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ABSTRACT

This document has been written to relate to the "Mathematics Framework for California Public Schools, Kindergarten through Grade Twelve" published in 1985. Part 1 of the document provides a brief summary of important characteristics of a strong elementary mathematics program. Part 2 of the document presents a portrait of a desired elementary mathematics program focused on the development of student understanding. Included in part 2 are discussions, lessons, and teaching suggestions related to certain basic, underlying mathematical ideas referred to in the "Mathematics Framework" as the "essential understandings." These are: (1) number; (2) measurement; (3) geometry; (4) patterns and functions; (5) statistics and probability; (6) logic; and (7) algebra. (RH)

Mathematics

MODEL CURRICULUM GUIDE

KINDERGARTEN THROUGH
GRADE EIGHT

CALIFORNIA STATE DEPARTMENT OF EDUCATION
BILL HONIG—SUPERINTENDENT OF PUBLIC INSTRUCTION
SACRAMENTO 1987

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FOREWORD

"... the first principles which are instilled take the deepest root ...."
Abigail Adams, Letter to John Adams, August 14, 1776

The importance of the elementary school experience cannot be overstated. The first connections that our children make with formal learning lay the foundation on which their future education will be built. If through their day-to-day school experiences students find that learning is important, interesting, and meaningful to their lives— and if we help students not only to want to learn but also to know that they can learn—we have put in place important building blocks for a solid academic foundation. Yet, to value learning and to want to learn are not the only ingredients of the foundation. The content of the curriculum—what we teach our children—ultimately determines how well they are prepared for the challenges of further education and for productive adult life.

California's attention to the content of the elementary school curriculum is part of the state's comprehensive curriculum improvement efforts. With the Hughes-Hart Educational Reform Act of 1983 (Senate Bill 813), California reinstituted high school graduation requirements, which increase for many schools the number of courses required for graduation. The legislation also mandated publication of the *Model Curriculum Standards, Grades Nine Through Twelve,* which is intended to help high schools improve the quality of academic coursework. The *Model Curriculum Guides, Kindergarten Through Grade Eight,* are aligned with the state's *Model Curriculum Standards,* as well as with our frameworks, handbooks, and assessment programs. Together, the *Guides* and the *Standards* are intended to engage each student from kindergarten through grade twelve in interesting, disciplined academic study. Their overarching purpose is to ensure that all California schools are able to address the state's most critical educational needs to:

- Prepare youth for productive adult employment in an economy increasingly dependent on highly skilled, highly literate employees.
- Graduate informed, thoughtful citizens who understand the shared values and ethical principles essential for a healthy, functioning democracy.
- Produce culturally literate adults who have learned not only basic skills but also a common body of concepts and central works of history, literature, English-language arts, science, mathematics, social science, foreign languages, and the arts, and through this study possess a strong sense of a shared tradition and cultural heritage.

The task of developing a curricular model appropriate for all children is enormous. Students in many of our schools come from widely diverse ethnic, racial, linguistic, and economic backgrounds. They also develop at very different paces academically, physically, and emotionally. To prepare so diverse a student population for the complex world of their future, educational leaders in California and nationally urge the adoption of a rigorous, integrated core curriculum that begins in kindergarten and builds from grade to grade through high school, as described here:

- A core curriculum of study validates each student's individuality and unique strengths while engaging all children in meaningful investigation of the common knowledge, values, and skills needed for productive adult life. Students study the beliefs which form the ethical and moral bonds of our nation. They develop an understanding of civic responsiblity in a pluralistic, democratic society and learn the technological literacy needed for our increasingly complex society. A common core curriculum ensures each student a sound educational foundation and develops fully the student's academic, ethical, and political potentials.

- In an integrated curriculum, teachers incorporate through the lessons they teach the knowledge and skills from two or more disciplines. In integrated units of study, teachers emphasize the rich connections among content areas, teach students the interrelatedness of knowledge and skills, and foster a holistic view of learning. Through the integration teachers also extend instructional time in subjects structured as integrated lessons. In a science lab in which students record their observations in concise, declarative sentences and read about one another's discoveries, for example, the students learn reading and writing along with science concepts.

At the beginning of this foreword, I quoted from a letter Abigail Adams wrote to her husband. Now I close with a most appropriate quote from a letter she wrote to her son, John Quincy Adams: "Learning is not obtained by chance; it must be sought for with ardor and attended to with diligence." As we build new curricula for our elementary schools, we need to keep both of Mrs. Adams's ideas clearly in mind: Those first principles that we teach our children lay a foundation that must not be permitted to assemble by chance. This guide and the other materials we have produced to support our educational reform efforts in California will help ensure that the education of our children will not be left to chance but will be pursued with the diligence and insight of an Abigail Adams.

Bill Honig

Superintendent of Public Instruction

PREFACE

The *Model Curriculum Guides* set forth the essential learnings for elementary and middle school English-language arts, mathematics, and science curricula. And guides for social science, history, fine arts, physical education, and foreign language are being prepared.

Although the *Guides* are not mandatory, they are intended as evocative models of curriculum content. Individual schools will probably modify and expand the content, as appropriate, for their particular student populations. For each subject, the *Guides* suggest a learning sequence, delineating concepts, skills, and activities appropriate for learners in the kindergarten through grades three, three through six, and grades six through eight. The sequences are suggestive. Teachers' judgments about a particular student's readiness for more advanced instruction will ulitmately determine when new concepts and skills are introduced.

Sequencing essential learnings for various grade levels is useful in organizing so large a body of information. Yet, the overarching message of the *Guides* is that learning is not linear. It is a process that involves a continuous overlay of concepts and skills so that students' understandings are ever-broadened and ever-deepened. The content and model lessons of the *Guides* are structured to help teachers lead discussions, frame questions, and design activities that contain multiple levels of learning. Examples indicate how knowledge at one level can be reinforced and expanded as students advance through the curriculum. The organization of material is intended to help teachers move each student quickly from skill acquisition to higher-order learning while, at all times, fully engaging the student in rigorous academic study.

The shift of emphasis from mastering basic skills to understanding thoroughly the content of the curriculum is intentional. Research indicates that children will learn more—and more effectively—if teachers focus lessons on content and the connections among subject areas. Students learn to apply skills by reading, writing, and discussing curriculum content. The essential learnings emphasize central concepts, patterns, and relationships among subject areas and reinforce inquiry and creative thinking.

Whenever possible, lessons include background information about the works, ideas, and leaders that have shaped the discipline. Such contextual knowledge helps students understand the way in which people discover and apply information under particular circumstances to advanced fields of knowledge. This fuller picture of academic content develops students' cultural and technological literacy —their ability to see both the content of the discipline and the broader context out of which facts and concepts evolve.

The *Model Curriculum Guides, Kindergarten Through Grade Eight,* will be successful if they help elementary and middle school communities shape an integrated, active core curriculum that prepares students for the challenges of secondary school and beyond. The building blocks of an academic foundation are in place in many of our elementary schools. Yet, so important is the foundation—not only to children and their families but also to our future communities and the nation as a whole—that we must make every effort to ensure that, in every school and for every child, the elementary curriculum is the best that we can provide.

"The beginning is the most important part of the work."

Plato, The Republic

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Deputy Superintendent
Curriculum and Instructional Leadership

FRANCIE ALEXANDER
Director
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and Assessment Division

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OVERVIEW

In the nation at large and in California in particular, the eighties have been a time of educational reform. There is a new, strong quest for educational excellence and equity in mathematics. New expectations have been established for California's students by the *Mathematics Framework for California Public Schools, Kindergarten Through Grade Twelve* and, for high schools, the *Model Curriculum Standards, Grades Nine Through Twelve,* both documents approved in 1985. This document, the *Mathematics Model Curriculum Guide, Kindergarten Through Grade Eight,* is written in relationship to the framework and the standards specifically as well as to the reform effort generally.

Part I of the Document

The *Guide* has been developed to serve different purposes for different readers. Part I provides a brief summary of a number of important characteristics of a strong elementary mathematics program. It is designed to aid teachers, principals, and parents in identifying program aspects which may require attention. For some readers it will provide an initial orientation to a strong mathematics program; for others, it can serve as a checklist for use in an intensive examination of a school's program. Part I is intended to identify hallmarks that a wide range of professionals and others can use repeatedly in planning and evaluation.

Part II of the Document

Part II presents a portrait of an elementary mathematics program focused on the development of student understanding. It is designed to be a guide and resource for persons with major mathematics curriculum or staff development responsibility. It will help them to establish specifications and assessment criteria for long-term improvement efforts. Elementary teachers can and should, over a period of time, read Part II carefully to acquire a tangible vision of the mathematics program that they should strive, with sustained support, to provide.

PART I

The early years in mathematics education are critical. Students must be both confident and comfortable with a full range of mathematical ideas as well as arithmetic operations if they are to succeed in future schooling. There are many ingredients of an elementary mathematics program. In this part of the document a number of hallmarks of excellence are identified and described briefly. For some readers they will simply be reminders; for others, they will help to focus attention on factors of program design and operation which they might otherwise have found difficult to discern.

DESIGNING THE PROGRAM

A commitment to high expectations for all students and staff is the crucial starting point. Regardless of the backgrounds of entering students or the career histories of the faculty, the school community must share the goal of the *Mathematics Framework* "... that students experience the joy and fascination of mathematics as they gain mathematical power." There are several issues of particular importance in designing a program to meet this goal.

1. Alignment of Expectations and Materials

"Tests drive the curriculum" and "textbooks are the curriculum" are two well known phrases. As propositions, they are oversimplified; yet they reflect the dismay that follows when tests and texts are not aligned with a school's intended curriculum. It is important for those who develop or choose tests (especially standardized tests) to compare what those tests measure with the school's learning objectives. It is equally important for those who select instructional materials to compare the content and instructional approaches of the materials with the curriculum. The greater the mismatch, the more it will be necessary to supplement or displace the tests or texts. The greatest possible alignment should be sought initially, and an ongoing willingness to modify or compensate is crucial.

2. Articulation: What Is Expected in Each Year

Perhaps more than in any other subject area, mathematics instruction suffers from low expectations of student knowledge and ability at the beginning of a school year. This problem tends to be compounded over the years, to the point that by seventh grade neither students nor teachers expect much more than a repeat of earlier years' curriculum. It is important that teachers at each grade level expect to *advance* the students a full year in the curriculum by the end of the school year. This advancement is possible only when each teacher knows what is taught the year before and the year after. There is no "perfect" sequence; there may, for example, be more statistics than geometry in fourth grade, or the other way around. What is important is that the expectations are shared.

3. Structuring a Year's Program

After establishing expectations for the year's work, it is essential for the teacher to design a structure for it. To do so requires decisions on both pacing and "mix." It's usually neither practical nor desirable to cover all the textbook pages. What chapters near the end are important for meeting the year's expectations? Which chapters or sections near the beginning could be omitted? When is it especially important to use supplementary materials? How will time be reserved for student projects requiring data collection, or for reteaching concepts to some while others pursue independent activities?

The teacher needs to establish a "flow" design for the year, so that he or she can make minor adjustments week by week to ensure that the major components of his or her program design for the year will be realized. Even when the textbook is relatively well aligned with the school's curriculum, it is important for the teacher to be in control of the instructional flow.

4. Providing for the Range of Students

The great majority of students can come to understand and apply much more mathematics than present performance measures show. They cannot all, however, learn in the same way and at the same rate. Thus, it is important to design the program to allow both variations in individual assignments and cooperative learning experiences in which students learn from and test ideas with one another. Such design features have a number of consequences, including a substantial reduction of traditional lessons in which students first observe the teacher and then individually complete twenty or more exercises as shown on the same two pages of the textbook. The teacher will plan a variety of embellishments, including not only mathematical games and puzzles but also interesting quantitative problems drawn from other content areas.

As students move to the middle grades, the spread of their knowledge and abilities inevitably grows. Teachers in sixth and seventh grade should plan increasing horizontal enrichment for their top students and, additionally, should expect many of them to be working with ideas and types of problems from the eighth grade course (as patterned after the framework). Seventh grade teachers especially should be well acquainted with the expectations for students in algebra so that they can prepare their strongest students to enter the college preparatory sequence in eighth grade.

5. Homework and the Parents' Role

Every student should regularly be assigned work to be done outside of class. Homework should not, however, consist of more of the same exercises performed in class. Instead, homework should add to the student's classroom program by leading him or her to explore situations, gather data, or interact with members of his or her family. Specific roles for family members to play often should be incorporated into the homework assigments, sometimes allowing three or four days or longer for a task to be completed. Information about the structure of the program and the function of homework assignments as well as information on their child's schoolwork should be regularly communicated to parents.

6. Support for Teacher Experimentation

It is, of course, necessary to give teachers time to design their programs, both for the year as a whole and for the daily lessons. It is also important for the principal and the school community to provide a climate of support, especially regarding the efforts teachers make to teach to the school's expectations rather than, for example, as laid out day by day in the textbook. It is not easy to delete, rearrange, or augment within an existing, textbook-driven curriculum, especially for teachers with limited mathematical preparation. There are always missteps when significant alterations or explorations are made, and teachers as well as students deserve encouragement rather than complaints when flaws are detected. Keeping focused on the goals for students learning will reduce the tendency to find fault with specific innovations being tested.

THE PROGRAM IN OPERATION

A strong elementary mathematics program is rich and diverse; it is attentive to programwide goals and to student differences; it involves students, teachers, administrators, and parents in a common academic focus. The remainder of Part I contains a selection, not a comprehensive listing, of significant characteristics common to the highest quality programs. Although the descriptions are brief, they can be used by teachers or other concerned members of the school community in asking questions about the program, in looking in on a few classrooms, or (with elaboration) in conducting a major evaluation of the program. The descriptions under each heading are written in terms of characteristics or qualities that a high-quality program has in operation.

1. The Beauty of Mathematics

Teachers demonstrate an appreciation of the beauty of mathematics. They enjoy engaging in mathematical activities and naturally project an expectation of enjoyment for students. Students are actively encouraged to discover the fascinating patterns and relationships that occur throughout mathematics. Lessons include puzzles and games that entertain as well as challenge. Students experience mathematical activities as pleasurable throughout the grade levels.

2. Mastering Single Digit Number Facts

As immediate recall of number facts is essential for student confidence as well as for efficiency in developing broader numerical capability, the program ensures that each student masters the single digit number facts and common number manipulations, such as doubling and halving and multiplying or dividing by ten or one hundred. Students practice in a variety of pleasant settings that involve them in real-life situations; for example, estimating and pattern recognition.

3. Arithmetic Operations

Students learn and can demonstrate or explain what is done in adding, subtracting, multiplying, or dividing. They are taught how the arithmetic operations are related to one another. They learn one or more algorithms for each operation and can apply the operations in such challenging aspects of computation as mental arithmetic, puzzles that require computational understanding and ingenuity, and alternative algorithms.

4. Reinforcement

Previously learned concepts and skills are reinforced regularly in each grade through problem assignments that require their use in a variety of new situations with real world settings. Explicit connections to what students already know are made as new or extended concepts are developed. Throughout the grade levels, relationships among concepts and skills are stressed. Students combine single or simple skills in order to solve practical problems, such as those involving ratio, proportion, and percent.

5. Mathematical Thinking

All students, especially those who are slowest to acquire abstract understanding, constantly have individual and group opportunities to explore, conjecture, test, discover, invent. Students are helped to approach mathematics with a common-sense attitude and to understand not only how but also why different procedures are applied in different situations. Teachers often pose What if? questions and encourage students to ask their own. Teachers frequently observe students at work and listen to their explanations, with primary concern for their reasoning rather than a particular result. Homework assignments include "thinking problems" that students and parents can discuss together.

6. Problem Solving

Problem solving abilities are deliberately and consistently developed throughout the program. Students are regularly presented problem situations which have not already been put into mathematical form. In selecting a solution approach, they consider whether the available information is pertinent and, if sufficient, what assumptions can or must be made, and what patterns or similarities there are with other problems they have encountered. They are challenged to consider whether an answer is reasonable, whether any key features were forgotten or ignored, whether simplifications that were made limit the validity of the solution, whether other approaches would work, and whether generalizations can be identified.

Students, through their own experience, come to understand that problem solving is a process, with solutions coming most often as the result of exploring situations, stating and restating questions, and devising and testing strategies over a period of time. They are supported in taking risks to help them realize that it is normal to try ideas and methods that turn out to be unsuccessful in solving a hard problem, and that by learning from their mistakes, they are able to increase their ability to choose an effective strategy.

7. Concrete Materials

In concept development students at each grade level work initially with concrete materials. In using these materials, students connect their understandings about the real objects to their own experiences and gain direct experience with the underlying principles of each concept. Because it is not sufficient to confine the use of concrete materials to teacher demonstrations, each student touches and moves his or her materials. Although the student may have been introduced to a concept in a previous grade level, he or she continues to use concrete materials until demonstrating a clear understanding of the concept. The teacher then assists him or her in making the connection between the concrete and the abstract. In the transition to abstractions, students work with pictures, drawings, diagrams, and other representations of the concrete objects.

8. Number Sense

The program intentionally has students work with numbers in many different contexts. Students are led to judge the size and precision of numbers in relation to the issue being considered. They have encounters with many different numbers that they can connect to personal experience.

Students learn estimation as a critical aspect of many mathematical activities, ranging from making a quick mental approximation before calculating exactly to solving a simpler variation of a problem before choosing an approach to solving the original, complex problem. Rather than blindly accepting calculator results, students are given enough practice in estimating to know when they have made a mistake in using the calculator.

Students consistently are asked to choose among estimation, mental arithmetic, paper and pencil, and the calculator as a method of computation. They are taught to consider how accurate a numerical result is needed; for example, is the advantage of knowing an answer to four significant figures over three worth the additional time/effort required?

9. Calculator Use

The hand-held calculator is fully incorporated into the program. From the primary grades on, calculators are used for exploratory activities. As students understand basic concepts and learn the basic arithmetic operations, they are taught how and when to use the calculator. As more experience is gained, the student's effective use of the calculator increases greatly.

In the upper elementary grades, students are regularly performing computations with calculators. At the same time they are learning that for some simple computations, the use of the calculator is cumbersome or, worse, can obscure the understanding of the calculation being performed. As they gain experience, they are expected to judge whether use of the calculator will be effective and efficient. Before the end of sixth grade, students have calculators continually available for use—in class, on homework assignments, and on tests.

10. All Strands for All Students

Major concepts or precursors of concepts from every strand—number, measurement, geometry, patterns and functions, statistics and probability, logic, and algebra—are incorporated and interwoven throughout the program at each grade level. Second grade students, for example, sorting geometrical shapes according to number of sides, are working with basic concepts of geometry and statistics as well as number. All students are instructed in the fundamental concepts of each strand, with no student limited to the computational aspects of the number strand. Students are regularly assigned problems that require them to apply concepts and skills from all the strands in a variety of practical situations.

11. Oral and Written Work

Students are involved regularly in activities that help them to communicate mathematically.

Students talk about or draw pictures or diagrams of the mathematics involved in order to test themselves and clarify their understanding. When students are problem solving in small groups, any member of the group is expected to be able to explain fully what the group has done and why.

Students sometimes give oral or written reports on mathematics homework or projects they have completed. At every stage of the problem-solving process, students are invited and encouraged to describe and discuss their mathematical reasoning, whether in one-to-one conversations with the teacher, in small groups, or in open class discussions.

12. Assessment

Students are tested on their ability to apply concepts and skills in situations that demonstrate their understanding of these concepts and skills. There is regular assessment of students' knowledge of mathematical concepts and skills, their ability to apply these concepts and skills appropriately and correctly in given situations, and their ability to identify appropriate procedures, explain reasoning, and demonstrate techniques for problem solving. Teachers often expect that students will be able to demonstrate understanding as they are engaged in hands-on activities.

Diagnosis and assessment of students' work continue throughout the year so that areas of difficulty can be identified and additional instruction provided as soon as it is needed. Assessment indicates the depth of comprehension as well as the pattern of errors. In analyzing a student's errors, the teacher listens to his or her reasoning before deciding on a plan for correcting the incomplete or erroneous understanding. Additional instruction uses interesting and fresh alternative approaches rather than repetition of the original instruction. Placement of a student in one type of group or the use of a particular method of instruction with the student is reviewed periodically.

PART II

INTRODUCTION

The *Mathematics Framework* declares that:

> *Every student can and should develop mathematical power. A commitment to this goal will enable each student to develop his or her ability to enjoy and use mathematics.*

The *Framework* goes on to say that:

> *Every student must be instructed in the fundamental concepts of each strand of mathematics, and no student should be limited to the computational aspects of the number strand.*

The nature of the instruction that students receive must enable them to appreciate and enjoy the inherent beauty and fascination of mathematics as they develop deep understanding of the fundamental concepts from each of its strands. When high priority is given to developing understanding, students will look for the sense in mathematics, recognize its applications, become persistent and effective problem solvers, and develop a positive attitude towards mathematics. These students, upon entering secondary school, will have a sufficient mathematical background to complete successfully those mathematics courses appropriate for their future.

In building on the *Framework*, Part II of the guide holds to the commitment of mathematical power for every student and focuses especially on what it means to instruct every student in the fundamental concepts. It deals with curriculum in the deepest sense—with the understanding of underlying ideas and relationships. It treats curriculum dynamically, providing illustrations of questions for students to encounter in developing their understanding rather than sample test problems to measure their acquisition of a traditional list of specific concepts and skills. It is written from the premise that *what* a student learns is intrinsically dependent on *how* he or she has learned it, and that curriculum and instruction in an empowering mathematics program cannot be separated.

The material in Part II responds to the following questions:

1. How do we teach for understanding? What are the basic principles that will guide us in planning this type of educational environment?

2. What are the fundamental, underlying mathematical concepts in each of the strands that we want students to understand? What kinds of activities and experiences will promote the development of these essential understandings?

3. How does teaching for understanding actually translate into the classroom? What will this kind of teaching be like?

Part II of the *Guide* is divided into four major sections. The first section considers the circumstances that have made it necessary to change the way we teach mathematics. The second section expands our view of what mathematics is and emphasizes the need to teach for understanding. The heart of Part II is presented in the third section, which describes those understandings in each of the strands of mathematics that are essential for students to develop. The fourth section contains examples of lessons illustrating the principles set forth in this document.

Focusing on the development of essential understandings rather than the learning of specific procedures may seem an unfamilar and uncertain approach to some readers. It is hoped that "understanding" will come to be seen as closely related to "common sense" and that mathematical reasoning can be the province of the many rather than the few: teachers, students, and parents alike.

THE NEED FOR CHANGE

The Effects of Technology

Technological developments are entering our daily lives to an ever increasing degree. We casually watch the evening news as live pictures from countries thousands of miles away are beamed into our living rooms. We pick up the phone and talk to a friend in Japan as though we were calling across town. We stand in line at the supermarket chatting with the checker while each item in our shopping cart is automatically priced and listed on our receipt with one motion of the checker's hand. We give messages to answering machines, receive phone calls from computers, and get money from the bank without ever talking to another person.

When we step inside a school to see how technology is affecting the daily lives of children, we notice something interesting. While the world around them has changed dramatically, children are still tediously working on page after page of arithmetic problems, filling in the answers in the same way their grandparents did. While Robert is figuring out the answer to a long division problem, his grandfather is using a calculator to balance his checkbook, and the cash register is telling his brother the amount of change to give a customer. Robert's parents are studying a computer printout with the financial information they need in order to make a decision about which IRA to buy.

The contrast between Robert's school experiences and the world in which he lives is striking. *The impact of technology and its implications for mathematics education can no longer be ignored.* The modern world demands an ability to think about and to use mathematical ideas to solve problems and to make decisions, not facility with rituals for getting answers. The time our students spend learning mathematics can no longer be limited to practicing long and tedious procedures which are more efficiently done with calculators. Now we can, and indeed we must, enrich and strengthen the mathematics education our children receive. Instead of requiring students to spend years studying arithmetic as a series of rules and rote procedures, it is time to provide the opportunity for all students to experience the richness and beauty of the study of mathematics as a whole.

The Teacher's Experience

For many years, teachers have been expected to get their students through the material in their textbooks rather than to focus on students' understanding of fundamental concepts. Much of the material in these textbooks has focused on step-by-step procedures for getting answers to computational problems. In classroom after classroom, math class has begun with the teacher demonstrating or explaining the particular procedures to be covered in the lesson plans for the day. Good teachers have explained these procedures as clearly as possible and have provided their students with the necessary practice and review. The same types of problems found in the textbooks have comprised the majority of the mathematics items on standardized tests. Teachers have felt a great deal of pressure to ensure that their students do well on these standardized tests and so have directed most of their mathematics instruction towards teaching their students to follow specific procedures.

Focusing on getting through the textbook and preparing for standardized tests has produced disturbing results. Year after year, students have entered their new classrooms unprepared for the work the teacher believed he or she was obligated to present. Teachers acknowledge that they have had to spend a great deal of time each fall reteaching and reviewing rules and procedures. This has often been frustrating for both teacher and student. Many students have felt a great deal of insecurity and anxiety. Other students have done very well on the particular exercises they have been required to do but have balked when asked to apply what they have learned in a new or unusual situation. Because of an overemphasis on speed of response, students have come to measure their success in terms of their ability to produce the right answer quickly. They have resisted listening to explanations and have urged the teacher just to tell them how. They aren't interested in why.

Regardless of the particular classroom situations in which teachers work, they have similar stories to tell about children who have trouble with mathematics. They see the same, familiar errors over and over again and soon are able to predict what students will do when given certain types of problems. Consider the following examples:

$2 + [9] = 7$ $\quad\quad$ $.8$ is less than $.78$

$$\frac{2}{3} + \frac{1}{2} = \frac{3}{5}$$

$$\begin{array}{r} 3.40 \\ \underline{x.15} \\ 1700 \\ \underline{340} \\ 51.00 \end{array} \qquad \begin{array}{r} 46 \\ +\underline{18} \\ 514 \end{array}$$

Too many students are not looking for the meaning in the mathematics they are learning. They are memorizing rules and procedures but have no way of making sense out of what they are doing if they forget or misapply the rules they have learned. Many have become passive and simply wait to be told what to do if they have forgotten.

Why Hasn't It Worked?

There are three major factors at work that contribute to the frustrations of teachers and students involved in the teaching and learning of mathematics. One contributing factor is that we teach as though symbols have obvious and inherent meaning. The second is that we too often teach without considering the students' level of cognitive maturity, not recognizing that what seems obvious to us as adults may not be obvious to the child. And the third is that in our search for the ever clearer explanation, we often overlook the importance of the students' need to construct their own understanding.

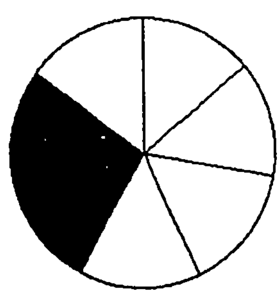
An analogy seems useful here to help us consider the problem of teaching as though symbols had meaning. If you were shown a picture of someone you had never met and were told the name associated with that picture, you could learn to say the appropriate name on cue whenever asked. However, you would not really know the person represented by that picture and would have difficulty if you were to describe the person's characteristics or be asked to predict that person's behavior. You would have no personal experience on which to base your answer. On the other hand, if the picture were of a beloved friend or family member, seeing the picture would be enough to remind you of many qualities the person had. So it is with the learning of mathematical symbols.

There is no meaning in the symbols themselves, only in the mind of the person perceiving the symbols. *If a child does not have an understanding of the concept represented by a symbol, no amount of practice working with that symbol will help to develop that concept.*

The interaction between the child's level of cognitive maturity and the need to construct his or her own understanding has implications for the way in which students receive the instruction they are given. As every teacher can testify, just because something has been taught does not mean it has been learned or, more to the point, that it has been understood.

Consider the following cases.

1. Bill has learned many things about fractions. He can name the numerator and the denominator. He can add $^{2}/_{8}$ and $^{1}/_{3}$ and get the correct answer. However, when he is asked to draw a picture of $^{2}/_{5}$, this is what he draws.



When asked to show what $^{2}/_{5}$ of a group of ten marbles would look like, he says you can't do that because you can't cut marbles. He has learned rules for working with the symbols without comprehending the nature of fractions.

2. James has been working with beans and cups to do place value problems. When given the problem, 42 minus 37, he builds 42 using 4 cups of 10 beans and 2 extra. He then dumps out one of the 4 cups and adds its 10 beans to the 2 extra in preparation for taking away 7 beans. Before he goes any further, his teacher asks, "How many beans do you have on your place value board now?" He expects James to say 3 tens and 12 ones, but instead James says, "52." James has focused on the action of adding ten to the 2 extra beans without considering that he has also removed 10. The fact that 52 is 10 more beans than he started with does not seem to disturb him. This is probably an indication that he does not yet realize that the quantity of a number does not change simply because it has been rearranged. His lack of understanding of the invariance of quantity has an effect on his understanding of the place value ideas his teacher is presenting.

3. Nina is trying to balance a group of blocks with a set of weights. When she adds the smallest weight to the weights already in the balance, the weights are now heavier than the blocks. She tries to remedy the situation by pushing the blocks closer together. Nina is still not sure what affects weight and what does not. Learning the number of grams in a kilogram will not help develop the understanding she needs.

4. Marjie has used the formula for finding the volume of a rectangular solid with complete assurance. She confidently gives the answer as 168 meters squared, not realizing that volume measures are cubed. When questioned, she reveals that she doesn't understand how a unit of measure is related to the dimensionality of the attribute being measured. She is simply applying a formula as she believes she was taught.

Learning the symbolic representations of concepts is an important part of learning mathematics. However, students are often required to deal with these symbols before they have full understanding of the complex ideas represented. Most mathematical ideas that children encounter in school can only be understood after many experiences confronting the idea in a variety of settings. Students need time to develop a full understanding of mathematical ideas before they are asked to deal with these ideas as symbols isolated from the real world. When students do not understand a mathematical idea, they have no way of making sense of the symbols and can complete their assignments only by rotely memorizing the procedures required.

No matter what the age or the ability of the students, their experiences with mathematics teach them something about themselves and their place in the world. The *way* we teach children mathematics has a profound effect on whether or not they think mathematics is something they can do. Therefore, it is vitally important that any changes in the teaching of mathematics with the aim of higher expectations for children's growth in understanding be carried out in light of what we know about how children learn.

TEACHING FOR UNDERSTANDING

Expanding Our View of Mathematics

The experiences with mathematics that many of our schoolchildren have can change only if we in the educational community expand our view of what mathematics is. Mathematics is not simply arithmetic procedures or algebraic formulas. It is not pages to complete in a mathematics textbook. It is not flashcards or timed tests or even drill and practice computer games. It is not a subject that inherently or inevitably breeds feelings of anxiety and incompetence. And neither is it just for the intellectually elite.

Mathematics is actually very much a part of life. It is present all around us. We see it in the grace and utility of the Golden Gate Bridge. We see it in the design and pattern in great grandmother's quilt. We sense it in the movement and predictability of the seasons. It is the everyday things, too. It is buying the right amount of wallpaper to decorate a bedroom; it is finding you're finally up to your Mom's shoulder; it is figuring out how much more money you need to save to get a new radio.

Mathematics is concerned with making sense of the world. It is natural for children to seek order and beauty and consistency. From the very beginning of a child's life, he or she is searching to organize and understand his or her world. Children notice size and shape and position. They wonder how long, how big, how much. Mathematical thinking is that which helps us to make connections, to see order and logic. It is seeing patterns and making predictions. It is estimating, proving, solving, and creating.

The mathematics that our children study in school should be consistent with the nature of the subject. That means students should be learning mathematics in ways that allow them to discover relationships and to develop understanding. **The fundamental premise on which this document is based is that every aspect of mathematics that students study should enhance their understanding of mathematical ideas and promote the growth of thinking.**

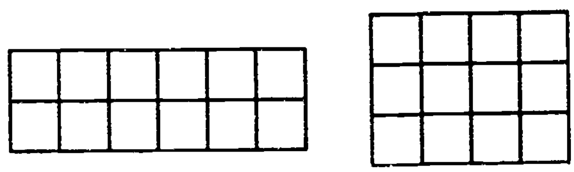
Teaching Mathematics Can Be Different

In spite of children's active interest in making sense of the world, we have, in the past, encountered many problems in teaching mathematics to children. Many children viewed mathematics as simply too hard to understand. They were frustrated and often developed feelings of math anxiety or avoidance. These problems arose, not because of the nature of mathematics itself, but because we attempted to teach mathematics in ways that did not take into consideration what we know about how children learn, and we did not give children the kind of support and tools they needed to develop understanding.

Mathematics instruction can be different. We can provide the kinds of experiences that will allow *all* students to make sense of mathematics and that will also deepen and enhance our best students' understanding.

Let's look at some examples that show what we mean:

1. Linda and Peter are working with 12 square tiles to find all the possible rectangles they can make with them. Each time they make a rectangle they record it by cutting out that same rectangle from graph paper. So far they have found the following rectangles:



   Working on this task contributes to Linda's and Peter's developing understanding of multiplication and factoring. Finding all the possible rectangles requires them to be actively engaged and to do some thinking rather than passively memorize multiplication facts or factors of 12.

2. Tanisha, Carlos, Eddie, and LeeAnn are working with various geometric shapes. Their challenge is to put the shapes in order

by area. Carlos is proving to the others in his group that the triangle they said was bigger than the parallelogram actually has the same area. He is proving his point by using a smaller triangle to measure the areas of both shapes. Rather than memorizing formulas for figuring the areas of geometric shapes, these students are able to use the figures to see for themselves and can thus correct their misconceptions and also gain an appreciation for the reasons certain methods are used for determining area.

3. Orlando, Cecilia, Gina, and Jason have been working a long time on this problem: "If an ice cream shop sells 31 different flavors of ice cream, how many different double dip cones could they make?" They decided to use colored cubes to represent the various flavors so they could act the problem out. They began their exploration by making the problem simpler and considering only three flavors. They then figured out what the number of double dip cones would be if they had four and five flavors. They organized their information and then looked for a pattern. By using materials and some problem-solving strategies, these students are able to cheerfully tackle a difficult problem and increase their understanding of patterns and functions. At the same time, these children are learning some of the basic ideas underlying the concepts of permutations and combinations.

These students are interested and involved *and* learning important mathematical ideas. Mathematics can be presented to children in ways that capture their imaginations, promote their intellectual growth, and foster their enjoyment of mathematics.

TEACHING FOR UNDERSTANDING: GUIDING PRINCIPLES

The type of mathematics instruction that involves students actively and intellectually requires much from the teacher. Without thoughtful decisions about the particular activities and without thoughtful interactions with students, potentially powerful mathematical experiences can become little more than interesting activities for students. The following basic principles are important to keep in mind when implementing a mathematics program that gives high priority to the development of understanding:

1. Our top priority should be the development of students' thinking and understanding. Whenever possible, we should engage the students' thinking and teach the mathematical ideas through posing a problem, setting up a situation, or asking a question.

2. We must know that understanding is achieved through direct, personal experiences. Students need to verify their thinking for themselves rather than to depend on an outside authority to tell them if they are right or wrong. We must see our job as setting up appropriate situations, asking questions, listening to children, and focusing the attention of the students on important elements rather than trying to teach a concept through explanations.

3. We must know that the understandings we seek to help the students gain are developed, elaborated, deepened, and made more complete over time. We must provide a variety of opportunites to explore and confront any mathematical idea many times.

4. We will not expect all students to get the same thing out of the same experience. What students learn from any particular activity depends in large part on their past experiences and cognitive maturity. We should try to provide activities that have the potential for being understood at many different levels.

5. To maximize the opportunites for meaningful learning, we should encourage students to work together in small groups. Students learn not only from adults but also from each other as well.

6. We must recognize that partially grasped ideas and periods of confusion are a natural part of the process of developing understanding. When a student does not reach the anticipated conclusion, we must resist giving an explanation and try to ask a question or pose a new problem that will give the student the opportunity to contemplate evidence not previously considered.

7. We must be interested in what students are really thinking and understanding. Students may be able to answer correctly but still have fundamental misunderstandings. It is through the probing of the students' thinking that we get the information we need to provide appropriate learning experiences.

8. We must be clear about the particular idea or concept we wish students to consider when we present activities or use concrete models. It is not the activities or the models by themselves that are important. What is important is the students' thinking about and reflection on those particular ideas dealt with in the activities or represented by the models.

9. We need to recognize that students' thinking can often be stimulated by questions, whether directed by the teacher or other students. We should foster a questioning attitude in our students.

10. We need to help students develop persistence in solving problems. Only in a learning environment in which mistakes and confusion are considered to be a natural part of the learning process can students believe they do not have to come up with quick, right answers.

11. We need to recognize the importance of verbalization. Putting thoughts into words requires students to organize their thinking and to confront their incomplete understanding. Listening to others affords them the opportunity to contemplate the thinking of others and to consider the implications for their own understanding.

12. We must value the development of mathematical language. Language should serve to internalize and clarify thinking and to communicate ideas and not be an end in itself. Memorizing definitions without understanding interferes with thinking. The emphasis is on developing a concept first, establishing the need for precise language, and then labeling the concept accurately.

THE ESSENTIAL UNDERSTANDINGS

The 1985 California *Mathematics Framework* states:

"Mathematical power, which involves the ability to discern mathematical relationships, reason logically, and use mathematical techniques effectively, must be the central concern of mathematics education and must be the context in which skills are developed."

Children may learn many facts and skills related to mathematics, but they will not be able to discern mathematical relationships, reason logically, and use mathematical techniques effectively unless they understand certain basic, underlying mathematical ideas. We are referring to these basic mathematical ideas as the "essential understandings". They are not a set of basic ideas which "come first" followed by more advanced concepts. These essential understandings bind together rather than precede all those specific concepts and skills which have traditionally been taught. They are the broad global ideas that expand or build, flower or evolve—that grow more complete and complex over time.

An important characteristic of the essential understandings is that they can be encountered at many different levels of complexity and abstraction. Therefore, they can be experienced in some way at all grade levels. They are those worthy mathematical ideas that can be explored by the five year old as well as by the thirteen year old. The nature of the activities that students of various age groups participate in will vary considerably, but the essence of the idea will be present at all levels.

These understandings are never totally mastered, but growth in students' understanding can be observed. Teachers can perceive the growth of their students' understanding by asking questions, posing challenges, and observing students at work. We must ask not only, "What can the student do?" but also, "What does the student understand?"

For their presentation to be consistent with the content organization of the *Mathematics Framework* and the *Model Curriculum Standards*, the essential understandings in this document have been organized by strands. The development of each understanding is illustrated through the presentation of examples of situations and questions that will give students opportunities to confront the important ideas that are embodied in the particular understanding.

NUMBER

Numbers are used to determine or define quantities and relationships, to measure, to make comparisons, to find locations and to read codes. We use numbers to interpret information, make decisions, and solve problems.

It is essential for students to understand that

Numbers can be used to describe quantities and relationships between quantities.

Numbers are not just symbols on a page to be manipulated according to sets of procedures. Students will become confident and capable users of number when they develop a sense of number and the relationships between numbers.

K-3

Most students come to school with some knowledge of the counting sequence and an idea that the sequence has something to do with quantities. However, their understanding of quantities is only beginning and needs to be further developed. It is important that early experiences in school give meaning to the counting sequence and expand on what the students bring to school. This can be accomplished through a variety of counting experiences that focus on the quantities and relationships among quantities rather than on the sequence of numbers or how they are written.

How many scissors do we need to get so that everyone at the table has one? How many containers of juice do we need to bring on our field trip so everyone in the class can have one?

About how many lima beans do you think are in this clear plastic cup? If it takes 13 beans to fill it up to here (half way), how many beans do you think the cup would hold if we filled it up? If the cup holds 24 lima beans, how many pinto beans do you think it would hold?

Consider the numbers: 6, 250, 15, 30, 3 $1/2$? Which number would describe a reasonable number of raisins to take in a lunch? Which could describe the number of students in a class? Which would be a reasonable number to describe the miles from here to your aunt's? Which could describe the number of hours it would take to drive there?

We see on the graph that 19 children like orange juice and 12 children like apple juice. Did more children like orange juice or more children like apple juice? How many more children like orange juice than apple juice? Did more than half or less than half of our class like apple juice?

Show 1 more than 7, 2 fewer than 9, 4 more than 6; double the amounts of 2, 4, 5, 10. Start with 10. What will happen if we add 1, add 10, add 2, double it, halve it? Try other numbers.

Explore several numbers to see which can be divided evenly into two piles. Circle the numbers on the 1–100 chart if they work. Is there a pattern? If they can't be divided evenly, how many leftovers are there? What happens if you try to divide those same numbers into three piles? Can you find a pattern on the 1–100 chart? How many leftovers? Try other numbers. What do you see?

3-6

Intermediate students should continue to develop an understanding of quantities and their relationships. They should be encouraged to use concrete objects when investigating number relationships if the concrete objects will help them understand and discover relationships otherwise not apparent to them. They can explore the properties of numbers through investigations which will serve as a foundation for understanding such mathematical ideas as prime and composite numbers, square numbers, factors, multiples, etc. It is important that they experiment with numbers, look for relationships, and come to generalizations from examining the results of their investigations. Their work with very large numbers should build on the number relationships they have come to know through their work with small numbers. They should move from developing a sense of whole numbers to developing an understanding of fractions and decimals.

When estimating or determining quantities, students should be looking for strategies that will help them be efficient rather than resorting to counting one by one. Estimating requires having or getting some information and using that information to figure out an approximate answer, not just to guess.

There are 64 pebbles in the pint jar. What is your estimate for the 2 pint jar? How many tiles in the classroom floor? in the cafeteria floor? In how many different ways can you estimate or approximate the number of beans in the jar, oranges in the crate?

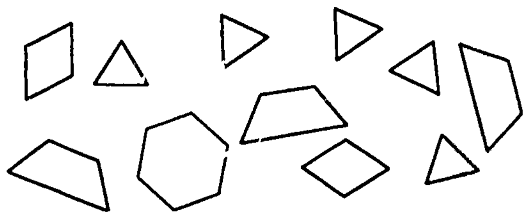
Given a handful of dried beans, can you find a way to estimate the number of beans in a pound?

Using dimes, pennies, nickels, quarters, fifty-cent pieces, and dollars, in how many ways can you make $1.25?

What happens if you add two odd numbers? Can you tell whether you will get an odd or an even number? Why or why not? If you add two even numbers together, can the sum be odd? Why or why not?

Get 24 square tiles all of the same size. Can you put the tiles into 2 equal rows? 3 equal rows? 4 equal rows? Try other numbers. Can you find any numbers of tiles that cannot be arranged in equal rows? What about 11? 23? As you explore making rows, notice which numbers make squares.

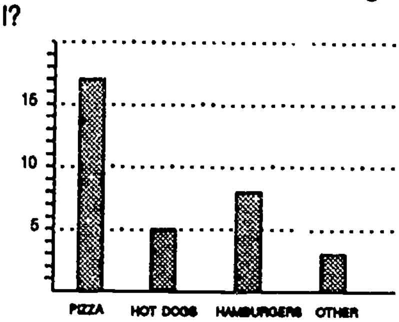
Consider these shapes:



If the area of the trapezoid is one, what is area of the hexagon? What is the area of the rhombus? If the area of the hexagon is one, what is the area of the triangle? If the area of the triangle is $^1/_6$, what is the total area of this particular set of blocks? If the trapezoid is considered to have an area of one, what is the total area of these blocks?

If we both have pizzas of the same size and I cut my pizza into 8 equal pieces and you cut yours into 10 equal pieces, whose pieces would be bigger? If I ate three pieces of mine and you ate 4 pieces of yours, who ate more pizza? Does it make a difference if the pizzas are different sizes?

We see .46 on the calculator. Is this more or less than a half? How can you find out? How could you show on a calculator which is larger: .01 or .001?

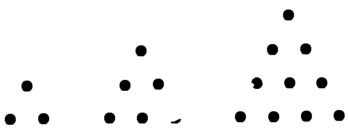


Look at the graph of favorite foods of the students in last year's class. What fraction of the class liked hot dogs best? Did more or less than 50% of the class like pizza? Is the percent who liked pizza closer to 50% or 75%? In this kind of a graph, is a person limited to one choice? Why?

6-8

The older students will continue to develop a sense of the quantities represented by very large and very small numbers and the relationships among them. They should continue to examine the properties of numbers. They will be able to look at collections of numbers and identify which are irrational, which are rational, which are integers as well as rational, and which are whole numbers as well as integers.

Which numbers form triangles when arranged in the following manner?



What kind of number do you get when you add two successive triangular numbers?

How many people could fit (standing up) in our classroom? How could we find out? What assumptions did you make?

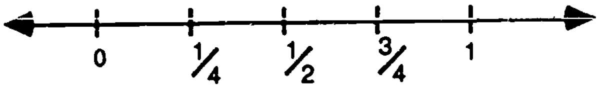
Compare a million with a billion. If you had a million dollars in thousand dollar bills, the stack would be 4 inches high. How high would the stack be if you had a billion dollars worth of thousand dollar bills? Could you carry it?

Take your pulse for 15 seconds. Based on this measurement, how many times would you say it beat since you got up this morning? This week? Since you were born? Do you think your answers are exact? Why or why not?

If the people in the United States were scattered evenly over the land, how close together would they be? Would they be within sight of each other? Within shouting distance? Within arms reach? What information do you need to solve this problem? Is one person per square mile more or less than the population density in our country?

How much milk is drunk each month at our school? Would it be enough to fill an Olympic-sized swimming pool?

Where would .34343434343 fit on the number line below?



Use your calculator to change $^2/_3$, $^1/_8$, $^1/_2$, $^2/_7$, $^4/_9$, $^5/_{11}$, $^5/_{12}$, ... into decimals. Which fractions terminate and which don't? Can you make any predictions from your results?

What determines whether you report information using fractions, percents, or decimals? If you knew 271 of the 952 students in the school rode bikes to school, would you say about 1/4 of the people rode bikes, 25% of the people, .25 of the people, or about 300? Would you report $^{271}/_{952}$?

Can you find a number between .245 and .246? If so, give one. Can you show why $9 < \sqrt{91}$?

Using your understanding of the Pythagorean theorem, can you construct a hypotenuse whose length is 13 cm? Whose length is 7 cm?

❷ *It is essential for students to understand that*

Any number can be described in terms of how many of each group there are in a series of groups. Each group in the series is a fixed multiple (the base of the place value system) of the next smaller group.

The place value system requires the act of counting groups as though they were single items. It is this organizational structure that gives us the power to deal with large numbers and small numbers in reasonable ways. Rather than endless, unfathomable series of numbers, we need only the digits from zero to nine. By grouping we can think of a hundred as a unit or a trillion as a unit; by subdividing we can think either of one thousandth or one millionth as a unit. We can record very large and very small numbers by using the position of the digit to indicate the group we are using as a unit.

K-3

Children in the primary grades need many experiences forming groups and looking for the patterns and relationships. The notions underlying the understanding of place value are complex, and only the foundation can be laid in the early years.

Here is a pile of counters (36 or so). How can you organize them in some way so that your friends can tell how many there are without counting by ones? Is there another way? Another way? Which way helped your friends tell the fastest?

Determine the number of objects in each of the jars (macaroni, beans, polished rocks, etc.). Organize as you count so that you and a friend can tell how many you have as you go along without having to count by ones. Arrange the objects in such a way that you can tell how many tens and ones you have.

Using the beans and cups, interlocking cubes, bean sticks, place value materials, show me 56, 32, 129.

3-6

Students in the intermediate grades will continue to explore the structure of the number system and the idea of place value, extending it to larger numbers than were dealt with in the primary grades. They will be able to deal with more abstract models for place value than primary students can.

If you have 4 hundreds, 2 thousands, 8 ones, and 0 tens, how can you write it? How could you model it with materials?

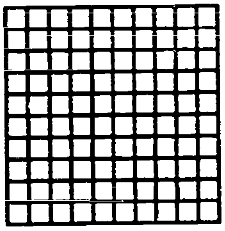
How many hundreds in a thousand? How many tens? Which is more: 10 thousand or 100 tens?

How many different ways can you display 342, using groups of tens, hundreds, and ones? Which way has the most groups, the fewest groups?

If the yellow counters are worth 1000, the red counters are worth 100, the blue counters are worth 10, and the green counters are worth 1, what are the following sets of counters worth? How can you write the value of each set?

12 blue, 16 green, 0 red, 2 yellow__________
0 yellow, 24 green, 7 blue, 0 red__________

Intermediate level students can also explore place value ideas as they relate to the decimal fractions. Partitioning into smaller and smaller groups is a more abstract concept than regrouping into larger and larger groups. It is important that students have models that will help them see how digits to the right side of the decimal point have decreasing values.



Use this 10 by 10 grid to stand for one. Show 1 tenth, 1 hundredth. How could you show .34, .26, .231? How do the squares on the grid relate to money? How would you show the same amounts with money instead of squares on the grid?

6-8

Middle grade students will continue to work with the decimal system, extending it to exploration of exponents, including scientific notation, and to integers. They should have experience relating fractions, decimal fractions, and percents. In addition, they should be exposed to other grouping schemes, such as grouping by twos or twelves.

We group by twelves in many instances: 12 months, a dozen eggs, 12 hours on a clock face, 360 degrees in a circle, 12 inches in a foot, 36 inches in a yard. Why? What would happen to our number system if we had base twelve instead of base ten? Would the same numbers be prime? Evenly divisible by three? Why?

Why does your science book list the speed of light as 1.86×10^5 miles per second? How far does light travel in one year?

10^3 is how much bigger than 10^{-3}?

Jane owes her mom $15; Jane's mom owes the bank $3000 on a car loan; the town of Springfield, with a population of 55,000, has borrowed $3.5 million to build a new school; the power company has just built a solar installation at a cost of $45 million. Who is deepest in debt? What assumptions did you have to make?

The pair of gerbils in Ms. Washington's room always have a litter of 4; the pair in Mr. Suarez's room always has a litter of 7. If each pair has litters every 40 days, how long will it take for Mr. Suarez to have three times as many baby gerbils in his room as in Ms. Washington's? How long will it take to have nine times as many?

Why is it that -8 is less than -5?

❸ *It is essential for students to understand that*

The operations of addition, subtraction, multiplication, and division are related to one another and are used to obtain numerical information.

Understanding operations involves knowing what actions are represented by the operations, how these actions relate to one another, and what operations are appropriate in particular situations. It is important that students understand when to use an operation as opposed to simply knowing how to perform a procedure to get an answer. Knowing how numbers are put together or taken apart can be very helpful in working with operations flexibly and allows students to figure out problems in a variety of ways.

For example: 26 x 8

26 is 25 and 1. 8 is 4 and 4. I see that 25 x 4 is 100, so 25 times 8 is 200. I still have one more 8 to add on, and that makes 208.

Another way:

Eight twenties is 160, and 8 sixes makes 48. I have one hundred and the 60 and the 40 make another hundred. Adding the 8 gives 208.

K-3

Children experience addition, subtraction, multiplication, and division situations naturally in their daily lives. They take 2 trucks outside and go in the house to get 2 more. They have 5 cookies and see they have 4 left after they have eaten 1. They have 9 crackers to share among their 3 friends. They have 4 packages of gum with 5 sticks in each pack. It is important that they learn to connect what they already know to the symbolic presentations. And they need to build a strong base of understanding about how numbers are put together and taken apart.

Act the stories out, using blocks to represent the horses. There were eight horses in the corral. Three of them were taken out to go to a horse race. How many horses are in the corral now? There were two horses in the field. Five more horses were brought to the field. How many horses are in the field now? How can we write these stories using numbers?

Make up stories that go with the following number sentences:

$8+2=10$
$9-4=5$

How many different ways can you show the quantity of ten? Can you think of any ways that use a minus sign?

If I tell you

$14+10=24$, what is $14+9$?
$6+2=8$, what is $36+2$?

Here is a pile of 29 beads. How many groups of ten do you think you could make with these? How many piles of 5? How many piles of 9?

Divide 9 pieces of construction paper equally among 4 people. How many pieces does each student get if you are allowed to cut up the paper? How can we write that number? Is the answer the same if we divide 9 pencils among 4 children?
Why or why not?

3-6

Intermediate students will deal with operations using the decimal number system and more complex situations. They should continue relating what they know in the real world with symbolic representations and exploring how particular numbers can be put together and taken apart.

Can you use this paper cup to help you figure out how many pieces of macaroni are in the bag? If the paper cup holds 35 pieces, how can you figure out how many the whole bag holds? Will your answer be the exact number?

Can you find the length of this room in interlocking cubes if you have only a few cubes?

What is the best answer for $398+299$? Why?
a. about 500
b. about 600
c. about 700
d. about 5,000
e. about 6,000

If I tell you...
$11.6+4.3$ is 15.9, what is $116+43$?
$6 \times 9=54$, what is 7×9?

How can knowing 2×6 help you figure out 2×60, 2×600, 2×6000?

Using graph paper, show how the result of multiplying 4×5 is the same as multiplying 5×4. Can you find more numbers that will work? Can you find any numbers where it will not work?

Use graph paper to show if $6+2$ is the same as $2+6$. Explain why the results are the same or different.

If you figure out how many weeks in 50 days, how does that help you know how many weeks in 100 days? 200 days?

Do you get the same answer if you add first and then multiply or multiply first and then add?

$4 \times 5 + 7 = ?$
$5 + 7 \times 4 = ?$

Our class of 30 needs four dozen buns. The price of buns is 89 cents for a package that contains 8 buns. How many packages do we need to buy? How much will it cost?

Our class of 30 is raising money to go to the amusement park. Individual tickets cost $2.80. We are selling ice cream bars for $.35 each. We make $.15 profit on each one. How many ice cream bars do we need to sell to raise enough money for our whole class to go to the park?

Demonstrate $^2/_3$ of 12 with counters, graph paper, symbols, drawing pictures, and words.

5 x 2.2 is 11; what is $^1/_5$ of 11? Why?

6-8

The students in the sixth through the eighth grades should be at ease with negative and positive rationals and realize that the operations and their properties extend to these number systems. The approach to operations should include estimating, working mentally, and finding a variety of procedures for the same situation.

If I multiply 35 by 5 and then divide the product by 5, do I get 35? Why? If I add 10% to 35 and then subtract 10% of the result from 35, do I get 35? Why?

If 13 x .17 = 2.21, what is 14 x .17?

If you got these answers on your calculator, would you think they were reasonable? Why or why not?

234 x 600 = 2408   90 x 20 = 1800

You earn $5.00/hour. Your supervisor earns $8.00/ hour. If you both get 5% per year raises, who gets the bigger increase in salary? How many years will it take to get $2.00 an hour more than you're receiving now? How long would it take you to double your salary? How long would it take your supervisor to double hers?

George said if he won the lottery he would share it with his friends. He would keep $^1/_3$, give $^1/_4$ to Diane, $^1/_5$ to Jose, and $^1/_6$ to Tony. Is this possible? If so, will there be something left? If not, why not?

What percentage of each type of nut is in this can of mixed nuts? Is that percentage true for all cans of this brand?

The same kind of chair is on sale at two stores. How could a store advertise 30% off and still make more money than the store down the street which is selling the chair for 20% off?

Compare the the following products. Are any of the products equal? Explain.

.24 x .84	24 x 84
.24 x 84	-24 x .84

❹ It is essential for students to understand that

The degree of precision needed in calculating a number depends on how the result will be used.

Exact answers are not always possible or appropriate in a calculation. It is important that students recognize situations in which approximation and estimation are the appropriate procedures. Sometimes an approximation is the reasonable choice and sometimes it is the only possible choice. Some situations require more precise answers than others.

K-3

Young children need practice determining what is meant by almost, nearly, a little more than, a little less than.

Which things are easy to count exactly, and which things are hard?

- Number of chairs at the table
- Objects in the jar
- Scoops of rice in the jar (Did everyone come up with the same exact count? Why or why not?)

Is it all right just to get close or do we need to find out exactly?

- Number of cartons of milks we need for the class
- Number of children on the playground
- Number of people coming to the party

3-6

Intermediate students will be determining which situations call for estimating and which require exact answers.

- How much your groceries will cost
- How long it will take you to get to the park
- How much it will cost to buy the red bike in the window
- How much tax you owe on the book you bought

6-8

Junior high students will be deciding how many decimal places to use in various situations.

The number of the decimal places needed depends on:

a) What you want to communicate

About what percent of students in our school wear glasses? We found 417 out of 1343. The calculator says 0.310498883. How do you report this?

b) How accurate your original measurements were

The odometer says you went 427.2 miles. You took 8 hours and 15 minutes. After you do some calculations, the calculator says 51.781818. How would you report the average speed? If the original measurements were in tenths, what degree of accuracy should you report?

c) What makes sense

Orange soda comes in six packs that are packed four to a carton. To make sure that all of the 247 students in the sixth grade will have a soda for the class picnic, how many cartons should we order? (Can you figure this out mentally?)

MEASUREMENT

Measurement is an important tool for learning about our physical environment.

❶ *It is essential for students to understand that*

W... n we measure, we attach a number to a quantity using a unit which is chosen according to the properties of the quantity to be measured.

A plowed field, the top of a picnic table, or a telephone pole has no obvious, visible numerical quantity. When we want to give the measurements of something, we need to find a way to attach numbers to it. The numbers we assign to it depend on the particular aspect we want to describe. For example, when we want to measure a telephone pole, we need to decide if we are interested in how tall it is, how heavy it is, how much wood is used, how long it would take to burn, or how much force it would take to knock it down. Whatever information we need requires a different unit of measure, none of which is an obvious characteristic of the pole itself.

K-3

Before young children learn to use units of measure to find out the size of a particular object(s), they engage in premeasuring activities in which they make direct comparisons. They stand back to back to determine who is taller. They match sticks to see which is longer. They lay objects on top of each other to determine which is bigger. The need for true measuring (i.e., using units rather than direct comparison) comes from the need to compare two or more objects which cannot be brought together. The use of a unit can help children determine not only which of two objects is bigger but also how much bigger one object is than the other. Children will come to a general understanding of what a unit is by working with a variety of nonstandard as well as standard units.

Which is longer, the table or the counter? What could we use to find out? Could we use string, paper clips, pencils, crayons, hands, rulers? Can we find out how much longer? Would it work if we measured the table with small paper clips and the counter with large paper clips? What if the pencils are various lengths? What happens if one child measures the table with his or her hand span and another child uses her hand span?

Which is bigger, the top of the table or the desk top? How can we find out? Can we use pieces of paper, books, circles, squares, triangles?

How many little blocks does it take to make one of the bigger blocks?

Put the following things in order by weight: the can of soup, the baseball, the rock, the cup, the can of tuna, the stapler. What if you wanted to know how much more one item weighed than another? What could you use to determine a number for the weight of each one?

3-6

In the intermediate grades, children should be put in situtations that require them to choose an appropriate means of measure according to the properties of the things they are measuring. Can they measure a particular object using units of length, square units, or cubic units?

Put 6 books in order by size. What are the different ways you could do that? By height? By weight? By thickness? By number of pages? By surface area of the covers? What different units would you use when measuring these different properties? Which would you measure with units of length such as centimeters? With units of mass such as grams? What kind of units would you use to measure surface area or volume?

Put these boxes in order by size. Estimate first and then measure to find out for sure. Is there more than one way to do that? Could you put them in order by the amount of wrapping paper it would take to cover them? By how much they can hold? How will you measure how much they hold? Would small wooden cubes help? Is there another way?

How can we find out how much wood is in the table tops? In these blocks of wood?

What kind of unit would you need to use to find out which drawer holds the most?

6-8

Older children are able to deal with more abstract properties and units as well as combinations of units such as miles per hour.

What do we need to consider to measure speed? What about the speed of a rocket, a bicycle, an earthworm, a car, the speed of lignt?

What do we consider when measuring a person's pulse? When measuring the amount of water that comes from the tap in one minute?

How can we measure the passing of time? How does the sundial measure time?

What does the report that "One inch of rain fell yesterday" mean? We often measure liquid in cubic units. Why do we measure the rainfall in linear units? How does the period of time affect this measurement?

How do we measure the windchill factor?

Compare the density of yogurt with the density of cottage cheese.

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❷ *It is essential for students to understand that*

Choosing an appropriate measuring tool requires considering the size of what is to be measured and the use of the measure.

K-3

Young children can be given opportunities to measure the same object in different ways so that they can compare their methods and determine which method is most efficient. They will find some choices to be more appropriate than others because of the size of the object(s).

Measure the length of the floor, a crayon, the record player, and a paintbrush using two-inch cubes, paper clips, toothpicks, and straws.
Would it be easier to measure some things if we strung some straws or paper clips together?
Why? Why don't we have inch rulers? Why do we put 12 inches together on one ruler? Why do some rulers have 36 inches on them?

3-6

Students in the intermediate grades can be asked to choose an appropriate tool. A variety of tools (both standard and nonstandard) should be made available: string, paper squares of various sizes, interlocking cubes, centimeter cubes, inch cubes, yard sticks, tape measures, rulers, scoops, measuring cups, scales, thermometers.

What would you use to measure the following items? Why?

The length and width of a piece of paper
The length of a fingernail
The length of the first base line on the baseball diamond
The area enclosed inside the baselines

The area of the table top
The area of the bottom of a cup

The capacity of various jars and bottles
The capacity of a bucket

When is it appropriate to use strides or footsteps to measure a room and when is is better to use standard measures? Which standard measures? What if you were trying to determine how to arrange the furniture? What if you were ordering carpeting? Would you need to know how carpet is purchased (e.g., square feet or square yards)? Why?

6-8

Students in the middle school will have more tools available to choose from. A variety of tools already familiar to them should be provided as well as protractors, clinometers, micrometers, and directional compasses.

How can you measure the following items?
The angle of a regular polygon
The direction the classroom is facing
The height of the building
The thickness of a piece of paper
The diameter of a straw

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❸ *It is essential for students to understand that*

Measurement is approximate because of the limitations of the ability to read a measuring instrument and the precision of the measuring instrument. The more accuracy you need, the smaller the unit you need.

K-3

Children need to consider how close is close enough to be accurate.

When you measure around the wastebasket with this string, how do you know if the string is too short, too long, or just right? How close is just right?

Find things in the room about as long as this string. The crayon. The cube.

How many paper clips long is the table? Did the measurement come out exactly even? If there is almost enough room for 24 paper clips, is the measure closer to 23 or 24? If it takes almost two paper clips to measure a crayon, is the measure one or two?

Are there some situations in which it is more important to be close than in others?

We need to get enough butcher paper from the office to make a sign for the assembly that includes all our names and some artwork from each of us. We need another piece of paper to make a mural for our bulletin board. How should we determine the amount of paper we need? How close do we need to be? Is it better to have too much or not enough or doesn't it matter?

3-6

Children in the intermediate grades can consider when it is appropriate to use small units, larger units formed by grouping together small units, or a combination of large and small units.

Do you want to measure the table in feet or inches or both? What about half or quarter inches?

The class is planning a Junior Olympics. The students need to make a decision about the standards for measuring for the broad jump. Does measuring to the nearest eighth of an inch make sense in this case? Why or why not?

What is the area of your handprint in square centimeters? How do you count the number of square centimeters when your handprint doesn't fit exactly on the squares? What about the extra spaces?

6-8

Older students need to consider when accuracy is important and when it is not important. When it is important, they need to decide what degree of accuracy is reasonable.

John says the distance between the school and the park is 6.2 miles. Anita says it is 6.17. What reasons for measuring might John and Anita have that would lead them to their different ways of reporting?

When do you need to be accurate to the nearest gram? Milligram? Second? Millisecond? Can you name something that we measure more accurately than we need to?

❹ *It is essential for students to understand that*

For an accurate drawing or model to be made, a constant ratio between the lengths of the model and the lengths of the real object must be maintained.

K-3

Young children first work with the idea of scale informally when they draw or build with blocks. As their awareness of proportion develops, they will become more and more aware of making it *look* right.

Situations which require them to consider relative size can be presented to them.

Here is a three dimensional model of our classroom made from a cardboard box. The windows are here; the doors are here. Can you make the tables and the shelves to fit?

3-6

In the intermediate grades, the students will take measurements and translate them in order to make smaller or larger representations.

Using this graph paper, draw a model of our classroom. What length do you want the side of each square to represent? Put in the windows, the doors, the shelves, and the tables.

Make a paper doll of yourself that is half as tall as you are. Try to make it as close to what you really look like as you can. What measurements do you need to make? Some parts of your body are curved, but your paper doll will be flat. Will that be a problem? How are you going to measure curved lines like your waist? Your head? What do you have to do to make the model look right?

6-8

The older students' work with scale will help develop a growing awareness of ratio and proportion. They will work with two and three dimensions, considering the way the parts relate to one another.

Which probably shows more detail? A map with a scale of 1 inch = 1 mile or a map with a scale of 1 inch = 10 miles? In order for both maps to show the same amount of detail, what would the physical characteristics of the two maps have to be?

Build a scale model of a building. Include the lot on which it is situated. What do you have to consider in deciding what scale to use?

GEOMETRY

The study of geometry builds an awareness of the physical world in which we live. We use geometry whenever we ask questions about size, shape, or position.

❶ *It is essential for students to understand that*

Real objects and abstract figures have one, two, and/or three dimensional features which can be examined, compared, and analyzed.

It is by experiencing and observing forms and shapes in the three dimensional world that students become aware of the properties and relationships of geometric figures. Looking at the various dimensions of physical objects is the starting point for learning about lines, surfaces, and volumes.

K-3

Young children first notice large obvious aspects of 3-dimensional objects. It is through handling and examining objects that they become more and more aware of finer differences and details.

Reach inside the bag and without looking, describe what you notice about the object in the bag. Is it round, flat, square, a cube, a ball, a can? Reach inside the other bags and see if you can find identical objects.

What can you do with these various blocks and containers? Do they roll, stack, fit together with no spaces between, with spaces between?

Can you find a line when you look at the table, the rug, the doorway? Where else do you see lines? How could you describe the edge of the round table? How could you describe the top of the table, the shape of the rug, the outline of the door? The table legs?

Place a cube on a paper and draw around the bottom face. What shape have you drawn? Cut this shape out and place it on each face of the cube. What do you discover? Get an empty matchbox. Draw around the bottom face. What shape is it? Tell why the matchbox is not a cube. Get a tin can. Draw around the bottom face. What shape have you drawn?

Take a sheet of paper. Roll it up. What shape have you made? What other things can you think of that are this shape?

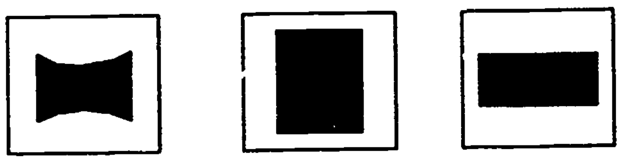
Children can become more aware of the differences between three dimensions and two dimensions by translating 3-dimensional objects to the two dimensions of paper and noticing what properties remain and which do not.

Have a friend help you trace around your body on large paper. What can you tell about yourself by looking at the paper model? Can you tell how tall you are? How far it is around your waist? Around your head? Around your wrist? How long your foot is?

3-6

Students in the intermediate grades can begin to notice how things look from different perspectives. What they see will depend on the direction from which they view things.

Using an overhead projector, project a variety of views of an object onto an overhead screen.



What three-dimensional figure could it be?

Draw what an ice cream cone would look like if viewed from the side, the top, sliced parallel to the base, sliced at an angle.

What shape would you get if you cut an oatmeal box down the side and spread it out flat? Try other containers. What shapes can you make? Can you make more than one shape with the same type of container?

How many different ways can you draw six squares connected together so that they would fold up into a cube?

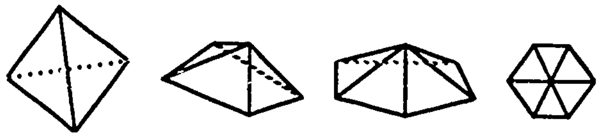
Build a structure with three or four blocks. Draw the side view, the front view, the top view. See if someone else can build your structure by looking at your drawings.

6-8

As older students observe and analyze two and three dimensional figures, and construct models of them, they can focus on the relationships between component parts and make generalizations about these relationships.

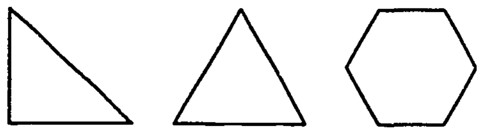
When you look at the corner of your classroom, what angles do you see? How many angles are there? Are all the corners alike? Are the corners you see at the ceiling the same as the corners you see at the floor? How do these represent planes and angles? Can you put three right angles together and not form a corner? What happens if you put four right angles together?

What happens when you tape three equilateral triangles together around the same point? Four? Five ? Six?



What shapes can be formed by "slicing a cube"? Can you figure out how to slice a cube so that you get a triangle, a rectangle, a pentagon, and a hexagon?

How many squares would you need to tape together to make a cube? How many of the following planar shapes could you tape together to make other geometric solids? Could you use a combination of shapes? Is there more than one way?



Look at a baseball, a basketball, a volleyball, a tennis ball, a soccer ball, and a football. How are the balls constructed? What pieces are used? How are they put together? If you took any of the balls apart at the seams, would the pieces lie flat?

Use uncooked rice, a hollow cylinder, and a cone with the same height and radius to find the ratio of their volumes.

❷ *It is essential for students to understand that*

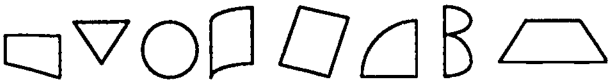
Geometric figures have specific attributes and properties by which they are identified, classified, and named.

Appropriate language for describing geometric figures and relationships can best be developed through hands on experiences that require students to identify similarities, differences and relationships. The investigation of properties progresses from whole figures or objects to attention to the components.

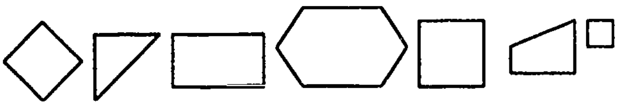
K-3

It is more important for children to recognize the attributes by which shapes are named than to learn the names of the shapes. Sorting a set of shapes or objects in several ways focuses the student's attention on various attributes and helps them realize that one particular shape can fit several categories. As children work with various geometric figures, they will become more and more discriminating. They will be able to sort by such attributes as the number of sides, the number of corners, the number of faces (flat surfaces), the relationship of the lengths of the sides, or whether the figure has straight lines or curves.

Their attention can be focused on the various attributes by questions such as the following:



Which shapes can you make with toothpicks only?
Which need yarn to complete?



The same shape can be included in many categories:

Find all the four-sided shapes
All the shapes with all sides the same length
All the shapes with four square corners and opposite sides parallel

In which categories would the square fit? Do any other shapes fit in more than one category?

3-6

Students in the intermediate grades will be able to consider more precise attributes of geometric figures using labels such as regular, irregular, symmetrical, equilaterial, equiangular, obtuse, acute, and oblique. After students have had many experiences recognizing attributes and classifying shapes by them, they will gradually come to the idea of a definition.

What information is enough to identify a figure?

Are all 3-sided figures triangles? Are all 4-sided figures rectangles? If a shape has 4 right corners, is it a rectangle?

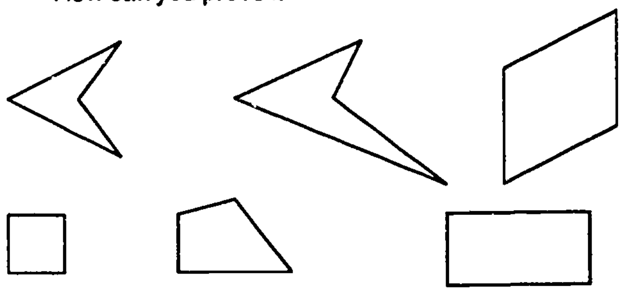
After which piece of the information that follows can you say for sure what figure is being described?

A figure has four sides. (What could it be?)
Not all of the sides are the same length.
(What have you eliminated?)
Two sides are parallel.
It has at least two square corners. (Could it be anything besides a rectangle? What else do you need to know to be sure?)

6-8

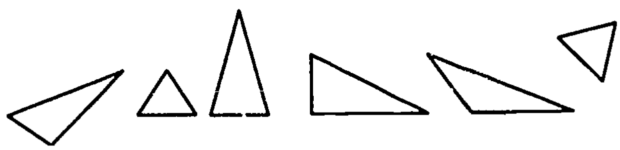
The older students begin to think in terms of what may be true or not true of a whole class of geometric figures rather than being concerned only with the specific figure with which they are working. Their classifications are applied to both two and three dimensional figures and become more refined.

Which quadrilaterals have at least one axis of symmetry? Which have a point of symmetry? How can you prove it?



Can you draw:
a rectangle with 4 sides of equal length?
a parallelogram with 4 right angles?
a parallelogram with 4 sides of equal length and no right angles?
a parallelogram with 4 sides of equal length and 4 right angles?

Which of these triangles is (are) acute? Obtuse? Right? Equilateral? Scalene? Isosceles?



Can you draw an obtuse equilateral triangle? An obtuse right triangle? An obtuse scalene triangle?

Describe a variety of polyhedra in terms of numbers of shapes and faces and number of edges and vertices. What pattern can you find for regular solids when you examine the number of faces, vertices, and edges?

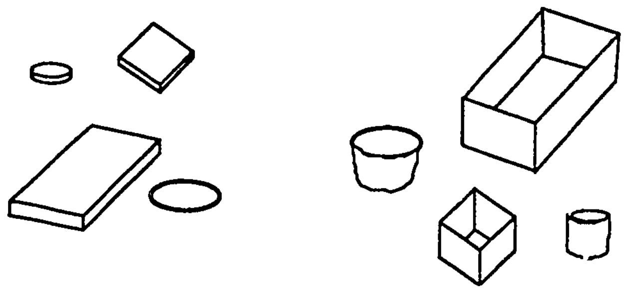
❸ *It is essential for students to understand that*

Geometric figures can be described in terms of their relationships with other figures. Important relationships include relative size, position, orientation, congruence, and similarity.

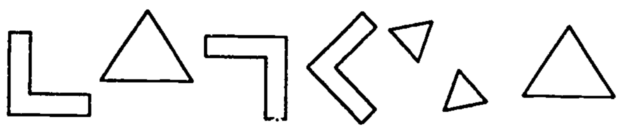
Sometimes shapes which appear to be different are different only in relative position or orientation.

K-3

Children in the primary grades will discover and confirm the relationships between figures by manipulating the shapes themselves.



Match the lids with the containers. How many different ways can you put the lid on a square jewelry box? A margarine tub? A shoe box?



Are the shapes above alike or different? How are they alike? How are they different? Are there any shapes that look the same but are different in size? Do any of them look like they will fit on top of each other? What if you turned them around or turned them over? Cut them out and see if you can make them fit exactly on one another.

3-6

Students in the intermediate grades will begin to make predictions about the relationships between and among shapes, but will still find using materials very useful for checking these predictions.

Which of these capital letters of the alphabet can be rotated 90 or 180 degrees and still look the same? Which can be flipped (top to bottom or side to side) and still look the same?

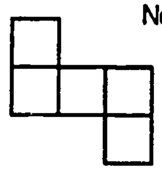
O B T L A M U I

Which capital letters, geometrical shapes, and other shapes can you cut from a piece of folded paper (a) folded in two, (b) folded in four? In how many ways can you put these shapes back into the holes from which they were cut?

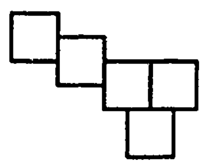
Students can create different shapes and then check to see if they are congruent through an activity like the following:

How many *different* shapes can you make by putting five squares together (full sides must be touching)?

This

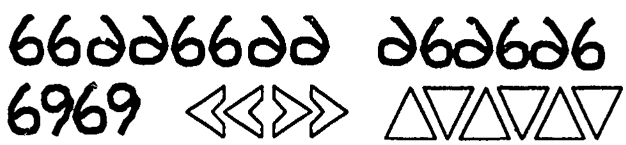


Not this



You can't count the shapes as different if they are congruent. Check your pieces. Do any of them fit on top of each other if you flip or rotate them?

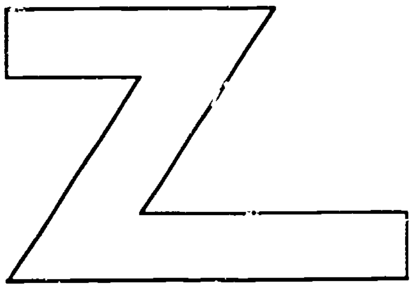
Make a border pattern by rotating, flipping, and sliding a single shape or figure. Analyze the border patterns made by other students and tell what transformations they used to create their patterns.



6-8

Older students will be able to make generalizations and analyze and describe various transformations.

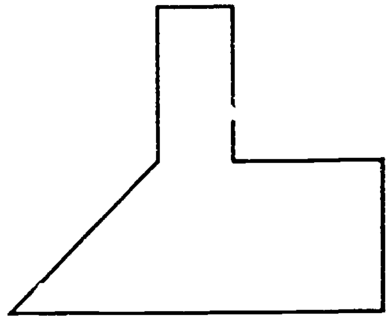
> Make a figure exactly like this but a different size. Measure the angles, the lengths, and the area. What stays the same? What changes? What are the relationships between the original measures and the measures of the new shape? What happens to the area of the figure when the lengths are doubled? Tripled? Halved?



> Which of the following sets of shapes are always similar? Why?
>
> Cubes, spheres, cones, rhombuses, parallelograms, quadrilaterals, regular hexagons, equilateral triangles, isosceles triangles

> Look at a globe. Are the lines of longitude parallel at the equator? Why do they then intersect at the poles? Why do latitudes lines *not* intersect?

> Draw this figure on a coordinate plane. Draw the same figure rotated 90 degrees; 180 degrees.



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4 *It is essential for students to understand that*

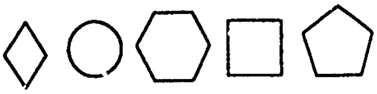
Geometric figures can be composed of or broken down into other geometric figures.

Analyzing geometric figures involves finding out how a certain shape can be made and finding out what can be made with shapes of particular kinds.

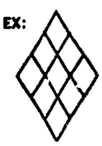
K-3

Students in the primary grades will explore shapes to see how they fit together and fill up space.

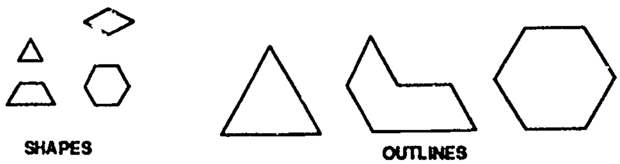
> Explore the following shapes. Which ones fit together and fill space with no spaces in between. Which ones don't?



EX:



> Using as many of the following shapes as you need, how many different ways can you fill in the following outline? Which way uses the most blocks? the least blocks?



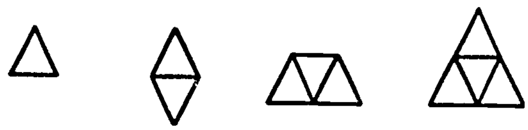
> What different shapes can you make if you cut a rectangle in half? What new shapes can you make with these halves? What different shapes can you make if you cut a circle in half? What new shapes can you make?

3-6

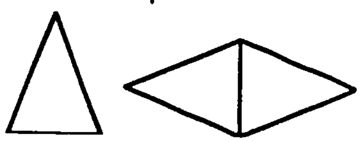
Intermediate students will be making note of the ways particular shapes combine or separate into other shapes. They will be becoming more systematic in their explorations.

Can you make a square using only triangles? A triangle, using only squares? A triangle, using squares and triangles?

How many different shapes can you make with all or some of these ten triangles?

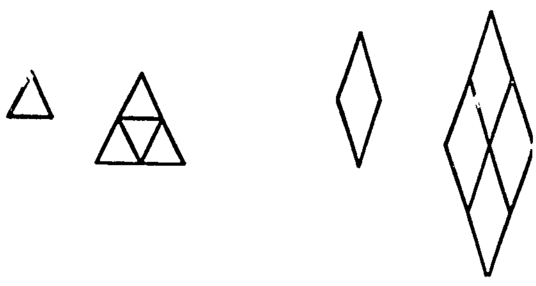


What about these triangles? Can you make the same or different shapes?



Which shapes and figures when fitted together fill space completely and which leave unfilled spaces? What combinations of figures can you use?

Which shapes can be fitted together to make similar but larger shapes?

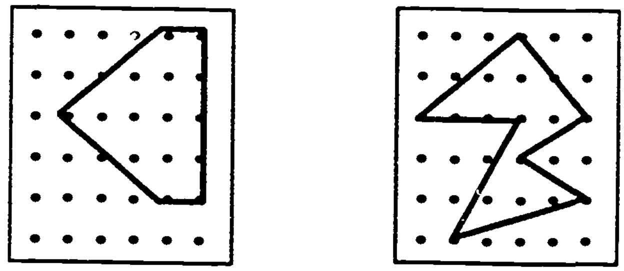


6-8

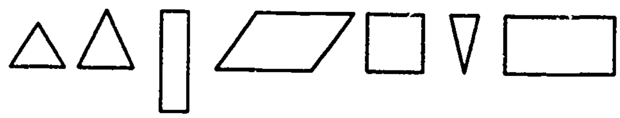
Students in the middle grades will be applying what they know about combining and partitioning figures in new situations.

Cut out a parallelogram. Can you cut it up and arrange the parts into a shape that makes it easy to determine the area? Can you do the same thing for a trapezoid? How does this help explain the area formula for these figures?

Here are some figures on geoboards. Can you use some familiar shapes to help you find the area?

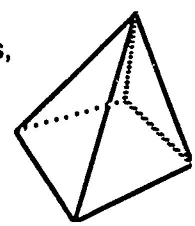


Put these shapes in order by area. Which shapes have the same area? How do you know?

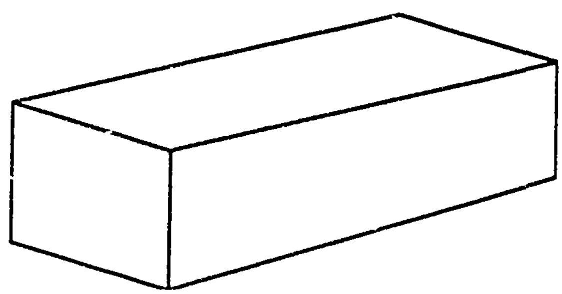


What is the smallest number of equilateral triangles you can use to make a closed three dimensional shape? What is the smallest number of squares you can use?

Using only figures like this,



Can you make a figure like this?



5 *It is essential for students to understand that*

Relationships within and between geometric figures can be revealed through measuring and looking for patterns. Constant relationships can be expressed as formulas.

There are many occasions when measuring discloses some of the properties of shapes and allows the development of a deeper understanding of otherwise unnoticed properties.

K-3

Primary students can measure common items in the room and look for relationships.

Measure one side of a table. Do you know how long any of the other sides of the table are? What if the table is a square? A rectangle? Measure one side of a book, one edge of the rug. What predictions can you make? Check your predictions.

If we had only five books, how could we tell how many books it would take to cover the table?

3-6

Measuring geometric figures and the growth of geometric figures can lead to discovering relationships. The exploration of these relationships will lead eventually to generalizations such as the formula for area, the definition of pi, and the relationships between area and perimeter.

Here is a piece of paper that it is 10 inches long and 5 inches wide. How many inch square pieces of paper does it take to cover it? What multiplication problem does this represent? How can we express that as a general rule? What is the relationship between the length and width of a rectangle and its area ?

Measure the circumferences and the diameters of all the circular things you can find. Is there a relationship?

Arrange 12 square tiles into all the possible shapes that have full faces touching and compare their perimeters. What is the largest (smallest) perimeter you can make? What if you could cut each tile in half?

6-8

Older students can look for more complex geometric relationships and use this information to form generalizations and to solve problems.

How many 1 cm squares does it take to cover a 2 cm cube? A 3 cm cube? A 4 cm cube? Is there a relationship? How many 1 cm cubes make up a 2 cm cube? A 3 cm cube? A 4 cm cube? Is there a pattern? Can you write a formula to determine the number of cubes it would take to fill any cubic box? What is the relationship between the surface area and the volume of a cube? How does that relate to the statement, "Parents should be advised that small children should not spend long periods of time outside in the extreme cold because their relative skin area is much larger than that of adults and, therefore, the heat loss would be greater."

What is the relationship between a 10 inch pizza and a 14 inch pizza. How much more pizza do you get when you buy a 14 inch pizza than when you buy a 10 inch pizza? If the 10 inch pizza costs $5.75, what would be a fair price for the 14 inch pizza?

What is the sum of the interior angles in a triangle? How can you prove it? What is the sum of the interior angles of a rectangle? Any quadrilateral? Can you show how to find the sum of the interior angles of any polygon?

Suppose you had to manufacture containers. What shapes would be most economical (i.e., would hold the most for a given amount of material)? Why are so many containers cylindrical in shape?

PATTERNS AND FUNCTIONS

Looking for patterns helps bring order, cohesion, and predictability to seemingly unorganized situations. The recognition of patterns and functional relationships is a powerful problem-solving tool that enables one to simplify otherwise unmanageable tasks and to make generalizations beyond the information directly available.

1 *It is essential for students to understand that*

Identifying a rule that could have been used to generate a pattern enables one to extend that pattern indefinitely.

The extension of a pattern is often based on a rule that is assumed from a few examples or a fragment of a sequence rather than based on information that is known with certainty.

K-3

The patterns that young children experience will be those that emerge from changes in motion, color, position, design, and quantity. The focus will be on copying, extending, and creating patterns of various levels of complexity using a variety of materials.

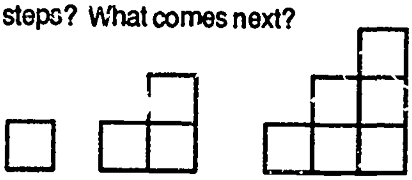
Join in when you know the patt[illegible]ap, clap, clap, snap, clap, clap,...." Joi[illegible]s pattern: "Snap, clap, clap, stamp, snap, [illegible]ap, stamp,...." How is this pattern like the first one? How is it different?

The sticks are arranged like this:

What comes next? How do you know?
What pattern can *you* make with the sticks?

Paste a sample of wallpaper in the middle of a sheet of paper. Use crayons to extend the pattern in all directions.

Build a staircase with the blocks. The first step has one block, the second step has two blocks, etc. How many blocks do you need to build seven steps? What comes next?



Circle the number of each step on the 0-99 chart. What do you notice?

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29

Children should confront situations where they need to consider the assumptions that are made when a pattern is extended.

I am thinking of a pattern. The pattern begins like this:

Red block, blue block, red block,....

What do you think comes next? The next block in my pattern is yellow. Did your idea work? Now what do you think comes next? Who has a different idea? What else could work? What else?

3-6

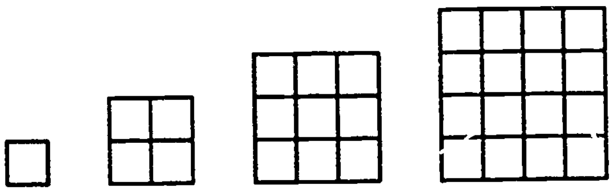
In the intermediate years, students will observe the number patterns that emerge from a variety of situations and explore the underlying structure of the number system. They will become familiar with commonly encountered number sequences such as the odd numbers, sequences of multiples, the square numbers, etc. The patterns students encounter will come from working with materials or from using the arithmetical operations with which they are familiar.

Start with the numbers 1,2,3,4,5,....
What is the pattern of the sequence if you
- a) Double each number?
- b) Halve each number?
- c) Add each number to the one before it?

What is the sequence of numbers on the 0-99 chart which is generated when you circle the multiples of
- four
- five
- two

What can you notice? Which multiples form columns? Diagonals? Why? What would happen if we used an 8 x 8 matrix instead of a 10 x 10? Would the same multiples form columns? Diagonals? If not, which ones?

Use squares to build larger squares.



1x1 2x2 3x3 4x4

How many squares does it take to build each square? What is the pattern? Can you predict the number of squares for a 9 x 9 square?

It is important that students have experiences that help them realize that a pattern can be extended in more than one way, depending on the rule that is assumed to govern the extension.

1	2
2	5
3	?

Explain why 8 could be the next number in the table. What would come after 8? Explain why 10 could work. What would be next? Can you find other rules you could use to add to the table?

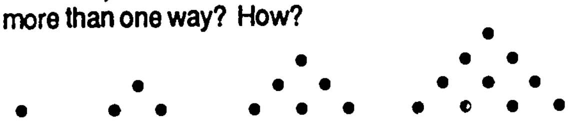
6-8

In addition to generating number sequences of their own, the students at this level can analyze number sequences to look for relationships.

Describe what is happening in the following patterns? Can you predict the next number?

2, 3, 5, 7, 11, 13,....
1, 4, 9, 16, 25,....
1, 2, 4, 8, 16,....
25, 23, 21, 19,....
1, 1, 2, 3, 5, 8, 13, 21,....

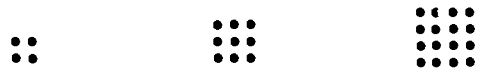
Could any of these sequences be extended in more than one way? How?



1 3 6 10

What are the next 6 triangular numbers? How many total dots will there be with a side of 10 dots? What comes next?

How do the squares increase? What is the pattern?



$2^2 = 1^2 + (2+1)$ $3^2 = 2^2 + (3+2)$ $4^2 = 3^2 + (4+3)$

What other patterns can you notice?

Use your calculator to explore the following patterns:

1 x 1 = 1 11 x 11 = 121 111 x 111 = 12321
Can you predict the middle number for 1111111 x 1111111?

4 + 99 7 + 99 13 + 99 24 + 99 Make predictions for other numbers. Can you find the answer without using your calculator?

5 x 101 9 x 101 11 x 101 15 x 101
23 x 101 Can you solve for other numbers without using your calculator?

Why do these calculations produce patterns?

❷ *It is essential for students to understand that*

When there is a functional relationship between two quantities, the value of the first quantity determines the corresponding value of the second.

The study of functions enables students to see relationships and to make predictions based on those relationships.

K-3

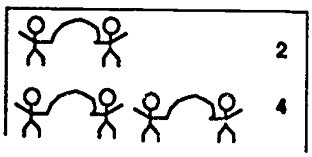
In the beginning students will look at a particular situation and determine the pairs of related numbers that arise from it. They will then look for a pattern in order to predict the numbers that come next in the list.

It takes two children to turn a jump rope. How many children would it take if we wanted turners for 15 jump ropes (with each child turning only one rope)?

Simplifying and organizing information and acting out the situation helps solve problems.

What is a simpler problem for us to solve? Can we figure it out for two jump ropes? Three jump ropes? Does that help us figure out how many we need for 15?

ROPES	CHILDREN
1	2
2	4
3	



What if you had 50 jump ropes? Could you figure out how many children you would need? How could you figure it out?

3-6

In the intermediate grades the students will begin working with functions by simplifying problems, by using tables, and by drawing and reading graphs.

How can we simplify the following problem and organize the information to help us solve it?

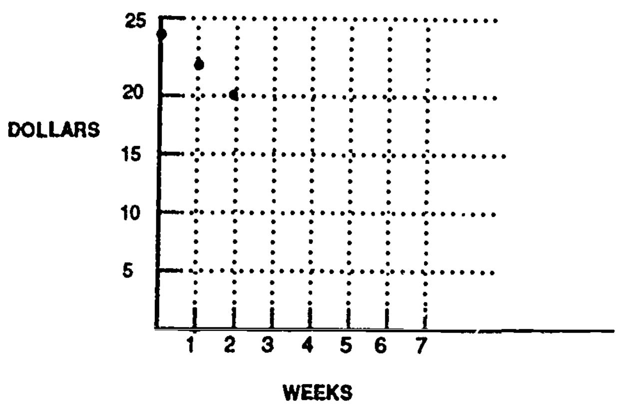
If we tear a sheet of paper in half, then tear each half in half, and continue doing this, how many sheets of paper will we have after 16 tears?

Tears	Sheets
1	2
2	•
3	•
•	•
•	•
•	•

How many pieces of paper would we have after one tear? Two tears? Three tears? Do you see a pattern? Could we figure out the number of pieces of paper no matter how many tears are made?

What information can we get by using a coordinate graph to help us solve the following problem?

In your bank you have $24. Every weekend when you finish your chores, your mom or dad gives you $3.00. On Saturday, when you go to the movies, you spend $5.00 for your ticket and food. If you don't spend any other money during the week, how long will it be until you're broke? If your mom or dad lets you borrow, how much will you owe in 6 months?



6-8

Students in middle school grades will be able to solve a variety of complex problems through the use of functions. The functional relationship can be expressed through a table, a set of points on a coordinate graph, or an algebraic formula.

> How many different double-dip cones can we make with 5 flavors of ice cream? What if we have 25 flavors? Will it be 5 times as many? Why or why not? Can you figure out how many cones we can make no matter how many flavors we have? Can you write an equation that describes this relationship?
>
> Maria leaves at 10 A.M. on her bicycle for her friend LaShana's house, which is 20 miles away. She arrives at 1 P.M. After spending the night, the next day she leaves at 10 A.M. and arrives home at 1 P.M. Can you draw a graph to show a point on the road home that she passes at exactly the same time as she did the day before?
>
> The Automobile Club has published this recommended schedule for touring Redwood Park:

9:00 AM	Leave campsite
10:00 AM	Stop at Fishing Bridge
10:20 AM	Depart for Canyon Path
11:00 AM	Arrive at Canyon Path and start hiking
12 noon	Return to beginning of Canyon Path and have lunch
12:30 PM	Leave for Green Gulch
1:30 PM	Arrive at Green Gulch and start nature tour
3:00 PM	Finish nature tour

> The Joneses had a flat tire and were delayed by an hour and a half in leaving the campsite. If they follow the same schedule at the same speed, what will their schedule look like?

---

❸ *It is essential for students to understand that*

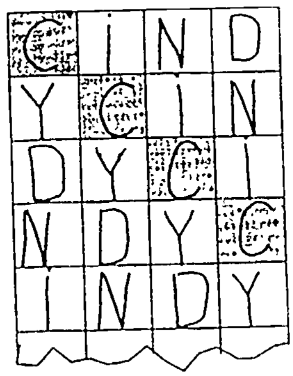
The same patterns can emerge from a variety of settings.

It is valuable for students to begin seeing the relationships among different occurrences of the same underlying patterns.

K-3

Students in the primary grades can translate patterns from one mode to another.

> Join in when you know the pattern.
>
> Jump, jump, clap, jump, jump, clap,....
>
> What is another way to show that pattern?
>
> Circle, circle, square, circle, circle, square,....
>
> Another way?
>
> Tall, tall, short, tall, tall, short,....
>
> Another way?
>
> A, A, B, A, A, B....



> Write your first name over and over in a 4 x 10 matrix. Color the first letter of your name. What pattern do you see? What other names have the same pattern. Why? If you put the counting numbers in the matrix and colored in the same squares as your name pattern, what numbers are colored in? Predict what numbers would be colored in if you continued the pattern. What if you colored in the last letter of your name?

Counting by two's gives the pattern 2, 4, 6, 8, 10....

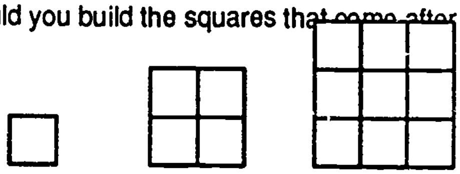
What other situtations can we find that give this same pattern?

3-6

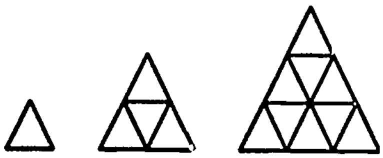
Students in the intermediate grades should begin to look at the various patterns they are discovering and notice when the same pattern occurs in different situations.

One of the patterns that will come up many times is evident in the following examples.

How would you build the squares that come after these?



How would you build the triangles that come after these?



What is the sum of the first two consecutive odd numbers? The first three? Four? Five? Can you predict what the sum of the first 8 consecutive odd numbers will be?

Why are these patterns the same? Why is this pattern called "square numbers"?

6-8

The older students should also be looking for the same pattern occurring in many different situations and contemplating the underlying reasons for the same pattern emerging.

For example:

If everyone in our classroom were to shake hands with everyone else in the room, how many handshakes would there be?

When you build the triangular numbers, how many counters do you need for each step?

You are setting up a tennis tournament. If you want every person to play every other person once, how many games of tennis will it take?

What pattern will help you find the sum of all natural numbers from 1 to 100?

Why do all these situations produce the same pattern?

STATISTICS AND PROBABILITY

Statistics allow us to make a summary of what we know of the world and to make inferences about what we don't know. Probability is used to indicate how sure we are about a prediction.

1 *It is essential for students to understand that*

When there is no direct observation that will answer a particular question, it is often possible to gather data which can be used to answer the question. Working with the data often generates additional questions.

The processes of questioning and predicting are cyclical. We begin with a question and some predictions about what we expect the data to reveal. Data are then collected, organized, displayed, and analyzed. This leads to more refined questions and new predictions.

K-3

The questions posed for or by young children should come from their interest in their immediate environment.

Do more children in our room wear shoes that tie or shoes that fasten with velcro?
How do the children in our class get to school?
How many people are in your family?
Do you live in a house, an apartment, or a mobile home?
What kind of pet do you have?

After the data have been gathered, organized, and displayed, various questions can be asked that help the children focus on the relationships revealed by the graph, chart, or table.

Descriptive questions
How many children have dogs?
How many children have goldfish?

Comparative questions
Are there more children who have dogs or more children who have cats?

It is important to get children thinking beyond the specific information presented on the graph to other ideas or implications.

We did a graph last week to answer the question, "Do you live in a house, an apartment or a mobile home?" Does that have anything to do with whether or not you have a pet or what type of pet you own? Do the people who have goldfish live in a house or an apartment? Do more people who live in houses have dogs than people who live in apartments?

3-6

Children in the intermediate grades can consider questions that concern their school or community but still are of personal interest to them. They can gather information in a variety of ways:

Taking surveys
What beverage do you drink with lunch?

Making observations
Watching to see what beverage the students are drinking.

Gathering real data
Counting the various beverage containers in the trash.

They can find answers to complex questions such as, "Do we need a safety patrol in front of our school?" by asking many related questions:

What percentage of our students walk to school?
How many come on the bus?
How many come in cars?
Are there certain periods of time when the traffic is heaviest?
Have there been any accidents or near accidents?

The questions they investigate will often be those that are raised during social studies and lead them to understanding more about their society.

How many of you were born in California? If you were born in this state, is it likely that your parents were too? What about your grandparents? Would the same be true if we asked students from another state?

6-8

By middle school, students are ready to evaluate the kinds of questions they wish to investigate to determine which will help them find out what they need to know and which will not be as helpful.

What kinds of questions would be helpful in determining why the traffic is heaviest at 8:00 A.M.? What do we need to know? Is the traffic heavy because of children being dropped off for school, commuters, delivery trucks, or something we haven't anticipated? Can we tell the difference between commuters and grocery shoppers? Would knowing the time the stores open help? How can we find out?

They also need to examine the kinds of questions that need interpretation before they are answerable.

Consider the question "What is the best brand of ice cream?" What does the term "best" mean? Does it mean best according to the opinion of the class? Or does it mean best selling in the town? Or the best deal for the money? Or best for your health in terms of being lowest in cholesterol?

---

❷ *It is essential for students to understand that*

You can gather data about every member of a group, or you can use a representative sample from that group.

When it is not practical to examine each member of a large population or group, the technique of sampling can be used. It is important for students to recognize both the usefulness of sampling, which can result in fairly accurate and complete information, as well as the limits of sampling and the care that must be taken to insure reasonable results.

K-3

A beginning notion of the idea of sampling comes from the daily life of the young child. After stirring the pot, one spoonful of soup is enough to tell you if the whole pot of soup is too salty or not; with some kinds of cereal, you take one handful out of the box and you know what the rest of the box contains; with other kinds (e.g. granola), it may take several handfuls to inventory all the ingredients.

Children's attention can be focused on the technique of sampling by posing questions such as the following:

Can you tell what is in this bag if you don't get to look inside and see everything in it? What if you could reach in and pull a few things out? Can you tell what is in the bag if you reached in several times and each time pulled out a red crayon?

In order to help children learn not to overgeneralize, it is important to present situations where the results of sampling are not reliable or consistent. Crayons of many different colors may be in a bag, even though the first two crayons taken out are red.

3-6

In the intermediate grades, the student's attention can be focused on the issue, "When is the sample large enough? "

> If five children are asked what their favorite color is and three of them say green, is that enough information to conclude that most children in the class like green? Would the fact that the room was decorated for St. Patrick's Day possibly affect the results?

> How many pages in a book need to be analyzed to determine the average number of words per page in the book?

> How many times would you have to measure the outside temperature before you could say you knew the average temperature in your town? Would it matter what time of day you measured?

Students should also begin to consider to what extent it is appropriate to generalize. A sample must be representative of a particular population.

> If our fifth grade class picks a particular movie as the favorite, does that mean we can assume all the fifth grade classes in the school like that movie best, too, or is there some special reason why our class likes that movie? What about first graders? What about the whole town?

> If I find out the average number of words per page in a book, does that mean I know the average number for any other book?

> What would be a fair way to choose the winner of a door prize?

> What if I only have time to interview five fifth graders about their favorite food? What is a fair way to insure that everyone has an equal opportunity to be interviewed?

6-8

Older students can go beyond personal experiences and begin to question the results of statistics reported in the media. They need to recognize that they must know something about the sample used before they can consider the implications of statistical information.

> If you wanted to predict the outcome of a vote for mayor, how would you gather information? What could you learn from asking students not old enough to vote? Would it matter if the adults you asked were registered voters? How many would you need to ask? Why is it that you can never predict with 100% confidence?

They need to consider when using a sample is appropriate.

> Would a sample be adequate for ordering class T-shirts of different sizes? Would a sample be adequate for ordering the kinds of pizza for a class picnic?"

❸ *It is essential for students to understand that*

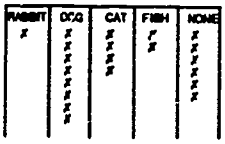
Data can be organized, represented, and summarized in a variety of ways.

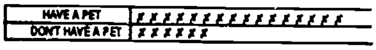
Graphs, tables, and charts are used to present information in organized ways that allow certain relationships to become apparent. A large set of data can be summarized or characterized through the use of measures of central tendency.

K-3

Children in the primary grades can explore the results of using different categorizations when presenting information.

The data gathered in response to the question "What pets do we have?" can be displayed in the following ways.





What kind of information do you get from the first graph that is not available from the second graph? What information is more obvious from the second graph?

Primary students' first experiences with a measure of central tendency will be that of the mode. They will generally summarize the information gathered in terms of what occurred most frequently.

More people in our room have dogs than any other kind of pet.

3-6

Students in the intermediate grades can explore the results of displaying information in a variety of ways and begin to consider the kinds of decisions that need to be made when constructing a graph.

In response to the question," How long does it take you to get to school?" the children in the class reported times ranging from six minutes to one hour and four minutes. The teacher displayed the information on a frequency table:

1		11	1	21		31		41		51		61	
2		12		22	1	32	1	42		52		62	
3		13	2	23	1	33		43		53		63	
4		14	1	24		34	1	44		54		64	1
5		15	3	25	2	35		45	1	55		65	
6	1	16		26		36		46		56		66	
7		17	1	27	1	37	1	47		57		67	
8	1	18	1	28	1	38	1	48		58		68	
9		19		29	1	39	1	49		59		69	
10		20		30		40		50	1	60		70	

How can we display this information so that we can easily see about how long most students take to get to school? How big should the intervals be? How would the information look if we used five minute intervals? Ten minute intervals? 30 minute intervals? Which would produce a flat graph? Which would show best where the most common times cluster together?

After the graph has been constructed, students should be able to answer the following questions.

What is the most common time (the mode) that it takes students in our class to get to school?

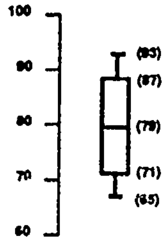
If we added up all the times and divided them by the number of students, we could find the average time (the mean) it takes students in our class to get to school. How does that time compare to the most common time?

6-8

Students in the middle school should explore various ways of displaying information and consider the appropriate uses of mean, median, and mode as ways of summarizing information.

After collecting data about pulse rate, represent the data on a

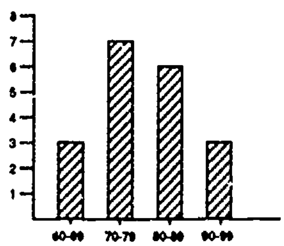
box and whisker graph



stem and leaf graph

6	5 7 9
7	1 1 1 2 3 7 9
8	2 2 3 3 7 9
9	0 1 3

histogram



Which graph gives a good represention of the the group overview? Which loses the individual data? Which highlights the mode? The median?

❹ *It is essential for students to understand that*

There are many reasons why an inference made from a set of data can be invalid.

The problem may be in the data itself (if the sample was not appropriate or if some error was made in the collection or measurement of the data) or the error may be in the logical reasoning used to make inferences.

K-3

Young children can look at their everyday experiences and can give possible reasons why events may be occurring.

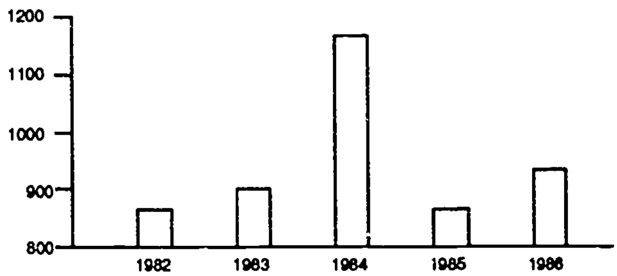
> The graph we made shows that most of the children came by car today. Usually most of us walk. Do you think more people got rides today than usual because it was raining very hard?

> A survey shows that the number of 3rd graders who bought spaghetti in the school cafeteria on Monday was very small. Does this mean 3rd graders don't like spaghetti or could there be another reason? Were they on a class trip that day or was some other special event going on?

3-6

Students in the intermediate grades can begin to interpret data. They should discuss which interpretations are plausible, which are possible, and which are impossible. They can also consider how some interpretations of the data may be misleading.

> In our school we have found a high correlation between the number of siblings and distance lived from school. What are some reasons this could be true? Could it have something to do with the type of housing available near the school? Can you think of any other reasons?



> What is your initial reaction to the numbers of traffic accidents in different years shown in the graph? What is misleading about the graph?

6-8

The older students should continue to consider various interpretations of data and determine which are plausible, which are possible, and which are impossible.

> Bring in newspaper articles containing conclusions based on data. Which conclusions are plausible? Possible? Impossible? What other questions would you ask in order to obtain data that would support or negate your reasoning?

5 *It is essential for students to understand that*

There are ways to find out why some outcomes are more likely than others.

Uncertainty is a part of our daily lives. Many of our daily decisions are based on our informal predictions concerning the likelihood or unlikelihood of certain events occurring. Should we bring a raincoat to school today? What will the traffic be like going to the game? What time should we leave the house to get to the game on time? We know with certainty that some things will happen and that some things will never happen, and there are many things that, with varying degrees of likelihood, may or may not happen. We can help students examine these ideas by looking at daily events to see patterns and trends and by setting up experiments which allow us to analyze probabilities.

K-3

Children in the primary grades should have opportunities to notice which things are sure to happen, which are sure not to happen, and which we can't be sure about.

> Can you be sure you will eat an egg tomorrow? That the sun will come up? That your baby sister will cry? Why or why not?

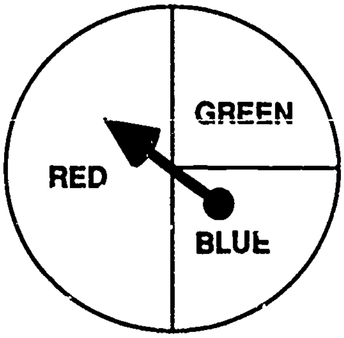
They can observe what happens during certain activities and notice that some events occur more often than others.

> I put three blue blocks in the bag. Can you be sure what color block I will pull out? I put one green block in with the blue. Now can you be sure which color I will get? Why?
>
> Toss a coin several times. What are the possiblities? Do heads or tails come up more often?
>
> Roll a die. What are the probable outcomes? Does any number come up more frequently than any other? Roll two dice. Add the numbers on the 2 dice and keep track of the sums. What sum comes up most often?
>
> What happens if you toss five lima beans which are painted on one side and left white on the other? Do you get all painted ones more or fewer times than you get some painted and some white?

3-6

Students in the intermediate grades should continue to observe what is happening in the world around them and also to begin to collect and record data that help them look at the frequency of events and begin to make some predictions.

> Estimate the likelihood of future events on this scale.
>
> |________|________|________|________|
>
> 0% 25% 50% 75% 100%
>
> All our classmates will be here tomorrow.
> A live elephant will visit our classroom this year.
> It will rain tomorrow.
> It will rain this year.
> I will drink some milk today.
>
> Each student in the class is to roll a die 6 times and record what comes up. When we look at the class results, how many ones? How many sixes? How many threes? Why? What would happen if we did this experiment again tomorrow? Would the results be the same or different?



If you spin this spinner 10 times, which color do you think will come up most? Why? Try it and see. Did red come up most times for everyone in the class? Why or why not? Can you change the spinner so that red will not come up more than the other colors?

What are all the possible outcomes if you roll two dice and add the numbers shown on the dice? Use a list and a table to show all the possibilities.

6-8

The junior high students can be more systematic in gathering information to make predictions. They should analyze situations and find all the possible outcomes and determine which are equally likely.

If we toss two coins, a penny and a nickel, what are the possible outcomes in terms of heads and tails? What about tossing three coins, a penny, a nickel, and a dime? What are the possibilities? What could happen most often? Why? Show the possibilities in an organized way.

When you do an actual experiment, does it always turn out the way you would predict? Why or why not? Is it different if the whole class contributes data? How? Why? How could you explain getting "heads" six times in a row?

If you had a chance to draw a lucky ticket, would you rather draw a single ticket from a box of ten tickets or ten tickets from a box of 100 tickets (putting back the ticket after each draw and shaking up the box before the next draw)? Would you prefer drawing one ticket from ten or 20 tickets from the box of 100? Why? Would you rather have the opportunity to draw one ticket from one box of ten or the opportunity to draw one ticket from each of ten boxes of one hundred tickets each?

The cycle for the traffic light on Main Street is green for 90 seconds, yellow for 3 seconds, and red for 12 seconds. What is the probability of having to stop at the traffic light?

LOGIC

The development of logical reasoning is closely related to intellectual and linguistic development. Children do not always reason in the same way as adults, and they should not be expected to deal with logical concepts they cannot understand. If presented with such concepts, they will repeat or respond to sentence patterns and structures in a rote manner and perceive certain activities to be "tricks." As much as possible, students should be encouraged to explain their thinking in their own language.

❶ *It is essential for students to understand that*

Classifying and sorting depends on the identification of a specific attribute or attributes.

Before sorting can take place, a particular attribute or attributes must be identified. It must then be determined if any particular item to be sorted has the particular attribute(s). Sorting can be more or less complex, depending on the nature of the attribute, the level of discrimination or abstraction required to recognize the attribute(s), and the clarity of the definition of the attribute(s). The sorting may depend on the simultaneous occurrence of two or more attributes (such as "must be a reptile" and "more than 10 cm long"), or the occurrence of either one or the other of any particular attributes (such as "is carniverous" or "has horns"). The nonoccurrence of an attribute may also be a basis for classifying or sorting.

K-3

Children in the primary grades will deal with the sorting of items physically present. They are, usually, not able to generalize beyond what they see in front of them. They may have difficulty dealing with abstract ideas such as "all the animals," or "all the red squares in the world," but will be able to consider relationships between the particular items being dealt with. An important idea for them to consider is that any particular group of objects can be sorted in more than one way.

Look at all the keys in this pile. Which keys can go together?
Can you sort them by size? By color? How else can you sort them? What's another idea?
Another idea?

Look at these blocks. Can you tell why they have been sorted into these groups?

3-6

As students enter the intermediate grades, they will be able to sort by more abstract attributes than primary children can. They will deal with problems that require more and more specific definitions of attributes. They will sort in situations that require recognizing more than one attribute at a time, and they will deal with situations that require them to consider attributes that are not present.

Are you a blond, brunette, or redhead? Are you sure? Could you put everyone in the class in one of those categories? Does each one of you agree where you belong?

Sort these pictures of animals according to the following categories: (a) live in water and eat plants only; (b) lay eggs and do not fly; (c) have feathers or scales.

Choose a set of numbers and sort them into these categories: (a) primes; (b) multiples of 3 and 8; (c) an odd number or a multiple of 3. Can a number belong to more than one of these categories?

6-8

The definition (or lack of clear definition) of attributes is an important idea for junior high students to consider. They will consider whole classes rather than only dealing with items in their immediate environment.

Is everything on earth either animal, vegetable, or mineral?

How many species of butterflies are there? What is it that distinguishes one species from another?

How would you classify different kinds of numbers such as even, odd, primes, composites, rationals, negative, factors, multiples? Can a number belong to more than one of these classifications?

---

❷ *It is essential for students to understand that*

Statements made precisely about what is known allow conjectures and conclusions to be examined logically.

If and only if you are precise in what you say is it possible to test the validity of your statement.

K-3

In the early grades, important to begin to use accurately such words as **all, some, none, every, or, and,** and **many** in speaking and writing.

Can you tell me something that is true about all of the buttons in this pile? Can you tell me something that is true about some of these buttons? Can you find any red buttons in the pile of buttons? Any small buttons? Are there any buttons in the pile that are both red and small? Give me the buttons that are red or small. Put the buttons in the box if they are not red and not small.

It is time to line up to go outside for recess. You may line up if you are wearing a belt. Line up if you have buttons on your clothing or if you have pockets.

You may go to the library if you have your library book at school and have finished your work.

Line up and number off. If your number is even, sit down. Would you stand next to your friend if you want her in your group? Why or why not?

Here is a box of assorted shapes. Are all the red shapes triangles? Are some of the red shapes triangles? Is none of the red shapes a triangle?

3-6

Continuing to use language accurately, more difficult concepts such as **if ... then** can be added to students' oral and written expression.

Fill in the blanks with **all, some,** or **none** to make the following statements true for our class.

_________ of the students are boys.
_________ of the students are either boys or girls.
_________ have two eyes.
_________ have one or two arms.
_________ have naturally blue hair.
_________ do not have blond hair.
_________ triangles have three sides.
_________ of the factors of 12 are 2, 3, and 4.

We have a rule for our class: If it is raining at recess, we stay inside; if it is not raining at recess, we go outside. We have another rule: If we want to finish our work, we do not go outside for recess. On Tuesday, we stayed inside for recess. Do you know whether it was raining on Tuesday?

6-8

The middle school students should begin to grapple with the idea of **if and only if**, even if that language is not used. They should also encounter more negatives.

A polygon with six sides is called a hexagon. Are all hexagons six-sided? Are all six-sided polygons hexagons?

A rectangle with equal sides is a square. Are all rectangles squares? Are all squares rectangles? Are all equal-sided figures squares?

A natural number is prime if and only if it has exactly two factors (which are natural numbers), namely, 1 and itself. What does that mean? From this definition, do you think that 1 is prime?

It is not the case that John did not get home. Did John get home?

Use an example to explain the difference between these two situations:

If it rains, then I'm going to take my umbrella.
I'm going to take my umbrella if and only if it rains.

❸ *It is essential for students to understand that*

Based on certain premises, a series of logical arguments can be used to reach a valid conclusion.

K-3

Children's early informal experiences help in their development of logic as they test their expectations against reality. They will begin to see that they can expect certain results all the time, but in other situations the results are unpredictable.

Look at these pictures: a bat, a bird, a mouse, a giraffe, a butterfly. I am thinking of one of these creatures. It can fly. Do you know for sure which one I am thinking of? Do you know what it might be? Do you know for sure what it is not?

Here are three marbles: two red ones and one white one. I hid all three marbles and now I am showing you the white one. I have another one of the three marbles in my other hand. Do you know what color it is? How do you know?

One of the bags contain triangles. Another bag contains circles. A third bag contains rectangles. If you reach in one bag and find out what it contains, what do you know about the other bags? What don't you know?

3-6

The students in the intermediate grades can consider how to show whether something is definitely false. They can begin to verbalize the steps in their reasoning to see if each step follows from the one before.

Suppose we have two positive integers and we know that:

1. The sum is odd.
2. Exactly one number is prime.
3. One number is larger than 15.

Can we tell if each of the following is definitely true, possibly true, definitely false?

a. Both numbers are odd.
b. The prime number is odd.
c. The sum is at least 19.

Suppose I tell you that at least 3/4 of the students eat corn and that at least 2/3 eat carrots. Can you conclude that (a) every student eats corn or carrots? (b) that at least one student eats neither? Why? How are the two questions related?

You have your choice of four pets. You are told that collies eat more than poodles, boxers eat more than collies, and boxers eat less than dalmatians. Which pet would eat the least amount of food?

Consider these sentences:
1. Five and six are even numbers.
2. Five is not an even number.

Is sentence 1 true or false? Is sentence 2 true or false? Can a sentence be both true and false?

6-8 [illegible]

In the middle school or junior high, students will deal with longer and sometimes more complex chains of reasoning. They will need to recognize that assumptions can be either true or false. They should begin to question the validity of the initial premises when examining or formulating a sequence of arguments. Students should also be given opportunities to examine and consider the distinction between necessary and sufficient conditions.

Given a set of parallel lines and a transversal, if you know an angle is 60 degrees, why do you know the measures of the other angles?

Mr. Derrin, Mrs. Scali, Mr. Kim and Ms. Charles are teachers in the middle school. No teacher teaches more than one grade level. Algebra and 6th, 7th, and 8th grade math are taught. Mr. Kim teaches a grade that is a prime number. Ms. Charles does not teach a grade level that has 2 as a factor. The 6th grade math teacher is a sister of the algebra teacher.

1) What is the first statement that helps you identify the 7th grade teacher?
2) Of what importance is the statement about the relationship of sisters?
3) How do you identify the algebra teacher?
4) What strategy is most useful in solving this situation?

Consider the following statements:
1) 2, 13, 16 are even numbers.
2) 27, 36, 45 are multiples of nine.
3) $2^2 = 4$, $2^3 = 16$, $2^4 = 32$
4) $\sqrt{144} = 12$, $\sqrt{49} \leq 7$, $\sqrt{5} < 2$

Why are some of the above statements false?

ALGEBRA

We can think of algebra as generalized arithmetic, so the understanding of algebra will grow out of the student's understanding of arithmetic.

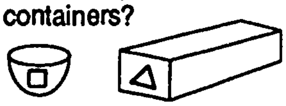
1 *It is essential for students to understand that*

A set of numerical relationships can be expressed through the use of variables.

K-3

Young children can use what they know about arithmetic and number sentences to explore the idea that various pairs of numbers can be used to make a number sentence with variables true.

How many different ways can you put ten counters into these two containers?

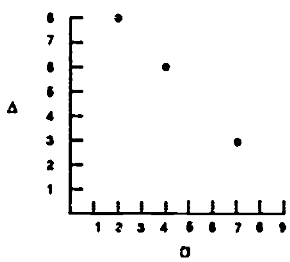


Here is a number sentence that can represent all the different ways that you get ten.
$\square + \Delta = 10$.

What number can you put in the box and what number can you put in the triangle to make the sentence true? Can you find two other numbers? Any more?

As students give pairs of numbers, the numbers can be listed in a table and then plotted onto a large coordinate graph.

□	Δ
2	8
4	6
7	3



Do you see a pattern? Can you use the pattern to mark another point that will work?

Consider $\square + \Delta = 10$.

If you put $3\frac{1}{2}$ in the $\square$, what would go in the Δ?
Where would the point be on the graph?
What other pairs and points are on the graph?

What would the graph look like if we changed the sentence to $\square - \Delta = 10$.

3-6

As the intermediate students' knowledge of arithmetic expands, they will be able to consider the possiblity of replacing the $\square$ and Δ with more types of numbers than the primary students can.

Students in the intermediate grades can use whatever arithmetic skills they currently have to consider the three types of sentences they will encounter in mathematics: true, false, and open.

Consider the following statements. Which are true? Which are false? Which are open? Why?

$3 + 11 = 986$ $\quad \square + 14.29 = \Delta$ $\quad (3 \times \square) + 2 = 14$

$\square + 2 = 7$ $\quad \frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ $\quad 2 \times 2\frac{1}{2} = 5$

For the open sentences, find values that will make them true and values that will make them false.

6-8

As the older students learn new sets of numbers, the graph of $\square + \Delta = 10$ should be extended and various other open sentences can be explored.

For $\square \times \square = \Delta$, what happens if you put -7 in the boxes? Find other pairs of numbers. What does the graph look like?

For $\square \times \square < \Delta$, find pairs of numbers. What does this graph look like?

After their introductory work with open sentences, the students will consider the three types of open sentences they will encounter in mathematics: sentences for which all numbers will work (identities), no numbers will work (contradictions), and for which some numbers will work (conditionals).

Consider the solution sets for each of the following:

$3\square + \square =$ $\qquad \frac{x}{x} = 10$ $\qquad 5 - x = x - 5$

$x + 1 = 3x + 3$ $\qquad 7 - x \geq 9$ $\qquad A + A = 2 \cdot A$

$B + 4 \neq 7 + 2B$

---

❷ *It is essential for students to understand that*

An equality relationship between two quantities remains true as long as the same change is made to both quantities.

K-3

Children in the primary grades will work with sets of objects to develop the idea of an equality relationship as a basis for understanding later work with equalities.

Each of you has ten raisins. When you have eaten half of your raisins, will each of you have the same number?

Linda has four pebbles in one hand and five pebbles in the other hand. Paul has two pebbles in one hand and seven in the other hand. Jane has four pebbles in one hand and three in the other. Steve has nine pebbles in one hand and none in the other hand. Which children have the same number of pebbles?

3-6

In the intermediate grades, the students can begin to work with the idea of an equality relationship when there is an unknown quantity involved.

Tara and Armando have the same number of peanuts. Tara has two unopened packages of peanuts which both contain the same number of peanuts and three loose peanuts. If Tara eats one of her peanuts and Armando eats one of his, does Tara still have the same number of peanuts as Armando? If Tara gets two more peanuts, what does Armando have to get so he still has the same number as Tara? If Armando's peanuts are doubled, how many more packages and loose ones does Tara need to get in order to have the same number as Armando? How can we use symbols to write down what is happening when we don't know how many peanuts Tara has?

If Armando has 35 peanuts, what can we do to both sets of peanuts (adding more? taking some away? taking half of both?) that will help us find out how many peanuts there are in one of Tara's packages?

6-8

Students in junior high school can begin to analyze two statements and determine whether they are equivalent (i.e., have the same truth set) and if so, how one can be transformed into the other.

$x + 4 = 7$ can be transformed into $x + 5 = 8$ by adding one to both sides. Is it possi[illegible] [illegible] transform $x + 4 = 7$ into each of th[illegible] following and if so, how?

$x - 5 = 2$ $\qquad 5x + 20 = 35$

$x - 10 = -7$ $\qquad \frac{1}{2}x + 2 = 3\frac{1}{2}$

$x = 3$ $\qquad .25(x + 4) = 7.25$

$2x + 4 = 14$

How many others can you find?

❸ *It is essential for students to understand that*

The properties of operations on variables are the same as the properties of operations on numbers.

K-3

Children in the primary grades need to explore and understand relationships among operations in specific instances using concrete materials in order to build the foundations they need later to understand the generalizations in algebra.

Get two counters and put four counters with them. How many do you have? Will the number be the same or different if you start with four and then add two?

Get six counters and take away two. What number is left? Is that the same number as you would have if you started with two and tried to take away six?

Show me three groups of four? Do you have the same number or a different number when you have four groups of three?

Divide nine counters into three piles. How many in each pile? Can you divide three counters into nine piles?

3-6

Students in the intermediate grades can begin to examine number relationships and look for generalizations. It is from the examination of these relationships that they will be able to form the generalizations required in algebra.

Consider these sentences: $20 \cdot 0 = 0$, $6 \cdot 0 = 0$, $5 \cdot 0 = 0$.

Which of the following sentences go with the sentences above?

$6+0=6$	$7+1=8$
$16+2=2+16$	$8+1=9$
$4 \cdot 0=0$	$0+7=7$
$22+0=22$	$9 \cdot 0=0$

Why do they go together? Can you make up more? Could you replace all the sentences that go together with one sentence with a variable in it?

6-8

When students move into the seventh or eighth grades, they can begin to build both on the arithmetic they know and their experiences with variables to establish the properties of algebraic relations.

Decide if each of the following is true for all numbers. Then find examples which illustrate whether they are true or false.

$a + 0 = a$

$a + b = b + a$

$a(b + c) = ab + c$

$a + 1 = 1$

$a \cdot 0 = a$

ELABORATED CLASSROOM EXPERIENCES

The purpose of the following descriptions of classroom experiences is to show how the programs envisioned by the *Mathematics Framework* and this curriculum guide might actually look in classrooms. There are three sets of lessons included in this section. Each set has one strand as its central focus; however, it is evident that many strands are integrated into each of the lessons. One set of experiences focuses on statistics; one on number; and one on geometry.

In each description, you will encounter three different teachers who work with children from different age groups: K-3, 3-6, or 6-8. Although the specific lessons and the expectations for the students' level of understanding vary with each teacher, many of their primary goals are the same.

The particular mathematical topic and the series of lessons designed to develop understanding of the topic are not as important as the reasons behind the decisions made by the teachers described here. Those reasons are highlighted and explained in a commentary which is written parallel to the description of the classroom experience.

EXPERIENCES EMPHASIZING STATISTICS

The three teachers described in the following lessons work with children of different age groups. They believe it is important to tie the world of mathematics to their students' personal experiences and interests, not simply because of the motivational value, but because the study of mathematics has real importance and application in everyone's daily life.

No matter what the age of the students, they will be exploring the following questions at different levels of complexity:

What do we want to find out?

How can we find out?

How can we organize the information we've gathered to make a graph that will make it easier to answer our questions?

What is average or typical?

Do we agree that the conclusions are valid?

At every grade level, the students will be engaged in making decisions, making predictions, and drawing conclusions.

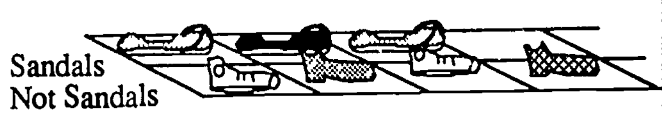
As you read the lessons, note how the gathering and analysis of statisical information automatically requires some sorting and classification, and some work with number relationships as well as logical thinking.

A K-3 Lesson Emphasizing Statistics

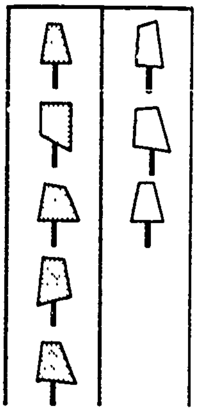
Mr. Silva teaches a class of second and third grade children. He sees his lessons on graphing as part of an ongoing year long plan. He wants to provide his students with many different opportunities to gather information about themselves and the world they live in and to organize that information in a variety of ways. He doesn't teach a unit on graphing but weaves graphing into his program all year. He is very clear about the purposes of the graphing activities. Because he knows his long term goals, he has the flexibility to respond to events in the classroom that suit his purposes as well as create those situations where he directs the students along certain lines.

In kindergarten and first grade, Mr. Silva's students had many experiences that helped them connect graphing experiences with the real world. Rather than dealing with symbolic representations, the children arranged and compared the actual objects or pictures of the objects under consideration.

Actual objects:
Are you wearing sandals or shoes that are not sandals? Place your shoe in the row where it belongs.



Pictures:
Did you choose a chocolate ice cream bar or a vanilla/orange bar? Draw a picture of your bar and place it in the proper colummn.



Mr. Silva has provided his students with many opportunities to collect and represent information. Some of the questions that sparked a great deal of interest and discussion were

Have you ever had gum in your hair?
Would you like to be older, younger, or the same age?
Have you ever had a broken bone?
Have you ever been lost?

These questions led to discussions concerning whether an experience was usual or unusual. Thinking about these kinds of ideas will help the children later to develop the idea of average or typical. When Mr. Silva poses questions that are of particular interest to the children, many of them choose to write about their experiences during the language arts time.

The following lesson is one that grows out of all the previous experiences the children have had in graphing. Sometimes the graphing experience is just a short activity that precedes a math lesson. Today Mr. Silva intends to go into more depth and use the graphing experience as the whole lesson of the day. His main purpose is to have the children explore the kind of decision making that is necessary to determine the categories on a graph.

The Lesson: Designing graphs

Mr. Silva's class has been working with the theme of night. One of the topics they have discussed is the importance of getting enough sleep. Mr. Silva has decided to do a graph about bedtimes. He presents the problem to the children. "We are going to make a graph about our bedtimes. We need to decide on the times to put on the graph. We don't want to pick just any numbers. How can we find out what to put in the first column of the graph?"

In the past Mr. Silva has made most of the decisions about the graphing topics and design, but now he wants to involve the children more. He does not really expect them to consider everything that is involved in designing a graph, but wants to make them more aware that many decisions must be made.

"I go to bed at nine o'clock," says Eddie. "We could put that number on."

"Should that be our earliest time, class?" asks Mr. Silva. "Does anyone have an earlier bedtime?"

"I have to go to bed real early," says Linda. "My mom makes me go to bed when my little sister does 'cause she doesn't like to sleep alone. I go to bed at 7:30 but I get to read for awhile if I want to."

"Does any one have an earlier bedtime?"

Bedtimes can be a sensitive topic for some children. They won't want to be called "baby" for having an early bedtime. This is not a problem in Mr. Silva's class because of the atmosphere of acceptance and respect that he has been fostering all year. He knows his students will probably not make negative comments, but is still sensitive to not putting any particular child on the spot.

There's a little rumble of noise from the children, and Mr. Silva hears a few "no ways" as they share bedtimes and their opinions of them with each other, but no one volunteers an earlier bedtime.

"We are trying to determine the range of our graph – the difference between the earliest time and the latest time – so now we need to figure out the latest bedtime. Who thinks he or she has a late bedtime?"

Mr. Silva uses the term "range" in a context that will mean something to his students. He will continue to use the word, but he will not expect his students to begin using it until they have heard it many times in many situations.

"I don't have a certain bedtime. I stay up as late as I want," states Chad.

"We'll have to talk about that later, but I can't really write a time for that situation," responds Mr. Silva. "Right now I need to know what the latest bedtime in our class is."

"Do you mean last night or the other nights?" asks Regina. "Last night we went to visit my grandma; she was sick and my mom was taking care of her. We went to bed real late."

Mr. Silva wants his students to face some of the issues about data collection that relate to the ambiguity of graphs. He wants them to recognize the fact that not everything is always clear-cut, so he welcomes comments like Chad's and Regina's. His goal is that eventually his students will learn to question the statistics that they hear and read and ask appropriate questions before just accepting someone else's interpretation.

"I guess we need to decide if this graph means the *usual* bedtime or the time you went to bed on a certain night."

Tim offers his opinion. "I think your bedtime means what you usually do. There's always going to be something going on that messes things up, but you still know what your bedtime is supposed to be."

Mr. Silva wants the notion of "usual" to be important in this discussion and will develop this idea further.

"We still haven't come up with the latest time for our graph. Does anyone think he or she has a late bedtime?" asks Mr. Silva.

"I do," says Ernie. "At least mostly. I get to stay up 'til my dad gets off work. The store closes at 9:30, but he usually doesn't get home until about 10 o'clock."

"Is ten o'clock the latest bedtime in the class?"

There is a murmur of "yeah." "I think so." "That's later than me."

"What about the intervals? Those are the times in between. What should we put? Do you think it should be hours or half hours or every 15 minutes or what? Talk to the group of students

Again, Mr. Silva uses a term he wants the students to become familiar with before he expects them to use it.

you are sitting with and see if you can decide what to put."

He intentionally includes some suggestions in the wording of his question so the students will have an idea of what some possibilities could be.

Mr. Silva wants *all* students involved and thinking about the concept under study. One way to maximize this approach is to structure the need to talk by telling the students to "discuss this in your groups."

Mr. Silva overhears some of the discussions.

"Well, one is 7:30 and one is 10:00, so I don't think it will work to have just hours. I guess we need all the half hours, too."

"I don't think we need 15 minutes on there. Who ever heard of going to bed at 8:45 or something? That would be right in the middle of a TV show."

After the small groups have discussed the issue for a couple of minutes, they share their conclusions with the class. There seems to be a consensus for using half-hour intervals.

"If some people go to bed at 8:15 or something, they could just put it on the time from 8:00 to 8:30," explains Tim.

Mr. Silva writes the intervals on the chalkboard: 7:30–8:00, 8:00–8:30, 8:30–9:00, 9:00–9:30, 9:30–10:00. He asks, "Are these the intervals you mean?"

"Hey, there's something wrong there. If you go to bed at 8:30, you could be in two places," observes Jue May.

Mr. Silva was not sure anyone would notice a problem Jue May pointed out until it was actually time to enter the information on the graph. If no one had anticipated the problem, he would have waited until the problem actually came up and would have had the class decide at that time what to do.

"Take a minute now to discuss that problem with your group. Decide what we can do about that."

Sylvia reports for her group. "Don't put the same numbers in the line. Leave off one of the numbers."

"We got a great idea in our group," reports Tim. "You could start each line with the number that is one minute after. Like you could have 7:30 to 8:00, and then one minute after is one minute after 8:00. But how would you write that?"

Mr. Silva expects the students to come up with solutions of various levels of sophistication. He did not know when he posed the question that anyone would come up with the type of suggestion that Tim did. He would have accepted whatever the children thought made sense. By posing this type of question, however, he presents opportunites for students like Tim to stretch their thinking.

Mr. Silva shows the class how the times could be written. He then quickly draws lines on a piece of paper to form the columns needed on the graph. With input from the class, he writes the various bedtimes in the appropriate columns. "We are going to have activity time now. Before the end of the period, I want each of you to come to the graph and mark the column that tells what your usual bedtime is."

Some of the children have a routine and go to bed at a regular time every night so it is easy for them to decide where to mark the graph. For others it is very difficult because their bedtimes vary so widely. A few children decide to pick the number that they think is the best bedtime and not worry about the actual bedtime. Often, young children reinterpret the question asked and come up with an answer that makes sense to them but may not be very precise. For some, this means choosing the time that they think will seem like the responsible choice and for others it is picking the latest time so that they can feel more grown up. A less ambiguous graph would be easier to do with the children, but Mr. Silva is intentionally raising the issue that statistics don't always tell the whole story. Data collection often involves decisions and judgments.

Later, when everyone has had the opportunity to enter his or her bedtime on the graph, Mr. Silva has the class discuss what conclusions they can make by looking at the graph.

Earlier in the year, Mr. Silva asked questions to get the students to focus on the relationships apparent in the graph. Now, he is able to simply ask the students to make observations.

"There are more people who go to bed from 8:31-9:00 than any other time," reports Paula.

"Three more people go to bed from 9:31 to 10:00 than from 7:31 to 8:00," says Nicholas.

Mr. Silva continues to focus the students' attention on information that can be determined by interpreting the information on the graph. "How many people go to bed before 9:30?" asks Mr. Silva. "Is that more or less than half the class?"

Mr. Silva poses a new type of question which asks the students to add up several columns when he asks, "How many go to bed before 9:30?"

When the discussion has ended, Mr. Silva makes an assignment. He says, "Today we made a graph that showed the times you thought were your bedtimes. Your homework tonight is to find out what your parents think your bedtime is. Write that down on a piece of paper. Then pay close attention to what time you really go to bed and write that down, too."

Mr. Silva wants to have the children think about bias in data collection. He will ask them if they think the time they went to bed last night was affected by bringing up the subject with their parents.

Later that week, Mr. Silva follows up this lesson with the question, "What time do you get up?" He has the students compare the range of times they get up with the range of bedtimes. He also challenges some of the students to figure out how many hours of sleep they get at night.

Extending the lesson: A typical day

After dealing with the notion of the usual bedtime, Mr. Silva is interested in extending the notions of usual or typical, which will become so important later when dealing with averages and other measures of central tendency. He will do this by having the children work with the idea of a typical or ordinary day.

One or two experiences with a concept is not enough, so Mr. Silva continues to provide opportunities for his students to deal with the notion of typical.

"Instead of thinking just about bedtime, now I would like you to think about the whole day and decide what you usually do during certain times of the day."

They discuss whether they should pick a school day or a weekend day to be the ordinary day. Since they go to school more days of the week, the class decides to pick a school day.

There are many issues the children might need to consider in order to complete this assignment:

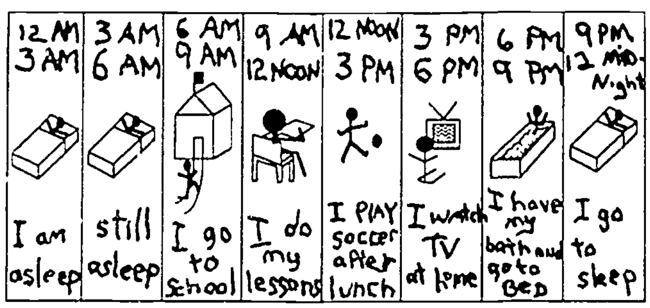
1. What constitutes a day?
2. Dividing a day into equal time segments (24 ÷ 8 = ?).
3. A.M. vs P.M.
4. Considering the proportion of time spent sleeping versus the time spent at school versus the time at home.
5. The idea that time sometimes seems to go fast and sometimes seems to go slow.
6. Recording time in several ways: drawing on clocks, using words, notations like 12:00.
7. Typical times/seasons for certain activities.

Mr. Silva is not going to teach directly to these issues, but he is alert to the possibilities and will deal with them as they arise for the individual students.

Mr. Silva models for the class the format he would like them to use to describe a typical or ordinary day. He shows them a long piece of paper that he made by cutting a piece of 12 by 18 newsprint in half and taping the ends together. As he folds the long piece of newsprint into an accordian book with eight sections, he raises several questions with the class. "How many hours are there in a day? If we divide our day into eight sections, how many hours will be in each section? When does the day start? When does it end?"

Mr. Silva makes sure that they are clear that their ordinary day book is to start at midnight and end at midnight. Then he instructs the children to pick something that they are doing at each of the 8 time blocks and to draw pictures of those activities and then to write something about them.

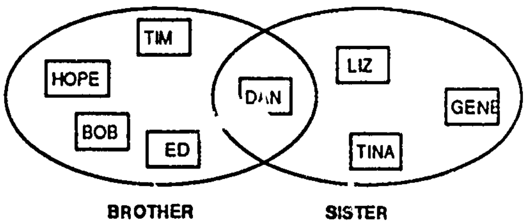
Mr. Silva does not expect all his students to notice the same things or come to the same level of understanding. However, this type of activity provides rich opportunities for all students, no matter what their level of ability or previous experiences.



A 3-6 Lesson Emphasizing Statistics

Graphing has been a regular part of Mrs. Stevens's math program. Several times during the week, she displays a question and asks the students to answer the question by placing their names in the appropriate spot on the graph. For example:

DO YOU HAVE A YOUNGER BROTHER OR SISTER?



Sometimes the students have contributed ideas for the topics or questions, but Mrs. Stevens has always designed the graphs. Now she wants the students to have the opportunity to make graphs independently.

They have brainstormed several things they would like to know about their class. Their list includes such things as color of hair, color of eyes, height, right-handed or left handed, favorite junk food, favorite ice cream, kind of pet, number of people in the family, favorite TV show, years lived in this town, and number of brothers or sisters.

Mrs. Stevens is planning to have the students work with partners to collect data, organize them, and display them on graphs. First, however, she wants to have the class experience a graphing task together so she can raise some of the issues that they will need to consider when designing their own graphs.

The Lesson: Choosing categories

"As you know, you and your partner will be choosing topics from our list and gathering and organizing the information you need to make a graph," explains Mrs. Stevens. "Before you work on that with your partner, the whole class will be working together to design a graph so that we can see all the things that need to be considered. I chose one of the topics from our list: What is your favorite ice cream flavor?"

"I want you to just think about the question first. Lots of times we assume that the answers to questions are obvious, but that is not always the case. If someone were to ask you right now 'What is your favorite ice cream?' could you tell them? What are some of the things you have to think about before you can answer that question?"

Several children volunteered responses.
"I think that's an easy question. Everybody knows their favorite ice cream."
"I don't like ice cream, so I don't even have a favorite."
"I have too many favorites."
"I can't remember the name of the one I really like. But I know it when I see it at the ice cream store."

Mrs. Stevens realizes th[illegible] question, What is your favorite ice crea[illegible] result in a great many different flavors that will need to be categorized in some way. She wants the students to be able to use this experience of forming categories when they have to make decisions about the graphs they will be making with partners.

While the main goal is designing a graph, she also wants her students to be aware that even a simple sounding question requires interpretation on the part of the person answering. Little in statistics is clear-cut.

"I'm always changing my mind, so I know what I pick today wouldn't be what I would pick tomorrow."

"What I would like you to do, even though it may not be your absolute favorite forever, is to pick one flavor right now that you could call a favorite and write it on a slip of paper," said Mrs. Stevens.

After each student has written his or her choice, Mrs. Stevens has the students share what they have selected. "Joe, what is your favorite?" she asks.

"I picked Rocky Road."

"Everyone else who chose Rocky Road raise your hand."

"Now, who picked a different one?"

"I like Burgundy Cherry," says Lisa.

"Did anyone else pick Burgundy Cherry?" asks Mrs. Stevens. "Who has a different favorite?"

Mrs. Stevens continues to call on the children until all have had an opportunity to say what their favorites are. As the children share, she writes the various flavors on the chalkboard and indicates by tally marks those flavors which have been chosen by more than one.

Rocky Road - ~~IIII~~ I
Burgundy Cherry - I
Chocolate - IIII
Quarterback Crunch - I
Fudge Brownie - III
Strawberry - IIII
Maple Nut - I
Pralines and Cream - II
Chocolate Chip - I
Vanilla - IIII
Raspberry Ripple - I
Strawberry Ripple - II
Almond Toffee - I

"Do any of you see any problems that might occur if you tried to make a graph right now with these data?" Mrs. Stevens asks.

"It looks like we might have too many different choices," says Jill.

"Yeah, our graph would just be long and flat," Peter notices.

The comments made by Jill and Peter are very insightful. Many of the children would have had to actually plan the graph and lay it out to see that it turned out very flat. For them, the need for categories would not be apparent as soon. Even after these comments, some children won't know what is being talked about until they are faced with making their own graphs. Mrs. Stevens knows the issues will come up for those children later when they are working on a graph so she feels no need to go into it any more now.

"Do you think there is any way that we could group any of the flavors together and put them into categories? Discuss this in your groups of four and see if you can find some ways to cut down the number of choices," the teacher says.

Mrs. Stevens has her class grouped in heterogeneous groups of four who stay together as a group about two weeks. She has the students discuss many things in their groups as it gives more students the chance to share their thinking.

After a few minutes of discussion, each group of four reports its idea to the rest of the class.

"If we were going to make a graph, we wouldn't just say, 'What's your favorite?' We would just pick the ones we wanted them to choose from. We would say, 'Do you like vanilla, chocolate, or strawberry?'" says Robin.

"Would that include all the flavors people picked in the survey we just did?" asks Mrs. Stevens.

"No, but there's just too many of them," replies Robin.

Robin's idea doesn't really fit the situation. The information has already been collected, and he can't change the question now. However, it was still an important observation.

Troy shares his group's idea. "We thought we could put a lot of different flavors together like the kinds that have fruit in the names and all the different ones with chocolate and the different ones with nuts. I don't think we really decided what to do if they had both chocolate and nuts."

"Our group thought we would just put them all together into two categories, chocolate or not chocolate," says Dino.

After each group had shared its ideas for categories, Ryan made an observation: "It seems kind of funny to me. I know we are talking about the same favorite ice cream flavors that we tallied on the board, but when we think about making different categories, it seems like that would make all the graphs look a lot different from each other. I wonder how they would all look."

Mrs. Stevens was planning to have the students work on different ways of displaying the same information and is pleased Ryan also thought of it. There are times when student suggestions actually are better than the plan she had in mind, and she is willing to respond to their ideas. In this case, if Ryan had not made the suggestion today, Mrs. Stevens would have posed the problem anyway.

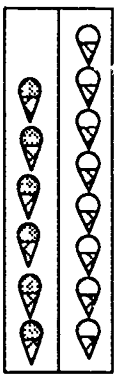
"It would be interesting to see all the different graphs that could come from the same information," says Mrs. Ste˙˙ns. "I'm even wondering if all the graphs would show the same ice cream being the favorite. Do you think you could make a graph that would make your own favorite come out better?"

"I am going to give you some time now to work on graphs in your groups of four. You all know what materials are available for you to use and I'll leave the information you need on the board. You may get started right away."

Most of the childr n probably did not understand Mrs. Stevens's comment about making a graph that would present information in a way that was more favorable to a particular point of view. Mrs. Stevens poses several questions to stimulate the children's thinking. She recognizes that not every question will interest or be understood by all the students. She is not planning on developing any of the ideas further at this time, but will watch to see if any of the students pursue these ideas as they work on their graphs.

The class spent the rest of that afternoon and part of the math period the next day working on their graphs. The graphs were different not only in the ways the students found to categorize the flavors but also in the different ways they used to display their information.

The class has reached a point where it can work with little direction. However, this is true only because Mrs. Stevens spent a lot of time earlier in the year preparing them to work independently.



CHOCOLATE · NOT CHOCOLATE

CHOC	XXXXXXXXXXXXX
FRUIT	XXXXXXXXX
NUTS	XXXX
OTHER	XXXXX

There was a great deal of discussion and decision making in the groups of four before they came up with the various ideas for the graphs. Working together on a project requires a great deal of cooperation. The students have been working in cooperative groups long enough to be able to handle this type of effort. This would not be the first kind of cooperative learning task Mrs. Stevens would give the students.

In the discussion of the graphs that followed, the children made several observations.

"The graphs sure look different from each other. Sometimes chocolate looks like the favorite, and sometimes it looks like people like the ones that are not chocolate more," said Dino.

"It's kind of confusing to me. I think if someone owned an ice cream store and looked at our graphs to see what people liked, he wouldn't know what to put in the store," remarks Troy.

"I think you could tell if you looked at some of the graphs. Some of them give better information than others. Some put so many different things together you don't know what you've got," says Jill.

Seeing the variety of ways that a single question and set of data could be represented is a powerful and effective experience for students. Mrs. Stevens wants her students to be able to question the statistical information they encounter, and this experience will certainly make them more aware of the issues.

Mrs. Stevens discussed with the children other problems that arose and decisions they made when constucting their graphs. She feels this experience has adequately prepared most of the students for making their own graphs on topics from their list. She knows that students will encounter difficulties, and she will be observing and interacting with them as they work.

Extending the Lesson: Introducing the idea of the mean

When the students in Mrs. Stevens's class worked with the topic of favorite ice cream, they were working with the measure of central tendency, which is the event occurring most often. This is called the mode. They have had many experiences working with mode, so Mrs. Stevens decides it is time for them to begin to explore the idea of an arithmetic mean. She chooses to introduce the idea with something common and familiar that can be physically divided, so she decides to use boxes of raisins.

Mrs. Stevens brought in $1/2$ oz. boxes of raisins and gave one to each student in the class. "How many raisins do you think are in a box?" she asks. "Write your estimate on a piece of scratch paper."

After the students have made their estimates, she tells them they need to find the range of their estimates – the difference between the highest and lowest numbers.

Mrs. Stevens is developing mathematical language in the context in which it is used rather than by definition.

"Does anyone have an estimate less than 50?" she asks. Several hands go up. "Less than 25?" A few hands go up. "Less than 20?" One hand goes up.

"I guess I have the smallest number. The box looked tiny to me, so I thought it would be 15," explains Peter.

"Who has a number more than 50? Than 75?" Three hands go up, and they find that 86 is the highest number.

"As you can see, our estimates are from 15 to 86. What is the difference between the two numbers?" Mrs. Stevens asks.

Mrs. Stevens has created an accepting environment in which it is safe for students to take risks. She doesn't believe anyone in the room would be embarrassed by sharing his or her estimate. They have had so many experiences like this that they no longer make comments or judgments about other people's ideas. They know there have been many times when the person who seemed the farthest off was really the closest.

Several children respond, "71."

"That difference is the range. What does that number tell you about our estimates?"

"Well," says Jill, "it looks to me like most of us had no idea. Fifteen and 86 are sure far apart."

"You are going to be working with your raisins." explains Mrs. Stevens. "Please don't eat or give any of the raisins away before you actually dump out the raisins and count them. Now, you are to look inside the box and count the number of raisins you can see on top and then make another estimate using that information to help you. I want you to talk over your strategies for making your estimate with the people in your group, and we will discuss them in a few minutes."

One of the important notions about estimating is being able to use information as you can to modify original estimates.

After the students have had time to make and discuss their estimates, Mrs. Steven asks them to share their thinking. Some of the responses were:

"We worked together in our group, and we decided there would be 24 raisins in our boxes. We counted the raisins on the top and then we multiplied by four."

"I counted my raisins on the top and there were 8 and it seemed like about 5 rows. Eight times 5 makes 40 raisins.

"It really helped to look in the box cause then you could see how they fit inside and at least get a little closer."

"Okay, now count your raisins," directs Mrs. Stevens. "Please organize your raisins in a way that helps you keep track of how many you have."

Organizational skills are very important in mathematics, and Mrs. Stevens wants to encourage her students to use some means of organizing whenever possible.

After the students counted their raisins, the class discussed the results.

"Did everyone in your group have the same number of raisins in his or her box?"

"No," says Dino. "None of us had the same number but they were all pretty close."

"What is your prediction about the range of the actual number of raisins in a box compared to the range of our estimates?" asks Mrs. Stevens.

"It will probably be *a lot smaller,*" several children respond.

"If we were to make estimates, I'd sure be a lot closer now even just looking at four boxes," says Jill.

"I was much closer just opening my box and taking a peek at the raisins on top. Those raisins were packed in there, and there sure were more than I thought there would be just looking at the outside of the box." says Peter.

"We weren't so close in my group. My box was full, but I had some big fat raisins and so I didn't have so many," said Jerry.

The students were interested in knowing what everyone in the class had in each of the boxes, so they called out their numbers and Mrs. Stevens tallied them on the board.

The students make the following comments:

"The range is much lower than for our estimates."

"The smallest number is 31 and the highest number is 42. That's only 11."

"Every box is almost the same except for Jerry's box."

Mrs. Stevens asks, "Suppose we wanted to share the raisins so that everybody in your group had the same number of raisins. What would you have to do? Could you predict the number you think it would be before you figured anything out?"

She wants her students to begin to consider ideas that will help them to understand later the idea of the arithmetic mean. She wants to raise issues and get childeren thinking rather than simply telling them procedures for getting answers.

After they had a few moments to discuss, Mrs. Stevens asks them to share some of their thinking.

"The main thing we noticed is that some of us would have to give some away in order to have the same."

"The ones with the smallest amounts would have to get some raisins, and the ones with the most would have to give some away."

"How did you feel about predicting a number? Was that hard or easy?" asks Mrs. Stevens.

"It was easy for our group. We all were so close we just had to think about giving away a couple of raisins to the other two, and we were set."

"Well, Jerry is in our group. It was harder to know exactly how many we would end up with if we each gave Jerry some. I had the most so I would have to give the most away," explains Paul.

"When you actually determine the number of raisins that each of you get if each one in your group is to have the same number, you are finding a type of average called the mean," explains Mrs. Stevens. "I would like you to find the average number of raisins for your group of four. After you figure it out, notice how close you came to the number that you predicted you would get."

Each group divides its raisins evenly and reports the mean for the group to the class, and Mrs. Stevens lists the numbers on the board:
35, 36, 39, 39, 37, 36, 38, 36.

"These 8 numbers tell us the mean for each of the groups. What is the range of these numbers?"

"Now the range is only four. That's even closer than our estimates were," comments Linda.

The range is so small that the students can tell just by looking at the numbers.

"If you had to predict the number of raisins in a box now, what number would you say? Why?" asks Mrs. Stevens.

The children look at the board and decide on a number.

"Does that mean you can be sure that the box you buy will have the number you said?" questions Mrs. Stevens.

"No, it's just that we would be pretty close and not way off like at first when we had no idea," says Peter.

"Your last job is to look at these numbers in your group of four and decide how many raisins each person in the class would get if we shared the raisins equally among all of us. The number that you come up with is the average or the mean for the whole class. Do you think you have a pretty good idea now what that number will be? When you finish finding the mean, you may eat your raisins," concludes Mrs. Stevens.

This one experience is only a beginning for the students. Mrs. Stevens will want to provide her students with many different ways of dealing with this idea in many situations.

A 6-8 Lesson Emphasizing Statistics

Ms. Shimamura is aware of how much our society is constantly bombarded by statistics. She knows that statistics are used (and sometimes, misused) by businesses and governments to make decisions that affect our daily lives. Yet few people question any data that they see in print. Nor do they question how the data were collected or organized. In this lesson, Ms. Shimamura wants to focus on questioning reported statistics and to also give her students practice determining the appropriate measures to describe data.

The Lesson: Gathering statistical data

Ms. Shimamura holds up a newspaper clipping and says, "This is an article about the amount of television that Americans watch. I am going to ask you a question about one of the statistics in the article; and, in your group of four, I want you to predict the answer. How many hours of television does the average teenager watch per day?"

Ms. Shimamura chooses a statistic about television because it is a part of her students' everyday lives and will be of interest to them. She chooses only one statistic because her primary objective for the lesson is to have the students begin to question data they see in print and how they might have been collected. She feels that if she has the students examine a variety of data (which she will do in the future), the impact of the lesson will be diminished.

The groups talk together a few minutes, and then Ms. Shimamura calls for responses, recording them on the board.

Group	Prediction
1	6
2	3
3	7
4	2
5	4

She has the students "predict" the answer to a question rather than presenting the statement, "The average American teenager watches television 3 hours per day," because this will heighten the student's interest and involvement and provide the opportunity to have them reflect on what information they are using to make their prediction.

The article says that the average teenager watches 3 hours of TV.

"Yea!" yells group 2. "Right on! We won!"

"Wait a minute," says Ms. Shimamura. "Do you believe that the average teenager watches 3 hours of TV a day?"

Even though these students have had considerable experience interpreting data, they still believe there are "winners." This is because much of their past experiences have emphasized success as getting one right answer.

"Of course," says Samantha. "It was printed in the newspaper, wasn't it?"

"I want you to pretend that you were the persons responsible for finding this out; and in your group of four, think about how you would collect the information to answer the question: How many hours a day does the average teenager watch television?"

After discussing this question in groups for several minutes, the class comes together to share ideas.

"It would be easy," says Fred. "We'd just go out and ask kids how many hours a day they watch TV."

"If I asked you how many hours a day you watch TV, do you know the answer, Fred?" questioned Ms. Shimamura.

"Well, I guess about, um, let me think, about ... oh, a couple hours, I guess," responded Fred.

"Are you sure about that answer?"

"Not really. I never really kept track."

"I think we know how they found out," says Leah. "They had people follow other people around and keep track of how much TV they watched."

"If you knew someone was following you around and counting up all the minutes you watched television in a day, do you think it would affect the amount you watched? Would you watch more or less TV under those circumstances?" asks Ms. Shimamura.

"Well, I probably wouldn't watch as much because I wouldn't want people to think that was all I did all day," replies Leah.

"We think they gave people questionnaires to fill out. They could just keep them by their televisions and mark whenever they were watching," reports Chuck.

"One thing we haven't talked about," says Larry, "is how you count the number of hours anyway. Does it count if you're sort of watching TV and doing your homework? What if you watch it kind of in between doing other things?"

"I watch a lot on Tuesday night cause that's when my favorite shows are on. I don't watch much the rest of the week," notes Todd.

"Does it count when I go get a snack? I don't watch any of the commercials," says Tyon.

At this point in the lesson, Ms. Shimamura realizes that she could have discussed the basis on which the groups made their predictions and also how they came to consensus within their groups, but chooses not to focus on these points in this lessons. She makes a mental note to discuss these issues more fully in similar situations in the future. She chooses to stay with her original objective and focus on having the students question the data and how they were collected.

Ms. Shimamura realizes that she is very muchin charge of what is happening in the classroom. However, she believes that students learn only when they personally confront and deal with inconsistencies and not when the teacher provides information in a neatly packaged form. She therefore tries to structure most of her lessons so that students will have opportunities to interact personally with other students, facing differences in knowledge and opinions. The questions she asks the group are phrased as much as possible so they facilitate further thinking on the part of the students. She tries to take "pat" answers students may give, such as Fred's, and ask additional questions so that the students may see some points they may have overlooked. At the same time that she asks additional questions, she is concerned that the students do not feel put down (or praised) by her response. Her questions show that she is carefully listening to what students are contributing but also responding in a manner that promotes further productive thinking.

Up until this point in the lesson, Ms. Shimamura has been posing most of the critical questions. Larry brings up an important question that has not yet been discussed in the full class.

"It sure would make a difference if they asked me on a weekend. And can you imagine what they'd get if it was in the summer?" remarks Casey.

"Who is the average teenager?" says Pang. "Do they find this person first and then count the number of hours he watches television, or do they collect information on the number of hours a lot of people watch and then find the average?"

"You're asking good questions," states Ms. Shimamura. "What seemed to be a simple, easy to accept, statement at first turns out to be quite complicated. I want you to go back into your groups of four, list the questions you now have about how these data were collected, and discuss ways our class could find out the average number of hours of TV watched per day by people in our class."

Ms. Shimamura recognizes these contributions by calling them "good questions." However, the questions are not "good" just because they were contributed by the students or that they "pleased" Ms. Shimamura. Her next sentence indicates that the questions help illuminate how complex the statistical statement really is.

The assignment she has given the students is similar to those that they have done before. This time, however, all groups of students will be seeking the answer to the same question. She hopes that some of the concerns that have come up in today's lesson will be reflected in how the students decide to collect and organize the data. She expects that the results the groups come up with will be quite different.

The groups discussed various ways the task could be done and shared their ideas with the whole class. Suggestions included asking each person to write down (1) the programs each student watched the day before; (2) the programs each student watches on Saturday; and (3) the number of hours each student watched the day before. After hearing the suggestions, the class decided on this plan. Tomorrow (a Wednesday which was chosen because they considered a weekday more "typical" than a weekend day) the students were to keep a log of when and what they watched on television. They decided that a program was being "watched" if they generally paid attention to what was occurring in the program.

On Thursday the students came in with their completed logs. Of the thirty-two students in the class, six had forgotten to collect their data. The class decided that, rather than omitting this information, those students should make a list of what they watched yesterday, recalling the hours, one by one, as they remembered what they had done. Several students had brought in TV schedules for Wednesday and let the six use them as a resource.

When the students reported the number of hours each watched, it was discovered that some had kept detailed records down to the minute and other students had kept track only of the total number of minutes in each program. In order that each person use comparable data, the class decided to count half hour segments and to define a "watched" half hour segment as watching fifteen or more minutes of the segment.

Ms. Shimamura says, "Now that we have agreed on how to count the number of hours each of us watched TV yesterday, I want each of you to make sure that you have calculated your TV watching time to the nearest half hour." After this was done, the groups of four compiled the totals and reported their results as Ms. Shimamura recorded the following frequency distribution on the board:

# HOURS	TALLY
0	/
.5	/
1.0	/
1.5	
2.0	////
2.5	𝍸 /
3.0	𝍸 //
3.5	//
4.0	
4.5	
5.0	
5.5	//
6.0	/
6.5	//
7.0	/
7.5	///
8.0	//

Ms. Shimamura asks, "Just by looking at the frequency distribution on the board, and without doing the computation, do you think the mean, median, and mode will be the same value?" Susan says, "Yes, because there are more tallies in the middle of the distribution." "No," says Joe, "because the tally kind of stretches out longer toward the higher numbers. I think the median and the mean will be higher than the mode, which I can see is 3.0."

The data that the students collected were "messy". The simplifications that the students are making by calculating TV watching time to the nearest half hour make the problem doable. Ms. Shimamura recognizes this and hopes that some of the students will bring up these points when they generalize what they have learned at the end of the lesson.

The students have done frequency distributions a few times before. This is not the focus of this lesson, but Ms. Shimarura knows that the review and use of terminology will reinforce the students' prior understandings.

Ms. Shimamura quickly reviews with the class how to determine the mean, median, and mode, and she asks each group of four to calculate all three. She reminds the students that they might want to use their calculators to calculate the mean. She also says, "After your group has found the mean, median, and mode for the data on the board, I want each group to discuss whether one or two of the measures of central tendency seems more appropriate to describe this set of data. Remember, your reasons will be as important as your conclusions."

Ms. Shimamura is aware that not all students have yet developed an understanding of the differences between the mean, median, and mode and how they are determined. The quick review of how each is calculated is left on the board and is meant to serve as a reference for those students who need it.

Ms. Shimamura is always looking for situations where using a calculator makes sense. Calculating means with paper and pencil for a large set of data is tedious. She also notices, though, that most of the students are doing many of the easy calculations mentally.

All groups find a median and mode of 3 and a mean of approximately 3.8. When asked to state which measure or measures seemed most appropriate to describe the data, the groups giv the following responses:

Though students are finding the values of the mean, median, and mode, Ms. Shimamura has asked them a more complex question about which seems more appropriate for the set of data. She wants the students to realize that finding the answer to a calculation does not mean that you have the answer to the question you originally asked.

Group 1: The median and the mode are most appropriate because they came out to be the same and the mean is different.

Group 2: The mean because it is found by doing arithmetic.

Group 3: The median and mode because there were some students who actually watched 3 hours of TV last night while no one watched 4 hours, which is what the mean rounds to.

Group 4: The mean because it gives more of an idea that there were a lot of kids in the class who watched a lot of hours of TV.

Ms. Shimamura realizes that the students still have a wide range of understandings of the concepts.

Ms. Shimamura says, "If you were a government agency that really believed that kids watch too much TV, which would you report: the mean, median, or mode?"

"The mean," choruses the class.

Ms. Shimamura brings in some examples so that students can see where reporting one measure of central tendency may make more sense for a particular group and for a particular reason.

"I wouldn't report the mean for a Wednesday " says Lupe. "I'd report the mean for a Saturday. Kids watch a lot more TV on a Saturday than a weekday."

"Can you think of a situation where the median and mode would make more sense to report than the mean for this set of data?" asks Ms. Shimamura.

"When we write a report to our parents telling them what we learned this week in school," says Laura.

"During the last three days we have questioned a statistic that was printed in the newspaper, and we have tried to collect some information about students in our class in the same area. In your group of four I want you to brainstorm some generalizations you have learned by doing this, write them on large sheets of paper, and post them around the room so we can discuss them," says Ms. Shimamura.

The students are used to writing generalizations at the conclusions of lessons. Ms. Shimamura views it as critical in helping students internalize some of the concepts. She has them work in groups, recognizing that the sharing of learnings will help one another. Learning mathematical vocabulary associated with statistics becomes a natural process as students have the opportunity to use it in meaningful situations.

Among the generalizations made are:

It's easy to come up with a mean, median, and mode, but it's hard to know if those numbers really make sense as answers to the question.

Just because something is reported in the newspaper, doesn't necessarily mean it's true, especially for you.

Sometimes the mean makes more sense, and sometimes the mode or median makes more sense to use.

Ms. Shimamura also uses these written generalizations as part of her assessment of learning. She walks around as the students are brainstorming, mentally noting who seems to have a good grasp of the concepts, and who is still working at a more intuitive level. Since all written responses are valued, the atmosphere in the class is not threatening.

Since it's hard to count the number of hours you watch TV, even when you try to do it carefully, the people who report statistics in the newspaper have also probably had a hard time counting.

Since the answer for our class came out pretty close to the answer in the newspaper, if we surveyed Mr. Koseburg's class, they probably would come out about the same. However, we can't be sure.

In order to give her students the opportunity to use the information they have been learning about and in order to gain more insight into the level of their understanding, Ms. Shimamura plans to follow up this lesson by asking the students to bring in advertisements from magazines or newspapers that use statistics. They will be asked to answer the following questions: Do you think this is fair advertising? Do you think the advertisers used the mean, median, or mode in reporting the data? Why do you think that? What questions would you like to ask the advertiser?

EXPERIENCES EMPHASIZING NUMBER

The lessons in this section will show how each of the teachers in the three grade level spans helps children develop one aspect of number. In the K-3 span, the focus will be on helping children develop place value concepts; in the 3-6 span, the emphasis will be on fractions; and in the 6-8, the lesson will be on relating fractions, decimals, and percents. While each of the teachers described also provides rich and varied opportunities for children to apply their developing number ideas in real life situations, the lessons described here focus on experiencing numbers in a way that enables students to gain a sense of the quantities being worked with. The particular concept being dealt with is different for each grade level, but the basic goal of developing a strong sense of number is the same. Students with a strong sense of the number system, fractions, decimals, and percents will be able to use these number concepts to solve problems with confidence and understanding. In order to develop complete understanding, students need to have a variety of experiences with the number concepts being studied. Concrete materials or models are often used to help children develop number concepts. When models are used, they are used as tools for thinking, not tools simply for getting answers to symbolic problems.

A K-3 Lesson Emphasizing Number

Mrs. Kahn wants to provide her young students with experiences that will give them a foundation for understanding place value. She wants her students to develop a sense of the size of numbers larger than ten, to begin to appreciate the convenience of grouping and counting by tens, and to recognize the patterns that appear because of the structure of our number system.

Mrs. Kahn uses a variety of classroom management systems, depending on her goals for her students. Sometimes she has the whole class meet together for some instruction or discussion. She sometimes works with small groups, and at other times has her students work in small cooperative groups. The activities she has set up in her classroom now require her students to work independently (or with others by choice).

The Lesson: Developing place value concepts

Mrs. Kahn looks around at her students busily working throughout the room. It is certainly not quiet, but everyone is working; and the noise is productive noise, at least for the moment. Half the class is involved with a project they can do at their seats. The other half are scattered around

For the most part, the students know what they are to do and are benefiting from this time to work on the activities without supervision or direct involvement from the teacher. Mrs. Kahn does not want to interfere with those productively at work. However, she sees her job as observing and learning as much about

the room involved in various activities. Those working at their desks are exploring the patterns that emerge from adding by 2's or 4's or 3's. No matter what number they are adding by, they are counting out beans and grouping them into cups of ten as needed and writing the numbers on a long strip of paper.

her students as possible. This will give her the information she needs to plan further lessons or experiences. She will also want to interact with those students who need some help, a challenge, or a chance to verbalize their thinking to her.

Those involved in the various activities around the room are working at many different counting tasks. Several children are counting to see how many rocks or beans or pompons are in a small jar. Paul and Dion are snapping together interlocking cubes to see how long the table is. CeCe is filling up an outline of her foot to see how many wooden cubes will fit inside. They are all recording the results of their investigations on a worksheet.

Everything is going as well as can be expected today. Yes, the beans spilled when Eddie set out the margarine tubs, and Yolanda couldn't seem to find her strip for writing her number patterns. But both those problems were solved by the students themselves. It doesn't always go this smoothly. Sometimes she misjudges and has tasks out for the students that they don't understand, and then she spends more time than she would like getting students underway. Other times, she has a set of activities out too long and the students lose interest and have some difficulty staying on task. Mrs. Kahn knows that she can learn from these situations, and when they arise she adapts her plans accordingly.

It is very busy in the classroom and could be chaotic, except that she has set clear goals and guidelines for behavior for her students. They know what is expected of them, where to get materials if they need them, and where to put things when they are finished. Because she has taken the time to teach her students to work independently, she can now focus on individuals, knowing the rest of the students can manage themselves very well.

Mrs. Kahn walks over to the table to see how the students working on their pattern strips are doing.

"Look," says ChaCho. "This is like the pattern we got when we counted eyes. See, 2, 4, 6, 8, 10, ..."

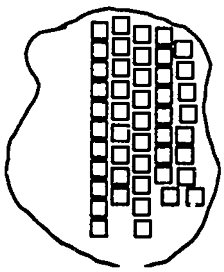
Carla looks up at Mrs. Kahn and says, "Something is wrong here, I don't get what this pattern is. I must be doing something wrong because it goes on and on and there's no pattern." She shows Mrs. Kahn her pattern. So far she has 0, 3, 6, 9, 2, 5, 8, 1,...

Mrs. Kahn wants the numbers the students work with to have meaning for them and not just be encountered as symbols on a page. She wants them to see that the numbers come from somewhere; so she is having them actually count out beans, form groups, and look for the patterns that emerge. Mrs. Kahn is pleased that ChaCho has made the connection between the pattern he is working with now and other experiences he has had.

"Keep going, Carla. Sometimes it takes a while before a pattern repeats," Mrs. Kahn reassures Carla.

Mrs. Kahn moves over to where she sees three children working with tiles and yarn.

She observes Marie, Dennis, and Jamie, who each have a piece of yarn which they have formed into different shapes. They are filling each of their shapes with tiles.



Marie has nearly completed the task. She has followed the teacher's instructions to organize the tiles into tens, so each time she gets to ten she uses a different color tile to make the next set of ten. It is easy for Mrs. Kahn to see that she has 4 groups of ten and 7 more.

"How many tiles are in your shape so far, Marie?" asks Mrs. Kahn. Marie ignores the groupings of ten and begins counting all the tiles, one at a time.

Dennis has also nearly completed the task. "I have 42 so far," he tells Mrs. Kahn before she has a chance to ask.

"How did you know without counting?" asks Mrs. Kahn.

"Because I have four groups of ten and 2 more," he replies.

Jamie is eager to share what he has learned. "I was sure I was going to have the same number of tiles for both shapes because it was the same yarn. But I found out I can make different sized shapes with the same length."

Mrs. Kahn knew that Carla needed a challenge so she had her add by threes. The pattern that emerges from adding by threes is an interesting one. It does not repeat quickly and has patterns within patterns to discover. Adding by threes also requires that Carla work with leftovers as she is forming groups. For example, when she has nine and adds three, she puts one bean with the nine to make a ten and has two left over. Experiences with this process will help her later on when she is regrouping in addition. Even though Mrs. Kahn has assigned the same basic task, she has provided for individual differences also. While Carla is adding by threes and noticing the patterns, Stacy, who is sitting next to her, is adding by ones to get the basic idea of forming and recording groups of ten and noticing the repeating pattern of 0,1,2,3,4,...

The various counting tasks provide for individual needs also. These three children are all doing the same task but getting different things from it. Marie is following directions and organizing the tiles into tens as she was directed to do, but as yet the organization does not mean anything to her. She still needs to go back and count by ones to be sure of what she has. Over time and with many experiences, she will begin to trust that counting the tens and ones will give her the same answer as counting each tile one at a time. Mrs. Kahn does not ask Marie to count by tens this time but lets Marie determine the number of tiles in a way that makes sense to her.

Dennis understands that organizing by tens and ones gives him information that saves him the trouble of counting each one.

Mrs. Kahn likes to provide the kinds of activities that have more potential than simply counting. She has asked the students working

with the yarn to make at least three different shapes with the same length of yarn. This provides an opportunity to notice something about the relationship between area and perimeter. Not all children will pay attention to this but Jamie has noticed that the same perimeter does not always enclose the same area.

At the end of the period, Mrs. Kahn comments to the class on the hard work she has seen. She gives the children an opportunity to tell the others what they did or what they learned, but she keeps the time allotted to class discussion quite short. She has found that young children are often more interested in telling about their own experiences than in listening to the other children talk. She finds it more effective to interact with individual children or small groups while they are working.

Mrs. Kahn has plans for many more place value experiences. She knows the concepts the students are dealing with are very complex and they need many opportunites to view the place value system from many different perspectives.

Mrs. Kahn wants all of her children to grow as much towards understanding as possible, but she does not expect the same growth or level of understanding from each child. Her task is to observe her students at work so she can provide the kinds of tasks that allow all of them to grow and be challenged at whatever level they are.

Mrs. Kahn is planning on setting up a store in her classroom so that her students can apply and expand their understanding of number and number relationships. She wants to tie what they are learning in math lessons to real world applications that also provide opportunites for growth. She can provide for many different levels of development because there are so many different situtations that can be set up in a store setting, depending on the size of the numbers used and the types of tasks she presents. A store can provide many opportunites for children to use what they know about equivalent amounts, adding, subtracting, as well as comparing when they are asked to buy and sell. However, Mrs. Kahn also knows that many of her children need many more experiences with counting, place value ideas, and making exchanges, so she is planning to provide some number exploration stations focusing on place value ideas before she asks her children to work independently in the store.

A 3-6 Lesson Emphasizing Number

Mrs. Chavez intends to give her fourth grade students a variety of experiences with fractions. It is through many different experiences using many different materials that her students will come to the generalizations that she knows are important. She had her students work previously with a model for fractions based on area. They were given several strips of construction paper (4 x 18). They left one strip of paper whole and cut another one in two equal parts. They described each of these two pieces as one of two parts and labeled each $1/2$. They cut another strip into four equal parts, described each small piece as one of the four pieces, and labeled each one $1/4$. The other two strips were cut into eighths and sixteenths and labeled. The students were able to use these pieces to find equivalent fractions and to add and subtract.

The idea of the "whole" is critical in the study of fractions. In addition to using a model where one rectangle or one circle is the whole, it is important that students also work with models where a group of objects is considered to be the whole. When students think about dividing a group of objects into fractional parts, they must keep in mind that the whole is the total set rather than an individual part. Mrs. Chavez has chosen to use egg cartons to introduce this concept to her students because it is easy to think of the egg carton as one whole composed of 12 cups.

The Lesson: Working with fractional parts of a group

The class began its work with the egg cartons yesterday. Their first task was to work together in groups of four to find all the possible ways that the egg cartons could be cut apart into equal sections.

Mrs. Chavez has her students work in groups of four. They are then able to discuss the concept being worked on and to help each other when necessary. When she poses an open-ended question such as How many different ways? all her students can be involved, no matter what level they are working on.

When they cut the sections out, they were to find ways to name these sections using fractions.

Each group was able to find all the possibilities:

12 single cups - each cup labeled $1/12$

6 sections of 2 - each section labeled $1/6$

4 sections of 3 - each section of 3 labeled $1/4$

3 sections of 4 - each section of 4 labeled $1/3$

2 sections of 6 - each section of 6 labeled $1/2$

The egg cartons are a somewhat more complex model than an area model. In many ways, working with the egg cartons is the same as working with a set of 12; but the fact that the sections are connected adds some structure.

They found they couldn't find ways to divide the egg cartons into 5, 7, 8, 9, 0 or 11 equal sections.

Today Mrs. Chavez asks her students to sit with the group of four they worked with yesterday and to get the egg carton pieces their group had cut out. After everyone had his or her egg carton pieces available, Mrs. Chavez presented her students with a question.

"For the next few minutes, I want you to find all the different ways to make a whole egg carton using the pieces that you cut out yesterday. Who can give us one example of a way to make one whole egg carton with our pieces?"

Instead of giving a lecture or demonstrating she poses a question. This gets her students actively thinking and talking when they are dealing with these concepts.

Several hands go up and Mrs. Chavez calls on Scott. "Here's one. You can use one of these sections of six, and then two of these threes," says Scott.

"How could we write that down?" asks Mrs. Chavez

Andrew volunteers to come to the board to demonstrate. "The first section is a half, and you write it like this: $^1/_2$."

"Yes," comments Mrs. Chavez. "This section is one of two equal parts."

Mrs. Chavez does not want the use of symbols to be the end goal for her students. Her goal is understanding fractions. She is, however, introducing the use of the symbols early in the activity, but she is careful to use language the students can understand when referring to the symbols. For example, she might sometimes refer to $^2/_3$ as "two of the three shares." She is careful to have the students dealing with symbols as labels for the materials they are using rather than as symbols that stand alone.

"Then the other section is one-third," begins Andrew. "No, it's not one third," Andrew corrects himself. "It has three cups, but it is one of the four pieces so it's one-fourth. There are two of them. So that's two-fourths. You write that like this: $^2/_4$."

Andrew's mix-up between the name for the fractional part and the number of objects in the part is common. Andrew's thinking out loud and correcting himself probably helped some of the other children to recognize the potential problem. Mrs. Chavez realizes that not everyone followed Andrew's thinking. She will be watching as the groups work to help any children who need clarification.

"I noticed the way you corrected yourself," says Mrs. Chavez. "It's easy to get mixed up when you see three cups but have to call them one-fourth."

"Since we are finding all the ways to make one whole egg carton, I would like you to write the information in the form of an equation like this."

Mrs. Chavez writes: $^1/_2 + ^2/_4 = 1$.

"Now, I would like you to work in your groups of four to find as many different ways as you can to make a whole. You will have about 10 or 15 minutes to work. Choose someone in your group to keep track of the ways you find."

Mrs. Chavez wants her students to figure out their own ways to get organized rather than doing the thinking for them and directing them to organize in her way. One of their tasks today is to find a way to manage all the pieces of egg cartons they have. She does not want to organize the recordkeeping for them either so she has not provided a worksheet but has asked the students to find their own way to keep records.

There is a lot of discussion and moving of pieces as the students explore the possibilties and discuss ways of keeping track.

After they have worked for a while, Mrs. Chavez has them stop to discuss their results. "Before we list any of the ways you found, I would like to know how you went about doing this job."

"When we started," Willis reports, "we just wrote down whatever we came up with. After we had found a few this way, Jan noticed we had $^{12}/_{12}$ and $^{6}/_{6}$ and said we ought to see if that works for the rest of the pieces."

"Yeah, that was neat," Jan adds. "We tried it for every section that we had and found out it took 2 halves, 3 thirds, 4 fourths, 6 sixths."

This is an example of something that seemed obvious to the teacher but was a new understanding for the students.

"Jeff got our group organized right away. He had us start with $^{1}/_{2}$ and found out what would go with it," says Addie. "We got $^{1}/_{2} + ^{1}/_{2}$, $^{1}/_{2} + ^{3}/_{6}$, $^{1}/_{2} + ^{2}/_{4}$."

Not everybody would have combined the three-sixths or two-fourths but may have written or said $^{1}/_{6}$ and $^{1}/_{6}$ and $^{1}/_{6}$ or $^{1}/_{4}$ and $^{1}/_{4}$. Either way would have been acceptable at this stage as she wants the students feeling at ease recording whatever they have done.

"That was sort of weird," Tim broke in. "All the numbers that went with the half were doubles."

"Could you explain that a little more?" ask Mrs. Chavez.

"Look, 3 and 3 are 6, 2 and 2 are 4, and 6 and 6 are 12."

Janelle raises her hand."That's not all the ways to make the halves, though. Our group found some different ways. $^{1}/_{4} + ^{3}/_{12}$, $^{1}/_{4} + ^{1}/_{6} + ^{1}/_{12}$, and $^{2}/_{6} + ^{2}/_{12}$."

After everyone has had a chance to share his or her results, Mrs. Chavez assigns the task she wants them to do for the rest of the period. "I will be giving you dice with fractions on them. Your job is to roll two dice to get two fractions. You then are to compare these two fractions to see which is larger and then to record that in some way. Look at the two fractions and predict which you think will be larger before you use the materials to check out your prediction. When you finish that, you are to add the two fractions and record that result."

Mrs. Chavez made the dice by writing fractions on the faces of plain wooden cubes. She included on this set of dice only those fractions that are possible using the egg cartons. (She will use this same set of dice later when she has her students work with fractions using 12 inch rulers.)

Later on Mrs. Chavez will present different models where it is possible to deal with other fractions like fifths, elevenths, etc.

"Let's do an example together first. Nina, please roll the dice and tell us what you got."

Nina rolls $^2/_3$ and $^5/_{12}$. The students begin finding the pieces in order to compare them.

"Wait a minute," says Mrs. Chavez. "Before you get your pieces out, see if you can tell which you expect to be bigger. Talk it over with your group, and then use the pieces to find out."

Rather than modeling certain procedures and rules for her students to follow, Mrs. Chavez is having them try to work out a reasonable solution on their own. They have enough experience with the convention for writing fractions to have something to start with. If any students come up with another way to record their work that communicates what they want, she will accept their ideas.

Mrs. Chavez overhears some of the conversations. "It seems like the $^5/_{12}$ should be bigger, but I know in fractions that doesn't always work. Seems like the pieces get littler and the numbers get bigger."

"I don't think I need the pieces. I know a third has four egg cups. Two of them would make 8 eggs, and that's more than 5."

When she sees that the groups have figured out which is the larger of the two fractions, she then asks them to find out what happens when they add the two fractions. She lets them work for a moment and then has them share what they did.

At this stage, she is not concerned about using the terms common denominator, nor is she imposing a rule that everything must be in simplest terms. After the students have worked with fractions and have attempted to make sense of them, they will have enough background for those ideas to be simply refinements for what they already know and understand.

Lucas says, "We put two of the thirds together and 5 of the 12ths and that makes one whole one and 1 twelfth left over. How can we write that down?"

Linda reports what her group did. "We think there's two ways to write it down. If you count up all the egg cups, you end up with 13. That could be $^{13}/_{12}$. Or if you put all the pieces together into one whole, you would have one and one twelfth left. We think you can write that 1 and $^1/_{12}$."

"That's right. Either way is correct; they both mean the same amount," says Mrs. Chavez.

Mrs. Chavez then gives them the rest of the period to do as many problems as they can. She will move around the room observing and will respond to the individual needs and concerns of the groups as they come up.

This is just the beginning of many experiences Mrs. Chavez wants her students to have with fractions. She will have them use many other materials, such as interlocking cubes, pattern blocks, graph paper, clock faces, and rulers. She wants them to be able to work with fractions with any denominator and eventually will ask them to choose the model they need to add or subtract unlike denominators.

A 6-8 Lesson Emphasizing Number

It is near the beginning of the school year in the heterogeneously grouped seventh grade class that Mr. Lee teaches. He knows that the students have had some experiences, both in school and out, in dealing with equivalent fractions, decimals, and percents. He also knows that there are tremendous variations of understanding of these equivalencies. Since many of the activities the students will do this year in the mathematics class involve rational numbers, he wants to help students make sense of the use of rational numbers. The purpose of this lesson is to help the students realize that they already know a lot about equivalent fractions, decimals, and percents by having the students identify some "reference points" that will be useful in estimating and operating with rational numbers. He will also use this lesson to do informal assessment of what the students understand about rational numbers.

The Lesson: Establishing reference points for equivalent fractions, decimals, and percents

"During the past week you have been bringing to class examples of fractions, decimals, and percents that you have found in newspapers and magazines. Each day we have taken a few minutes to share what you have found. Today we are going to use some of those examples, discuss what they mean, and find other ways of saying the same thing," Mr. Lee said.

Mr. Lee has used informal activities the past week to prepare the students for this lesson. He believes that students intuitively make sense of rational numbers in "real life."

"I have collected three statments that include the use of a fraction, decimal, or percent from articles that you have brought in during the past week. In your group of four, I want you to discuss what each one means."

Mr Lee has chosen the examples from the articles brought in by the sutdents. He wants to use examples that are probably familiar to the students, and ones that involve numbers close to one half.

1. 48% of California eighth graders watch television three hours or more on a typical weekday.

2. The winning percentage for the Padres this season was .524.

3. All athletic shoes are $^1/_2$ off.

The groups spent a few minutes discussing the meaning of these statements and then shared. Mr. Lee recorded their interpretations of the fraction, decimal, or percent under each statement.

1. 48 out of 100 students watch ...
   48/100th watch ...
   About one half of the 8th graders watch ...
   A little less than 50% of the 8th graders watch ...

2. The Padres won 86 games.
   The Padres won 52.4% of the games.
   The Padres won more than one half of the games.
   The Padres won about $^1/_2$ of the ...

3. The shoes are on sale, and the sale cost is half of the regular price.
   The shoe sale is 50% off.
   When you buy athletic shoes, you will get only one shoe.

As the group stated the last comment, the class laughed. Mr. Lee asked why the statement was funny.

Mr. Lee notices that most of the interpretations are sensible. He likes to promote a classroom atmosphere in which students are invited to look for the humor or absurdities in things. He knows that when students can see when something is absurd, they have some understanding.

"Ev yone knows the shoes are on sale," said Sandy. "When we were talking over what the $^1/_2$ meant, Jack made a crack about getting just one shoe. We thought the whole class would like his joke."

Mr. Lee smiled and then went on with the lesson. "Look at the three numbers in the statements, 48%, .524, and one half. Compare their values. Are they close or not so close?"

"Well, we used $^1/_2$ in our comments about each statement," said Bill. "They all are about $^1/_2$."

"How much is a half dollar?" asked Mr. Lee, "and how do we write it with decimals? How does a calculator show $^1/_2$?"

Several students responded with the correct answer.

Mr. Lee continued his discussion with the students, helping them focus on the meaning of words and references they hear as a part of daily conversation. "How much is one-fourth of a dollar, and how do we write this as a decimal? What is the name for the coin that is $^1/_4$ of a dollar? How much is a quarter of a pie? What do you do if you quarter an apple? What percent would we use if $^1/_4$ of our class is going to the game?

"How much is three-fourths of a dollar? How do we write this as a decimal? As a fraction? If $^3/_4$ of the class were going to the game, what percent of the class would be going?"

"If you get all the questions correct on a test, how do you express your score as a percent? If you missed 99 out of 100 questions on a test, how many items did you get correct? What percent of the items did you get correct? If 100% of the class were going to the game, how many would be going? How do we write one dollar with decimals?" Mr. Lee asked.

"What amount of money is close to being broke? Name some fractions that are so small that they are close to zero."

Mr. Lee passes out a worksheet and explains the assignment. "Let's look at some of the other examples of fractions, decimals, and percents from the articles you brought in. I want you to decide whether each one is closer to 0, $^1/_4$, $^1/_2$, $^3/_4$, or 1. Write each example under the most appropriate heading on the worksheet. Write examples you're not sure of in the not sure column." After they have done a few examples in the whole class, he asks them to work in small groups to complete the task.

Mr. Lee realizes that students often deal with mathematics as though it were a collection of rules that have no part in their daily lives. They miss making connections and seeing relationships and approach everything as though it were a new task to be learned. He is trying to help them associate what they already know and are able to use in their daily lives with the mathematics they will be using in school.

Mr. Lee has his students work together because he views this assignment as a learning opportunity for all students rather than as a test of what they already know.

This is only a beginning lesson on rational numbers. Mr. Lee plans to do many lessons using manipulatives, grid paper, and applications to help students further develop their understanding of the meaning of and operating with rational numbers. At all times in these future lessons, however, he will try to keep the focus on making sure the students can relate their work to what they already know and understand.

EXPERIENCES EMPHASIZING GEOMETRY

The three teachers presenting this series of lessons in geometry recognize the importance of providing their students with opportunities to study the structure and form of the three-dimensional world in which we live. The study of geometry in the elementary school not only gives students the background they need for more formal analysis in later course work but also helps them understand and appreciate their environment and provides the kinds of experiences that will be important to them in their daily lives. As adults they will be frequently faced with spatial problems in their jobs whether they be architects, bricklayers, or dress designers, and in their every day experiences, whether it be parking cars, playing tennis, or putting up shelves. Developing spatial awareness is a major goal of beginning geometry. Students cannot learn spatial awareness through memorizing definitions. Rather they need active involvement of an exploratory nature. A teacher's major responsibilty is to set up experiences which cause the students to sharpen their observations, to focus in on certain characteristics and attributes, and to develop the language necessary to talk about what they are seeing.

A K-3 Lesson Emphasizing Geometry

Mr. D'Amico's goals for his students in geometry include much more than learning the names of the basic shapes. He wants them to learn to look at the structures in their world and to become aware of shape and form. He wants them to recognize the attributes of shapes, to notice how shapes are alike and how they are different, to learn how they fit together and how they fill up space. He realizes that awareness of shape and form evolves over time as the result of many experiences. One way in which Mr. D'Amico has provided his students with these experiences is through allowing them time to explore shape and structure through building with blocks of all kinds. The blocks he has his children work with have mathematical relationship built into them. Children can discover such things as two short blocks are just as long as one long block and that three blue pattern blocks fit on top of one yellow block. The way each child gains in his or her ability to work with spatial tasks is very individual, so Mr. D'Amico provides open-ended tasks that can be approached in many different ways. He has chosen tasks very carefully so they provide an opportunity for every child to enter at any level and move to a higher level of competence.

The Lesson: Exploring shapes in a variety of ways

Mr. D'Amico has decided to spend several days focusing on geometry. Today the children in his class have a variety of tasks from which to choose. Most of the tasks are those that have been introduced previously and need little explanation or monitoring. He has one new task that he plans to introduce to the whole class, and then he plans to spend most of the time monitoring this new task while the others work independently on the more familiar activities.

The basic materials that the students will need to do each of the tasks are in plastic tubs. Each tub has a label which matches a label on a table or area in the classroom. The children can independently set up the stations by delivering each tub to its appropriate place in the room.

Before he asks some children to set up the math stations, Mr. D'Amico gathers the children around him on the floor and reviews what is in each tub and what they are to do at each station.

Mr. D'Amico has found that things run much more smoothly in his classroom when he gradually replaces familiar tasks with new tasks rather than introduce many new tasks all at once. His students are able to work well independently because most of them know exactly what to do and only a few children at a time need his help.

One task that the children have worked on several times before requires the use of pattern blocks. The children are to build a design with the blocks and then to copy the design by gluing down paper shapes that match the blocks.

In the second task, the children will work with the basic shapes to create figures or designs. They are provided with paper rectangles (including squares) and triangles of various sizes which they can cut apart to make new shapes if they wish. They can make their own circles by tracing around lids of various sizes.

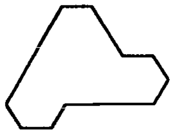
The children may choose to work with geoboards at the third station. Their task is to make as many different triangles as possible on the geoboard and then to record these triangles on the geoboard dot paper.

A fourth task that the children may choose to do is wallpapering a box. They have a collection of small jewelry boxes, tissue boxes, shoe boxes, and cereal and cracker boxes in the art corner. Their task is to cut pieces of paper that fit on each face of the box and then glue the paper to the box. They are then to make a simple recording, telling what shapes they used and the number of each shape.

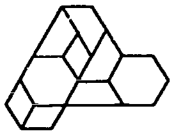
The tasks Mr. D'Amico presents his children often require some problem solving. This is true of the wallpapering boxes activity. The children must figure out a way to cut out pieces of paper the right sizes to cover the faces of the box. Mr. D'Amico has observed the children in earlier experiences and knows how difficult this is for them. They often are able to cut out papers that are the right shape but too small. They then cut out various little patches to cover the empty places. Mr. D'Amico must constantly decide whether to give his students more help by modeling some possible techniques or letting them work out their own ways to solve the problem. Today he has decided to wait and see how the students approach the task and talk to individuals rather than model anything for the group.

After Mr. D'Amico has reviewed the familiar tasks, he introduces a new task to his class.

"I have several cards here that have pattern block puzzles on them." He picks one of the cards and says, "I want to see how many different ways I can fill this puzzle with the pattern blocks."



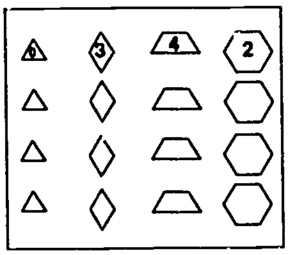
He places blocks on the card and then asks the children to notice which blocks he used.



"Since I am going to be filling this puzzle in many different ways, I want to keep track of the different blocks I use. This paper will help me do that."

He shows them an example of the recording sheet they will be using.

Mr. D'Amico spends quite a bit of time explaining the new task to his class. He knows the task is open-ended and that his students will each approach the task in an individual way. This does not mean, however, that they are free to do anything they want when they get to that station. He expects his students to respond individually but within the limits of the procedures he has established.



"How many yellow hexagons did I use?"
"Two."

"I am going to write a 2 right here on the yellow hexagon shape on my paper."

He knows that when he models something for the whole class, not everyone will be sure what to do. The majority of the class will know and will be able to help the others.

"How many red trapezoids did I use?"
"Four."

"So, I will write a 4 right here on the trapezoid shape."

"How many blue diamonds did I use?"
"Three."

"Where will I write that?."
"Right there on the diamond shape."

"And how many green triangles did I use?"
"You didn't use any. Zero."

"Where will I write that?"
"On the triangle."

"Good. Now, we know what pieces we used when we filled the puzzle one way. We need to fill the puzzle a different way. What do you think I should use this time?"

Mr. D'Amico knows that some students have trouble dealing with zero. It is hard for them to understand that you can write a number to stand for something that is not there. This particular activity is a good one for helping children make sense of zero. He makes sure that he has included zero in the example he is using when he models so that the children will know how to deal with it when it comes up.

Mr. D'Amico fills the puzzle a second time and shows his students how to record the pieces they used on the second row on the recording sheet.

He then tells the children that when they have their own record sheet, they will need to fill one puzzle four different ways and record those ways on their sheet before they choose a new card.

Mr. D'Amico then has his helpers deliver the tubs to the stations and begins dismissing the children to go to work. He first asks, "Who would like to work with these new cards today?" A reasonable number of hands go up, and he sends all of them to the station to begin work.

He dismisses a few more children at a time to choose a place to work.

He waits to see that everyone has made a choice and has begun work. Then he moves over to the new station so that he can make sure the children really understand what their task is. They are busy filling in the puzzles.

Mr. D'Amico allows his students to choose the tasks they wish to work with for the day. He knows they often have better insights into what they need than he does. Also, if they are allowed to choose, they usually are more involved than if someone chooses for them. Mr. D'Amico is comfortable allowing his students this freedom because he knows they have learned to handle it. They know he expects them to get right to work and to work hard.

Lina is having some difficulty. She is not keeping the blocks inside the puzzle outline. Mr. D'Amico focuses her attention on the lines and she is able to find the blocks that fit. He then helps two of the children find the right line on their recording sheet, but the others seem to understand what to do without any help.

Mr. D'Amico has learned to help his students by asking them questions which often get them to look again or notice something they have not focused on before. Whenever possible, he wants them to correct themselves rather than to look to him for the answers or the right way.

He looks over at Juan, who has picked a new card. "How many yellows do you think you could fit in that card?" he asks.

Juan studies the card for a moment and replies, "Well, maybe three." He then reaches for the yellow blocks and begins to arrange them in the outline. "Look, if I put a yellow here, then I can fit only two, but if I move it here, I can fit three."

It is important that the tasks that Mr. D'Amico presents his students have the potential for meeting the needs of individuals. Lina and Juan are doing the same task but in very different ways.

Mr. D'Amico sees that Juan is intrigued with arranging the yellows in many different ways and needs no more help from him so he moves over to watch Regina. Regina has nearly filled one of the puzzles. "What blocks do you think you could fit in that space?"

"I don't know. I didn't do it yet," responds Regina.

Maria looks over at Regina's card and says, "She could put a red or a blue and a green."

Dom is working rapidly, already filling a second recording sheet. In order to make the task more challenging to Dom, Mr. D'Amico poses a problem. "Could you find out the fewest number of blocks it would take to fill this card?" Dom starts piling up a stack of yellow hexagons and then realizes the hexagons will not fit on the card he has chosen. He slows down for a minute to study his card more carefully.

The children seem to understand the procedure for filling the puzzles and recording, so Mr. D'Amico decides to move around to see how the other children are doing.

He walks over to the station where the children are building with pattern blocks and copying their designs with paper pattern block shapes. He notices that some designs are quite simple and others complex. One child has the right paper shapes to go with each block but has not arranged them to match his original design. Another child, Barry, is just gluing the paper shapes down and has no blocks. Mr. D'Amico stops him and asks where his design is. "Right here," Barry says as he points at his paper. Mr. D'Amico tells him he is to make a design with the blocks first and then copy it with the shapes.

There are two reasons why Mr. D'Amico insists on Barry working with the blocks first. One is that Mr. D'Amico wants the children to understand that there are times when they must follow his directions. The other reason is that the purpose for the task is not simply to create a design but to analyze that design in order to copy it.

He then moves over to the table where children are working with the geoboards. Several children are making the same triangle shape over and over and just rearranging it on the geoboard.

He sits down to join the group and gets his own geoboard to work with. He makes one triangle pointing down, one tall and narrow, and one right triangle.

When he sees that the students have a misconception, he chooses to model what he wants them to do rather than tell them what to do. Those who are ready to gain in understanding by watching him will get the information they need. Those who are not ready will have no pressure to try to do what they believe he wants them to do before they understand.

Marciano looks over at Mr. D'Amico's geoboard and says, "Those aren't triangles. They look funny."

"Yes, they are," says Chris. "Now I'm getting the hang of this. Just look at this one!"

Mr. D'Amico notices that many still think there is only one way to make a triangle. He doesn't try to develop that understanding now but plans to teach a lesson in which that will be the main focus.

Chris didn't think there was anything to the task; he knew what a triangle was and thought he had thought of everything. Seeing the triangles that Mr. D'Amico made was enough to help Chris expand his idea of triangles and to stimulate his interest and creativity.

Mr. D'Amico continues to move around the room observing, modeling, and intervening only as needed to focus attention and stimulate thinking and creativity.

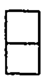
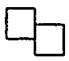
A 3-6 Lesson Emphasizing Geometry

Mrs. Olson knows that students need many informal experiences in geometry in order to develop the concepts and the ability to work spatially that is required to work with geometry in a more formal way later. She wants to provide experiences for her students that will help them understand how shapes and forms fit together to create structures. Activities that require students to build structures or to visualize geometric forms also require them to notice particular attributes and to begin to notice relationships.

Mrs. Olson's students have not had many previous experiences with geometric concepts. These concepts are very difficult for some students, but for others, they seem to come quite easily. She is careful to pick tasks that aren't tests that students can either do or not do. Instead, she wants to assign the types of activities that allow students to figure out what they don't already know and give them opportunities to improve no matter what level they are on. Mrs. Olson has seen that even those students who thought they couldn't learn to do spatial tasks really do improve if given the practice they need.

The Lesson: Developing spatial visualization

Mrs. Olson began this week's work in geometry by asking her students to work in their groups of four to find all the different ways to arrange five square tiles of equal size. The shapes that result from arranging five tiles are called pentominoes. Mrs. Olson explained that the tiles in each of the arrangements had to have full sides touching.

This  , not this 

As they arranged the tiles into various shapes, the students drew these shapes on graph paper and cut them out. They checked each shape carefully to make sure each was different from all the rest. If it was possible to lay one cutout shape on top of another and have it fit exactly, it could not be counted as different.

Mrs. Olson provides many opportunities for her students to work with mathematical ideas cooperatively in groups of four. She knows that students learn much from working together. Their thinking is stimulated by interaction with each other, and they are often willing to tackle problems that would intimidate them if they were working alone.

Mrs. Olson was very pleased with all that happened during this lesson. She could see

The students sometimes had to rotate or flip the shapes to see if they matched. The students worked until all the groups were convinced that they had found all the possible pentominoes.

Mrs. Olson wants her students to work with a variety of geometric tasks, so she is following the pentomino task with several different activities. Her students will be working for several days on these tasks. The tasks are listed on a chart that is posted in the front of the room. The chart is referred to as the MENU. The students may work alone or, if they wish, with others from their group of four. The tasks listed under DESSERT will be worked on if any students finish early.

MENU
Bird's eye view cards
The Factory (computer)
Folding boxes
Milk carton pentominoes
Mystery shapes (overhead projector)

DESSERT
Factory box problem
Pentomino puzzle

Today, Mrs. Olson looks around the room to see how the students are doing. This is the second day that the students have worked on this menu, and they seem to know what they want to work on and are getting to work quickly.

Nettie, Casey, and Joel are working with the bird's eye view cards. Each card has a picture of the side view of a structure made from various blocks and pieces of wood. The students are to draw a picture of what they think the structure looks like if viewed from above. Each of these students is approaching this task in his or her own way.

Nettie is very confident and quickly sketches out the top view of the structure on the card she is using. She then searches through the stack of the remaining cards to find one that looks hard to her and presents more of a challenge.

Casey is looking at a card and digging through the box of blocks to find the ones pictured on her card. "I need to build it first and then look at it,

that the students were really developing the idea of congruent shapes. Many were quite surprised when they discovered that shapes they thought were different actually were just flips of shapes they already had cut out. She was also very interested in the process the groups went through in deciding whether they truly had found all the ways. At one point, one group was ready to give up; but when they heard another group had found more, they persisted a little longer and found another shape. She asked the group that finished first to consider the possibility that there were more ways they hadn't found yet; but they were sure they had considered all the possibilities in a systematic way and were not pursuaded to keep trying.

and then I can draw what I see. I have no idea what it will look like if I don't," she explains to Nettie. Although Casey is aware that she is very good in other mathematical tasks, especially those requiring logical thinking or strategies, she knows she is not good yet with spatial tasks.

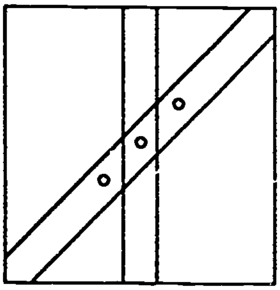
Mrs. Olson is glad to see that Casey is so open about her need to use the materials. She has spent a lot of time helping students learn to accept their various abilities and to recognize that everyone has some things that are relatively easy for them and other things that are more difficult.

Joel is sketching out what he thinks will be the top view of the structure on his card. "I think this is what it is going to look like, but I am going to check and see." After he has drawn what he thinks is the top view, he checks his prediction by building the structure and comparing it to his drawing. "Hey, I did it!" he exclaims.

Mrs. Olson notices that Joel is improving and is now able to visualize some structures from various points of view. She also sees growth in Joel's willingness to take risks. Earlier in the year, Joel did not want to try anything unless he knew ahead of time that he would do it right. Mrs. Olson sees two reasons for this newfound confidence. First of all, Joel worked with these cards yesterday and brings some experience to the task today. Even with relatively little experience, Joel has become better at visualizing. Mrs. Olson also knows that the change in Joel's attitude comes in part from the conscious effort Mrs. Olson has made to establish a safe environment for students to work in. Joel has learned that it is okay to take a risk because if he is not right, there will be no penalty or disapproval on the part of his teacher.

Mrs. Olson smiles as she observes the students working at the computer. Three students are very intent on watching the computer screen and involved in a lively conversation. These students are working with a program called The Factory, which was given to their classroom as part of a set of materials provided to all schools in the state. In this program, the students are attempting to reproduce a board shown them by the computer.

For example:



They do this by directing the "factory" to go through a series of actions. The actions include such things as rotating the board, punching round or square holes in the board, or painting wide or thick stripes on the board. In order to reproduce a board, the students have to direct the computer to take the appropriate steps in the correct order.

Mrs. Olson is relieved to have found an appropriate use for her one classroom computer. She has been searching for effective ways to use it so it would not just become an expensive alternative to a workbook. The program, The Factory, really takes advantage of the power of the computer. There is no way that Mrs. Olson could provide this type of experience for her students without the computer. It is really a good way for students to improve their visualization skills. Having the computer as one of the Menu tasks is an easy way to give all students a chance to use it.

Mrs. Olson overhears some of the conversation.

"No, we have to turn the board 45 degrees first and then paint the stripes."

"I think we have to paint the stripes before we turn it."
"That won't work. If we do that, the stripes will be going the wrong way."

Mrs. Olson moves away to observe another group. She knows that whatever the students decide to try they will be able to see the results instantly when the computer follows their commands and produces a "board." Then, they can make any corrections that are necessary.

Jake and Paul are working on a task which is a follow-up to the pentomino task the students worked with earlier in the week. They have a sheet that has all the pentominoes on it. Their task is to look at the pentominoes and predict which ones can be folded into boxes with no tops. After they have made their predictions, they will check those predictions by cutting out the shapes and actually folding them.

Mrs. Olson selects tasks that allow her students to use the materials to validate their thinking rather than always looking to her for the answers.

"I can't believe this one folds into a box. It sure didn't look like it would to me," says Dino to Mrs. Olson.

Other students are using small milk cartons with the tops cut off for another task that follows-up the work with pentominoes. They are to decide how to cut the cartons so that when they flatten them out, they form each of those pentominoes that fold into boxes. Junior is studying a pentomino shape very carefully before making any cuts in a milk carton.

Mrs. Olson allows enough time at the end of each math period for the students to discuss their work for the day. This gives the students an opportunity to bring up problems and concerns as well as insights and accomplishments. They know they are not to give away anything that would rob other students of new learning and discoveries. Usually, however, these discussions motivate students to try tasks they haven't started yet. It also creates an atmosphere of productivity in the classroom. This is important because Mrs. Olson wants her students to take the responsibility for planning their time carefully and getting everything done that she has assigned on the menu.

A 6-8 Lesson Emphasizing Geometry

Ms. Lightfoot believes that the successful study of geometry in seventh and eighth grades builds on informal experiences. When she begins work with her students, she does not know how complete or accurate their understanding of geometric concepts is. She does not assume understanding just because they can perform certain procedures and arrive at correct answers. She finds that asking her students to use concrete materials to explore particular questions often reveals the depth of their understanding and provides opportunites for her to extend their understanding and challenge their thinking at the same time.

Ms. Lightfoot is going to begin a unit on surface area and volume by having her students observe the patterns and relationships that emerge from building cubes of increasing sizes. Before she gets into the concepts she wants to teach, she will have her students actually build cubes. She believes this is necessary as she knows from past experience that her students will likely have many incomplete ideas or misconceptions about the structure of cubes.

The Lesson: Determining volume and surface area of cubes

Ms. Lightfoot has her students working with partners. Each set of partners has a box of approximately 50 identical cubes.

Ms. Lightfoot had to borrow cubes from other classrooms in order to make sure she would have enough for these experiences. It was worth it to her because she has become convinced, by watching her students, that they need to actually build cubes to really understand. She used to think that because they had had experiences in earlier years with models and because they could learn the formulas for volume, they didn't need these types of experiences anymore.

"In a minute I am going to ask you to open your box of cubes. Before you take any of the cubes out, predict the dimensions of the largest cube you think you can build with your box of cubes. You all have approximately 50 cubes in your box. Make your predictions and then build the largest cube you can."

Ms. Lightfoot listens to the conversations the students are having as they discuss their predictions.

Another reason that Ms. Lightfoot is beginning with this exploration of cubes is to find out what the students already know and understand (or do not understand). She will listen to the their use of vocabulary and watch for any misconceptions that they bring to the task. With this information she will be better able to provide the appropriate experiences and pacing. She will also be able to see which students need to be challenged.

"I bet the cube will be 7 x 7. That would be close to 50," said Peter.

Jessica tells her partner, "Well, cubes have length and width and height, so I think we need to divide 50 cubes by three. That will make about 15 on each side."

Ms. Lightfoot sees that many of the students used their prediction as a starting point for building the base of their cube. Peter and his partner began building a base of 7 x 7 and soon realized they would just barely have enough to build one layer and not a whole cube.

She notices that several of her students started with a 2 x 2 x 2 cube and added on to it. Two of these students started adding layers and ended up with a structure that was 2 x 2 x 4. It didn't take them long to realize that what they were building wasn't a cube any longer. In their attempt to fix it, they ended up with a 3 x 3 x 4. They laughed when they realized the number of cubes they needed to build a 4 x 4 x 4 and that they didn't have enough.

When Ms. Lightfoot sees that several of the students have finished building their cubes, she stops the class so that they can discuss what happened and whether their predictions were reasonable or not.

During the discussion the students agree they were surprised by two things. One was by how fast they used up their cubes, and the other was by how tricky it was to keep the cube growing correctly.

"There are some other interesting things about cubes and relationships between cubes that I would like us to explore and think about," says Ms. Lightfoot. "We will be building cubes of various sizes and recording information about these cubes on a chart like this. Let's begin by looking at a cube that is made from just one of the blocks."

Ms. Lightfoot is going to build on her students' interest in building cubes and will pose several questions for them to explore. Through these explorations she wants them to think about where the formula for determining volume comes from, to appreciate the growth of cubic measures, to learn about surface area, and to consider the idea of relative surface area. She will expand on all of those ideas later; but for now she wants them observing, wondering, and noticing.

Each of the terms on the chart is clarified by the students or Ms. Lightfoot, and the information is recorded on the chart.

Edge of one side	*1*	*2*	*3*	*4*	*5*
Area of one side	*1*				
Volume	*1*				
Surface area	*6*				

Some of the students had difficultly understanding what was meant by surface area. Ms. Lightfoot explained that it may help to think of the surface area as the "skin" of the cube.

Ms. Lightfoot goes on with the lesson. "Now I want you to build a cube with an edge of 2."

The class examines the cubes they built, and the second column of the chart is filled out.

Edge of one side	*1*	*2*	*3*	*4*	*5*
Area of one side	*1*	*4*			
Volume	*1*	*8*			
Surface area	*6*	*24*			

"Your task for the rest of the period is to work together in your groups of four to construct successively larger cubes with edges of 3, 4, and 5. Fill in the information on a chart, including the information for 1 and 2 that we filled in on the class chart. Once you have the information recorded, look to see what kinds of patterns and relationships you can discover."

Ms. Lightfoot is confident that all the patterns and relationships she wants her students to know will come up through the group of four work and class discussion. She prefers the students to be actively searching for these relationships rather than passively listening to her lecture or point them out.

"There is one more thing I would like you to look at when you build the cubes. Look at the cube constructed from just one block. How many of its surfaces are showing?"

Several students respond, "Six."

"Now look at the cube with the edge of two. Are there any blocks that have 6 surfaces showing?"

"No," says Andrew. " Each of the blocks in this cube has 3 surfaces showing."

"Do you think that will be true for all the other cubes? Look at the other blocks as you build them and keep a separate record of the numbers of surfaces showing. When you have looked at the information you gathered about these cubes, write at least 4 summary statements about what you learned. "

Asking students to write what they noticed is an effective way to get them to refine and clarify their thinking. It will take a lot of productive discussion for a group of four to decide what conclusions it can reach from the information the students gathered.

The students get busy working. There is much discussion as they figure out how to work together to build the cubes and record the information. Sometimes they find it necessary to join with another group in order to have enough cubes.

Ms. Lightfoot moves around the room as the students work, joining in the conversations, posing questions, and redirecting if needed.

Ms. Lightfoot will follow up this lesson in many different ways. After she reads the summary statements, she will choose several of them

(including both accurate and inaccurate statements) and post them in the classroom. She will ask the groups of four to decide whether they agree or disagree with each statement. This will give the students another opportunity to look at the relationships which appeared on the chart and to discuss and clarify them. She will also share the following newspaper article with them and ask them to consider whether or not any of the information they gathered could help them understand the concept of relative skin area.

> *Parents were being advised that small children should not spend long periods of time outside in the extreme cold because their relative skin area is much larger than that of adults and the heat loss would therefore be greater.*

She views this introductory lesson as being one that asks the students to begin thinking about surface area and volume, but which will need to be followed up with many additional experiences.

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