from that they experienced themselves as learners and different from that which they have been enacting as teachers. As teachers attempt to enact this vision, they will be reinventing for themselves what it means to teach and learn mathematics. In this context of reform, studies of teacher change can illuminate the problems, issues, and dilemmas faced by teachers as they attempt to change. Such studies can uncover consistent themes and patterns of teachers' learning and change that might enlighten further growth in teachers' learning and further attempts at mathematics education reform. Just as researchers and educators learned and benefited by studying teacher change during earlier reforms, such as the "New Math" of the 60s (Blair, 1989; Romberg & Carpenter, 1986) and later reforms of the 70s (Romberg, 1988), so also might they learn from studying teacher change within the context of the current mathematics education reform.

Participants in this group were united by a common interest in the study of teacher change, by a common belief in the importance of what can be learned from such studies, and by a common focus on teacher change as a major part of their on-going research. Participants also had in common taking a "case approach" in their studies of teacher change--they attempt to understand the knowledge, beliefs, and thinking of individual teachers who are engaged in attempts to reform their practice, and they attempt to learn from their analyses of these individual cases. Each participant brought significant knowledge and experiences to the discussion.

Perspectives and Experiences that Participants Brought to the Discussion

Hilda Borko is co-principal investigator on "The Learning to Teach Mathematics" project funded by the National Science Foundation (MDR 8652476). In this study, the researchers

are examining the process of becoming a middle school mathematics teacher by following a small number of novice teachers through their final year of teacher preparation and first year of teaching. Their major goal is to describe and explain the novice teachers' beliefs, knowledge, thinking and actions over the two-year course of the study. Additional goals are to describe and explain the contexts for learning to teach created by the novice teachers' university teacher education experiences and by their experiences in the public schools where the novices teach and hold their first teaching jobs. The researchers hypothesize that these experiences will be a major source of influence on the process of learning to teach. Secondary sources of influence are expected to be the novice teachers' personal histories and the research project itself. In this study the researchers rely primarily on interviews and observations to gather information pertinent to the participants' learning to teach experience. For example, they observe the novices' teaching, the mathematics methods course, and the teaching of the cooperating teachers during the final year of teacher preparation. They interview participants, as well as numerous people in the novices' university teacher education program and the public schools. They hope that this in-depth study of a small number of novice teachers will reveal the complexity of the learning-to-teach process and the influences on that process.

In her research, Susan J. Russell has been looking at the way that students and their teachers in grades 4 through 8 understand a domain of mathematical content that has not previously been taught at these grades--the domain of statistics. Part of that work is to try to understand the development of students' ideas in statistics. Some of the questions that she is striving to answer are: In the area of statistics,what do students' mathematical ideas look like in fourth grade? What do they look like in sixth grade? What do they look like in eighth grade? Are there age-related differences or does the same range of conceptions occur in all of these grades? By the end of the project, Russell hopes to have good descriptions of how people describe distributions of data, what features they see as salient, how they think about measures of center, what kind of measures of center they invent, or if they don't know any, how they understand the traditional measures of center they have learned, how they relate that or often don't relate it to the data that it represents, whether they can compare data sets and what they use as a basis for that comparison.

Andee Rubin spent seven years working with researchers at the Center for the Study of Reading in Champagne, Urbana, to develop self-awareness curriculum for teaching writing in upper elementary schools. Although in some ways, this research on the teaching of writing may seem far from the teaching of mathematics in elementary schools, Rubin sees important parallels when she examines the process of teacher change and focuses on how classrooms change when an innovative way of thinking about a subject is introduced. Some of the issues she sees as important arose out of her study of a number of classrooms that were in an experiment to use computers in the teaching of writing. The desired change was toward classes oriented toward a lot of collaborative writing, the use of the computer not just as an editing environment, but also as a way for students to send mail to one another, and to plan their compositions as well as to write them. After the classroom teachers had gone through training sessions, Rubin and her colleagues went into the classrooms over the school year, and studied what happened to their idealization--what was actually realized in those classrooms. What came out then was very different in each classroom--the idealization was reflected or refracted through a lens that was the environmental situation and the teacher in each of those classrooms. From looking at some of the differences and similarities among classrooms, Rubin gained an understanding of what were the difficult areas for teachers to change and what were the easy places for teachers to assimilate what they were told into their prior practice. Rubin has seen these things happen in mathematics classrooms as well. When teachers are faced with a radical change, they segment their practice and they engage in a process of

assimilation of new practi s to earlier practices. The small changes that result may look like they are changes, but they may not be full changes or even changes at all.

With several colleagues (Deborah Ball, David Cohen, Richard Prawat, and Ralph Putnam), Penelope Peterson and Suzanne Wilson are engaged in a study of teacher change related to implementation of the state-level "Mathematics Framework" in California. In this study, researchers attempt not only to document effects of state education reform in elementary mathematics curriculum on teaching and learning in elementary mathematics classrooms but also to understand how and why certain effects occur. Additionally, they are exploring the processes that lead to certain effects on teachers' classroom practice of teaching and learning mathematics. To do this, researchers collect data at four levels--state, school district, school, classroom (teachers and students). In the Fall 1990, issue of Educational Evaluation and Policy Analysis (EEPA) the group has published five case analyses of these teachers.

In the EEPA volume, Wilson (1990) describes the case of a teacher, Mark Black, as he struggles to adapt the calls for the reform of teaching in California. Wilson explores how Mark enacts the curriculum of Real Math, the textbook that Mark's school district recently adopted. Through the lenses of his beliefs about the nature and structure of mathematical knowledge, his beliefs about how students best learn mathematics, and his beliefs about his role as a teacher, Mark transforms the innovative textbook into a more familiar, traditional elementary mathematics curriculum. Wilson discusses four real and perceived constraints that influence Mark's ability to enact the curricular policy proposed by the Framework and argues that teachers are themselves learners who need to be supported and nurtured as they try to change their practice.

In that same volume, Peterson (1990) examines the perspectives and practice of elementary mathematics teaching of Cathy Swift, a second-grade teacher in a low-socioeconomic status (SES) school in a large California city. Cathy's mathematics lessons are smoothly and swiftly paced lessons in the tradition of effective teaching for basic skills. Yet from her perspective, Cathy is implementing a "new" mathematics program connected with the state-level Mathematics Framework. Cathy's view of the state policy is through her textbook, one of several approved by the state, and through district-level Achievement for Basic Skills (ABS) materials developed specifically for low-SES schools. The ABS model includes components of content coverage, pacing, mastery testing and reteaching, maximizing students' time on task, and use of direct instruction. To these, Cathy has added new elements--using manipulatives, using partner and group work, and emphasizing problem solving. Exploration of Cathy's perspectives and practice reveals powerful effects of knowledge and beliefs, tangled influences of layers of policy, and multiple uncertainties and conflicts.

The final participant, Nancy Knapp, served as the recorder for the group's discussion. Knapp is a former adult education teacher, who after years of experiences teaching basic literacy skills to adults, began to wonder about what had happened to these adults during their early years in school that influenced their learning in powerful ways, but in ways that did not facilitate their development of literacy and numeracy as children and youth so that they now sought to learn these basic skills as adults. She decided to return to graduate school at Michigan State University in order to seek out some answers to her questions about why some students are not learning in our nation's schools and how schools might be changed to help students learn.

Major Questions Addressed

Although the group's discussion was far-ranging, participants tended to make statements that reflected one of the following three things: what they knew and had learned from their case analyses or others' case analyses of teacher change; what they were uncertain about and felt needed further study and research; and what they thought could be done to improve future research. Thus, one might say that, in a tacit way, the group addressed three major questions: (1) What have we learned from studies and case analyses of teacher change? (2) What have we yet to learn? and (3) What can we do ensure that we learn more from our future research on teacher change?

The group spent most of its time conceptualizing and thinking about teacher change. But the group also addressed the issues of (a) the relationship of teachers' knowledge and beliefs to teacher change and (b) levers, opportunities for, and constraints on teacher change. And finally, the group briefly discussed methodological issues.

Conceptualizing Teacher Change

The group began by noting that researchers' and teacher educators' remarks to teachers too often have a tone of "Why don't you teachers change more!" and that researchers may underestimate how hard it is to change. Andee Rubin pointed out that just as some teachers think that "identifying the misconception (held by the student) is the new way to teach," teacher educators often try to do this with teachers--you find the teacher's misconception and correct it. Hilda Borko agreed and noted that as researchers, we tend to focus more often on questions that teachers are unable to answer, rather than on questions than would reveal teachers' knowledge and understandings.

Group members noted with irony that many researchers espouse the value of teachers recognizing and valuing students' partial learning, but they don't always value teachers' partial learning. Researchers tend to underestimate how difficult change may be for the teacher, and often the tone in which researchers or teacher educators describe the changes in teachers makes it sound like the change is insignificant, but for the teacher something significant is happening for him professionally. These things that researchers may see as little changes, a teacher may see as enormous changes. One implication is that in studying teacher change, researchers need to try to understand change from the teacher's perspective. Peterson and Wilson noted that this has been the aim of their team of researchers studying elementary teachers' enactment of the California Mathematics Framework in their practice. A second implication is that researchers need to begin to develop ways of talking about teacher change that illuminate the stages and varieties of teacher change and lead to a greater understanding both of the nature of the change as well as how and why the change came about. The kinds of changes that are possible in teaching vary in terms of their magnitude and focus. The possibility for change derives from the source of the change, the power of the experience, and the demands of the job--what the teacher is capable of doing or responding to in a particular context.

The group then spent some time trying to invent ways that they might talk about teacher change that begin to illuminate possible stages and varieties of teacher change. Peterson noted that one way that teachers might begin to change is through an "Ah-hah!" that they experience about their children's mathematics knowledge or learning.

The Nature of an "Ah-hah!" Experience

Peterson commented that she was really intrigued with the question of "How far can the typical elementary teacher go following from an "ah-hah!" experience in which the

teacher suddenly understands what is meant by students' "mathematical understanding?" For her this question emerged from her study with Liz Fennema and Tom Carpenter of the first grade (CGI) teachers in Wisconsin. For these teachers what was more significant than anything else was their discovery that their own first-grade kids had a lot of their own mathematical knowledge of how to solve addition and subtraction word problems. For many teachers part of this discovery also involved a sort of "Ah-hah! Now I understand what conceptual understanding in mathematics really is." For example, one teacher (Ms. J.) did not herself understand how to solve some kinds of addition/subtraction word problems before the CGI inservice workshop. When she gained this conceptual understanding, Ms. J. understood how she really needed to get inside her kids' heads and think about their understanding. Ms. J. significantly changed her views and beliefs about mathematics learning such that out of 40 first-grade teachers in the study, she became the most constructivist in her beliefs about children's mathematics learning (see Peterson, Carpenter, & Fennema, 1991).

Russell described the case of a staff developer in mathematics for a K-5 school, who in the midst of an interview, discovered the structure of the "mean." The staff developer started out with an interesting misconception that children often have. She was working on the following problem: "If you know that the average allowance that a child gets is $1.50, then what might the data set look like that produced that average?" When children try to construct the data set with tiles or with a graph, they often balance the amount of money on each side of the average--they are balancing the total amount of money, not looking at deviations. During the interview, the staff developer did this. Then she went through that wonderful state of knowing it wasn't quite right. She persisted and tested her own notions by simplifying the problem and employing wonderful strategies. Finally, she realized that it was the *differences* from the mean that mattered, not the absolute total. She turned to Russell, her interviewer, and commented that she had never had the opportunity to do that--to delve into one small piece of mathematics deeply in a way that got her past the surface understanding that she had developed from her previous experiences.

Borko suggested another way in which a teacher might have an "ah-hah!" experience--listen to what children don't know right after the teacher has "explained it all" to them. Such an experience serves as a graphic demonstration that, in this case, "telling" didn't work. Indeed, telling teachers how to teach for conceptual understanding may be an oxymoron.

Not to say that "telling" never works, but that it is only one strategy for teaching--one that is often overused and inappropriate for the audience due to their lack of prior knowledge. The dynamics of teachers being responsible for a lot of kids often leads teacher to choose "telling" as a way to reach the most kids at once. Suzanne Wilson noted that, as a teacher, sometimes you have to move the whole group along to something else.

Stages of Teacher Change

Russell described a "lesser stage" of this "ah-hah!" experience for teachers that takes a different form. In this form the nature of the "ah-hah!" experience has less to do with the teacher realizing that, "I need to pay attention to my students' understanding" and more to do with "I want math to look different in my classroom, I want it to be more active, I want kids to be more involved and interested." From interviewing teachers, Russell has found that teachers often have images of how they want mathematics to be different, but the images do not involve a deeper understanding about really building and paying attention to student thinking. Rather, the teacher has an image of activity and concrete material--they know that they ought to be using concrete materials to teach mathematics. Further, they want children to like mathematics and to have more fun doing mathematics, and they

want to have more fun teaching mathematics. Perhaps these different images lay the groundwork for openness to change, or perhaps they serve as intermediate steps along the route.

Research Questions on Stages of Teacher Change

As a result of their attempts to begin to conceptualize teacher change, the group members (indicated by their initials) generated a number of questions that they thought researchers need to address. Among these were the following:

1. When do "ah-has" happen? Under what circumstances? What fosters them? Can working with other teachers in inservices bring on the "ah-ha"? (SR) Once teachers observe kids actually thinking, can this bring about the "ah-ha"? (PP) What are components of this "ah-ha" experience? Is it different for different people? (HB)

2. Can we find patterns of how change occurs? What is the time frame for an "ah-ha" experience to influence a teacher's classroom behavior? What are the intermediate steps toward change? Is there an intermediate period of frustration? What support is needed during this process? (SR)

3. What different kinds of teacher change are possible--of what magnitude and focus? (SW) Are there different qualities of "ah-ha's," and are some deeper than others? (SR)

4. Do teachers need models of what to do instead of "telling" in order to change? (HB) How come there are so many teachers still "telling" if all we have to do is show them that it doesn't work? (PP)

Considering Teachers' Knowledge and the Intellectual Mess

An important question is: Now having had a conceptual breakthrough or an "ah-hah!" experience, how far can teachers like these go before they come up against the limits of their own mathematical knowledge? Both Ms. J. (the CGI teacher described by Peterson) and the K-5 staff developer (described by Russell) had come to realize that they themselves had only superficial understanding of the mathematics that they were to teach their elementary children. These two teachers are typical of elementary teachers. Suzanne Wilson noted that when you ask questions about "simple" topics like addition, you see troubling gaps in elementary teachers' mathematical knowledge, and when you get to multiplication and division, you find that elementary teachers' mathematical content knowledge is not very deep.

One possibility is that the teacher will recognize the need to develop a deeper knowledge of the subject--in this case mathematics-- and will seek out ways to learn more about the subject herself. For example, by the end of the second year, one CGI teacher, Ms. J, went to Liz Fennema and said she realized that she needed to learn more mathematics herself, and she asked for help in doing so.

Another possibility is that, *without deep knowledge of the subject, the teacher as a naive constructivist will get lost when (s)he attempts to teach for understanding*. At the very least, the teacher will fail to realize which ideas are key ideas that come up in during classroom discourse and will miss important opportunities to follow up on these ideas and to develop them. For example in a recent paper entitled, "Tolerating Intellectual Mess: Data Analysis in the Elementary Classroom," Russell (1989) described the difficulties of Mr. K, a teacher who was struggling to replace traditional arithmetic with a new approach to mathematics. In Mr. K's classroom the students had been measuring and representing

information about their bodies. The curriculum materials that Mr. K was using suggested the idea of building on students' informal ideas about *middle*, that were not yet connected explicitly to the statistical use of *median*. As Russell described it:

> These 8- and 9-year-olds had been very intrigued with finding the middle-sized foot or the middle-sized height of a third grader. As students looked at a representation of their height data, which they had just constructed (the data ranged from 47 to 58 inches, with a median of 52 inches), the following conversation ensued:
>
> Mr. K: Who can answer my question? What's the middle-sized third grader? (A chorus of "ooh's." Many hands raised.)
>
> Anita: 53.
>
> Mr. K: 53. How do you know? So I can this is the middle-sized (he draws a vertical line between 52 and 53 on the graph)? How can you tell me that's the middle-sized?
>
> Tony: Between 52 and 53.
>
> Mr. K: Between 52 and 53. But what does that mean to be middle-sized? What does that mean to be in the middle?
>
> Josie: You count, um . . . (long pause). . .
>
> Mr. K: That on each side, what?
>
> Josie: You count how many there are.
>
> Mr. K: And they should be the . . . ?
>
> Many students: The same (i.e., the same number of data points on either side of the middle value).
>
> Mr. K: Let's do that then. (He counts, gets 11 on one side of his line, 10 on the other, and marks these counts on the diagram.) Is that about right? There's 11 that are less than 52 and 10 that are more than 52. So are we about in the middle? (upbeat, hopeful voice)
>
> Several students: No.
>
> Mr. K: No? Where should we be?
>
> Jose: Between 52 and 53.
>
> Mr. K: Between 52 and 53. Here. There are 11 less than that (pointing at his line on the graph) and 10 more. Isn't that about the middle?
>
> Several students: Yes. No.
>
> Tania: Because 53, you get to put it by 53 and then it's 11.
>
> Mr. K: But then this will be [illegible]. So what should I do?

Luis: Put the 11 there (on the right side of the graph).

Mr. K: I can change this back and forth, but this would become 10, and this would be 11. Does it really matter?

Several students: Yes.

Mr. K: We're about in the middle here, aren't we? (At this point Mr. K changes the focus to describing what else the graph shows about class heights.) [From Russell, 1989, pp. 3-4.]

Although it is difficult to tell exactly what some of these students mean, it is clear that there is disagreement about where the "middle" is and how to find it. Some students advocate shifting the line which is dividing the data in two parts to the left in order to increase the count of 10 on one side to 11, apparently not realizing that shifting the partition of the set will affect *both* sides of the partition. It is also unclear whether some students are looking at the middle of the range while others are actually focused on the middle value in the data set. In any case, finding the middle is an interesting mathematical problem for these students--as we have seen in other classes of 8-10-year-olds--which is cut short because *Dave does not recognize that the students are on the verge of engaging in important mathematical investigation* (Russell, 1989, p. 4).

So an important question is, what do teachers need to know about the subject matter in order to not only recognize opportunities to develop powerful mathematical ideas but also to do something productive in the intellectual mess? Suzanne Wilson described her own struggle with this kind of problem in her teaching at the elementary school level:

I have a lot of beliefs about mathematics. I've done a lot of mathematics on the way to studying teaching, and I've studied Magdalene Lampert's teaching, and I've read a lot of philosophy of mathematics. I think I have a pretty OK conception of mathematics, but I don't know the subject matter. I haven't studied the subject matter of mathematics, but I have a very well developed conception of the nature of social studies and of history.

I teach a social studies class, and we're doing map construction right now. And map construction has a lot to do with mathematics! And so, I have good images in my head for what it would mean to teach mathematics in meaningful ways. I've seen good mathematics teaching, and I know a lot about the subject matter, but I don't know the stuff, so that when kids say things that I recognize as opportunities to do stuff, I understand that I am not in a position to do anything at all with what they've said.

For example, yesterday we were discussing the length of the hallway. I was trying to get the students to grapple with the idea of standard units of measurement on maps. We had been measuring the classroom, and we were going to go into the hallway. I asked the kids about strategies for measuring the hallways, and lots of interesting ideas came up such as, "Let's use all the rulers in the classroom--all the brown ones, and all the white ones." But John says, "We can't do that, it will mess up the problem." I say, "Why is that John?" John replies, "Well, the brown ones are inches

and the white ones are centimeters, and if we use both that will mess up the problem."

I recognized this as an opportunity! Another student says, "Well, we can count up the number of squares and then we can multiply it by the number of inches the squares are." So I send some kids out to the hallway, and they come back and report that there are 348 squares. Well, there *are* 348 squares, but the problem is that there are 248 squares that are nine-inch squares, and there are 100 inch squares that are one-inch squares. And, the students have come back to report that if we multiply 348 by nine, we'll get the length of the hallway. I recognize this as another opportunity!!

Now, this is in the course of teaching social studies, so there are lots of reasons that I choose not to follow up on some opportunities. But, some of the reason why I can't, and will not follow up on some of those opportunities is because there are things I don't know about the subject matter, nor about how kids are thinking about the subject matter. So, here I am--a smart person in the classroom who has the right images, the right models, and probably even a reasonable conception of the subject matter--still, I don't know what it is I need to know in order to be able to teach in a meaningful way.

The case descriptions of Mr. K and Suzanne Wilson show how when discourse gets very diffuse, it may be hard for teachers to identify quality remarks and to figure out what ideas to follow up on. In referring to her study of elementary teachers trying to carry on discussions of "data analysis with their students, Susan Jo Russell put is succinctly, "Finding the quality in the intellectual mess--that's one of the things that doesn't happen very easily."

Russell went on to say that she knows some features of discourse that work--valuing students' partial answers and using students' own words to reflect their thinking back to them:

For example, I think there are two kinds of things. I think there are mathematical characteristics that are characteristics that have to do with how you see the mathematical process. And then there's characteristics that have to do with how you interact with students. An example of the first would be that partial solutions to problems are valued, and supported, and worked with, and seen as necessary and central to the whole enterprise. And an example of the other would be a technique that you use students' own words to reflect back to them what they've said to help them clarify and extend what they're thinking about. That's not mathematics specifically, you could do that in social studies, but it's one of the features of the dialogues that you see that are examples of good mathematical discourse. But I certainly don't have the definitive list, and I really expect that we and the teachers together are going to figure out what seem to be the characteristics of mathematical discussion when it really works to engage kids in mathematical thinking.

Thus, group members agreed that teachers will need both substantial subject matter knowledge and substantial pedagogical knowledge in order to teach mathematics for understanding. Yet they also agreed that much remains to be determined about how much and what kind of knowledge teachers need and how teachers might best develop the kinds of substantial subject matter and pedagogical knowledge that they will need to teach mathematics differently than they have been teaching.

Research Questions on Teacher's Knowledge and Beliefs as Related to Teacher Change

As the above discussion illustrates, the group members viewed teachers' knowledge and beliefs as important foci for research on teacher change. Accordingly, they suggested the following questions as worthy of inquiry:

1. How far can the typical elementary teacher go with "ah-ha" experience before needing to gain more mathematical knowledge him/herself? How much of this additional mathematical can teacher learn on his/her own, given books/materials? (PP)

2. How important is the teacher's image of him/herself as a mathematician to change toward conceptual teaching of math?

3. Does a teacher's teaching of a process approach to writing facilitate changing his/her math teaching? (PP) Are teachers aware of noncongruity between teaching more conceptually in one subject while teaching more traditionally in another? (SR)

4. Is there a pattern of breakdown in change from substantive to merely procedural? Is this pattern similar across subject matter? (AR & HB)

5. What causes this breakdown? Messiness? Control issues? (PP) Lack of knowledge of subject? (HB)

6. Without content mapping, how can we say we have a full view of teachers' ideas in an area of content? (HB) How can we avoid deciding what is important a priori if we use content maps? (SR)

Levers, Opportunities for, and Constraints on Teacher Change

Borko noted that in their research they have found that the cooperating teacher has great influence on change in preservice teachers. The supervising teacher has significantly less influence because the supervising teacher is present in the student teaching setting for only a small amount of time. Borko suggested that one implication of this finding is a need to move to a professional development school model where novice teachers learn to teach and are supervised by "clinical faculty" who are adjunct university faculty as well as classroom teachers in the professional development school. She said that she was building on an idea that Sharon Feiman-Nemser and Margret Buchman have talked about--the notion that cooperating teachers have to think of themselves as teacher educators. Here Borko seemed to be outlining a model of facilitating teacher change through teacher learning in a professional development context that might be created in the school where the teacher is teaching. Such a model would be appropriate for educating both preservice teachers and experienced teachers who need to learn and change their thinking and mathematics teaching in ways suggested by the NCTM Standards.

Although teacher education and professional development opportunities constitute one important mechanism for facilitating change in practicing teachers, most policymakers appear to be intent on pursuing other less costly mechanisms to effect change in teachers. Textbook adoption is the lever that some states, such as California, are trying to use to effect changes in teachers' mathematics teaching. In their study of teachers' implementation of the California Mathematics Framework, the Michigan State research team found that most teachers thought their textbook was the Framework (See, for example, Ball, 1990; Cohen et. al., 1990; Peterson, 1990). Further, when presented with the new texts, teachers

first picked up on a lot of new buzz words, such as "using manipulatives; and doing cooperative learning," without catching onto the substantive ideas behind the new terms (Ball, 1990; Cohen, 1990; Peterson, 1990). In California, there was too little money, time, and personnel for educating teachers about the ideas in the Framework. The Michigan State researchers noted that the several inservice activities that they observed or that were described to them by teachers did not usually emphasize mathematical understanding, but rather focused on mathematical activities of the "make-and-take" type. Such inservice activities did not seem to provide the California teachers with the learning opportunities needed for more substantive change (see, for example, Heaton, in press).

Borko argued that it's almost unethical to cause experienced teachers to realize the limitations of the way they have been teaching if the teachers are in a situation where they feel they can't change or if teachers do not have access to the kind of knowledge and support that they will need to learn to teach differently. It can be very painful for teachers who say "Now I can't help but pay attention to kids, but I can't do anything about it."

Russell agreed that support is very important to facilitate teacher change, but she has found it to be difficult to sort out the effects of support from the effects of other variables such as the degree of teacher's commitment or teacher's knowledge. For example, she ran into this problem in a three-year study of how teachers in four middle schools were integrating technology into their teaching for special needs children. Russell found that the most committed teachers were learners themselves and they put time and energy into their own development, but these were the same teachers who were in the most supportive contexts. However, it is impossible to sort out causation here because teachers who are inclined to be learners may themselves seek out school contexts that would support their professional development.

Andee Rubin argued that to facilitate change, teachers have to be given something to work with, not just modeling but discussions about how to implement this type of teaching in their classrooms. She also noted that congruence of goals and congruence of meaning are very important in determining degree of change In a study of implementing innovations in science instruction with bilingual students, researchers Chip and Sarah Michaels found that the word "literacy" meant very different things to different people. For some teachers literacy meant only knowing English, knowing the grammar of English, or being able to write coherent sentences. The researchers had a very expanded notion of what the word literacy meant, including science literacy. Rubin noted that if the teacher's goals are really different from the researchers' or reformers' goals of change, then they are going to rub up against each other.

Research Questions on Levers, Opportunities, and Constraints

In the course of group's discussion, the following emerged as questions that need to be addressed in future research:

1. What facilitates teacher change? Inservices (how many, how long, what kind)? What support systems would facilitate continued incremental change? (SR) At what stage in the development of change are inservices needed/effective? (AR) Study groups? (HB) Does discussing specific techniques for implementing new teaching in the classroom facilitate change? (AR) Can teachers generalize these techniques to other content areas?

2. What are barriers to teacher change? (SR) How does the number of children taught as a group or in a class influence teacher change toward conceptual teaching? How do demands of the job delimit teacher change? (SW)

3. What do those responsible for causing change (superintendents, etc.) have to know about subject matter? About teacher beliefs? About student thinking? (SR)

4. What support is needed for teachers who must face years of having taught without insight if they adopt change? (SR)

5. How can we avoid confounding job demand variables with teacher dedication variables, since the most dedicated teachers seem to be found in the most supportive environments? (SW)

6. How and how much do teachers learn informally from peers? (PP)

7. How much should text/materials prescribe teaching procedures? Do we need a phone book or a 10-page pamphlet?

Methodology of the Study of Teacher Change

The group concluded with a discussion of methodological issues. Hilda Borko commented that one of the things Ralph Putnam said to the group yesterday that really struck her was: "If we're committed to doing research in classrooms and to trying to at least look for evidence of change in classrooms, how do we do it?" She suggested that, in studying teacher change the sampling issues are particularly problematic. Some of the questions that arise about sampling issues are:

1. If we want to track change in actual classroom teaching, must we be there every day like Paul Cobb? Can we trust people's recollections of their own development and thinking, or do we actually need to see it happening? (HB & PP)

2. How can we pinpoint places/people in which change is about to occur, so that we can use our limited resources wisely? Can we develop "hunches" based on experience about what is likely to precipitate or impede change so we can go there to document it? Does doing clinical interviews before observations help to target important things to observe? (HB) We need to straighten out the sampling issue. We need long-term, longitudinal research to study these issues. (HB)

The group also wrestled with methodological issues of how much involvement the researcher needs to have in order to document and understand teacher change. Is it essential for the researcher himself or herself to go into the classroom to document and understand teacher change? Moreover, to study and to understand inservice effects, do we need to provide inservices ourselves, or can we just study the effects of an inservice provided by someone else?

Group members felt that they themselves could learn a lot about the substance and methodology of the study of teacher change by assembling and analyzing as set of cases of "teachers en route to change." Some of the questions that might be asked of all the cases are the following: Is this a case of a particular kind of change? Is this a case concerning barriers to or supports for change? Is this a case illustrating a particular vehicle for change? How do particular methodological choices made by the researcher affect what can and can not be learned from this case? The group then compiled a beginning list of their own and others' cases that might form the beginnings of such a casebook. The group concluded that such a casebook would be helpful to advance their own and other researchers' thinking both about substantive issues of teacher change as well as methodological issues.

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FACILITATING SHARED ROLES AND UNDERSTANDING BETWEEN RESEARCHERS AND TEACHERS

Participants: Audrey Jackson, Perry Lanier, Richard Prawat, Tom Romberg, Karen Stoiber, Janine Remillard, and Linda Weisbeck

Summary prepared by Karen C. Stoiber

We began with the assumption that the recently released Professional Standards for Teaching Mathematics (National Council of Teachers of Mathematics, 1991) have little meaning unless they are embraced seriously and implemented enthusiastically by classroom teachers. Related to this assumption is a central belief: Whether and how the NCTM Teaching Standards are implemented has as much to do with what researchers and trainers in mathematics education bring to this task as with what classroom teachers bring to it. The group from which I am reporting, then, operated from the proactive stance that as responsible members of the mathematics education community, researchers and trainers need to identify roles and strategies that will contribute to successful implementation of the NCTM Teaching Standards in mathematics classrooms across the United States. We recognized this almost immediately as a task involving paradoxical concerns and diverse themes. Realizing, also, the complex nature of our task, we resisted a push for closure, for producing a set of principles. Rather, we adopted a future-oriented goal--to catalyze further dialogue and to open debate on what researchers and trainers need to do to facilitate greater understanding about teaching mathematics in well-defined content domains. Hence, we also had an immediate objective--to uncover understanding leading to improving research on, and teaching in, mathematics. The overriding theme, then, was one of cautious but constructive action, rather than reaction.

In the final hour of our discussion, we began to sense the need to provide some structure or scaffolding to our thoughts. Our high level of urgency produced a set of organizers as a basis for approaching the goal of sharing responsibility for the successful implementation of the NCTM Teaching Standards among researchers, trainers, and teachers. The structure of this document corresponds to our dialogue on what we determined to be important issues in enhancing our functioning as professionals engaged in research on teaching in mathematics. Our dialogue focused not only on the broad issue of roles but also dealt with concerns related to the objective of optimizing the outcomes of our interactions and collaborations with teachers of mathematics. The concluding pages deal with my thoughts on authentic collaboration.

Roles

With regard to what our role should be, we discussed the paradox of how to establish professional identities separate from classroom teachers but also to transcend the professional boundaries to which individuals from each professional group usually adhere. In the Introduction to the NCTM Teaching Standards, the authors state, "creating an environment in which children have access to experiences and challenges that lead to the acquisition of mathematical power . . . calls for a teacher who is educated, supported, and evaluated in ways quite different from most training models of today" (NCTM, 1989, p. 2). Romberg reminded us of this point in emphasizing that "in essence, the Teaching Standards are arguing for empowerment of teachers in certain ways: in making decisions about what to teach, in judging students' [mathematics], in sharing ideas with other teachers, in becoming part of a colloquial group." Romberg further stated that in order for teachers to be empowered in ways suggested by the Teaching Standards, they need to become "confident and comfortable with what they are dong and to share that with parents, administrators, and other important individuals." The group extended

Romberg's notions in endorsing that researchers and trainers need to define ways in which they themselves can be responsive to this call for alternate models in educating, supporting, and evaluating mathematics teachers.

Specific suggestions of ways researchers could be responsive were made. We agreed that mechanisms of discourse between academic and school communities are essential to ensure that the spirit of the NCTM Teaching Standards reaches classrooms. We also concurred that the Teaching Standards provide good reason to increase our dialogue with teachers, to share ideas, to discover new ways of interacting. Romberg pointed out that the historic role of a researcher as "an observer, a gatherer of information, as a reporter" no longer makes sense. If we want the NCTM Teaching Standards to have an effect, to paraphrase Romberg, we need a new conception of what is desirable as the "work" of researchers and trainers. As he put it, we should not be above or behind the initiative to reform mathematics in the schools but rather an integral part of it. Prawat added that we need to develop prototypes or schemas for becoming better integrated with schooling practices and processes. He believed that researchers "help frame the discourse," but are lacking procedural and conceptual maps to follow.

Strategies

Our discussion moved to the topic of what researchers can do to facilitate implementation of the NCTM Teaching Standards in our schools. Jackson questioned our responsibilities. Do researchers have a responsibility to educate the community? Do researchers have a responsibility to communicate to parents the reasons for teaching understanding in mathematics? Stoiber questioned what our priorities should be. Should we attempt to alter the values implicit in local newspapers that report the ranking of community schools in standardized achievement tests? Or is it more important to attempt to understand the ideas an individual teacher has about how math should be taught? Several group members supported the viewpoint that mathematics education is not limited to what happens within classroom walls.

Stoiber made a plea for researchers to turn attention to parents' beliefs and conceptions about mathematics so as to understand children's thinking better. She reported a study that she conducted that examined what parents think and do with their preschool age children. One finding was that parents emphasize traditional mathematics learning activities with their children, such as number recognition, counting, and computing. Consequently, parents were much better at predicting their child's performance on these tasks than on reasoning or problem-solving tasks. Furthermore, parents showed limited awareness of the conceptual structures or representations children use to think about and understand mathematics. Lanier added that the popular press makes our job of altering parents' perceptions of mathematics more difficult. For example, the kinds of workbooks on mathematics sold in book stores, grocery stores, et cetera, mislead parents by stating that they explain new conceptual approaches to mathematics but then emphasizing counting or learning of multiplication facts. Jackson suggested that perhaps the notions and methods used in engaging teachers should be modified for use with parents.

Jackson argued for using naturalistic approaches on the ways we conduct research. One paradox prevalent in most current research in children's learning of mathematics is that although we espouse an intent to access real representations of how children think and learn, we usually access this knowledge in contexts other than the classroom. Though interviewing students outside of the classroom may be more convenient, by so doing, researchers fail to incorporate critical aspects that may impact on students' learning. Romberg stated that researchers should also assist in suggesting how to gather that information, by specifying "what kinds of things to look for, what kind of questions to

raise." Romberg seemed to be stressing that, as researchers, we bring certain kinds of expertise to our work with teachers that need to be recognized. Other group members asserted that, although the skills and expertise researchers hold should perhaps be recognized, they also need to be evaluated and modified.

Romberg identified several preservice and inservice teacher training models that incorporate the "wisdom in practice" rather than solely the expertise from trainers and researchers. One example comes from his experience with schools in Australia, called a "sandwich approach" to teacher training. The sandwich approach involves having a group of "apprentice" teachers observe another teacher teach a unit for several days. Each apprentice teacher then teaches the same concepts in his/her own classroom while the "master" teacher observes. The master teacher provides feedback to the apprentice teacher and the apprentice uses this information in modifying his/her instructional methods. Another example is having a peer teacher, research assistant, or curriculum specialist observe a teacher while teaching lessons in specific content domains, such as ratios, measurement, or fractions. At the end of the day, the teacher and observer discuss their own perceptions of what happened. Romberg pointed out that by having teachers collaborate with others in examining their instructional practice, they have the opportunity to draw on others' knowledge bases in content areas that teachers often find challenging to teach.

While all of us identified with a desire to find new ways of becoming responsive to the NCTM Teaching Standards, we also identified barriers to being more responsive. Both external and internal barriers surfaced in our discussion.

External Barriers

Economics was a problem raised by several group members. Prawat stated that most school boards will not readily buy the kinds of resources that Romberg described as important to make available to teachers. Lanier pointed out that more school-based support systems for teachers would place a financial stress on existing school budgets. Stoiber argued, however, that "poor" mathematics instruction is currently occurring in our nation's schools and that situation should not be ignored. Prawat questioned whether there are data based on cost-benefit analysis of teaching mathematics for understanding. Romberg surmised that the costs for the teaching resources that he advocates would be offset by the long-term benefits, " because you are going to have less kids dropping out of school, less needing public assistance." He stressed that we need to shift the focus from immediate costs to long-term benefits. Lanier expressed concern that the benefit to teachers as well as to students be considered in the equation. We agreed that more emphasis should be placed on the value of solid problem-solving and thinking skills for later real world functions. In addition, we supported more systematic accountability for initiatives aimed at improving students' thinking.

A barrier related to economics is the constraint of time. Several group members pointed out that the amount of time that teachers have to plan, to reflect, and to make sense of their instructional practices will influence their capabilities to teach for understanding. Although the group shared the sentiment that much of teachers' time is probably taken up by activities not requiring a teaching professional (e.g., supervising lunch, recess duty), the group did not have specific recommendations for changing teacher task assignments in schools. However, they thought that research might help alter current practices by demonstrating the value of teachers' use of time for planning and evaluating their instruction for specific teacher and student outcomes.

Another external barrier to teachers' adoption of the NCTM Teaching Standards is that mathematics is not the only curriculum area affected by the new ideas of teaching for

understanding. Jackson reminded the group that in most elementary schools, teachers are responsible for teaching the subject areas of reading, writing, science, social studies, and mathematics. Although she agreed that this arrangement limits teachers' potential for becoming specialists in a subject domain such as mathematics, she argued that the chances of changing this situation are dubious, at best. Romberg cautioned that, because teachers are experiencing many of the same pressures to apply notions of inquiry and exploration in all subjects, not just in mathematics, teachers' energies will not be focused solely on mathematics. As a result, teachers may depend on more heavily on the textbooks while teaching. Romberg viewed this as a major drawback because he felt that when teachers' attention is directed to reading lessons from texts, they attend less to understanding how their students are thinking.

Several group members expressed related concerns about textbook use in schools. Jackson expressed concerns about textbook adoptions at district and state levels without any teacher input. Romberg emphasized that, although leaders in mathematics education might be dissatisfied with the content of existing textbooks, they face a dilemma because textbooks that emphasize understanding in mathematics are lacking. One group member made the controversial suggestion that perhaps instruction in mathematics would improve if no textbooks were available.

Several members reiterated that an additional external barrier continues to be a lack of incentives for teachers to emphasize reasoning processes and problem solving in their mathematics teaching. Jackson provided an example from her experience of attempting to promote teaching for understanding in the teachers she supervises. She reported that a common response by teachers is that reasoning and problem solving pale as important outcomes for teachers when they feel they are being judged on their abilities to practice good standardized achievement test scores by their students.

Internal Barriers

A second set of barriers confronting efforts to facilitate acceptance of the NCTM Teaching Standards might be conceptualized as internal. Internal barriers refer to constraints that might thwart our own initiatives as individuals or as research professionals who share a common goal: to establish guidelines for studying the teaching of mathematics in well-defined content domains.

Group members referred often to a need for models of several kinds: (a) models of how research might be conducted to promote greater understanding in teachers' beliefs, planning, and thinking; (b) models to define how we mean by teaching for understanding in one content domain, such as addition, is similar and different from teaching for understanding in another domain, such as multiplication; and (c) models for engaging dialogue and sharing ideas between educational researchers and educational practitioners. A theme related to a need for models is a need to evaluate more specifically what we can learn form models currently in place. One group member commented that the Institute for Research on Teaching at Michigan State University has initiated a researcher-as-teacher model in which faculty teach a class in one subject area throughout the school year as part of their teaching assignments. It was pointed out that the researcher-as-teacher at MSU has many opportunities and resources available that are not available to the typical classroom teacher. For example, the researcher-as-teacher has other professionals who are interested in examining ideas, she has time to reflect on decision-making subsequent to teaching, she has a scholarly forum for writing about instructional practices, and she has a teaching assignment in only one subject area. Because of the unique qualities surrounding the MSU researcher-as-model, several group members suggested that we need to be cautious about generalizing from these researchers'

experiences to typical classroom teaching. Lanier commented that the MSU model should be viewed as one model, that might provide a basis for evaluating components of teaching and learning processes. He added that this model might be adapted and developed so that the knowledge gained be more generalizable.

The Cognitively Guided Instruction Project was mentioned as an additional model that has value. One group member pointed out that a distinguishing quality of CGI is that it does not prescribe instructional techniques or methods but rather it provides teachers with knowledge of how kids think in two content domains--addition and subtraction. One group member questioned whether the teachers in CGI are given knowledge about the"conceptual properties of knowledge" or strategies and procedures in addition and subtraction. Discussion ensued about whether researchers have the conceptual understanding in content domains other than addition and subtraction that would be needed to empower teachers to teach with understanding. Prawat commented that teachers must have conceptual structures or mechanics for understanding a concept as a basis for "figuring out how to probe to assess whether the child really has it."

The discussion of CGI led to a debate on the current status of knowledge on students' thinking in domains other than addition and subtraction. In specific, Prawat questioned, "What knowledge about students' thinking in mathematics do teachers at grade levels beyond first grade need to be empowered?" This question was viewed by most group members as very complex and as having no easy answers. The question precipitated a prolonged discussion about what we know and don't know about mathematics, in general, and about what we know and don't know about well-defined content domains specifically. Although group members differed in viewpoints, terminology, and reasoning, the group attempted to identify important issues and topics, as well as to appreciate what might be learned from these differences.

Status of Understanding in Mathematics

To evaluate our knowledge about mathematics in well-defined content domains, we decided it was necessary to evaluate the more basic notion of what is meant by "mathematics." Romberg provided his definition: "Mathematics is a language used to abstract from problem situations, a language to represent situations." He emphasized that this conception of mathematics is fundamental to our notions about children's thinking at any developmental stage or grade level. For example, when presented with a situation involving three objects, an individual must eventually make sense of it. The number 3 is a symbol that we use as a starting point for representing a collection of three objects. But that is only one situation, one starting point, one representation. Children need to be exposed to different problem situations in order to realize that they need another representation to understand "threeness," (e.g., to understand the relationship of 3 to 2, or the idea that 3 is 2 and 1.) Romberg explained that we "invented signs and symbols and rules for their use." "Romberg explained similarly how fractions are symbols for representing. Fractions are used to represent part-whole relationships and to represent precision in measurement, such as a half, a quarter, a tenth. These two representations provide a basis for the rules for the use of fractions.

Romberg continued by discussing how a "specific set of signs and symbols and rules for their use" define different content domains. For example, a rule for adding fractions is to "get like denominators before you add." He cautioned, however, that the features distinguishing content domains are often ambiguous because the functions in different content domains might be interconnected. Again, he used the example of fractions. After following the rule in adding fractions of "getting denominators the same, you only add numerators, which is essentially whole number addition." Romberg added that one of the

difficulties in understanding what we mean by "well-defined content domains" is that it involves a network of situations, as well as representations, ideas, and procedures to give meaning to these problem situations.

Prawat proposed what he considers to be a more conceptual approach to defining mathematics. To understand children's thinking in mathematics, he suggested a focus on "the conceptual structure in the child's head." Also, in teaching mathematics, teachers "need to start out with a sense of what the ideas are, the big ideas that people have developed for thinking in a specific discipline." According to this approach to mathematics education, the goal is to teach children "how to search out situations for which the ideas (for thinking in discipline) are relevant." Prawat stated that he thinks his view of "big ideas" as the basis for mathematics teaching is very different from an approach that conceptualizes problem situations as the starting point from which representations or ideas are created. Prawat argued further that teachers need to know the "big ideas" connected to specific content domains to be truly empowered in teaching mathematics. He believes that knowledge of these "big ideas" is essential in guiding a teacher's instruction. A teacher needs to plan her instruction by thinking about the ideas or important aspects of an idea that she wants her students to consider. These ideas, then, guide the problem situations that a teacher chooses as well as the specific techniques of her pedagogy, such as her instructional questioning. Prawat questioned whether we know enough about children's learning and thinking in well-defined content domains to facilitate big idea-driven" mathematics instruction. Prawat urged that we create a learning community in which mathematics educators, teachers, researchers, and curriculum specialists jointly pursue understanding the "big ideas" connected to specific content domains.

Stoiber pointed out that perhaps one way to facilitate greater understanding in well-defined content domains might be for a teacher to analyze carefully her teaching in a specific domain. By giving thought to children's thinking in one content domain--how children make sense of a concept, the misconceptions they have, what facilitates their understanding--teachers will develop conceptual structures that they can use for developing understanding in other content domains. The role of the researcher in facilitating a teacher's in-depth analysis might be to interview regularly the teacher about her teaching. Examples of questions the researcher might ask include, "How do you know whether a student understands?" Why did you choose that problem situation?" and "How might your students begin to understand a concept?" Perhaps by coming to understand better their ideas about students' thinking and by teaching mathematics in one content domain, a teacher might be more attuned to mathematical structures and properties in other content domains.

Several group members raised the concern that much of what happens in preservice education is incompatible with the kinds of thinking processes that we desire in practicing teachers. In particular, group participants recognized that the traditional teacher education curriculum often fails to promote reasoning and problem solving in prospective teachers. One paradox in our conceptions of teachers is that we expect them to facilitate higher order, thinking in their instruction, but we fail to recognize similar expectations for teachers' own learning. The problem between expectations held traditionally for teachers and the expectations implicit in the NCTM Teaching Standards of teacher-as-problem-solver forced us to examine our own practice and involvement with teachers. This examination pushed us beyond platitudes to the ethical dilemmas inherent in altering current practices in teacher education.

Ethical Dilemmas

A theme of social consciousness became the framework which the group used to consider dilemmas. The group recognized dilemmas wide and vast in working toward a goal of improving the understanding that researchers, teachers, and students bring to mathematics in well-defined content domains. Researchers face procedural dilemmas, such as whether to take a collaborator, a participant, or an observer role when attempting to study classroom learning. Researchers confront credibility dilemmas, such as how to substantiate and document their findings. Faculty face instructional dilemmas, such as how to convince faculty colleagues who teach traditional research approaches to appreciate and to teach nontraditional research approaches.

Prawat presented one cogent example of a dilemma, that stemmed from his attempts to alter a teacher's conceptions of traditional teaching to a more understanding based approach. Prawat described this teacher as demonstrating exemplary traditional "effective teaching" practices--instructional clarity, rules and procedures, immediate feedback, rapid pacing and snappy transitions. Then the teacher participated in a summer program aimed at improving her understanding of students' thinking in mathematics and at promoting her use of information about students' thinking to teaching mathematics. Rather than empowering this teacher to teach with understanding, however, the program seemed to foster in this teacher a sense of helplessness, lack of confidence, incompetent self-perceptions, and guilt. She revealed to Prawat her dawning awareness that she had not had been teaching for understanding in mathematics during her 20 years of previous teaching experience. This new awareness made her feel as though she had done a disservice her students all those past years. She also revealed doubts about her ability to apply conceptual notions in mathematics to her teaching. She wondered whether she had the knowledge and understanding in mathematics to facilitate understanding in her students. In Prawat's example, the teacher recognized she was not teaching in a manner consistent with the NCTM Teaching Standards, but her transition from traditional teaching methods to teaching for understanding was laden with fear, trepidation, feelings of incompetence, and resistance. Hence, the ethical case of this teacher presents a dilemma for educators and researchers who strive to alter the practices of teachers in mathematics.

Lanier offered another compelling example of an ethical dilemma. He described a third-grade teacher who was attempting to use a more conceptually based approach to teaching mathematics. She reported feeling alienated from administrators, parents, and other teachers because they compared her to their more traditional teaching standards. In one case, the teacher was compared to another third-grade teacher at her school who followed religiously the instructional lessons accompanying the mathematics textbook. The other teacher had covered 17 chapters compared to her 5 chapters. The criterion for success emphasized at her school was tied primarily to quantitative dimensions of learning (e.g., the number of pages covered), rather than qualitative dimensions (e.g., the students' level of understanding). Hence, the teacher who was attempting to change reported being questioned constantly about why she had made so little progress with her students. The paradox in this example is that, although researchers might extol, and believe, that their involvement with teachers will improve the teachers' experiences as teachers, the involvement might also be frustrating.

The group discussed a final dilemma--how to collaborate with teachers who have a different vision of what is meant by teaching mathematics for understanding. Jackson expressed the concern that individual teachers will likely interpret and apply the NCTM Teaching Standards differently. For example, a teacher might believe she bases her instruction on children's natural development of mathematical understanding in a

content domain but actually organizes her instruction around her own conception of understanding and knowing. Although she might "think" she listens to her children and follows children's thinking in her teaching, she might actually use methods that are more directive than reflective. Several group members questioned what criteria are available to distinguish "those teachers who teach mathematics for understanding" and what researchers' roles should be in identifying those teachers.

These questions brought the group full circle to the issue of what roles to pursue and how to pursue them. Group discussion also returned to the need for more models and more knowledge to enhance our activities as researchers and trainers engaged in the study of teaching mathematics. In sum, the group's discussions reflected a complex and interconnected series of issues and themes. Clearly, some of the issues raised were not resolved, but served as cases for catalyzing reaction and reflection. Although the perspectives that group members brought to discussion often differed, the group was united in its goal to continue to find ways to engage in productive dialogues with teachers, and to work together in pursuing understanding of mathematics in well-defined content domains.

THOUGHTS ON AUTHENTIC COLLABORATION

Karen C. Stoiber

Two reasons that guided the discussion in our group can be distinguished. The first is that for the NCTM Teaching Standards to be implemented optimally, university researchers and trainers, along with classroom teachers and other school personnel, should be involved. The second reason, which bears on the first, is that the involvement of each professional group would be enhanced by collaborating on implementation strategies. Our recognition of researchers and teachers as viable partners in facilitating the use of NCTM Teaching Standards delineates an important priority. Unfortunately, establishing priorities is only one component in a multifaceted task. For researchers and teachers to engage in authentic collaboration they need to develop and share well-defined conceptual meanings and strategies in mathematics. What remains to be clarified, then, are the concepts and methods of collaboration. A framework for authentic collaboration is necessary to help us figure where to begin and how to proceed.

One source of direction might come from the modus operandi that typically characterize exchanges between researchers and teachers. In recent years, the influences and functions of educational researchers in school settings have been diverse, including information giver, classroom inquirer, classroom consultant, and policy decision maker (See Brown, Pryzwansky, & Schulte, 1987; Jackson, 1990; Weinberg, Fishhaut, Moore, & Plaisance, 1990). Teacher roles that correspond to these include knowledge-recipient (learner), research subject, consultee, and program implementor. Most of these researcher-teacher functions can be further categorized into different theoretical or methodological approaches. For example, several models of consultation exist in the literature (e.g., behavioral, mental health, organizational) and each model ascribe varying types and levels of expectations about the functioning of the consultant and consultee. The consultant might function as a process observer, a problem solver, a collaborator, or an expert. Similarly, researchers have assumed roles ranging from behavioral observer to study participant as classroom inquirers, depending upon their theoretical and methodological perspective.

In my view, one way in which we might begin to find an authentic method of collaboration is to critique each type and form of traditional researchers-teacher roles. A primary focus of critique should be to evaluate the capacity of typical modi operandi to provide a collaborative structure in which teachers and researchers might engage conceptions about mathematics. Obviously, some of these former modi operandi are better suited to authentic collaboration. For example, drawing again from the literature in consultation, certain models of consultation dictate different role expectations for the consultant and consultee, which vary in their capacity for collaboration. The beliefs and values of the consultant will mostly dominate when a consultant identifies with an expert orientation. In contrast, beliefs and values owned by consultant and consultee should have an equivalent, though perhaps different influence, when the consultant functions as a collaborator (Brown, Pryzwansky, & Schulte, 1987). The critique of methods, then, should evaluate factors that determine the degree and level of collaborative activity provided by traditional modi operandi. Clearly, the role assumed by the researcher and teacher is one factor that influences whether or not these professional functions as collaborators. Other factors that might also influence exchanges between researchers and teachers are the skills, knowledge, and needs they bring to the task of implementing the NCTM Teaching Standards. By understanding the factors that have catalyzed and maintained existing researcher-teacher roles--information giver/knowledge recipient, classroom inquirer/research subject, classroom consultant/consultee, policy decision maker/program implementor--we should become more aware of considerations and

circumstances that need to be addressed to propel teachers and researchers toward improved collaborative exchanges about mathematics. Engaging in authentic collaboration is quite likely a change in functioning for most researchers and teachers, which cannot occur effectively without a critique of present practices of each professional group.

A second and related approach that should improve the quality of collaborative efforts between researchers and teachers is to consider documented outcomes of present researcher-teacher roles and practices. One documented influence of university researchers and teacher trainers on practitioners is the suggestion that teachers do something other than they are currently doing (Richardson, 1990). As Richardson observed, a critical aspect of this formidable influence is that "someone outside the classroom decides what changes teachers will make" (p.11). It should, therefore, not be surprising that much of the literature on teacher change and consultation indicates another common outcome: Teachers often resist implementing or fail to adopt the recommendations made by researchers. Recently, various researchers have examined the structure of the school organization and personal attributes of teachers to explain why research-based suggestions are not implemented (See Donmeyer, 1987; Guskey, 1988; McLaughlin, 1987). In investigating the school organization, researchers have focused on various school-level features such as norms of collegiality and experimentation and instructional coordination that affect whether new programs are adopted by teachers. In contrast, other researchers have examined the affects of various individual teacher factors, such as teachers' beliefs, values, knowledge, and intentions. It seems likely that another variable, which has received little attention, concerns the university researcher. Some issues or factors related to the researcher that might affect the likelihood with which teachers accept ideas from researchers include, Does the researcher solicit and value the teachers' own conceptions of teaching practices? Is an assessment of the teachers' needs and priorities conducted to assure the content of the conservations are meaningful for the teacher? and Does the teacher perceive the researcher as holding useful knowledge or as having a useful purpose? Unless researcher factors are given consideration, we will probably fail to understand a set of variables that enter importantly into the equation of whether authentic collaboration is operative. Furthermore, for a paradigm to address factors having significance, the researcher as well as the teachers' perceptions of the researcher must be evaluated. Otherwise, the paradigm is not collaborative in design and it limits the potential for improving the outcomes of collaboration.

This leads to a final notion regarding the method and content for authentic collaboration in mathematics between researchers and teachers. During the discussion in our group, Romberg stated that one of the difficulties in understanding what is meant by well-defined content domains is that it involves a "network of situations, as well as representations, ideas, and procedures to give meaning to these problem situations. Perhaps exchanges between researchers and teachers might be improved by uncovering and confronting the meanings held by a community of educators of mathematical situations, representations, ideas, and procedures. All members of the education community--teachers, researchers, administrators, and trainers--would share thoughts focused on a well-defined content domain. Specific mathematical content domains, such as proportions, subtraction, fractions, or equivalence, would be chosen as a site for the reflective and collaborative dialogue. In discussing a particular mathematical situation, all the elements that affect its conception and meaning--representations, ideas, and procedures should be contemplated by the collaborative team. It involves evaluating our representations, justifying our ideas, and comprehending the procedures we apply to learn and do mathematics. The purpose of this activity is to develop and share the meaning that is embedded within a mathematical concept.

By soliciting and examining systematically the conceptions each member holds about well-defined mathematical content, we might also uncover and confront critical misconceptions. For example, in a recent exchange with a teacher she revealed that she used a "knowledge construction" approach to teaching subtraction. When I asked her what she meant by knowledge construction, she replied that children construct knowledge about mathematics by using manipulatives. She could give no further examples of knowledge construction in mathematics. In an exchange with a different teacher, she expressed difficulty with comprehending how the problem 50 ÷ 1/2 = 100 because in her mind 50 divided into half is 25. She stated that, though she knew how to apply the algorithm to obtain the correct answer, she worried that she would not be able to explain to her students why 50 ÷ 1/2 = 100. These examples illustrate that, unless teachers are given the opportunity to interject their conceptions, the basis on which shared conceptual understandings about mathematics are formed might be faculty. Using well-defined content domains as a context for researchers to articulate the representations, ideas, and procedures that they link to a mathematical situation should provide a framework for filtering their misconceptions as well.

The misconceptions of researchers and teachers about mathematics and the teaching of mathematics may be the barrier to authentic collaboration. However, these same misconceptions provide the most critical reason for creating opportunities wherein researchers and teachers think and act in collaboration.

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