DOCUMENT RESUME

ED 303 369 SE 050 359

| | |
|---|---|
| AUTHOR | Goldenberg, E. Paul; Kliman, Marlene |
| TITLE | Metaphors for Understanding Graphs: What You See Is What You See. |
| INSTITUTION | Educational Technology Center, Cambridge, MA. |
| SPONS AGENCY | Office of Educational Research and Improvement (ED), Washington, DC. |
| REPORT NO | ETC-TR-88-22 |
| PUB DATE | 88 |
| CONTRACT | OERI-400-83-0041 |
| NOTE | 30p.; Drawings may not reproduce well. |
| PUB TYPE | Reports - Research/Technical (143) |
| EDRS PRICE | MF01/PC02 Plus Postage. |
| DESCRIPTORS | Computer Assisted Instruction; *Computer Graphics; Computer Oriented Programs; *Graphs; *Mathematical Concepts; Mathematical Enrichment; *Mathematics Instruction; Mathematics Materials; *Metaphors; Scaling; Secondary Education; *Secondary School Mathematics |

ABSTRACT

Computer graphing makes it easier for students and teachers to create and manipulate graphs. Scale issues are nearly unavoidable in the computer context. In interviews and protocol analysis with six students from grade 8, and 12 students from grades 11 and 12, it became apparent that some aspects of scale are clearly understood very early while other aspects remain confusing to even some of the most successful students in pre-calculus and calculus, and that there is a consistency and meaning in metaphors which students invoked in explaining their ideas to themselves and teachers. Three metaphors inferred from students' words and one metaphor supplied by the authors are discussed. These are: (1) the computer as automatic paper and pencil; (2) scaling is like using a magnifying glass; (3) scaling as a rubber sheet (supplied by the authors); and (4) the mathematical curve as a bead necklace. Some implications concerning the curriculum are discussed. (YP)

ED303369

TR88-22

# METAPHORS FOR UNDERSTANDING GRAPHS: WHAT YOU SEE IS WHAT YOU SEE*



Educational Technology Center
Harvard Graduate School of Education
337 Gutman Library Appian Way Cambridge MA02138

SE 050 359

Metaphors for Understanding Graphs:
what you see is what you see*

Prepared by:
E. Paul Goldenberg and Marlene Kliman
Education Development Center, Inc.

The work discussed in this paper was conducted at the Center for Learning Technology, Education Development Center, Inc., under a subcontract from the Educational Technology Center at the Harvard Graduate School of Education. Preparation of this report was supported in part by the U.S. Office of Educational Research and Improvement (contract # OERI 400-83-0041). Opinions expressed herein are not necessarily shared by the OERI and do not represent Office policy.

# Metaphors for Understanding Graphs:

## what you see is what you see*

E. Paul Goldenberg and Marlene Kliman

Education Development Center, Inc.
55 Chapel Street
Newton, MA 02160

Graphing has traditionally been a paper-and-pencil activity and, ' ecause of the time and effort involved, has not constituted a large part of mathematics curricula. This is changing as greater classroom access to computer graphing makes it easier for stud ts and teachers to create and manipulate graphs. The growing interest in the curricular use of graphing raises a need for insight into students' approaches to the graphing process.[1]

Insights are necessarily conditioned by the context in which they are developed. Therefore, if we wish to understand *computer* graphing, it is essential that students' approaches to graphing be studied, at least in part, in a *computer* context, which brings its own particular set of issues to the fore. For example, issues of *scale*—of paramount importance in understanding both graphs and the concept of *significance* as it is used in mathematics—are typically not noticed and may even be deliberately avoided in paper-and-pencil work.[2] By contrast, scale issues are nearly unavoidable in the computer context.

* The research reported here was conducted by the Center for Learning Technology, Education Development Center, Inc. (EDC) under a subcontract from the Educational Technology Center, Harvard Graduate School of Education, supported by the United States Office of Educational Research and Improvement (Contract # OERI 400-83-0041). The original draft of this report was prepared under the same contract. Revision for publication in the *Journal of Mathematical Behavior*, 1989, was supported in part by EDC. Opinions expressed herein are not necessarily shared by OERI and do not represent Office policy.

1. When we first began our work (Goldenberg, 1988), neither research nor teaching literatures had dealt much with this issue. The research literature was almost nonexistent (but see Goldenberg, 1988, for a list of what we had found at that time), and the teaching literature, also fairly small, focused almost exclusively on ways to teach graphing skills in a paper-and-pencil context.

2. Because of scale's nearly total absence from most curricula, its mathematical importance is generally overlooked. The value of scale is discussed in Goldenberg (1988), and summarized here for convenience:

a. Attention to scale is essential in reading any graph accurately.

b. Scale is at the root of understanding significance in its various mathematical senses: statistical significance, significant figures, etc. It also has a special importance in graphs, in that the usefulness of a graphic (as opposed to tabular or formulaic) representation of a mapping resides primarily in the gestalt that it shows. Scale affects which aspects of that gestalt are emphasized and de-emphasized, strongly influencing the impression one has of the relationship between the variables.

Scale and its interaction with function in generating graphs captured our interest quite early and seemed particularly worthy of study. But scale issues are subtle. In working with students from 8th to 12th grades, it became apparent that some aspects of scale are clearly understood very early while other aspects remain confusing to even some of the most sophisticated and mathematically successful students in pre-calculus and calculus.

Scale is also difficult to isolate from the mathematical context in which it is encountered. What emerged from our study was nothing like a sequence in the development of scale ideas or a taxonomy of scale issues. Rather, we began to see the most consistency and meaning in what we shall call the "metaphors" students invoked in explaining to themselves and to us their ideas about the scale-related graphing problems we posed.

We discuss four such metaphors here, three inferred from our students' words and actions and one supplied by us.

- The paper-and-pencil metaphor is, at one level, a denial of dynamic scale change. As paper may be cut, graphs may be cropped, but stretching and shrinking are not natural operations if one thinks of graphs only as they exist on paper.
- The magnifying-glass metaphor is a concretization of dynamic scale change. It treats mathematical objects under magnification as if they were physical objects, and allows them, therefore, to appear rougher or grainier when sufficiently enlarged.
- The rubber-sheet metaphor, supplied by us, is also a concretization of dynamic scale change. It substitutes concepts of stretch and distortion for the magnification, resolution, and closeness that are fundamental to the magnifying-glass metaphor.
- The bead necklace metaphor allows ideas of scale to be applied to points, which are unscalable. Students seem generally to regard points as *extremely small* rather than dimensionless; the attribution of any size whatsoever to points allows them to be conceptually magnified, lined up in a row, and so on.

These metaphors are of central importance because they served to focus, aid, direct, and misdirect our students' explorations of graphs, and to inform and limit their understanding of the mathematics behind those graphs. An exposition of these metaphors therefore became the major product of our study.[3]

Any discussion of student metaphors makes it hard not to sound as if our study were about "how students think about graphs." But claims made with integrity about "how students think" tacitly promise to be about "the typical student," or "a great many students,"

c. In order for students to generalize appropriately from experiences with graphs, they must learn to see essential features and ignore inessential ones. Our early work (Goldenberg, 1988) suggested that student manipulation of scale as well as manipulation of function is critical to this process.

3. The metaphors we speak of are similar in several respects to the components of intuitive knowledge about physics that diSessa (e.g., diSessa, 1983) postulates. diSessa's phenomenological primitives, or p-prims, function as explanatory constructs that guide approaches to and explanations of physical phenomena. Like the metaphors we have just enumerated and describe further below, p-prims originate with our interpretations of our interactions with the physical world.

or at least "a reasonable diversity of students" and are, therefore, essentially statistical in nature. By contrast, our study focussed on a small number of selected students and endeavored to learn how they mobilized the knowledge and strategies that they built out of their everyday real-world experiences, and how they applied these strategies in the abstract visual world of graphed functions. Our goal was therefore not to make claims about the typical or even atypical student's approach to the problems, but to understand what complexities lurk within the mathematical problems we pose.

## *METHODS*

In all interviews that we conducted—twelve with juniors and seniors taking mathematics at least at the pre-calculus level, and six with eighth graders taking first year algebra—we chose to work with bright and articulate students. Our purpose in selecting the most successful students was to eliminate from our study, insofar as such a thing is possible, results that could be attributed primarily to a particular student's gross lack of interest or broad mathematical incompetence. By interviewing students who had, for the most part, done well in mathematics, we could feel relatively assured that the confusions *they* evidenced were probably widespread, and reflected areas of genuine difficulty.

Of the twelve high-school students, only four had previously used computers much in any context except for (limited) word-processing. The four who had had significant contact with computers were programmers with varying degrees of experience, and three of those had spent some of their programming time teaching computers to graph functions. This latter experience seems to have been influential in their responses, as we shall explain later in this paper. No other experiences with graphing programs of any type were reported by any of the students or their teachers.

Five of the interviews were videotaped, and provide the central data set in this study. The pre-calculus students in these interviews created functions on the computer using an experimental prototype of a program called *The Function Analyzer*.[4] They observed the graphs, manipulated them in various ways (e.g., by changing the function or by changing the scale of the graph), and explained what they saw and did. Two video cameras, one trained on the screen and the other on the participants, recorded each interview.

Transcriptions of the videotaped interviews were heavily annotated with visual information about context (such as the state of the computer screen), writing or drawing

[4] *The Function Analyzer*, one component in *EDC's Algebra Series*, has since been published by Sunburst Communications, Inc. It allows one to explore relationships between symbolic expressions and their graphs by manipulating either representation and tracking the effect on the other. Students may freely mark reference points on the graph and may also create, edit, and study tables of values associated with their marked points, graphs, or symbolic expressions. Scale change, the issue we chose to explore in this research, can be accomplished either through a non-numeric stretch-shrink operation, or through explicit numeric setting of coordinate boundaries. For the purposes of this research, we felt the latter command structure would give us a richer picture of students' strategies, although, as we shall speculate later, it conditioned, in part, the kinds of responses we would be likely to see.

that the student or interviewer might have done, typing, pointing or other communicative gestures, and other clarifications or supplements to the spoken parts of the dialogue.

The other thirteen interviews—five exploratory interviews with juniors and seniors on computer, two similar interviews off computer, and six interviews with 8th grade students using work-sheet material on various graphing issues—served as preparation, comparison, background, and follow-up data. Audiotapes of eight of these were transcribed.

Students in all eighteen interviews had plain or horizontal-lined paper on which to perform computations or sketch graphs. No graph paper was used. In the off-computer interviews, we tried whenever possible to follow the same general format that we followed on computer. We asked the same kinds of questions and encouraged students to express and explain their reasoning.

Our strategies in these interviews and our interpretations of the new data are both strongly influenced by data we collected in our first study (Goldenberg, 1988), which included other computer-based interviews, paper-and-pencil tests with larger groups of students, formal experiments with small groups of students, and interviews with teachers.

From this various research, we selected the metaphor data as the most significant and informative for future curriculum and materials design.

## Analysis

The gestures and casual language our students used as they worked on the scale-related problems we presented became increasingly important in our analyses, as important as the mathematical manipulations that the students performed.

While we are reluctant to label our insights too broadly by claiming they represent students' thinking, we do find the language and actions of our students very revealing about the complexity of the problems we have posed, the nature of the graphing process, and the kinds of knowledge that bear on graphing—a complex of characteristics we call the "problem space."

As mentioned earlier, we also allowed ourselves to postulate metaphors—were it not so bulky, we might prefer to call these "experience-based conceptions of graphs"—from the language and actions our students use. One such metaphor—interpreting a graph as if it were a partial view of a scene one might observe while looking out of one's office window—was reported in Goldenberg (1988). Interviews with high school students using graphing software lead us now to add to the office window metaphor three others—the paper-and-pencil, magnifying glass, and bead necklace metaphors mentioned above—and to use them as a basis for considering the problem space and implications for the teaching of mathematics.

The metaphors are, of course, constructs of our own; we do not ascribe them to our students *thinking*. Nevertheless, these metaphors are such close derivatives of our students' *communicative acts*—words, gestures, drawings, and computer-aided manipulations—that it feels entirely sound to treat them as the tacit organizing ideas behind our students' attempts, using the means they had available, to describe their thinking. The difference is subtle, but important to our purpose. It is the communicative acts and our perception of the organizing theme behind them, and not the thinking itself, that we claim to know. But because these acts represent *our students'* attempts to communicate, we also assume that they choose a means they feel is likely to have shared meaning with an interviewer who is previously unknown to them. As a result, we believe that the metaphors are not largely idiosyncratic and would communicate with a far broader population of students than our bright and articulate sample represents.

## Interview approach and interpretation

We deliberately chose problems that we felt were likely to elicit conflict and discrepancies in students' approaches to graphing. The nature of the conflict itself was of interest to us, and the students' strategies for resolving the conflict helped to tell us about the relative strength and resiliency of their approaches. Most of our problems were designed not to pinpoint a single issue we wished to study while excluding others, but rather to embed the issue we wished to study in what we felt was a reasonably natural context. This added complexity contributed, in part, to the richness and variety in the protocols, though it also made the protocols more difficult to interpret.

Having students confront and reason about conflict is both a research tool and a pedagogical philosophy. Conflict evokes strong reactions: it makes the interview more interesting and elicits more vivid language and gesture. As long as the students do not feel threatened in the interview situation, the puzzle also builds investment in the interview and pulls quite naturally for an explanation of both the similarities and discrepancies between students' predictions and observations.

Typically, we began by inviting students to describe the graph of "a quadratic function." They all said without hesitation that it would be a parabola. Presented with the task of describing the graph of a specific one like $x^2+7x+1$ without using the computer, the students varied; some worked analytically while others generated points. Still, all expected that the graph would be a parabola. We then showed them the graph in figure 1, asserted that it was an accurate graph[5] of $x^2+7x+1$, and asked them to explain it.

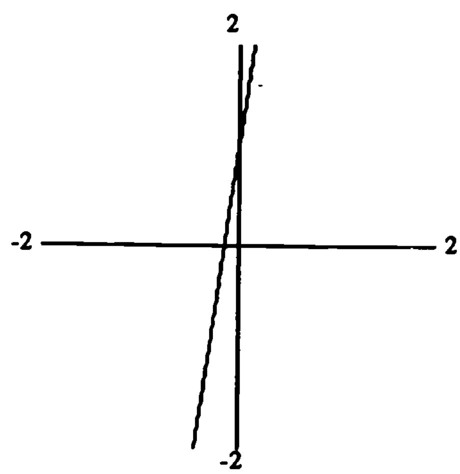



Figure 1

After students discovered (always with help, in our sample) that scale changes could make a parabola look like a line, they were asked to explore that idea in various ways. Could a graph that they identified as a parabola be made to look like a straight horizontal

5 What constitutes an "accurate graph" is a very interesting issue raised by our research but beyond the scope of this paper. It will be touched upon briefly, however, later in this paper. See, in particular, footnote 7

line? A vertical line? Both diagonals? Despite having once seen the parabola rendered as a straight line, many students—*good* students—would continue to deny that it would be possible to show it as a different straight line, or would say things like "maybe but I don't think so."

The following extended excerpt from an interview illustrates such a confrontation with conflict and something of our questioning and interpretive style.

Intrvwr I'm going to put in the parabola $x^2+7x+1$. And I wonder if you would sketch [on paper] more or less what you think that will be.

Syl[6] (laughs) OK. ... OK. OK. (after about 15 seconds) Do you want me to draw like spaces and stuff? Like...

Int However it would make sense....

After some thinking and hesitation, she drew the graph shown in figure 2—no numbers, but two axes, seven ticks along $x$ and two along $y$, a big dot at the vertex, and the curve swinging upward. Though incorrect in many respects, her graph showed that she had a clear idea of what a parabola should look like. She explained that she did not evaluate the function expression to generate points, but rather tried to analyze the equation to determine what transformations from "a regular parabola" she would need to show in her graph.

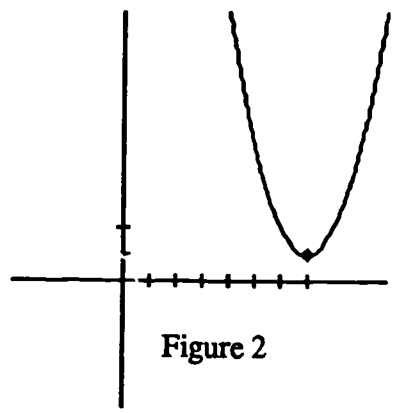


Figure 2

Although she was generally wrong about how those transformations could be derived from the symbolic expression, her concept of the transformations that might be performed was clear and correct. She was certain the parabola would not go through the origin, but she used the coefficients 7 and 1 as coordinates of the vertex. She knew that "there'd be a negative sign [somewhere in the expression]" if the parabola opened downward, but didn't indicate that she knew that it mattered where that negative sign was. About the width of her parabola she remarked:

Syl I didn't change it, but I made it like a regular parabola.

Int ...What would *that* be?

Syl At the origin and then at points 1 and 1 *[at point (1,1)]*, it'd be $y=x^2$

6 Students' names have been changed.

She admitted uncertainty about the placement, but she was quite certain that the graph should "look like a parabola." Having established this certainty, the interviewer showed her the graph in figure 3, raising the conflict toward which the rest of her thinking and analysis would be aimed.

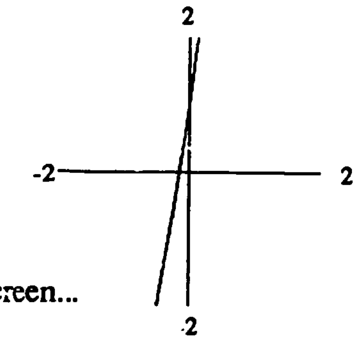


Figure 3

Int ...OK. I have a graph of it up on the screen...

Syl That's it?! *(laughing)*

Int That's it.

Syl That's not a parabola!

Int It doesn't look like a parabola at all! Um ... But how does this differ from what you're... first of all, how is it *similar* in any ways to what you were expecting?

Syl Um, no. *(laughs)* Where's the origin? You're saying this is the middle, like a regular graph?

Note that her notion of a "regular graph" includes symmetry around the origin. After receiving confirmation that the origin is in the middle, she answers how the computer graph is similar to what she had predicted.

Syl ...The line, OK. Um ... Well, it's not on the origin, which I didn't think it would be. I don't know, I've never seen anything like that...

The interviewer then suggests she draw a box on her paper-and-pencil graph (figure 2) that contains the same space as is indicated on the computer screen (figure 3), a square space centered around the origin and measuring 2 units out from the origin.

Syl *(Immediately.)* Well, then my graph won't be in it at all...

The interviewer confirms her understanding—the box won't even touch the graph she had sketched (figure 2)—but explains that the computer's graph does pass through the box. The interviewer then sketches in the segment that the computer shows and asks the student to sketch how she thinks the graph might continue beyond the confines of the boxed area.

Int OK. ... What might that look like if you looked beyond the edges of this box?

Syl The rest of the graph?

Although the interview does not capture it, we infer from the rest of Syl's knowledge that she is well aware that "the rest of the graph" is infinite, and cannot be drawn. Rather, her question reveals her sense that something about *this* graph is *peculiarly* incomplete.[7]

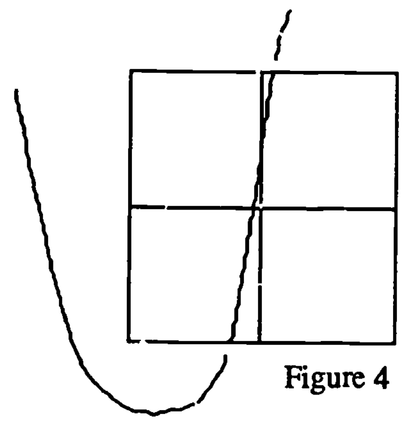


Figure 4

At this point in the interviews, some of our students were trapped by the perception of the graph as a straight line, and drew extensions to it as straight lines also. But when Syl sketched her idea of the behavior of the curve outside of the boxed region, she made it clearly parabolic (figure 4).

Syl You mean like this? Something like that?

...

Int Again, you went back to this idea of parabola, why are you holding on to it?

So firmly is she convinced that this must be a parabola that she assumes the interviewer's question is directed not at why she has chosen a parabola at all, but rather at why she has chosen the upward rather than downward orientation.

Syl I don't know, that's 'cause, I don't know, I guess in class you always do them like that, I mean, for the most part. *She gestures a U-shape.* Exceptions are going another way. *Gestures an upside down U.*

Int Oh, I see, you mean the orientation.

Syl Yeah.

Int But you really want it to be a parabola despite this straight line.

Syl Yeah.

---

7 This interesting issue—What constitutes *enough* of a graph?—is beyond the scope of this paper, but it, too, suggests the significance of scale. Even when one is fully aware that no graph can show the entire parabola, one almost automatically regards this graph as showing only "part of" the parabola, and is thus incomplete by comparison with a conventional presentation. Critical features of the function, and even canonical form (in conflict with "symmetry around the origin"), seem to be part of the casual definition of "a graph of the function" although special purposes may lead one to choose other graphic presentations—"portions of the graph."

Int Why?

Syl Because usually a parabola, that's what we learn, is that one side of it's squared.

This extended excerpt illustrates several features of our interview and analytic style. Principally, it shows how we try to construct situations that ask the students to resolve conflicting ideas. For example, we did not present our oddly scaled graph until *after* Syl had declared what a quadratic should look like and identified those characteristics that might distinguish one quadratic from another.

Also, the interviewer allowed himself to be quite active during the interview, and to teach as well as to question. For example, when he asked Syl to draw a box on her paper-and-pencil graph that contained the same space as was indicated on the computer screen, he was consciously suggesting a strategy for interpreting the screen graph. Although such interventions would require us to qualify our interpretations, they served us in other ways. In this case, we were able to see that Syl took no time at all to find the scale information on the screen, and were therefore confident to infer that it had simply not been salient to her earlier—it was not something she was in the habit of attending to. We also felt that appropriately timed teaching was part of our responsibility to these cooperative and interested students.

## *MOBILIZING KNOWLEDGE FROM REAL-WORLD EXPERIENCES: THREE METAPHORS*

Over the course of the interviews, students seemed to invoke metaphors as "tools" for making predictions about, explaining, and creating graphs. Metaphors did not appear as internally consistent, carefully thought out, and well articulated models of graphs, but rather as a relatively fluid and freely shifting set of analogies between graphic situations and real-world experiences that seemed to share some features. A single student might shift from one metaphor to another in the course of an interview, using whichever best suited the specific graphing issue of the moment.

In this section, we describe three of the metaphors we derived from the interviews. To give a sense of the ways in which we inferred these metaphors from students' words and actions, we begin each description with an illustrative anecdote. With many examples of each metaphor to choose among, we have deliberately chosen those that suggest the complexity of interaction between the metaphor to be illustrated and other aspects of the student's thinking.

As in this kind of work it is easiest to see metaphors when they lead the student astray, we therefore demonstrate the existence of the metaphor when it leads to a mathematically less than optimal approach or answer. We do not contend, however, that having or using metaphors is itself a problem. In our conclusion we in fact discuss issues of enriching student metaphors as a way of augmenting their mathematical ability and suggest further research in the area of metaphor enrichment.

## Metaphor 1: Computer as automatic paper and pencil

Computer graphs are treated like paper and pencil graphs that can be cropped, but not stretched or shrunk.

Later in the same interview excerpted above, Syl changed the scale of her on-screen graph to show the parabolic shape she expected (figure 5). The interviewer asked her if some region of it could be displayed in a way that makes it look like a horizontal line.

She picked the vertex region on which to experiment, but despite having just seen and undone a similar scale manipulation in which this parabola appeared as a slanted line (see above, figure 3), she expressed doubt that it could be made to look horizontal.

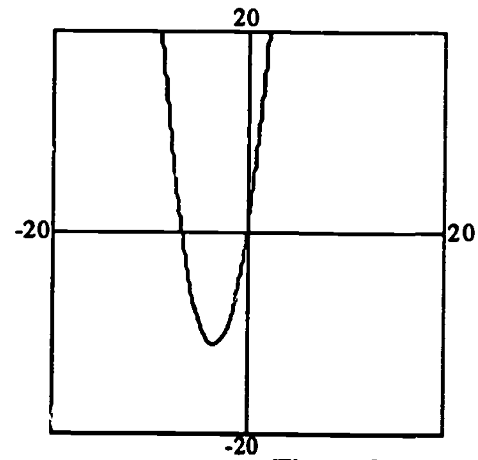


Figure 5

> Syl I think it's going to be impossible, just because a parabola doesn't really have [a horizontal line at the vertex]; it just has a point. But I'll try it.

Visually, she estimated correctly that the vertex would be found within $-7 \leq x \leq -2$, and (incorrectly) within $-5 \leq y \leq -4$. When she indicated to the *Function Analyzer* that she wished to change the coordinate values, a small copy of the graphing window appeared below the screen on which her graph was drawn.

Figure 6 shows the screen after she has reset the *x* coordinates, and as she is about to set the upper extent of the *y* coordinate (highlighted). The *y* axis does not appear on the scaling window because the subdomain $-7 \leq x \leq -2$ does not contain it. After making all of the changes she wishes to make, she presses RETURN to graph her function at that scale.

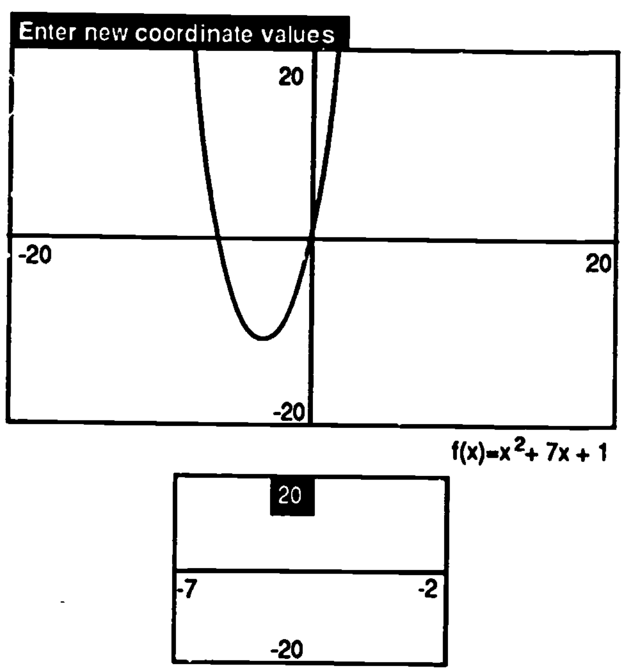


Figure 6

After some experimentation, she lowered the bottom extent of the *y* coordinate to -12, producing the graph in figure 7. From this graph, she further refined her estimate of the *x* extents, producing the graph in figure 8. Notice that this represents her choice to *narrow* the graphed domain to exclude the left-side hill, without changing the *y* scale, even though both scales may be changed at the same time and in the same way.

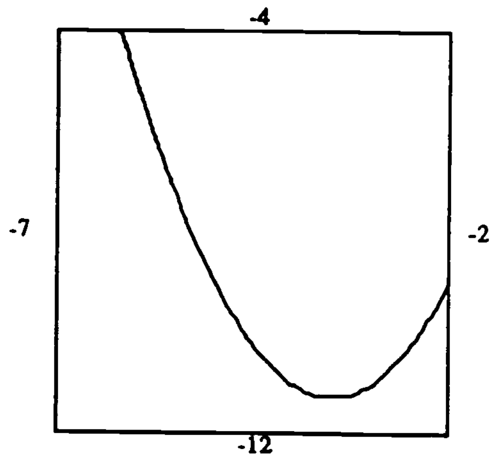


Figure 7

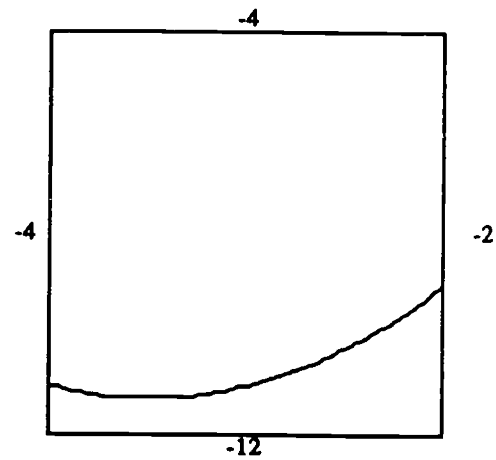


Figure 8

She then set about excluding the right-side hill, and at the same time tried to center the graph vertically (figure 9). Note that in the latter move, she *added* to the bottom exactly what she removed from the top, performing a translation and not a dilation as her previous moves had been. The window on the graph was moved downward without reducing its scope at all. It still measured 8 units from top to bottom. Her next step was to eliminate curves on both sides by moving both sides toward the center, again without any change in vertical dimension (figure 10). Her line is now nearly perfect: a horizontal line with a single jog in it.

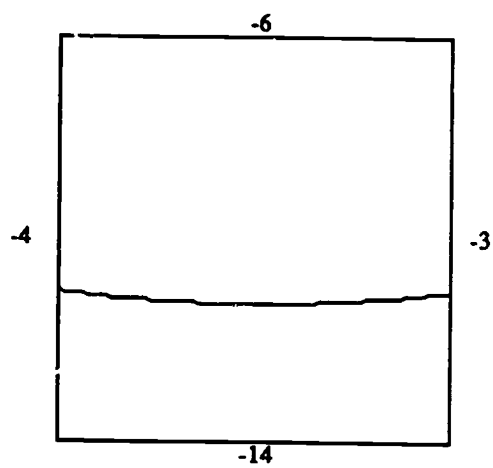


Figure 9

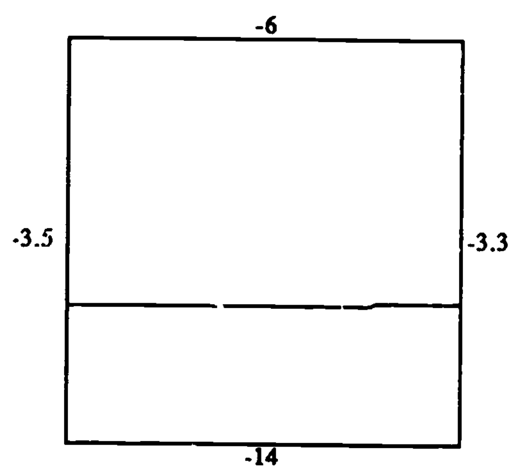


Figure 10

These last two experiments can be graphically summarized as shown in figure 11. Superimposed on the graph shown in figure 7 are a dotted box, representing the region graphed in figure 9, and a narrower solid box representing the region graphed in figure 10. The fact that she successively narrowed her domain while maintaining the same "height" as the original graph suggests that she was attending to domain only, and not to scale.

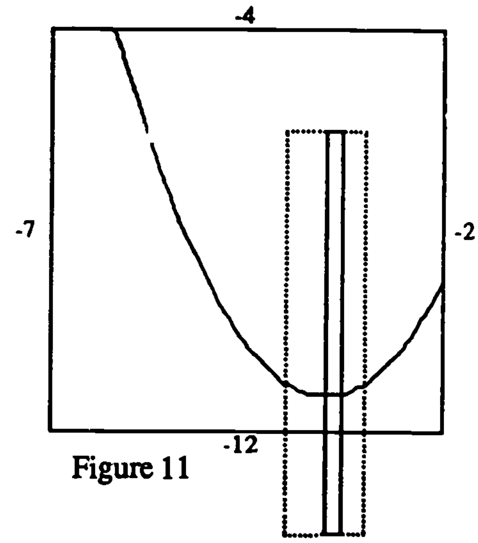



Figure 11

She performed a final step to cut off the jog on the right. The interviewer then asked her to explain a contradiction. She had asserted not long ago that "a parabola doesn't really have [a horizontal line at the vertex]; it just has a point." Nevertheless, she succeeded in producing a graph on the screen that clearly showed a line.

Syl *(puzzled and amused)* ...It doesn't make sense!

Int But you reasoned it out. I mean, it's not as if it was just an accident. You actually made that happen.

Syl *(laughs)* I don't know how, though.

This response is, itself, something of a puzzle.

Recall that although she doubted it would be possible to make the graph look like a horizontal line, she knew exactly how to go about doing it, if it could be done. She *created* this transformation entirely on her own: she conjectured that a small neighborhood around the vertex was what she wanted, estimated the size of the neighborhood, performed successive adjustments, and had her method confirmed by achieving the desired result. Because she invented the method, we might therefore expect that she would understand *why* it worked. Yet she acted stunned by the contradiction between a firmly held belief, that parabolas do not have flat bottoms, and the observation that she, herself, was able to create a graph of a parabola that was a horizontal line.

Perhaps more puzzling is why she would have made an attempt to find a horizontal line in the first place. She clearly did not believe it existed, even after she saw it.

The interviewer then suggested she draw on her paper and pencil graph a box representing the region displayed on the screen—in a style similar to that shown in figure 11. She did so with no difficulty. To help her consider relationships between the shape of the region on paper and the appearance of the graph that was displayed on the computer, the interviewer asked her what difference it might make if the graph she showed on the screen represented a shorter, squarer region (figure 12) than she was currently showing.

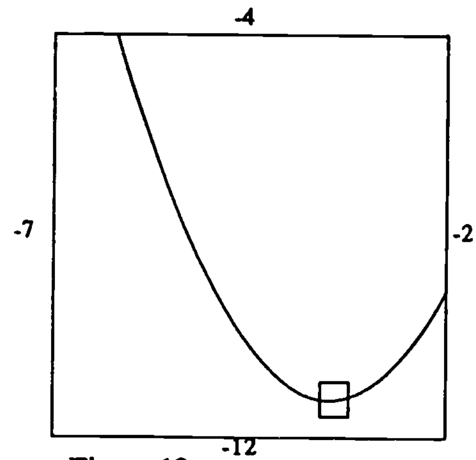



Figure 12

Syl (*after a short pause)* I don't think it would matter ... as long as you don't make it [the region of the graph] wider [again].

Int Explain what you mean by you don't think it would matter.

Syl I don't think the straight line [on the screen] would change—the horizontal line—as long as you keep the width [the $x$ extents] the same.

Int Why not?

Syl Because, um, [if you change the vertical dimensions, too] you're just looking at the [same] view from a shorter— I mean, you're not looking at anything different on the parabola; you're looking at the same thing but from a smaller box.

A heavy dependence on experience in the paper-and-pencil graphing world would explain all three puzzles: why she would seek a horizontal line that she thought did not exist, why she could not explain a successful procedure that she invented herself, and why she expected that no alteration of the graph would result from this latest experiment that suggested making a shorter, squarer box around the segment she was graphing.

Although she believed that no *truly* straight and horizontal segment existed, she *saw* an *almost* straight and horizontal segment of the parabola—the best she could find. In a paper-and-pencil world, she could "cut away" the unwanted portions of the parabola, as if with scissors, to show her chosen segment by itself.[8] Thus, her willingness to offer a tiny

[8] There are two kinds of flatness to be understood in this problem: the mathematical flatness that is reflected in the limiting nature of $\Delta y/\Delta x$ as $\Delta x \rightarrow 0$, and the suppression of change in $y$ value by measuring it on a much grosser scale than that used for changes in $x$, effectively magnifying more along the $x$ axis than along the $y$. It is interesting to consider what Syl's response might have been to a graphing utility that used a cutting and cropping metaphor as part of its scaling interface. In some utilities, the user may construct a rectangle and slide it around on the graph to indicate what portion of the graph is to be magnified, which is to say scaled up *while maintaining the existing aspect ratio, or ratio between* x *and* y

chunk of curved line as an approximation to the desired straight line led her to apply a technique—cutting away distracting features—that she understood in a concrete way.

She is a bright student, and if she had been asked to *anticipate* what the computer would display as the result of her first such cut, she might well have figured it out. But she was not asked to predict, and most likely she *saw* the result before thinking much about it.[9] This allowed her to be seduced perceptually into repeating the process without ever quite noticing where she was being led. When, after several iterations, she saw a truly straight, horizontal line, and not a tiny curved approximation, she did not know what to make of it.

If our conjecture is correct—if, that is, her strategy is based on some notion like cutting off unwanted paper with a scissors and if she never accounted for the unexpected expansion of what was left—then it would also explain why, once she had succeeded in displaying a horizontal line, she could not see how snipping off blank space could change anything about its appearance.

This is exactly the behavior we would expect if we were performing the operations on paper. If we snip away portions of a paper and pencil graph, the size and shape of the remaining piece changes, but the curve drawn on that piece is not altered at all. Syl had not expected that any amount of surgery on figure 5 would reduce the parabola to a horizontal line. The paper-and-pencil metaphor cannot explain her observation that the surgery did make a change, but she appears to cling to this metaphor because she has nothing to replace it. Consequently, she approaches the removal of the blank space in the same way.[10] Cutting off all but a very narrow band above and below the horizontal line on a paper-and-pencil version of figure 10 would result in an unconventionally shaped graph, but would, as she maintained, produce just as horizontal a line as the graph began with.

The effect on the computer screen was quite different from what the paper and pencil metaphor would suggest. "Snipping away" a portion of the graph did not change the size and shape of the window in which the remaining portion would be displayed. Therefore, choosing to display only a very narrow band above and below what appeared to be a horizontal line would stretch the band to the full height of the display window. If there was, in fact, any curvature to the line, this magnification might be enough to show it.

## Metaphor 2: Scaling is like using a magnifying glass

> As one looks closer at a curve, one sees its true nature and composition better. Thus, if it is curved, one sees the curves better. As with a physical object, magnification shows roughness that may not otherwise be visible.

---

*measures*. Software using such a metaphor favor the separation of these two sources of flatness, but at a sacrifice of some clarity about scale changes that expand rather than contract the viewing window.

9 The role of doing the computer's work in one's head—both prediction of and reflection on the outcome of an experiment—will be explored later.

10 Notice that it is her continued *surprise* at the results that gives us a sense for the strength and tenacity of the metaphor. This will be apparent in later examples, as well.

When Dan first saw a graph of $x^2+7x+6$ scaled to make it look like a line, he rejected his original prediction that it would be a parabola, explaining the discrepancy with the disclaimer that "because you add the new, other $x$ term [$7x$], it changes from a parabola to a line... you're changing the whole equation, which changes the ultimate graph."

He then set about computing the formula for the line that he saw on the screen, using the knowledge from the formula that the $y$-intercept was 6, and the observation from the screen that the $x$-intercept was -1. He compared the two graphs—$f(x)=x^2+7x+6$ and $f_1(x)=6x+6$—by displaying one on the screen, tracing over it with a felt-tip pen, and then displaying the other. They were indistinguishable by this method of comparison.

Despite having so quickly capitulated to the image and given up his original prediction that the graph would be curved, and further, despite having apparently proven that the quadratic *was* equivalent to a line, Dan maintained a sense that something was amiss. At the mere suggestion that things might be different outside of the small territory he was viewing—a suggestion like the one to Syl that he draw a box around the graphed region—he spontaneously returned to his expectation that these functions should be different and, also spontaneously, explained why.

> Dan  Well just the range that we're using, ultimately...because of the squared term, the larger number you use, the greater it [the squared term] is going to get, and that's going to throw the 6x.

He then changed the scale (figure 13) to show the parabola in the form he had anticipated.

The interviewer asked him to explain the new graph that he produced.

> Int  OK. Now what are you seeing?
>
> Dan  We're seeing that *this* [neighborhood of -1] is the part that I was seeing [in the earlier graph]. [When the parabola just looked like a line, we were] seeing just part of the parabola. And then when you expand it [you see the parabola].

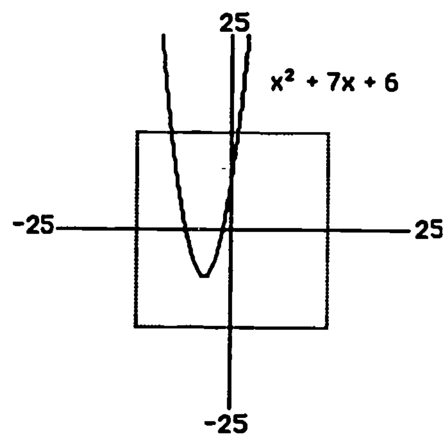



Figure 13

> Int  ... Let's see if we can get this thing to change appearance even more. Before, it was a straight line; now it's a curve. Can you get it to be a different straight line, like, for example, a straight line going *that* way? [Gesture indicates NW-SE diagonal.]

Dan pointed to a short segment of the parabola just left of the vertex and said he was "trying to target that section down there." As it was then displayed, the segment was quite small and appeared very straight, partly due to its size and partly due to the pixel approximation that represented it. When he did succeed in viewing just that section (figure 14), it became apparent that the segment was curved. He did not use this new figure as a basis for further transformation as Syl did, but remarked immediately on its *non*-satisfaction of the request to generate a straight line.

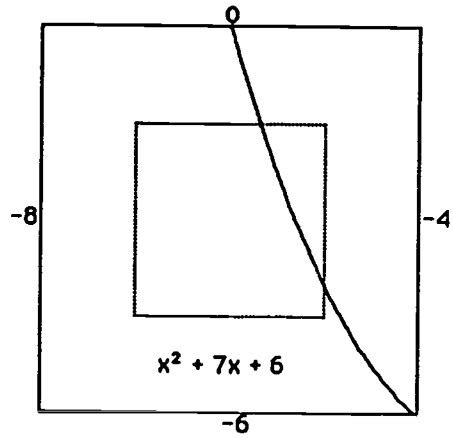



Figure 14

Dan That's slightly —— *gesture indicates a curve*

Int That's slightly curved. What would you have to do to make that look very straight? Or is it possible?

Dan ... Well, the smaller you get, the more it looks straight.

Int How do you know? You said that awfully fast, without experiments, what do you mean?

He seemed to have said the correct thing instantaneously and so the interviewer wanted to know how he figured it out. But, in fact, the interviewer misunderstood his meaning. From the interviewer's notes:

> I assumed that he meant "the closer in you zoom" or "the smaller the segment you look at under magnification," the more it looks straight. In fact, what he means is something quite different. As he perceived the phenomenon, what had looked straight to him in the original parabola turned out to be slightly curved under magnification, and so what he means by "the smaller you get, the more it looks straight" is "the less magnified the segment is (i.e., the smaller it is), the less well you are able to see whatever curvature it has." As he indicates below, this belief is quite firm with him.

Dan Well, just because, you know, when you're looking at a bigger... like a magnification of something, you can see the imperfection, say, of the line a lot better than when it's smaller. I mean, if you were to take this part *[points to a part of the curve currently on the screen]* and magnify it, then it would— it would, you know, look curved, more curved than if you shrunk it down into a smaller piece.

...

Dan Like, if you move the scale back to the 25s *(figure 13)*—you know, 25s all around—this segment right here, the segment that I was targeting, looks like a straight line... Because of just the resolution or whatever. It looks like a straight line... But when you zoom in, and you get a magnification of that area, *(figure 14)* it's actually curved. It will be curved.

Here, too, is a puzzle. Dan had already seen a parabola so convincingly represented as a line that he actually computed a formula for that line. Furthermore, in his explanation of figure 13, he seemed to comprehend immediately what transformation had been responsible for the straight line he had first been shown. Can it be, then, that he now believes such a transformation could not be applied a second time or to a different portion of the parabola?

Even the wording of the challenge—"Before, it was a straight line; now it's a curve. Can you get it to be a different straight line?"—suggests such a thing can be done. Why does he now seem to reject the possibility?

One interpretation is that he is *not* rejecting the possibility, but having difficulty divorcing himself sufficiently from what he perceives as the mathematical "reality"—that there are no straight lines in a parabola—to consider appearances alone. He might, for example, be trying to say something like "If one looks closely enough at a parabola, one can see that it is never quite straight"—a statement about some kind of *mathematical* looking closely, and not about what one *sees* on a graph. But everything about his language suggests instead that he is indeed talking about appearance, and not about the mathematical reality.

The interviewer tests the strength of Dan's beliefs first by trying to convince him to take the next step, as Syl did, and repeat the scale change and display an even smaller segment of the curve, but Dan is so convinced that he knows what will happen that he merely explains his reasoning and does not perform the experiment.

Dan — I mean, there you can see that it is slightly curved, if you magnified it a lot more, you'd see there's even more of a curve....

Int — So if we were to look at, let's say, just *that* part [inside the dotted box in figure 14], it would seem *more* curvy?

Dan — If you were to take this part [inside the dotted box] and magnify it more, it *would*. ...If you take [it] where it is now, it looks like a line... But if you take [it] and magnify it... putting, you know, more dots in for the screen, you'd be able to see that it is curved.

So, the interviewer pushes harder, even arguing outright.

Int — I don't believe that... I mean if it were more curved, wouldn't that mean that this whole thing is like full of ripples? If there's a lot of curve here and a lot of curve here and a lot of curve here. *(Interviewer points to contiguous segments of curve and tries* very *hard to argue him out of his position.)*

Momentarily shaken, Dan's argument becomes incoherent, but he quickly recovers and returns to his original position.

Dan — ... Well, it's the curvature in, in, you know, total, the total curvature. Whereas, whereas, when you're looking at here, I mean, if you take this segment, and, you know, take it to there, I mean that looks like a viable line.

Int Well it certainly looks like a line.

Dan On a screen, but when, you know, the more you look at it, I mean, if you were to shrink this down to that, change the scale, it would look a lot more like a line than if, than if you took this part and magnified it to there.

Behind Dan's use of "magnified" and "resolution" and "zoom in" is an optical metaphor, a notion that a dilating scale change on a *mathematical* curve behaves somehow like looking at a *physical* object in a magnifying glass. It certainly *is* true that when we look at something smooth under strong enough magnification, we see curves and holes and bumps that aren't apparent without magnification. This experience is generalized inappropriately to the mathematical object.[11]

This is a particularly powerful metaphor for several reasons. For one thing, until a student has much experience with "magnifications" of mathematical objects—or, in place of the concrete experience, sufficiently clear formal understanding of such difficult ideas as continuity and infinity—physical magnification seems the only available analogue in the real world. Further, the vocabulary of optics seems naturally to be used by just about all the students and teachers with whom we have spoken. Also, there is ample confirmation on the computer of the observation that some *very small* segment that appears straight at one scale can appear quite curved when enlarged just enough. For a single example of that, compare the zigzag straight-line look of the "distance view" (left) in figure 15 with the smoother, more curved representation of those hills and valleys in the enlargement (right).

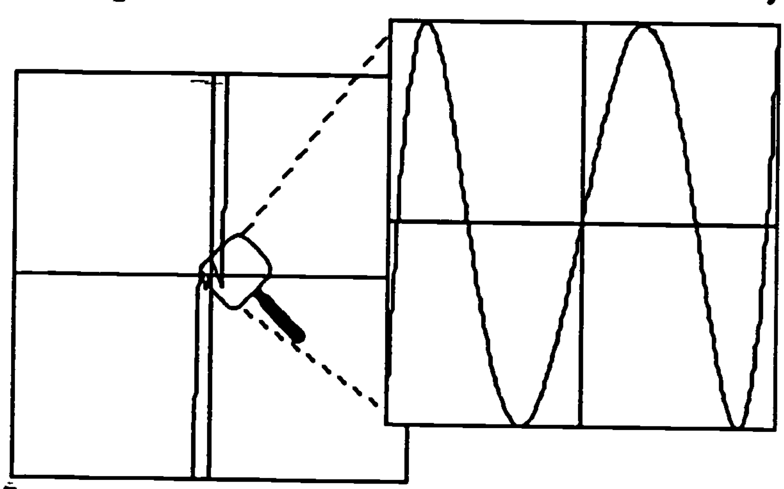
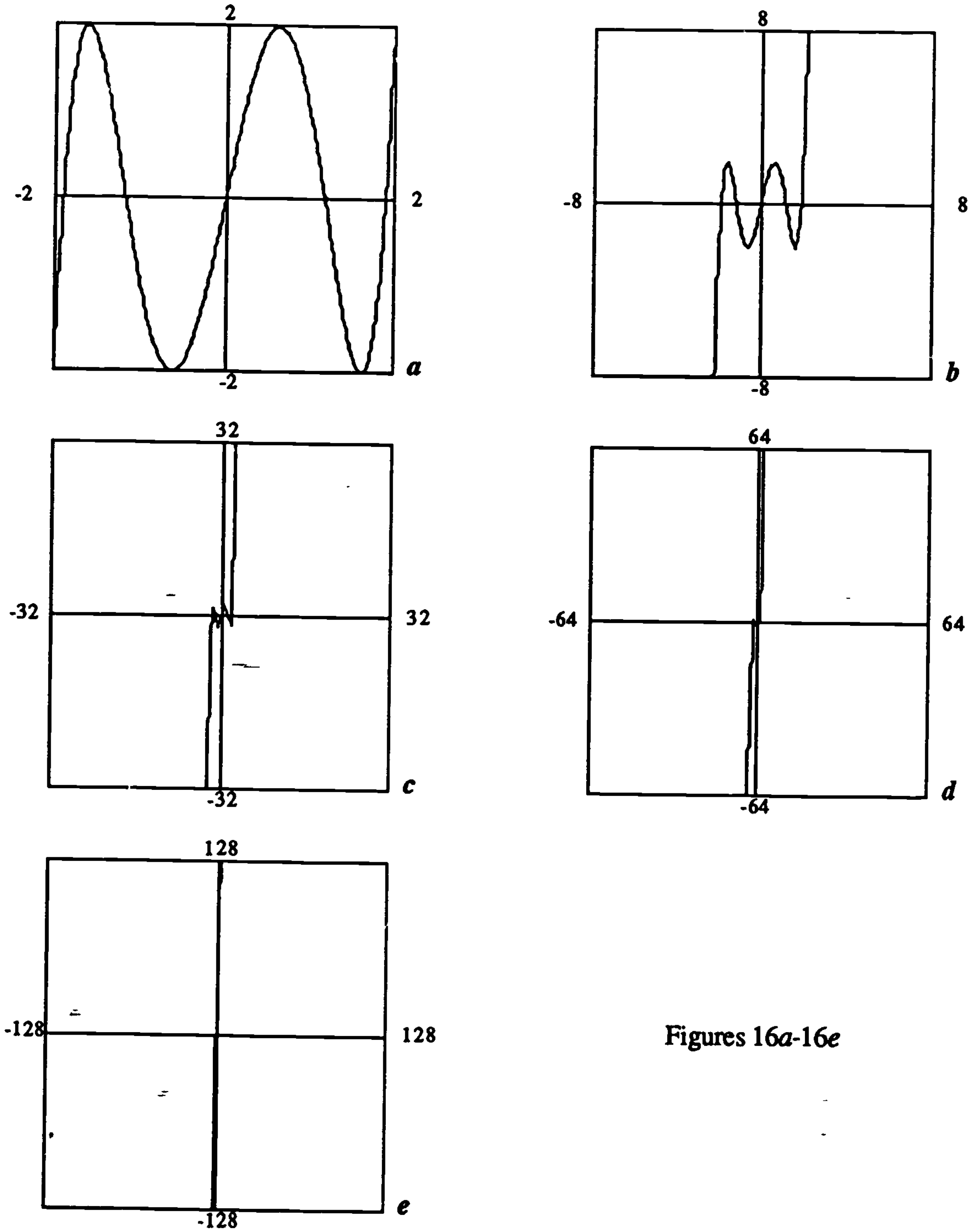


Figure 15

Finally—and perhaps most seductive of all—the metaphor is backed up by what one already *believes* about the mathematical reality. That is, although Dan seemed (until much later in the interview) to be talking strictly about appearances, it became clear that he did believe "If one looks closely enough at a parabola, one can see that it's never straight."

[11] Except that students do not fully extend the idea and imagine that a computer-"magnified" mathematical line would appear wider, as would, say, a graphite line under a real magnifying glass.

Although it is not an experiment Dan had tried, there is even a mathematical experience with scale that may strengthen this metaphor. Critical features—e.g., a local bump—on a *mathematical* curve can be missed when one views the curve on too large a scale. For example, in the sequence shown below, the function $f(x)=x^5-3x^3+3x$ varies from a roller coaster (figure 16a) through a slightly broken line (figure 16d) to something that cannot be distinguished visually from a nearly vertical straight line (figure 16e).




Figures 16*a*-16*e*

Just as Syl held two opposing operating principles—one that told her that parabolas contained no horizontal line segments, and another that nevertheless allowed her to design and perform a successful procedure for displaying such a segment if it existed—Dan worked from two such opposing principles. Alongside his magnifying glass metaphor resides a sufficiently clear understanding of scale to let him perform a correct transformation (producing figure 13) and explain it well.

## Metaphor 3: Mathematical curve as a bead necklace

> Points in a curve, like beads in a bead necklace, line up "next to each other." A radical enough scale can magnify these points so that they can be seen, and certain scale changes can distort the appearance of these points.

When Syl had succeeded in showing a horizontal line at the bottom of the parabola, she claimed that she did not know how she had done it. She tried to explain the result anyway:

Syl I guess ... um, the points are... I don't know... The computer doesn't show the points small enough or something?

Int What do you mean by that?

Syl Well, a parabola is only supposed to have one point [at the "bottom"—the vertex], but I've blown it up big enough so that the point looks longer than, longer than, like in reality.

Several students invoked notions of the computer not showing points "small enough," as if points, themselves, had a physical existence. Syl's description of a point that "looks longer than in reality" suggests a distortion of an otherwise more symmetric point, again attributing shape and size—albeit very small size—to the point. Several students spoke as if very small scales allowed them to see (or the computer to represent or misrepresent) individual points, and further, to alter the appearance of these points with scale changes.

The bead necklace metaphor treats curves as if they are strings of extremely tiny but still physical beads. Enlarged sufficiently, these individual bead-points can be seen (Figure 17) and can even be distorted (Figure 18). Like the magnifying glass metaphor, the bead necklace may be to some extent encouraged by limitations of computer representation. The "steps" that appear when a section of a curve is enlarged may be construed as (perhaps somewhat distorted) "points" (Figure 19).

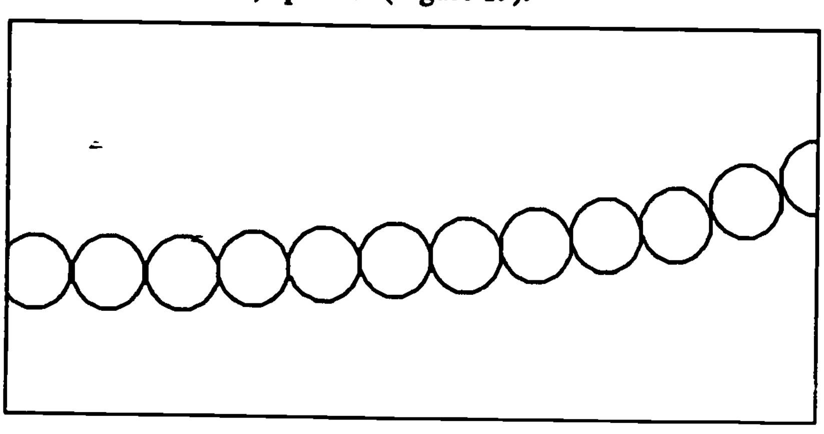


Figure 17

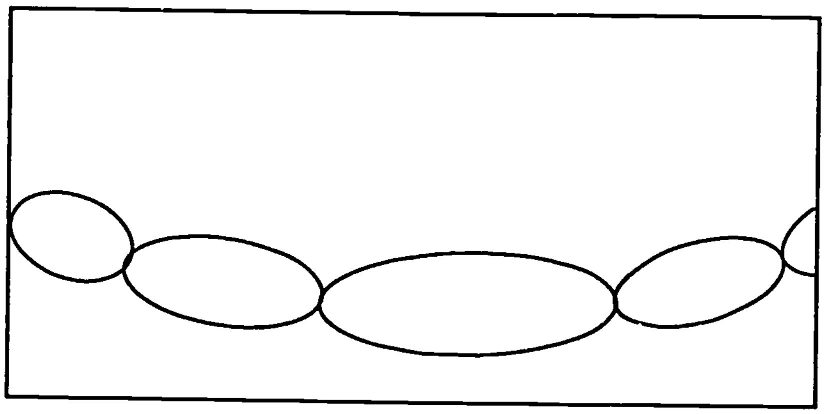


Figure 18

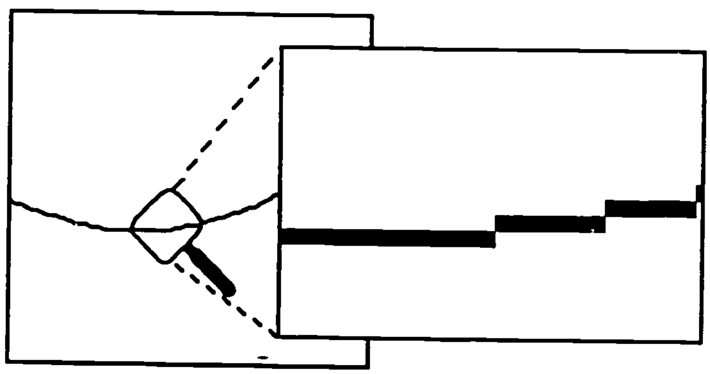


Figure 19

Further contributing to the "bead necklace" metaphor is the impossibility of representing a point visually without treating it as a dot with finite (in fact, visible) size. Visual representation of the "absence" of a point—e.g., the hole in the graph of $(x^2-4)/(x-2)$—also feeds the metaphor. But there is even a subtlety buried right in that statement that attests to the insidious nature of this metaphor. The point is *not* absent. A point is a location, not the "ink" in the location. The point is present. Rather, because the function is undefined at the value $x=2$, it does not pass through, or exist at, the point (2,4). This kind of language is so casually and widely used that we suspect this metaphor to be pervasive. Graphing software might help students recognize the special characteristics of functions like $(x^2-4)/(x-2)$ by representing the hole as a gap in the curve, but such a representation also contributes to the notion that such a hole is visible, which it cannot be at *any* scale.[12]

Corollaries of the bead-necklace metaphor are the notion of a finite number of points in a given section of a curve and the notion of "adjacency" of points. This latter idea was invoked by a few of our students. For example, in trying to explain why he thought he

12 Of course, the graph itself feeds the metaphor. Not only are points invisible at all scales, but so are one-dimensional objects, including the graphs we are examining.

could not display a parabola in a way that would look like a horizontal line, Dan argued that parabolas do not have two consecutive points in a horizontal line.

### *GRAPHING IN ONE'S HEAD*

Because we had observed so often that the students who were experimenting with graphs on the computer were seduced by what they saw, we conducted a preliminary interview with two high school juniors--both sophisticated computer users—without providing a computer on which to work.

There are too many differences between the treatment and experiences of the on- and off-computer students to draw undisputable conclusions from a comparison. However the unanticipated differences in treatment and the magnitude of the differences in results make such a comparison thought provoking nonetheless.

When the students working off-computer were first asked to sketch $x^2$, and were then shown a straight-line graph that they were told was also a graph of a parabola (recall figure 1), they immediately assumed the graph was a parabola at a scale that gave it a non-standard appearance. When the students working on-computer were given a similar problem, *no one* seemed even to consider without the aid of the interviewer that scale might be responsible—even though the scale was on the graph!

One very different way in which the absence of graphing software may have influenced students' approaches centers around dimensionality. The computer screen, paper and pencil, and rubber sheet are all two-dimensional. During the off-computer interview, one of the students supported an explanation he gave with the fact that the parabola may be generated from the slicing of a cone. No one in the on-computer interviews mentioned the parabola as derived from a cone. Is the fluency with which one may shift between two- and three-dimensional imagery reduced when one faces a two-dimensional image or when one faces a piece of relatively unfamiliar technology?

## A metaphor for talking about scale: The rubber sheet

In the absence of a computer on which to perform experiments and see the results, we felt it necessary to provide a common language for talking about the mental experiments the students would be asked to perform.

Early in the off-computer session, we asked the students to determine whether it was possible to stretch a section of a parabola drawn on a rubber sheet so that it looked horizontal. The students readily adopted this "rubber sheet" metaphor and continued to use it throughout the rest of the interview. They sometimes referred to it in describing their methods of problem solution and defending their solutions (e.g. "you can stretch it out so much in the $x$ direction that it appears horizontal"), and produced suggestive gestures such as using their hands to stretch an imaginary rubber sheet while solving problems.

The rubber sheet view of scaling is primarily an action on the plane. As the plane is stretched, *all* of its points—both those identified as the graph and the non-graph points—are moved. Computer rescaling looks like an action on the graph alone and not on the plane because its action on non-graph points is invisible and the stand-in for the plane—the computer screen—suggests rigidity with a flow of movement *on* it. The importance of the rubber-sheet metaphor is the explicitness of the action on the plane.

This metaphor seemed to contribute to several fundamental differences between the on-computer and the off-computer interviews. Perhaps the most striking of these concerns students' use of numbers. Since the rubber sheet is essentially qualitative, the students had little need to find specific coordinates. Perhaps the critical element here is the introduction of a new metaphor. Would the introduction of new metaphors have aided the computer-using students as well? This is an important pedagogical as well as future research issue.

## Absence of numbers may free students to think qualitatively

Although the problems we posed were essentially the same for students on and off of the computer, the way the students navigated through the problems was not. All problems were qualitative in nature, but the on-computer students communicated scale-changes to the computer only through *numbers*, while, partly as a result of the rubber-sheet metaphor, the off-computer students could focus exclusively on the qualitative aspects of the problems.[13]

## Inability to experiment may free students to predict and reason

Although no computer was available, lined paper was provided during the off-computer interview, and the interviewer made use of it a few times in posing problems. Remarkably, the students made very little use of paper, despite an invitation to do so; and when they used it, it served primarily for communicating to the interviewer or to each other (e.g., for indicating a particular part of a curve) and not for working out details of the problem. Unlike the students with access to computer graphics, these students solved problems almost exclusively "in their heads." Even when a problem was posed with a paper and pencil graph, the students chose to stretch, cut, or otherwise manipulate the graph mentally, rather than draw or sketch graphs.

Because no computer was used, they had not only to devise the experiment, but also to imagine the result. By contrast, the on-computer students often saw results of their experiments before they had a chance to make formal predictions. Recall, in particular, our conjecture that that may have weakened Syl's performance. They also tended to iterate their experiments in a successful direction without necessarily taking the time to reflect on why that direction was successful. In fact, on-computer interviews occasionally adopted an almost laborious "trial and error" flavor as students went through a series of coordinate changes in an attempt to fine tune an idea or to recover from earlier misjudgments in estimation. Precise answers, rather than general ideas, sometimes became the focus for the on-computer students where off-computer work precluded such precision.

[13] As mentioned earlier, *The Function Analyzer*, has a non-numeric stretch-shrink operation, but we felt that the greater freedom provided by the option to set coordinate boundaries freely would give us a richer picture of students' strategies. A mouse-driven system would combine the flexibility of setting coordinates at will with the qualitative sense we lost by specifying coordinates numerically.

### Absence of external pictures may free students to create internal ones

The impetus to conduct these interviews was our observation that students are often trapped by the images they see in front of them. The picture is already there and they have no room to imagine it another way. By contrast, the off-computer students, as a result of shunning paper as well as having no computer, had little or nothing relevant to look at while solving graphing problems.

The absence of numbers, the absence of experimental checks, and the absence of external pictures may all help bright, confident students focus on generalizable ideas rather than on the particular data of the moment.

Here, of course, is where our selection of very capable students may be most strongly biasing our results. Less able students may depend more on experimentation and pictures than these students did and therefore fare particularly poorly without the computer.

### Advantages of extensive computer experience *outside of* the interview

Among the differences between the on-computer and the off-computer interviews is the nature of the students. Their style, and their facility with the problems we posed, reminded us of Che, a remarkable student in our early exploratory interviews who did use the computer during his interview but primarily to demonstrate the validity of predictions he made. Beyond doing our problems mentally, Che and the off-computer boys had something else in common. All three were long-time computer users who had considerable experience with graphics and programming.

Che readily explained some of his extraordinary clarity and quickness as the direct result of his having worked on a graphing program in Pascal for his school computer center. Although he had never encountered scale in mathematics, he had had to think about how to pick values of $x$ to evaluate and to maximize the speed of plotting while maintaining reasonable resolution in the graph. It was this experience that he invoked several times in explaining ideas he had. Experimenting with and reflecting on this issue gave him tremendous mathematical insight which he brought to bear on the problems we presented.

## *THE INTERACTION OF EXPERIMENTATION AND REFLECTION: SOME THOUGHTS ABOUT CURRICULUM*

It is interesting to speculate about the off-computer group, but let us not overclaim. With an $n$ of 2 and at least six differences from the on-computer interviewees (in addition to the five differences mentioned in the previous section, the two off-computer students worked with a different interviewer!), even our speculations about what made these students so much more clear and successful must be attributed more to our biases than to our data.

But having presented the relevant caveats, it would then seem just plain miserly to withhold our speculations, especially as they point to rich areas for research and plausible areas for curriculum development.

The two things that Che and the off-computer group had in common were their extensive prior experience on the computer and their inclination to work on problems mentally.[14] It is quite plausible that these are not merely coincidental with their extraordinary performance.

*All* of our students found their work on the computer very thought-provoking; most, in the course of the computer exploration, confronted and sorted out one or more misconceptions that they had brought into the interview. If such clarification of mathematical ideas could be observed in a single 40-minute problem-solving session on the computer, how much more expectable it must be when students regularly engage in similar, but perhaps more didactically organized, thoughtful exploration over days, months, or years! For the three computer-experienced students, such a long-term exploration was at their own initiative and largely extracurricular. If the kind of exploration that our on-computer interviewees performed for 40 minutes and that the computer-rich students performed for months can be incorporated into a curriculum, it might provide some of the experience that we hypothesize was so contributory to the success of the computer-rich three.

But the other advantage that we think may have helped the off-computer group was the fact that their work *was* off the computer. Pictures can be overpowering, and the ease of performing what was, in our study, a quantitatively accurate computer-aided experiment can seduce one away from performing a qualitative version of the same experiment mentally. Che, too, worked off the computer to the extent that he invariably chose to predict the results and then use the computer to demonstrate his claims to the interviewer or to check them quantitatively. The off-computer students had no quantitative check available and so relied solely on analogy and reason. And we had conjectured much earlier that if Syl had had the opportunity to think about what she *expected* before she was overwhelmed with what the computer claimed was Truth, she would not have been led quite so far astray by her experiments.

In other words, both computer experience and reflective internalization of that experience, especially *away* from the computer, seem to be important ingredients.

And what about the fact that graphing in the computer context seems inevitably to raise a complex tangle of issues—the relation between the continuous and the discrete, the behavior of the infinite and the infinitesimal, the interaction between scale and function in generating the images we call graphs? Isn't it better to present students with a somewhat more tame and restricted experience first?

Certainly, there is a level of complexity which just becomes overwhelming to a student and is therefore not helpful. But, as a general pedagogical principle, we feel that presenting students with (real and appropriate) complexity is not as harmful as protecting them from it. Through careful selection, one may generate sets of classroom exercises that avoid certain

---

[14] We have not overlooked the fact that these were also unusually bright students, perhaps even by comparison to the rest of our sample. But we choose to speculate that the behaviors that we observe as "bright" are at least as much the *result* of opportunity and experience—such as exploring on the computer—as they are the prerequisites to such experience.

complexities, but the selection process imposes a pattern that may be so strong that it distracts students' attention from the pattern in the underlying mathematics or obscures that pattern altogether. We are all familiar, for example, with students' assumptions that if the division doesn't "come out even" they must have made an error somewhere in their computation. If students are going to be looking for some kind of structure in their classroom—"psyching out the teacher" or finding the "trick" in the lesson—then we might as well make that structure the mathematical message rather than the criteria by which problems are selected or presented.

Last year (Goldenberg, 1988), we convinced ourselves that the fact that computer graphing opens up a can of worms may be a blessing—in fact, the appropriate treatment—rather than a problem merely to be coped with. This year we have seen still more worms, and with them still more fertile soil in which to grow mathematics. When conflict is avoided, misconceptions have time to ossify. When students are given the opportunity to confront their own misconceptions and work out the conflicts between incompatible theories or images, we believe they deepen their understanding.

# REFERENCES

diSessa, Andrea, 1983. Phenomenology and the Evolution of Intuition. In Dedre Gentner and Albert Stevens (eds.), *Mental Models*. Hillsdale, NJ: Erlbaum

Goldenberg, E. Paul, 1988. Mathematics, metaphors, and human factors. *Journal of Mathematical Behavior*, in press (August, 1988).