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ABSTRACT

This paper is an attempt to draw on the thinking of a group of secondary school geometry teachers who are participants in the Laboratory Sites Study of the Educational Technology Center (ETC) at Harvard University. The purpose of the Lab Sites Study is to understand the process of implementing technology-enhanced guided exploration in school classrooms. Teachers and researchers are working together to understand what it takes to use computer-based technology to support student exploration of mathematical and scientific ideas in ordinary classrooms. The data analyzed in this paper was collected as a substudy of the Lab Sites project, which looked at comprehensive questions of implementation in relation to materials produced at ETC for teaching science, mathematics, and programming. The substudy reported herein was concerned with teachers' points of view about using one piece of educational technology--The Geometric Supposer--to substantially change the way they teach geometry. The Supposer is designed to fundamentally change the way instruction is delivered in classrooms by enabling students to engage directly in the exploration of subject matter. What is reported here, therefore, is the teacher-users' thinking about that broader change in the way they do their work, as well as their thoughts about the technology. (TW)

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TEACHERS' THINKING ABOUT STUDENTS' THINKING ABOUT GEOMETRY:
THE EFFECTS OF NEW TEACHING TOOLS

Technical Report

January 1988

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TEACHERS' THINKING ABOUT STUDENTS' THINKING ABOUT GEOMETRY: THE EFFECTS OF NEW TEACHING TOOLS

TECHNICAL REPORT

January 1988

Prepared by:

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Educational Technology Center

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Preface

This paper draws on the thinking of a group of secondary school geometry teachers who are participants in the Laboratory Sites Study of the Educational Technology Center (ETC) at Harvard University. The purpose of the Lab Sites Study is to understand the process of implementing technology-enhanced guided exploration in school classrooms. A group of diverse school sites was chosen for the study, including an inner city magnet school, a large urban comprehensive school, a suburban school and a small rural school. In each of these sites, teachers and researchers are working together to understand what it takes to use computer-based technology to support student exploration of mathematical and scientific ideas in ordinary classrooms. The teachers range in years of experience in teaching geometry from two to twenty. They also range in their post-secondary educations from a minimum of mathematics courses to a masters-degree-plus in mathematics education.

The data analysed in this paper were collected as a substudy of the Lab Sites project, which looked at comprehensive questions of implementation in relation to materials produced at ETC for teaching science, mathematics, and programming. The substudy reported herein was concerned with teachers' points of view about using one piece of educational technology -- *The Geometric Supposer* -- to substantially change the way they teach geometry. The *Supposer** is designed to fundamentally change the way instruction is delivered in classrooms by enabling students to engage directly in the exploration of subject matter. What is reported here, therefore, is the teacher-users' thinking about that broader change in the way they do their work, as well as their thoughts about the technology.

The *Geometric Supposer* teachers were chosen from among all of the Lab Sites teacher-participants for this study because their decision to use the *Supposer* meant that they would be revising curriculum and instruction across the entire school year in a course that is a required part of the secondary school program for many students. Using the approach that the technology supports thus required a more substantial reorientation for these teachers than it did for the other Lab Site participants.

The teachers who participated were interviewed during the fall and spring of the 1986-87 school year, observed as they taught with the *Supposer*, and observed as they participated in monthly "Users' Group" meetings with ETC researchers. Common themes in their thinking were identified during the fall, and in the spring, observations and interviews were used to further probe issues of concern to the teachers.

* The *Geometric Supposer* was developed at Education Development Center by Judah Schwartz and Michal Yerushalmy. The software is published and distributed by Sunburst Communications.

INTRODUCTION

Why and how is geometry taught in secondary schools? The answer to this question depends on how one thinks about what it means to know geometry and what one assumes about how that knowledge is acquired. Most commonly, coming to know geometry is thought to be an occasion for students to learn to use deductive thinking, that is, to have the experience of arriving at mathematical knowledge by starting with axioms and thinking through their logical consequences. But from another perspective on teaching and learning in school, knowledge about geometry is assumed to be acquired by listening to teachers and studying textbooks. What is learned is a kind of indoctrination of novices to important elements of a subject like definitions, algorithms for solving "proof" problems, which are needed in preparation for further math courses or college entrance exams. (Cf. Green, 1971, for a comparison between indoctrination and instruction; Schoenfeld, 1988, on "good teaching.") A third, and somewhat less common epistemology which could characterize the teaching and learning of geometry is induction; in this view, knowledge is supposed to be acquired empirically. Teachers direct students to perform geometrical operations, like constructing, measuring and comparing, and students draw conclusions by thinking about their observations. What students know as a result of this process are principles they have constructed from their own experience of trying to make sense of data. Each of these epistemologies -- deductive, indoctrinating, and inductive -- implies a different pedagogy and a different kind of relationship among what teachers do, what students learn, what is mathematically proven, and what comes to be "known."

A recent innovation in the materials available to high school geometry teachers provides an opportunity to look at these relationships from a fresh perspective. New computer-related tools have been developed at the Education Development Center and the Educational Technology Center at Harvard University which enable students and teachers to quickly construct and measure geometrical figures: *The Geometric Supposer* is a set of software to use on classroom computers for the inductive exploration of geometrical relationships. Print materials that accompany the software pose problems that lead students to make conjectures based on the data they collect, and students can return to the *Supposer* to search for counterexamples which determine the generality of their conjectures. As with other materials being developed at the Educational Technology Center, the *Supposer* materials are designed to make the classroom computer a tool that can be used to support "guided inquiry" as a method of teaching and learning in ordinary schools.

Case studies of seven teachers who experimented with the *Supposer* in their high school geometry courses during the 1986-87 school year suggest that this teaching tool has the potential to change the way teachers think about what it means to know geometry, to affect what they believe about how that knowledge can be acquired in classrooms, and to change their teaching practice. Those developments will be reported in this paper. Because the *Supposer* is designed to have students use induction to learn geometry before they learn to prove theorems deductively, the materials are a serious intervention in what has become common practice in high school geometry courses. In the process of negotiating between the kinds of teaching and learning intended by the Supposer materials and that intended by the conventional curriculum, the teachers who experimented with the Supposer expressed and changed their beliefs about what it means to know and learn geometry.

In a sense, this paper is about the effects of a new technology on teaching and learning in schools. But it is also about the fact that subject matter, and the social order in classrooms, are at least as much of a factor in determining what gets taught and learned as the available technology. The power of the Geometric Supposer lies not in what it can do, but in what it enables teachers to do if they are both able and disposed to use it in the way it was intended. (Cf. Cohen, in press; Resnick, April 1987.) The technology enables students to engage in inductive geometrical inquiry; what occurs in the classrooms of the teachers who use the technology results from their attempts to integrate this inductive inquiry with traditional curriculum and instruction. It would be difficult, but not impossible, to teach geometry as a process of "guided inquiry" without the Supposer technology, and the technology would be used in ways very different from those intended by its designers if the subject matter and social order of the classroom were not redefined to give meaning to "guided inquiry".

GEOMETRY TEACHING AND LEARNING

The content of high school geometry is conventionally organized around students' learning to prove the theorems of Euclidean geometry. Before they get to the tenth grade, these students have been taught the definitions of various kinds of plane figures and they should know how to draw and label them according to mathematical conventions. They may also have been introduced to the idea of "proof" in a course in introductory algebra, but in most secondary school curricula, that topic is reserved for the Geometry course. This pattern of instruction has changed little in the last fifty years, in spite of "the new math," technological developments, and changes in the field of geometry itself (Usiskin, 1987). Comparing this conventional pedagogy with the sort of reasoning that is involved in doing the mathematics is a necessary backdrop to understanding what using the Supposer meant to Geometry teachers.

"Doing a Proof" in mathematics and in high school geometry

Within the world of mathematical discourse, establishing a geometrical truth such as "intersecting lines form opposite angles that are equal" begins with the definitions for "intersecting lines" and "opposite angles." In conventional deductive form, these definitions and some algebraic manipulations using supplementary angles*

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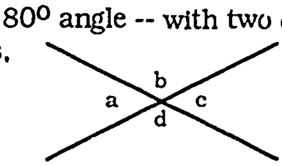
* Supplementary angles are two angles that, when placed next to one another, form a straight line -- a 180^0 angle -- with two of their sides, e.g., in the case of these two intersecting lines,

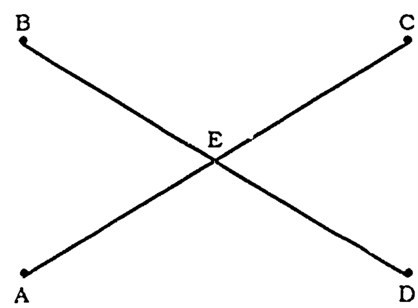


(angles a and b, b and c, c and d, and d and a are <u>supplementary</u>. b and d are <u>opposite</u> angles, as are a and c).

can be logically related to produce the conclusion that whenever two lines intersect, the opposite angles formed will be equal. This must be true simply because opposite angles are supplementary to the same angle. The conclusion can be reached by deduction, and generalized thereby to all pairs of intersecting lines; it does not depend on actually drawing any lines or even looking at two lines that intersect.

But much more than this simple deductive argument is required to prove this assertion in the form that is typically taught, and sometimes learned, in high school Geometry courses. A student who wanted to demonstrate that he or she had learned how to prove that intersecting lines form opposite angles that are equal would need to produce something like this:

Theorem: If two lines intersect, then the opposite angles formed are equal.



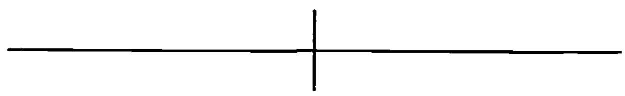
Given: Line BD intersects line AC at point E.
To prove: Angle AEB = Angle CED.

Statements	Reasons
1. Line BD intersects line AC at point E	1. Given
2. Angle AEB + Angle BEC = 180°	2. Supplementary angles
3. Angle BEC + Angle CED = 180°	3. Supplementary angles
4. Angle AEB + Angle BEC = Angle BEC + Angle CED	4. Quantities equal to the same quantity are equal to each other
5. Angle AEB = Angle CED	5. Subtracting the same quantity from both sides of the equation

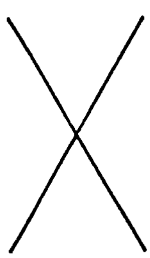
Q.E.D.

Considering what kind of knowledge it might take to construct such a proof produces a useful analysis of the content that is taught, and usually learned, in a high school Geometry course. The "basics" are definitions and labeling conventions. Students have to learn those first before they can go on to the reasoning process that is involved in proofs.* Another "basic" is what is usually referred to as the "two-column form" for proof. To generate a geometrical argument in this form, the theorem must be written in what teachers call the "if-then" form, and then the "if" part of the hypothesis must be translated into a statement of what is "given" and the "then" part into a statement of what is "to prove." This is done simultaneously with creating the diagram to illustrate the conditions that are given. This illustration turns the general condition being stated into a claim about somewhat more particular entities: in the illustration, the intersecting lines become lines BD and AC which intersect at E. They form angles BEC, CED, DEA, and BEA,** so the opposite angles which must be shown to be equal are either BEC and DEA or CED and BEA.

Of course, a pair of intersecting lines could also look like this:



or like this:



or like any other configuration of two lines meeting; and any letters, in fact any symbols, could be used for labeling them. But in any illustration of the general condition, the lines and their point of intersection must be specified in a way that is a concrete exemplar of the given condition. In the activity of creating the diagram, general terms in the hypothesis, like "intersecting lines," become specific: two actual lines are drawn on paper and labeled.

In a secondary school geometry lesson, some attention might be given to the meaning of this mathematically significant translation from universal logical conditions to a particular empirical illustration, but whether the "proof" is about the particular elements of the diagram or the more general elements of the theorem becomes somewhat ambiguous. Whether students come to know that the proof is universal while the drawing that illustrates it is particular probably depends on the

* In less "academic" Geometry courses, it would not be unusual for students never to get beyond definitions, drawing digrams, and labeling.

** Some teachers and textbooks require that one always distinguish between lines and line segments by higlighting the points which are labeled on a line and putting arrows at either end, but in most situations, this formalism falls away consciously or unconsciously early on in the course. Here it is assumed that BD is a segment of line BD and AC is a segment of line AC.

authority asserted by the teacher rather than on their appreciation of the subtleties of logic.* (To wit: Once the diagram is drawn, it would not be surprising if a student suggested that it would make sense to <u>measure</u> as a way to prove what is listed on the "to prove" line.)

The list of statements and reasons make up the body of the proof. They must always be written in parallel columns, and they must always begin with what is "given," and proceed through a series of steps so that the last is a statement of what one set out to prove. The proof then ends with the triumphant "Q.E.D."-- a claim in authoritative Latin that one has demonstrated what was to be demonstrated. There is never any question that what needs to be proved <u>will</u> be proved; typically students will write the first and last steps before going on to fill in the middle ones.

What goes in between the first and last statements is more of a problem for both students and teachers. That is where "logical thinking" comes in, or doesn't. In the case of the proof that the opposite angles formed by intersecting lines are equal, one begins with definitions and manipulates them with a bit of algebra to get from the given lines to the conclusion. In other proofs, one might also need to use previously proven theorems as "reasons" for the statements in the argument. ** As with arithmetic word problems, it is useful to know the topic of the chapter in which a proof problem is given in order to narrow down the possible choices for "reasons." Many students admit to memorizing the steps in proofs that are given in their books or demonstrated by their teachers. Teachers rarely ask students to write a proof that they have not seen before, except in "honors" or "advanced placement" sections.

A typical scenario for a Geometry class of average ability would be to have the teacher "go over" the proof as written above in class, and then give a homework assignment which would have students writing the proof to show that Angle BEC = Angle AED. To do this correctly, students would reproduce the steps as above, but change the angle names to match the new set. They would not be expected to do much in the way of reasoning.

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* This ambiguity becomes significant if one wants students to be able to distinguish between deduction and induction as sources of mathematical certainty. For those initiated into mathematical conventions, there is no question but that proof is about deduction. But what is being taught -- and presumably learned -- in high school geometry is not just how to do a proof. Students are also learning how to learn mathematics: "don't do what seems to solve the problem (i.e., measure to see if two angles are equal), but do what the teacher tells you is the right thing to do."

** One teacher reported to me that she had a problem with students who wanted to put <u>everything they could remember</u> somewhere in the proof they were working on. This is a wonderful case where the culture of the classroom, in which the idea is to let the teacher know what you remember, and the culture of mathematics, in which the idea is to bring relevant definition and theorems to bear on a logical argument, come into an unproductive conflict!

What is learned in learning "a proof"?

Presumably, after seeing the logical steps in a proof like the one given above, students should be convinced that whenever two lines cross, the angles that are across from each other will be equal, even if they did not invent the reasoning that leads to that conclusion themselves. That is the way acquiring geometric knowledge goes if one believes that acquiring such knowledge follows the form of using deductive reasoning to arrive at mathematical truth. Starting with definitions and using algebraic principles, geometric elements are combined to generate new relationships which are known to be true by virtue of being logically inevitable. The conclusion is known because it follows logically from what is given.

But is that what happens to produce students' knowledge in high school Geometry classes? It might be. But there are at least two other possibilities. One is that students know that intersecting lines form opposite angles that are equal because the teacher said so, or because it is in the book. They can demonstrate that they have this knowledge on a test, and perhaps even call it forth when it is useful in another proof. But they have acquired it through acceding to authority and memorization, not because it has been logically demonstrated to be true. Another possibility is that students know that opposite angles are equal because whenever they have drawn two lines that cross, the angles that were across from each other looked equal. They might even have drawn some intersecting lines and measured the angles with a protractor to find that every time they do it, the angle measures are the same. They do not need to draw too many different pairs of intersecting lines to be sure that the next time they do it, the results will be the same, as they would also be for anyone else who tried the experiment. (If the measures did not come out the same, they would be more likely to think that a measuring error had been made then to have their faith in the conclusion shaken.)

Both of these scenarios lead to students wondering why they should bother to "prove" what they already know is true. But students soon stop raising this question, if they ever bother to raise it in the first place, because what they have learned is to go along with what the teacher says to do. Geometry is a required course in the college preparatory track, and especially for the ambitious, that is enough incentive to learn about proofs if that is what will get you a passing grade. What students learn about proof from this experience seems incongruent with the role of proof in mathematics, and they are learning little from the proof about the logical consequences of axioms and definitions.

Teaching "proof" and teaching mathematics

Many geometry teachers and high school curriculum developers continue to assert, however, that learning to prove geometrical theorems will help students to "think logically" in math and in general. Some teachers teach proof because they want students to appreciate the particular mathematical beauty of Euclidean geometry, which can generate a great deal of mathematics from a few axioms and definitions, using deduction. Others simply recognize that their students will not pass the final exam and be able to go on to the next math course unless they can write proofs in the conventional form. So teachers of geometry cajole and convince, they require and explain, they do whatever it takes to get their students to be able to "do proofs." And they divert students away from measuring actual geometric figures--that is not the way Euclid did it, and it will not help on exams. By participating in this kind of learning, students get ideas about what mathematics is together with whatever other knowledge they acquire. What they acquire is a very limited perspective on mathematical thinking, however. Common methods of teaching geometry, and the assumptions they express about student learning, mix a formalist philosophy of mathematics with a reliance on teacherly authority as the source of mathematical knowledge and truth. Formalism defines mathematics as a set of logical procedures by which conclusions are deduced from abstract propositions. In this view, everything that is known is derived by rules of logic, which is thought to give mathematics its special claims to certainty. Formalism came upon a significant challenge early in this century, however, when some mathematicians established that they could produce contradictory sets of results by applying legitimate logical procedures to the same set of axioms. (Cf. Brouwer, 1964; Kline 1980.) This bit of logical acrobatics inspired all sorts of arguments and generated several alternative theories about the nature of mathematical knowledge, and it shifted attention in the field from theorems to the axioms from which they were deduced. Several non-Euclidean geometries developed from different sets of axioms, and it seemed as if the certainty of mathematical knowledge might be lost forever in a sea of relativism (Kline, 1980, Polanyi, 1960).

The identification of mathematical creativity with the deductive process has also been more recently challenged by philosophers taking a more careful look at how mathematical knowledge has developed over the centuries. Accounts that mathematicians have left of their work do not give the impression that it all just "flowed" logically from a small set of axioms. (Cf. Hedemard, 1975; Polya, 1954; Davis and Hersh, 1981; Lakatos, 1976.) Instead, we see mathematicians making leaps into "truth" using induction or-- even worse from the formalist point of view--intuition, and only resorting to proof when they suspect that something they have asserted is not true.

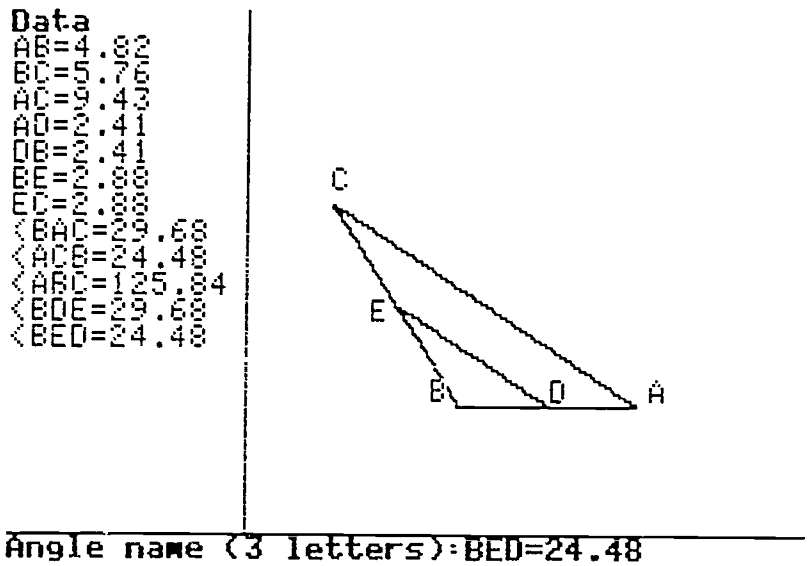
These developments in thinking about the nature of mathematical knowledge do not seem to have had much affect on the practice of secondary school mathematics teachers. Before they began working with the *Supposer*, all of the teachers in the experimenting group taught geometry as the logical development of theorems from the Euclidean axioms. They followed textbooks which assumed this perspective on the subject and expected their students to learn definitions, labeling conventions, and the two-column form for proof. They felt strongly that theorems needed to be taught in a particular order so that each new proof could be built on previously established truths. They controlled the agenda according to which knowledge was acquired, with the help of their textbooks, and they told students what they needed to know and in what order it made sense to know it. Sometimes, they did activities that would suggest that they thought students ought to "discover" mathematical theorems empirically, such as having them draw intersecting lines and measure the angles that are formed to find that in every case, the opposite angles are equal. These activities supplemented "teaching as telling" and they were done primarily to motivate students and vary the routine.

In these classrooms, and in many others that have been observed by educational researchers, the relationship between what the teachers do and what the students learn is defined not only by the course content, but also by the particular kind of social interaction that is a routine part of the culture of schooling. (Cf. Jackson, 1910; Green, 1971; Doyle, 1985; Powell, Farrar, and Cohen, 1985; Florio-Ruane, 1987.) This culture interacts in complex ways with the kind of mathematics that is, and can be, taught. Teachers tell things to students. Sometimes they even "explain" them, although most teachers I have asked are hard pressed to tell me what that entails. Teachers give students assignments, students do them, and they are checked. If students don't do their assignments correctly, teachers do more telling or explaining, and then they give a test. If students "know" what they have been taught, they will do well on the test. That is just the way it is in schools. That is the way the culture of the classroom defines "knowing" and students and teachers participate together in that culture. Teachers are party to a social contract with students and their parents that binds them to current representations of what is to be taught and learned and current procedures for measuring those accomplishments. It makes sense in this culture to memorize what the teacher says you need to know, even if it does not make logical sense to you. Using the *Geometric Supposer* to support "guided inquiry" over the course of a year in the midst of this culture was disconcerting to both teachers and students; it caused the renegotiation of common understandings about what was going to be taught and learned, and how students' knowledge was going to be measured.

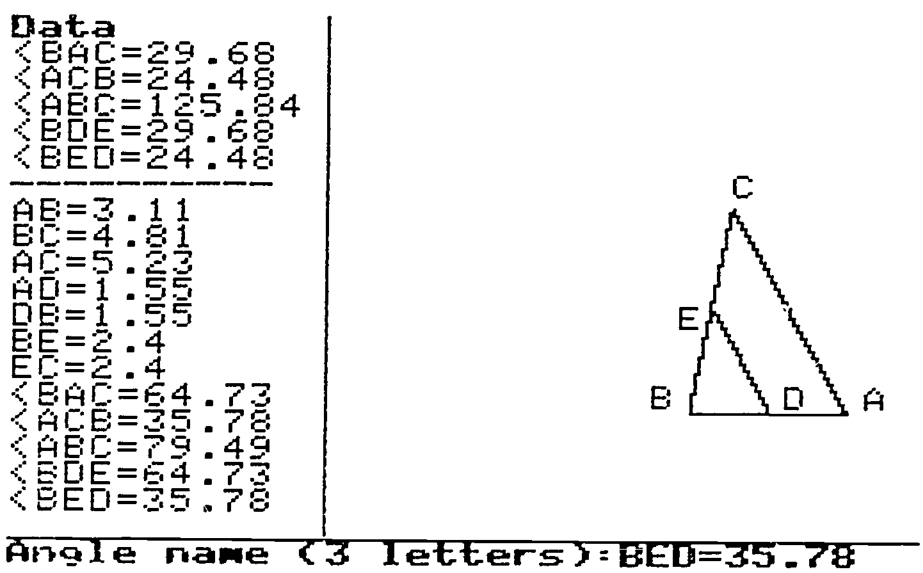
THE *SUPPOSER* INTERVENTION

The "Lab Problems" that accompany the *Geometric Supposer* software are designed to have students consider geometrical relationships inductively before they are exposed to deductive proof. (Cf. Yerushalmy and Houde, 1986.) Students are to use the technology in a laboratory-like setting to construct and measure many specific figures before the classroom discourse turns to a discussion of what the patterns in these measurements might imply for relationships in what one teacher calls "the universal triangle." The Lab Problems are designed to lead students to the same abstract geometrical theorems as the more conventional curriculum, but the activities they entail are such that they can lead in many other interesting directions as well.

An example of a Lab problem students might be given to explore on the *Supposer* is, "On different kinds of triangles (acute, obtuse, equilateral, scalene) ...ct a line that is parallel to one side and bisects one of the other two sides. ...ou conjecture about this line in relation to the sides and angles of the ...angle?" The software enables the production and measurement of ...nstantiations of each kind of triangle, the instant construction of the ...isector in each, the speedy measurement of all lines and angles, and the automatic computation of ratios between pairs of measures. Students work on problems at the computer in pairs, and they are provided with charts on which to record whatever measurement data they choose to collect. One pair of student might be looking at a figure and data like this:



While another pair would be looking at this:



and several more variations would be produced by each pair in a lab session. The software is designed so that students do not need to remember the definitions of terms in order to produce these figures. In the class following a lab session, students report and discuss their conjectures. Some of their conjectures are chosen by the teacher for the development of deductive proofs, first in paragraph form, then in the traditional "two-column" form. Initially this development is led by the teacher, and later it is primarily the independent work of students.

In the Lab Site classrooms, the software, the computers, and the commitment to spend regular class sessions doing "*Supposer* Lab Problems" functioned together to redefine both the nature of geometric knowledge and the teachers' and students' assumptions about their respective roles in the acquisition of that knowledge. The presence in the classroom of the *Supposer* technology as another "authority" on what it means to do geometry was a substantial intervention in the interactive routines that defined what teachers needed to teach and what students needed to know and know how to do. The technology was in no sense a "replacement" for either the teacher or the textbook, however. It was an additional resource in the classroom that put both of those more familiar elements in a different perspective.

In the early weeks of their attempts to use the *Supposer* technology, the phenomena of most concern to the teachers was sharing their control of the intellectual agenda with students-- they both wanted students to engage in the kind of independent exploration that would lead to the invention of conjectures and worried about the consequences of this kind of activity for "covering" the curriculum. As students were taught to collect data about geometrical relationships, look for patterns, and make conjectures about what sorts of relationships might be "always true" of figures in a certain class, they became engaged in working on problems that were posed neither by the teacher nor the textbook. As they proceeded to make and test conjectures, they were able to take off in many different directions through the subject matter. In contrast to theorems in the textbook which are chosen by the teacher for the teaching of proof, *Supposer*-related conjectures were experienced by students as geometric relationships which <u>needed</u> to be proven before they could be accepted as true. Thus, the work of proof was "owned" by students in a way that it had not been in the traditional curriculum. They constructed their own intellectual roadmaps through geometry as they asserted geometrical relationships and tested their reasonability. This student "ownership" of the agenda was in radical contrast to what had occurred in these classrooms before the *Supposer* became a tool for teaching and learning, when the roadmap that defined the content of the Geometry course and the journey a class was going to take through it was determined by how the teachers decided to proceed through the textbook.

The teachers who experimented with the *Supposer* were faced with a serious conflict. The culture of the schools in which they were working had not prepared either the teachers or their students to feel secure that, if students followed their own intellectual roadmaps, they would learn what they were supposed to know. Yet the new technology was seductive-- to both students and teachers-- and drew them along mathematically interesting paths that did not coincide with the routes through the subject defined by the textbook. Students went off on mathematically productive tangents that no one was able to keep track of. As they made conjectures that they wanted to prove, teachers were barraged with questions. Even teachers went off on tangents as the *Supposer* captured them in some interesting mathematical puzzle. This was both exhilarating and frustrating; the participants enjoyed what was happening, but they were not sure what connection it might have with what they had come to know as learning high school geometry. They were faced, at a very practical level, with figuring out how to "guide" the inquiry process once students were engaged in it. The situation the teachers experienced was something like that of a tour operator whose bus has been replaced with a collection of glitzy motor scooters in the middle of the Place de la Concorde. Many of the "tourists" could see places they wanted to explore, but did not know how to get to them. Others did not even know where to go or how to begin to make choices. Everyone was complaining because their expectations for the tour were not being met, while at the same time they were anxious to jump on a scooter and take off down the road. To make matters worse, the cameras that were to serve to record where everyone had been were left on the bus. The theme that dominated the teacher's thinking over the course of the year was how to make a situation like this into one that would fit into the culture of schooling without losing the excitement and engagement the *Supposer* engendered.

What follows is a report on patterns of development in the teachers' thinking during this experimental period based on an analysis of the following data: two series of interviews, one conducted in the fall and one in the spring; visits to each teacher's school, including observations of *Supposer*-related lessons in their classrooms; interviews with the teacher-advisor whose job it was to facilitate each teacher's work with the project; and notes from "Users' Group" meetings conducted once a month by the developers of the *Supposer* classroom materials.

The approach taken to the analysis of this data assumes that the teachers studied were in a period of "liminality"; this anthropological concept has been used by von Gennep (1975) and recently elaborated by C.A. Bowers to describe how people think in the midst of a period of cultural change:

> ...liminality designates those moments in cultural transition where the individual is 'betwixt and between' established patterns of thought and behavior... In being 'betwixt and between' accepted definitions that will serve as the basis of social experience, new definitions can be presented, and the conceptual foundations of authority renegotiated. (1987, pp. 6-7)

The *Supposer* innovation brought about a situation in the classrooms of the Lab Site teachers that was analagous to a cultural transition. During the year that the study was conducted they were "betwixt and between" taking the meaning of their activities for the "accepted definitions" of conventional geometry curriculum and instruction and the "new definitions" of geometry, teaching, and learning that would emerge from inductive inquiry into the subject in the classroom. The authority to define what is to be learned and how it is to be taught was being negotiated on a daily basis, over the course of the school year, putting previously unquestioned practices into a new perspective.

Although it would be reasonable to imagine that changes in teaching practice <u>follow</u> changes in teachers' beliefs and attitudes, the *Supposer* innovation did not work that way. Teachers' beliefs and attitudes changed over the course of their year's work with the *Supposer*, being strongly affected by what they observed their students doing as they used these new teaching tools. This sequence has been proposed as a "model" for teacher development to follow on the notion that teachers will embrace a change in their practice if they perceive it as practical, i.e., if it produces student learning (Guskey, 1986). This cause-effect explanation does not take full account, however, of the complexity in how a teacher defines student learing. Changes in their ideas about what should be learned in a G netry course left the teachers with several questions at the end of the experimental year about whether the *Supposer* tools could be considered "practical."

Teacher thinking about subject matter: Finding a new way to map the terrain

Since the textbook has generally played such an important role in the organization of mathematics curriculum and instruction, and since there is little to challenge its authority in defining high school geometry, it was not surprising to find that when teachers began using the *Geometric Supposer* they merged their definitions of the subject of geometry with the contents of the book they had been using to teach it. For all of them, the measure of what students were supposed to know at any point in the course was the book, and the order in which knowledge was supposed to be aquired was the order in which it appeared in the book. But their use of textbooks for guidance about content was not a simple acquiescence to a "higher" pedagogical authority. They assumed that the order in which material was presented in the book was congruent with the order in which geometrical knowledge logically develops.

Euclidean geometry is a wonderful example of mathematical elegance: beginning with very few unproved assumptions (called axioms or postulates), one can generate a large body of geometrical truth using deduction. This logical order in the subject makes it difficult for teachers to accept a diversion from what is in the book: they worry that students will get confused, or that what they are learning is not really geometry if it comes out of the proper sequence. (Their position is not at all comparable to that of an English teacher who is faced with deciding whether to teach novels first, or poetry, in a literature course.) The order of topics, arranged around axioms and theorems, is meant to teach students something <u>about what geometry is</u>.

One teacher in the group put this quite plainly:

> The textbook is a logical progession, it absolutely is. As a matter of fact, the nature of geometry is sequential; that's how Euclid did it. You accept some statements as true because they make logical sense and then you deduce things from them. That's how geometry progresses. (G.S.int1*)

Another teacher, who would agree with this description of geometry, said that the *Supposer* had thrown off her sense of how the material she was supposed to teach

* Teacher talk is designated by the speaker's initials. "Intl" means that the quote was taken from interviews done in Fall, 1986; "int2" quotes are from Spring, 1987.

should be sequenced, and that she did not yet have a sense of how to construct a new sequence to replace the one that was familiar:

> In terms of specifics, I'm very confused, because it's so nitty gritty in the book [what students need to know to do the problems, that is],and then when you go to use the *Geometric Supposer*, it's kind of "Well, they'll get that idea when they do problem #18, so do I really even have to discuss it?" I mean in terms of the postulates you need for doing that proof or whatever. (A.D.int1)

For all of the teachers in this study, the development of geometry in the classroom had proceeded according to what they thought to be the internal post hoc logic of the discipline. They have all taught geometry several times, and they have a realistic calendar in their minds about where they should be in this progression of axioms, definitions, and theorems at any given point in the school year. When they began using the *Supposer* at the beginning of the year, they combined lab work on the computer with textbook excercises so that they could maintain the familiar schedule, rather than giving up any of what they had done before:

> I did very traditional stuff the first couple of weeks, so they have learned their vocabulary fairly traditionally and we also did several things [with the *Supposer*] in small groups in my class. I still feel comfortable they've gotten everything we'd normally give a class up to this point. And then its been enriched by the *Supposer*. I'm concerned and wondering if the stuff we're doing with the *Supposer* can be transferred to a more traditional testing and textbook oriented curriculum, if you could teach the subject purely from the *Supposer*. (G.S.int1)

What this teacher is reflecting on here is not only how to teach or when to use the new technology; he is raising a more fundamental question about how order defines the content of a high school Geometry course.

Although each of the teachers found ways to maintain this order throughout the school year, using the *Supposer* "radically altered" their views of what geometry is, of what the aims of teaching it are, of and the way it should be taught. One of the teachers said she felt personally threatened by her perceptions of what she saw students doing with the *Supposer* because one of the reasons she liked teaching geometry was her appreciation of the "beautiful logical development of the whole corpus of Euclidean geometry." She wanted her students to like geometry for its "self-contained completeness." But her goal was at risk as she watched her students conjecture about what might be true of the shapes they had generated on the *Supposer*. Her students were "making generalizations all over the place that they did not have the background to prove":

> We did some of these projects, just to get a sense of the *Supposer* and collecting data. We made conjectures and then just threw them in there [without proving them] to explore what similarity was... They've got a lot of neat conjectures, and they can't do anything with them because they don't have the theory yet, that is, they can't do anything in the sense of proof. This bothers me because one of the things that is

> neat about geometry for me is that you can really do a deductive process with it. My teaching the deductive process has gone out the window... It's like I'm in a totally new ballgame and I don't know what the rules are. (J.H.int1, emphasis mine)

To this teacher, "not having the theory yet" means that students do not have a corpus of already proven theorems that they can use to go on to prove the conjectures that result from their observations of the measurements produced by the *Supposer*. This contradicts what it has meant to her to "do geometry", because the ideas that students are encountering are not coming in logical deductive order. In her view, doing geometry means:

> You have to be clear about what you know is true at the start and where you want to end up; you have to have that set out in your mind and you have to have available some place up here [in your head] what relationships exist that you know about and then be able to put them in the right places to apply them to this situation. <u>And that comes from where you are in the course of the year</u>. You can begin with something as simple as proving triangles congruent and wind up with proofs about the internal and external triangles on a circle. The progression is really remarkable. (J.H.int1, emphasis mine)

Again, there is a merging of the track one takes through the course syllabus and how the nature of the subject is defined.

Near the end of the school year, I asked this teacher how she was thinking about the relationship between students going off on *Supposer* explorations and the elegant logical corpus of Euclidean geometry. Her first response was to talk about the difficulties she was having trying to <u>both</u> give time for exploration <u>and</u> complete the syllabus. She was still associating the syllabus with the logical progression of postulates and theorems, and still feeling something of a tension between that progression and the process of inductive inquiry intended by the *Supposer*. She said that another teacher in her school had done better at integrating the two, but she was still going back and forth in a "linear" way, feeling that although she wanted to integrate, it would be "much more of an effort for me to think that through." She hoped to do a better job on that the following year. Then she spoke about her "sabotage" of the deductive process she enjoys so much:

> I've sabotaged it all, I give the kids all kinds of axioms, but-- it's because I so value the problem-solving experience and I don't think I have lost a lot in the deductive process. I provide them with more axioms than before, and that is not a small thing. But it isn't the same kind of issue as I thought it was earlier because now I am able to enjoy the course in a little different way. There is no question but that the deductive process is still extremely important.
>
> The thing is that we have in no way lessened that. And in some ways, maybe we've strengthened it. [With the *Supposer*, now] you've got to be able to say <u>why</u> something is true. (J.H., int2)

The meaning of deduction has broadened for this teacher, and it seems as if her students are also having a more personal experience of the deductive process because they are having to state their own conjectures and they are expected to be able to say, from a logical standpoint, why those conjectures might be true or not.

Another significant change in this teacher's thinking is in the way she finds her way around in the subject matter as she attempts to guide students from their conjectures to the knowledge she believes it is important for them to acquire. After *Supposer* labs, she admits to spending class time discussing conjectures that are not related to the topic on her agenda because, "some of the things they come up with are just remarkable, and we need to get all that up there." Discoveries are written on the blackboard; they deserve a place in the public discourse whether or not they are part of her agenda. But she also "makes sure that the things that are important to me get stated." She attempts to draw many connections between students' discoveries and the curriculum in any given lesson, so that "kids can begin to appreciate that everything matters." The theorems she believes are important are ones that will be used at some other point during the year to prove other theorems. But now, because of the *Supposer*, she moves around in the subject toward these important ideas along a web of relationships that follow her students' thinking, rather than following the classic agenda from axioms to theorems that is often believed to represent Euclid's thinking.

Where do teachers get this idea that doing geometry is a logical progression from a few axioms and definitions to theorems and more theorems, all of which follow one another in a neat deductive sequence? One explanation is that they have never done geometry as it is done by geometers, or even seen it done. There is not a required college course in geometry per se for math majors, so it is likely that their model for doing geometry comes from their own high school course, heavily supported by the textbooks they are using. As one of the teachers commented:

> The way you were taught geometry is not always the way it's taught now, but I don't know which is best. With the *Supposer*, there is this terrific pressure on you to just jump ahead and show them things or let them discover things in the *Supposer* that actually you shouldn't because the theorems are supposed to follow a certain rigid order. (B.L.int1)

Another said, "I know what they're supposed to know in terms of the curriculum I'm supposed to cover." (A.D. int) In both of these comments, the process and sequence in which one acquires the knowledge that is geometry in school is entangled with the presumed "rigid order" of the subject matter.

After several weeks of experiencing a tension between the ideas that students were coming up with in *Supposer* Lab and the ideas that were contained in successive chapters of the book she was using, one of the teachers decided to abandon the book for what she considered to be the "more natural" order of students' thinking as it was generated by the *Supposer* problems. In the process she gained some insight about how the order of the theorems and axioms in Euclidean geometry could be rearranged and still follow a "logical" sequence:

> I've really gotten away from the book in terms of the specific topics in the chapter. I found it to be better if I did the order which went with

> the *Supposer*. But I still use the textbook if there are appropriate homework problems... I could see from the problems I asked them to prove at night (based on work in the *Supposer* Lab) that it was very easy, for example, to use the first three postulates and then just go right into CPCTC [Corresponding Parts of Congruent Triangles are Congruent], whereas the book waits for CPCTC... it logically followed; it was a real nice step to do just then, and the kids would understand it really easily. (A.D., int2)

This teacher's decision about the agenda is a negotiation between what she saw that "the kids would understand really easily" and what she knew they needed to learn at some point in the year.

Other teachers made less radical adjustments in their use of textbooks but they all commented on the amount of extra organizational, intellectual work that was involved in diverging from the order of postulates and theorems presented in the book.

> It's an awful lot of work from the teacher... just sitting down and trying to organize the unit. In a geometry book, the order does make a difference, let's face it. And if you're going to go away from that order, it takes a lot of thinking and a lot of work to organize the unit, a lot of thinking ahead. (A.D., int2)

"Reorganization of the subject matter" was mentioned by all of the teachers as one of the most difficult tasks that had resulted from their experiments with the *Supposer*. Each of them talked of plans to spend time over the summer going over the Lab Problems and figuring out where what students could do with them would fit into the corpus of material that was supposed to be taught in a Geometry course. They all seemed to be disposed to reorder the material in response to students' thinking, but they felt constrained, both by time and by their familiarity with the conventional curriculum. One teacher said:

> My mind is so straight and all I see is just what I know and my mind doesn't seem to go off on all the tangents like the kids do... You've got to take the time to study a lot and you have to think a lot before you give the projects out, and that's what I have to do more of... I never realized the kids will come up with so many more ideas, or so many different things than what I thought of. (P.O., int2)

It is this kind of work that J.L. felt she did not have time for, and this kept her from integrating her *Supposer* classes with the syllabus she was expected to cover. She said that when she did have enough time to read over the papers on which students recorded their data and made their conjectures, she felt as if she could produce a lesson that was both responsive to the syllabus and attentive to their discoveries. Whereas she had been able previously to rely on the textbook, knowing that its order was logical, now she had to figure out what theorems should follow what axioms on her own.

As the *Supposer* teachers have reflected on the tension between celebrating what students discover inductively while they are using the *Supposer*, and being responsible for students' knowing the postulates, definitions, and theorems that are presented in order in the textbook, they raise fundamental questions about what it means to know something in mathematics, how much of that knowledge is internally generated by the knower ("discovered"), and how much comes down from the authority of the discipline in the sense that it has been "proven" according to acceptable methods of logical deduction. What is the relationship between the process of individual inquiry and already existing bodies of knowledge? To what extent can we trust that engagement and exploration in a subject will result in students learning what it is important for them to know? (Cf. Petrie, 1981, Kitcher, 1984.)

These questions arise because of the potential of the *Supposer* technology to tap students' capacity to aquire geometric knowledge on their own-- i.e., without getting it from a teacher. The teachers who use the *Supposer* are thus challenged to expand their sense of their role to include directing student inquiry and legitimizing the knowledge that is aquired in that process.* They were also faced with the need to construct a new kind of social interaction in the classroom so that they could pay close attention to students' thinking and follow their leads through the subject without wholly giving up the authority to determine what would be learned.

The role of induction in changing teachers' ideas about their role in geometry teaching and learning

There seems to have been a common belief among this group of teachers before they began to use the *Supposer* that geometry could be "done" without any data collection and conjecturing on the part of learners; they saw doing geometry as a process of moving from teaching definitions and axioms and posulates to proving theorems, and using those theorems to prove more theorems. But they also felt that this process did not "work" for many of their students; whether students aquiesced to it or not, they were alienated by it. The "thinking" belonged to teachers and texts. In the *Supposer*, teachers saw a tool which could counter this intellectual alienation. As one teacher said, it provides "the opportunity for students to do their own thinking and not just a lot of responding to what the teacher is asking for." (J.H., int2)

The *Supposer* is seen by these teachers as a "motivator" both because the technology has some appeal and because students can do lots of things without teachers always telling them what their next move should be. The teachers are not unrealistic about how much work they still need to do to, especially to convince students of the need for proof,

> but at least now it is about something they [i.e., students] have shown some interest in exploring, whereas before it was all coming from me

* Comparing this teaching task to the more typical kind of teacher work that Putnam, 1987 describes as "construcing curriculum scripts" gives one a sense of the magnitude of the problem they faced.

> and it seemed totally contrived. The motivation for proof--that still, most of the time, comes from me. If I never threw out the idea of "Let's prove this now," I doubt very much if many of them would try it. But at least they're tuned in...
>
> It motivates them to be interested in the problem for the problem's sake more than teaching it traditionally would. And that is a motivation that is absolutely critical in education, particularly in math education. This stuff just doesn't seem relevant to the overwhelming majority. But [with the *Supposer*], they've invested the time in exploring, and they enjoy working with the computer. I think there's that empowerment there. (G.S., int2)

Watching what their students come up with on their own, once they are educated to collect data and look for patterns in it that lead to conjectures, has been a surprising experience for the teachers in this experiment. They see that their students are able to construct new knowledge from what they have been taught before and that they can actually generate new knowledge by induction. The more they see their students do on their own, the more willing teachers are to let students go. What results is an interactive process of empowerment: students take charge and teachers trust them to do so because they recognize capacities that they did not know were there.

But how much mathematical knowledge can students acquire without being taught? How much of it comes from a process of observing patterns in physical phenomena and inventing concepts to describe them? These deep philosophical questions have become matters for everyday consideration as the *Supposer* teachers try to figure out how to incorporate students inductive discoveries into lessons. One of them commented on her reaction to what she had observed in a lab sessions in which her students were experimenting with the *Supposer*:

> I see that they know an amazing amount. Not only do they know stuff I haven't taught them, but they know stuff it never occured to me to teach them, and they've got it <u>in their gut</u>. (J.H.int1, emphasis mine)

Allowing the *Supposer* into her classroom has revealed to this teacher that students can come to know a lot of geometry "in their gut." Students make discoveries with the *Supposer* that are not related to what they are being taught. These discoveries may derive in part from geometry that was learned in earlier courses, but they are also attributable to the students' intuitive appreciation of mathematical relationships. At the end of her year of experimenting with the materials, J.H. said:

> The amazing thing about [the *Supposer* problems] is that they are open-ended. The kids can go in so many directions and so they're coming up with stuff that <u>I</u> never thought about before... It gives them the opportunity to create their own mathematics, to go where other people haven't necessarily forged before you...
>
> For many folks, this is the very first time they have a chance to create their own mathematics, to do the kind of thinking required to justify their conclusions, to do a logical connecting. (J.H., int2)

Where do these intuitions fit in the school learning and teaching of geometry? And where do they fit in the process of doing formal mathematics?

Geometry has always been thought to be about "teaching thinking," but what the users of this phrase had in mind was deduction, not induction. The teachers using the *Supposer* are not clear about how to relate the process of inductive inquiry to traditional content. But they observe that when students "discover" a Geometrical relationship from their observations of patterns in numerical data, they get excited about it, and maybe they are even more likely to remember it.

This kind of "thinking" both excites and troubles the teachers. For one thing, it makes the job of keeping track of <u>who</u> has learned <u>what</u>, and <u>where</u> they should go next, quite difficult. But also, the teachers see the process of inductive discovery as threatening their capacity to teach students the conventions of logic, which require that a conjecture be proven before it is "true." The teachers say they teach students that "it's not true until you prove it" because they believe that is the way mathematics is supposed to work. But they also acknowledge that discovering a relationship for oneself (or "constructing it" as cognitive scientists would say), makes it true in a way that may be more powerful than truth that is accepted because the teachers says it is true, or because of the way truth is defined in an academic discipline.

One teacher commented :

> I don't find any problem with [the *Supposer* using numbers and measurement]. I think its good, at least for this level. Giving them specific problems and allowing them to work out problems, that's the only way they're going to get <u>their own conclusions</u>. I think the universal triangle works with very intelligent kids, but not with average kids. (A.D.int1, emphasis mine)

She thought that the students she was teaching might be more likely to learn geometry from trying to find patterns in visual and numerical data than from trying to deduce relationships among abstract figures.

But another teacher worried that she was not spending enough time "teaching proof" because of doing *Supposer* Labs, even though she thought students were getting a deeper understanding of the relationships among figures.

> Maybe proof isn't everything. Maybe that isn't all there is to geometry. Maybe knowing that equilateral always means all sides and all angles equal, and what an isosceles triangle is, is more important than doing proofs. They are coming up with the terminology... They are talking more geometry... They are knowing the difference between a rhombus and a square now because we did that [*Supposer*] project on quadrilaterals. And they realize that a square and a rhombus have some of the same properties, but they are not the same. They were able to come up with conjectures like that. They were able to come up with a lot of things on their own before they were taught the difference between them... They learned it better because they did it on their own. (P.O., int2)

Induction and deduction: Complementary or contradictory?

At the same time that they celebrate its results, the teachers see the inductive discovery process as something that they have to monitor very closely so that students do not get the wrong idea about the nature of their knowledge:

> We had constructed a figure [on the *Supposer*] so that these two base angles here were made by perpendicular lines. They both measure 90 degrees. The students did that with a lot of similar triangles and analyzed all the ways that it will be true. Well that's kind of a danger, and you have to keep cautioning them that it's not always going to be true. They have to prove it by using a proof of it. It is beginning to dawn on them that "just because I observe it doesn't mean that it's true" <u>because we're harping on that quite a bit</u>...We have to keep telling the kids that just because you discover something doesn't mean its true unless you can prove it and that proof has to be without measurement, without reference to any kind if measurement. (P.O.int1, emphasis mine)

This teacher believes that she must regularly exert her authority to teach students that the relationships they observe to be true are <u>not</u> true until they are proven to be so by a "proof." It was not clear what this teacher meant when she said "they...analyzed all the ways it will be true," but this process was clearly not a substitute in her view for doing a "proof."

"Doing a proof" has both a mathematical and an organizational function in the high school Geometry classroom. It demonstrates to students that inductively derived conjectures are not enough to establish geometric proof. But it also serves as a public synthesis of the diversity of discoveries that students come up with as they work at Lab Problems on their own. It is the teacher's opportunity to take charge again of what is being learned and to feel secure that students will not miss an important point.

The *Supposer* Lab Problems have been designed with this kind of balance between students' discoveries and "covering the curriculum" in mind. Working backward from the theorems students are expected to know, the problems are devised so that students will inductively see the reasonablity of a relationship before they are faced with constructing a logical argument for its applicability to all figures of a given type. Although it was not always accomplished in practice, this balance was appreciated by the teachers, and often enabled them to integrate *Supposer* work with the rest of their agenda:

> What we're doing is we're picking our [*Supposer*] project by what we're teaching next. We had just come through teaching side-angle-side, angle-side-angle, and side-side-side, and the next lesson was going to be isosceles triangles, so we picked a project that would tie together formal proof with introducing some of the isosceles theorems.
>
> It worked beautifully because <u>they were already assuming the theorems from their work on the *Supposer*</u>. They had already said,"Wait a minute, the bisector of the vertex angle is always going to be the perpendicular bisector of the base," and I said, "Not always. You

> can't do that, you can't assume that until you can prove it." And we did. So now they can use it. But they had already discovered it. So it's really neat because you can have them discover and then say, "Okay you have to prove it before you can use it." And then you come to the point where you can use it all the time because you can prove it. (B.L. int1, emphasis mine)

Working on the *Supposer* gives students some investment of their own in the material teachers want them to learn. At this level, however, they still need to be "indoctrinated" to the need for a deductive proof.

THE *SUPPOSER* IN THE SCHOOL CULTURE

Given all the tensions they are feeling between the sort of learning that the *Supposer* intends and the curriculum they are supposed to be teaching, it seems appropriate to ask why these teachers are willing to take the trouble to put everything that is familiar in a jumble and experiment with a new approach. Besides the external incentives (being associated with a prestigious research project, wanting to please outside observers, etc.), the teachers have regularly spoken about the ways in which working with the *Supposer* enhances their capacity to engage students in the process of doing mathematics, and that seems to make what might otherwise look like intellectual chaos into a worthwhile activity. But the culture of the school does not "naturally" lend itself to supporting this kind of divergent mathematical discovery. So the teachers who want to use the *Supposer* have to figure out ways to negotiate between the sorts of independent student activities they want to encourage and the familiar social and intellectual expectations everyone has about what should go on in school.

One of those expectations is that there will be a common curriculum for all and that everyone will follow it in the same order. Another is that math problems have right and wrong answers, and the more right answers you get, the more you know. And, of course, everyone assumes that teachers already know everything that students will learn, and that information will pass from teachers to students. These expectations were not always met in *Supposer* classrooms, and both teachers and students needed to invent new schemes for making sense of what was happening instead.

The sequences of teaching and the sequence of learning

Although the teachers in this group saw geometry as a logical sequence of deduced theorems, the *Supposer* challenged them to think about whether students actually learn it that way. One teacher described an experience of communicating with a student about a *Supposer* project that put her sense of the learning sequence in a new perspective:

> There is a *Supposer* project where there is a triangle that has parallel lines drawn on it this way, and your job is to duplicate the figure without using the parallel option. My God! They were just coming up

> with all sorts of ideas about how to do it; they were doing it, and one kid subdivided the sides and connected them and he got parallel lines this way, but when he connected them this way, they weren't parallel, and it was just driving him crazy.
>
> Why did it work this way, and why didn't it work this way? All I could say was "In two weeks, I'll be able to tell you." But it really bothered him. He came up to me the next day and said "I really need to know why." So poor old Jackie had to do a fade while we wait for the theory. (J.H. int)

What "Jackie" wants to learn at this point is not what his teacher has in mind to teach. Because of his engagement with a *Supposer* problem, he wants to go off in a different direction from the one that follows the Euclidean-geometry-oriented curriculum.

One important aspect of this experience is that the *Supposer* seems to have taken the ordering of the learning agenda away from the teacher. Another is that the *Supposer* seems to have the capacity to engage the student directly in wondering about the generalizability of an observed geometrical relationship. When this teacher says she needs to "wait for the theory" in order to answer the student's question, what she means is that the theorems that he could use to prove his conjecture about parallel lines had not been proven yet in this class, and so they were not to be considered part of the public store of "reasons" that students could call upon in developing proofs for conjectures. There was no deductive basis for arguing the truth or falsity of conjecture he had discovered inductively, and so the teacher was stymied about how to answer the student's question. On another occasion, this same teacher said she found herself "just giving them a lot of axioms so they could proceed." That is, she taught provable theorems as though they did not need to be proved (i.e. as axioms) so that students could use them as reasons in their arguements without knowing the proofs. This is a serious diversion from the classic course in Euclidean geometry, which is special because it relies on only a few axioms. By adding more axioms to this classic list, the teacher supports the process of "discovering" geometry, but she also redefines what geometry can mean. She is diverging from the expectation that the teacher and the book will determine the agenda, and adjusting her lessons to students' "maps" for exploring the territory of geometry.

Another teacher talked about "all the tangents kids can go off on" when they are working on a *Supposer* Lab Problem that diverge from the planned substance of a lesson, and he tried to figure out how to mediate between this kind of learning and his curriculum:

> What tangents do you take off on? That's where the curriculum can bog you down. Or you can eliminate parts of the curriculum and still be able to proceed, it just means you take a slightly different path. There are all of those considerations, and then there are the considerations of whether they'll need it in some other mathematical area down the line. In fact you do, you need a big chunk of geometry. (G.S. int)

This teacher is concerned that if the corpus of geometry does not get communicated to his students in a particular sequence, and to all of the students at the same time, they'll miss something in the curriculum or something they will need to know in a more advanced math course. If students "miss something" he sees it as the teacher's (i.e. ,his) responsibility. Yet he also seems to be putting the curriculum in some larger perspective by recognizing that it is only "one path" through the subject matter.

The textbook has offered teachers a way to keep a path through the subject clear in their minds, and the *Supposer* is challenging that security. But it is not only teachers who use the book to feel secure about the legitimacy of what is being taught and learned; in the culture of the classroom, the book also functions as a treaty between teachers and students about what they are supposed to be doing together in school (Powell, Farrar, and Cohen, 1985). Another of the teachers in the group reflected on the difficulties she had disrupting this long-standing agreement:

> I have always tried to set up problems so that kids can draw their own conclusions. The problem is that the Geometry textbook is so structured, and that's where I get all the problems... Richard (the advisor from the Lab Sites Project to the *Supposer* Group) said you have to pick and choose as far as what you want to discuss from the book. I'm not sure the kids are going to be happy about that.
>
> At this level, the students really like to have things spelled out and if you don't go over all fifteen problems in the book, they're very anxious about whether they did them correctly, whether they got the right answers... What I hope to do is to take a day every couple of weeks and go over anything we haven't talked abou t that they're uncomfortable with, just putting the pieces together, and we'll see how that goes. It's going to be tough. It's going to be very different. (A.D int)

Another teacher also said she "went over the problems in the book so students would feel secure." (J.H., int2) They would have to take an exam at the end of the year, and both she and they felt more trust in the book than in the *Supposer* to prepare for that exam.

The question of what students are learning about geometry from their own independent explorations on the *Supposer*, apart from how the teacher manages to incorporate those explorations into the whole class' agenda, is salient to the teachers as they try to decide how much time should be spent on divergent inquiry as opposed to the common study of the material by the whole class. Most of the teachers are willing to experiment, however, and see what comes of it later on in the course. But they do not yet know how their students' self-directed explorations will fit in with what they are supposed to teach and students are supposed to learn. As one teacher said:

> I don't think they're going to get much out of [doing their own explorations after they finish the assignment] now. I think once they get to learn more geometry and we do the other projects they'll say "Oh, we did that before," and I think they'll remember them, but

> they're not saying too much because they don't know what is ahead of them. (B.L. int)

What this teacher says about the student's explorations here raises questions about how she defines "learning geometry"; she seems to be saying that although something is contributed to that process by independent exploration, the explorations will only become knowledge once the students "learn more geometry" according to her planned agenda. Like G.S., quoted above, she is trying to figure out where the diversions from her agenda fit into the learning process.

When I asked the advisor to the teachers in the *Supposer* group what he thought about the problem of students' divergent learning paths, he said:

> You have to come in and teach from the philosophical view that not all learning happens when it is supposed to happen. Some people teach kids how to solve linear equations and they'll show them how to do it and they'll give them review sheet after review sheet and at the end of those two days, dammit, those kids are supposed to know it. But maybe in two months, after they've had to do a more advanced topic, usually with equations, they say, "Oh yeah," and they learn how to solve linear equations. The same kind of thing, I think, happens with all kinds of learning. (R.H. int)

Both of R.H. and B.L. see an interaction between the path a student takes through learning a subject and the path that is chosen by the teacher, but they seem to value them differently. The difference may be simply attributable to experience, however, in that the teacher advisor has been through an entire year with students using the *Supposer*, while the other teacher was just beginning to see how it might fit in to her way of teaching.

When is conjecture a right answer?

It was not only in the area of following the order of the problems in the book that the *Supposer* teachers needed to renegotiate their treaties with students. The routines for using the *Supposer* to come up with conjectures before proving them deductively meant that another essential aspect of classroom culture needed to be redefined. When students are given the assignment to produce conjectures from the data they have collected, the idea is that they should come up with <u>as many</u> conjectures as their data can reasonably support without concern for whether they are provable. Many of these will later be discussed in class, and the ones that can be proven will be retained as theorems. Others will be refined or rejected because counterexamples can be produced which demonstate that they do not apply as generally as the student may have initially assumed. (For example, if a student conjectures that a line parallel to the one side of a triangle is perpendicular to the bisector of the angle opposite that side, the teacher or another student might show that although this is true in equilateral triangles, and on one side of isosceles triangles, it is not true of triangles with sides of three different lengths.)

In mathematics, the process of creating plausible conjectures and then finding the limits of their applicability is the heart of the matter, and it is this experience that the *Supposer* is designed to give to teachers and students. But in school, a rejected or corrected conjecture becomes equated with a "wrong answer"-- and students are reluctant to produce those for fear of negative teacher evaluations. This fear is disabling when lessons are based on deciding what the limits are of a particular conjecture and students are to be rewarded for risking lots of conjectures for discussion.

One teacher said of her students, toward the end of their year of working with the *Supposer*:

> As far as coming up with conjectures, they still have a sense that there's a correct answer there. They still search, you know, with the idea that there are correct answers and incorrect answers to those. I don't think you could ever get away from that. [But] I think it's a real negative kind of, um, emphasis, if they always think "well just because they say something, it's wrong." And it's not. ...I think it's something, as an adult we understand, and I'm not sure kids can. Because in every other subject they're, you know, most of the time you're either right or wrong. That's the kind of tests we give, objective tests. Um, so I'm not sure, I think it would be kind of losing battle to even try at this age.

Learning from teachers and learning from peers

When they come to school, students usually behave as if their teachers possess a body of knowledge that they are to acquire. They expect to learn <u>from</u> their teachers, whether or not what they know can be attributed to what they have been taught. Their knowledge is legitimized by the teacher knowing that they have it. Students do not come to school expecting to learn from their peers or to have what they know confirmed in conversation with their peers. But that is what often occurred as students worked together on *Supposer* Lab problems.

The *Supposer* Lab is a classroom that has microcomputers all around the walls, one for every two students. During a lab session, students are generally working with a partner on a project that they have been assigned by the teacher from the *Supposer* collection of problems. Since each computer is separately booted, each pair of students may have a different figure in front of them, even though they are working on the same project. After they have taken the measures of one figure, they can call up another figure of the same type and measure its parts as well. The software is carefully designed so that not all of the figures will have a "base" that is parallel to the bottom of the screen, and so the work students are doing can actually look quite different from one pair to another. Usually, after some initially more structured activities, students can decide for themselves what parts of a figure they want to measure as they pursue patterns in the data they are collecting, and they can decide to test conjectures by getting the sum, difference, product, or ratio of measures. This further diversifies the potential paths they can take through the given problem. The software is designed so that there are very few stopsigns relating to terminology or the technicalities of labeling figures in the paths of these explorations.

This is an unusual structure for academic tasks, and it means there is a greater potential for peer communication than there would be if everyone were facing the teacher and/or the blackboard. (Cf. Doyle and Carter, 1984; Morine-Dershimer, 1983) All of these structural innovations in the way geometry is communicated to learners can result in a substantial increase in students' capacities to learn mathematics independently from their teachers.

Most teachers will tell you that they would like to see students "work more independently," and the teachers in this group are no exception. But with the *Supposer* in their classroom, they are seeing what one version of such student independence looks like, and often they are surprised at what students are curious about and what they are able to do. One teacher said of her first lab session:

> I'm blown away! I didn't give my kids a menu or anything and they took off! (J.H. meeting, 10/9/86)

She was pleased, of course, but she was also a bit shocked by how quickly things were out of her hands. Her students had already had half a year of geometry without the *Supposer* (because of the way the curriculum is structured at her school), so she felt her class might be more "bold" than the other classes that were starting out with both the *Supposer* and geometry at the same time. But in another school, with the traditional sequence, pairs of students also reportedly "dove in" to doing geometry with the same enthusiasm, "even though they were strangers to the *Supposer* and to each other." In this school, the teacher observed that:

> There was a great deal of interaction between pairs of students seated at the same machine, and the discussions the kids were having was actually about geometry... Word travels fast in the lab as kids make discoveries-- like that you can add angles or that you need to put the vertex letter in the middle when you label angles. (B.L. meeting 10/9/86)

Another teacher said that students "who don't talk to one another in the halls" were talking about geometry, and that a lot of teaching was going on in that talking. In her school, they had at first decided they would have students share a computer terminal because they did not have enough for every student to have one, but then they decided this was a good arrangement because of all the peer teaching that was occuring.

Several other teachers commented that definitions and the conventions of notation were "learned" in a functional manner as students worked together on their *Supposer* problems. One said:

> Studying notation and labeling on the *Supposer* is more interesting than on paper; it's a pick and shovel job by hand, but with the *Supposer* they could just see that ABC and CBA were the same." (L.H. meeting 10/9/86)

Another said that one effect of the *Supposer* lessons was:

> ...to immerse kids in geometry situations which as they explored them, caused kids to want the language and definitions in order to

> describe what they were discovering. The would ask, "Isn't there a word for these angles?" (M.S. meeting, 11/3/86)

On the occasion of one of the *Supposer* Users Group meetings, another of the teachers described a lesson she did with her class when the same sort of "need" arose:

> I asked them, after they had been working with the *Supposer* for a while, "What do you think a right triangle is?" and wrote their definitions on the board. I was excited that the kids had worked out for themselves how to label angles and that they realized which letter had to be in the middle. (B.L. meeting 9/17/86)

Another teacher commented that it was hard at first for her students to figure out how to "read an angle" on the *Supposer*, but they were able to figure it out "by trial and error" (A.D. meeting 9/17/87). This information about what things are called and how they are labeled was formerly the property of the teacher to be doled out according to a careful agenda to students. With the *Supposer*, students no longer needed their teachers to get access to these basics.

One thing the students are learning from one another are the mechanical conventions that the software uses to do things like label angles and add angles, but they are also confronting the idea that you <u>can</u> add angles, and that the way angles are labeled is functional as you move around from one part of a figure to another. And they are learning these things from one another because <u>they</u> need to know them to pursue the questions <u>they</u> are working on.

What role does technology play

The idea that students are actually <u>doing</u> geometry independently as they work with the *Supposer* was appreciated without reservation by all of the teachers interviewed. They seem surprised at this development, and even more surprised by the fact that students were working together with their peers <u>on mathematics</u>. As one teacher described a lab session:

> The thing that is most amazing to me is that mathematics is going on all the time, all the time. I assigned kids very arbitrarily to partners, and rather boy-girl, and with some eye to balancing personalities, but in most cases, these kids don't know each other. But the very first day, they were in there working together! It was just wonderful...
>
> They were really caught up in whatever was happening in the problem itself. It was a good open-ended one and it caught their imaginations. (J.H. int)

I asked her if she thought it was the technology that was responsible for their immediate engagement in the problem and she replied:

> Well, its the technology that makes it possible for them to check it out themselves. But... the technology also grabs them in a way that they wouldn't be, even if I gave them a really good problem at their desks. (J.H. intl)

From this teacher's perspective then, the technology makes it possible for students to carry on their own mathematical inquiries, "to check it out themselves," and this potential draws them directly in to the process of discovering geometrical relationships. The mathematical appeal is complemented by the general appeal of having a novel technological tool available for their use in the classroom.

Another teacher commented on the appeal of the software, and her comments also expressed the idea that a substantial part of the appeal for students seemed to be what the technology enabled them to do with mathematics rather than its "bells and whistles":

> It's not animated, it doesn't make any sound, it's not appealing in any other way other than that it is just another way of looking at the subject. To someone who likes it, this is a fantastically interesting subject, but to someone who doesn't get into this cut and dried stuff, it's dull and boring. This is a better way of enticing the students to stick with it. (B. L. intl)

The teachers are seeing that their students are "enticed" by the software, that they don't "fool around with it," but they use it to do mathematics, even with the potentially distracting arrangement of working with a partner on an independent project. But the mathematics they are doing is different from what it would be if they were not using the *Supposer*, and that difference is yet to be fully apprehended by the teachers. The teachers' appreciation of what <u>students</u> are doing with the software mathematically has been related by them to the fact that they are actually doing mathematics with the *Supposer* <u>themselves</u>. They talk about this as an unusual opportunity and a renewing experience. And some of them believe that if they are actually doing mathematics in the classroom, they will be communicating with their students in a different way. The technology is a tool that enables this to happen. It frees up <u>both</u> teachers and students to engage in a new kind of relationship with their subject.

A department head who is one of the teachers in the Users' Group wrote to me before the school year began:

> I chose to use the *Geometric Supposer* for much the same reasons I have varied classes, textbooks, approaches, manipulative aids for the past twenty years-- a need to keep vital as a teacher.
>
> My hunch is that this software will empower my students early on and they will likely come up with several hypotheses that I have never entertained. They will thus <u>see me playing with their hypotheses</u> in an attempt to convince myself of their truth or falsity. They will also work very hard (more so than usual) <u>to convince me</u> of the truth of their arguments. (M.S. letter, 9/2/86)

The teacher advisor on this project--who is also head of a high school mathematics department and a geometry teacher--also feels that it is important that teachers look at what students are doing on the *Supposer* with mathematical as well as pedagogical eyes:

> It's really important that the students think you are seeing conjectures for the first time and considering their reasonability. It makes them think you find it interesting too, that math isn't something that is all finished being discovered. (R.H. intl)

Another teacher in the group, who is also a department chair, expressed a different sort of connection between his own mathematical interests and his students 'work with the *Supposer*:

> I skipped over parallel lines into triangles because I was fascinated by them and wanted to get started. It took me a little away from the curriculum, but it's still tied in...
>
> The kids are noticing that the angles add up to 180 degrees without my telling them. It's hard for me to get used to. But the kids like the lab and feel more comfortable with it. (L.H. intl, emphasis mine)

That the teachers are working directly on mathematics rather than only on the problems of how to teach mathematics as they grapple with this innovation is further evidenced by how they choose to spend their time at User Group meetings. They spend a substantial period of their time together working at terminals in small groups on geometry problems, and a larger fraction of this time is spent on the doing of the problems than on talking about how to "use" them in the classroom.

The kinds of problems that are posed by the *Supposer* can be challenging at many different levels, and the teachers obviously enjoy this opportunity to work on mathematics with other adults. One of the teachers said that these meetings were a stark contrast to department meetings at his school, where "we never come anywhere near mathematics." He talked about his work with the *Supposer* Users' Group as an inspiration to go back to school to study more mathematics, "and really become a mathematician." (G.S. int) Another teacher in the group said that she enjoyed doing the problems on the *Supposer* herself because:

> There comes a time as a teacher when you have to do something for yourself to keep your mind active. This is just a great opportunity for me. (B.L. meeting 9/17/86)

It seems as if these teachers are recognizing that sharpening their own mathematical skills may be good preparation for coping with the unpredictability of students' independent explorations on the *Supposer*.

Who is responsible if a learner takes a wrong turn?

The power that the *Supposer* gives students to take off on their own and teach one another makes the teachers in this group anxious at the same time that it excites them. At an early meeting of the *Supposer* Users' Group, one teacher said he was uneasy about whether the conventions for labeling figures that students developed while using the *Supposer* would be "correct" in the terms that might be expected on standardized tests. He also had a deeper worry about the broad course of students' independent inquiry:

> What if kids are doing this off by themselves on the computer, what would happen if they got the wrong idea and went off on a wrong tangent and you weren't there to go with them? (L.H.meeting 9/17/86)

His anxiety hit a responsive chord in most of the teachers in the group, and as the discussion continued, the staff entered with some reassuring comments. One of the researchers said that the software was designed in such a way as to help students correct their own mistakes:

> They might just go down a wrong path, but eventually they notice things aren't working and they correct themselves. (D.C. meeting, 9/17/86)

The teacher advisor to the group added:

> It is frustrating and worrisome initially, not knowing what students are doing. The payoff of teaching with the *Supposer* comes later in the course and you just have to have faith in the early weeks. (R.H. meeting 9/17/86)

"Having faith" that students are going to learn what they need to know from the *Supposer* technology and from one another does not come easily to teachers, since they are responsible to students and parents for the outcome of this experiment. If the teacher takes charge of the agenda, at least he or she can claim that material was "covered" even if it was not learned.

The teachers' anxiety about whether students will learn what they should "independently" exists side by side with their appreciation of the direct line from students to the subject matter that is opened up by the presence of the *Supposer* in their classrooms. One teacher said that she gave up an earlier idea that she could use the *Supposer* at a large monitor in front of the classroom (the way she uses the blackboard) for teacher directed lessons. She saw that her students came up with lots of different ideas that they all wanted to pursue at once:

> I put the problem on the monitor and these kids, granted, knew a lot of geometry. We started some things, started to generate some pictures, and so on... We had fights going on in the classroom. It was a talky group in that class, very verbal, but they were arguing and screaming with one another about the input that I was going to put on the computer. Right then, I said to myself, "We're going to need more than one of these, because certain kids want to take some idea and go with it in their direction and if you don't have enough computers and software to do that, <u>you are really robbing that student from the thought that he or she wants to continue</u>." So immediately after I had exposed a set of students to it, I knew that the only way to teach it was to give them their own individual pieces of software.
>
> Then we brought the students in here to the computer lab, and they just went crazy. They were all doing their own problems and their own discoveries, and it was wonderful. (P.O. int, emphasis mine)

But despite this enthusiasm for the students' independent explorations and her appreciation of their importance, this teacher also said that she was quite nervous about letting her students "go on the computer" right from the beginning of the year. She said that if she had not been receiving regular support from the teacher advisor in the Lab Sites project, she would have begun the year using the *Supposer* only in teacher-directed lessons with a large monitor at the front of the room:

> After my first experience. I knew that if I were to do the *Supposer*, I would have wanted to try it this way [with students having their own computers and software], but at this point, I think I would have backed off from that and waited until they knew a lot more geometry... I would have used it more as a teaching tool right now rather than have discovery methods. It's not as good a method of teaching it, but I think that's what I would have done until I was more comfortable with the software. (P.O. int)

Her reasons for this hesitation had partly to do with her worries about her knowledge of the software, but they also had to do with the management problems raised by having students going off in all different directions while she was struggling to figure out the mechanics of the technology. Her feelings about this were shared by the other teachers in the group; they did not often separate their concerns about the classroom management problems associated with the new technology and the more substantial content control issues. This same teacher recounted a particular example of her frustration, and compared herself to the teacher advisor who had had more experience teaching with the *Supposer*:

> When the problem just tells the kids to "generate an isosceles triangle," you've got narrow ones, wide ones, and it's hard because the students are concerned. They're saying "I have a question" and you're going from one real specific problem to another very specific problem. And you as a teacher have all that general information in your head and you're listening to what they're saying and you find yourself feeling very fractured. (P.O., intl)

All of these changes in the classroom culture--students learning important material from peers, ambiguity about right and wrong answers, and negotiating the sequence of lessons around student discoveries mean both more and different kinds of work for teachers who support the kind of inductive inquiry that the *Supposer* is designed to enable. Even as students enjoy their independence and engage with geometric problems on their own, teachers must guide their activities toward mathematically productive ends. This is both easier and more difficult than getting students to do the problems at the end of each textbook chapter successfully.

The sort of teaching that the designers of the *Supposer* materials had in mind is what Thomas Green (1971) calls "instruction" as opposed to the more commonly found "indoctrination." The distinction rests on students having reasons -- both inductive and deductive -- for accepting the truths of geometry, rather than accepting them because they are in the book or because they have been expounded by a teacher. Having reasons gives students a degree of intellectual and social autonomy that they would not otherwise possess, and it brings the work they do in classrooms closer to the work of the scientist and mathematician. (Cf. Polya, 1954.) But teaching in a way that guides students' inquiry, and giving up the assumption that what is taught is the same as what is learned, cannot but make a teacher feel "fractured." It requires a different kind of energy than indoctrinating students to the important facts and skills in a discipline according to a neat and comprehensive agenda. In the words of Robertson Davies:

> To instruct calls for energy, and to remain silent, but watchful and helpful, while students instruct themselves calls for even greater energy. To see someone fall (which will teach him not to fall again) when a word from you would keep him on his feet but ignorant of an important danger, is one of the tasks of the teacher that calls for special energy, because holding in is much more demanding than crying out. (The Rebel Angels)

It also takes a lot more time, and a more demanding relationship with students, than teaching that is telling. But the *Supposer* teachers, among others, believe it is a worthwhile way to spend their teaching energy. Although they all have reservations, each of the tcachers said he or she would use these tools again the following year and help other geometry teachers to use it as well.

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