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ABSTRACT

Beneath educational pedagogies lie philosophical assumptions about the nature of learning, knowledge, truth and morality. These different philosophies form the foundations of a variety of instructional programs in all academic disciplines. This paper addresses constructivism, a recent attempt to provide a philosophical pedagogy which affects classroom instruction, teacher training, curricululm development, and educational research It specifically looks at constructivist theory as it relates to mathematics and science education. In so doing, the paper examines: (1) epistemology in the classroom; (2) epistemology in education research; (3) epistemology in mathematics and science (faith and skepticism); (4) Piaget's constructivist epistemology; and (5) implications for education. A bibliography is included. (TW)

CONSTRUCTIVISM IN MATH AND SCIENCE EDUCATION

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Constructivism in Math & Science Education

Beneath all educational pedagogies lie philosophical assumptions about the nature of learning, knowledge, truth a morality. These different philosophies form the foundations of a variety of instructional programs in all academic disciplines. Although the theoretical perspectives of the disciplines have evolved historically and are chronicled in survey courses on the history of their respective fields, in methods courses (qualitative and quantitative) and in the respective academic journals, far less attention is devoted to the study of the methods of transmission of knowledge within the disciplines. Transmission and appropriation of knowledge are two defining operations of education. There are important philosophical and psychological assumptions attached to the terms "transmission", "appropriation", and "knowledge" which require careful examination for education, as a discipline, to achieve a coherent theoretical perspective.

Constructivism is one recent attempt to provide a philosophical pedagogy which affects classroom instruction, teacher training, curriculum development, and education research. Constructivism also questions the methods and goals of the various academic disciplines as well as redefining the social roles and responsibilities of teachers and students. Philosophically, constructivism advocates an epistemology which views knowledge as belief, and truth as anything but absolute. With respect to the sciences, major advocates of this persuasion are Karl Popper (1963), Thomas Kuhn (1962), and Paul Feyerabend (1975). In mathematics, the constructivist position is characterized by the work of Ludwig Wittgenstein (1956), Imre Lakatos (1976), and Morris Kline (1980). These writers advance the thesis that knowledge in mathematics and science is a product of history and culture. The scientific process is manifest in the dialogue between people committed to the rationalization of a world they will not and cannot know in any absolute sense. Given the dialectical characterization of knowledge, the constructivist must reevaluate the function of teacher as transmitter and student as appropriator. Knowledge can no longer be the medium of exchange from the former to the latter. Instead, knowledge must be constructed from human mentation through the active communication between people with equal authority in their claims to truth.

The importance of epistemology in education is apparent in the classroom, in the educational research laboratory, and in the academic disciplines of math and science. An epistemology of truths-as-facts generally produces students who view learning as memorizing rather than as the search for relationships between concepts. This is an epistemology antithetical to the goals of "critical thinking" and "higher order thinking" which are so recently acclaimed the most needed and deficient skills of American youth (Resnick, 1986). Cognitive process researchers studying problem-solving and conceptual development in mathematics and physics are beginning to acknowledge the epistemological assumptions inherent in the very concepts they study with their subjects (Vernaud, 1983; Kaput, 1979; Minstrell, 1984). Within the sciences, constructivism critiques the "inductive habit of thought" which is the hallmark of scientific method. A school of mathematicians calling themselves Constructivists criticize the Platonic epistemology they see as prevalent, though generally not acknowledged by most mathematicians. Even "logical truths" such as the principle of contradiction; "a statement cannot be both true and false" are questioned by mathematical constructivists.

Many constructivist arguments stem from the problem of philosophical skepticism. Since the time of the Pre-Socratics, philosophers have struggled with questions about the veridity of the senses, of logic and of mental models as conduits of or correlates to reality. A contemporary group of philosophers known as Radical Constructivists reject even the possibility of this correlation.

But not all of constructivism is nihilism. The aim of this epistemology is not to destroy for the sake of destruction, but to expose the myth of reality so that attention may be focused on the social and psychological processes of individuals together defining not one world, but many worlds --- the worlds individually constructed.

A pioneer in this endeavor is Jean Piaget (1970,1971) who attempted to answer the epistemological question: "What do we know and how do we know it?" with a careful analysis of the intellectual development of the human child. His conclusions support the thesis of constructivism generally. They also direct educators toward a practical pedagogy which has as its highest value the individual's capacity to construct knowledge and to accept responsibility for those constructions.

The Role of Epistemology in Education

Epistemology in the Classroom

In an article about examsmanship in the liberal arts, William Perry (1963) distinguished two opposing epistemologies with ramifications for education in the liberal arts and in the sciences. While appearing somewhat apocryphal, the following anecdote, as recounted by Perry, will serve to illustrate these two epistemologies, and so is allotted some space.

A student at Harvard took an exam in a course he was not enrolled in, and on a topic which he was almost completly unfamiliar with. He succeeded in his attempt to fool the system with an expert display of "bull". Having made several inferences based on the title of the course - social anthropology, the titles of the books mentioned, and conjectures about the authors' ethnic background from their names alone, the student patched together an essay for which he earned an A-. Supposing that the author of the text was an anthropologist studying his own culture, [Geoffrey Gorer, author of The American People], the student elaborated on his conjectures about the methodological difficulties facing the scientist. This he did with no facts! A colleague of his, who attended class all term, had studied hard, memorizing many facts, earned a grade of 'C'. The 'A-' student admitted to having written "just a lot of bull", while the 'C' student was outraged.

Perry insists that the grading indeed was not fair. The student who was awarded a grade of 'C' for having only recounted facts should have instead been given an "F". The other grade stands as marked. His reasoning for this judgement rests on the assumption that these two students typified two epistemologies which may be characterized with the following definitions:

cow (pure): facts, no matter how relevant, without relevancies

bull (pure): relevancies, no matter how relevant, without facts

Allowing the tongue-in-cheek quality of his definitions, Perry makes an important observation about the role of epistemology in education. An appreciation for a theoretical perspective, and for the problems associated with developing and using one, demonstrates a far greater understanding of an academic discipline than does a mere recitation of facts. Perry (1970) conjectured that students developed progressively more sophisticated epistemologies. They advance from the naive epistemology of learning, as typified by the memorization of facts, toward a "higher level" epistemology which views facts as constructions which stem from theories which are themselves constructions. Students in the early stages of intellectual development criticize their teachers' theorizing in the following manner:

> "If teachers would stick more to the facts and do less theorizing, one could get more out of their classes... A certain amount of theory is good, but it should not be dominant... The facts are what's <u>there</u>. And I think that should be.... the main thing." p. 67

DiSessa (1985) makes a similar distinction in the sciences, particularly from the study of students in college physics courses. Two epistemologies with radically different implications appear in their respective adherents. One prevalent epistemology is described by DiSessa: "[there is a] traditional view that learning physics is acquiring new knowledge specifically located in the laws, principles, and equations of textbooks, understood essentially on the surface level of knowing those principles by name and statement, and those equations by the letter.", p. 4. This description is typical of students he refers to as "results men". Learning, for these students, consists in the match of problem to equation to produce an answer. The student must substitute the facts (data) given in the problem statement, into the proper variables in the appropriate equation (also a fact). He then performs the indicated algebraic and arithmetic operations (more facts) to produce a correct answer which is true by virtue of all the above facts - and if by chance it is an odd-numbered problem, the answer may be verified in the back of the book.

Another epistemology, more rare, but certainly more powerful, understands physics as a way to view the world. In this case, the student realizes that in order to "get the point" his intuitions must be substantially reorganized. This is an epistemology in search of conceptual understanding rather than just facts and answers.

Clearly, epistemology plays a crucial role in the classroom. It will determine not only what students believe they are learning, but also how they must proceed to learn it. The types of questions teachers ask and the concepts they hope to teach will affect their students' epistemologies in ways which may foster or suppress conceptual understanding.

Epistemology in Education Research

It is the task of education researcher to identify the cognitive processes needed to understand concepts as well as to identify cognitively important concepts. As may be expected, they too must be sensitive to their own epistemological perspectives. They must also be sensitive to the epistemological assumptions implicit in their own understanding of particular concepts.

Researchers in mathematics and science education have recently addressed the problem of identifying epistemologies in student subjects and in the methods and contents of cognitive studies.

Vernaud's (1983) attempt to define and analyze a mathematical concept illustrates his concern for explicating the epistemologies implicit in the researcher's work. For example, many mathematical concepts, particularly geometrical concepts, involve both physical and mathematical constructions. Volume is such a concept. Children, aged 5 to 11 years, can distinguish the relative sizes of volumes. They demonstrate an understanding sufficient to compare two volumes, to approximate a volume, to coordinate unit systems, and so on. They can compare volumes with a consideration of a comparison of lengths and areas, and they can evaluate one volume by combining information about lengths, areas, angles, etc. All of these skills involve what Vernaud calls a "unidimensional conception of volume" - volume as quantity. However, these same children, and also adolescents, demonstrate little understanding of volume defined as a product of measures, as in a straight parallelepiped:

$$V = L \times W \times H$$

A "tridimensional conception of volume" requires an understanding that length, width and height are independent variables, and that volume is proportional to a change in one of those variables when the other two are held constant. Although volumes may be studied concretely, i.e. empirically, as a unidimensional concept, the peculiarly mathematical tridimensional concept of volume requires a different epistemology. Carpenter (1983) also found evidence in his own research in children's understanding of addition and subtraction to corroborate the thesis that the researcher must examine his own mathematical conceptualizations to effectively study the students'.

An important epistemological task for mathematics education research is defining what is meant by the term "mathematics concept." Vernaud (1983) suggests that a concept consists of a triplet of three sets:

$$\text{Concept} = (S, I, L)$$

S: set of situations that make the concept meaningful; the referent

I: set of invariants that constitute the concept (different properties, different levels); the signified.

L: set of symbolic representations that can be used to represent these properties and the situations; the signifier. p. 19

In the previous example, the referents are any of the particular, concrete instances of volume that may be experienced: cones, cups, boxes, spherical balls, cylindrical straws, etc. The unidimensional, conservation properties of volume were much discussed by Piaget, and will not be recounted here. However, one example of a unidimensional property of volume is the addition axiom of the theory of measure:

Measure (A • B) = Measure (A) + Measure (B)
provided A and B have no common part

The tridimensional properties of volume represent a different level of conceptualization. Some symbolic signifiers for the tridimensional conception of volume are the symbolic representations:

parallelepiped:	$V = L \times W \times H$
sphere:	$V = 4/3 \pi r^3$
prism:	$V = 1/2\ b\ h\ l$
cylinder:	$V = r^2 h \pi$.

By identifying the triplet of sets which define a concept, the researcher makes explicit the epistemology contained in the concept.

Cognitive scientists studying physics education have identified student epistemologies which are at variance with the epistemology of physicists. Even the most elementary distinction between the positions and velocities of two objects is problematic. Minstrell (1984) reports that 50 percent of high school physics students believed that when two objects are at the same position, they have the same speed. Students also have difficulty distinguishing between the physicist's conceptions of acceleration, average velocity, instantaneous velocity and change in velocity. (McDermott and Trowbridge, 1980). The study of 'naive physics' has uncovered a number of

Pre-Newtonian conceptualizations, revealing a physical epistemology similar to Aristotle's. Aristotelean notions are still quite prevalent in our society. There are common beliefs that heavy objects fall proportionally faster than lighter ones, and that objects require constant force for continued motion. Furthermore, there is much evidence that these conceptions are resistant to the benefits of instruction (Clement, 1982; McClosky, Green and Caramazza, 1980). Simply telling the physicist's conceptualizations is not sufficient for the student to discard a strongly held intuitive belief. Frequently, students memorize definitions and operations by rote in order to pass exams. However, when asked for qualitative responses to conceptual physics questions, students exhibit the same misconceptions they had prior to instruction. Researchers in this field concur that for instruction to succeed, students must first be permitted to articulate their naive epistemologies, re-evaluate them in the light of new evidence and arguments, and finally construct a new epistemology that more closely resembles the physicist's.

Unfortunately, the method of instruction of the scientist and the mathematician is further complicated by a mathematical symbol system which cloaks the actual epistemology of invention of concepts. Typically, students are introduced to new concepts via formalisms which deny the initial cognitive processes responsible for the creation of the concept. Kaput (1979) argues that many symbolic representations in mathematics are filled with anthropomorphisms and physical metaphors which are "disacknowledged" by mathematicians. He cites an important example from calculus. The symbol for the concept of a limit reflects the underlying formative conceptualization of the discipline. This fundamental concept is symbolized with a motion metaphor, an arrow:

$$\lim_{x \to \alpha} f(x) = L$$

is to be read: "As x moves towards , f(x) moves toward L." This relationship is sometimes written:

as $\quad x \to a\alpha, \quad f(x) \to L$

Mathematicians seldom report the motion metaphor in their presentation of calculus. Instead, the metaphor was replaced with the logical formalism of the familiar $\epsilon - \delta$ proof. According to Kaput (1979):

> "... the attempt to wring out from the calculus its motion-content was:
> 1. historically very irrelevant to the stupendous success of calculus,
> 2. historically very difficult to achieve, and
> 3. s today disastrous in the teaching of calculus." p. 215.

Most cognitive process researchers hold the position that the cognitive processes responsible for the initial invention of concepts be reconstructed anew by education researchers and by students. While the reconstruction can never be identical to the initial conceptualization, the similarities will surely manifest. New insights ma also occur. For the constructivist, there can be no purely distilled knowledge that stands true and perfect. Knowledge is a human construction, and learning is a creative activity. Any presentation of education which misses this epistemology is dishonest and destructive. Kaput (1979) says:

> "Our failure to acknowledge the acts of knowing and learning is analogous to the Victorian attitude toward sex. One cannot develop mathematical conceptions without engaging in the torrid act of learning. There is no such thing as immaculate conception!" p. 290

Epistemology in Math and Science: Faith and Skepticism

For more than 2000 years, philosophers have struggled with problems in epistemology. Many of the difficulties discussed in antiquity are still debated by philosophers of mathematics and science. The debate is focused around the investigation of what humans can and cannot know. There are arguments for absolute faith in the veridity of human knowledge against the assertions of skepticism - that truth is either beyond human thought or manufactured temporally by each individual thinker. This last conjecture is closest to a constructivist position, while the position of faith is closer to the beliefs of most mathematicians and scientists throughout history.

Segal (1986) suggests that the language and logic of Western culture reflects a "wish for reality". The dimensions of this wish are four-fold:

1) We wish reality to exist independently of us.

2) We wish reality to be discoverable; to reveal itself to us.

3) We wish that the workings of reality be lawful so that we can predict and control reality.

4) We wish for certainty; we wish that what we have discovered about reality is true. p. 3

The first of these observations is characteristic of Platonic philosophy. Plato's thesis requires that truths exist somewhere in the universe for humans to discover. In the dialogue, Meno, Plato argued for the absolute nature of truth by means of a geometrical demonstration. Euclid's geometry had been the greatest evidence for this hypothesis for more than 2000 years. The postulates (axioms) were considered self-evident and true beyond any doubt. The theorems which followed by logical construction were also considered absolutely true. Mathematicians believed they were discovering truth. Davis and Hersh (1981) state that at least 65% of working mathematicians are Platonists who believe that mathematical knowledge is discovered and not invented. There is the additional implication that there exist more mathematical truths "out there" to be discovered.

Few mathematicians believe that their discoveries are made through empirical observations. Plato doubted the reliability of the senses completly, maintaining that truth was accessible to reason only. This rationalist position has been the bulwark of mathematical philosophy. Scientists, on the other hand, need the senses for their empirical observations. They assume that the objects they measure also exist independently of themselves as observers. Using mathematics and inference they also purport to discover reality. In both math and science, faith is crucial; faith in reason and faith in the senses. Historically, the faith is shaken.

Long before the advent of Non-Euclidian geometry and quantum mechanics, an attack on the verities of reason appeared in the form of a paradox. In the

6th century B.C., the Cretan, Epimenidies, questioned the Aristotelian fundamental assumption that all statements were either true, or they were not true, i.e. false. He asked for a logical analysis of the following statement:

"I am from the island of Crete, and all Cretans are liars."

This paradoxical statement raises doubts about the status of the entire logical program. It surfaced again in the 19th century to undermine a rationalist proposal to place arithmetic on firmer ground than previously had been thought possible. Given several basic assumptions about sets, Gottlob Frege developed a logically consistent arithmetic, with all the desired properties and operations, without recourse to a postulate which requires the existence of numbers. As a logical system, it was to arithmetic what Euclid's Principles was to geometry. By the time his work, The Fundamental Laws of Arithmetic, had gone to press, Bertrand Russell wrote to him indicating a paradox in Frege's set theory. The paradox arises with a problem in self-inclusive sets. Consider: "the set of all sets which do not contain themselves as an element". The statement is paradoxical in that it seems impossible to determine whether the set is itself an element of the set it defines. Frege had no solution to the paradox, and Russell's solution in effect disallows self-inclusive sets. The similarity of this paradox to that of Epimenidies did not escape Russell, even if a clear solution did.

Subsequent attempts to place mathematics on firm epistemological ground failed for very similar reasons. Having abandoned the Platonic assumptions implicit in Euclidian geometry, mathematicians like Frege, Russell, Whitehead, Hilbert, Godel and others, attempted to provide a foundation for mathematics in formalized logic. In a paper entitled "On Formally Undecidable Propositions of Principia Mathematica and Related Systems", Godel (1931) proposed his Incompleteness Theorem. This theorem dealt a fatal blow to the epistemological gainsaying of all widely accepted axiomatic systems. He successfully argued that any formal theory which is consistent, and may include a theory of whole numbers, must be incomplete. By assigning numbers (Godel numbers) to numbers, operators, equal signs, letters, and other symbols, Godel was able to show that there exist meaningful statements in number theory that are neither provable nor not provable. Kline's (1980) description of Godel's theorem is remarkably similar to the liar's paradox:

> "...Godel showed how to construct an arithmetical assertion G that says, in the verbal meta-mathematical language, that the statement with Godel numbers m, say is not provable. But G, as a sequence of symbols has the Godel number m. Thus, G says of itself that it is not provable. But if the entire arithmetical assertion G is provable, it asserts that it is not provable, and if G is not provable it affirms just that and so is not provable. However, since the arithmetical assertion is either provable or not provable, the formal system to which the arithmetical assertion belongs, if consistent, is incomplete." p. 262

Simply stated, Godel considered the self referential statement, "This statement is not provable." The paradox is nearly identical to the Liar's Paradox.

Not all attacks on Platonic Rationalism stem from this one paradox. There are laws in mathematics, albeit analytical laws, which may be doubted for very different reasons. In 1908, the Dutch mathematician, Brouwer, presented a counterexample to the "law of trichotomy" which ushered in a new school of mathematics called "Constructivism". The law of trichotomy is not only intuitively appealing, it also plays a fundamental role in calculus and analysis. The law states that every real number is either zero, positive or negative. The proof of this law requires another fundamental law in mathematics, the law of the excluded middle, which asserts that a statement is either true or false. Both laws are inappropriate to an analysis of Brouwer's counterexample and therefore unacceptable in constructivist proofs.

Brouwer's counterexample may be derive from any real number whose decimal expansion is infinite. Any irrational or transcendental number will serve as an example. Brouwer considered the number pi. Using the algorithms for decimal expansion of π, he postulated a second number $\hat{\pi}$. This new number is generated by the same rule used to expand π, and is identical to π to some arbitrary degree of precision; say for example, to the first billion decimal places. To this rule another arbitrary rule may be added: expand $\hat{\pi}$ until a row of say 100 successive zeroes is reached or until the desired precision for $\hat{\pi}$ is reached, whichever comes first. If in the expansion, a row of 100 zeroes is reached and starts on the nth digit, then terminate the expansion with the following rule: If n is odd, let $\hat{\pi}$ terminate in its nth digit. If n is even, let $\hat{\pi}$ have a 1 in the n+1 digit, and then terminate. According to the law of trichotomy, any real number must be positive, negative or zero. Both π and $\hat{\pi}$ are real numbers, and their difference, $\pi - \hat{\pi} = Q$, is also a real number. But unless the expansion of has within it a string of 100 zeroes, the sign of Q cannot be determined. The statement, "in the expansion of π there nowhere appears a row of 100 successive zeroes", cannot be proved or disproved until <u>such time</u> that it is calculated. This example illustrates the time dependent and subjective character of mathematical truth. (Davis and Hersh, 1981).

Soon after philosophers of mathematics had reached their crisis in foundations, the philosophers of science brought about a foundations crisis of their own. The inductive method of science, which relies on inferential reasoning, has been attacked as an effective epistemology for logical reasons (Popper, 1963), for historical reasons (Kuhn, 1962) and for pedagogical and methodological reasons (Feyerabend, 1970).

Scientific method is characteristized by induction---generalized statements about accumulated observations. D'Amour (1979) described five tenents of inductivist science:

1. Science begins with a solid base of "brute facts", basic statements that are justified by observation.

2. These basic statements are logically prior to and independent of the theories inferred from them.

3. Inferences from such statements are made in accordance with an ideal calculus - for example, the probability calculus.

4. Making inferences in accordance with an ideal calculus assures the attainment of the primary aim of science, namely, reliable theories.

5. Science arrives at theories of increased reliability in a step-by-step fashion; the scientific method is cumulative. p. 184

The first two of these tenets were criticized in the work of science historian Thomas Kuhn (1962) who argued that facts are theory dependent. His analysis of the history of science describes not a cumulative process [note tacit assumption #5], but a process of revolution and subsequent revolution. Each successful revolution entrenches with its theories, methods, instrumentation and data. It becomes a paradigm from which all accepted practitioners draw guidance and recognition. A successful revolution replaces the previous paradigm only when all of the old practitioners either die or retire. Each new paradigm describes the type of observations which may be accepted as data. According to Kuhn:

> "... paradigm changes do cause scientists to see the world of their research-engagement differently. In so far as their only recourse to that world is through what they see and do, we may want to say that after a revolution, scientists are responding to a different world." p. 111

D'Amour's tacit assumptions 3 and 4 have been attacked by the "critical fallibist" program of Karl Popper (1963) which questions the logical validity of inferential science. In a continuation of the arguments advanced by the 17th century philosopher, David Hume, Popper advanced a new theory of scientific theory construction which avoids the logical problems of inductive reasoning.

Hume argued that no inductive argument could provide a sufficient confirmation of a scientific theory. The following logical argument is typical of inductive science (Garrison, 1986):

Premise 1: If the hypothesis is true, then some specific empirical-experimental observation is also true (i.e. may be observed as predicted).

Premise 2: The predicted result is observed

Conclusion: The hypothesis is true.

Note that in propositional logic, the statement "A implies B" is logically dissimilar to the statement "B implies A". If in the statement "A implies B", B is true, nothing at all may be concluded about A.

This problem of verification (verificationism) was addressed by Popper (1963) in his reformulation of scientific method. Asserting that "there is neither a psychological nor a logical induction" p. 54, Popper proceeds by stating, "Only the falsity of the theory can be inferred from empirical evidence and this inference is a purely deductive one." p. 55. The only logically valid theory construction is that which relies on the logic of modus tollens - the logic of refutation. The following argument illustrates modus tollens applied to scientific theory refutation:

Premise 1: If the hypothesis is true, then some specific empirical/ experimental observation is also true (i.e. may be observed as predicted).

Premise 2: The predicted result is not observed.

Conclusion: The hypothesis is not true.

The legitimate method of science, according to Popper, requires that science must be deductive instead of inductive. Science is a continuing series of conjectures and refutations. Popper (1963) writes: "... there is no more rational procedure than the method of trial and error - of conjecture and refutation: of boldly proposing theories; of trying our best to show that these are erroneous; and of accepting them tentatively if our critical efforts are unsuccessful." p. 51.

Imre Lakatos (1976) extended Popper's philosophy to mathematics. In his book, Proofs and Refutations, he recreated in dialogue form, the history of the mathematical proofs of Descartes' (1635) theorem of polyhedra which states:

$$V - E + F = 2$$

for all polyhedra where:

V = Number of Vertices

E = Number of Edges

F = Number of Faces

The dialogue, supported with historical documentation, illustrates the simultaneous search for proofs and counterexamples to the proofs. The formalistic interpretation of truth as the result of an unbroken and unbreakable chain of logical reasoning from assumptions to conclusions is credibly undermined. For Lakatos, proof is explanation, justification, elaboration and persuasion; not to achieve truth but to gather credibility.

The method of science and mathematical discovery by refutation is not without adversaries. Besides dogmatists who adhere to Platonic forms, to formalist-logical systems and to inductive science, there is yet another attack from those who are closer in sympathy to the Critical Fallibists, but more radical. Paul Feyerabend (1978) opposes all methodologies. His thesis is that theory and fact have historically been "pushed" into relation. In practice, scientists believe what they want to believe regardless of confirmation or refutation. Feyerabend states that within the confines of methodological rules, no science is possible. (Feyerabend, 1978):

> "...science as we know it can exist only if we ... revise our methodology, now admitting counterinduction in addition to admitting unsupported hypotheses. The right method must not contain any rules that make us choose between theories on the basis of falsification. Rather, its rules must enable us to choose between theories which we have already tested and which are falsified " p. 65-66.

To argue skepticism beyond Freyerabend, one would have to doubt not only the methods of rationalizing experience, but also doubt experience itself. Radical Constructivism is an epistemology which doubts the possibility of correlating human experience of the world with the objective world. Without this correlation there can be no proof of objective reality - that is, reality apart from the individual's mental construction. The argument advanced by the Radical Constructivist is essentially the paradox of self-referential statements evident in the paradox of the Cretan Liar. This paradox appears with the logical analysis of any self-referential statement. The judgement of the veridity of a mental model is unacceptable from a self-referential frame. There is no authority to evaluate the match between mental model and external reality. Therefore, there can be no certainty in our mental representation of the world. The consequence of the paradox of self-referential statements is the basis for an epistemological perspective called "philosophical skepticism." The argument of "philosophical skepticism" is a fundamental principle of Radical Constructivism. It is stated concisely by Von Glaserfeld (1983):

"If experience is the only contact a knower can have with the world, there is no way of comparing the products of experience with the reality from which whatever messages we receive are supposed to emanate. The question, how veridical the acquired knowledge might be, can therefore not be answered. To answer it, one would have to compare what one knows with what exists in the "real" world - and to do that one would have to know what "exists". The paradox, then, is this: to assess the truth of your knowledge you would have to know what you come to know before you come to know it." p. 47.

This apparently nihilistic aspect of constructivism is not its only characteristic. It is, however, the starting point for an epistemology which values above all human thought and actions and the moral responsibility which is commensurate.

Piaget's Constructivist Epistemology

Piaget was perhaps the first constructivist who after criticizing all attempts to establish formalized theories of knowledge advanced the position that the only valid study of epistemology should come from psychology. Most modern epistemologies are formalistic systems which attempt to prove either that language or logic is primary. For Piaget, the debate is unresolvable through analytical formalisms. The position that logic is a product of linguistic convention is argued by the Logical Positivists. In deriving logic and mathematics from general rules in the use of language, i.e. general syntax, general semantics or general pragmatics, the positivists argue that language precedes logic. Proponents of Chomsky's "deep structure" argue the contrary position that language is based on logic, i.e. reason, which is innate. Both epistemological schools rely on formalized logical programs to substantiate their claims. Piaget (1970) objects to a strictly formalized study of knowledge for 3 reasons. (1) No single logic is adequate to the task of formalizing the construction of human knowledge. The development of many logics is inevitable, as has occurred, and considering the diversity of these many logics, no synthesis appears possible to establish a "single value basis for knowledge." (2) Godel's Theorem proves the limitations of formalization. Any system which is consistent, and which contains the operations of arithmetic, cannot prove its own consistency. Piaget is prompted by the Incompleteness Theorem to ask "what does logic formalize?" (3) Historically, most attempts of epistemologists to explain knowledge appear in the philosophy of science. Piaget would agree with Feyerabend in his observation that knowledge in the sciences is not purely formal. Indeed, there are other aspects to it.

Having argued against the formalized search for absolute truth, Piaget prefers to think of knowledge as a continuum which varies from lesser to greater validity. The task of the epistemologist is to search for a link between a model of the construction of knowledge and the judgement of the validity of knowledge. The investigation may proceed using one or several of three proposed models: the historico-critical approach, the psychogenetic approach and the biological approach. The biological approach is the least relevant to the topic of this paper, and so will be excluded from this study.

The historico-critical approach is essentially the method of historical analysis of intellectual historians in mathematics and science. Kuhn (1957), Feyerabend (1975), Lakatos (1976), and Kline (1980), all employed this method for their respective analyses. Piaget's use of the historical approach is somewhat different from the others. His view of history supposes a direction and a development quite similar to his developmental theory of intellectual growth in the human child. The following example illustrates his prejudice.

Boutroux's description of the history of mathematics was analyzed by Piaget as a history in three stages:

Level 1: This is the contemplative period of Greek mathematics. Mathematicians are unconscious of operations as activities performed by the subject (person). Instead, operations are viewed as features of the mathematical "objects". According to Piaget, this epistemology is characterized by a refusal to accept algebra.

Level 2: Operations become conscious. This is manifest in the growth of algebra, analytic geometry, calculus, etc.

Level 3: This level is marked by the search for foundations. It is characterized by the conscious construction of structures as manifest in the theory of groups and the work of Frege, Whitehead, Russell, and Hilbert.

The levels progress from the concrete to the abstract in stages. It is a conception of history that is not marked by revolutionary paradigm shifts a la Kuhn, nor is it the random process of idiosync a ic persuasion t'at Feyerabend reads from history. Although Piaget acknowledges the importance of historical analyses in the constructivist epistemological program, he is far from his fellow constructivists in the substance of his attempts in this field.

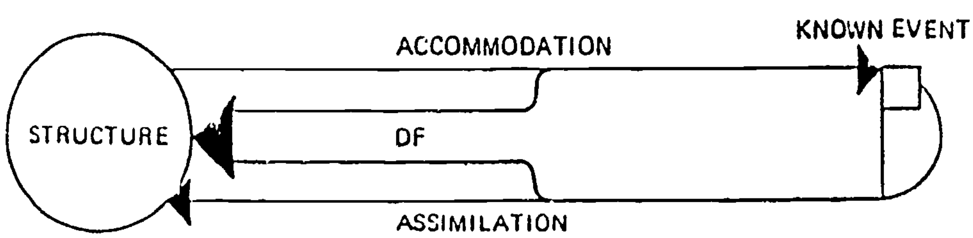
But Piaget is not convinced that the historico-critical approach is sufficient for a ' ,rough epistemological study. It is not enough to only examine history for an understanding of the validity of knowledge. According to Piaget, epistemological theories cannot be experimentally verified. (According to Popper, no theory can be experimentally verified. It may be possible, however, to refute such theories.) Piaget also points to the additional limitations of the historico-critical approach. Pre-historic epistemology may be assumed, but is unknowable from a historical perspective. His most powerful criticism of the historical approach stems from his observation that history is universally adult history. There are few histories of children. Piaget (1967) states that an understanding of adult cognitive structures is impossible without an understanding of their development: "The child explains the man as well and often better than the man explains the child." p. IX

Piaget's most important contribution to epistemology comes from his psychogenetic approach. From the study of the development of knowledge in the human child a constructivist theory of epistemology emerges.

All theorists make certain methodological assumptions; so too does Piaget. His first assumption is that behavior contains knowledge. Inferences about the internal structures of the mind are possible and necessary for a proper study of the psychogenesis of knowledge. This assumption leads Piaget to the logical assumption that there is a dichotomy in the operational definition of knowledge. In knowledge one can distinguish <u>form</u> and <u>content</u>. "Content" is the observable manifestation of events to which knowledge is directed. Some synonyms are: facts, information, and stimuli. "Form" is the unobservable internal structure. It is the mental representation that, though not observed, may be inferred from the observation of content. Some terms for form used synonymously by Piaget are "general framework", structure, meaning, understanding, and essence. Inferences about the development of knowledge require observations of change in the forms of knowledge. This is Piaget's final methodological assumption. While forms are unobservable, the <u>change in forms is observable</u>. Formative change is the reflection of development which defines the making of intelligence. Because these changes of form are observable, the study of change is amenable to controlled observation. For this reason, Piaget feels justified in his claim to having established a scientific epistemology and not merely a philosophical one.

Piaget's Constructivist Epistemology requires a dynamic interpretation of the development of knowledge. A recursive cycle of observations, operations, comparisons and evaluations occur with constant self-regulation. Says Piaget (1970): "I think that human knowledge is essentially active. To know is to assimilate reality into systems of transformations. To know is to transform reality in order to understand how a certain state is brought about..." p. 15

The following diagram will serve as a focus for a discussion of Piaget's model of the psychological process of the "transformation of reality." The diagram is due to Hans Furth, whose paper appears in Silverman (1980).



Developmental feedback (DF) as source of structural growth.

PIAGET, PHILOSOPHY & HUMAN SCIENCES, p. 9

Briefly, the stages of development of transformation occur through the following processes: (1) Assimilation means incorporation into an existing system. Biologically, assimilation consists of the intake of substances (food or energy) from the environment with the goal of continued survival of the living system. Psychologically, assimilation consists of the integration of objects, empirical or mental, into schemes of actions. (2) Structures are the totality of transformations. "In cognitive structures," writes Rita Vuyk (1981), "the elements forming the content of the structure are perceptions, memories, concepts, operations, structures or "any object whatsoever" in mathematics and logic. The relations between the elements giving the form to the structure can be spatio-temporal, causal, implicative, etc." p. 54. (3) Accomodation is the outgoing process of applying general schemes to particular contents. If an object cannot be assimilated, it may be either ignored by the subject or accomodated in the sense that the structure may change <u>so that</u> assimilation is possible. This inter-relationship between assimilation and accomodation is also responsible for the construction of new sub-structures. Vuyk (1981) translates Piaget (1978) from <u>The Development of Thought</u>: "every assimilatory scheme has to accomodate to the elemer it assimilates, that is, to change as a function of the characteristics, but without losing its continuity [i.e. its closure as a cycle of independent processes] nor its former powers of assimilation." p. 66 in Vuyk (1981). This recursive cycle of knowledge formation involves "feedback" which is both self-referential and self-monitored. The self-referential characteristic of Piaget's theory stems from psychological investigations that are the result of an empirical study of children's behavior. As such, it is not merely the result of formalistic philosophy. The self-monitoring aspect of his theory is evident in all mental processes though it may not be conscious. When this process is conscious it is referred to as "reflexive-abstraction"; a concept bearing many similarities to the more recent psychological investigation of "metacognition". [This topic will be expanded in another paper. See "Metacognition", Narode, (1987)].

As "systems of transformations" is the key concept in Piaget's psychogenetic epistemology, further elaboration is in order. Another term, used synonymously for "systems of transformations", is "operations". Piaget describes operations as actions which are:

1. internalized - carried out in thought as well as being executed materially.
2. reversible
   a) reversible by inversion or negation; example: $+ A - A = 0$
   b) reversible by reciprocity as in the reversibility of order; example: $A = B$ is equivalent to $B = A$
3. invariant in the sense that something is conserved; example: $5 + 1 = 4 + 2 = 3 + 3$ in that the sums are invariant.
4. Related to a system of operations. No operation exists alone. Every operation is related to a system of operations called a structure. This feature is similar to Vernaud's (1983) description of concepts appearing in fields (conceptual fields) rather than as isolated and individual mental constructions.

Employing the psychogenetic method to a study of epistemology, Piaget hypothesizes the psychogenesis of logico-mathematical structures from the coordination of actions which occur prior to the development of language. The child at the sensory motor stage of its development (typically less than one year) gives evidence of developing pre-operational understandings that will later become operations. The most fundamental of these pre-operational schemes form the basis for logico-mathematical thought which requires: 1) the logic of inclusion, 2) the logic of order, and 3) the logic of correspondence. These principles are the basis for logical thinking.

For Piaget, actions precede all epistemology. The coordination of actions provides the material and impetus for the development of logical structures. An infant may be observed to use a stick to move an object. The scheme, "use the stick to move the object," involves at least two subschemes: hand/stick and stick/object. The scheme includes the subschemes. This is evidence of a working knowledge-in-action which uses a type of logic of inclusion. Also evidenced in this example is the logic of order. The causal connection implies that the infant understands the order in which the actions must occur for the object to be moved.

The logic of correspondence requires the understanding of one-to-one correspondence as demonstrated in counting. Assigning a number to each object, knowing not to double count the objects, and when to stop counting, requires the logic of correspondence. Piaget observes that infants exhibit this logic during imitation. When an infant imitates a model, he knows that the model corresponds to his imitation - even if he imitates himself when repeating an action.

In addition to these logico-mathematical principles, infants at the sensory-motor stage exhibit behavior that is evidence for the formation of operations. At least two essential characteristics of operations are observed at the formative stage; conservation and reversibility. The recognition of the permanence of objects is a form of conservation which appears in most infants at the end of the first year. A 7 to 8 month infant, who witnesses an object being placed behind a screen, behaves as though the object no longer exists once it is removed from sight. However, by the time a child is 1 year, not only does the screen get pushed aside to get at the object, but if the object is placed in a box behind a large chair, the child will still locate the object. Reversibility is exhibited in detour behavior; a movement in one direction can be cancelled by a movement in another direction. According to Piaget, 2-year-olds and chimpanzees understand that a point in space can be reached through a number of different routes.

From the many cognitively rich observations of children answering questions from ingenious problem tasks, Piaget has developed a persuasive argument for a genetic-epistemology. Knowledge is constructed through actions in a cycle of assimilation and accomodation to pre-existent structures. Since knowledge is not absolute, it must be a relative measure of an organism's ability to adapt to its environment. It appears that Piaget's method of psychogenetic analysis is one very useful means to link the relative validity of knowledge to the subjective internal model of reality.

Implications for Education

The following points summarize some contributions that a constructivist epistemology has to offer educators.

1) Epistemology matters in the classroom. A student's epistemology shapes the attitude toward, and conceptions of, both the content and process of learning. It determines whether the student is a rote memorizer or a conceptualizer. Furthermore, the teacher's epistemology has direct bearing on the classroom he or she creates and on the epistemologies of the students.

2) Concepts and their symbolic representations contain hidden epistemologies which must be elucidated by education researchers and then communicated to educators and to students. A description of the historical and cognitive genesis of concepts would contribute greatly to conceptual understanding.

3) The academic disciplines are steeped in epistemological assumptions which their practitioners should acknowledge. Assumptions about the truth-value of authoritative knowledge are open to question from a number of perspectives.

4) All knowledge is ultimately self-referential and all self-referential knowledge is relative -- not absolute. It is constructed individually. Consequently, students need individual attention.

5) The construction of knowledge is subject to a psychogenetic study of the change in knowledge structures in humans in the course of their development. Cognitive scientists would benefit from developmental studies of children and adolescents.

6) Actions contain knowledge. Many logical structures may be traced to goal-oriented actions. Knowledge, therefore, is fundamentally active and dynamic. Conceptual learning, as opposed to factual memorizing, is best facilitated with the type of goal-oriented actions required in problem-solving.

7) There being no absolute knowledge, responsibility for the construction of knowledge lies ultimately with the learner. The teacher may facilitate this process by providing conceptual problems, and actively engaging in a dialogue in which both student and teacher learn. In this manner, individuals may reach at least a temporary consensus as to whether their knowledge has become more valid.

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