DOCUMENT RESUME

ED 315 277 SE 051 147

AUTHOR	Sworder, Steven C.
TITLE	The Pendulum and the Calculus.
PUB DATE	Dec 89
NOTE	12p.
PUB TYPE	Reports - Descriptive (141) -- Guides - Classroom Use - Guides (For Teachers) (052)
EDRS PRICE	MF01/PC01 Plus Postage.
DESCRIPTORS	*Calculus; *College Mathematics; Higher Education; *Laboratory Experiments; Laboratory Procedures; Mathematical Applications; *Mathematical Formulas; *Mathematics Materials; *Mechanics (Physics); Physics

ABSTRACT

A pair of experiments, appropriate for the lower division fourth semester calculus or differential equations course, are presented. The second order differential equation representing the equation of motion of a simple pendulum is derived. The period of oscillation for a particular pendulum can be predicted from the solution to this equation. As a laboratory experiment, a pendulum can be constructed, the period measured, and the experimental results compared with the analytic solution. In addition, the system of two second order differential equations representing the equations of motion of a pair of spring coupled pendulums is derived; the time between stops for a particular pendulum can be predicted from the solution to this equation. As a laboratory experiment, the coupled pendulum is constructed, the period measured, and the experimental result compared to the analytic solution. (YP)

The Pendulum and the Calculus

by

Steven C. Sworder
Mathematics Department
Saddleback College
Mission Viejo, California
92692

Abstract

Presented here are a pair of experiments that are appropriate for the lower division fourth semester calculus or differential equations course. The second order differential equation representing the equation of motion of a simple pendulum is derived. The period of oscillation for a particular pendulum can be predicted from the solution to this equation. As a laboratory experiment, a pendulum can be constructed, the period measured, and the experimental results compared with the analytic solution.

The system of two second order differential equations representing the equations of motion of a pair of spring coupled pendulums is derived. The time between stops for a particular pendulum can be predicted from the solution to this equation. As a laboratory experiment, the coupled pendulum is constructed, the period measured and the experimental result compared with the analytic solution.

Simple Pendulum

A simple pendulum is a well known device and an easy one to construct. Simply tie one end of a string to a weight and the other end to an elevated point from which the weight can swing freely. For small amplitudes of oscillation (i.e. less than thirty degrees or so the equation of motion of the pendulum can be approximated quite nicely by an ordinary homogeneous linear second order differential equation with constant coefficients. This equation is readily solvable by methods developed by Leonhard Euler (April 15, 1707 to September 18, 1783) and presented in the lower division fourth semester calculus or differential equations course. From this solution, the period of oscillation of the pendulum can be predicted. This prediction may be easily checked by constructing the pendulum and measuring the period.

Derivation of the Equation of Motion

A simple pendulum is illustrated in figure 1. The force of gravity acting on the pendulum mass resolved into components along the pendulum string and along the x-axis is shown if figure 2. The force of gravity, F_g, acting on the mass is $-mg$ (where g is the gravitational constant). The force along the pendulum, $F_{||}$, is $F_g\cos\theta$ and so $F_{||} = -mg\cos\theta$. The force perpendicular to the pendulum, $F_{\perp}$, is $F_g\sin\theta$.

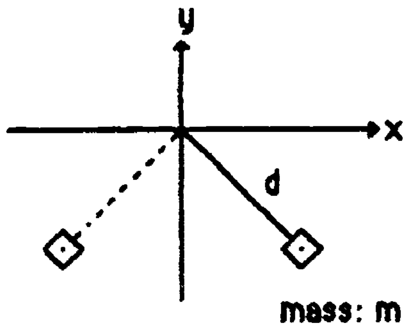

Figure 1
Simple Pendulum

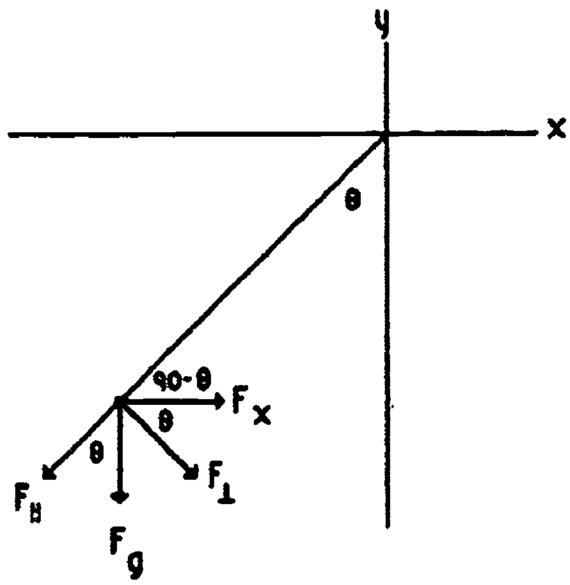

Figure 2
Resolution of Force of Gravity
on the Simple Pendulum

The force along the horizontal axis, F_x, is $F_{\perp}\cos\theta$ and so

$$F_x = -mg\sin\theta\cos\theta = -\frac{1}{2}mg\sin(2\theta).$$

From Newton's Laws, looking at the motion of the x coordinate of the center of gravity of the mass,

$$m\ddot{x} = \sum F_x \quad = -\frac{1}{2}mg\sin(2\theta)$$

$$(1) \quad m\ddot{x} + \frac{1}{2}mg\sin(2\theta) = 0.$$

For sufficiently small angles, θ: $\sin(2\theta) \approx 2\theta$ and $x \approx d\theta$. Substituting these quantities into equation 1 gives the following

$$m\ddot{x} + \frac{1}{2}mg(2\theta) = 0.$$

$$m\ddot{x} + mg\left(\frac{x}{d}\right) = 0.$$

The approximate equation of motion of the x coordinate of the center of gravity of the pendulum mass is given in equation 2.

$$(2) \quad \ddot{x} + \frac{g}{d}x = 0.$$

Let $\omega_o^2 = g/d$. If g is measured in feet per second squared and d is measured in feet, then ω_o^2 is expressed in units of per second squared. Thus, ω_o has units of per second or equivalently radians per second. With this substitution, equation 2 becomes

$$(3) \quad \ddot{x} + \omega_o^2 x = 0.$$

The pendulum can be set into motion by pulling the mass center of gravity to an initial x coordinate of A feet, while keeping the string. If the mass is then allowed to fall with zero initial velocity, the initial conditions paired with equation (1) are $x(0) = A$ feet and $\dot{x}(0) = 0$ feet per second. Solving equation 3 yields $x(t) = A\cos\omega_o t$. The x coordinate of the center of gravity of the mass of the pendulum exhibits simple harmonic motion with amplitude A, frequency ω_o radians per second or $\omega_o/2\pi$ Hertz, and period $2\pi/\omega_o$ seconds.

Laboratory Exercise

The student is given equation 3 along with the derivation and asked to solve the differential equation. The student should determine the period of oscillation expected for the pendulum based on this analytical solution. This result is compared with the value determined from an actual simple pendulum that the student constructs from provided materials. All that is needed is a tape measure, a fishing sinker, a ball of string, and a watch with a seconds display.

The mass of the sinker in slugs (weight in pounds divided by 32.2 feet per second squared) can be determined by the student if desired but will require that a scale be made available. The student selects a desired length for the pendulum, cuts an appropriate length of string, hangs the fishing sinker, sets the pendulum in motion and measures the period of oscillation. The theoretical period for the given physical parameters may be calculated based on their solution to equation 3 and compared with the observed period. The origin of any discrepancies could be discussed as part of the laboratory write-up.

COUPLED PENDULUM

As an extension of the work with the simple pendulum in the previous section consider two identical simple pendulums connected together by a spring as illustrated in figure 3. For small amplitudes of oscillations the equations of motion of this system can be approximated by a pair of coupled homogeneous linear ordinary second order differential equations with constant coefficients. These equations are readily solvable using methods presented in the lower division fourth semester calculus or differential equations course. As is shown below, this system exhibits the phenomenon of beats. From the analytical solution the beat period can be determined and checked against observations made on an actual physical system.

Equations of Motion

Consider a pair of identical simple pendulums coupled together by a spring whose unstretched length is U and whose spring constant is k. This is illustrated in figure 3.

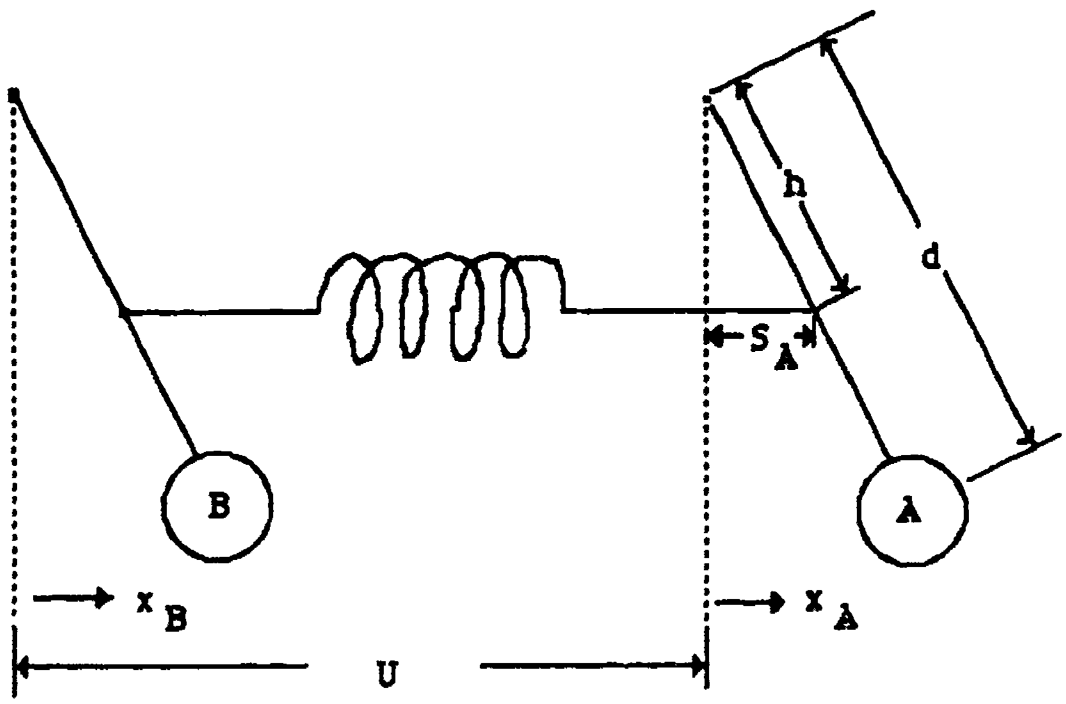

Figure 3
Coupled Pendulum Geometry

When mass A is at x_A the right end of the spring is at S_A and

$$\frac{S_A}{x_A} = \frac{h}{d} \Rightarrow S_A = \frac{h}{d}x_A.$$

When mass B is at x_B the left end of the spring is at S_B and

$$\frac{S_B}{x_B} = \frac{h}{d} \Rightarrow S_B = \frac{h}{d}x_B.$$

Thus when mass A is at x_A and mass B is at x_B the stretch of the spring is

$$S_A - S_B = \frac{h}{d}(x_A - x_B)$$

and the restoring force is

$$k(S_A - S_B) = \frac{kh}{d}(x_A - x_B)$$

Combining this coupled restoring force with the forces on the simple pendulum we have the following equations of motion:

$$m\ddot{x}_A + \frac{mg}{d}x_A + \frac{kh}{d}(x_A - x_B) = 0$$
$$m\ddot{x}_B + \frac{mg}{d}x_B - \frac{kh}{d}(x_A - x_B) = 0$$

With the variable definitions $\omega_o^2 = g/d$ and $\omega_c^2 = kh/dm$, the equations of motion become:

$$\ddot{x}_A + \omega_o^2 x_A + \omega_c^2(x_A - x_B) = 0$$
$$\ddot{x}_B + \omega_o^2 x_B - \omega_c^2(x_A - x_B) = 0$$

or equivalently,

$$\ddot{x}_A + (\omega_o^2+\omega_c^2)x_A - \omega_c^2 x_B = 0$$
$$\ddot{x}_B + (\omega_o^2+\omega_c^2)x_B - \omega_c^2 x_A = 0$$

This system of two second order differential equations can be written as a system of first order differential equations by making use of the following substitutions:

$$x_A = x_1$$
$$\dot{x}_A = x_2$$
$$x_B = x_3$$
$$\dot{x}_B = x_4$$

The initial conditions for the experiment described here become:

$$x_A(0) = x_1(0) = A$$
$$\dot{x}_A(0) = x_2(0) = 0$$
$$x_B(0) = x_3(0) = 0$$
$$\dot{x}_B(0) = x_4(0) = 0$$

The equations written in terms of the new variables are:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(\omega_o^2+\omega_c^2)x_1 + \omega_c^2 x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -(\omega_o^2+\omega_c^2)x_3 + \omega_c^2 x_1$$

Written in matrix form the system of first order differential equations is as follows:

(4) $\dot{x} = Px$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(\omega_o^2+\omega_c^2) & 0 & \omega_c^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_c^2 & 0 & -(\omega_o^2+\omega_c^2) & 0 \end{bmatrix}$

with initial conditions $x(0) = \begin{bmatrix} A \\ 0 \\ 0 \\ 0 \end{bmatrix}$

This differential equation can be solved using techniques learned in the lower division differential equations course and emplcying either Laplace transformations or fundamental matrices. When this work is completed it is found that

$$x_1(t) = \frac{A}{2}\left(\cos\sqrt{(\omega_o^2+2\omega_c^2)}\ t + \cos\omega_o t\right).$$

Using the trigonometric identity:

$$\cos A + \cos B = \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

the solution becomes

$$x_1(t) = x_A(t) = A\cos\left(\frac{\sqrt{\omega_o^2+2\omega_c^2} - \omega_o}{2}\right)t\ \cos\left(\frac{\sqrt{\omega_o^2+2\omega_c^2} + \omega_o}{2}\right)t.$$

Similarly it can be shown that

$$x_3(t) = \frac{A}{2}\left(\cos\omega_o t - \cos\sqrt{(\omega_o^2+2\omega_c^2)}\ t\right)$$

and consequently,

$$x_3(t)=x_B(t)=A\ \sin\left(\frac{\sqrt{\omega_o^2+2\omega_c^2}-\omega_o}{2}\right)t\ \sin\left(\frac{\sqrt{\omega_o^2+2\omega_c^2}+\omega_o}{2}\right)t.$$

If the following substitutions are defined

$$\lambda = \frac{\sqrt{\omega_o^2+2\omega_c^2}-\omega_o}{2} \text{ and } \gamma = \frac{\sqrt{\omega_o^2+2\omega_c^2}+\omega_o}{2}$$

it is clear that $\lambda < \gamma$ and $x_B(t) = A\sin\lambda t\sin\gamma t = M(t)\sin\gamma t$. The mass B exhibits sinusoidal oscillation with frequency γ and time varying amplitude M(t) that is sinusoidal of frequency λ. A sketch of this type of motion is given in figure 4.

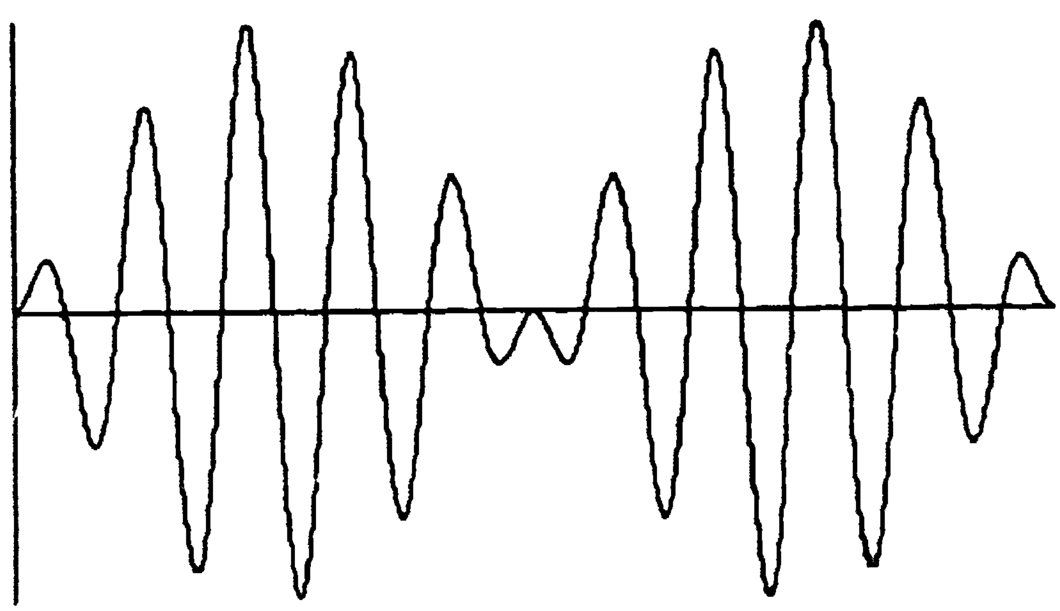

Figure 4

Graph of $A\sin\lambda t\sin\gamma t$

This implies that the mass B will appear to stop moving when M(t) = 0 (i.e. amplitude of oscillation is zero) This is referred to as a "stop." The time between stops is one-half the period of $M(t)=A\sin\lambda t$. Thus the frequency of stops of mass B is

$$2\lambda = \sqrt{\omega_o^2+2\omega_c^2} - \omega_o \text{ radians per second}$$

$$= \frac{\sqrt{\omega_o^2+2\omega_c^2} - \omega_o}{2\pi} \text{ Hertz.}$$

The time between stops, T, is

$$T = \frac{2\pi}{\sqrt{\omega_o^2+2\omega_c^2} - \omega_o}$$

Laboratory Exercise

The student is given equation 4 along with the derivation and asked to solve the differential equation. The student should determine the period of the stops expected for one of the pendulums based on this analytical solution. This result is compared with the value determined from an actual coupled pendulum that the student constructs from provided materials. All that is needed is a tape measure, two fishing sinkers, a ball of string, a weak spring borrowed from the Physics Department, and a watch with a seconds display. The mass of the sinker in slugs (weight in pounds divided by 32.2 feet per second squared) can be determined by the student if desired but will require that a scale be made available. The spring constant can be determined by the student if desired but will require that a small weight be also available. Alternatively, the students may be provided with the measure of the masses and the spring constant. The student selects a desired length for the pendulums, cuts the appropriate lengths of string, hangs the fishing sinkers, sets the pendulums in motion, and measures the time between stops for one pendulum. The theoretical period for the given physical parameters may be calculated by the students based on their solution to equation 4 and compared with the observed period. The origin of any discrepancies could be discussed as part of the laboratory write-up.