DOCUMENT RESUME

ED 295 826 SE 049 207

AUTHOR	Kerslake, Daphne
TITLE	Fractions: Children's Strategies and Errors. A Report of the Strategies and Errors in Secondary Mathematics Project.
REPORT NO	ISBN-0-7005-1006-0
PUB D*"E	86
NOTE	139p.; Some drawings may not reproduce well.
AVAILABLE FROM	NFER-NELSON Publishing Company, Ltd., Darville House, 2 Oxford Road East, Windsor, Berkshire SL4 1DF, England.
PUB TYPE	Reports - Research/Technical (143)
EDRS PRICE	MF01/PC06 Plus Postage.
DESCRIPTORS	Arithmetic; *Fractions; Mathematics Achievement; Mathematics Curriculum; Mathematics Education; *Mathematics Instruction; Mathematics Skills; Number Concepts; *Problem Solving; *Rational Numbers; Secondary Education; *Secondary School Mathematics

ABSTRACT

This report describes an investigation in England of some of the problems children experience with fractions. In particular, the restricted view that some children have of fractions is investigated. The methodology involved a large number of interviews with secondary school children (aged 12-14). The results of these interviews led to the design of teaching experiments which involved both small groups and whole classes with a pretest, immediate posttest, and delayed posttest. Results of the study address student understanding of models of fractions, the division aspect of a fraction, fractions as numbers, and equivalent fractions as well as the teaching module that was developed. Implications for the teaching of fractions include: (1) the learning of fractions would be facilitated if teachers developed their skills of listening to children and of encouraging them to talk about their interpretation of fractions and associated problems; (2) a greater emphasis could be placed on the division interpretation of fractions; (3) more attention needs to be given to the basic ideas of equivalence of fractions; (4) more attention needs to be given to the limitation of the "part of a whole" model of a fraction; and (5) more attention needs to be paid to the importance of the transition from the realm of counting numbers to that of rational numbers. Appendices include the interview and testing materials. (PK)

fractions:

A REPORT OF THE STRATEGIES AND ERRORS IN SECONDARY MATHEMATICS PROJECT

Fractions: Children's Strategies and Errors
A Report of the Strategies and Errors in Secondary Mathematics Project

Author/Researcher: Daphne Kerslake

Based on research carried out at
Bristol Polytechnic in conjunction
with the Strategies and Errors
in Secondary Mathematics Project
at Chelsea College, University of London

NFER-NELSON

Published by The NFER-NELSON Publishing Company Ltd.,
Darville House, 2 Oxford Road East,
Windsor, Berkshire SL4 1DF

First published 1986

ISBN 0-7005-1006-0
Code 8199 02 1

Library of Congress Cataloging in Publication

Typeset by Illustrated Arts, Sutton, Surrey
Printed and bound in Great Britain by
Dotesios Printers Limited, Bradford-on-Avon, Wiltshire

Contents

List of Tables

List of Figures

List of Appendixes

Preface

The aims of the research project 'Strategies and Errors in Secondary Mathematics (SESM)' were to investigate in depth some of the problems commonly experienced by secondary school pupils in the area of mathematics and to examine to what extent these could be alleviated by specially designed teaching modules. The result, it was hoped, would thus be greater insight, for both teachers and researchers, into the process of learning mathematics and the interaction between what is taught and what is learned.

The SESM project, funded by the Social Science Research Council (SSRC), at the Centre for Science and Mathematics Education, Chelsea College, University of London, from 1980–3, was a sequel to the 1974–9 programme 'Concepts in Secondary Mathematics and Science (CSMS)', also SSRC financed. One feature of the CSMS mathematics results (reported fully in Hart, 1980, 1981) was an unexpectedly high incidence of certain wrong answers which suggested the presence of misconceptions leading to inappropriate strategies. The present SESM project focused on a small number of these errors, drawn from various topic areas within secondary mathematics, and investigated them in considerable depth.

The methodology involved a large number of individual interviews with secondary school children. The results of these interviews led to the design of teaching experiments which involved both small groups and whole classes.

The errors investigated by the SESM project were in the following areas:

Ratio
Algebra
Graphs
Measurement
Fractions

The research team consisted of the directors, David C. Johnson and Margaret Brown, and two full-time researchers, Kathleen Hart and Lesley Booth. Three research students were associated with the project, Shiam Sharma, Timothy Burns (both SSRC supported) and Daphne Kerslake (who was assisted by funds from Bristol Polytechnic research committee). Work on the first two topic areas listed above is reported in two separate monographs in this series.

Further work is already in progress at Chelsea College following on from SESM (supported by SSRC, 1983–4) which has as its focus the development of 'Children's Mathematical Frameworks' in the age range 8 to 13 years.

All three research projects have depended heavily upon the cooperation of pupils, teachers and local education authorities and a special note of thanks is extended to all those who gave so willingly of their time.

1

Background to the study

This report describes the 'Strategies and Errors in Secondary Mathematics (SESM)' investigation of some of the problems that children experience with fractions. In particular, the restricted view that some children have of fractions is investigated. The results from the project 'Concepts in Secondary Mathematics and Science (CSMS)' which was based at Chelsea College, London University, from 1974 to 1979 are used as a starting point. This chapter gives a brief description of that project; in particular, some of the results concerning fractions are examined.

The CSMS (Mathematics) project

The 'Concepts in Secondary Mathematics and Science' project set out to investigate the difficulties that children experienced in a number of areas of mathematics, of which fractions was one. The aim of the project was to develop a hierarchy of understanding in mathematics in the secondary school curriculum. As it was a large-scale investigation, written tests were used, and the items were designed to elicit children's understanding of a topic rather than their ability to apply skills. The test items were tried out on individual children in a series of interviews. The children were aged 11 to 15 years and the interviews were tape-recorded and transcribed. Particular consideration was given to the methods the children used to solve the problems and to the errors that were made.

Based on the results of the interviews, written tests were produced and given to some 10,000 children who were representative of the English population of 11- to 15-year-olds. A hierarchy of levels of understanding was established.

The CSMS fractions results

Four tests on fractions were used in the CSMS project. The first two contained items that were presented in problem or diagrammatic form. One of these was a version that was given to the 12- to 13-year-olds and contained items on recognition of fractions, equivalence, addition and subtraction. The other test also contained items on multiplication and division, and this was used with the 13- and 14-year-olds. The items on these tests were also matched by a parallel set that were presented with no words or diagrams but in a straight computational form. The tests were given to 246 first-year, 309 second-year, 308 third-year and 215 fourth-year pupils. Some of the conclusions drawn from the results (Hart, op. cit.) are now given.

Comparison between problem and computation questions

There seemed to be no connection in many children's minds between the problem version and the computation version of a question. The children could often solve the problem but not the computation. This suggests that they used other than taught algorithms when given a problem. For example, the responses to the question:

> Three bars of chocolate are to be shared equally between five children. How much should each child get?

are shown in Table 1.1.

Table 1.1: Facilities for problem version of 3 - 5

	3/5
12 yrs	65 9
13 yrs	63 4

However, when presented with 3 ÷ 5 on its own, without a context, the scores were much less high. They are shown in Table 1.2.

Table 1.2: Frequency of correct or incorrect responses to computation version of 3 − 5

	3/5 or 0.6	*1 2/5*	*1 rem 2*	*5/3 or 1 2/3*
12 yrs	35	5 3	18.3	3 3
13 yrs	31	9 4	17 5	8.7

The results show that the item was considerably more difficult than the problem version, with the proportion of successes being reduced from about two-thirds to about one-third.

The nature of the incorrect responses is also of interest. The fourth column shows the percentage of children who interpreted 3 ÷ 5 as 5 ÷ 3, giving an answer of 5/3 or 1⅔. However, the answers in columns two and three are also greater than one, suggesting that these children, too, divided the larger number by the smaller. Thus nearly 27 per cent of the 12-year-olds and 35.6 per cent of the 13-year-olds appear to have divided five by three instead of three by five. The matter of the interpretation of the division sign in relation to fractions will be investigated further in this study.

Fractions and whole numbers

The children were seen to feel relatively secure when working within the set of whole numbers and when bound by the restrictions imposed by them. The fact that some of these restrictions do not apply within the set of fractions, or, indeed, that fractions are needed in order to solve some problems for which whole numbers provide no solution, seems to have escaped them.

When asked 'How many fractions lie between 1/4 and 1/2?', 29.5 per cent of the 14-year-olds and 30.2 per cent of the 15-year-olds gave the answer 'one': this 'one' could refer to the fraction 1/3, where the children were looking at the denominators only, or it could have referred to a fraction half-way between. The form of the question suggested that there was at least one such fraction and the children who made no response – 13 per cent of the 14-year-olds and 16.7 per cent of the 15-

year-olds – may have thought there were none. This is an important notion, as it is the basis of the 'filling out of the number line' and illustrates the first extension to the number system that children meet. It is investigated later as part of this study.

Avoidance of fractions

A sign of the children's reluctance to accept fractions is their choice of an answer in remainder form rather than fractional form.

An illustration of this [illegible] by the children's responses to:

> A piece of ribbon 17 cm long has to be cut into 4 equal pieces. Tick the answer you think is most accurate for the length of each piece:
>
> (a) 4 cm remainder 1 piece
> (b) 4 cm remainder 1 cm
> (c) 4¼ cm
> (d) 4/17 cm

The percentage of children who chose (b), that is 4 cm remainder 1 cm, was 37.4 per cent of the 12-year-olds, 29.8 per cent of the 13-year-olds, 26.6 per cent of the 14-years-olds, and 27.4 per cent of the 15-year-olds. These children avoided using fractions by reverting to a system acceptable before they had learned of the existence of fractions. This tendency of children to give answers that do not use fractions is clearly of importance and will be considered later.

The use of diagrams

It was observed that diagrams often helped towards the solution of a problem. During the interviews, children sometimes needed a diagram to help in the interpretation of a word problem. The relationship of a diagram to the fraction that it represents is an interesting one, and it is possible that, while a diagram can help in the understanding of certain aspects of fractions, other aspects may be made more elusive. For example, the 'part of a whole' aspect of a fraction is well illustrated by the shaded part of a circle, while the notion of a fraction as a number is not. This study will include research into the use of diagrams in the understanding of fractions.

Equivalence and the addition of fractions

A very common error in the addition of fractions was found to be the adding of numerators and denominators. This occurred in each computation involving the addition of two fractions and was more prevalent where the denominators were different. The actual frequency of each error is shown in Table 1.3. The entries are given as percentages of the total sample.

Table 1.3: Frequency of correct and incorrect responses to fraction addition

	12 years	*13 years*	*14 years*	*15 years*
3/8 + 2/8 = 5/8	77.6	66.3	71.8	67.9
1/10 + 3/5 = 7/10	54.5	38.2	48.7	45.1
1/10 + 3/5 = 4/15	15.9	27.2	23.7	21.[illegible]
1/10 + 3/5 = 4/10	1.2	3.6	1.9	2.8
1/3 + 1/4 = 7/12	54.1	37.[illegible]	35.1	44.7
1/3 + 1/4 = 2/7	18.3	29.1	21.8	19.9

About one-quarter of the 13-, 14- and 15-year-olds added numerators and denominators for 1/10 + 3/5, the 11-year-olds being more successful. There was also a number who used the same strategy for 2/3 + 3/4. The method of adding numerators and denominators is clearly an attractive one for many children. Apart from the case of the 11-year-olds, considerably fewer than half of the children found a common denominator for 2/3 + 3/4. This does not seem to be because the children were unfamiliar with the idea of equivalence, at which they were rather more successful. For example, one item was:

Two boys have the same amount of pocket money. One decides to save 1/4 of his pocket money, the other decides to save 5/20 of his pocket money. Tick the answer you think correct:

(a) 5/20 is more than 1/4
(b) 1/4 is more than 5/20
(c) 5/20 and 1/4 are the same

The percentage of children who chose each statement is shown in Table 1.4.

Table 1.4: Results for equivalence item

	12 years	*13 years*	*14 years*	*15 years*
(c)	76.0	71.0	81.8	84.2
(b)	10.2	12.0	11.4	9.8
(a)	10.6	9.1	5.5	4.7

So it seems that there are many children who recognize equivalent fractions but do not use them to add fractions with different denominators or to insert a fraction between two given fractions. A similar reluctance to use equivalence is seen in the results of the item which asked for a fraction between 1/2 and 2/3. Only 13 per cent of the 14-year-olds and 12 per cent of the 15-year-olds gave the answer '7/12' and were clearly using equivalence.

Suggested reasons for the errors observed in CSMS fractions

It appears that the errors observed in the CSMS results are not so much the result of wrong strategies, but rather stem from a lack of understanding of the very idea of a fraction beyond that of a 'part of a whole'. In particular, the following three problems emerge.

The existence of fractional numbers

It has already been observed that many children appear to avoid using fractions altogether. This avoidance occurs in a variety of ways. Brown, in Hart (1980) comments on children's readiness to equate expressions like 391 ÷ 23 and 23 ÷ 391. She found that, when asked to 'divide by 20 the number 16', the percentage of children who gave the response 'There is no number' was 51 per cent of the 12-year-olds, 47 per cent of the 13-year-olds, 43 per cent of the 14-year-olds and 23 per cent of the 15-year-olds. She suggests that many children regard the division of a number by a number larger than itself as illegitimate. This could be one way in which fractions can be avoided. Hart, in Hart (1980), observed that children avoided using multiplication of fractions in the test on ratio, but used a building up by addition instead. These results, together with the instances in the fractions test, to which reference has already been made, lead one to the view that many children do not treat fractions in the same way as natural numbers. It may well be that such children are not able to think of fractions as numbers at all. This contention will be investigated in this study, and if it proves to be the case, methods of encouraging children to change their way of thinking of fractions so that they are seen as numbers will be tried.

The use of diagrams to illustrate fractions

The second problem concerns the part that diagrams and other models play in learning about fractions. The diagrams used in the CSMS test were of the 'part of a whole' type that constitute the most commonly found model in textbooks and other teaching material. The universality of this particular model results in the observation from Silver (1981), for example, that when 20 interviewees were asked to report the images they had for the fraction 3/4, 15 of them reported that they 'saw' pies or circles, and ten of these were unable to think of any other image even when asked to think of a different one. However, this 'part of a whole' model has severe limitations. Kieren (1980) suggests that it limits the development of the idea that a fraction can be greater than one. Certainly the procedure of starting with one whole that is then split into several equal parts of which some are taken does not adapt easily to the fraction 5/4, for example. More importantly, though, this 'part of a whole' model does not illustrate the operations on fractions: the addition of 2/3 + 3/4 is not helped by the image of 2/3 of one circle being added to 3/4 of another circle, and multiplication creates even worse problems. Kieren (1976) argues that rational number concepts are different from natural number ones in that they do not form part of a child's natural environment. Moreover, the variety of possible interpretations of rational numbers makes a corresponding variety of experiences necessary. Thus, for example, in order to appreciate that rational numbers can be thought of as measures, work with fractions on a number line is necessary. Similarly, for the understanding of rationals as elements of a quotient field, the opportunity of experiencing the partition aspect of division is required.

One aspect of this study is the investigation of children's models or images of fractions. The need to extend the 'part of a whole' model to include the quotient aspect of a fraction, and that of fractions as points on a number line is discussed.

The equivalence of fractions

The third problem is that of equivalence. One aspect of rational numbers is that they are equivalence-classes of fractions. The notion that fractions can be said to be equivalent underlies the methods used for the addition and ordering of fractions. The CSMS results suggest that children are fairly competent at recognizing equivalent fractions, or giving an equivalent when the multiplying factors are easy. However, they were less successful at adding fractions or at finding a fraction between two given fractions. So the ability to recognize or construct simple equivalent fractions does not reflect itself in a readiness to apply equivalence in order to solve other problems. This could be because the children were able to recognize the skills required in a question on equivalence but did not have a sufficient depth of understanding to enable them to apply their knowledge.

These three areas form the basis for further investigation as part of the SESM project. The framework of this project is first described briefly.

Framework for the SESM research

Theoretical perspective

While a specific theoretical perspective had not been rigorously decided upon for the research, it was nevertheless intended to work broadly within a Piagetian framework. However, mindful of the criticism levelled against the Piagetian theory, particularly in terms of such issues as the invariance of the 'stage' construct across different tasks and contexts and a too-strict adherence to the purported correspondence between age and cognitive stage, it was considered inappropriate to adopt the theory *in toto* as the guiding paradigm for research into children's understanding of mathematics. Discussing these questions, the Research Proposal for SESM suggested that:

> with respect to the proposed research, the intention is not to assume the total validity of Piagetian theory; rather, the theory and in particular the equilibration model, is being regarded as sufficiently meaningful and fruitful to inform the investigation of children's understanding in the complex area of secondary school mathematics.

Thus Piagetian notions such as the equilibration model and the possible role of 'conflict' (Inhelder, Sinclair and Bovet, 1974) in bringing about cognitive and conceptual change were regarded as of potential importance to the study, although the possible viability of alternative approaches was not ruled out. In addition, note was also made in a working document produced subsequently (see SESM, 1983), of the need to consider a 'framework of knowledge' of concepts, relationships, procedures and cognitive viewpoints relating specifically to mathematics and which the child constructs as the result of instruction (see Ausubel, Novak and Hanesian, 1978).

Research methodology

The influences noted above are also to be seen in the choice of research methodology. The emphasis on analysing children's errors is based upon the Piagetian view that a consistently made error to a given problem reflects a way of viewing that problem or handling its solution, which is consonant with the child's cognitive structure. Analysis based upon the child's perspective and way of functioning with respect to that task rather than upon the logic of the task therefore provides insight into the child's cognition, and the clinical interview procedure developed by Piaget was adopted as the best way of achieving this analysis. Having elaborated a hypothesis concerning the child's conceptual and procedural structures with regard to the topic under investigation, however, ways must be found of both examining this hypothesis and helping the child to construct the kind of knowledge framework which is necessary to a correct handling of the problems in question. This may be done by means of the teaching experiment, by which the child's interaction with each instructional step is monitored as closely as possible as the treatment proceeds. The procedural model for the teaching phase is based on good teaching practice and 'cognitive instruction' as defined by Belmont and Butterfield (1977). In this model the child's thought processes and the use made of the instruction are monitored as the treatment progresses. The experimenter must observe as directly as possible how the child is thinking while performing a criterion task, having identified the nature of successful reasoning on that task. The important feature of this model is that the researcher's task is to help the child to build up a particular cognitive framework.

The overall methodology thus adopted involved the interviewing of a large number of students followed by instruction of brief duration, and was addressed primarily to pupils making the errors in question, rather than to a selection of different ability groupings.

The products of the research were anticipated to be of two kinds, namely an improved understanding of the difficulties children have in attempting to learn the various topics under investigation, and an actual teaching programme in each topic designed to alleviate some of these difficulties which could be used as it stands, but which would preferably form the basis for future curriculum development work in that topic.

The approach adopted by the team was thus to divide the research programme into three main sections:

(1) an investigation into the causes of the errors under study by means of individual interviews with children identified as making the errors,
(2) the conduct of small-scale teaching experiments based on this analysis, and
(3) the development of prototype teaching modules for trial with whole classes.

Form of the report: SESM fractions

This report describes the investigation into children's difficulties with fractions. The research comprised three major stages, which are each described here. The first was the use of a series of individual interviews, described in apters 2 and 3, followed by the construction of a small-scale teaching modu apter 4) which was further tested (Chapter 5). The report concludes with cussion of the findings and their implications for the teaching of fractions.

2

SESM interviews (Phase One)

The aim of the next stage of the research was to gain more information on the way in which children think of fractions and, in particular, to investigate further three specific aspects which emerged from consideration of the CSMS results. These were:

(1) to see whether children were able to think of fractions as numbers, or whether they think that the word 'number' implies only 'whole number';
(2) to discover what models of a fraction the children had at their disposal;
(3) to determine how the children viewed the idea of equivalence.

The method chosen consisted of a series of individual interviews with children who displayed some insecurity with fractions. It was important for the interviews to be carried out in a relaxed manner, in which the child could feel confident that his opinions would be treated with respect. However, it was sometimes helpful to introduce an element of conflict into the interview, in the sense that the children might be encouraged to be aware of any inconsistencies in their argument, or to respond to an alternative argument. When this was the case, it remained essential that the child did not feel under attack, and that the friendly atmosphere was maintained.

While it was desirable that the interviews should be kept as flexible as possible so that any interesting contribution could be explored, it was felt necessary to have a basic structure so that general observations could be made. For the first phase of the interviewing process, the structure consisted of a fairly large number of opening questions, with the intention that the more useful ones could be examined further at a second phase.

The interview sample

Twenty-three children were interviewed at this stage. As in the case of the CSMS research, the pupils were all aged 12 to 14 years. They attended a mixed comprehensive school, and came from three middle ability classes which consisted of pupils who were likely to be entered for a CSE in mathematics in due course. In the case of the first group the whole class was given the CSMS test on fractions in order that a group of children who were making errors could be identified. However, as most of the children made many errors, the group was selected randomly, and it seemed unnecessary to test the other two groups. Instead, their teachers were asked to nominate children who would be likely to be willing to be interviewed. There were 12 boys and 11 girls interviewed. The interviews all took place individually in a separate room, so that it was as quiet and free from interruption as is possible in a large school. The atmosphere was kept informal, and every attempt was made to make the child feel at ease. It was obvious that the children enjoyed the experience of talking individually. The interviews were tape-recorded, with the children's permission, and lasted about 30 minutes.

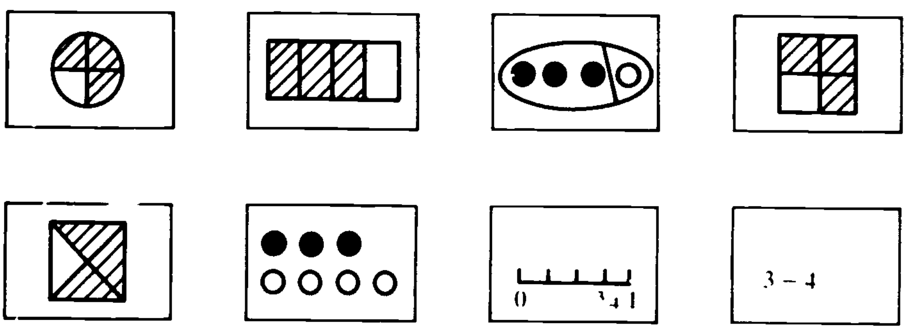
The structure of the interviews

The interview questions related to the three aspects of models of fractions, fractions as numbers, and equivalent fractions. There were three or four questions concerning each aspect, and they were given orally, although some referred to diagrams that were presented as a set of cards. Although questions concerning the three aspects were distributed throughout the interview, they will be considered separately for the purposes of description. The actual interview schedule can be found in Appendix 1; the questions are grouped differently here for the purpose of analysis.

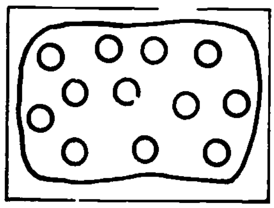
Models of fractions

There were three questions that attempted to discover what models of fractions the children were familiar with. These were:

1. How would you explain to someone, who didn't know, what a fraction is?
2. Which of these cards would help someone to understand what the fraction 3/4 is?



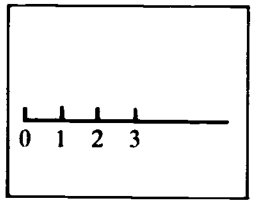
3. How would you find 3/4 of this collection of counters?



Fractions as numbers

The three questions in this group attempted to establish whether the children were able to accept that fractions are numbers:

4. Where would the number 4 go on this number line? And the number 2/3? And the number 1/2?



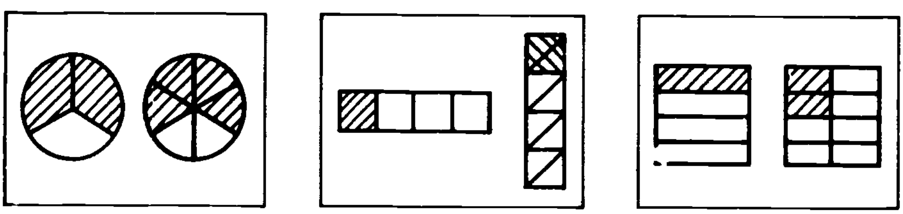
5. How many numbers are there between 2 and 3? And between 0 and 1?

6. Two numbers add up to 10: What could the numbers be?

Equivalent fractions

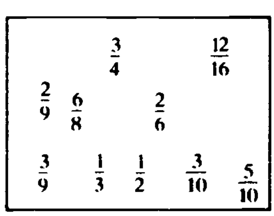
There were four questions about the equivalence of fractions, presented in a variety of forms:

7. What could we find out from these cards?



8. Two boys have equal amounts of pocket money. One decides to save 1/4 of of his pocket money, the other decides to save 5/20 of his pocket money. Is 5/20 more than 1/4, is 1/4 more than 5/20 or are 5/20 and 1/4 equal?

9. Can you find some fractions that are the same in this collection?



10. Would you try these: 2/3 + 3/4; 3/8 + 2/8; 1/10 + 3/5? Would you explain what you did?

Results of phase one interviews

The results make use of extracts from the transcripts made during the interviews. The children are referred to throughout by two initials – e.g. KH – while the interviewer is denoted by I.

Models of fractions

1. How would you explain to someone, who didn't know, what a fraction is?

The responses given by the children included references to 'parts of a whole', 'part of a number' and 'two numbers, one over the other'. The number of responses in each category is shown in Table 2.1.

Table 2.1: Children's definition of a fraction

Choice of model	*Number of children*
Part of a whole	10
Part of a number	3
One number over another	6
Don't know or couldn't say	4

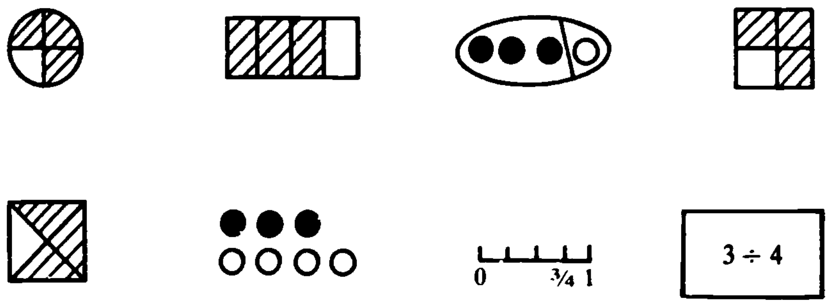
The 'wholes' in the first category ranged from actual examples like a cake, or pie or an orange, to rather vague 'whole ones' or 'whole things' or 'it', as in the case of HW who said 'I'd say it's like splitting it up into four, and giving them three pieces.'

Examples from the third category include:

VK: 'A smaller number on top of a bigger number.'
PM: 'I'd probably say it was two numbers with a line through the middle.'
GP: 'If someone asked me what 3/4 was, I'd say 3 and 4 with a dash in the middle.'
TH: 'I would tell them it was a number below 1, and it would be something like 3/4, which would be a 3 on top of a 4.'

It can be seen that many of the responses consisted in giving instructions for a procedure – either of splitting up a whole, or writing one number over another. Some children found it difficult to produce any explanation, and as it was the opening question, it was not pursued further. But it does illustrate the fact that the only model produced was the 'part of a whole' one, although the response 'part of a number' might suggest that some of those children see a connection between fractions and numbers. Those who said 'one number over another number' show that they recognize the way fractions are written, but do not give any evidence as to whether they understand what they represent.

2. Which of these cards would help someone to understand what the fraction 3/4 is?



The geometric models were accepted by all the children. The others were less popular, and were often strongly rejected. The results are shown in Table 2.2.

Table 2.2: Children's choice of models of the fraction 3/4

Model	*Accepted by*	*Rejected by*
	23	0
	15	8
	3	15
0 1	19	4
3 ÷ 4	3	20

Some of the reasons for accepting or rejecting a model are described below:

Model: 

Those who accepted this version of 3/4 usually explained it in terms of 'four things of which three are taken', e.g.:

JC: 'Yes, the blue one is to tell you that one's separate.'
KP: 'Yes, that blue don't count.'
SB: 'Yes, that [the blue] would be the one that is left.'

However, others rejected it because there was no 'whole' of which to take a part, e.g.:

CF: 'No, it wouldn't be right, it's not a whole thing.'
HH: 'No, that's just got circles. I don't think it would be any help.'

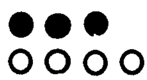
Two children thought it was a fraction, but not 3/4:

TH: 'No, that would only be 1/3.'
MP: 'No, they'd think it was 1/4.'

One child thought this the best model, although his enthusiasm for it appeared to be qualified:

BW: 'Yes, I think that one's best. That's good. That's the way we had it in the Juniors. I didn't do very well in the Juniors.'(!)

Model:



Only eight of the 23 children accepted this. Again, some rejected it because of the absence of a 'whole', e.g.:

MT: 'No, it's got no shape.'
TE: 'No, there's three of those and four of those.'

Like TE, others could only see it as seven circles:

GB: 'No, there are seven.'
SB: 'No, I don't like that, 'cos it looks like a 7.'

Even those who accepted the model were rather reluctant:

GP: 'Well, if they were representing quarters it would. It's all right, as long as you know how it's explained – it's quarters.'
HW: 'That'd be more difficult – they'd get the two sets muddled up. You've got the four in a row, and the three red ones are just like shading them in, to show how much you are left.'

Model:

Although 19 of the 23 children liked this model, it became apparent that it was not a very useful illustration, as most of the children treated it as another geometric model, seeing the line as a 'whole', whereas the intention was to see if the children thought of 3/4 as a point on a number line. For example:

GB: 'Yes, it's four divisions, and that's the third.'
TE: 'Yes, there's three parts here and one part here.'
BW: 'Yes, it's 4 cm take 3 cm.'
KP: 'Yes, 'cos there's four along there, and that's three.'
TH: 'Yes, that would be 1/4, 2/4, 3/4.'

It was rejected by four children for reasons such as:

MT: 'No, I wouldn't be able to do that one.'
CF: 'No, I don't like that.'
CP: 'No, 'cos there's five of them.'
T: 'So do you think that's the right place for 3/4?'
CP: 'No, I'd put it about there [between 1/2 and 3/4].'

Model: $3 \div 4$

In many ways this was the most interesting example. Only one child immediately saw the connection between $3 \div 4$ and 3/4, while two more eventually talked their way round to the idea. The other 20 children firmly rejected the idea that there was any connection, often laughing at the absurdity of the suggestion! However, when they were asked to explain the meaning of $3 \div 4$, many interpretations emerged.

Thirteen of the children proceeded to attempt to divide 3 into 4, although some of them had actually said that it was '3 shared by 4' or even '3 divided by 4'. For example:

KP: '3 shared by 4 . . . that's 3s into 4 . . . 1 remainder 1.'
CD: '3 divided by 4 . . . 1 remainder 1. 3 goes into 4 once, and there's 1 left.'
GB: '3 divided by 4 . . . I don't know . . . 3 goes into 4, 1 left over.'
TE: '3 divided by 4 . . . 3 goes into 4 once, 1 over.'
MT: '3 shared by 4, 3s into 4, isn't it?' (Writes $3 \div 4$)

These responses were often characterized by a pause between the initial reading and the restatement, suggesting that, realizing they could not deal with the division the first way round, they deliberately reversed it. Other children reversed the order immediately. For example:

VC: '3s into 4. 3s go into 4 one, and 1 remainder.'
GP: '3s into 4. One remainder, 1 over 3 . . . or 4.'
JC: '3 divided into 4. There's 1 left. [Writes $3 \div 4 = 1$ r. 1.] 3⅓, I think. Something like that. No, 1⅓ . . . or something.'

Three other children read the division the correct way, but thought that it was impossible to divide the smaller number by the larger:

TH: '3 shared by 4. You can't do that. 4 is bigger than 3.'
SB: '3 shared by 4. You couldn't do that. Well, it wouldn't . . .'

The intention was to present those children who saw no connection between $3 \div 4$ and 3/4 with the task of sharing three cakes between four people, to see whether the connection then became obvious. However, because of the ambiguity in the way many children read $3 \div 4$, it was first necessary to establish which way round they thought it was, so that the illustration of the smaller number divided by the larger could be presented to them. Where this ambiguity existed, the children were asked to consider the pair $12 \div 4$ and $4 \div 12$.

It emerged that some children appeared to think that division is commutative.

KP, for example, said of 12 ÷ 4 and 4 ÷ 12 that 'they're exactly the same', while MT also thought it acceptable to reverse the order:

MT: (Points to 4 ÷ 12) 'That's the wrong way round . . . 12 won't go into 4.'
I: 'So what do you think we should do about it?'
MT: 'Change it round – 12 divided by 4.'
I: 'Is it all right to choose which way you do it?'
MT: 'Yes, because you'd get that one otherwise – 12 into 4, and that's the wrong way.'

Other children realized that they were different, but thought that one could not be done, such as:

JC: (Points to 12 ÷ 4) 'That one you can't do, 'cos 12 is bigger than 4.'
VK: (Points to 4 ÷ 12) 'That's a 3. [Points to 12 ÷ 4] You can't share 12 by 4.'
MT: '4 shared by 12. 4s into 12. But 12 shared by 4, you can't do, as 12 is bigger than 4.'

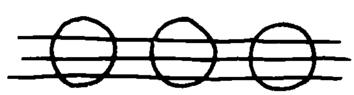
GP had an ingenious way round the problem:

GP: (Points to 12 ÷ 4) 'That's 12s into 4 – you can't do. I'm not sure what you do. (Points to 4 ÷ 12) That's 4s into 12 . . . goes three. Um . . . 12s into 4 . . . you borrow from the 12, I think, and it turns into 14. That'd be 12s into 14 go once, remainder 2. 2 over 14. Yes, 12 ÷ 4 is 1 and 2 over 14.'

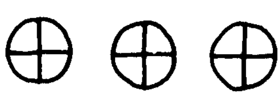
The problem of interpreting the division sign only became apparent after interviewing the first five children. For these, their reasons for accepting or rejecting 3 ÷ 4 as a model for 3/4 were not investigated. Once the problem was recognized, it became necessary first to find out which way the children were interpreting the division sign, and then to use their version of 3 divided by 4. Then, those who still maintained there was no connection with the fraction 3/4 were presented with the following task:

> Here are three cakes. There are four children who all want a fair share. How would you do that?

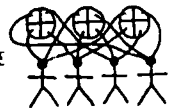
The intention was to see if the process of carrying out the sharing would enable the children to see the connection between 3 ÷ 4 and 3/4. Only one child (KP) had difficulty with the task itself, and used the following method of sharing:



When asked if all the pieces were the same, he said. 'The drawing's not very accurate.' He could not think of another way of doing it, but when reminded of Card 1 (), he then drew:



Eleven children quickly saw the connection as soon as they had carried out the sharing. For example:

SB: (Draws )

SB: 'If you had it like that. Each . . . each person gets 3/4.'
I: 'Would you look at this [3 ÷ 4] again?'
SB: 'Yes, that's three cakes, and four children, and everyone gets 3/4.'
I: 'So 3 ÷ 4 is . . . ?'
SB: '3/4. I couldn't do it before, but thinking of it like that, I see it now.'

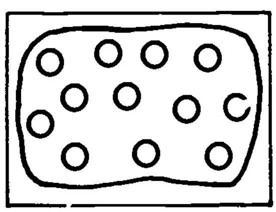
GP: (Draws )

GP: 'Well, you could use two cakes, divide them into halves. That would be half a cake each. Have you got to use all three?'
I: 'I think they're going to want to, don't you?'

GP: 'Yes.' (Draws )

GP: 'After they've had their halves, they could have 1/4 each of the other one.'
I: 'Fine, so everyone has had . . . ?'
GP: '3/4 of a cake.'
I: '. . . Would you like to look at this [3 ÷ 4] again? Does it describe what we have done?'
GP: 'I suppose it is, really.'

3. How would you find 3/4 of this collection of counters?



This was a question taken from the CSMS test. All the children interviewed were able to do this without any difficulty.

Fractions as numbers

There were three questions in the interview schedule concerning fractions as numbers. These concerned the use of a number line, the existence of numbers between 2 and 3 and the use of fractions in finding pairs of numbers that add up to 10.

4. Where would the number 4 go on this number line? And the number 2/3? And the number 1/2?

Only one (HW) of the 15 children asked the question was able to place the fraction 2/3 correctly. Thirteen of the others placed it at 2, which was, of course, two-thirds along the line, while the remaining child put it between the 2 and the 3.

Table 2.3: Response to number line question

	Fraction 2/3		Fraction 1/2
Placed at 2/3	1	Placed at 1/2	3
Placed at 2	13	Placed at 1½	9
Placed at 2–3	1	Placed at 2	3

When asked to position 1/2, some placed it correctly, some at 1½, the half-way point, and others at 2, the same as they had chosen for 2/3. The numbers of children in each category are shown in Table 2.3.

Clearly, most of the children interpreted the question as 'find the point two-thirds the way along the line'. The confusion is illustrated by the extracts from the interviews with MT and GB:

I: 'Where would the number 4 go on the line?'
MT: 'Here'. (Points to 4
I: 'And where would the number 2/3 go?'
MT: (Points to just beyond the 2 – judging two-thirds of the distance on the line)
I: 'What about 1/2?'
MT: (Points to correct point)
I: 'And what about 2/3?'
MT: 'Well, would it be just past the 1?'

Here she seems to have used what she knew about the position for 1/2 to wonder whether her position for 2/3 should move to the left:

I: 'Would you show me where the number 4 would go on this number line?'
GB: (Places 4 correctly)
I: 'And where would the number 2/3 go?'
GB: (Points to the 2)
I: 'And where would the number 1/2 go?'
GB: (Points to half way between 1 and 2)
I: 'Are there any numbers between these two – 0 and 1?'
GB: '. . . A tenth?'
I: 'Fine. What about the point in the middle?'
GB: 'That would be five tenths.'
I: 'So the number one-half would go . . . ?'
GB: 'What, one?'
I: 'No, just one-half.'
GB: 'There' (Points to correct place)
I: 'So the number 2/3 would go . . . ?'
GB: 'There.' (Points to the 2 again)

The invitation to consider the existence of numbers between 0 and 1 helped GB to place the fraction one-half correctly, as he thought out the difference between 'half-way' and 'one-half'. It had been thought that, the fraction 1/2 being more familiar than 2/3, the request to place 1/2 would help the children to place 2/3 correctly, but this was certainly not the case for GB.

The confusion between measuring off a fraction of the line and placing a point that represents a fraction was present in 14 of the 15 children in the case of the fraction 1/2, and in 12 of the 15 children for the fraction 2/3. This appears to be another example of the children perceiving the line as a 'whole' of which the fraction is a part. The problem only became apparent during the course of the interviews, and further information was seen to be required from the next set of interviews.

5. How many numbers are there between 2 and 3? And between 0 and 1?

Seven children were asked how many numbers they thought there were between the numbers 2 and 3. BW and HW both had the idea that there were a very large number. HW said, 'as many as a thousand or more – it depends how big the line is.' Four others gave small finite numbers – 2 or 3, 4, 6 and 25. HW and RW both said there were none:

I: 'How many numbers are there between 2 and 5?'
HH: 'In the middle? Just two, 3 and 4.'
I: 'What about 3½?'
HH: 'Well, if you say 3½, you'd have to have another half to make one whole one.'
I: 'So is 3½ a number between 3 and 4?'
HH: 'Well, I don't think so.'
I: 'What about between 1 and 2 – would 1½ be a number between 1 and 2?'
HH: 'Um. . . . It could be, but I don't really think so.'

HH was confident at finding numbers between 2 and 5, but rejected the idea that any existed between 2 and 3. She appears to have thought that the word 'number' could only be applied to whole numbers.

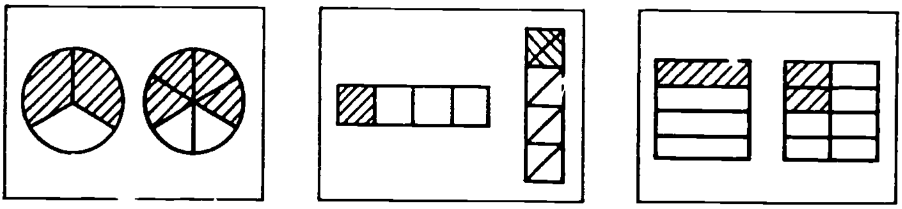
6. Two numbers add up to 10: What could the numbers be?

Because of shortage of time, only eight children were asked this question. Two of them actually suggested some pairs that included fractions – 9½ and 1/2, for example. When the others were asked if they could also use fractions, three then found some, but two (HH and LM) rejected the idea. LM tried '1/5 + 1/5', but then said 'no'. The sixth child, JE, tried '1/6 + 1/4', which she said did add to 10. Unfortunately, insufficient time was allowed in the interviews to explore this aspect of fractions, and there was not enough information to draw any firm conclusions. Seven children were asked all three questions, and of these, only one, HW, correctly answered all three. BW, who said there were 'loads' of fractions between 2 and 3, and also accepted fractions adding up to 10, could not place either 1/2 or 2/3 on the number line. The other five could all be said to have found it difficult to think of fractions as numbers. It was considered important to obtain more information on this aspect in the second stage of the interview programme.

Equivalent fractions

There were three questions on the interview schedule that concerned equivalence directly, and also some addition questions that, of course, made implicit use of equivalence.

7. What could we find out from these cards?



When shown the diagrams, all the children readily produced some equivalent fractions, and appeared very familiar with the idea. Two of them explained that if some of the lines were removed from one diagram they would look the same. Two observed that they were the same, but one had smaller pieces, e.g.:

KP: (Who saw the two diagrams as 'cakes') 'This one is 2/3, and that's 4/6. Four pieces of cake . . . you'd have two bigger pieces, which is the same, but these are cut up more.'

Three made reference to cancelling, e.g.:

MT: 'That's 1/3, and there's six on there . . . 2/6. They're both the same. You go "2s into 2 goes 1, and 2s into 6 goes 3".'

or

LC: 'The two orange ones are 2/3, and that's 4/6. You cancel 4/6 down to get 2/3.'

8. Two boys have equal amounts of pocket money. One decides to save 1/4 of his pocket money, the other decides to save 5/20 of his pocket money. Is 5/20 more than 1/4, is 1/4 more than 5/20 or are 5/20 and 1/4 equal?

Of the 17 asked the question, 13 said they were the same, while the remaining four said they were not the same. Some of the reasons for this response were:

JE: '1/4 is bigger than 5/20.'
I: 'Could you explain why?'
JE: ''Cos at our old school, they said the lower the number looks, the more it is.'
I: 'If I asked you to compare 1/3 with 1/5, which is bigger?'
JE: '1/3.'
I: 'And if I asked you to compare 1/3 with 4/5?'
JE: '. . . 1/3 again.'
I: 'If you ate 1/3 of a cake . . . which fraction is bigger?'
JE: '4/5.'
I: 'So you've changed your mind? You said that the one with the smaller number at the bottom was the bigger.'
JE: 'Because you get 1/4 left with that one . . . no, that one's got 1/5 left over, and that's got 2/3 left over.'
I: 'Now, back here, you said 1/4 is bigger than 5/20. What do you think now?'
JE: 'Yes. It is.'

HH: '5/20 is more than 1/4.'
I: 'Right. Why do you think that?'
HH: 'Because you'd draw a diagram into 20, and shade in five . . . that would be more than 1/4.'
I: 'Would you like to try that?'

HH: (Draws )

I: 'And you think there are more than 1/4?'
HH: 'Um . . . no . . . I think 1/4 might be bigger. Hard to tell.'

The four children who thought that 1/4 and 5/20 were not the same when applied to an amount of money had all recognized equivalence in the geometric diagrams, suggesting that as soon as one gets away from the 'part of a whole' model again, equivalence becomes more difficult to understand.

9. Can you find some fractions that are the same in this collection?

$\frac{3}{4}$ $\frac{12}{16}$

$\frac{2}{9}$ $\frac{6}{8}$ $\frac{2}{6}$

$\frac{3}{9}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{3}{10}$ $\frac{5}{10}$

Sixteen of the 17 who were asked the question were able to pick out the set 3/4, 6/8 and 12/16.

10. Would you try these: 2/3 + 3/4; 3/8 + 2/8; 1/10 + 3/5? Would you explain what you did?

The answers given by the children are shown in Table 2.4.

Table 2.4: Responses to addition items

Question	Number interviewed	Response	Number of children
2/3 + 3/4	23	1 5/12	5
		5/7	9
		1	3
3/8 + 2/8	19	5/8	13
		5/16	3
1/10 + 3/5	16	7/10	3

Other answers to 2/3 + 3/4: 7/12; 5/4; 2/3; 9/12; 17/24
Other answers to 1/10 + 3/5: 4/10; 4/15; 4/5; 2/10, 8/5; 2

Much useful information was gained about their understanding of equivalence both from those children who were able to perform the addition correctly and those who were not, by asking for an explanation of how their answers were obtained. Of the five children with the correct answer to 2/3 + 3/4, none was able to give any explanation of why they used the denominator 12. Typical responses were:

BW: '2/3 + 3/4 = 8/12 + 9/12 = 1 5/12.'
I: 'Would you explain how you did that?'
BW: 'I said, "What's the next thing 3 will go into?" . . . that's 12. Then, "What do you do to 3 to make 12?" . . . Times it by 4. Two 4s are 8. "What do you do to 4 to make 12?" . . . Times it by 3. Three 3s are 9. Add the two together.'
I: 'Why did you do this bit with the twelves?'
BW: '. . . I'm not sure.'
I: 'It's difficult to explain?'
BW: 'You're taught something, you're never taught why.'

BW is clearly repeating a routine that he has used many times, and one can hear the dialogue between the prompt of 'What do you do to . . .' and the reply 'Times it by . . .'. For example:

CF: '2/3 + 3/4 = 8/12 + 9/12 = 17/12 = 1 5/12.'
I: 'Would you tell me how you did it?'
CF: 'I saw what the 3 and the 4 both go into; that would be 12. Then I saw how many times 3 goes into 12 . . . four times . . . and timesed the top number by the bottom number. That's four 2s are 8. Then I saw how many times the 4 went into the 12, that's three, and three 3s are 9. Then I added it up, and 17/12 is more than a whole one, so it's 1 5/12.'
I: 'Would you tell me why you used the 12?'
CF: 'Um . . . I think it's because, if you had the bottom number divided, you'd have to have the top number timesed, otherwise it wouldn't be right.'
I: 'So you said . . . ?'
CF: '3 goes into 12 four . . . and four 2s are 8.'
I: 'And why did you do that?'
CF: '. . . I don't really know. It's just the way I was taught.'
I: 'Suppose I said "2 and 3 is 5, and 3 and 4 is 7, so the answer is 5/7"?'
CF: 'I was told you couldn't do it that way. I used to do it that way, and I was told . . . I used to do them that way, but I got them wrong, and then I was shown how to do them properly.'
I: 'So what's wrong with this way, then?'
CF: 'Well, 5 and 7 make 12 . . . I'd say you got it wrong, because the 12 should be underneath, if you did it properly.'
I: 'And why have I got it wrong?'
CF: '. . . I don't really know why you've got it wrong.'

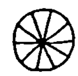
Here, CF sees her taught method just as a means of obtaining the right answer, and has no idea of why the 'adding numerators and denominators' method, which one feels she really still prefers, is not appropriate.

It is certainly not wise to assume that ability to add fractions implies an understanding of equivalence.

Some of the children who did not obtain the correct answer appeared to try to use a common denominator:

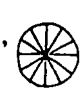
GB: '2/3 + 3/4 . . . Oh dear! I can't remember how to do it. I think you just add the two top ones and the two bottom ones . . . No, I'm not completely sure how you do it. Oh, you've got to find a common denominator, I think . . .'
I: 'Would you have a go at that, then?'
GB: 'Well, 12 would be the most likely one. But I can't remember what you do to the top . . . I think you have to turn one of them upside down. 2/3 + 4/3 = 2. I think that's the way you do it . . . But I think you've got to find a common denominator No, that's with timesing, I think.'

GB does not link the idea of a common denominator with that of equivalence. LC, on the other hand, realizes that she needs equivalent fractions, but has difficulty in finding them:

LC: (Writes 1/10 + 3/5) 'The easiest way to do it is to make that [1/10] into fifths. 2/5 is the same as 1/10, so if we add 3/5, we make a whole one. 2/5 + 3/5 = 1.'
I: 'Right. What do we think about that?'
LC: 'I think that's right.'
I: 'Could we use this drawing ()?'
LC: '1/10 is the same as 2/5, so I think that's right.'
I: 'You're sure?'
LC: 'Oh, hang on . . . no, it's not. No, 3/5 and 1/10 would be . . . Can't be, it's smaller. Unless you make it into halves . . . 2/15, because 5 and 10 will go into 15 . . . oh, I've done it wrong, 5 and 10 will go into 20 . . . (Writes 1/10 + 3/5 = 12/20 + 2/20 = 14/20) . . . 7/10.'

So we find children who can add fractions but who don't see the need for equivalence, and children who can't but who do seem to have some understanding of equivalence.

Since the children interviewed all found the geometric instances of equivalence easy to interpret, it seemed sensible to present some of the children who had incorrect answers with a diagram of the fractions to be added, in order to see if they could, at least, see the inconsistency of their result. Some examples of the use of this strategy are:

PM: (Writes 2/3 + 3/4 = 4/12 + 3/12 = 7/12)
I: 'If you had 2/3 of a cake, and added an extra 3/4, you'd expect to get 7/12?'
PM: 'I think so.'
(Pause)
I: 'If I draw this circle and divide it into 12 pieces . . .' 
PM: (Shades in four pieces)
I: 'You've shaded in four pieces. Do you think that's the same as 2/3?'
PM: '. . . Looks like 1/3 . . . 2/3 would be 8 . . . 8/12. So that's wrong. . . . 3/4 is 9/12 . . . 1 and 5/12.'

GB: (Writes 1/10 + 3/5 = 2/5 + 3/5 = 1)
I: 'Do you think that's all right? You have added 1/10 and 3/5 and got 1?'
GB: 'Yes, I think so.'

I: 'Could we have a drawing of 3/5?'

GB: (Draws )

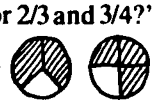
I: 'And now we're adding on 1/10.'
GB: 'That's 7/10.'
I: 'Does that help with the other one, 2/3 + 3/4?'
GB: 'You have to find something that's the same as 2/3 but a bigger number. Um . . . I can't find one.'

GB illustrates the fact that the use of a diagram for addition is helpful if one knows the common denominator, but it is of no help in finding one. JE illustrates the same problem:

JE: (Writes 2/3 + 3/4 = 5/7)
I: 'Do you think that's a reasonable answer?'
JE: '. . . You could cancel it down. . . . No.'
I: 'Could you use a drawing to help?'

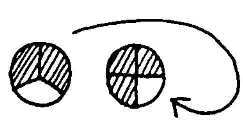
JE: (Draws )

I: 'And for 2/3 and 3/4?'

JE: (Draws )

I: 'So does that seem all right?'
JE: '. . . It should be a whole one, and a little bit over.'
I: 'That would be a better answer?'
JE: 'Yes.'

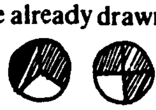
VK suggested drawing a diagram for herself:

VK: (Writes 2/3 + 3/4 = 2/1 = 1/2)
I: 'Does it matter which – 1/2 or 2/1?'
VK: 'No. . . . Yes, it should be 2.'
I: 'Is 2 a sensible answer?'
VK: 'I don't know. Can I draw it?' (Draws )

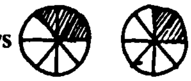
VK: 'No, because if I put this shape in here, it would be 1 and a bit. It'd be smaller than 2.'

Other children used the diagram to confirm their errors:

MT: (Writes 2/3 + 3/4 = 5/7)
I: 'You've already drawn a diagram for 3/4. Would you try one for 2/3?'

MT: (Draws )

MT: 'There's three shaded in there, and two shaded in there. . . . That's five and there are seven parts altogether.'
I: 'Would you try this – 3/8 + 2/8?'
MT: '5/16.'
I: 'Let's draw a diagram.'

MT: (Draws )

MT: 'Five shaded, 16 altogether.'

JC: (Writes 2/3 + 3/4 = 5/7)
I: 'Do you think that's a sensible answer?'
JC: 'Yes. If it was the other way round it would be top-heavy.'
I: 'We have a picture for 3/4 already. Could you draw one for 2/3?'

JC: (Draws )

I: 'So we have 3/4 and then we add 2/3 on.'
JC: 'Yes. There are seven pieces, and there are five shaded in, two and three.'
I: 'O.K. Could we look at this one? – 3/8 + 2/8?'
JC: '5/8. They're all eighths.'
I: 'Could we have a drawing for that?'

JC: (Draws )

I: 'Do you think that is the same sort of drawing that you used for that one – 2/3 + 3/4?'
JC: 'No, 'cos they're different sizes . . . on here the shape is quarters, and here they're thirds and they're bigger.'
I: 'So you think 5/7 is right?'
JC: 'Yes. I think so.'

The interview questions on equivalence suggest that the geometric illustration of simple equivalence was well understood, and 13 of the 17 children were able to say that 1/4 and 5/20 are the same. Sixteen of the 17 were also able to say that 3/4 and 12/16 are the same or equivalent when seen in symbolic form only. However, no child was able to explain the use of the idea of equivalence in adding fractions with different denominators. The lack of other examples of the application of equivalence was noted, and more will be needed for the next stage of interviews.

Other observations on the addition of fractions

Many idiosyncratic methods of adding fractions were observed, among them the following:

MT: $\frac{{}^{1}\cancel{2}}{\cancel{3}_{1}}+\frac{\cancel{3}^{1}}{\cancel{4}_{2}}=\frac{2}{3}$

KP: $\frac{{}^{1}\cancel{2}}{3}+\frac{3}{\cancel{4}_{2}}=\frac{4\ +\ 5}{12}=\frac{9}{12}$

LC: $\frac{2}{3}+\frac{3}{4}=\frac{6\ +\ 6}{12}=1$

HW: $\frac{2}{3}+\frac{3}{4}=\frac{8\ +\ 4}{12}=\frac{12}{12}$

CD: $\frac{2}{3}+\frac{3}{4}=\frac{5}{3}\text{ or }\frac{5}{4}$

GB: $\frac{2}{3}+\frac{3}{4}=\frac{2}{3}+\frac{4}{3}=\frac{6}{3}=2$

Some children adopted different strategies for the three additions 2/3 + 3/4; 3/8 + 2/8 and 1/10 + 3/5. Three children added denominators and numerators consistently, while the others that did so for 2/3 + 3/4 completed 3/8 + 2/8 successfully. For 1/10 + 3/5, they gave 7/10, 4/10, either 4/5 or 4/10, either 2/10 or 4/10 and either 4/10 or 4/15. Many of the children appealed to general precepts that they seem to have acquired. Some of them are true, as in (for 3/8 + 2/8) 'Because the bottoms are the same, they stay the same' or 'The bottom number stays the same.'

For 2/3 + 3/4 the rule becomes 'You've got to try to change it' or 'You've got to find a common denominator' or 'You have to find what both 3 and 4 go into.' Sometimes the rule gets confused, as in:

MT: (Writes $\frac{2}{3}+\frac{3}{4}=\frac{{}^{1}\cancel{2}}{{}_{1}\cancel{3}}+\frac{\cancel{3}^{1}}{\cancel{4}_{2}}=\frac{2}{3}$) 'Find the common denominator between those two [2 + 1].'

This was also noticeable earlier when the children were talking about division – 12 ÷ 4: 'You can't do it because you can't divide by the bigger number' and 'You always divide by the lower number.' Some children seemed to think that there was more than one possible method, e.g.:

LC: 'There are several different methods, so as I got it wrong with the plus, I think I should do it like this.'

MT: 'Find the common denominator . . . Sometimes you do it that way.'

It seemed to be a case of trying various methods until the correct answer emerged.

This concludes the major findings from the first stage of the interviews. The remaining questions were asked of so few children that they are not worth reporting here. The conclusions so far are summarized below.

Summary of findings from phase one interviews

1. Models

The only model of a fraction that was widely accepted was that of the geometric 'part of a whole'. Not only was it the only universally accepted model of 3/4, but children referred to parts of circles or parts of cakes when trying to explain other problems during the course of the interviews, such as the addition of fractions, or whether 2/3 was bigger or smaller than 3/4.

The main problem with placing the fraction 2/3 on the number line seemed to lie in the children's insistence on seeing it as two-thirds of the way along the line, making the line a 'whole' to be divided.

However, it seemed relatively easy to extend the children's view of fractions to include the division aspect when a concrete example was presented to them.

In considering modifications in the questions for use in the next interviews, it was thought worth changing the choice of the fraction 3/4 for the items relating to models, to see whether a less familiar fraction would produce different results.

2. Fractions as numbers

Most children found it difficult to think of fractions as numbers, particularly when asked to place them on a number line. It was felt worth repeating the number line question with different fractions in the next set of interviews, including a fraction greater than one in the fractions to be plotted.

The need for more questions that explore this concept of fractions as numbers was also noted.

3. Equivalent fractions

The children were able to recognize equivalent fractions when presented in geometric 'part of a whole' form, but were less successful when the fractions were presented in symbolic form, and no child showed evidence of an understanding of the application of the idea of equivalence to the addition of fractions. It was felt that the interviews did not really elicit what the children understood of the concept of equivalence. Most of the questions were answered in terms of cancelling or multiplying numerators and denominators, and it was not clear whether the children realized why this resulted in equivalent fractions. It seemed important that more time should be devoted to trying to discover more of the children's understanding of the relation between pairs of equivalent fractions. It would also be helpful to have examples other than addition in which the idea of equivalence is needed to solve problems.

Conclusions from interviews: elaboration of initial hypotheses

The results of the first stage of the interviews suggest a general hypothesis that the problems children have with fractions are due to their restricted view of a fraction. In particular, the children interviewed appeared to find it difficult to think of a fraction as other than a part of a whole. It seems likely that this identification of a fraction as part of a shape makes it difficult to make any sense of addition or of placing a fraction on a number line. Similarly, although the diagrammatic approach may well assist in the identification of equivalent fractions, it does not necessarily help with the algorithm for finding a fraction equivalent to any given fraction. The second phase interviews are used to elaborate these ideas.

3

SESM interviews (Phase Two)

Although the first phase interviews produced much useful information, there were certain aspects which warranted further investigation.

Description of interview sample

Fourteen children were interviewed, all 13-year-olds from two comprehensive schools, one an inner city urban school, and the other on the outer edge of a city.

The children were again in middle ability groups in their schools, likely to take a CSE in mathematics in due course. The arrangements for the interviews were just as before, with the children having individual interviews that lasted about 30 to 40 minutes.

Design of interview schedule

A number of changes were made in the interview schedule. For example, the questions used to investigate the children's image of fractions all referred to the fraction 3/4, which is, of course, a familiar fraction, so it was felt necessary to repeat the questions with a different fraction to see if the responses were noticeably different. In the first set of interviews, the geometric illustrations of fractions were all of fractions less than 1: illustrations of fractions greater than 1 should appear in the second phase of interviews.

Some problems that emerged during the interviews, such as the children's difficulty in interpreting $3 \div 4$, had not been anticipated, as a result of which a number of useful lines of inquiry were not used with the children interviewed early on and could be written in to the next interview schedule. This was particularly true of the questions on equivalence, which did not appear to allow for the difficulties experienced by the children to be adequately investigated.

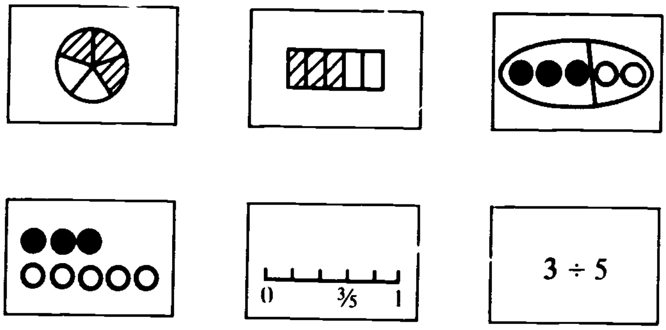
Necessary modifications to the questions on fractions as numbers include changing the fractions on the number line by substituting fifths for quarters and adding a fraction greater than 1. The item concerning the existence of numbers between 1 and 2 required rewriting, as, in its original form, the children were not given sufficient opportunity to say there were none at all.

The complete schedule for the next set of interviews can be found in Appendix 2, and the individual questions are now described. The schedule is only a guide, and the questions are grouped and re-numbered for convenience.

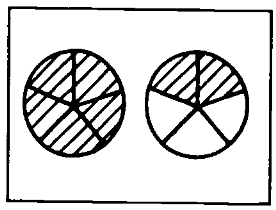
Models of fractions

The first question concerning the children's recognition of models of fractions was very similar to the corresponding one in the earlier interviews, but the fraction 3/4 was replaced by 3/5. Only two geometric examples were kept, as they presented no difficulty.

1. Which of these cards would help someone who didn't know what the fraction 3/5 is?

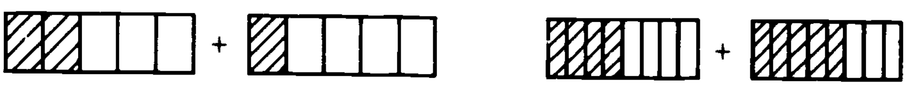


2. Here are 3 cakes. Could you share them equally between 5 people? Do you see any connection between what you have done and 3 ÷ 5?



3. If you took all the red pieces from these circles, how much would you have?

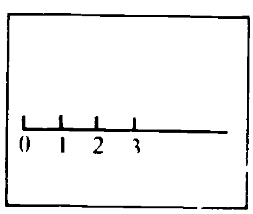
4. What do these diagrams tell us?



Fractions as numbers

Three questions were included in this set:

5. Where would the number 4 go on this number line? And the number 3/5? And the number 1⅕?



6. Can you find a number between the numbers 1 and 2? How many are there?

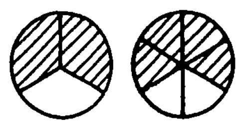
7. Can you find two numbers that add up to 10? Are there any fractions?

Equivalent fractions

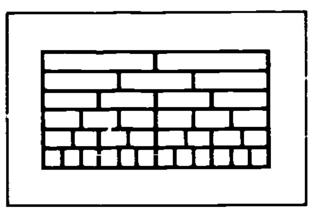
The next set of questions concerned equivalence of fractions:

8. Would you rather have 2/3 or 10/15 of a cake you particularly like?

9. Suppose you saw these diagrams in a textbook. What could you tell from them?



10. What does this diagram tell us?



11. What can you say about these two fractions: 3/4 and 12/16?

12. Could you complete these?

$$\frac{3}{4} = \frac{?}{12} \qquad \frac{9}{12} = \frac{12}{?}$$

$$\frac{5}{3} = \frac{15}{?} \qquad \frac{14}{16} = \frac{?}{24}$$

Would you explain what you did?

13. Could you find a fraction between 1/2 and 1/4?

14. (i) Which is the biggest fraction: 3/8, 3/7, 3/9? (ii) Which is bigger, 3/4 or 4/5?

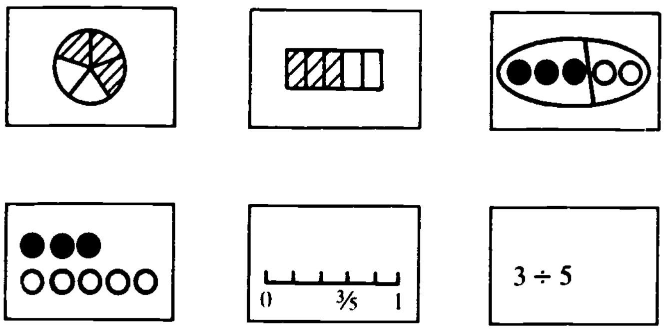
15. Would you try these: 2/3 + 3/4; 3/8 + 2/8; 1/10 + 3/5? Would you explain what you did?

That concludes the description of the questions put to the children in the interviews.

Results of second stage interviews

Models of fractions

1. Which of these cards would help someone who didn't know what the fraction 3/5 is?



In order to compare the children's reaction to this question with that for the previous one, which used the fraction 3/4, the number of children accepting each model is expressed as a percentage of the total number of children in each case. There were 23 children in the first interviews and 14 in the second. These results are shown in Table 3.1.

Table 3.1: Response to models for 3/4 and 3/5

Fraction 3/4 *Model*	*(n = 23)* %	*Fraction 3/5* *Model*	*(n = 14)* %
	100		100
	62 5		71 4
	34 8		14 3
0 3/4 1	82 6	0 3/5 1	71 4
3 ÷ 4	13 0	3 ÷ 5	0

Given the small numbers, it is unwise to attempt to draw many conclusions from these results. The order of popularity of choice of model remains virtually the same, and all children accepted the geometric one in both cases. The ratio aspect was less popular for the fraction 3/5, and no child accepted 3 ÷ 5 as being helpful. The reasons for rejecting it were much as before. Twelve of the 14 children said that it was no help because it was division, e.g.:

TH: 'That's 3s into 5 . . . and that's division, not a fraction.'

LG: 'That means you've got to divide the 3 into 5, and it doesn't go. You'd get a remainder, and you'd have to go into long division.'
I: 'What's the connection between that and 3/5?'
LG: 'None, as far as I can see.'

This problem was further investigated in the next question:

2. Here are 3 cakes. Could you share them equally between 5 people? Do you see any connection between what you have done and 3 ÷ 5?

Ten of the 14 children were able to carry out the sharing successfully and arrive at the fraction 3/5. Of these, six were then able to see the connection between 3 ÷ 5 and 3/5, although they had denied its existence in question 1. The other four still saw no connection, one such child being AE:

AE: 'You could divide each cake into 5, which would be 15 pieces, and divide it by 3 . . . each child gets 3/5.'
I: 'Would you think about this [3 ÷ 5] again?'
AE: 'Um . . . I think that's different. Well, a bit different, anyway.'
I: 'Does that mean they are a bit the same?'
AE: 'They've both got divide in them. And you've got to divide 15 by 3. You can do that with the cakes, but you can't really do 3 ÷ 5.'
I: 'You could argue that this [3 ÷ 5] is three cakes being divided?'
AE: 'Ah, I see what you mean. It would be the same that way. But you said these were cakes. But if it was 3 ÷ 5 you couldn't do it.'

AW was so sure that the division 3 ÷ 5 was impossible that she was unable to reconcile this with the fact that she had been able to divide the cakes. When the fraction used was 3/4, all the children were able to share the three cakes and arrive at the fraction 3/4. However, with 3/5, there were four children who first gave each person half a cake, which left one-half over, which would then be split into five equal parts. So they had one-half plus one-half of a fifth each. As they were unable to say that was the same as 3/5, it was impossible to pursue the connection between 3 ÷ 5 and 3/5 with them.

The response of the children in the two sets of interviews is shown in Table 3.2.

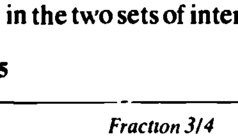
Table 3.2: Response to 3 – 4 and 3 – 5

	Fraction 3/4	*Fraction 3/5*
Number of children	14	15
No able to obtain fraction by sharing cakes	10	15
No who saw connection with the fraction	6	9

So the proportion of children who changed their minds and were able to say that the fraction and the division were connected was the same for both sets of interviews.

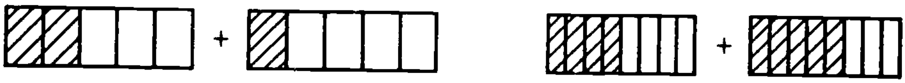
The next two questions were new ones:

3. If you took all the red pieces from these circles, how much would you have?



Eight of the 14 children said that the red pieces came to 7/5, while the remaining six said it was 7/10. Those in the second group counted the ten small sections and saw that seven of them were red. When, later in the interview, they were shown question 4, the results were very similar.

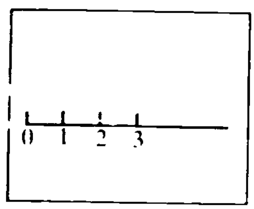
4. What do these diagrams tell us?



In the first case, the same eight children said '3/5' and the other six said '3/10'. For the second diagram, which produced a mixed number, nine chose '9/8' and five chose '9/16'. Just one child used a different method for the two diagrams. No child expressed any doubt as to the meaning of the diagrams or appeared to see any ambiguity. However, the strength of the support for counting the total number of parts in both shapes and then the shaded ones, obtaining the results 3/10 and 9/16 (7/10 in question 4) as opposed to 3/5 and 9/8 (7/5 in question 4), must throw doubt on the effectiveness of the use of such diagrams to illustrate the addition of fractions. The results suggest that, for many children, the two diagrams become a new 'whole', of which some parts are shaded. The second interviews confirmed the general conclusions of the first set, namely that the geometric 'part of a whole' was universally accepted and the division aspect generally rejected, although a practical example helped some children to accept this too. The diagrams that illustrated two fractions being added together were shown to be ambiguous.

Fractions as numbers

The first question made use of the number line:



5. Where would the number 4 go on this number line? And the number 3/5? And the number $1\frac{1}{5}$?

The results are shown in Table 3.3.

Table 3.3: Results for number line question

Fraction 3/5		*Fraction 3/4*	
Correctly placed	2	Correctly placed	12
Placed at 3.5	5		
3/5 of length of line	7		

The response to the fraction 1⅕ was very much better than for the fraction 3/5, which five children confused with 3½, and three interpreted as three-fifths of the way along the line. Nine children who were successful with 1⅕ but not with 3/5 were asked to look back to the fraction 3/5 again, in order to see if placing 1⅕ had caused them to change their minds about 3/5. Three children did so, e.g.:

AE: '3/5 . . . that's just past 3, more than half-way . . 1⅕ . . . just past 1.'
I: 'Fine – that's 1⅕. And 3/5 you said was . . . ?'
AE: 'Oh, 3/5 . . . between 0 and 1. I thought you said 3⅗.'

At first, AE was measuring three-fifths of the way along the line, a similar interpretation to that of several children in the first interviews. Then, having correctly placed 3/5, she realized her error. PL made the same explanation for the same error:

PL: 'I thought you meant 3⅗.'

It was interesting to note how much more successful the children were with 1⅕ than 3/5. In the case of the mixed number, the interpretation of finding that fraction of the whole line was abandoned, and the children appeared to switch methods depending on whether the fraction was more or less than one.

6. Can you find a number between the numbers 1 and 2? How many are there?

One child said there were millions, and one said there were none. The others chose numbers ranging from three to 'more than twenty'. TH specified her three as '1/4, 1/2 and 1/3', while LG said 'I don't know . . . not many.'

7. Can you find two numbers that add up to 10? Are there any fractions?

The results were again very similar to those of the earlier interviews. No child offered fractions immediately, and only two agreed that there were some when it was suggested to them. One of these was PE:

I: 'Could we have two fractions that add up to 10?'
PE: 'I don't think so.'
I: 'Suppose one was 1½?'
PE: '1½ plus 8½.'
I: 'Could you think of another pair?'
PE: '5½ plus 4½.'
I: 'So we could. When I asked if we could have fractions, what did you think?'
PE: 'No. I just couldn't think of any.'

Three other children tried 5/10 + 5/10 and rejected it, while others tried 1/5 + 1/5, 5/5 + 5/5 and 2/5 + 3/5. All of these used either two tens or two fives as the denominator in trying to make up ten. Three children made no attempt to find any fractions.

These interview questions contribute to the opinion that the children found it very difficult to think of fractions as numbers. They were much more able to plot a mixed number on a number line than a fraction less than one, and this was probably because in this case it was difficult to think of it as part of the whole line.

Equivalent fractions

The first question in this section was:

8. Would you rather have 2/3 or 10/15 of a cake you particularly like?

Nine of the 14 children said that the two fractions would give the same, although some, like BP, were not sure at first:

BP: 'I'd say 2/3.'
I: 'Could you say why?'
BP: 'You'd get more, because with 10/15 you'd only have little pieces.'
I: 'Yes, but you would have ten of them.'
BP: 'You'd probably get sick of them after a while.'
I: 'Why?'
BP: 'Well, can you imagine eating ten pieces of cake?'
I: 'What, rather than 2/3?'
BP: 'Yes.'
I: 'But, do you get more cake that way?'
BP: '. . . You'd get more. . . . No, it's the same!'
I: 'What's the same?'
BP: 'They're exactly the same.'
I: 'Aha! Why?'
BP: 'Because 2 goes into that five times, and that goes into that five times, so they're the same.'

So BP was first concentrating on the denominator of the fraction 10/15 and the fact that the pieces would be small. Then, when her attention was drawn to the fact that there would be more of them, she thought only of the ten, and changed her mind. Finally, she was able to reconcile both views, and concluded that the fractions were the same, quoting the multiplying factor in support.

The reason given by most of those who said the fractions were the same was based similarly on the use of dividing or multiplying. GW is an example:

GW: 'They're both the same.'
I: 'What makes you think they are the same?'
GW: 'Because 3 goes into 15 five times, and 2 goes into 10 five times.'
I: 'Why does that make them the same?'
GW: 'Because that one's lower and that one's higher, but that goes into that.'
I: 'So, couldn't you say that 10/15 is bigger?'
GW: 'Yes, it is.'
I: 'But you told me they were both the same?'
GW: 'Yes, 'cos 3 goes into 15 five, and 5 goes into 10 twice.'
I: 'But you said it is a bigger number?'
GW: 'Yes!'
I: 'But if it's bigger, why don't you choose 10/15 of the cake?'
GW: 'Because they're both the same.'
I: 'Although it's bigger?'
GW: '. . . Yes.'

This extract shows GW's difficulty in reconciling what he knew about equivalent fractions and his feeling that 10/15 must be bigger because of the separate numbers being five times bigger.

LG tried to explain the apparent conflict by appealing to a multiplying rule, but encountered further problems:

LG: 'You'd get the same amount.'
I: 'Could you explain that?'
LG: 'Well, you have to times them . . . if you timesed them, that by 5, you'd get the same anyway.'
I: 'If you timesed it by 5, how could it be the same?'
LG: 'Um . . . 'cos you times it by 5, and say 2/5 equals 1/10, and 5 there equals 15. So it's about the same . . . '
I: 'Could you say that again?'
LG: '. . . I don't know . . If you divided a circle into that [2/3], and divided a circle into that [10/15], you'd find they were the same.'

Realizing his explanation of the use of 5 was not convincing, he produced the idea of using a diagram with a flourish.

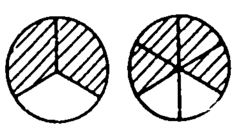
Three children thought that 2/3 was bigger that 10/15, and these all referred to the fact that there was only one piece left if one took the 2/3. PE is an example:

PE: 'I'd say 2/3 is bigger.'
I: 'Because . . . ?'
PE: 'Well, with 2/3, there's one piece left until you get the whole one, and with that one [10/15], there'd be five pieces left.'
I: 'The one piece left here is . . . ?'
PE: '⅓. And they are 5/15.'
I: 'Does that make any difference?'
PE: 'No.'

The remaining two children thought that 10/15 was bigger than 2/3.

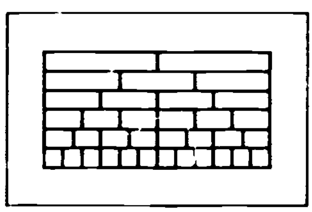
The next question used geometric illustrations of equivalence:

9. Suppose you saw these diagrams in a textbook. What could you tell from them?



All 14 immediately saw equivalent fractions, and this confirmed the observation of the first interviews that children were familiar with these diagrams. The fraction chart diagram was used in the next question:

10. What does this diagram tell us?



This time, no child offered any statement about equivalent fractions, limiting their observations to picking out halves, thirds and so on, and saying, for example, that three thirds made a whole one. However, when it was pointed out that some of the sections lined up, 12 of the 14 children did find some equivalences. But the diagram did not seem to suggest equivalence without some prompting.

The next two questions presented equivalent fractions in symbolic form:

11. What can you say about these two fractions: 3/4 and 12/16?

Ten of the 14 children said the fractions were equivalent, while four did not. The results were similar for question 12:

12. Could you complete these:

$$\frac{3}{4} = \frac{?}{12} \qquad \frac{9}{12} = \frac{12}{?}$$

$$\frac{5}{3} = \frac{15}{?} \qquad \frac{14}{16} = \frac{?}{24}$$

Would you explain what you did?

The number of children able to complete each is shown in Table 3.4.

Table 3 4: Results for equivalence questions (n = 14)

Question	*No. with correct solution*
3/4 = ?/12	11
5/3 = 15/?	7
9/12 = 12/?	1
14/16 = ?/24	1

The explanations of their methods given by the children were all based on an application of a rule concerning multiplying or dividing, and there was no evidence that the children knew why they were doing it. Two examples are:

SE: '3/4 = 9/12.'
I: 'Could you tell me how you did it?'
SE: 'To make it into 12, you times it by 3, so you times that by 3, which makes it 9.'
I: 'It could look as though 9/12 is three times as big as 3/4?'
SE: 'Yer, it could . . . you've got to times that by the number you timesed that.'
I: 'Yes, I see. But I wonder what you'd say to someone who said that 12/16 is three times as big as 3/4?'
SE: 'Um . . . yes, I think they'd be right, but it's not really. They're equal, aren't they?'
I: 'What's equal about them?'
SE: 'Those two there [3/4] will go into those two there [9/12].'
I: 'But they don't look equal, do they?'
SE: 'No. That one looks as though it's three times as big, but it's not.'

AH was successful eventually, but found it difficult to explain:

AH: '3/4 . . . 16/12?'
I: 'How did you do that?'
AH: 'I'm trying to remember . . . Hang on . . . 9/12?'
I: 'How did you get that?'
AH: 'I said that . . . um . . . I said how many times did 3 go into 12 . . . no, how did I do that? Oh, I think I know. What I done is . . . three 4s is one less than four, so I said 12, and if you had, say, one less than that, if you were adding up, you'd have 9.'
I: 'I think that one less than 12 is 11!'
AH: 'Yes, but that's how we done it.'

GW tried to use the dividing technique, but failed to complete the question successfully:

GW: '3/4 = 6/12.'
I: 'Would you tell me how you did it?'
GW: '4 goes into 12 three times, and you add 3 onto that 3 and it makes 6.'
I: 'Why did you add 3 onto 3?'
GW: 'I don't know!'

The method of dividing numerators and denominators seemed to work well enough when there was a factor, but not in the case of the third and fourth examples:

PL: '3/4 = ?/12 . . . um . . . 9. Yes, 9/12. 4 goes into 12 three times, and 12 times 3 is 9.'
I: 'And this one? – 9/12 = 12/?'
PL: 'You can't do that one. 9 don't go into 12.'
I: 'Do you mean that there is no number – that it can't be done – or that *you* can't do it?'
PL: '*I* can't do it, but I think it could be done.'
I: 'You said that 9 doesn't go into 12.'
PL: 'I can't see any way. . . no, no way . . .'
I: 'So do you think it can't be done?'
PL: 'Well, I can't do it, but I think it could be done.'
(Pause)
PL: 'It could be 12/3. That's four whole ones.'
I: 'Is that right?'
PL: 'No, it could be 12/12 . . . 12 goes into 12 once . . . oh no! . . . ! . . . 12/4 . . . That looks all right.'

The child seemed convinced that an answer was possible, but he was unable to think of a fractional divisor. LG, on the other hand, was prepared to adapt this method and made a promising start:

LG: '5/3 = 15/? . . . You times that [the 5] by 3, so you times the 3 by 3 which is 9 . . . 15/9.'
I: 'Fine. What about this one: 9/12 = 12/? ?'

LG: 'Ooh, crumbs! 3 goes into 9 goes three, and 3 into 12 goes four . . . so 12 into . . . it would be nine. 9/12 = 12/9. That's top-heavy . . .'
I: 'And was the 9/12 top-heavy?'
LG: 'No . . . Um, you could divide it again . . . you could cancel it down. 3 into 12 goes four, and 3 into 9 goes three. 12/9 = 4/3. That's still top-heavy. You could cancel 9/12 down.'
I: 'Does that help?'
LG: 'It can't be right. It could be 12/3. That's four whole ones.'
I: 'Is that all right? 9/12 equals four whole ones?'
LG: 'No! It could be 12/12 . . . 12 goes into 12 once . . . oh no! . . . 12/4. That looks all right.'

On reflection, it is unfortunate that LG was not encouraged to develop his first suggestion, as it is not clear which '12' he is referring to. His later suggestion of cancelling 9/12 down would have been more helpful earlier, before he had become thoroughly confused. This was the method used by the one child who successfully completed the last two questions. The others all found that their method failed them when they could not find a common factor. The idea of a fractional multiplier did not appear to be considered by any child, and was explicitly rejected by PE, for instance:

PE: '9/12 = 12/? . . . I don't know.'
I: 'What's the difficulty this time?'
PE: 'Well, um . . . you couldn't multiply anything . . . there isn't anything . . .'

The last three questions concerned the application of equivalence in the solution of other problems:

13. Could you find a fraction between 1/2 and 1/4?

Seven of the 14 found an acceptable fraction, but only one child, AE, actually used equivalence. She observed, '1/2 is 4/8 and 1/4 is 2/8, so it's 3/8.' The others all chose the fraction 1/3, and demonstrated that the choice of fractions was ill-considered, as it was possible to look at the denominators only. The next question was more helpful:

14. (i) Which is the biggest fraction: 3/8, 3/7, 3/9? (ii) Which is bigger, 3/4 or 4/5?

The first part, of course, did not require the use of equivalent fractions. Six of the 14 children correctly chose 3/7. The others all chose 3/9, and these were then asked to order 1/8, 1/7 and 1/9. Three of them were able to do this, and then to correct their answer of 3/9. For example, AE responded to a prompt:

AE: '3/9 is the biggest and 3/7 is the smallest.'
I: 'Let's think of these: 1/7 and 1/9. Which is bigger?'
AE: '1/9 is bigger.'
I: 'Think of finding a ninth of something.'
AE: 'Oh, 1/7 is bigger, because they're not so big. 3/7 is bigger than 3/9.'

In response to the two fractions 3/4 and 4/5, only two children made equivalent fractions, both using the denominator 20. HW and AE both correctly chose 4/5, but both made use of shading in fractions of a circle, and comparing them by eye.

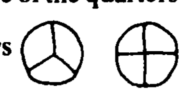
PL and BP both chose 3/4 first, but then, again after referring to circles or cakes, changed their minds:

PL: '3/4 is bigger.'
I: 'How did you know that?'
PL: 'If it was 3/4 of a cake, you'd have one piece left . . . oh, it's the same.'
I: 'You said, with 3/4, there'd be one piece left. What fraction would be left?'
PL: '1/4 . . . and fifths . . . 1/5 left. It isn't the same! 1/4 is higher than 1/5.'
I: 'So, what do you think now – which is bigger, 3/4 or 4/5?'
PL: '4/5.'

Others remained convinced that 3/4 was the bigger:

LG: '3/4 is the biggest.'
I: 'How do you know?'
LG: 'I don't know . . . I just do.'
I: 'How could you convince me?'
LG: 'Draw a diagram?' (Draws )
LG: 'Yes, 3/4 is the biggest.'
I: 'Certain?'
LG: 'Yes.'
I: 'What about the bits that are left over – not shaded in?'
LG: 'They're about the same, but I think that [1/4] one's a bit bigger.'
I: 'They're quarters and fifths, remember.'
LG: '. . . I still think 3/4 is bigger. Yes.'
I: 'But you did say that the bit left over with the quarters was bigger . . . ?'
LG: 'Yes. Oh! That is bigger than that, so the 4/5 must be bigger.'

Eventually, LG was convinced by his diagram, although his reasoning became confused. TH, though, maintained that 3/4 was bigger:

TH: I'd draw a circle, and divide it into 3/4 and 4/5.'
I: 'How would you decide which is bigger?'
TH: 'Count how many . . . no, the smaller the number the bigger it is. A fifth goes more into a circle than a quarter. So 3/4 is the biggest.'
I: 'But you have got four of the fifths, and only three of the quarters . . .'
TH: 'I still think 3/4 is bigger . . . can I draw it?' (Draws )
I: 'Why is that 3/4?'
TH: 'Because there's four . . . no . . . three of them . . . I just don't know.'
I: 'But it is 3/4?'
TH: 'Yes. I still think 3/4 is the biggest.'

Five children thought the two fractions were the same, all appealing to the one piece that was left. For example:

GW: 'They're both the same.'
I: 'Sure?'
GW: 'Positive.'
I: 'Because . . .'

GW: 'Because 4/5 means . . . one section left . . . and 3/4 means one quarter left as well.'
I: 'And 4/5 leaves . . . ?'
GW: 'A fifth.'
I: 'And do you still think they are the same?'
GW: 'Yes.'

The main conclusion from the children's response to this question, however, remains that only two children thought of producing equivalent fractions to solve the problem.

15. Would you try these: 2/3 + 3/4; 3/8 + 2/8; 1/10 + 3/5? Would you explain what you did?

This was the same set of additions that were used in the first interviews. Many of the responses were very similar, and so this account will concentrate on certain new features that emerged as a result of the closer questioning used this time. First, in trying to explain why they used the denominator 12 for 2/3 + 3/4, five children gave the impression that they realized that the purpose was to make the addition easier, but were not able to justify it. For example:

I: 'Why did you use the 12?'
SE: 'Because these two [2/3 and 3/4] don't add up together. So you've got to find a number what they both go into.'
I: 'Would you explain why that does make it work?'
SE: 'Because a 12 and a 12 are the same, and a third and a fourth aren't. So I timesed the 3 by 4 to get 12.'
I: 'O.K. Now, someone might say that you've done a different question?'
SE: 'Yes, but you wouldn't be able to do that properly, 2/3 + 3/4 . . . without doing complicated sums and everything, but with that, it's a lot easier.'

SE seemed to think that the fractions could be added as they were, but the use of 12 made it easier. But PE, for example, thought they couldn't be added without using the 12:

I: 'Could you explain why you did that?'
PE: 'To make the bottoms the same fraction.'
I: 'And why do you have to make the bottoms the same?'
PE: 'Because you can't add them if they are third and fourths.'

This observation that the fractions cannot be added as they are suggested that the children might be thinking that the 'sum' has actually been changed, so some were questioned on this point. GW had no answer to this suggestion:

GW: 'You have to find the number the 3 and the 4 will both go into. And then how many times 3 will go into 12 – four – and then 4 times 2 is 8 . . . that's 17/12 . . .'
I: 'Fine. Instead of writing 2/3 + 3/4, you wrote 8/12 + 9/12. Somebody could think that was a different sum you're doing?'
GW: 'Yes.'
I: 'What would you say to them?'
GW: 'You have to put them in twelfths if they're not the same.'
I: 'But someone might think you're doing a different sum?'

GW: 'Yes . . . it's just the way we've been taught.'

SE, when asked why he thought he could change the fractions, agreed that one couldn't usually do it, but 'you can do it with fractions'. Then, asked later whether he thought it all right to change the fractions, he said, 'No, but if you can't do it, you've got to, haven't you?'

All but three children made some attempt to make the denominators the same, but, as in the first interviews, this appeared to have little connection with the idea of equivalence. In five cases, the method used seemed just the application of mis-remembered rules, e.g.:

LG: 'You've got to make them the same. I think you turn one over . . . Oh, but that's not right . . . they've got to be same at the bottom. It's made it top-heavy. Um . . . you could cancel down first . . . 2 goes into itself once, and 2 goes into 4 twice . . . 3 goes into 3 once, so that makes that 2/3.'

I: 'What do you think about that?'

LG: 'Yes.'

Three children made no attempt to use a common denominator. The conclusion drawn from these questions is that, while all the children recognized equivalent fractions in geometric form, and most had some facility with fractions in symbolic form, only one child was able to apply equivalence in only one of the last three questions.

Conclusions from interviews: the basis for development of the teaching module

The data of the two sets of interviews suggest three aspects of the nature of pupil's difficulties with fractions.

1. Children's perception of a fraction. There was considerable evidence to suggest that the only model of a fraction with which children felt comfortable was that of a fraction as part of a whole. In particular, they found it difficult even to extend this view to include the division or sharing aspect: that is, for example, that the fraction **a/b** can be interpreted as '**a**' things shared between '**b**' people. Although this aspect appears in textbooks or work-card-based courses, and is the basis for the method commonly used to turn fractions to decimals, pupils were very reluctant to acknowledge any connection between a/b and $a \div b$.

2. The children interviewed did not appear to think that fractions are numbers. It seems that they are so confused by their part-whole view of a fraction that they cannot adjust their mental constructs so as to accommodate the notion of a fraction as a number.

3. While most of the children interviewed were able to identify or construct simple examples of equivalent fractions, there was little evidence of any ability to connect the diagrammatric illustration of equivalent fractions with the algorithm for finding an equivalent by multiplication or division.

The next phase of the research was to try to produce a teaching module which would help pupils to overcome some of the difficulties with fractions indicated above. The teaching module concentrated on providing activities and experiences which relate to the three aspects of fractions which were seen to cause problems. The teaching programme was planned first for a small-scale pilot study which could then be modified and tested on a larger scale. This teaching module is discussed in the next chapter.

4

Teaching experiments

Having identified certain aspects of fractions with which pupils appeared to have difficulty, the next stage of the research was to investigate whether a period of teaching intervention would help to alleviate these problems. The difficulties seemed to stem from a lack of understanding of the fundamental nature of fractions: this resulted in the pupils relying on the strategy of searching through their memories for a previously taught algorithm. It was decided that the teaching intervention should take the form of a series of activities which presented fractions in a wide variety of ways, which would afford the pupils the opportunity to obtain a firmer basis for their understanding. It was also felt to be important that the style of the teaching intervention should make it possible for pupils to discuss their incorrect strategies and any misconceptions which they had. In the first instance, the intervention took the form of a set of worksheets which pupils worked at in small groups, with the researcher as observer and helper. This teaching sequence was carefully monitored and the worksheets were modified where necessary. The material was then used with whole classes under the direction of the researcher, but with the pupils working in groups as before. The results of these experiments form the body of this chapter. It was hoped, however, that the material might also suggest ways in which fractions could be introduced to pupils in the first place, so that the problems and misconceptions which have been identified might be avoided. Thus the teaching sequence was designed so that it might reasonably be used by all teachers. It was felt necessary to restrict the amount of time to about six 40-minute sessions, as it was felt unlikely that schools would feel able to offer more time than this. The teaching sequence was then tested by teachers working with their own classes: these results are reported in Chapter 5.

The teaching sequence

General principles

It was decided that the teaching sequence should focus on three areas of difficulty which emerged during the interviews. These were (1) that the fraction a/b can be interpreted as $a \div b$, (2) simple equivalent fractions and (3) that fractions are numbers.

The nature of the work to be presented should provide a wide variety of instances of fractions, using as many different models and types of activity, and presenting fractions in as many different ways as possible. Thus it was intended that there should be both practical work and games. Calculators were also used, partly because it was felt that, being usually used in the context of numerical work, they would reinforce the idea that fraction work is also about numbers. They also offer opportunities to present ideas of equivalence, using the connection of fractional notation with division.

The existence of opportunity for the pupils to discuss their work, both among themselves and with the researcher, was felt to be very important. This was in order to increase the level of awareness of the work and so that any incorrect strategies or misconceptions could be made apparent and resolved. Discussion also helped in the general monitoring of the experiment.

It was important that there should be a relaxed style of working if the pupils were to feel able to discuss their misconceptions and difficulties. Exposition, as a teaching style, was not thought to be appropriate: it was important for the pupils to be able to work independently in small groups, so that the researcher or teacher could be free to offer encouragement, ask questions or make suggestions where appropriate. The use of group-work also made it more possible for the researcher to monitor the research more effectively, in addition to the use of more conventional testing techniques.

A pilot study was first carried out, in which the teaching sequence or module was tried with children in three classes. The module was evaluated by giving the children a pre-test, a post-test immediately after the teaching, and a delayed post-test taken some six weeks later. The tests reflected the work of the teaching sequence, but also contained conventional context-free items, such as those on addition and equivalence of fractions. As a result of the pilot experiment, certain changes were made in the worksheets and in the tests used to assess the children's performance, and these were used for the main experiment. It is the results of this main experiment that are presented in this chapter.

The design of the teaching module

Having decided on the teaching style to be used, it was decided that the teaching module should take the form of a series of worksheets. This enabled the teaching to be informal, with the children encouraged to work in pairs or small groups. The need for the children to be able to discuss their work, both among themselves and with the teacher, has already been stated. The worksheets were designed so that they included tasks that were as practical as possible, and that provided a variety of models and approaches. They also invited the children to make observations and generalizations based on their answers to some of the questions.

There were three groups of worksheets, each relating to one of the aspects of fractions outlined above. The first set dealt with the division aspect: that the fraction a/b can be interpreted as $a \div b$. The activities of the worksheets included:

(i) a number of sharing activities such as those in which five cakes have to be shared between four people, two bars of toffee between five children, two pints of milk between three cups and three pints of water between two jugs;

(ii) a systematic sharing of 1,2,3. . .10 chocolate rolls between three children;

(iii) the use of calculators to find $1 \div 2$, $1 \div 4$, $3 \div 4$ and to observe their connection with the fractions 1/2, 1/4 and 3/4 respectively. (The interviews and pilot study had indicated that similar pupils were familiar with the result that 1/2 is 0.5 and that 1/4 is 0.25.);

(iv) the use of a calculator to find decimal equivalents of pairs of fractions such as 3/15 and 15/3.

The worksheets concerned with equivalence included:

(i) a game called 'Loot' in which 48 one hundred pound notes are shared between four thieves using fractions such as 2/4, 1/4, 1/4 and 6/12, 3/12, 2/12, 1/12, with questions concerning equivalence and ordering of the fractions;

(ii) the use of calculators to evaluate sets of fractions such as 3/4, 6/8 and 33/44;

(iii) the use of a number line to order fractions and to find equivalents.

The number worksheets included:

(i) the use of number patterns which extend to fractions;

(ii) the use of a calculator to multiply by numbers just bigger and just smaller than 1;

(iii) the calculator game 'Target';

(iv) the game 'Numbers-on-a-line'.

The worksheets, incorporating the revisions arising from the pilot study, can be found in Appendix 3.

As the worksheets asked the children to make observations and generalizations, the need for discussion both among pupils on their own and with the teacher was an important feature. Three specific points of discussion were introduced by the teacher if they had not arisen during the course of the work. These related to each of the three aspects of fractions and were:

(1) that the idea of division does not only apply to the case of dividing a larger number by a smaller, as many of the children interviewed appeared to think, and that it is possible to divide, say, 3 by 4;

(2) the relationship of equivalent fractions, 8/12 and 2/3 for example, and that though, in a sense, some 'multiplying by 4' has taken place, 8/12 is not four times as big as 2/3;

(3) that the word 'number' does not mean 'whole number' only, and that it is possible to have other sorts of number.

The tests

As no test which was concerned specifically with the particular aspects of fractions under consideration was found, it was necessary to design some new ones. In some cases, questions from the CSMS fractions test were used. The other items reflected the problems encountered during the interviews. For instance, in the case of the division aspect, some examples were set in a 'sharing' context, and items such as $4 \div 12$ and $12 \div 4$ appeared in pairs so that pupils could not evade the fraction by reversing the order of the division. The equivalence items included ones which explored the relationship between, say, 2/3 and 8/12, rather than the ability to find an equivalent fraction. Similarly, the items on fractions as numbers allowed for the possibility that some children do not see fractions as numbers. The tests can be found in Appendixes 4 and 5, and details of the individual items appear later in the text of this chapter. As three tests were needed, for a pre-test and two post-tests, the first task was to find items of comparable difficulty. This was not easy, as the number of fractions which are in common use is quite small. It was felt that only fractions using thirds and quarters would be comparable, and that in the case of equivalence, multiplying factors of three and four could be exchanged. Because of the restrictions, it was decided that the delayed post-test should be the same as the pre-test, and that some items should be the same for all three tests. Thus the test called T1 was used as the pre-test and the delayed post-test and T2 as the immediate post-test.

The sample

Children from three comprehensive schools took part in the experiment. The schools were urban, with pupils from a wide range of socio-economic backgrounds. The children with whom the teaching module was used were 13 to 14 years old, and were members of middle-ability classes, who were likely to be entered for a CSE in Mathematics in due course. There were over 90 pupils in the three classes, but only 59 of them attended all the teaching sessions and were present for the three tests. It is the results for these 59 pupils that were used in evaluating the experiment.

Procedure

A group of six children was extracted from each class to work with the author in a separate room. In each case, the rest of the class was taught by a student who carried out the same procedure. The three students were spending the term on teaching practice at the schools being used, under the author's supervision. They had already taught the classes concerned for a few weeks before the experiment took place. The students had volunteered to take part in the experiment. They were invited to send any six pupils to work with the author, and encouraged to send those whom they regarded as the most 'difficult'. This allowed the students to concentrate on the presentation of the work rather than on behaviour problems. The students were very competent and their professional skills enabled them to organize the work without any difficulty.

Marking the tests

Each item in the tests was marked individually and a record of incorrect answers was kept. These results will be found in the detailed analysis later in this chapter. In order to compare overall scores between the three sets of items, and in order also later to compare results between schools, it was helpful to be able to present total scores for items. It was not feasible to mark all the items by a simple 'right/wrong' approach, as in many cases the items allow for a variety of responses. There has been an attempt to order the results in the sense of 'Mathematical Correctness', and in particular to take into account the existence of incorrect answers as well as the correct ones. This is well exemplified by question 7:

Put a ring round each of the statements below that you think are other ways of writing the fraction $\frac{3}{4}$:

3×4 $\quad 3 \div 4 \quad$ 3/4 $\quad 4 \div 3 \quad$ three-quarters

The aim of the questions was to see whether the children could recognize '3 ÷ 4' as an acceptable representation of 3/4. However, quite a number of the children who ringed '3 ÷ 4' also ringed some incorrect forms, such as '4 ÷ 3' or '3 × 4'. It was felt that these children displayed less insight than those who chose 3 ÷ 4, together, possibly, with other correct forms, such as 'three-quarters' or '3/4', but who avoided the incorrect ones. So children in this latter category scored 2, while those who chose incorrect answers as well scored 1. Those who did not ring 3 ÷ 4 scored zero. Such a scoring system is, of course, an ordering device and it does not imply that one answer is twice as good as another. The total number of marks for the test is 46.

Because the items have different totals, each item score has been expressed as a percentage of the total score, as in the pilot study. For example, question 1 has a possible score of 3, so the possible total score is 59 × 3, or 177. The actual total achieved was 71 on the pre-test, and this is recorded as (71/177) × 100, or 40.1 per cent.

Results of the teaching experiment: general observations

The pupils appeared to enjoy working from the worksheets, which, on the whole, they were able to read and understand. The style of the presentation caused no problems, although the children from two of the schools were not used to working in this individual or small-group way. The children were mostly happy to join in discussion, although one or two of the girls were reluctant to say much at first. While the work was not found to be difficult, pupils were reluctant to make many of the observations or generalizations which were invited. This is, perhaps, not surprising, since it was not normally expected of them in their mathematics classes. Children were rarely found to comment on the non-commutativity of 12 ÷ 3 and 3 ÷ 12 when evaluated with a calculator, although this was found to be a problem during the interviews.

Many of the activities made use of calculators; the pupils were found to be familiar with them, but not very experienced. While they all knew the decimal equivalents for 1/2 and 1/4, they were very surprised to find that they obtained the same results by dividing 1 by 2 and 1 by 4 respectively. They were amused by their results for 2/3, comments being 'Oh, it goes on and on and on' or 'Oh my God, it's nought point six six six six . . . !'

Several of the worksheets were in the form of games, and the pupils appeared to enjoy these. They took on the roles of Alf, Bert and so on in the game 'Loot'. Although some found difficulty in working out 1/16 of the 48 one hundred pound notes, the use of the notes helped. Observations on the equivalence and relative sizes of the fractions emerged during discussion.

The pupils did not appear familiar with the use of number lines in which fractions are marked as points on the line. While they found no difficulty in filling in the missing fractions, they were not all able to use the lines for the comparison of fractions, as the following extract shows:

LG: 'It says "Which is bigger, 2/3 or 3/4?" That's 3/4.'
MP: 'No, 2/3.'
LG: 'If you look on the line for 2/3, here, then you look for 3/4 . . .'
MP: 'I still think 2/3 is the biggest. (Draws a circle and shades in 2/3.) Oh, it's 3/4.'
LG: 'He's back to the cakes again!'
MP: 'Oh, I've got it . . . it's further along.'

Other discussion points were emphasized during the teaching sequence. In one example, children used calculators to help in establishing the equivalence of frac- s. Most were then asked to relate this to the use of a multiplying factor, but not all ound it easy to articulate. This extract is from the discussion of the set 3/4, 9/12, 30/40 and 33/44 with their teacher (T):

FH: '3 will go into all the top numbers and 4 will go into all the bottom numbers.'
T: 'So would 9/80 give the same answer?'
FH: 'Yes.'
T: 'Let's try it with the calculator.'
FH: 'Oh, no!'
T: 'What about 9/16? 3 goes into 9, and 4 goes into 16 . . .'
KJ: 'Yes . . . Oh, no! . . . 3 goes into 9 three times . . . it's times by three.'

Similarly, those who gave 5/7 for 2/3 + 3/4 were asked to find 2/3 and 3/4 with the calculator and to add the results. FR, finding this to be 1.416666, said 'Oh, 5/7 is wrong, 'cos it should be more than a whole one.'

In terms of general interest and ease of use, the teaching module appears to have been successful. However, the monitoring of the programme depends on the use of the pre-test, post-test and delayed post-test, and the outcome of these is now discussed.

Results of the testing

The results will be considered in two stag...

Overall results

The scores for the division, equivalence and number items are shown in Appendix 6. As the totals are not the same for each group, the means of the scores for each group of items, expressed as a percentage of the possible total, were calculated, and these are shown in Table 4.1.

Table 4.1: Mean scores for division, equivalence and number items (n = 59)

	Pre-test	*Post-test*	*Delayed post-test*
Division	29 0%	64.6%	51.2%
Equivalence	39.6%	59 8%	60 9%
Number	34.8%	57.9%	61.7%

It can be seen that the scores for each group of items has increased after the use of the teaching module, with an increase of approximately 22 per cent, 21 per cent and 27 per cent for the division, equivalence and number results respectively from pre-test to delayed post-test. The scores, though, never reach two-thirds of the total scores for the groups of items Possible reasons for this are discussed later. One immediate observation is that the results for equivalence and number are similar in that there is a considerable increase in scores from pre-test to immediate post-test, with a further slight increase by the delayed post-test. The continued growth over a period of time suggests that the worksheets have enabled the pupils to incorporate their new experiences of equivalence and of fractions as numbers into their existing mental structures.

The division results, on the other hand, show a greater increase from pre-test to immediate post-test, but this increase was not sustained at the delayed post-test. It seems that some children successfully extended their 'part of a whole' view of a fraction to include that of the quotient aspect in the short term, but reverted to their more restricted view after a few weeks. Piaget argued that when new material is assimilated into existing cognitive structures, this can involve too great a cognitive conflict, and that this is only resolved by a change in the cognitive structure, an accommodation. Skemp observes that if an existing schema is such that a new idea will not fit into it, then the new idea is only learnt temporarily and is soon forgotten. This would seem to be the case for some children in the case of the division items. This will be investigated further, by examining the children's responses to the division items in more detail.

The scores obtained by individual children for the three groups of items are now discussed. Of the 59 children, the number whose scores either increased from immediate post-test to delayed post-test or remained the same for the equivalence and number items was 34 and 37 respectively, while for the division items it was 21. In fact, the scores increased from the immediate post-test to the delaved post-test for 25 children for the equivalence items and for 26 children for the number items. There had been no teaching of fractions at the schools in the intervening period, so it seems that for these children the teaching module had brought about the perceptual reorganization which the Gestalt psychologists, for example, suggest continues to operate upon information stored in the long-term memory.

The results for four children are of particular interest. The possible score for the whole test was 46. One child, who scored 17 at the pre-test, 19 at the immediate post-test but 30 at the delayed post-test, reported afterwards that he had been 'having a bad day' on the day of the immediate post-test. The other three showed very little improvement after the teaching. Their scores for the whole test at the pre-test, post-test and delayed post-test were (i) 11, 16, 15, (ii) 11, 18, 13 and (iii) 10, 19, 13. These children were somewhat disenchanted with school in general, and although they worked quietly at the worksheets, they were helped by a more able pupil. They appeared to make little effort at the tests, when, of course, they had to work on their own. Their low scores have depressed the scores generally. For the most part, the other children showed considerable improvement in their scores after the teaching. Although only five children scored more than half the possible marks at the pre-test, 18 scored over two-thirds of the marks and four over three-quarters of the marks at the delayed post-test. An examination of which questions remained difficult after the teaching will be made later.

It was not the case, in general, that those children with low scores at the pre-test showed less benefit from the teaching than the others. The overall results suggest that the intervention of the teaching module resulted in increased scores in the three aspects of fractions. While the scores for the equivalence or number ite s remained at their higher level or improved between the immediate post-test and the delayed post-test, those for division items showed a greater increase from pre-test to immediate post-test, which was not sustained at the delayed post-test. These results were also reflected in the scores for individual children. It seems that the part of the teaching module that concerned the division aspect of fractions did not enable the children to restructure their view of fractions sufficiently to accommodate the new aspect of a fraction as a quotient.

The second part of the analysis of the results consists of an examination of the scores for individual items.

Detailed item results

The results of the individual items on the tests are now discussed in detail.

DIVISION ITEMS

The division items were those that concerned the interpretation of a fraction a/b, say, as the division $a \div b$. The items were numbered 2/3, 16/17, 12/13 and 7, items being linked in pairs where they were interrelated, and they will be discussed in that order. Differences between the two tests, T1 and T2, will be indicated where they exist.

T1. 2. $12 \div 4 = \ldots\ldots$
    3. $4 \div 12 = \ldots\ldots$
T2. 2. $15 \div 3 = \ldots\ldots$
    3. $3 \div 15 = \ldots\ldots$

The number of children with the correct response to each is shown in Table 4.2.

Table 4.2: Frequency of correct responses to questions 2 and 3

Question	*Pre-test*	*Post-test*	*Delayed post-test (n = 59)*
$12 \div 4$; $15 \div 3$	51	58	54
$4 \div 12$; $3 \div 15$	9	26	8

Question 2, in which the division produced an integral answer, was correctly answered by almost all the children. The children who did not give the correct answer invariably misinterpreted the order of the division and gave the answer '3' or '5' to question 3 instead of to question 2. Far fewer children were able to give a correct answer when the division gave rise to a fraction. The scores for question 3 increased markedly after the teaching, but decreased again at the delayed post-test. However, it is more interesting to consider the two questions, either $12 \div 4$ and $4 \div 12$, or $15 \div 3$ and $3 \div 15$, as a pair. Some children gave the same response to both, and so gave the answer '3' to $12 \div 4$ and also to $4 \div 12$, or '5' to $15 \div 3$ and to $3 \div 15$. Others gave '3' or '5' to one of the divisions, but '0' to the other. A few children gave '3' and '0.3' or '5' and '0.5', or made no response to one or other of the questions. The number of children in each category is shown in Table 4.3.

Table 4.3: Frequency of different responses to questions 2 and 3 (n = 59)

	Pre-test	*Post-test*	*Delayed Post-test*
Both correct	8	24	10
Same for both (3 or 5)	25	21	27
One '0' – (3,0) or (5,0)	15	2	9
3 or 5 and no response	3	3	1
No response to either	3	0	0

The number of children with both answers correct increased considerably immediately after the teaching. The commonest single error was to give the same answer, either '3' for 12 ÷ 4 and 4 ÷ 12 or '5' for 15 ÷ 3 and 3 ÷ 15. Thus there was a substantial number of children who treated division as though it were commutative both before and after the teaching. In this way they avoided reference to the existence of numbers less than 1. Other children managed to avoid using fractions by giving the answer '0' when the smaller number is divided by the larger. It is suggested that this strategy reflects the verbal constructs '12 and 4 you can't' or '12 into 4 won't go' that are heard when long-division algorithms are stated. To children familiar with such phrases, the numeral '0' would seem a reasonable way of representing the absence of the number of times 12 can be divided into 4. Other children avoided reference to fractions by not answering the questions 4 ÷ 12 or 3 ÷ 15. Taking these three strategies together, it can be seen that 46 children avoided the use of fractions at the pre-test, 26 at the immediate post-test and 37 at the delayed post-test.

The next pair of questions, 16 and 17, also presented two division items in symbolic form, but this time both gave rise to fractional answers.

16. 3 ÷ 4 = ...
17. 4 ÷ 3 = ...

There were fewer correct responses to these items than to the previous two. The number of correct responses to each is shown in Table 4.4.

Table 4.4: Results for questions 16 and 17

Question	*Pre-test*	*Post-test*	*Delayed post-test*
3 ÷ 4	12	37	22
4 ÷ 3	17	29	17

As in the case of questions 2 and 3, one error was the reversal of the order of the division, so that '1⅓' was given for 3 ÷ 4 and '3/4' for 4 ÷ 3, but this time fractions could not be avoided by this strategy. However, fractions were avoided either by the use of a remainder rather than fraction, so that the answer became '1 remainder 1'. Other children gave the answer '0' to one or both questions, this again reflecting the view that '4 won't go into 3', and this is also regarded as an 'avoidance of fractions' strategy. There were also children who made no response to the question. As this was uncommon for the test as a whole, it is considered that these children, too, were avoiding the use of fractions. The frequency of the common errors is shown in Table 4.5.

Table 4.5: Frequency of incidence of avoidance of fractions in questions 16 and 17

Question	*Response*	*Pre-test*	*Post-test*	*Delayed post-test*
3 ÷ 4	1 rem. 1	8	1	4
	0	6	3	5
	No response	9	6	11
	Total	23	10	20
4 ÷ 3	1 rem 1	7	1	5
	0	4	2	3
	No response	12	4	7
	Total	23	7	15

The number of children who avoided fractions was reduced considerably in the short term, but, as with the previous question, there was a reversion by the delayed post-test. It may well be that this reluctance to acknowledge the existence of fractions is the cause of the difficulty in accommodating the idea of the division aspect of fractions. This matter will be discussed again later.

As with the previous two questions, it is of interest to consider the questions as a pair. There were some children who gave a correct answer to one question (invariably 3/4) and '0' for the other, some who gave the same response to both questions and others who gave no response to one or both questions. The number of children in each category is shown in Table 4.6.

Table 4.6: Frequency of different pairs of response to questions 16 and 17

Response	*Pre-test*	*Post-test*	*Delayed post-test*
Both correct	10	24	11
One zero	10	3	7
Both the same	12	10	13
No response	18	9	14

The number of children with both answers correct has more than doubled from the pre-test to the immediate post-test, but reverts again at the delayed post-test, so the teaching module produced a short-term effect only. This result is similar to that obtained with questions 2 and 3. There are, again, a number of children who think that the answers to both $3 \div 4$ and $4 \div 3$ are the same. When one looks at how individual children respond to the item at each of the tests, certain patterns emerge. A number of children change from the incorrect 'same for both' or 'one zero' response to the correct response at the immediate post-test. However, an appreciable number of these revert to their original response at the delayed post-test.

In the case of both of the previous pairs of questions (2 and 3; 16 and 17), the questions were presented in symbolic form. The first three division worksheets of the teaching module were, on the other hand, practical in nature. They related to sharing cakes, pints of milk and so on, and did not make explicit reference to the connection between division and fractions. The calculator activities did draw attention to the connection between fractions such as 2/3 and the expression $2 \div 3$, and the last worksheet also made the connection between the sharing activities and the division symbol. The next item from the test to be discussed had a much closer connection with the activities of the worksheets, the hypothesis being that this would be found easier.

12. 2 pints of milk are divided equally between 3 cups, all the same size. How much does each cup hold?
13. 5 pints of milk are divided equally between 3 jars, all the same size. How much is there in each jar?

The number of correct responses for each question is shown as a Venn diagram in Figure 4.1.

After use of the teaching module, the number of children with correct responses to both questions at the immediate post-test was approximately doubled. It can be seen that question 12 was easier than question 13, but that for both questions there were children with one right and the other wrong.

It was certainly the case that these questions, in which a context was supplied, were easier than the other division items, as can be seen in Table 4.7.

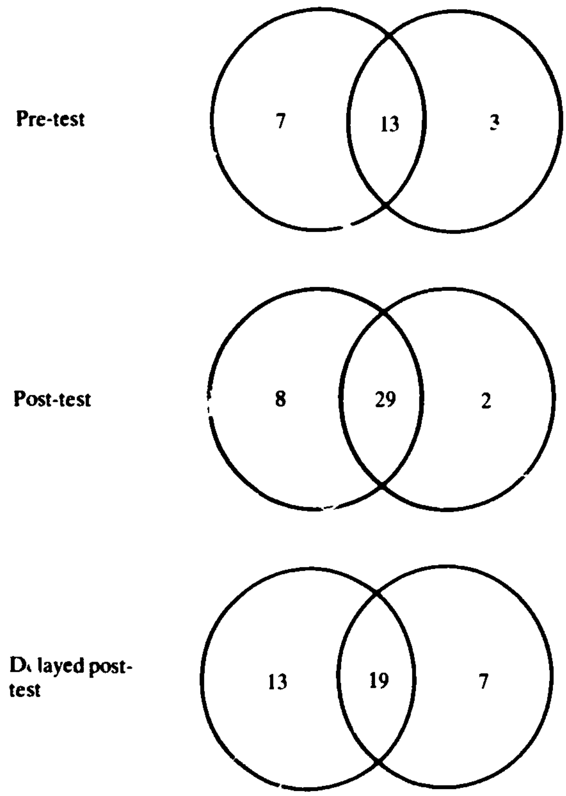


Figure 4.1: Number of correct responses to questions 12 and 13

Table 4.7: Number of correct responses to division items (n = 59)

Question	*Pre-test*	*Post-test*	*Delayed post-test*
4 – 12	9	26	8
3 – 4	12	37	22
4 – 3	17	29	17
2 pints, 3 cups	20	37	32
5 pints, 3 jars	16	31	26

The existence of a context to the question also affected the errors made by the children. Treating division as commutative had been a common error in the questions that made use of a division sign, and three children reversed the order to give 1½ as an answer to question 12 and 3/5 for question 13. The other common error of giving the answer '0' when a smaller number was divided by a larger did not arise in the case of '2 pints and 3 jugs'. It seems that this is because of the context of the question. Experience of similar situations would, at least, have led the children to realize that there would be some milk in each jug, even if they did not know how much. So the argument '3 into 2 won't go' does not apply. There were no errors that were made frequently, but there was an unusually large number of children who left the question unanswered. These children are categorized with those who avoid using fractions where possible. All the children had been able to answer question 11, which referred to 6 pints being shared between 3 jugs, but it was not scored as it did not concern fractions. This indicates that the children understood the nature of the task, but were not prepared to give an answer when the quotient was a fraction.

The frequency of the incorrect responses is shown in Table 4.8.

Table 4.8: Frequency of certain errors in questions 12 and 13 (n = 59)

Question	*Error*	*Pre-test*	*Post-test*	*Delayed post test*
2 pints, 3 jugs	3/4	6	6	6
	1½	3	5	9
	No response	12	5	2
5 pints, 3 jars	3/5	3	3	4
	No response	17	5	4

The last question that concerned the division aspect of fractions was question 7:

7. Put a ring round each of the statements below that you think are other ways of writing the fraction 3/4:
   3×4 $\quad$ $3 \div 4$ $\quad$ 3/4 $\quad$ $4 \div 3$ $\quad$ three-quarters

The reason for including this question was to see if the children would select $3 \div 4$ as a way of writing 3/4. Two other acceptable forms were included, and $4 \div 3$ was an obvious incorrect one. 3×4 was included to give more than one incorrect example. The number of children choosing each option is shown in Table 4.9.

Table 4.9: Frequency of choice of each representation of 3/4 in question 7 (n = 59)

Choice	*Pre-test*	*Post-test*	*Delayed post-test*
$3 \div 4$	19	47	27
3/4	42	48	50
Three-quarters	49	48	49
$4 - 3$	17	16	19
3×4	7	7	12

It is worth noting that although most children chose the word form 'three-quarters' or the form using the solidus, '3/4', there were a few who did not. There was a group of children who chose the option '3×4'. The choice of '$4 \div 3$' was probably made by the group of children who think that the order of division can be reversed. This matches the results for questions 16 and 17, in which the same answer was given to $3 \div 4$ and $4 \div 3$ by 12 children at the pre-test. The objective for this item was to see whether the children could connect the idea of a fraction with that of division. The correct division form '$3 \div 4$' was considerably more popular immediately after using the teaching module, but was rejected again by some children at the delayed post-test. However, the results are of more interest when looked at collectively, as it was observed that some of the children who correctly chose '$3 \div 4$' also chose '$4 \div 3$'.

Taking the responses as a whole, the following groups are produced:

1. $3 \div 4$, other correct choices but not 3×4 or $4 \div 3$
2. $4 \div 3$, other correct choices but not 3×4
3. $3 \div 4$ and $4 \div 3$ chosen with other correct choices but not 3×4
4. $3 \div 4$ but also 3×4
5. 3×4 or $4 \div 3$ and no correct choice
6. No response

The number of children in each group is shown in Table 4.10.

Table 4.10: Grouped responses to question 7 (n = 59)

	Pre-test	*Post-test*	*Delayed post-test*
$3-4$ & others correct only	4	30	11
$4-3$ & others correct only	3	0	2
$3 \div 4$ & $4 \div 3$ & others correct only	13	13	14
$3-4$ & 3×4 & others correct	2	4	4
3×4 or $4 \div 3$ & none correct	5	3	9
Three-quarters or 3/4 only	32	9	19

Of the 19 children who chose $3 \div 4$ at the pre-test, 15 also chose $4 \div 3$ or 3×4. The number who chose $3 \div 4$ and no incorrect ones increased greatly after the teaching.

SUMMARY OF DIVISION RESULTS

All the division item results displayed the characteristic of showing a marked increase immediately after the teaching intervention that was not sustained at the delayed post-test.

It is suggested that the 'part of a whole' model of a fraction is so firmly held by the pupils that it is necessary, in Skemp's terms, for a major accommodation of their existing schema if they are also to be able to see the fraction a/b as $a \div b$. It seems that although the children were able to carry out the tasks of the teaching module with apparent ease, the intervention was sufficient only to cause a temporary disturbance.

A number of strategies were observed that resulted in the use of fractions being avoided. One such was to treat division as commutative, and always to divide the larger number by the smaller if one divides the other exactly. When reversing the order would still give a fraction, as in the case of $3 \div 4$, the answer '0' was given. If neither of these strategies was appropriate, as in the case of 2 pints and 3 jugs, the fraction was avoided by making no response at all to the question, although the incidence of 'no responses' was low for the test as a whole. The tendency of some children to avoid fractions will be referred to again later.

EQUIVALENCE RESULTS

The next group of results to be discussed is of those test items that concerned the idea of equivalent fractions. Four of the items, numbers 5, 8, 14 and 22, tested the children's ability to recognize or produce some simple equivalent fractions. In the other three questions, 9, 15 and 19/21, the notion of equivalence is not explicit, but they need the application of ideas of equivalence. The first group is now considered.

T1. 5. Jane has 3/4 of a bar of chocolate, and John has 9/12 of a bar the same size. Tick *all* the statements below that you think are true:
- (a) John has more chocolate than Jane
- (b) Jane has more chocolate than John
- (c) They have the same amount of chocolate
- (d) John has 3 times the amount of chocolate as Jane
- (e) John has smaller pieces of chocolate but they have the same amount

(T2 was similar, but used the fractions 2/3 and 8/12.)

Statement (c) was, of course, the correct one, but statement (e) could also be true, and has been included among the correct results. The number of children choosing each option is shown in Table 4.11.

Table 4.11: Frequency of choice of options in question 5 (n = 59)

	Pre-test	*Post-test*	*Delayed post-test*
(c)	39	49	52
(e)	31	37	36
(b)	17	6	8
(d)	7	3	4
(a)	3	3	2

However, some of the children who ticked statement (c) also ticked incorrect ones. So it is more useful to look at the results as a whole and distinguish between correct answers only and a mixture of correct and incorrect ones. These are shown in Table 4.12.

Table 4.12: Frequency of groups of response to question 5 (n = 59)

	Pre-test	*Post-test*	*Delayed post-test*
(c) only or (c) & (e)	31	48	46
(e) only	2	0	0
(c) & (a) or (b) or (d)	8	2	6
(e) & (a) or (b) or (d)	4	4	1
Incorrect only	14	5	6

After the use of the teaching module, the number of children who chose the correct statement increased and the number of children who chose only incorrect responses decreased. The question was found to be easier than most of the other items on the test, scoring relatively highly at the pre-test and very highly at the delayed post-test. It was placed in the familiar context of sharing chocolate, and the fractions 3/4 and 9/12 could easily be visualized using the 'part of a whole' model, with the bar being broken into 4 parts of which Jane takes 3 or into 12 parts of which John takes 9. It is of interest to note that the teaching module did not use this model for equivalence, concentrating instead on the use of calculators and the number line. The activity that was the most similar was the sharing out of money in the game 'Loot'. The fact that nearly 25 per cent of the children were more successful at the delayed post-test suggests that their generally increased familiarity with the idea of equivalence helped them to do the question successfully. The next question was similar in style, but this time the equivalent fractions were not placed in any context, except in the case of part (f):

T1. 8. Tick *all* the statements below that you think are true about the fractions 2/3 and 8/12:
(a) 8/12 is 4 times as big as 2/3
(b) 2/3 and 8/12 are equivalent
(c) 2/3 is smaller than 8/12
(d) 8/12 is found by multiplying 2/3 by 4
(e) 2/3 and 8/12 are the same
(f) If you had 8/12 of a bag of sweets you would get more than if you had 2/3 of the same bag of sweets

(T2 was similar, but used the fractions 3/4 and 12/16.)

The number of children who chose each option is shown in Table 4.13.

Table 4.13: Frequency of different responses to question 8 (n = 59)

		Pre-test	*Post-test*	*Delayed post-test*
Correct	(b)	43	50	49
	(e)	41	46	52
Incorrect:	(a)	23	21	16
	(c)	6	5	3
	(d)	49	40	42
	(f)	13	8	8

Looking just at the number of times the two correct statements were chosen, it might seem that the results are very similar to those of question 5. However, when three groups of results are constructed, taking (1) the correct only responses, (2) a mixture of correct and incorrect responses and (3) incorrect only responses, certain differences emerge. It is apparent that a large number of children chose (d) - the option that 8/12 is found by multiplying 2/3 by 4. It is perhaps unreasonable to classify this as 'incorrect', since there is a sense in which multiplication by 4 does take place. The results are grouped thus:

1) Correct only: (b) and/or (e)
2) Correct plus (d): (b)/(e) and (d)
3) Correct plus (a): (b)/(e) and (a)
4) Correct plus (d) plus (a): (b)/(e) and (d) and (a)
5) Incorrect only

There is also an interesting group of children who ticked at least one correct statement and (f), the option that referred to bags of sweets as well, with various combinations of others. So we have, as another category:

6) Correct plus (f) plus . . .

The numbers in each category are shown in Table 4.14.

Table 4.14: Frequency of grouped responses to question 8 (n = 59)

Response	*Pre-test*	*Post-test*	*Delayed post-test*
Correct only	6	10	12
Correct plus (d)	15	25	24
Correct plus (a)	0	4	2
Correct plus (d) plus (a)	13	8	11
Incorrect only	12	7	5
Correct plus (f) plus	11	5	4

It will be noticed that in addition to choosing the correct statements (b) and (e) a number of children also selected (d), the option that '8/12 is found by multiplying 2/3 by 4'. If this response is included with the correct ones, then the number of correct responses at the three tests is 21, 35 and 36 respectively. The percentage increase in the number of correct responses from the pre-test to the delayed post-test was 25.4 per cent and this compares with the increase of 23.1 per cent for question 5, so the teaching appears to have had some effect. However, considerably fewer children were successful at question 8. The existence of the context in question 5 was, no doubt, the reason for this. Another interesting feature of question 8 was the number of children who chose statement (a) – '8/12 is 4 times as big as 2/3'. The number of children who chose this was 23, 21 and 16 at each test. However, the corresponding statement in question 5 – 'John has 3 times as much chocolate as Jane' – was chosen by only 7, 3 and 4. So again the context seems to have been of considerable help. The number of children who chose only incorrect statements was much the same and decreased as before, from 12 at the pre-test to seven at the immediate post-test and five at the delayed post-test.

14. Put the missing numbers in the boxes. If there is no number, write 'no' in the box.

(a) $\frac{3}{4} = \frac{\square}{12}$ (b) $\frac{5}{3} = \frac{15}{\square}$ (c) $\frac{9}{12} = \frac{12}{\square}$ (d) $\frac{14}{16} = \frac{\square}{24}$

This was a conventional equivalence item, using pairs of equivalent fractions with one numerator or denominator missing. The number of correct responses is shown in Table 4.15, together with the numbers of children who wrote 'no' (there is no such number) or made no response.

Table 4.15: Frequency of various responses to question 14 (n = 59)

Question	*Response*	*Pre-test*	*Post-test*	*Delayed post-test*
3/4 = ?/12	Correct	40	49	48
	'No'	0	0	0
	No response	3	3	5
5/3 = 15/?	Correct	35	41	45
	'No'	5	1	3
	No response	6	3	4
9/12 = 12/?	Correct	4	4	4
	9	12	8	9
	15	10	12	9
	'No'	17	25	29
	No response	6	7	5
14/16 = ?/24	Correct	6	3	3
	22	1[illegible]	14	12
	'No'	13	25	14
	No response	8	7	5

The last two items are seen to have been very difficult, and the number of children who said there was no such number was much larger than for the other two. If the children were unable to find a simple multiplying factor to complete the equivalence, it seems that they rejected the very existence of a solution. The teaching module did not refer to examples as difficult as these, so it is not surprising that the post-tests showed no improvement. However, the use of calculators did encourage the children to discover which fractions were not equivalent, and so the continuing number of children who said that 12/9 was equal to 9/12 was disappointing, particularly as the division worksheets also made use of calculators to show that division is not commutative. The other error that occurred in the last two items involved adding rather than multiplying, so that 9/12 was said to be equal to 12/15 and 14/16 equal to 22/24. This addition strategy has been documented in Hart (1982).

The existence of these two questions with an extremely low facility is one of the reasons why the total scores for the test and for individual children, discussed in the first part of this chapter, are depressed.

The last question (22) in the first group on equivalence was more open, and asked the children to produce their own equivalents, rather than to recognize given instances:

22. Write down some fractions that are equivalent to (or the same as) the fraction 3/8.

The children are classified according to whether they gave more than one correct equivalent, one equivalent or none. The number of children in each group is shown in Table 4.16.

Table 4.16: Frequency of responses to question 22 (n = 59)

	Pre-test	*Post-test*	*Delayed post-test*
More than one	24	30	37
One only	5	11	7
None	30	18	15

This question appears to lie between questions 5 and 8 in difficulty. Taking the percentage who found one or more equivalents and no incorrect ones and comparing them with the percentages for 5 and 8 (with (d) included), the results are as in Table 4.17.

Table 4.17: Comparison of results (percentaged) for questions 5, 8 and 22 (n = 59)

Question	*Pre-test*	*Post-test*	*Delayed post-test*
5	55.9	81.4	78.0
22	49.1	69	74.6
8	35.6	59.3	61.0

The fraction most commonly given as equivalent to 3/8 was '6/16', and was chosen by 47 of the 59 children. Thirty-four children gave '12/32', while '9/24' was given by 29 children, '24/64' by 23 children, '15/40' by 15 children, '48/128' by 11 children and '1½/4' by three children. The powers of 2 were particularly popular, 24 of the children giving denominators of 16, 32, 64 . . . only. Indeed, ten of these children went as far as 256, three as far as 512 and one to 1024. Some of the incorrect responses displayed a misplaced use of pattern. This was, in the case of three children, by doubling the numerator and halving the denominator, to give '3/8', '6/4', '12/2' and '1/24'. Others doubled the denominators, but added equal amounts to the numerators, to give, for example, '3/8', '4/10', '5/12', '6/14', '7/16', '8/18' and '9/20' or '3/8', '6/16', '9/32' and '12/64'. Another example was '3/8', '6/16', '7/32', where the pattern started after '6/16'. There were some eccentric choices, such as, for one child, '5/16', '1/3', '6/7', '10/12', '11/20' and '10/22', and for another child '4/9', '2/16' and '3/27'.

The next questions to be considered are those in which the idea of equivalence was not referred to explicitly, but for which the use of equivalent fractions was the obvious method. The first involved finding a fraction between two given fractions.

T1. 9. (a) Find a fraction between 13/16 and 1/2
(b) Find a fraction between 5/8 and 7/16
(c) Find a fraction between 9/4 and 15/8
(d) Find a fraction between 2/3 and 4/5
(For T2 the pairs of fractions were 11/16, 1/2; 5/8, 13/16; 9/4, 15/8 and 2/3, 3/4.)

The obvious method for this question is to make the denominators of the two fractions the same. The number of children successful at each item is shown in Table 4.18.

Table 4.18: Number of correct responses to question 9 (n = 59)

	Pre-test	*Post-test*	*Delayed post-test*
(a)	15	34	32
(b)	10	32	[illegible]
(c)	5	15	21
(d)	4	5	13

This was considerably more difficult than the previous questions. Less than a quarter of the children were successful at (b), (c) and (d) at the pre-test. Although the final percentages were still quite low, there was an improvement. The teaching module included only four examples of finding a fraction between two given fractions, and these were in the context of work on a number line. It seems that this was insufficient to allow most children to establish a general method. On reflection, the absence of items in the test in which equivalence was presented in the number line context is regrettable. It had been hypothesized that the instances presented to the children during the teaching would be sufficient to allow them to produce general methods. Given that this proved on the whole not to be the case, it would have been interesting to see whether the children would have been more successful if the number lines had been given.

There were several errors that occurred. In most cases the children chose one or other of the denominators and put a number between the numerators on top. The frequency of each incorrect response is shown in Table 4.19.

Table 4.19. Frequency of certain incorrect responses to question 9 (n = 59)

	Error	*Pre-test*	*Delayed post-test*
(a)	6/8	11	11
	6/16	0	4
(b)	14/8	1	5
	10/4	6	9
(c)	2/4	1	1

It is possible that the children who obtained the correct answer, 3/4, to part (d) used the strategy of choosing a numerator between those given, in this case 4, but also chose a denominator between the 3 and the 5.

An interesting feature of the results for this question was the increase in the number of correct responses between the post-test and the delayed post-test – from 15 to 21 for (c) and from five to 13 for (d). A possible reason for this is that the children had reflected on the notion of equivalence and were in a better position to apply it in an unfamiliar situation.

The next question involved ordering fractions, for some of which the idea of equivalence was needed.

15. Put a ring round the *bigger* fraction in each of these pairs. If they are the same, write 'same' by them.
(a) 1/4, 1/8; (b) 3/7, 3/9; (c) 3/8, 6/16; (d) 13/10, 7/5; (e) 1/4, 7/32; (f) 3/4, 7/9

The number of children who chose the correct fraction is shown in Table 4.20.

Table 4.20: Number of correct responses to question 15 (n = 59)

	Pre-test	*Post-test*	*Delayed post-test*
(a)	52	52	54
(b)	42	43	42
(c)	34	40	44
(d)	43	42	41
(e)	41	42	46
(f)	7	10	13

The children were much more successful at this question. In the case of the first two items, equivalence was not needed, and it should be remembered that the children were, in any case, making a choice between two options. The intervention of the teaching has had little effect, but the comments about the teaching module made in connection with the previous question apply here also. The process of ordering fractions used in the teaching module was applied to fractions marked on a number line, whereas the fractions in question 16 were presented in symbolic form. The children were successful at the number line activities in the teaching module, but the experience does not appear to have helped the children to move to what Skemp (1979) refers to as the 'second stage of generalization', in which pupils reflect on a schema and formulate its essential features in a form independent of the examples.

In considering the next questions, on the addition of fractions, it should be remembered that such questions were familiar to the pupils, while the previous two were not.

19. 3/8 + 2/8 = ...
20. 3/5 + 1/10 = ...
21. 2/3 + 3/4 = . . .

The number of children with correct answers to each question is shown in Table 4.21.

Table 4.21: Frequency of correct responses to questions 19, 20 and 21 (n = 59)

Question	*Pre-test*	*Post-test*	*Delayed post-test*
19	33	49	47
20	13	34	31
21	13	25	29

It can be seen that, although the questions were familiar, they were still found to be difficult. The results were considerably improved after the teaching, with an increase in facility of about 30 per cent, but only about half of the children were successful at adding the fractions when the denominators were different. The first two questions, 19 and 20, were taken from the CSMS tests. It is not possible to make direct comparisons as the children who took part in the CSMS research constituted a normally distributed sample, while the present children were all from middle-ability classes. However, it is worth noting that the CSMS results showed facilities of 38.2 per cent for 13-year-olds and 48.7 per cent for 14-year-olds for the question '3/5 + 1/10', and so that they too found the question difficult. The facility for the same item for the present experiment was 22.0 per cent at pre-test and 52.5 per cent at the delayed post-test, so the improvement is perhaps more remarkable than it seems.

Most of the errors came, predictably, from the addition of numerators and denominators. The frequency of the errors is shown in Table 4.21.

Table 4.22: Frequency of certain errors to questions 19, 20 and 21 (n = 59)

Question	*Error*	*Pre-test*	*Post-test*	*Delayed post-test*
19	5/16	7	5	5
20	4/15	14	7	6
	4/10	8	1	5
21	5/7	11	9	6
	5/12	5	5	2

The results show that adding numerators and denominators occurred in all three fractions, although the first, in which the denominators were the same, was considerably easier. The number of other eccentric errors increased with the complexity of the example, as did the number of children who gave no response.

RECOGNITION OF EQUIVALENCE AND ITS APPLICATION TO PROBLEMS

It has already been stated that the equivalence questions were of two types:

1) those involving the recognition of simple equivalent fractions and the ability to find a fraction equivalent to a given fraction (*E*);
2) those presenting simple tasks in which the need for equivalents was implicit (*A*).

The question of whether satisfactory performance in the *A* questions implies satisfactory performance at the *E* questions is now posed. It would seem natural to suggest that the ability to recognize and recall simple equivalents is a necessary precondition for the ability to apply ideas of equivalence to solve other tasks. Equally, if one cannot do the *E* items it should not be possible to do the *A* items. In order to test this proposition, it is necessary first to define success at the *E* items 5, 8, 14 and 22. Children were said to have succeeded at 5 and 8 if they ticked correct statements only, at 14 if they got two or more parts right and at 22 if they produced at least one correct equivalent. If they were successful at two out of three of these they were said to have succeeded at *E*. In the case of the *A* items, 9, 15, 20 and 21, the scoring was simpler as there were no multiple-choice items. Question 9 had four parts, question 15 six parts, of which only four needed equivalence, and questions 20 and 21 one each, making a possible total of ten. Success at the *A* items was defined as scoring six or more. The results for the pre-test and the delayed post-test were compared, and the results are shown in Table 4.23.

Table 4.23: Comparison of results for equivalence and application of equivalence items

Pre-test		A items			Delayed post-test		A items		
		Pass	Fail	Total			Pass	Fail	Total
E items	Pass	3	19	22	E items	Pass	17	23	40
	Fail	2	35	37		Fail	5	14	19
		Total	5	54		Total	22	37	

The success rate for the E items has been raised from 37 per cent to 67 per cent from the pre-test to the delayed post-test, and for the A items from 8 per cent to 37 per cent.

There is support for the argument that it is necessary for the children to have been successful at the E items in order to succeed at the A items. Whereas, at the delayed post-test, 23 children were successful at the E items but failed at the A items, only five were successful at the A items but not at the E items. This group of children is interesting in that it seems they must have used some other strategy than equivalence to solve the problems. A closer examination of these children's results shows that their success at the A items related partly to their ability to do questions 20 and 21. All five were successful at 20, and four of them also at 21. Since the addition questions were familiar to the children, it is suggested that they might have been using an algorithm they had been taught, and so they did not need to use equivalence. In order to see whether there were other instances of children being unsuccessful at the E items but successful at the A items, the A items are considered separately. The number of children who passed and failed at the E items and at each A item is shown in Table 4.24.

Table 4.24: Comparison of results for E items with individual A items (n = 59)

	A9		A15		A20		A21	
	Pass	Fail	Pass	Fail	Pass	Fail	Pass	Fail
E items	P17	23	P22	18	P22	18	P19	21
	F 3	16	F 4	15	F 8	11	F 7	12

It can be seen that, for each A item, there are a few children who were unsuccessful at the E items but successful at the A item. It seems that the children must have used some other strategy than the use of equivalence for solving each of the problems. There was a large number of children who were successful at the E questions but not at each individual A item. However, it seems that the ability to answer the equivalence questions is neither a necessary nor a sufficient condition for success at those questions for which the use of equivalence was implied.

OBSERVATIONS ON THE EQUIVALENCE RESULTS

The teaching intervention appears to have resulted in markedly increased scores in most of the equivalence items. There was a 25 per cent increase in facility in question 5 (equivalent fractions of bars of chocolate), question 8 (equivalence in symbolic form), question 22 (finding equivalents), question 9 (finding a fraction between two other fractions) and questions 20 and 21 (addition of fractions). Increases in facility of over 10 per cent were also observed in the two parts of question 14 (3/4 = ?/12 and 5/3 = 15/?), and the first part of question 15 (3/8 and 6/16 are the same).

On the other hand, the teaching module had little or no positive effect on the other items of question 15, in which pairs of fractions had to be ordered. The starting facilities for most of these items were high, but as they involved a three-way choice, of ringing the bigger of the two fractions or saying they were the same, guesswork may have had an influence. The other two items for which the teaching was unsuccessful were 9/12 = 12/? and 14/16 = ?/24.

The worksheets of the teaching module used three approaches to the idea of equivalence: sharing money in different ways, plotting fractions on a number line and using a calculator to compare fractions. The test items were conventional items, in which the fractions were for the most part devoid of any reference to context. The evidence of the interviews was that the only common model of a fraction was that of a part of a whole. While this model helps the idea of equivalence when diagrams or concrete objects are present, it is not necessarily helpful in recognizing equivalence when no such visual aid is present, or for other tasks such as finding a fraction between two given fractions or adding fractions. The results would suggest that extending the children's view of equivalence has enabled them to operate more successfully with the symbolic form.

Two other observations were made. The first concerns the notion that, given two equivalent fractions, one is a multiple of the other. Questions 5 and 8 offered the children a choice of statements about the pairs of equivalent fractions 2/3, 9/12 and 2/3, 8/12. A number of children maintained, both before and after the teaching, that 9/12 was three times as big as 3/4, and that 8/12 was four times 2/3. In the first case, there was a context to the question and the number of children who gave this response was seven at the pre-test, and down to four at the delayed post-test. For the second example, 23 children said that 8/12 is four times as big as 2/3 at the pre-test, and 16 at the delayed post-test. There were another 42 children who, at the delayed post-test, said that 8/12 is found by multiplying 2/3 by 4. Many of these also said the fractions were the same or equivalent. These children appear to believe that one fraction is multiplied by a number to produce an equivalent fraction, but that, in some way, the fractions remain the same as each other. This phenomenon, also observed during the interviews, suggests a conflict between how the children think equivalent fractions are constructed, namely by multiplication, and their awareness, reinforced by geometric illustrations, that they are the same. They attempt to resolve the conflict by attaching an interpretation of multiplication that they apply only to fractions, namely that it is possible to multiply a fraction by a number such as 4 without altering the value of the fraction. The idea of multiplying by the identity element is, of course, the underlying principle behind the statement that 2/3 = 8/12. This entails the acceptance that the multiplying factor of 4/4 is a replacement for the identity element, and this may be said to be a complex system. The evidence suggests that, unable to operate at this level, some children are left with an unresolved dilemma: they know that in order to construct equivalent fractions they must multiply, but also that when equivalent fractions are illustrated geometrically they cover the same space.

Finally, there were children who again denied the existence of fractions. This occurred in question 14, in which missing numbers had to be given to complete four pairs of equivalent fractions. The children were given the option 'if there is no number, write "no" in the box'. Although no child chose this option for the first, 3/4 = ?/12, there remained a number who did so for the other three, even at the delayed post-test. As many as 29 said there was no number for 9/12 = 12/?.

In general, though, it seems that the process of extending the children's view of equivalent fractions by using a variety of practical instances of fractions had the desired effect of increasing the children's ability to succeed at the exercise of simple skills involving equivalence.

NUMBER RESULTS

There were five questions that related to the idea that fractions are numbers. They all presented fractions in the same context as whole numbers. The first offered the children a collection of number words and symbols:

T1. 1. Underline the numbers in this set:
$\{4$ some $\frac{3}{4}$ 5.7 ten lots $2\frac{1}{2}$ $\frac{13}{5}$ 17 $\frac{19}{23}$ $7/3\}$

T2. 1. Underline the numbers in this set:
$\{4$ many $\frac{2}{3}$ 5.9 five few $2\frac{1}{4}$ $\frac{17}{5}$ 13 $\frac{19}{27}$ $8/3\}$

The number of children who underlined each fraction option is shown in Table 4.25.

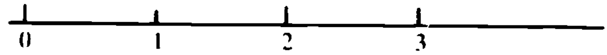
Table 4.25: Frequency of choice of each fraction option in question 1 (n = 59)

Fraction	*Pre-test*	*Post-test*	*Delayed post-test*
3/4 or 2/3	22	34	39
2½ or 2¼	34	43	50
13/5 or 17/5	22	35	42
19/23 or 19/27	22	33	39
7/3 or 8/3	21	22	37

It will be seen that, with the exception of 2½ on T1 and 2¼ on T2, less than half the children included fractions among the numbers they underlined at the pre-test. In the case of all five fractions, there was an increase in the number of children who acknowledged that they were numbers at the immediate post-test and this number increased further at the delayed post-test, although the alternative version for T2 was constructed to be as similar as possible to that of T1. The mixed numbers, 2½ or 2¼, were the fractions most often underlined. It seems likely that the presence of the whole number part was the reason for this, and that the children, seeing the '2 first', immediately realized that it was a number. After the teaching module the fraction 2½ was underlined by 85 per cent of the pupils, while 63 per cent underlined the least popular fraction 7/3, which was the only one printed with the solidus. It seems unlikely that the children were unfamiliar with the solidus form, as 50 of the 59 children accepted 3/4 as a representation of the fraction $\frac{3}{4}$ in question 7. It might have been thought that the fractions 3/4 or 2/3 would have been most often selected, being the most familiar ones. This was not the case. At the delayed post-test, more children chose both 2½ and 17/5 than 3/4. Reference has already been made to the use of geometric models to illustrate fractions. One possible interpretation of the fact that 3/4 was not underlined by as many children is that it could be more readily seen in geometric terms as part of a whole, and the existence of this model made it more difficult for the fraction to be seen as a number. It is suggested that there is a real cognitive conflict in having to move from the idea of fractions such as 3/4 as three-quarters of a shape to that of 3/4 as a number between 0 and 1.

Question 1 concerned terminology only: examining whether fractions are classified by the word 'number'. In the next question there is the implicit suggestion that fractions are numbers, and the children are asked to use them in a context that is familiar for whole numbers, namely of plotting them on a number line.

4. Mark and name the numbers 4, 3/5, 1⅕, 9/5 on this number line:



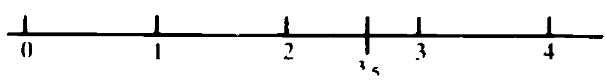
(For T2 the numbers were 4, 3/4, 1¼ and 9/4.)

The degree of accuracy expected was not high, as the important feature of the question was the notion that there are numbers between the whole numbers. The fractions 3/4 and 3/5 were said to have been correctly plotted if they were nearer to 1 than 0, and, similarly, 1⅕ and 1¼ if they were nearer 1 than 2, and 9/4 and 9/5 if they were nearer 2 than 3. The number of children who successfully plotted each point is shown in Table 4.26.

Table 4.26: Frequency of correct numbers plotted in question 4 (n = 59)

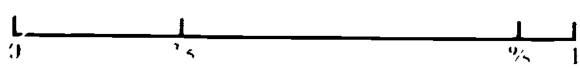
Number	*Pre-test*	*Post-test*	*Delayed post-test*
3/5, 3/4	25	46	43
$1\frac{1}{5}$, $1\frac{1}{4}$	35	53	53
9/5, 9/4	18	36	32

At the pre-test, less than half of the children correctly plotted the numbers 3/5 or 9/5. The mixed number, 1⅕, is seen to be easier, perhaps because the '1' directed the children's attention to the point '1' and they realized that 1⅕ would be just to the right of it. Again there was considerable improvement by the delayed post-test. There were some interesting errors, including two identifiable strategies. The first was to view the line as a 'whole' and to mark in the point 3/5 three-fifths of the way along the line, so that it appeared thus:



This error will be referred to as the 'fifths' error. It was made by seven of the 59 children at the pre-test and six at the delayed post-test. Four of the children who made the 'fifths' error at the pre-test were able to place 3/5 correctly at the delayed post-test.

Another error was to divide the part of the line between 0 and 1 into ten equal parts, and to plot 3/5 three parts along from 0 and 9/5 nine parts along, appearing thus:



This error will be referred to as the 'tenths error'. Six children made this response at the pre-test and they all placed the fraction correctly at the delayed post-test. At the delayed post-test, six children likewise made the 'tenths' error. Four of these not having attempted the question at the pre-test.

The way in which children changed categories between the pre-test and the delayed post-test is shown in Table 4.27.

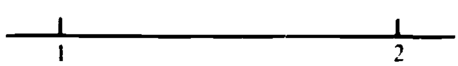
Table 4.27: Frequency of 'fifths' and 'tenths' errors in question 4 at pre-test and delayed post-test (n = 59)

		Delayed post-test			
		'Fifths' error	'Tenths' error	Correct	Total
Pre-test	No attempt	3	4	0	7
	'Fifths' error	2	1	4	7
	'Tenths' error	0	0	6	6
	Correct	1	1		
	Total	6	6	10	

One child in the 'fifths' category at the pre-test moved to the 'tenths' category, but no child moved from the 'tenths' category to the 'fifths' category. All the children in the 'tenths' category at the pre-test were correct at the delayed post-test. It is tempting to suggest that the 'tenths' response is a higher order response than the 'fifths' response, but the numbers in both groups was small. The 'tenths' response can be regarded as more sophisticated in that an attempt to plot the fractions as numbers seems to have been made, whereas the children in the 'fifths' category treated the fraction as a part of a whole, the 'whole' in this case being the whole line.

The next question to be discussed also made use of a number line:

18. Mark on the line, and label, 3 numbers between 1 and 2. If you think there are none, write 'none'.



How many numbers do you think there are between 1 and 2? . . . . .

The number of children who did the first part successfully was comparable to the number of children who successfully plotted the number $1\frac{1}{5}$ in question 4. The results are shown in Table 4.28.

Table 4.28: Frequency of responses to question 18, first part (n = 59)

Response	*Pre-test*	*Post-test*	*Delayed post-test*
Correct	36	50	50
'None'	10	3	7
No response	9	5	3

There was a significant increase in the number of children who correctly plotted three points. The worksheets in the teaching module made use of number lines, but the lines were sectioned, with marks where fractions had to be named. It seems that this experience was sufficient for most children to be able to section the line and mark in some fractions. It should be said again that a high degree of accuracy in the placing of the fractions was not required. There remained a group of children who maintained, even after the teaching, that there are no numbers between 1 and 2, and who still seemed to think that the word 'number' means 'whole number'.

For the second part, those children who said 'loads', 'too many to count' or 'as many as you like' were classed as correct. There were many other responses, some of which are shown in Table 4.29.

Table 4.29: Frequency of different responses to question 18, second part (n = 59)

Response	*Pre-test*	*Post-test*	*Delayed post-test*
Correct	6	9	10
'None'	8	5	6
3	12	13	11
6	4	1	2
1	4	0	1
No response	18	10	9

Although the teaching module appears to have increased the children's ability to find points on a number line between two integers, it does not seem to have helped more than four children to see that there are infinitely many such points. This was not an objective for the worksheets, but the children did work with lines divided into thirds, fourths, eighths and twelfths, for example. The jump between working with several different families of fractions and saying that there are infinitely many such fractions seems to have been too big a one for these children. This problem relates to the larger issue of the distinction between countable discrete points on a line and the idea of a continuous quantity, and this will be discussed further later. The reply '3' is probably related to the three points asked for in the first part of the question, for which most children chose 1/4, 1/2 and 3/4.

The number of children saying there were no numbers between 1 and 2 is the more surprising since the question, in asking 'how many?', implied that there were some and so was loaded against the reply 'none'. These children are again denying the existence of fractions between 1 and 2.

The last two questions in this section placed fractions in the context of ordinary arithmetic. Some solutions in each case were whole numbers, but others were not.

T1. 6. Find 5 pairs of numbers that add up to 3: . . . .
T2. 6. Find 6 pairs of numbers that add up to 6: . . . .

By being asked for five or six pairs of numbers, the children were really forced into producing some examples with fractions and the number who did so was 34 at the pre-test, and 49 at delayed post-test, suggesting that the teaching module had resulted in another 15 children being prepared to offer fractions in response to a question that asked for 'numbers'. Forty-five children used 1½ + 1½ as their example, while 21 used 1⅓ + 1⅔ and three used 2⅓ + ⅔. There were some eccentric offers: 9/3 + 0, 3/3 + 3/3, 1/3 + 2/3, 4/2 + 2/2 and just 12/4 or 9/3.

T1. 10. Put the missing numbers in the boxes. If there is *no* number, write 'no' in the box.

(a) $5 \times \square = 15$ (b) $3 \times \square = 18$ (c) $2 \times \square = 7$

(d) $4 \times \square = 10$ (e) $2 \times \square = 1$ (f) $8 \times \square = 5$

(For T2, part (c) was: $2 \times \square = 9$.)

The first two examples were of whole numbers and will be ignored. The number of children with each of the parts (c) to (f) correct is shown in Table 4.30.

Table 4.30: Number of correct responses to question 10 (n = 59)

	Pre-test	*Post-test*	*Delayed post-test*
(c)	28	35	36
(d)	16	22	25
(e)	24	43	40
(f)	0	1	0

There was a considerable increase in the number of correct responses from pre-test to post-test for parts (c), (d) and (e), and this was sustained at the delayed post-test. By contrast, the teaching intervention did not help the children to solve $8 \times \square = ?$. This is not surprising since the teaching module was concerned only with presenting the idea to children that it is possible to multiply by a fraction, and did not attempt to develop any skills in multiplying by a fraction. The examples in the teaching module were of multiplying by one-half or of using a calculator to multiply by numbers just more than or less than one. So, on the test paper, only parts (c) and (e) reflected the teaching of the module, and even here the questions did not present the visual models used by the worksheets. Some nine children appear to have transferred what they had learnt to part (d) of the question, in which they had to find a multiplying factor of 2½. It is interesting to note that the number of children who said that there was 'no number' that could be found was larger for this part than for parts (c) and (e) and very much larger for part (f). The actual numbers are shown in Table 4.31.

Table 4.31: Frequency of response 'none' to question 1

'None'	*Pre-test*	*Post-test*	*Delayed post-test*
10(c)	28	23	21
10(d)	32	33	26
10(e)	27	6	12
10(f)	42	22	27

There is thus quite a body of opinion that no number exists that could complete the equations. In other words, almost all those children who were not successful at answering the question said that it was impossible to find such a number. These, again, appear to interpret the word 'number' as meaning 'whole number'.

The reduction in the number of responses of 'none' for 10 (c) corresponds to the increase in correct responses. But with 10 (f), where the number of 'nones' is also reduced, there was no improvement in the number of correct answers. In this case the question was ignored.

OBSERVATIONS ON THE NUMBER RESULTS

The teaching module does seem to have been successful in helping many of the pupils to realize that fractions are numbers. More than three-quarters of the children were successful in six of the items at the delayed post-test, this representing an increased success for at least 14 of the 59 children. The items for which the teaching intervention seems to have been successful are shown in Table 4.32.

Table 4.32: Number items with increased scores at delayed post-test

Question	*Pre-test*	*Delayed post-test*	*Increase*
2½ is a number	34	50	16
13/5 is a number	22	42	20
3/5 plotted on line	25	43	18
1⅕ plotted on line	35	53	18
3 fractions between 1 & 2	36	50	14
Fractions that add to 5	34	49	15

The idea that fractions are numbers was implicit in the activities of the worksheets, depending largely on instances in which fractions are treated in the same way as integers. Thus fractions and integers were plotted on number lines, visual models of multiplication used pictures of 'halves' as well as 'wholes', and calculator activities used integers and fractions. It was hoped that such experiences would enable the children to infer that fractions can be seen as an extension to the number system, and not just as 'parts of a whole'. The teaching module appears to have been successful in enabling these 14 or so children to extend their view of fractions and to be able to think of them as numbers.

However, there were children who, after the teaching, remained committed to the idea that when the word 'number' was used it meant 'whole number' and who refused to acknowledge the existence of numbers other than integers. In answering each of the number questions there were children who either ignored fractions or who said there were none when asked to provide fractions to replace 'missing' numbers. These results are collected in Table 4.33.

Table 4.33. Number of children avoiding or denying the existence of fractions

Question	*Pre-test*	*Post-test*	*Delayed post-test*
1 No fractions underlined	24	15	6
4 No fractions marked on number line	18	5	5
11 No numbers between 1 and 2	10	3	7
No response	9	5	3
6 No fractions given that add up to 5	25	25	10
10 There is no number 2 × □ = 7	28	23	21
4 × □ = 10	32	33	26
2 × □ = 1	27	6	12
8 × □ = 5	42	22	27

Mixed numbers were more easily accepted as numbers. At the pre-test, some ten to 12 more children accepted 2½ as a fraction and plotted 1 on the number line than accepted fractions like 3/4 or 13/5. The delayed post-test showed similar results. There were also ten fewer children who said that there was no solution to 2 × □ = 7 than to 8 × □ = 5.

Although, in general, there were fewer children who avoided fractions after the teaching experiment, there remains a substantial number who rejected the notion of fractions as numbers and denied their existence where possible.

This phenomenon was observed in the results for the division and equivalence also. The implications of this are now discussed.

General implications

The teaching module, as a whole, had as a major objective the intention of extending children's view of a fraction beyond that of a 'part of a whole'. The interviews described in Chapter 4 suggested that the only view of a fraction that was familiar to all the children was that of a geometric part of a whole such as, for example, a circle cut into '*b*' parts of which '*a*' are taken. It is suggested that, for these children, this view of a fraction was incompatible with that of the division aspect, in which '*a*' things are shared between '*b*' people. The teaching module appeared to enable all the children to carry out such sharing activities successfully, and to be more successful at the immediate post-test. But, in Piaget's terms, it seems that there was too much of a cognitive conflict for these children to resolve. Fischbein (1976) refers to the inhibiting presence of 'false primary intuitions', and the geometric 'part of a whole' view of a fraction would seem to fit into this category. It is argued that this is also the reason why many children were unable to think of fractions as numbers. When asked to plot the fraction 3/5 on a number line marked from 0 to 4, seven children placed the fraction three-fifths of the way along the line, treating the line as a 'whole' of which 3/5 was needed. When asked to underline the 'numbers' in a set that included the fractions 3/4, 2½ and 13/5, the numbers 2½ and 13/5 were more often chosen than 3/4. It was argued that it was the familiarity of the fraction 3/4, and the likelihood that this was thought of as three-quarters of a circle, for example, that made it more difficult to think of it as a number. The reluctance of children to think of fractions as numbers produced a strategy of avoiding them when they appeared in other than their more familiar form. This strategy is now described.

There were instances, in all three aspects of fractions studied, of children who appeared to avoid fractions or deny their existence wherever possible. A number of children remained, even after the teaching, who were reluctant to think of a fraction as a number, and who interpreted the word 'number' to mean a whole number. Children avoided using fractions in all three aspects of fractions studied. In the division items this was effected by reversing the order of the division, writing or thinking of 4 ÷ 12 as 12 ÷ 4 About one-quarter of the children appeared to think that division is commutative, and the teaching experiment had little positive effect in altering this view. These children gave the same answer to 12 ÷ 4 and 4 ÷ 12, to 3 ÷ 4 and 4 ÷ 3 and chose both 3 ÷ 4 and 4 ÷ 3 as a representation of 3/4. Other children avoided fractions in the items 12 ÷ 4 and 4 ÷ 12 by giving answers '0' or '1 remainder 1' or making no response to 3 ÷ 4 or 4 ÷ 3. In the item on equivalence in which a missing number had to be found to complete a pair of equivalent fractions, children avoided fractions by saying that there was no such number, or making no response. Fractions were also avoided in the two number line questions in which children were unable to plot fractions on a number line or said that there were no numbers between 1 and 2. The missing number items which made use of fractional multipliers also produced from many children the response that no such numbers exist. These instances are collected in Table 4.34.

Table 4.34: Instances of the 'avoidance of fractions' strategy

Question	*Response*	*Pre-test*	*Post-test*	*Delayed post-test*
12 − 4, 4 − 12	Same answer for both	25	21	27
3 − 4	1 remainder 1	8	1	4
	0	6	3	5
	No response	9	6	11
4 − 3	1 remainder 1	7	1	5
	0	4	2	3
2 × ? = 7	'None'	29	25	21
4 × ? = 10	'None'	32	33	26
2 × ? = 1	'None'	27	6	12
8 × ? = 5	'None'	42	22	27
5/3 = 15/?	'None'	5	1	3
9/12 = 12/?	'None'	17	25	29
14/16 = ?/24	'None'	13	25	14
Mark in 3 numbers between 1 and 2	'None'	10	3	7
How many numbers between 1 and 2?	'None'	8	5	6

The number of instances of children's avoidance of fractions is remarkable. One explanation has already been suggested: that children's existing model of a fraction is based on the 'part of a whole' exemplified by parts of circles or squares. The teaching module sought to extend the children's view to include the division aspect and the idea that a fraction is a number. That this was successful for many children has already been demonstrated. But the results of Table 4.33 show that this approach was not sufficient for others. Skemp (1971) argues that a major accommodation of the number schema is required before fractional numbers can be understood. Fraenkel (1958) said that 'Bridging the abyss between these two heterogeneous domains (the discrete nature of number in the 'combinatorial' domain of counting and the continuous nature of points in space in the 'analytical' domain of measuring) is not only the central, but also the oldest problem in the foundations of mathematics and in the related philosophical fields.' It is perhaps not surprising that the intervention of a short teaching episode could not achieve this bridge for all children.

Further investigation

The next stage of the research was to test the use of the teaching module by teachers working in their own classrooms. Before moving to this stage it was deemed necessary to make both procedure for using the module, and, in particular, the place of discussion in the module.

5

Class trials

The previous chapter describes the way in which a teaching programme was devised with a view to alleviating some of the probl ms experienced by pupils with regard to fractions. The teaching programme was tested in three classes each split into two groups, one supervised by the researcher and the other by a student under the direction of the researcher e main aim of this exercise was to gain information as to the effectiveness of the teaching material. The next stage of the project was to assemble this teaching module into a 'kit' form which could be used by other teachers in their own cl sses. This 'kit' contained some teaching notes, the worksheets and necessary materials. Teachers from six schools volunteered to try out the kits and thus to take part in the class trials. They studied the teaching module and found classes for whom they considered the work would be appropriate. As before, the class trials were evaluated by the use of a pre-test, an immediate post-test and a delayed post-test. The marking of the worksheets was carried out by the researcher. The major reason for using class trials was to determine whether the teaching module was usable in a normal class situation. The results of the class trials were compared with those for the main experiment. It might be hypothesized that there would be a different outcome when the module was used independently, instead of being used by the author or her students.

The sample

Six teachers from different schools (designated by their initials S,C,H,A,F and M) agreed to take part in the experiment. The schools were comprehensive schools from various parts of the country – Cambridge, Essex, London and Lancashire. The classes who took part in the experiment were classified by the schools as of middle ability, containing pupils who would, in due course, be entered for a CSE in Mathematics. The pupils were, therefore, selected in the same way as those for the main experiment, but they were drawn from a wider age-range. Two of the classes were second-year classes, two third-year and two fourth-year. The number of children in each class ranged from 18 to 27, but there was a considerable degree of absenteeism in each class, so that some children missed one or more of the tests or the teaching sessions. The results of only those children who attended for the three tests and completed the worksheets were used, and this reduced the number of children for the experiment to 81. The number of children in each class whose results were used is shown in Table 5.1.

Table 5.1: Number of results used from each of the six schools constituting the class trials

Scho	*Number*	*Year group*
S	7	4
C	14	3
H	13	3
A	21	4
F	8	2
M	18	2

Procedure

The teachers received a short briefing as to the nature of the research and for the management of the teaching module. In particular, the philosophy behind the use of the worksheets was explained in some detail, as some of the teachers were not used to this style of teaching. The procedure for administering the tests was also explained. In specific instructions, the teachers were asked:

(a) to see that the tests were to be carried out formally;

(b) to encourage co-operative activity during the teaching sessions, with the pupils to work in pairs or small groups;

(c) to encourage discussion between pupils amongst themselves and with the teacher;

(d) to collect in the worksheets at the end;

(e) to make their own comments on the module.

In addition, the teachers were asked to discuss – when and how they wished – the following three points with their classes:

(f) that the word 'division' does not necessarily imply that the larger number has to be divided by the smaller;

(g) that, given a pair of equivalent fractions such as 3/4 and 9/12, the second is not three times as big as the first;

(h) that the word 'number' does not imply 'whole number' only, and that fractions are also numbers.

The teachers carried out the teaching and testing with the class they had chosen.

The tests were marked by the author at the completion of the trials, using the same marking scheme as that used in the main experiment. The results for the 81 pupils are first considered together. It might have been thought that, as there were second, third and fourth-year classes involved, the results would first be grouped by age. The CSMS longitudinal survey suggested that, generally, only small differences in attainment were observed, and it will be seen later that the results for this experiment are in agreement with that observation. Differences between the results of the six schools will be discussed later. The responses of the 81 children to the questions at the pre-test, immediate post-test and the delayed post-test are now discussed.

The items are, as before, grouped into those that concern division, equivalence and number and are scored by expressing the score obtained as a percentage of the total possible score. Figure 5.1 shows the results for the class trials and the main experiment at pre-test, immediate post-test and delayed post-test.

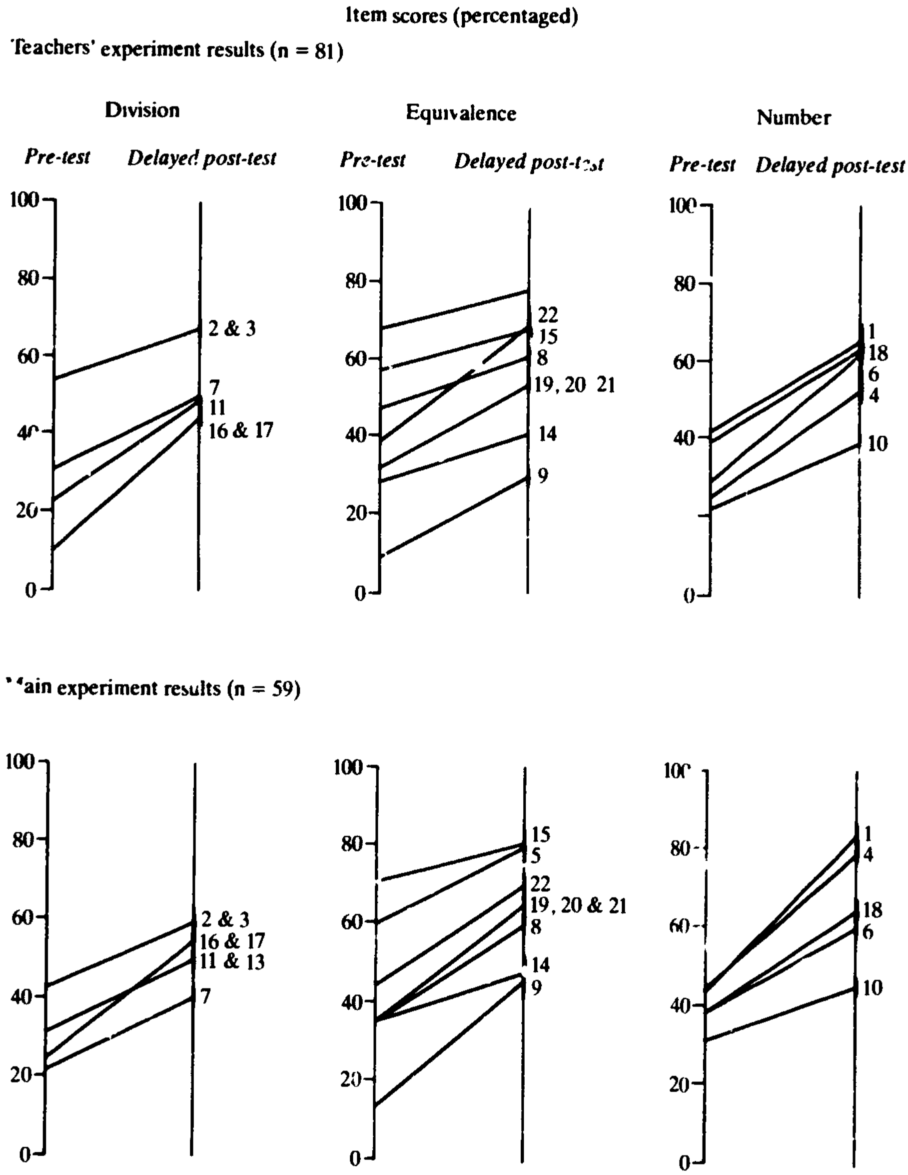


Figure 5.1: Comparison of teachers' experiment results for items grouped by division, equivalents and number with those for main teaching experiment

The results for the two sets of children can be seen to be comparable. The mean score for each group is found by expressing the total score for the group of items as a percentage of the possible total for the group. Table 5.2 shows the means for the class trials and for the main experiment.

Table 5.2: Comparison of mean scores for division, equivalence and number between class trials and main experiment

	Pre-test	Post-test	Delayed post-test
Division			
Class trials	30.5%	59.6%	54.0%
Main experiment	29.0%	64.6%	52.2%
Equivalence			
Class trials	38.8%	52.1%	54.2%
Main experiment	39.6%	59.8%	60.9%
Number			
Class trials	31.3%	49.7%	54.6%
Main experiment	34.8%	57.9%	61.7%

The results, generally, display the same characteristics. There was a significant increase at the 5 per cent level between the pre-test and the delayed post-test for each of the items in both the main experiment and the class trials. The results for the division items show a considerable increase at the immediate post-test that is reduced somewhat by the delayed post-test, whereas those for the equivalence and number items continue to increase between the immediate and the delayed post-tests. This suggests that there is, again, a difference between the effect of that part of the teaching module concerned with the division aspect and those parts relating to equivalence and number. In the case of the latter two, it seems that the worksheet activities have allowed the pupils to form the structures that allow for cognitive growth over a period of time.

The results for the three aspects are now considered in more detail. It will be remembered that the items on the tests had possible totals ranging from one to five. In order for comparisons between the class trials and the main experiment to be made, the actual total scores for the items at pre-test and delayed post-test, together with the total possible score, are now considered. This information for the division items is displayed in Table 5.3. The results are given as percentages of the possible score in each case.

Table 5.3: Results for division items for class trials (n = 81)

Item	Pre-test	Delayed post-test	Increase
2 and 3	54.7%	67.1%	12.3%
7	31.5%	51.9%	20.4%
11–13	22.8%	50.6%	27.8%
16 and 17	10.7%	44.4%	33.7%

The increase in scores for all items other than 2 and 3 is slightly greater for the class trials than for the main experiment, but the differences are not significant at the 5 per cent level. Although there were considerable increases for all items between the pre-test and the delayed post-test, it will be observed that the final scores were not high: only for items 2 and 3 ($12 \div 4$ and $4 \div 12$) was the score more than 66 per cent. The most noticeable increase was for items 16 and 17 ($3 \div 4$ and $4 \div 3$), but, of course, the initial score was much lower. The equivalence results are examined similarly, and are shown in Table 5.4.

Table 5.4: Results for equivalence items (n = 81)

Item	Pre-test	Delayed post-test	Increase
5	67.3%	76.5%	9.3%
8	46.9%	61.7%	14.8%
9	10.1%	31.2%	21 3%
14	29.3%	41.0%	11.7%
,5	56.5%	65.7%	9.1%
19–21	32.9%	53.9%	21.0%
22	39.5%	64.8%	25.3%

The scores at the delayed post-test are somewhat higher for several of these items; only those scores for items 9 (Find a fraction between two given fractions) and 14 (completing pairs of equivalent fractions) were less than 50 per cent. The increases in scores for the class trials are lower than those for the main experiment this time, the differences being significant at the 5 per cent level for items 5, 8, 9 and 19–21. There is no obvious reason why this should be the case. Table 5.5 displays the similar result for the number results.

Table 5.5: Results for number items (n = 81)

Item	Pre-test	Delayed post-test	Increase
1	42.0%	65.0%	23.0%
4	24.7%	51.2%	26.5%
6	34.0%	63.0%	29.0%
10	21.3%	39.5%	18.2%
18	39.5%	64.2%	24.7%

The percentage increase in the scores was quite high for all the items in this group. Item 10 (completing equations such as $2 \times \square = 1$) proved difficult, even after the teaching intervention, with many of the pupils opting for the response 'none'.

In the case of the first two items, the results for the main experiment were significantly better at the 5 per cent level. For the three other items, the class trials results were slightly better, but not significantly so.

There was no significant difference at the 5 per cent level between the main experiment and the class trials in ten of the 16 items on the tests. In the remaining six items, the results for the class trials were not as good as those for the main experiment, although there was still a significant improvement from pre-test to delayed post-test in these items There is no obvious reason why there should be any difference between the sets of results. The degree of commitment on the part of the experimenter and the students working with her might be thought to balance the desire of the class teachers for their children to do well at the tests. The fact that there was no significant difference between the two experiments for ten of the 16 items suggests that the class trials have largely confirmed the results of the main experiment. In the analysis so far, the results from the six schools have been combined and treated as a group. In the next section, the differences between the individual schools are examined. It should, however, be remembered that the number of children in each case is small. In order to compare the samples for each school, the performance of the individual children at the pre-test is first considered. Figure 5.2 shows this information. The total possible score for the test is 46.

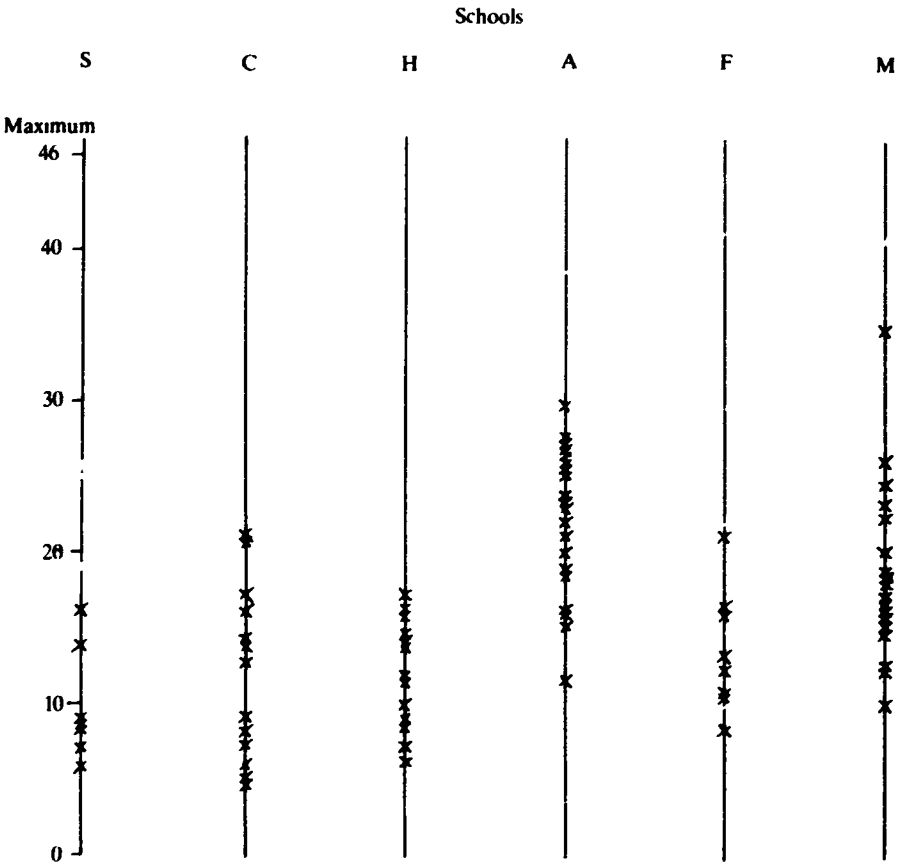


Figure 5.2: Scores at pre-test for the six teachers' experiment schools

It might have been thought that the older children would have been more successful at the pre-test than the younger ones, but this was not so. For example, all but three of the children in the second-year class of school M were more successful than the highest scoring child of the fourth-year class of school S. The performance of the second-year class of school F was comparable with the third-year classes of schools C and H. The classes had all been designated by their teachers as 'middle ability' classes and likely to take a CSE in Mathematics by the teachers, but, of course, schools' standards differ, and this may be one reason for the disparities in the scores. Another possible explanation as to why the older children were not necessarily more successful than the younger ones is that repeated lack of success with fractions had produced a depressing effect. There are children who meet ideas of fractions in each school year from the age of ten onwards, and experience a sense of failure each time. One of the teachers who took part in the class trials reported that the reaction of her class was 'Oh no, not fractions again!'. It is of interest to compare the effectiveness of the teaching module with each class. Figure 5.3 shows the scores of the children from each school at pre-test and delayed post-test.

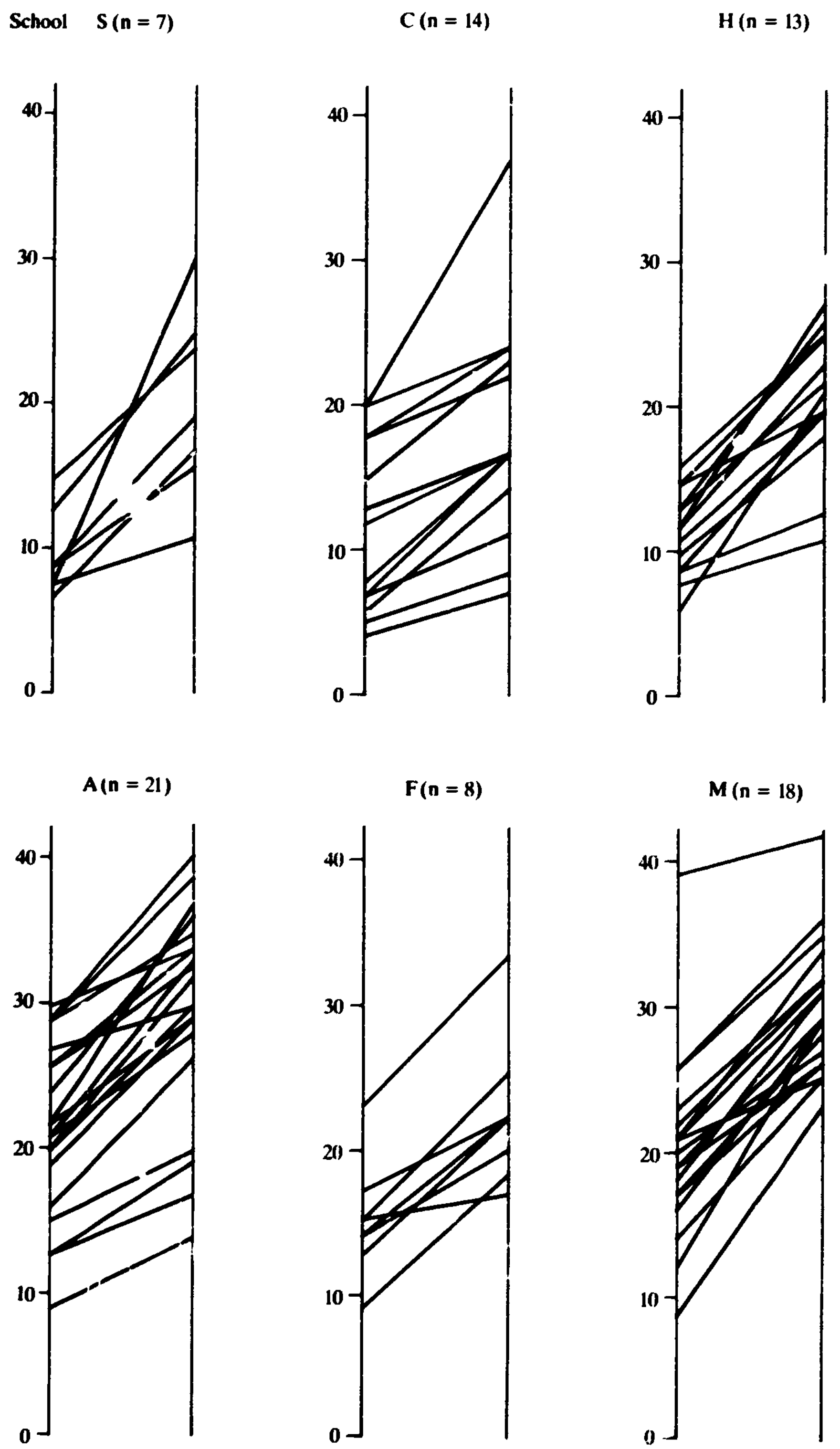


Figure 5.3: Scores at pre-test and delayed post-test by children at the six class trial schools

The mean score for each school at the two tests is shown in Table 5.6, but it should be remembered that the number of results for each school is small.

Table 5.6: Mean scores for each school at pre-test and delayed post-test (maximum score = 46)

School	*No. of children*	*Pre-test*	*Delayed post-test*	*Increase*
S	7	9.9	20.1	10.2
C	14	11.9	18.6	6.7
H	13	11.5	20.8	9.3
A	21	21.8	30.4	9.4
F	8	15.0	22.4	7.4
M	18	20.0	29.7	9.7

The increases in scores for the six schools seem comparable, school S showing the greatest increase and school C the least. The increases in scores from pre-test to delayed post-test for the individual children from these two schools are shown in Table 5.7.

Table 5.7: Comparison of increases in scores of individual children in schools S and C

	School S (n = 7)	*School C (n = 14)*
Mean increase	10.3%	6.71%
S.D.	5.5%	3.6%

$$t = \frac{10.3 - 6.7}{4.55\sqrt{\frac{1}{7}+\frac{1}{14}}} = 1.72$$

With 19 degrees of freedom, the difference between S and C is not significant at the 5 per cent level. Similarly, no significant difference was found between the other schools.

There was a notable increase in score of 22 for one child in school S, from 8 at the pre-test to 34 at the delayed post-test, out of a possible 46. Two children from school H and two from school A improved their scores by 15 marks. At the other end of the scale, there was one child in each school whose increase in score was only 2 or 3.

It was not the case that those children who scored a low mark at the pre-test improved their score by a smaller amount than those who scored better at the pre-test. There are several examples of children who had relatively high scores at pre-test who did not improve much and of children who had a low score at pre-test who made a considerable improvement.

Division results

The way in which the children from the six schools responded to the individual items on the test is now examined. Figure 5.4 shows the results for the division items.

Some interesting differences emerge, particularly related to the scores at the pre-test. School M obtained scores of about 80 per cent at items 2/3, and 11/13 at the pre-test, while schools S and C scored less than 30 per cent for the same items. Questions 2/3 were 12 ÷ 4 and 4 ÷ 12, while 11/13 concerned the sharing of milk between jugs. School A did relatively well at item 16/17 (3 ÷ 4 and 4 ÷ 3), at which the children of schools S, C and H failed to score. Question 7, in which the

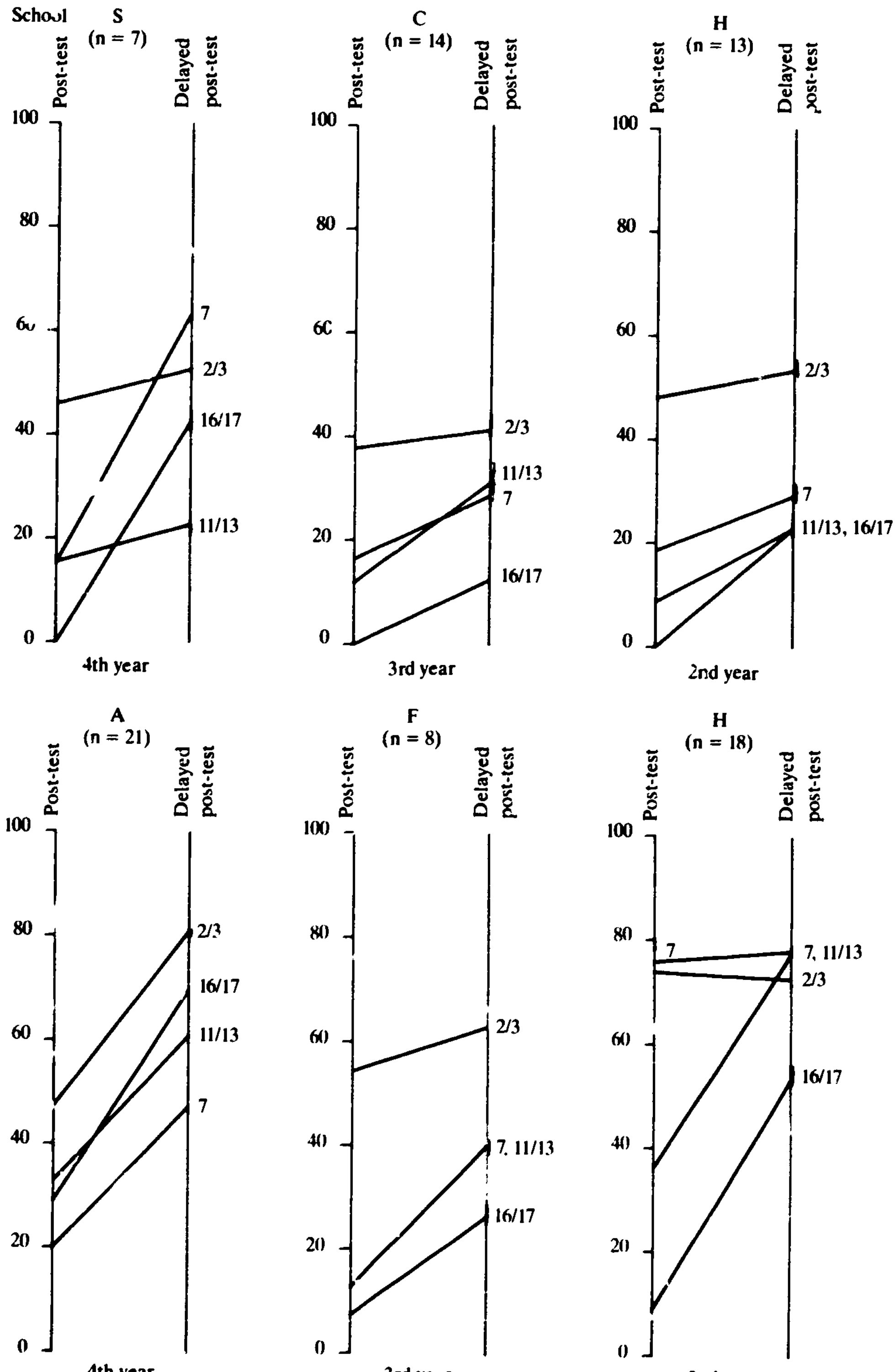


Figure 5.4: Results for division items for the six class trial schools

children were asked to indicate acceptable forms of the fraction, including 3 ÷ 4, showed the greatest disparity, varying from a facility of less than 15 per cent for both schools S and C to one of over 80 per cent for school M. One possible reason for these variations could be differences in the ability of the children. Although they were all in what were classified as middle-ability classes and likely to be entered for a CSE in Mathematics in due course, estimates from schools vary. It is also possible that the variations in the results reflect a difference in the syllabus adopted by the schools, or in the emphasis placed on the division aspect by the individual teachers.

There was an increase in the score at the delayed post-test for each division item for all six schools. Three schools, S, A and F, showed more than a 25 per cent increase for item 7, school M having a facili of about 80 per cent for this item both before and after the teaching. The teaching also appeared to have been successful with item 16/17 (3 ÷ 4 and 4 ÷ 3), for which three schools, S, A and M, also showed an increase in facility of over 25 per cent. There was only a very small increase in the scores for item 2/3 (12 ÷ 4 and 4 ÷ 12) for five of the schools, although school A showed an increase of 25 per cent. School C appears to have accrued the least benefit from the teaching, but, in general, the division results are comparable to those of the main experiment.

Equivalence results

Figure 5.5. shows the results for the equivalence items.

The results at the pre-test for the equivalence items are more spread out than those for the division items. School A, for example, scored over 90 per cent for item 5 and only just over 10 per cent for item 9.

Schools C and H performed relatively badly at item 5 (Jane has 3/4 of a bar of chocolate, John has 9/12), at which the other schools scored well, but school C showed an increase in score of over 25 per cent at the delayed post-test. School S, which had a score of nearly 80 per cent for the item at the pre-test, showed a fall of nearly 20 per cent by the delayed post-test. The same school also showed a decrease, albeit much smaller, for question 18 (Mark in 3 numbers between 1 and 2 on the number line). The apparent negative effect of the teaching is, in part, attributable to one child whose scores declined for both items and also for 19/21. Her total scores for the test were very low – 8, 13 and 11 for the pre-test, post-test and delayed post-test. There was another child in the group whose delayed post-test score was less than that of the pre-test. Both children were described by the teacher as 'not very co-operative'. There was another instance of a decrease in sc re by the delayed post-test: item 8 for school F. Here there were two children whose scores at the three tests for this question were both 3, 3 and 1. In the absence of any information about these children, one can only comment on the similarity of their results. The worksheets seem to have been particularly successful with respect to question 9 (Find a fraction between two given fractions), for which three schools showed an increase of over 25 per cent. Similar increases were shown by two schools, S and F, at item 15 (ordering fractions), and by schools H and A for item 22 (find fractions equivalent to 3/8). The only conclusion that can be drawn about the equivalent results is that, while there was an overall increase from 38.8 per cent to 54.6 per cent from the pre-test to the delayed post-test and ten individual increases of more than 25 per cent, there were substantial differences between schools both in their performance at the pre-test and their reactions to the teaching module.

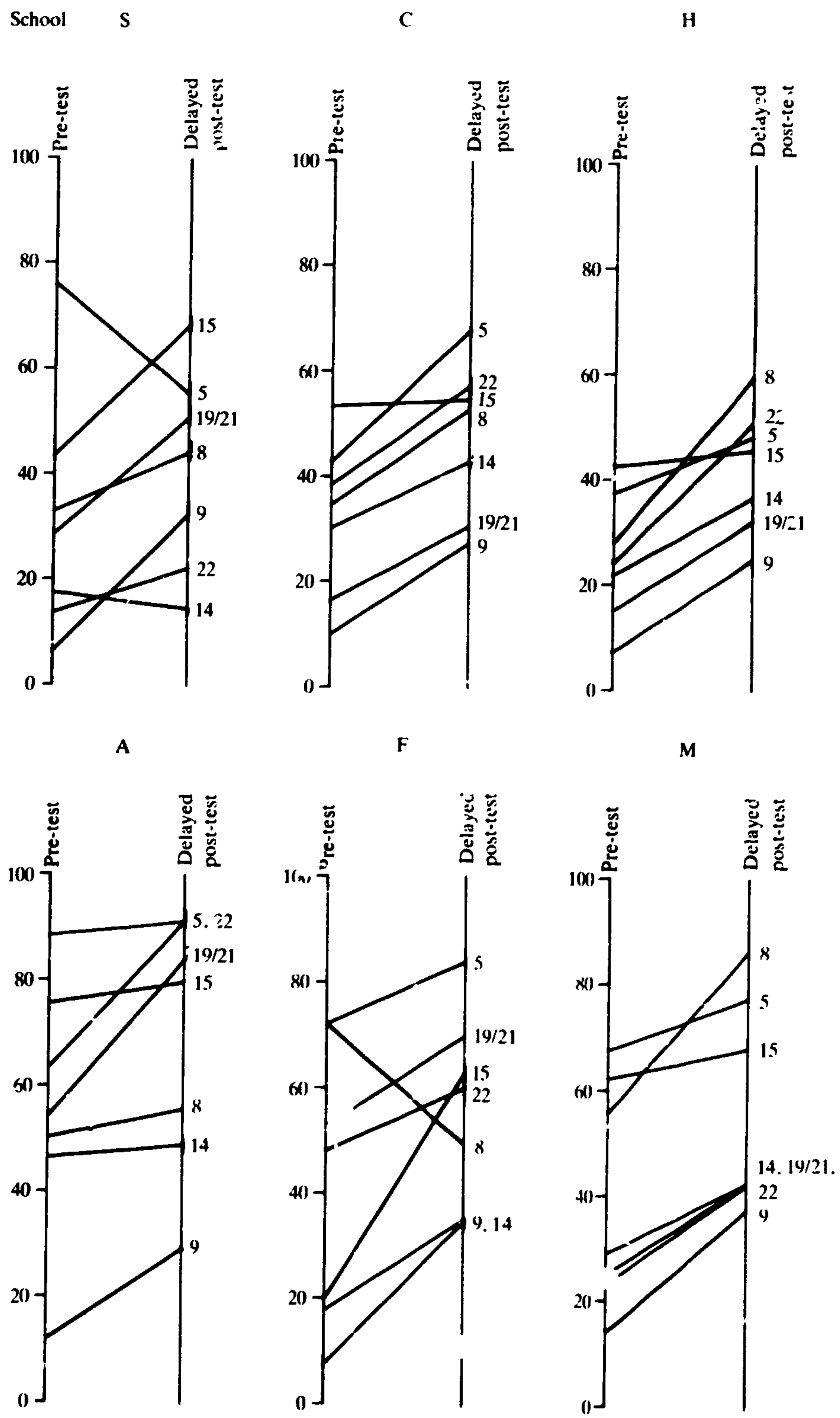


Figure 5.5: Results f.. equivalence items for the six class trial schools

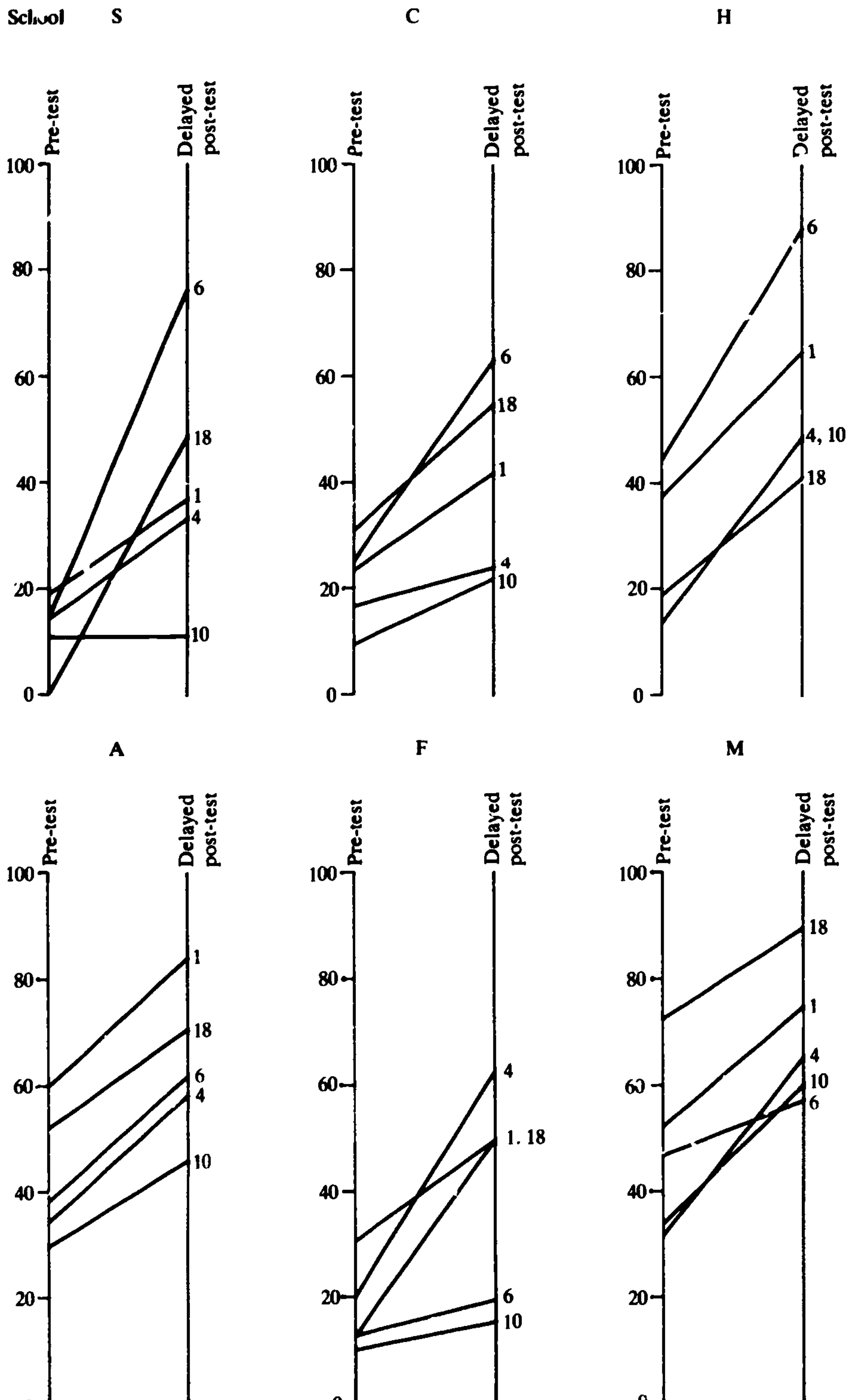


Figure 5.6: Results for number items

Number results

The results for the six class trial schools for the items that concern the fact that a fraction is a number are now discussed. Reactions to the individual number items are shown in Figure 5.6.

Schools A and M performed relatively well at all the number items at the pre-test. Three of the schools, S, C and F, scored less than 30 per cent at any of the five items at the pre-test, while two scored more than 30 per cent at all five. All the items showed an increase in score at the delayed post-test. The increases for item 1 (Underline the numbers in the set . . . .) and 10 ($2 \times \square = 9$ . . . .) were fairly modest, but in item 4 (Mark in fractions on the number line), four of the six schools showed an increase of more than 25 per cent. This was also the case for item 6 (Pairs of numbers that add up to 5), and in one case the increase was more than 50 per cent. Three schools showed an increase of more than 25 per cent for item 18 (Numbers on number line between 1 and 2), one of which was an increase of over 50 per cent.

It appears that the worksheets on the number aspect of fractions have increased the children's performance at the number items in the test in the six schools, and, in several cases, the increases were large.

Feedback from teachers

The teachers who took part in the class trials were invited to make any comments they wished both on the presentation and the suitability of the material in the teaching module. Only two of the six schools made more than superficial comments on their reaction to taking part in the experiment. One suggested that the worksheets should include some worked examples, and that 'answers and explanations' of the points suggested for class discussion should have been given. The other teacher said that her class had previously experienced repeated failure with fractions, and, not surprisingly, were not keen, at the outset, to spend further time on them. However, she reported that the children enjoyed the worksheets, although they found the style rather unfamiliar. The class had been used to a formal style of working, very much teacher-directed, and were rather hesitant to respond to the questions which asked them to say what they noticed about a result or pattern of results.

Bearing in mind the fact that the teachers had agreed to take part in the experiment, and so might be regarded as relatively interested in the teaching of Mathematics, their comments suggest that the documentation accompanying the teaching module was insufficient It had been thought that the style of the worksheets and their approach to the teaching of Mathematics would be familiar to the teachers, but this seems not to have been the case. Certainly, the teacher who wanted worked examples on the worksheets has misinterpreted the author's intention that the worksheets should present the children with experiences that would help them to restructure their thinking about the nature of fractions. It is the author's opinion that 'worked examples' will, in general, offer pupils short-term methods of getting similar problems right, rather than the opportunity to accommodate new ideas. It may be that this philosophy needs to be explained to teachers using the module. Similarly the comment by a teacher that 'answers and explanations' of the discussion points should have been given suggests that the purpose of the discussion was not understood. This was to explore some of the misunderstandings that had been observed during the interview stage of the research, such as, for example, the belief that, in division, the larger number is always divided by the smaller. The worksheets produced several pieces of evidence to the contrary, by making use of sharing activities and calculators to illustrate the division of a smaller number by a larger. The purpose of the discussion in this case was to help those pupils for whom this evidence was in conflict with their beliefs about the nature of division to resolve that conflict. The discussion point on equivalence concerned the conflict, also observed during the interview stage, between the idea that two fractions are equivalent but that one fraction is a multiple of the other. It was hoped that the discussion would relate to the experience of the worksheets, so that again pupils might be able to resolve their conflict. The last discussion point related to the view that the word 'number' means 'whole number', and how this relates to the experiences of the worksheets. The whole purpose of the discussion was not to give explanations, but to allow pupils the opportunity to talk about some aspects that have been seen to cause difficulty.

Summary

The main reason for testing the teaching module in more schools was to see whether it could be used effectively in any classroom, with the pupils being taught by their regular teachers. The comparability of the results with those of the main experiment suggests that it is possible to use the teaching module effectively by presenting it as a package for teachers together with some supporting information. Another reason for carrying out the class trials was to try the teaching module on a larger sample of children. Taking the 81 children from the six different schools together, the results of the teachers' experiment are comparable with those of the main experiment. There is a wider range of age-group in the children who took part in the teachers' experiment, but it did not follow that the older children were always more successful than the younger ones. As in the case of the main experiment, some of the children with the lowest scores at the pre-test obtained higher scores at the delayed post-test than those who had been more successful at the pre-test. This was particularly the case for schools S, C, H and M. There were considerable differences between the results for specific items at the pre-test and in the increase in scores at the delayed post-test.

The teaching module certainly seems to have had some success when used by the six schools involved in the class trials. The comments of the teachers suggest, however, that if it is to be used on a wider scale fuller documentation will be needed. This should include more information on the reasons for the choice of style of the worksheets and the nature of the discussion that is expected.

6

Summary and implications

This research was based on a series of interviews with children of 13 to 14 years of age who were in middle-ability classes for mathematics. These interviews indicated certain areas of difficulty concerning fractions. A teaching module, consisting of a set of activities which attempted to help children overcome these difficulties, was then constructed and tested. The effectiveness of the teaching module was evaluated through observation by the researcher as participant/teacher in the trials and by means of a pre-test, an immediate post-test and a delayed post-test taken some six weeks later.

A summary of the main findings of the research appears in this chapter, together with a discussion of the results. In addition, some suggestions are given concerning the implications of these findings for the teaching of fractions and for further research.

The nature of children's difficulties with fractions

Research evidence from both CSMS and from other parts of the SESM project suggests that many children do not use formal 'taught' methods in mathematics, but use, instead, their own informal methods (see Booth and Hart, 1982 and Booth, 1984). In the case of fractions the position appears to be somewhat different: here, children are seen to rely on rote memory of previously learned techniques. Teachers will be familiar with the problems that ensue when half-remembered rules are inappropriately applied, and the present study has highlighted some of these. The underlying problem appears to be a lack of any attachment of meaning to the notion of a fraction (see Hasemann, K., 1981). With the exception of certain simple examples such as 1/2 and 1/4, fractions do not form a normal part of a child's environment, and the operations on them are abstractly defined and not based on natural activity. Thus this research has concentrated on difficulties in interpretation of certain aspects of the idea of a fraction.

Summary and discussion of findings

1. Models of fractions

The only model of a fraction that was familiar to all the children who took part in this study was the 'part of a whole' one.

SUMMARY

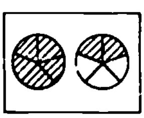
The research showed that, while the interpretation of the fraction *a/b* as a geometric region which is divided into '*b*' parts of which '*a*' are taken was acceptable to all the children who took part in the interviews or the teaching experiments, other possible interpretations were not. During the interviews, children were invited to consider a number of models of the fraction 3/4 or 3/5 for discussion. These included a part-group and a ratio model: not only were both of these rejected as possible instances of the fractions by most children, but the reason given was, invariably, that there was no whole shape to be divided up. An illustration of three red counters and one blue counter was said by some not to illustrate 3/4 because 'it's not a whole thing', and a ratio model of three red and four blue counters was rejected because 'it's got no shape'. A number line, marked from 0 to 1, with the fraction 3/4 marked on it, was more often accepted as a model of 3/4. It seemed, however, that the diagram was given a 'part-whole' interpretation, in which the line was split into four parts, with the fraction 3/4 marking the third part, rather than being seen as a point on the line. For example, TE said 'There's three parts here and one there.' Children referred to the 'part of a whole' model when trying to explain other problems during the course of the interviews, such as adding the fractions 2/3 and 3/4, or deciding which of two fractions was the bigger. GP thought that 3/4 and 1/3 were 'the same', but then drew diagrams for both and said 'That's not right.' VK, having produced an answer of '2' for the fraction 2/3 + 3/4 said 'Can I draw it? . . . No, because if I put this shape [a circle with 2/3 shaded] in here, it would be one and a bit.' Others used similar diagrams to confirm their errors, so that JC, for example, shaded 2/3 of one circle and 3/4 of another and said 'Five-sevenths. There are seven pieces and there are five shaded in.'

The results of the teaching experiment confirmed those of the interviews in showing that no child had any difficulty in making the connection between the fraction 3/4 or 3/5 and a circle or square in which three-quarters or three-fifths was coloured. The ubiquity of the 'part of a whole' model is also reflected in the textbooks examined; nearly all were seen to make use of it. Similarly, of the teachers who took part in the survey, about five in eight of them said that they made use of this model in teaching fractions. A reason offered by the others for not so doing was that such models were unnecessary as the children were already familiar with them.

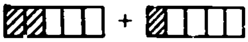
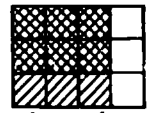
DISCUSSION

The familiarity of the 'part of a whole' model found in this research supports the outcome of other studies, such as that of Hart, 1981, in that the ability to shade in two-thirds of a rectangle marked in three sections was classed as a Level 1 activity for 12- to 13-year-olds, with a facility of 87–93 per cent. Silver (1981), found that most of the young adults with whom he worked had a circular region as their only model of a fraction.

It must be said that the emphasis on this particular model is not necessarily helpful. The interviews suggested that the use of geometric 'part of a whole' models to explain the addition of fractions is open to misinterpretation. Of the 14 children shown the diagram



, eight said that it represented the fraction 7/5 while the others said it was 7/10. A similar response was found to the diagram

 , with the same eight children interpreting it as 3/5 and the others as 3/10. The use of such diagrams seems, at least, ambiguous. It could further be argued that it is misleading even to suggest that a fraction of one shape can be added to a fraction of another shape: it is comparable to trying to add apples and pears. The problem is compounded further in the use of diagrams to illustrate the multiplication of fractions that were found in the contemporary textbooks described in Chapter 2. They were of the form  for the product 2/3 × 3/4; this was also the model most used by the teachers who took part in the survey. In this diagram, the 3/4 is seen as part of a rectangle, while the first fraction, 2/3, is treated as an instruction to take two-thirds of the resulting shape. The two fractions are thus interpreted differently, since it is patently absurd to try to multiply a part of one shape by a part of another shape. It may well be that the notion of operating on fractions should be delayed until the child no longer needs to think of a fraction as a part of something else, but can see it as a number in its own right. This would enable the treatment of fractions to be conducted without recourse to 'part of a whole' diagrams.

Not only are 'part of a whole' diagrams possibly misleading, but, more seriously, it will be argued later that their use may well inhibit the development of other interpretations of a fraction; that, for example, a major accommodation is required before a fraction can be thought of as a number or as the result of dividing the numerator by the denominator.

The worksheets that made up the teaching module made no reference to 'part of a whole' diagrams, but, instead, were designed so as to provide a number of other models and approaches. The particular aspects of fractions with which they were concerned will be discussed in subsequent sections.

2. *The division aspect of a fraction*

The division aspect of a fraction was not familiar nor readily accepted.

SUMMARY

Although, as already seen, the 'part of a whole' model was readily accepted by all the children who took part in the research, only a very few of them were familiar with the interpretation of the fraction a/b as 'a' things shared between 'b' people, or as $a \div b$. The teachers who took part in the survey also seemed reluctant to acknowledge this aspect: more than one-half of them said that they never made reference to it in their teaching. The suggestion, made during the interviews, that there might be such a connection between the fractions 3/4 or 3/5 and 3 ÷ 4 or 3 ÷ 5, was, in the main, firmly rejected by the children. The problem of which way round to interpret the division sign soon emerged, but the advantage of the interview style of investigation was that it was possible to discuss this, and to focus on the instance of the smaller number being divided by the larger, whichever way the child read the division symbol. Common responses that were given were 'You can't do that, four's bigger than three', 'That's the wrong way round' or 'Four shared by twelve . . . you can't do that, twelve is bigger than four.'

When presented with the actual task of sharing three cakes between four children, many were then able to see the connection between 3 ÷ 4 and 3/4. Others were not: AE, for example, suggested that it was possible to do it with cakes, but not with numbers. The teaching module invited children to perform similar tasks, given pictures of objects such as cakes, bars of toffee and jugs of milk, which were to be shared in different ways. In addition, the worksheets contained examples that made use of calculators; the children were asked to use the calculator to find, for example, $1 \div 2$, and to notice the connection between $1 \div 2$, 0.5 and 1/2. The calculator was also used to evaluate pairs of fr ctions such as 3/15 and 15/3, to emphasize the non-commutativity of division.

The results from the main teaching experiment showed that many children can, with only a short period of time devoted to such activities, extend their view of fractions to include the idea of a division activity: the mean scores rose from under 30 per cent at the pre-test to nearly 65 per cent at the immediate post-test. The fact that the mean score fell again to just over 50 per cent by the delayed post-test suggests that some children were able to change their view of a fraction to include the division aspect in the short term only. A similar pattern was observed in the results of the class trials. The items which caused the children problems in the longer term were, invariably, those that presented pairs of divisions in symbolic form. The number of cnildren, out of the 59 who took part in the main experiment, who were able to give the right answer to both $12 \div 4$ and $4 \div 12$ rose from eight to 24 immediately after the teaching, but fell back to ten at the delayed post-test. When the question concerned two pints of milk being divided equally between three jugs, the increase in the number of successful children was largely sustained at the delayed post-test. The existence of a context helped more children to keep hold of the notion that division and fractions are connected.

An interesting observation was the number of children who employed strategies which resulted in the absence of any reference to fractions. Nearly one-half of the children who took part in the main experiment gave the same answer, 3, to both $12 \div 4$ and $4 \div 12$ at the pre-test: when different answers were given, about one-quarter gave the response '0' for $4 \div 12$. Similar results were found for the pair $3 \div 4$ and $4 \div 3$, for which responses '1 remainder 1' and '0' were given. The teaching module had little positive effect on these children.

DISCUSSION

It is perhaps surprising that the interpretation of a fraction a/b as $a \div b$ was found to be so unfamiliar. A review of some contemporary textbooks showed that, for the most part, mention was made of this idea. The fact that the presentation of the idea was not, on the whole, followed up with examples for the pupils to try may well have something to do with the problem. It has been observed that children seem to ignore the explanations and move straight to the exercises. Activities which lead to the notion of a/b as $a \div b$ are more practical in nature than the comparable shading-in tasks associated with the 'part of a whole' model, and this may be a reason why they are not so often found in textbooks. They are dynamic activities, involving the notion of cutting and sharing out; the process is not as easily illustrated on the printed page as the shading of parts of a whole shape, in which the end state indicates very clearly what has taken place.

The ready availability of the 'part of a whole' model may itself be the inhibiting feature. If, in thinking of the fraction 3/4, say, the image that immediately springs to mind is that of a circle split into four parts of which three are shaded, then it may prove too difficult to adjust to an alternative image of three circles and four people. The fact that there were children who were able to carry out the activities of the worksheets and to meet with a considerable success at the division items in the short term that was not sustained by the delayed post-test suggests that necessary readjustment of thinking was beyond the scope of these children, given only a limited period of teaching intervention.

Another reason why some children find it difficult to make a connection between, say, $3 \div 4$ and 3/4 may stem from the way in which they are introduced to the idea of division in the early years at school. Children are heard to make general statements of strategy in dealing with division problems, such as 'You always divide the larger number by the smaller' or 'Fours into 3 won't go so you bring down a nought.' When the divisor is not a factor of the dividend, the remainder usually consists of indivisible objects, such as sweets, which are placed on one side and recorded as 'remainder x'. It seems that some children fail to appreciate that although these general 'principles' were acceptable earlier when dealing only with integers, they are no longer valid when working with the set of rationals. Thus the 'errors', such as $12 \div 4 = 4 \div 12$ and $3 \div 4 = 0$, can be regarded as instances of a failure to appreciate that fractions exist in this context; such items are dealt with according to the rules for integers. These children are operating a system in which fractions are rendered unnecessary: given the rule of 'divide the larger by the smaller', the expression $4 \div 12$ is restated as $12 \div 4$ and the need for a fraction eliminated. Similarly the rule '4 doesn't go into 3' results in the answer being recorded as '0', since in this case the strategy of reversing the order no longer helps.

3. *Fractions as numbers*

Many children found it difficult to accept the fact that a fraction is a number.

SUMMARY

Instances of children's reluctance to acknowledge that fractions are numbers were found during all the phases of the research.

Only about one-quarter of the children who took part in the survey described in Chapter 1, in which they were invited to express their view of what fractions are, made any reference to numbers. Where such reference was made, it was of the form of statements that fractions are 'broken up numbers' or 'not quite whole numbers' or 'split from numbers' or 'two numbers just put on top of each other'.

One of the interview questions invited the children to find some pairs of numbers that add up to 10. When no fractions were offered, which was invariably the case, the children were asked if they could find some that included fractions. This they almost all found impossible; in most cases the idea was emphatically rejected. Such attempts as were made included 5/5 + 5/5, 1/5 + 1/5 or 5/10 + 5/10. Difficulty was also experienced in placing fractions on a number line: this was the subject of one of the items on the tests used in the teaching experiment. In the main experiment, less than one-half of the 59 children were able to plot the numbers 3/5 or 9/5 on a number line at the pre-test: one strategy used by a group of seven children was to treat the line as whole, and place the fraction 3/5 at a point three-fifths of the way along the line. Another, more explicit, instance of children's reluctance to regard fractions as numbers was seen in their responses to items such as 'find the missing number in $2 \times \square = 7$'. Although 28 of the 59 children gave the correct response at the pre-test, the same number of children said that no such number exists. Similarly, when asked to plot three numbers between 1 and 2 on a number line, ten children said that there are none, and a further nine children made no response. After the teaching intervention, which included calculator activities, work with number lines and some number patterns, the number of children who were successful at these items was increased significantly: the mean scores for all the 'number' items rose from 38 per cent at the pre-test to 57.9 per cent at the immediate post-test and 61.7 per cent at the delayed post-test. There remained, however, a group of children who still consistently rejected fractions after the teaching: just over one-third of these children said, at the delayed post-test, that there was no solution to $2 \times \square = 7$; similarly, there was a group of ten children who said that there were no numbers between 1 and 2 or made no response. When asked how many numbers there are between 1 and 2, six children made the response 'none'.

There is evidence that some teachers also find it difficult to think of a fraction as a number. In the survey of mathematics teachers in secondary schools, nearly one in eight of the teachers, when asked whether they thought of a fraction as one number, two numbers or not a number at all, chose the latter option.

The teaching module activities which related to this basic concept of a fraction as a number focused on the presentation of fractions in the same context as whole numbers. It appears to have had some success, in that more children then accepted the notion of a fraction as a number. The degree of success was slightly higher at the delayed post-test than at the immediate post-test. At the same time, the number of children who avoided fractions, even when their existence was implied in the question, was decreased.

DISCUSSION

The difficulty experienced by some children in interpreting expressions such as $3 \div 4$ was reported in the previous section, and this was attributed to their inability to extend their view of numbers to include fractions.

The results of that part of the research which concerned fractions as numbers suggests that the problem is more deep-rooted than this, and that there are children who reject the notion that a fraction is in any way a number. This is not to say that such children are unaware of the existence of fractions, for it has already been seen that the attachment of a fraction to a part-shaded diagram was within the capability of nearly all children in the secondary age-range. It is suggested that the failure is in their inability to connect the two ideas: the geometric illustration does not seem to lead readily to the notion that a fraction is a number. It appears that the definition of the idea 'number' has become inextricably linked, for some children, with that of 'whole number'. It is fair to say that, in the majority of cases, 'how many?' questions usually imply whole number answers. Most examples in, for instance, the introduction of graphs ask for points with integral co-ordinates to be plotted. Equations in algebra usually have integral solutions. Work on number patterns invariably concerns integers. Thus it may well seem to some children that, unless fractions are specifically mentioned, they need not be considered. The teaching experiment, however, concentrated on the use of instances in which the word 'number' very definitely did include fractions, and, indeed, many children appeared to adjust their thinking about numbers sufficiently to absorb this view. There remained some children who were not able to make this step. It seems that they are not able to detach themselves from the idea of numbers as counting-numbers which describe actual groups of objects and so to move to the more abstract mathematical system of which counting-numbers are but a part.

4. Equivalent fractions

There was some confusion about the relationship between pairs of equivalent fractions.

SUMMARY

The evidence gained from the interviews was that children were well able to recognize instances of equivalence when presented in geometric form. They were also able, in the main, to complete simple statements of equivalent fractions with a missing number in one numerator or denominator, when the multiplying factor was 2, 3, 4 or 5. It became apparent during the interviews, however, that there was sometimes a conflict between the awareness that, for example, 2/3 and 10/15 were the 'same', and the feeling that the multiplication by 5 had made 10/15 bigger than 2/3. This phenomenon was investigated further in the main teaching experiment: at the pre-test, 13 of the 59 children said that the fraction 2/3 was the same as, or equivalent to, 8/12 and also, at the same time, that 8/12 was four times as big as 2/3.

A difference was observed between children's ability to construct equivalent fractions and their ability to apply this skill to the solution of simple problems such as the ordering of two fractions, inserting a fraction between two fractions or adding two fractions. Conversely, there were children who could add fractions but were not successful at finding simple equivalents.

There were children who, as in the case of the 'number' items discussed in the previous section, rejected the existence of fractions. For example, 29 of the 59 children said at the delayed post-test of the main teaching experiment that there is no number that could replace the missing number in the equation 9/12 = 12/?. This was not a matter of failing to answer the question, which was the case for another five children; they were making a positive statement that no such number exists.

The rationale for the teaching module was based on the provision of a variety of instances of equivalence. These included sharing activities with coins of different denominations, and the use of sets of number lines on which families of fractions were plotted and equivalent fractions observed. Calculators, which had already been used to illustrate the division aspect of a fraction, were employed to evaluate fractions such as 3/4, 6/8 and 30/40, and so to emphasize equivalence. The geometrical 'part of a whole' model was not used. Most of the items showed a marked increase in facility after use of the teaching module, with increases of over 25 per cent in five of the seven questions on equivalence. The teaching intervention had little positive effect on that group of children who displayed ambivalence as to whether equivalent fractions were equal or whether one was a multiple of the other.

DISCUSSION

While many children readily accept the notion of equivalence when presented with geometric areas partitioned in different ways, and are also able to apply multiplication (or division) techniques to find equivalent fractions, they may well not see the connection between the two activities. Thus they feel ambivalent towards pairs of fractions such as 2/3 and 8/12, for example. On the one hand, they may be able to visualize rectangles divided into thirds or twelfths, and say that they are 'the same'. On the other hand, they may be aware that 2/3 is 'turned into' 8/12 by a process of multiplication by 4, and feel that 8/12 must be four times as big as 2/3. In order to understand the multiplication aspect, it is necessary to recognize the function of the identity element for multiplication, in this case 4/4. The idea that successive multiplication and division by 4 leaves the value of the fraction unaltered is quite sophisticated, and it seems likely that, for many children, it is obscured by the algorithmic approach of 'multiply top and bottom by 4'. The algorithm, having no basis in meaning, leads children to believe that the whole fraction has been multiplied by 4.

The lack of meaning which children give to the idea of equivalence may explain why those who can construct equivalent fractions were unable to add the fractions 2/3 + 3/4, or, even if successful at this, to explain their method. Certainly, if the only view of equivalence that makes sense is one which relates to two rectangles or circles differently partitioned, then operations with fractions such as addition or finding a fraction between two fractions are meaningless. Even if any meaning can be attached to the addition of 2/3 of one circle to 3/4 of another circle, the drawing   does not lead even to the idea of twelfths. Indeed, for some children it reinforced their opinion that the answer was 5/7, as they compared the number of shaded parts to the total number of parts.

The results of the teaching experiment suggest that the use of a variety of instances of equivalence had some success in increasing the children's ability to deal with simple tasks involving equivalent fractions.

5. *The teaching module*

A short period of teaching intervention can prove beneficial.

SUMMARY

The teaching module, although it occupied only five or six sessions, appears to have achieved some success in each of the three areas of difficulty which were identified earlier. Increases in facility of 25 per cent, 21 per cent and 36 per cent were observed between the pre-test and the delayed post-test for the division, equivalence and number items respectively. Similar patterns in the results were observed in the three phases of the research, namely the pilot study, the main experiment and the class trials. In the case of the equivalence and number items, a further increase in the scores was observed between the immediate post-test and the delayed post-test, suggesting that the teaching intervention had allowed the children to form the structures that allow for cognitive growth over a period of time. The division results, on the other hand, showed a greater increase at the immediate post-test that was reduced somewhat by the delayed post-test. There were considerable differences in the performance of the children who took part in the teaching experiments. In the main experiment, three of the 59 children showed very little improvement; these three had worked quite well during the teaching sessions, but were somewhat disenchanted with school in general. The class trials also included a few children whose test scores improved by only one or two marks. There were also instances of children whose scores decreased at the immediate post-test but increased by the delayed post-test. It was not the case that the children with low scores at the outset derived any less benefit from the teaching than the others. Some of the lowest attainers at the pre-test achieved high scores at the delayed post-test. In the class trials, pupils from second-, third- and fourth-year classes took part, and, again, it was not the case that the younger children were less successful than the older ones although they were all in classes designated as middle ability.

DISCUSSION

The teaching module adopted the use of worksheets, focused on three specified areas of difficulty. The material was designed in this form as it was felt that an informal style of working was important, particularly given the evidence that many children disliked fractions. It was designed to encourage as much active participation as possible, and presented a range of types of activity, including the use of calculators, games and the opportunity to make generalizations. Discussion was also encouraged, both between pupils on their own and between pupils and teacher. The class trials indicate that the material can be used by teachers with their own classes, but observations have already been made about the difficulty those teachers experienced in appreciating the underlying philosophy of the style of working. If the teaching material is to be more widely used, then better documentation would be necessary, particularly on the importance of discussion.

The work in each area, namely the division aspect of a fraction, fractions as numbers and equivalence, made use of several different examples; however, the geometric 'part of a whole' model was not used. This omission was deliberate, since it seems likely that the three areas of difficulty with fractions referred to above are a result of the failure to make a connection between the geometric model and the abstract symbolic form of a fraction.

The suggestion is now made that it is the 'part of a whole' view of a fraction that inhibits other aspects of work with fractions. First, it is argued that if a fraction is seen in geometric, 'part of a whole' terms, then a major readjustment is required if a fraction is also to be thought of as a number. The research shows that this readjustment was not possible for many children even after the intervention of the teaching module. This problem could explain the many instances encountered during the research in which children denied the existence of fractions when found in a numerical context.

The conclusion has to be that, while the geometric 'part of a whole' model may well be a useful one in establishing some of the basic ideas about fractions, serious consideration is necessary as to its limitations and to the need for presenting the idea of a fraction in a wider context.

Implications for the teaching of fractions

One might consider, first, whether it is necessary for children such as those who have been the subject of this study, namely children in middle-ability classes in secondary schools, to study fractions. The survey of the reports reviewed in Chapter 1 suggests that a basic facility with fractions is required by the young school leaver entering many forms of employment. The evidence is that the level of skills needed is mainly limited to the manipulation of simple fractions having denominators 2, 4, 8, 16, 32 and 64. The processes used consisted of ordering, comparison and addition of fractions, and the recognition of equivalents. The multiplication of a simple fraction by an integer or by another simple fraction was also mentioned. So it seems likely that pupils in the 11–16 age group and who are of middle ability will continue to encounter fractions in their school curriculum in the foreseeable future, but that, from the point of view of future employment, the level of complexity need not be high. It remains, therefore, important to try to find ways in which fractions can be made easier for such children to comprehend. This project has drawn attention to some of the difficulties children encounter and has indicated ways in which these difficulties can be reduced. Some of the major implications for the teaching of fractions are now discussed.

1. One of the most important features of the CSMS and SESM projects has been the use of interviews with individual children. It was only by talking and listening to children that the nature of the errors that they made and the strategies that they used were made apparent. Diagnostic tests may give information as to the type of errors made, but do not give insight into the way in which the child is thinking. It

seems likely that the learning of fractions would be facilitated if teachers developed their skills of listening to children and of encouraging them to talk about their interpretation of fractions and associated problems. Some specific areas of difficulty are discussed in the next paragraphs.

2. A greater emphasis could be made on the division interpretation of a fraction: that a/b can be thought of as $a \div b$. The results of the teaching experiment suggest that the use of practical activities can help children to recognize that fractions can suggest a division or sharing activity. Even the use of pictures of cakes, for example, which have to be partitioned so that they can be shared between a number of children appears to be helpful. It is important, however, that children make their own partitioning; the activity is not easily described on the printed page. For this reason, the use of worksheets on which children can draw is suggested. An alternative approach is by the use of a microcomputer, employing its ability to provide the active demonstration very effectively. A program concerned with the division aspect of fractions is being devised for use on a BBC microcomputer. This will enable children to see a number of objects being shared in a variety of ways. with ideas of equivalence also emerging.

Another feature of children's difficulty in associating the fraction a/b with the form $a \div b$ seems to be associated with the language used in describing division. Phrases such as '4 into 3 won't go', or 'you always divide the larger number by the smaller', were encountered frequently in the research. It is important for children to be aware that these phrases related to a stage in their development when they were working within the field of integers only and that they are no longer applicable; that, indeed, fractions are introduced so that equations such as $3 \times \square = 4$ have solution .

3. Discussion with children revealed that more attention needs to be given to the basic ideas of equivalence of fractions. The children who took part in the research were familiar with the use of geometric diagrams to illustrate equivalence, and were also, for the most part, able to apply the technique of multiplying the top and bottom of a fraction by a factor. However, there was little evidence that the one illuminated the other. Many children appeared to feel ambivalent as to whether the fractions, once the 'multiplying' had been done, remained the same or whether one had been made bigger than the other. The idea of multiplication by the identity element, in this case 4/4, may be a difficult one for the children in middle-ability groups with whom we are concerned here. It is suggested that the use of other illustrations of equivalence, in addition to that of the geometric part-whole one, would have the effect of making the idea of equivalence more meaningful. More use could be made of sets of parallel number lines with fractions marked as points on the lines, so that equivalences can be observed. Calculators, used to allow the children to investigate the results of dividing 2 by 3, 8 by 12, 22 by 33 and so on, should provide extra experience of equivalence.

4. More attention needs to be given to the limitations of the 'part of a whole' model of a fraction. In particular, distinction needs to be drawn between the embodiment and the idea. For many children, it appeared that the idea of a fraction was inextricably linked with a picture of a partly shaded shape; this is a very limiting view of fractions. Shapes are not fractions; they merely illustrate them. Moreover, it seems likely that the use of the 'part of a whole' model can inhibit the development of the more general idea of a fraction. Two instances of this have already been indicated. The difficulty experienced by children in think-

ing of the fraction a/b as $a \div b$ can be attributed to their unwillingness to abandon the 'part of a whole' model. Similarly, the use of shapes partitioned in different ways to illustrate equivalence does not appear to help in the understanding of the usual arithmetic way of constructing equivalent fractions.

The use of shaded circles to illustrate the addition of fractions was found to be ambiguous, in that the children took different views as to what was the 'whole'; what is more, it suggests that a fraction of one shape can be added to another fraction of a second shape.

Most importantly, it seems that the dependence on diagrams inhibits the appreciation of the idea that fractions are numbers. This is discussed in the next section.

5. The research revealed a group of children who used a variety of strategies which resulted in them avoiding the use of fractions in all but the simplest cases: this was exemplified by their refusal to accept, for example, that there are any numbers between 1 and 2, or any solution to $2 \times \square = 7$. It seems that these children have failed to appreciate that the introduction of fractions results in an extension of the number system. Teachers need to be aware of the fact that a major adjustment of the meaning of number has to be made. This means that it is necessary to emphasize the importance of the transition from the realm of counting numbers to that of rational numbers. It seems likely that this transition takes time, and there is a need for the child to encounter many examples of instances which result in fractional answers. The results of this research suggest that more use could be made of the plotting of fractions as points on a number line, and that calculators help to establish a link between integers and fractions. A program, similar to one used by Wachsmuth, Behr and Post (1983), suitable for the BBC microcomputer is being devised, in which the learner can aim at balloons placed on a number line. This should also help with the understanding of equivalence and in the ordering of fractions, since it is possible for balloons to be burst by more than one fraction. Immediate feedback can be given as to whether the suggested fraction is too big or too small, or to show the appropriate number of divisions on the line. The pocket calculator, too, could be used more frequently to reinforce the idea that expressions such as $3 \div 4$ do exist and to explore the whole field of multiplication and division by numbers less than one. More generally, fractions could be included in examples used in other aspects of the mathematics syllabus. While it is obviously sensible to use simple numbers when introducing a new idea, continued use of integers in, for example, the solution of equations could lead pupils to discount the possibility that non-integral solutions can exist.

It may well be that the introduction of the arithmetic operations on fractions should be delayed until pupils have grasped the idea that fractions are numbers. This implies that formal work on fractions should commence somewhat later than is generally the case at present. There would seem to be no advantage in pursuing consideration of addition or multiplication of fractions while they are still thought of in the context of shaded parts of geometric shapes. The attempt to recall algorithms has been shown to result in absurdities such as these found during the interviews.

VK: (Writes 2/3 + 3/4) '... 2 goes into 4 twice, 3 goes into 3 once ... that's 2 over 1'

LG: (Writes 2/3 + 3/4) '... You've got to make them the same. I think you turn one over ... No, 3/2 + 3/4 ... Oh, that's not right they've got to be the same on the bottom ... 2/3 + 3/4 ... 2/3 ... No. I don't think you can add them straight away ... you have to do something to them first ...'

TH: (Writes 2/3 + 3/4) '. . . that's $\frac{\cancel{2}1}{\cancel{3}1} + \frac{\cancel{3}1}{\cancel{4}2}$ . . . 2/3 . . . I don't think that's right . . . you could times them . . . 17/17 . . . I don't think that's right either . Take them away? . . . You could say 2 from 4 – no, that doesn't sound right.'

Such examples are, no doubt, familiar. They indicate a lack of understanding of what fractions are, the consequence of which is a search through the memory through a series of techniques.

6. More generally, the research suggests that an approach to the teaching of fractions that is more practically based and that provides a wide experience of instances of fractions is likely to be more successful. The teaching experiment also emphasized the need for opportunities for children to be able to discuss the difficult ideas and this would seem to be an important feature of successful learning about fractions.

Suggestions for further investigation

This present study has focused on only a small area of difficulty with fractions, and there are a number of other investigations which suggest themselves. Some of these include:

1. An investigation into the way in which ideas relating to fractions are introduced in primary schools.

The initial ideas about fractions are, conventionally, introduced to children while at the primary school, and it is here that the foundation for further work is laid. It would be of interest to discover whether this early work is based entirely on the 'part of a whole' model of a fraction, or whether the initial teaching provides the opportunity to widen the view of a fraction later. More particularly, the present research has indicated two limitations: that children find it difficult to think of a fraction as a number, and that they do not make a connection between a/b and $a \div b$. Thus it is desirable for more information into the way in which both numbers and division are introduced in the primary school, to see whether restrictive notions such as '4 into 3 won't go' and 'you always divide the larger number by the smaller' are actually suggested by teachers, and if so how they can be avoided. This, of course, is a large task, given the number of primary teachers in this country. However, some useful insight could be gained by the use of a questionnaire in which teachers were asked to indicate whether they used such expressions and what models they use for the introduction of fractions. This could be coupled with a period of close observation with a few teachers.

2. A long-term investigation to see whether time spent on the type of activity employed in the teaching module of this study does result in an increased likelihood of success later in carrying out operations with fractions and in improved attitudes to fractions. There has been an implicit assumption in this study that time spent in establishing a secure understanding of the concept of a fraction and widening the pupils' views of the nature of fractions will result in improved performance in later years. A longitudinal study, taken over the years of secondary schooling, could provide evidence as to the truth of this assumption.

3. An investigation of the effect of continued use of calculators on the understanding of fractions.

As calculators become more generally used in schools, the link between fractions and decimals is likely to become more important. The present research has suggested that their use in emphasizing the division aspect of a fraction can assist with ideas of equivalence and the ordering of fractions, and further research in this direction is indicated.

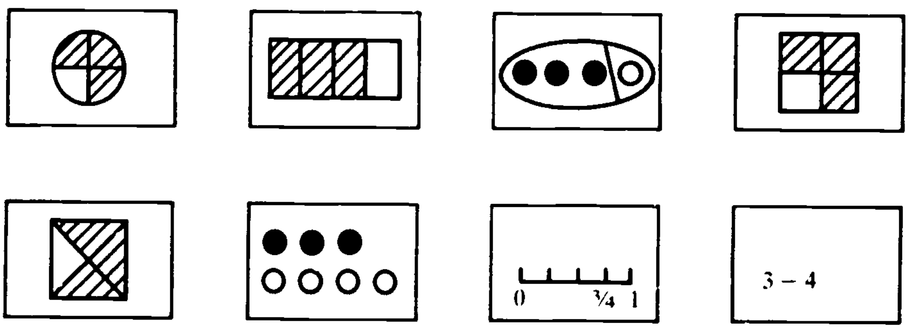
4. An evaluation of computer programs concerned with fractions. The microcomputer seems likely to have much to offer in the learning of fractions. Firstly, with animated diagrams, it can easily provide the active demonstration of the various aspects of fractions which this study has indicated to be of importance. Secondly, the microcomputer, with its immediate feedback and capacity for generating as many examples as the learner requires, could do much to remove the fear of fractions expressed by many of the children who took part in this study.

Appendix 1

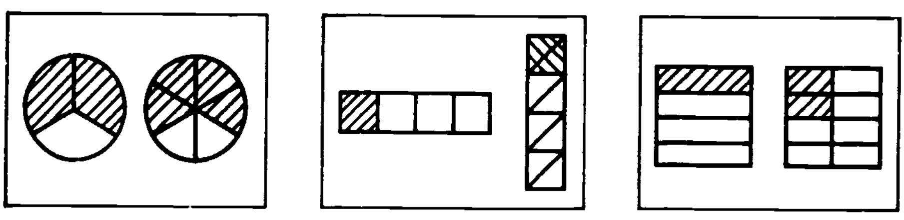
Framework for Phase 1 interviews

1. How would you explain to someone, who didn't know, what a fraction is?

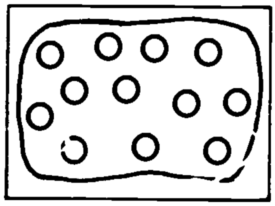
2. Which of these cards would help someone to understand what the fraction 3/4 is?



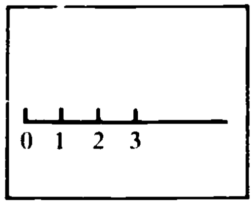
3. What could we find out from these cards?



4. How would you find 3/4 of this collection of counters?



5. Where would the number 4 go on this number line? And the number 2/3? And the number 1/2?



6. How many numbers are there between 2 and 3? And between 0 and 1?

7. Would you try these: 2/3 + 3/4; 3/8 + 2/8; 1/10 + 3/5? Would you explain what you did?

8. Two boys have equal amounts of pocket money. One decides to save 1/4 of his pocket money, the other decides to save 5/20 of his pocket money. Is 5/20 more than 1/4, is 1/4 more than 5/20 or are 5/20 and 1/4 equal?

9. Mary and John both have pocket money. Mary spends 1/4 of hers and John spends 1/2 of his. Is it possible for Mary to have spent more than John?

10. Can you find some fractions that are the same in this collection?

$\frac{3}{4}$ $\frac{12}{16}$

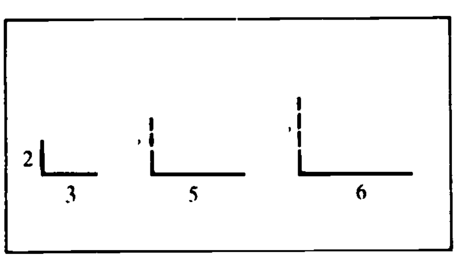
$\frac{2}{9}$ $\frac{6}{8}$ $\frac{2}{6}$

$\frac{3}{9}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{3}{10}$ $\frac{5}{10}$

11. Two numbers add up to 10. What could the numbers be?

12. A recipe for 8 people says you need 1/2 pint of cream. How much cream would you need if you were making the recipe for 4 people? How much for 6 people?

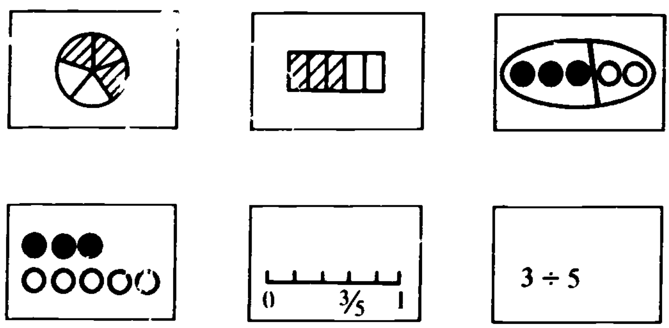
13. This drawing has to be made the same shape but bigger. What should the heights be?



Appendix 2

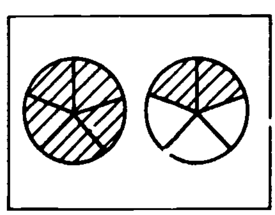
Framework for Phase 2 interviews

1. Which of these cards would help someone who didn't know what the fraction 3/5 is?

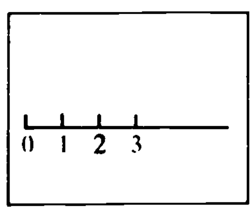


2. Here are three cakes. Could you share them equally between 5 people? Do you see any connection between what you have done and 3 ÷ 5?

3. If you took the red pieces from these circles, how much would you have?

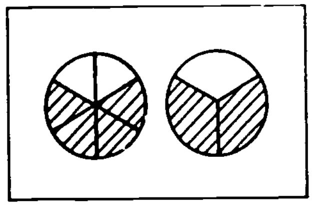


4. Where would the number 4 go on this number line? And the number 3/5? And the number 1 1/5?

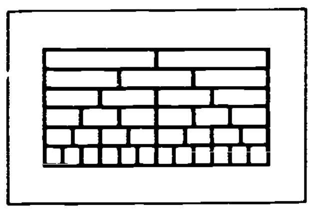


5. Would you rather have 2/3 or 10/15 of a cake you particularly like?

6. Suppose you saw these diagrams in a textbook. What could you tell from them?



7. What does this diagram tell us?



8. Can you find some fractions that are the same in this collection?

$\frac{3}{4}$ $\frac{12}{16}$

$\frac{2}{9}$ $\frac{6}{8}$ $\frac{2}{6}$

$\frac{3}{9}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{3}{10}$ $\frac{5}{10}$

9. Could you complete these:

$\frac{3}{4} = \frac{?}{12}$ $\frac{9}{12} = \frac{12}{?}$

$\frac{5}{3} = \frac{15}{?}$ $\frac{14}{16} = \frac{?}{24}$

Would you explain what you did?

10. Could you find a fraction between 1/2 and 1/4?

11. Which is the biggest: 3/8, 3/7, 3/9?

12. Which is bigger, 3/4 or 4/5?

13. Would you try these: 2/3 + 3/4; 3/8 + 2/8; 1/10 + 3/5? Would you explain what you did?

14. Can you find two numbers that add up to 10? Are there any fractions?

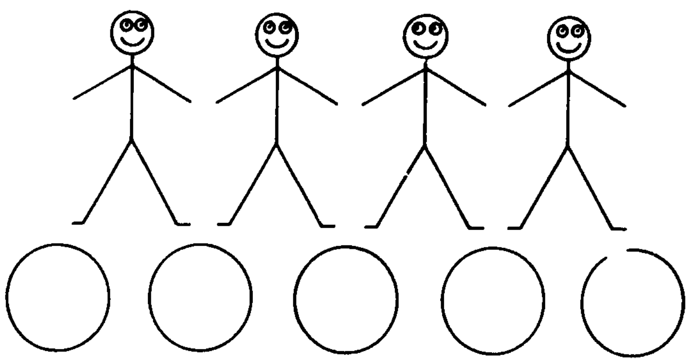
15. What do these diagrams tell us?

16. Can you find a number between the numbers 1 and 2? How many are there?

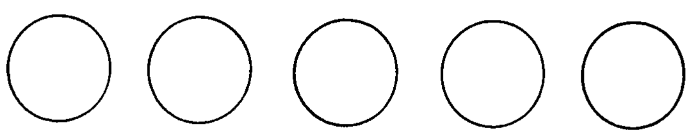
Appendix 3

Worksheets

D1. a) These 5 cakes are to be divided equally between 4 people. Shade in the amount one person would get

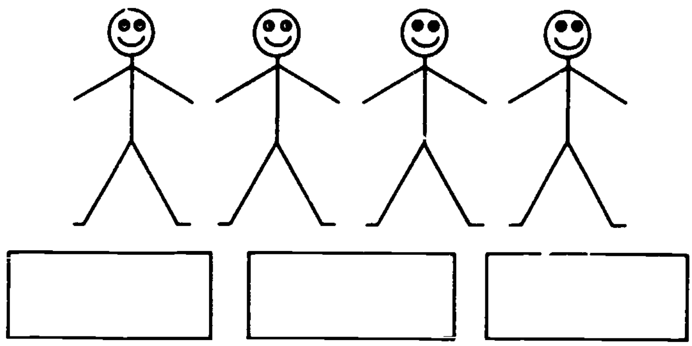


Can you find another way of dividing up the cakes?



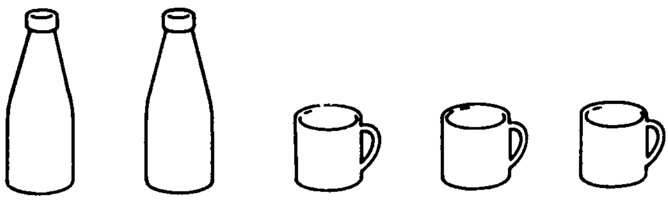
How much cake does each person get? . . . .

b) These 3 cakes are to be divided equally between 4 people. Shade in the amount each person would get.



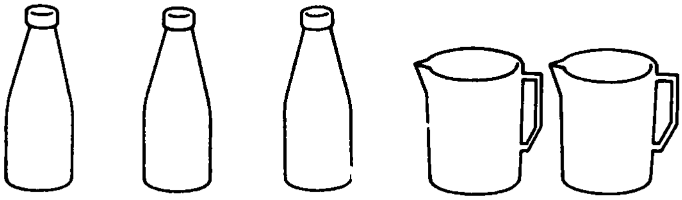
How much cake does each person get? . . . .

c) These 2 pints of milk are divided equally between 3 cups.



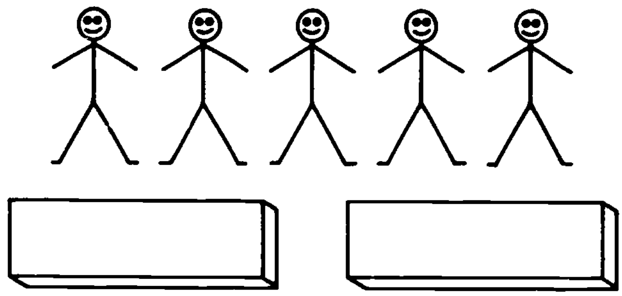
How much milk is there in each cup? . . . .

d) These 3 pints of water are divided equally between 2 jugs.



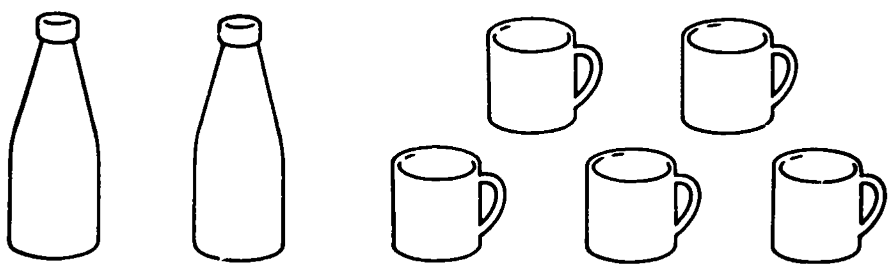
How much water is there in each jug? . . . .

e) These 2 bars of toffee are divided equally between 5 children.



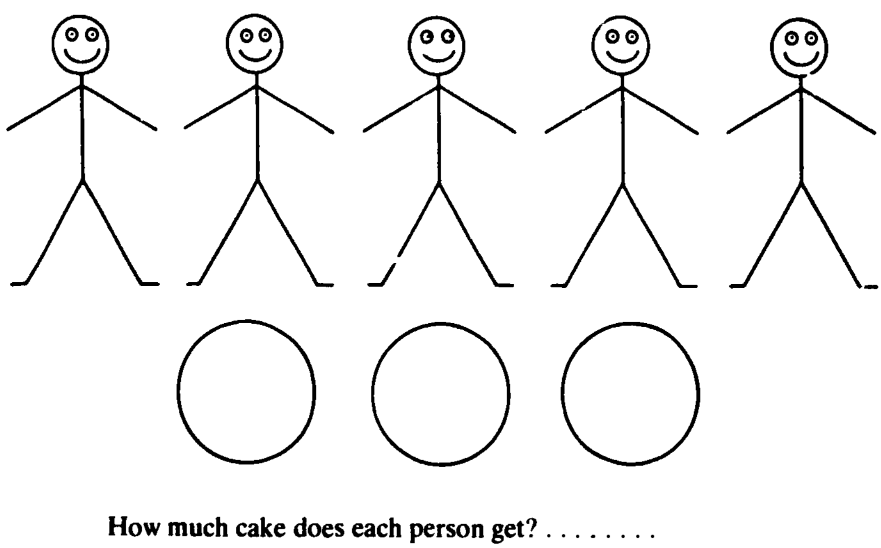
How much toffee does each child get? . . . .

D2. f) Two pints of milk are divided equally between 5 cups, all the same size. Imagine how you would do this.



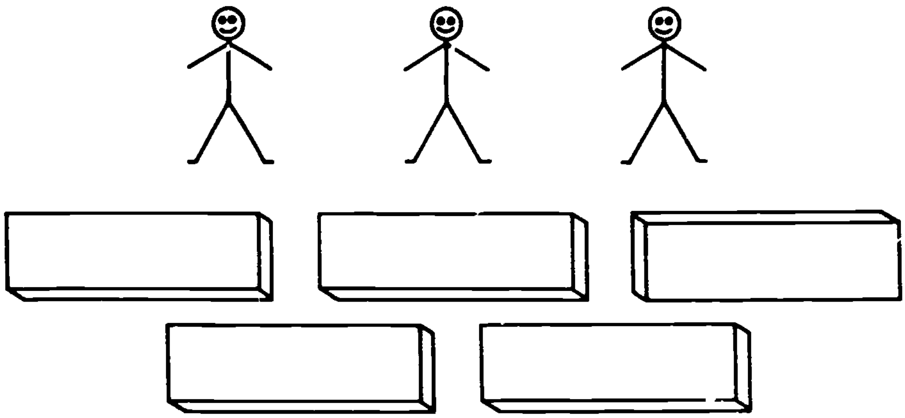
How much milk is there in each cup? . . . .

g) Three cakes are divided equally between 5 people. Imagine how you would do this.



How much cake does each person get? . . . .

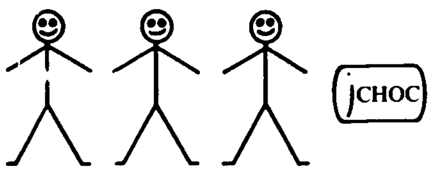
h) Five bars of toffee are divided equally between 3 children. Imagine how you would do this.



How much toffee does each child get? . . . .

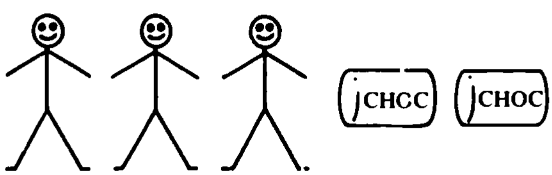
If you are not sure whether your answers are right, draw some pictures and see if that helps.

D3. This choc roll is to be shared equally between 3 children.



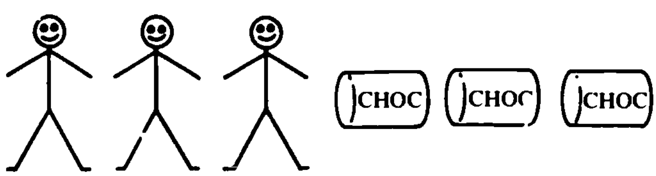
How much does each child get? . . . .

These 2 choc rolls are shared equally between 3 children.



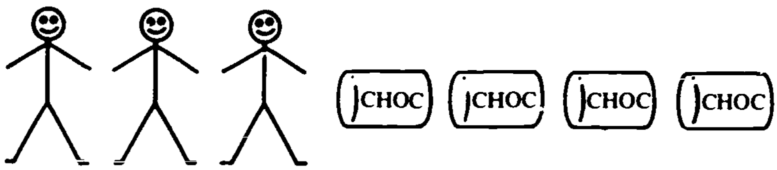
How much does each child get? . . . .

These 3 choc rolls are shared equally between 3 children.



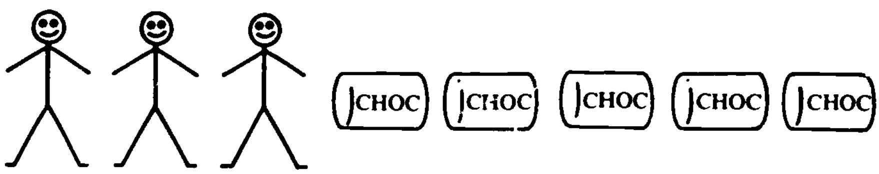
How much does each child get? . . . .

These 4 choc rolls are shared equally between 3 children.



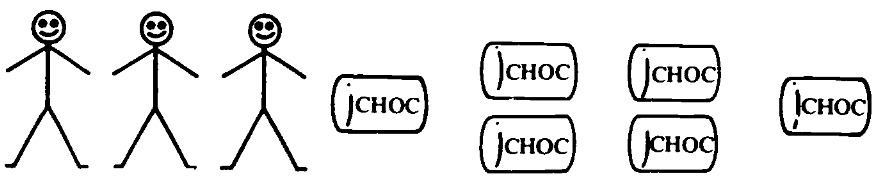
How much does each child get? . . . .

These 5 choc rolls are shared equally between 3 children.



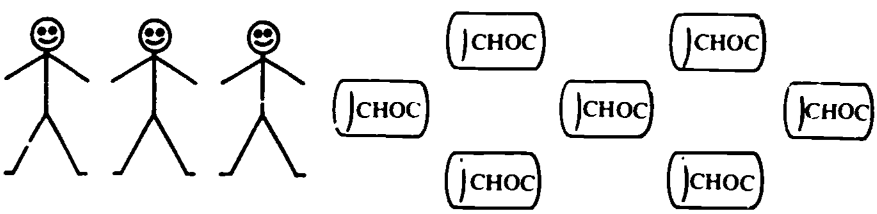
How much does each child get? . . . .

These 6 choc rolls are share equally between 3 children.



How much does each child get? . . . .

These 7 choc rolls are shared equally between 3 children.



How much does each child get? . . . .

How much would each child get if there were:

8 choc rolls? . . . .

9 choc rolls? . . . .

10 choc rolls? . . . .

D4. What is $\frac{1}{2}$ as a decimal? . . . .

What is $\frac{1}{4}$ as a decimal? . . . .

What is $\frac{3}{4}$ as a decimal? . . . .

Use the calculator to find $1 \div 2 =$ . . . .

Use the calculator to find $1 \div 4 =$ . . . .

Use the calculator to find $3 \div 4 =$ . . . .

What do you notice? . . . .

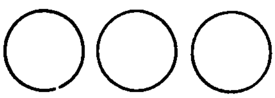
$\frac{2}{3}$ is the same as $2 \div 3$. Use the calculator to find $2 \div 3 =$ . . . .

$\frac{3}{5}$ is the same as $3 \div 5$. Use the calculator to find $3 \div 5 =$ . . . .

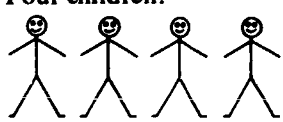
Use the calculator to find the following fractions as decimals:

$\frac{2}{5} =$	$\frac{3}{5} =$	$\frac{1}{6} =$ . . . .
$\frac{5}{16} =$	$\frac{7}{8} =$	$\frac{16}{3} =$ . . . .
$\frac{3}{16} =$	$\frac{12}{3} =$	$\frac{3}{12} =$ . . . .

D5. Three cakes:



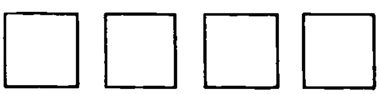
Four children:



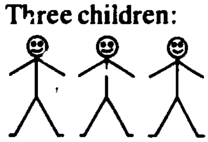
Each child gets:  or 

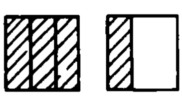
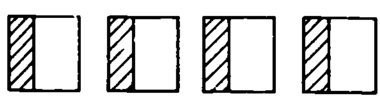
$3 \div 4$ is the same as $\frac{3}{4}$

Four cakes:



Three children:



Each child gets:  or 

$4 \div 3$ is the same as $\frac{4}{3}$

$\frac{3}{4} = 3 \div 4$ | $\frac{2}{5} = 2 \div \square$

$\frac{4}{3} = \square \div \square$ | $\frac{5}{2} = \dots\dots\dots$

$\frac{2}{12} = \dots\dots\dots$ | $\frac{12}{2} = \dots\dots\dots$

$\frac{3}{5} = \dots\dots\dots$ | $\frac{5}{3} = \dots\dots\dots$

$2 \div 3 = \frac{2}{3}$ | $3 \div 2 = \frac{3}{\square}$

$3 \div 4 = \frac{\square}{\square}$ | $4 \div 3 =$ ______

$4 \div 5 =$ ______ | $5 \div 4 =$ ______

$12 \div 3 =$ ______ | $3 \div 12 =$ ______

E1. **LOOT**

You need a bag of 'loot', containing some £100 notes. There are 3 plans for sharing tne loot:

Plan 1 It is to be shared between Alf, Bert and Charlie, so that Alf gets 2/4, Bert gets 1/4 and Charlie gets 1/4. How many notes does each man get?

Plan 1	Alf	Bert	Charlie

Plan 2 It is to be shared between Alf, Bert, Charlie and Dave, so that Alf gets 6/12, Bert gets 3/12, Charlie gets 2/12 and Dave gets 1/12. How many notes does each get, now?

Plan 2	Alf	Bert	Charlie	Dave

Plan 3 It is to be shared between Alf, Bert, Charlie and Dave, so that Alf gets 8/16, Bert gets 4/16, Charlie gets 3/16 and Dave gets 1/16. How many notes does each get, this time?

Plan 3	Alf	Bert	Charlie	Dave

Which plan is best for Alf? . . . .
What fraction of the loot did he get each time? . . . .

Which plan is best for Bert? . . . .
What fraction of the loot did he get each time? . . . .

Which plan is best for Charlie? . . . .
What fraction did he get in plan 1? . . . . . .
What fraction did he get in plan 3? . . . . . .
Why is plan 1 better for him than plan 3? . . . .
Which plan is best for Dave? . . . .
What fraction did he get in plan 2? . . . . . .
What fraction did he get in plan 3? . . . . . .
Why is plan 2 better for him than plan 3? . . . .

E2. Remember that $\frac{3}{4}$ is the same as $3 \div 4$.

Use the calculator to find:

a) $\frac{3}{4}$ = ... $\frac{9}{12}$ = ... $\frac{30}{40}$ = ...

$\frac{33}{44}$ = ... $\frac{60}{80}$ = ...

b) $\frac{5}{8}$ = ... $\frac{10}{16}$ = ... $\frac{15}{24}$ = ...

$\frac{50}{80}$ = ... $\frac{25}{40}$ = ...

c) $\frac{8}{12}$ = ... $\frac{12}{18}$ = ... $\frac{10}{15}$ = ...

What do you notice? . . . .

Which of the following pairs of fractions will give the same answer on a calculator? Write 'yes' or 'no' after each pair. Check with the calculator if you are not sure.

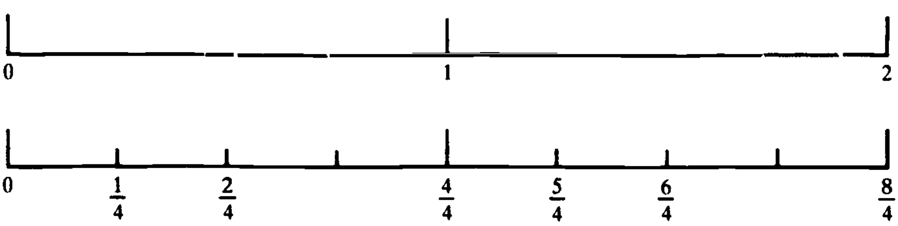
$\frac{2}{3}$ and $\frac{4}{6}$ . . . . $\frac{4}{5}$ and $\frac{6}{7}$ . . . .

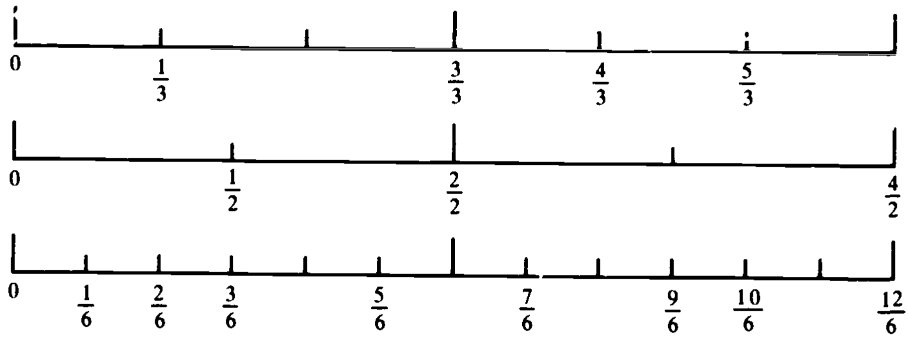
$\frac{4}{5}$ and $\frac{8}{10}$ . . . . $\frac{1}{4}$ and $\frac{4}{16}$ . . . .

$\frac{8}{40}$ and $\frac{2}{5}$ . . . . $\frac{1}{?}$ and $\frac{11}{13}$ . . . .

$\frac{4}{3}$ and $\frac{16}{12}$ . . . . $\frac{15}{10}$ and $\frac{3}{2}$ . . . .

E3. Complete these number lines up to the number 2:





Use them to answer these questions:

Which is bigger, $\frac{2}{3}$ or $\frac{3}{4}$? . . . .

Which is bigger, $\frac{7}{4}$ or $\frac{5}{3}$? . . . .

Which is bigger, $\frac{3}{2}$ or $\frac{10}{6}$? . . . . . .

Which numbers are directly above $\frac{3}{6}$? . . . .

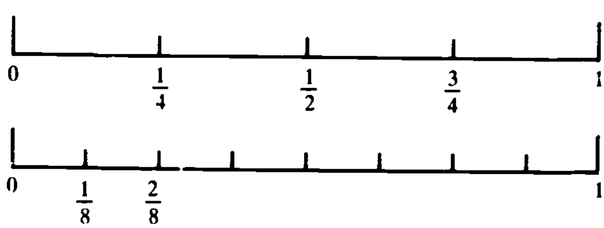
Which numbers are directly below 2? . . . .

Which numbers are directly below $\frac{6}{4}$? . . . .

Which numbers are directly above $\frac{8}{6}$? . . . .

Which numbers are directly below $\frac{2}{3}$? . . . .

E4.

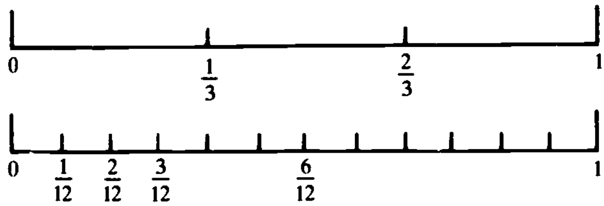


$\frac{3}{4}$ is the same as $\frac{\square}{8}$

Which is bigger, $\frac{3}{4}$ or $\frac{7}{8}$? . . . .

Find a fraction between $\frac{3}{4}$ and $\frac{1}{2}$ . . . .

What is $\frac{3}{4} + \frac{1}{8}$? . . . .



$\frac{2}{3}$ is the same as $\frac{\square}{12}$

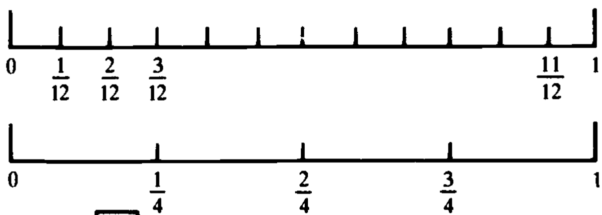
Which is bigger, $\frac{2}{3}$ or $\frac{9}{12}$? . . . .

Find a fraction between $\frac{2}{3}$ and $\frac{11}{12}$ . . . .

What is $\frac{2}{3} + \frac{1}{12}$? . . . .

What is $\frac{2}{3} + \frac{5}{12}$? . . . .

E5.



$\frac{3}{4}$ is the same as $\frac{\square}{12}$

Which is bigger, $\frac{3}{4}$ or $\frac{8}{12}$? . . . .

Find a fraction between $\frac{3}{4}$ and $\frac{7}{12}$ . . . .

What is $\frac{1}{2} + \frac{1}{12}$? . . . .

What is $\frac{3}{4} + \frac{1}{12}$? . . . .

What is $\frac{3}{4} + \frac{5}{12}$? . . . .

$\frac{1}{4}$ is the same as $\frac{\square}{12}$

$\frac{1}{3}$ is the same as $\frac{\square}{12}$

What is $\frac{1}{4} + \frac{1}{3}$? . . . .

$\frac{3}{4}$ is the same as $\frac{\square}{12}$

$\frac{2}{3}$ is the same as $\frac{\square}{12}$

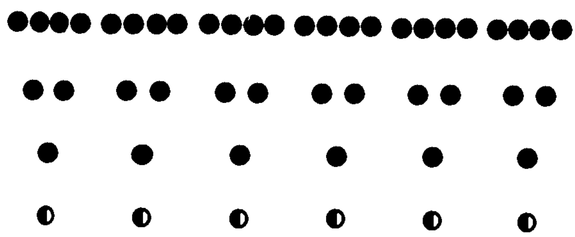
What is $\frac{3}{4} + \frac{2}{3}$? . . . .

$\frac{5}{6}$ is the same as $\frac{\square}{12}$

Find a fraction between $\frac{5}{6}$ and $\frac{7}{12}$ . . . .

What is $\frac{5}{6} + \frac{\cdot}{12}$? . . . .

N1. Fill in the missing numbers:

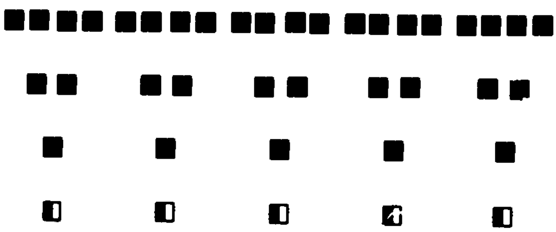


$6 \times \square = 24$

$6 \times \square = 12$

$6 \times \square = 6$

$6 \times \square = 3$

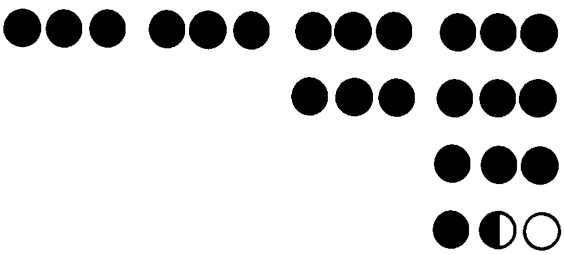


$5 \times \square = 20$

$5 \times \square = 10$

$5 \times \square = 5$

$5 \times \square = 2\frac{1}{2}$



$\square \times 3 = 12$

$\square \times 3 = 6$

$\square \times 3 = 3$

$\square \times 3 = 1\frac{1}{2}$

Fill in the missing numbers:

$10 \times \square = 5$	$16 \times \square = 8$
$\square \times 12 = 6$	$\square \times 7 = 3\frac{1}{2}$
$8 \times \square = 4$	$9 \times \square = 4\frac{1}{2}$

N2. You will need a calculator.
Enter the number 54 and do the following multiplications:

$54 \times 2 =$
$54 \times 1.5 =$
$54 \times 1.3 =$
$54 \times 1.1 =$
$54 \times 1.05 =$
$54 \times 1.001 =$

What do you notice? . . . .
What is $54 \times 1 =$ . . . . ?
Now try these:

$54 \times 0.95 =$
$54 \times 0.9 =$
$54 \times 0.8 =$
$54 \times 0.6 =$
$54 \times 0.5 =$
$54 \times 0.1 =$

What do you notice? . . . .

Without working it out, what can you say about 62×0.9? Will it be bigger than 62, or smaller? Much bigger or much smaller?

. . . .

Without working it out, what can you say about 45×1.1? Will it be bigger than 45, or smaller? Much bigger or much smaller?

. . . .

Check with the calculator.

N3. **TARGET** – A game for two players.
You need one calculator.

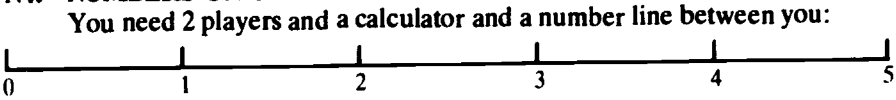
First player: Enter any number you like on to the calculator. Pass the calculator to your partner.

Second player: Multiply this number by any other number, trying to get the calculator to show a number as close as possible to the target – 100. Pass the calculator back to your partner.

First player: Multiply this new number by any other number, again trying to get as close as possible to the target – 100.

The game continues until one player wins by getting the calculator to show 100.*****.

N4. **NUMBERS-ON-THE-LINE.**
You need 2 players and a calculator and a number line between you:



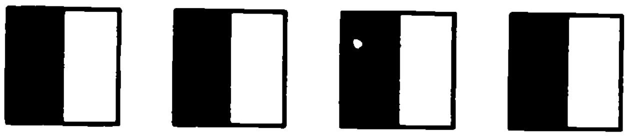
First player: Choose 2 numbers from this list:
5 3 107 13 2 24 1 8 25
Use the calculator, if you need to, to multiply them or divide them so that the answer will fit on the line. Write down what you did in the space below, and mark your point, if you can, on the line.

Second player: Choose 2 numbers from the list – it doesn't matter if they have already been used – and multiply them or divide them so that you get a different point on the line. Write down what you did in the space below, and mark your point, if you can, on the line. If your point has already been marked, you miss a turn.

The first person to get 6 points marked on the line wins.

1st player	
2nd player	

N5. This is a picture of 4 halves:



How many whole ones? . . . .
We write $4 \times \frac{1}{2} = 2$.

This is a picture of 7 halves:

How many whole ones? . . . . How many halves over? . . . .
We write $7 \times \frac{1}{2} = 3\frac{1}{2}$

Find the answers to the following questions. Draw pictures if it helps.

1. $3 \times \frac{1}{2} =$ . . . .
2. $6 \times \frac{1}{2} =$ . . . .
3. $5 \times \frac{1}{2} =$ . . . .
4. $8 \times \frac{1}{2} =$ . . . .
5. $5 \times \frac{1}{4} =$ . . . .

Now try these:

1. $10 \times \square = 5$
2. $6 \times \square = 3$
3. $8 \times \square = 4$
4. $9 \times \square = 4\frac{1}{2}$
5. $5 \times \square = 2\frac{1}{2}$
6. $20 \times \square = 10$

Appendix 4

SESM Fractions T1

NAME.......................... SCHOOL ....

TODAY'S DATE.......... DATE OF BIRTH....................

---

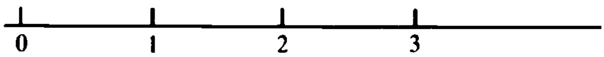
1. Underline the <u>numbers</u> in this set:

   {4 some $\frac{3}{4}$ 5.7 ten lots $2\frac{1}{2}$ $\frac{13}{5}$ 17 $\frac{19}{23}$ 7/3}

2. $12 \div 4 =$ ....

3. $4 \div 12 =$ ....

4. Mark and name the numbers 4, $\frac{3}{5}$, $1\frac{1}{5}$, $\frac{9}{5}$ on this number line:



5. Jane has $\frac{3}{4}$ of a bar of chocolate, and John has $\frac{9}{12}$ of a bar the same size. Tick <u>all</u> the statements below that you think are true:

   (a) John has more chocolate than Jane
   (b) Jane has more chocolate than John
   (c) They have the same amount of chocolate
   (d) John has 3 times the amount of chocolate as Jane
   (e) John has smaller pieces of chocolate but they have the same amount

6. Find 5 pairs of numbers that add up to 3: ....

   ....

   ....

   ....

   ....

7. Put a ring round each of the statements below that you think are other ways of writing the fraction $\frac{3}{4}$:

3×4 $\quad$ $3 \div 4$ $\quad$ 3/4 $\quad$ $4 \div 3$ $\quad$ three-quarters

8. Tick <u>all</u> the statements below that you think are true about the fractions $\frac{2}{3}$ and $\frac{8}{12}$:

   (a) $\frac{8}{12}$ is 4 times as big as $\frac{2}{3}$

   (b) $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent

   (c) $\frac{2}{3}$ is smaller than $\frac{8}{12}$

   (d) $\frac{8}{12}$ is found by multiplying $\frac{2}{3}$ by 4

   (e) $\frac{2}{3}$ and $\frac{8}{12}$ are the same

   (f) If you had $\frac{8}{12}$ of a bag of sweets you would get more than if you had $\frac{2}{3}$ of the same bag of sweets.

9. Find a day of the week between Monday and Thursday . . . .

   Find a fraction between $\frac{13}{16}$ and $\frac{1}{2}$ . . . .

   Find a fraction between $\frac{5}{8}$ and $\frac{7}{16}$ . . . .

   Find a fraction between $\frac{9}{4}$ and $\frac{15}{8}$ . . . .

   Find a fraction between $\frac{2}{3}$ and $\frac{4}{5}$ . . . .

10. Put the missing numbers in the boxes. If there is <u>no</u> number, write 'no' in the box.

    a) $5 \times \square = 15$ $\quad$ b) $3 \times \square = 18$ $\quad$ c) $2 \times \square = 7$

    d) $4 \times \square = 10$ $\quad$ e) $2 \times \square = 1$ $\quad$ f) $8 \times \square = 5$

11. 6 pints of milk are divided equally between 3 jugs, all the same size.

    How much is there in each jug? . . . .

12. 2 pints of milk are divided equally between 3 cups, all the same size.

    How much does each cup hold? . . . .

13. 5 pints of milk are divided equally between 3 jars, all the same size.

    How much is there in each jar? . . . .

14. Put the missing numbers in the boxes. If there is <u>no</u> number, write 'no' in the box.

    a) $\frac{3}{4} = \frac{\square}{12}$ $\quad$ b) $\frac{5}{3}$ $\quad \frac{15}{\square}$ $\quad$ c) $\frac{9}{12} = \frac{12}{\square}$ $\quad$ d) $\frac{14}{16} = \frac{\square}{24}$

15. Put a ring round the <u>bigger</u> fraction in each of these pairs. If they are the same, write 'same' by them.

a) $\frac{1}{4}$ $\frac{1}{8}$ b) $\frac{3}{7}$ $\frac{3}{9}$ c) $\frac{3}{8}$ $\frac{6}{16}$

d) $\frac{13}{10}$ $\frac{7}{5}$ e) $\frac{1}{4}$ $\frac{7}{32}$ f) $\frac{3}{4}$ $\frac{7}{9}$

16. $3 \div 4 =$ . . . .

17. · = ...

18. Mark on the line, and label, 3 numbers between 1 and 2. If you think there are none, write 'none'.

How many numbers do you think there are between 1 and 2? . . . .

19. $\frac{3}{8} + \frac{2}{8} =$ . . . .

20. $\frac{3}{5} + \frac{1}{10} =$ . . . .

21. $\frac{2}{3} + \frac{3}{4} =$ . . . .

22. Write down some fractions that are equivalent to (or the same as) the fraction $\frac{3}{8}$ . . . .

Appendix 5

SESM Fractions T2

NAME.......................... SCHOOL....................

TODAY'S DATE........... DATE OF BIRTH....................

---

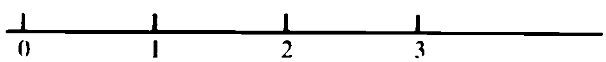
1. Underline the <u>numbers</u> in this set:

   {4 many $\frac{2}{3}$ 5.9 five few $2\frac{1}{4}$ $\frac{17}{5}$ 13 $\frac{19}{27}$ 8/3}

2. $15 \div 3 =$ ....

3. $3 \div 15 =$ ....

4. Mark in and name the numbers 4, $\frac{3}{4}$, $1\frac{1}{4}$, $\frac{9}{4}$ on this line:



5. Jane has $\frac{2}{3}$ of a bar of chocolate, and John has $\frac{8}{12}$ of a bar the same size. Tick <u>all</u> the statements below that you think are true.

   (a) John has more chocolate than Jane
   (b) Jane has more chocolate than John
   (c) They have the same amount of chocolate
   (d) John has 4 times as much chocolate as Jane
   (e) John has smaller pieces but they have the same amount of chocolate

6. Find 6 pairs of numbers that add up to 4: .. ....

   ....

   ....

   ....

   ....

   ....

7. Put a ring round each of the statements below that you think are other ways of writing the fraction $\frac{3}{4}$:

   3×4 $3 \div 4$ 3/4 $4 \div 3$ three-quarters

8. Tick <u>all</u> the statements below that you think are true about the fractions $\frac{3}{4}$ and $\frac{12}{16}$:

   (a) $\frac{12}{16}$ is 4 times as big as $\frac{3}{4}$

   (b) $\frac{3}{4}$ and $\frac{12}{16}$ are equivalent

   (c) $\frac{3}{4}$ is smaller than $\frac{12}{16}$

   (d) $\frac{12}{16}$ is found by multiplying $\frac{3}{4}$ by 4

   (e) $\frac{3}{4}$ and $\frac{12}{16}$ are the same

   (f) If you had $\frac{12}{16}$ of a bag of sweets you would get more than if you had $\frac{3}{4}$ of the same bag of sweets.

9. Find a day of the week between Tuesday and Friday . . . .

   Find a fraction between $\frac{11}{16}$ and $\frac{1}{2}$ . . . .

   Find a fraction between $\frac{5}{8}$ and $\frac{13}{16}$ . . . .

   Find a fraction between $\frac{9}{4}$ and $\frac{15}{8}$ . . . .

   Find a fraction between $\frac{2}{3}$ and $\frac{3}{4}$ . . . .

10. Put the missing numbers in the boxes. If there is <u>no</u> number, write 'no' in the box.

    a) $4 \times \square = 12$ b) $3 \times \square = 15$ c) $2 \times \square = 9$

    d) $4 \times \square = 10$ e) $2 \times \square = 1$ f) $8 \times \square = 5$

11. 6 pints of milk are divided equally between 3 jugs, all the same size.

    How much is there in each jug? . . . .

12. 2 pints of milk are divided equally between 3 cups, all the same size.

    How much does each cup hold? . . . .

13. 5 pints of milk are divided equally between 3 jars, all the same size.

    How much is there in each jar? . . . .

14. Put the missing numbers in the boxes. If there is no number, write 'no' in the box.

a) $\frac{2}{3} = \frac{\square}{12}$ $\frac{5}{3} = \frac{20}{\square}$ c) $\frac{9}{12} = \frac{12}{\square}$ $\frac{14}{16} = \frac{\square}{24}$

15. Put a ring round the bigger fraction in each of these pairs. If they are the same, write 'same' by them.

a) $\frac{1}{4}$ $\frac{1}{8}$ b) $\frac{3}{7}$ $\frac{3}{9}$ $\frac{3}{8}$ $\frac{6}{16}$

d) $\frac{13}{10}$ $\frac{7}{5}$ e) $\frac{1}{4}$ $\frac{7}{32}$ f) $\frac{3}{4}$ $\frac{7}{9}$

16. $3 \div 4 =$ . . . .

17. $4 \div 3 =$ . . . .

18. Mark on the line, and label, 3 numbers between 1 and 2. If you think there are none, write 'none'.

How many numbers do you think there are between 1 and 2? . . . .

19. $\frac{3}{8} + \frac{2}{8} =$ . . . .

20. $\frac{3}{5} + \frac{1}{10} =$ . . . .

21. $\frac{2}{3} + \frac{3}{4} =$ . . . .

22. Write down some fractions that are equivalent to (or the same as) the fraction $\frac{3}{8}$ . . . .

Appendix 6

Individual children's scores – main experiment

	DIVISION			EQUIVALENCE			NUMBER		
Child	*Pre*	*Post*	*Del*	*Pre*	*Post*	*Del*	*Pre*	*Post*	*Del*
JE	4	8	7	4	15	15	8	5	10
SB	2	5	3	5	14	13	2	5	10
AB	6	9	10	10	17	17	3	9	12
PH	2	7	3	12	15	14	4	8	10
ML	6	9	6	14	18	19	9	6	12
IF	3	6	9	7	19	14	4	4	11
AF	1	2	3	5	16	14	3	4	6
TB	2	6	3	3	16	18	5	6	4
SK	6	7	8	8	11	12	7	10	8
SW	3	5	5	5	15	12	5	5	5
MK	3	6	7	5	9	19	4	5	7
GO	3	4	3	12	15	14	6	8	7
PW	3	6	2	9	17	14	5	11	8
DB	2	5	4	2	14	14	4	5	5
PT	4	5	4	5	10	15	3	7	9
JF	2	4	3	9	18	17	4	5	4
MR	2	1	1	5	13	12	2	6	5
CW	3	10	6	8	18	14	4	11	11
AC	6	6	9	10	15	12	3	4	10
KR	3	8	7	3	11	11	4	7	6
SC	4	9	9	4	10	7	4	11	11
PW	2	5	5	7	13	15	2	9	10
SD	3	10	3	14	18	20	5	10	11
RW	3	3	5	13	12	19	1	4	2
MF	1	6	2	16	18	20	5	9	9
JH	2	9	10	14	22	21	6	6	7
GN	2	7	8	4	12	8	2	12	12
SA	6	4	6	14	17	16	7	10	11
JH	3	10	3	10	16	16	1	6	9

CL	2	5	4	7	8	5	2	5	4
ML	2	5	1	6	7	4	2	7	8
AM	2	10	7	11	13	18	6	6	8
LS	2	10	6	12	14	15	6	9	10
AC	4	8	7	15	19	20	8	10	11
HD	0	5	5	6	15	14	3	9	8
CW	0	6	6	6	14	19	3	10	10
JC	2	10	4	14	15	18	3	11	6
TB	4	6	8	11	12	14	3	8	8
CP	3	2	2	8	10	14	5	7	6
MC	5	6	1	8	16	12	5	7	6
LB	4	8	6	9	3	12	7	9	8
LG	5	10	8	14	14	17	7	7	4
LE	2	10	4	12	13	17	4	6	3
SB	2	3	3	11	15	15	3	3	5
AS	4	9	8	13	17	15	7	9	6
WG	4	6	4	8	14	12	8	10	8
RW	3	3	5	13	12	19	1	4	6
SW	6	6	6	13	16	16	3	9	9
SD	5	6	6	10	15	17	7	9	8
JR	1	5	5	11	13	13	7	7	9
DT	2	6	9	16	18	15	8	11	12
RH	4	5	5	6	9	9	6	9	9
ML	3	8	5	10	14	17	8	6	10
SH	4	7	4	4	16	10	6	8	10
SH	0	6	5	13	16	17	3	5	9
SH	0	6	3	14	12	10	4	11	11
IP	3	6	2	8	7	12	5	5	7
MT	0	5	5	4	16	10	8	6	10
DP	2	5	3	5	6	10	3	6	5

Bibliography

BEHR, M J., POST, T. R. and WACHSMUTH I (1982). Children's acquisition of order and equivalence concepts for rational numbers: report on a clinical teaching experiment. Paper presented at 1982 Annual Meeting of American Educational Research Association, New York City, 1982.

BOOTH, L. R. (1981). 'Child methods in secondary mathematics', *Educational Studies in Mathematics*, 12, 29–40.

BOOTH, L. R. (1984). *Algebra: children's strategies and errors* Windsor: NFER-NELSON.

BOOTH, L. R. and HART, K. M. (1982). Child methods in mathematics. Paper presented at the 9th Annual Conference of the Research Council for Diagnostic and Prescriptive Mathematics, Buffalo, U.S.A.

FRAENKEL, A. A. and BAR-HILLEL, Y. (1958). *Foundations of Set Theory*. Amsterdam: North Holland.

HART, K. M. (Ed) (1981). *Children's Understanding of Mathematics: 11–16*. London: John Murray.

HART, K. M. (1980). In: JOHNSON, D (Ed) *Secondary School Children's Understanding of Mathematics*. London: Chelsea College, Centre for Science and Mathematics Education Research Monographs.

HASEMANN, K. (1981). 'On Difficulties With Fractions', *Educational Studies in Mathematics*, 12.

KIEREN, T.E. and NELSON, D. (1981). 'Partitioning and Unit Recognition in Performances on Rational Number Tasks'. In: *Proceedings of the 3rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Minneapolis.

KIEREN, T. E. (1972). 'Perspectives on Rational Numbers'. In: LESH, R. A. (Ed) *Number and Measurement: Papers from a Research Workshop*, ED 120 027.

KIEREN, T. E. (1981). *Five Faces of Mathematical Knowledge Building, Occasional Paper No. 20*, Department of Secondary Education, University of Alberta.

KIEREN, T E. (1983). Constructive Mechanisms and Informal Symbolism in Rational Number Building: Partitioning. Paper presented at Annual Meeting of American Education Research Association, Montreal.

SILVER, E. A. (1981). 'Young Adults' Thinking about Rational Numbers'. In: *Proceedings of the 3rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Minneapolis.

SKEMP, R. R. (1971). *The Psychology of Learning Mathematics*. London: Penguin Books.

SKEMP, R. R. (1979). *Intelligence, Learning and Action*. Chichester: John Wiley & Sons.

Index

Fractions: children's strategies and errors

Daphne Kerslake

Fractions: Children's Strategies and Errors is the outcome of three years of research into the difficulties experienced by secondary school pupils in using fractions. These difficulties were identified in the earlier research project 'Concepts in Secondary Mathematics and Science' (CSMS).

During the first part of the study, individual children were interviewed in order to establish their perception of fractions and the approaches they chose to use. The interview data was used to develop and to trial a short teaching module, included in the book, which is designed to extend pupils' interpretation of the meaning of fractions.

Written for teachers, teacher trainers, developers of mathematical curricula and researchers, this book provides many insights into pupils' interpretation of basic ideas concerning fractions.

Dr Daphne Kerslake has taught mathematics in secondary schools, and has a wide experience of both initial and in-service education of teachers. She is a Principal Lecturer in Mathematics Education in the Education Department at Bristol Polytechnic.

NFER NELSON

The NFER-NELSON Publishing Company Limited,
a joint venture of the National Foundation
for Educational Research in England and
Wales and Thomas Nelson and Sons Limited
Educational Publishers.

ISBN 0-7005-1006-0
Code 8199 02 1