DOCUMENT RESUME

ED 277 573 SE 047 673

AUTHOR	Verhille, Charles, Ed.
TITLE	Canadian Mathematics Education Study Group. Proceedings of the 1985 Annual Meeting (Universite Laval, Quebec, Quebec, June 7-11, 1985).
INSTITUTION	Canadian Mathematics Education Study Group.
PUB DATE	7 Jun 85
NOTE	176p.; Much of document contains small type.
PUB TYPE	Reports - Research/Technical (143) -- Collected Works - Conference Proceedings (021)
EDRS PRICE	MF01/PC08 Plus Postage.
DESCRIPTORS	Attitudes; Calculus; Computer Software; Educational Research; *Educational Theories; Elementary Secondary Education; Error Patterns; Higher Education; *Mathematical Applications; *Mathematics Education; *Microcomputers; Problem Solving; Sex Differences; *Teaching Methods
IDENTIFIERS	Mathematics Education Research

ABSTRACT

These proceedings include two lectures followed by working group and topic group contributions in reduced format. The two lectures were by Heinrich Bauersfeld, speaking on contributions to a fundamental theory of mathematics learning and teaching, and Henry Pollak, speaking on the relationship between applications of mathematics and the teaching of mathematics. Working groups concerned lessons from research about student errors, Logo activities for the high school, the impact of symbolic manipulation software on the teaching of calculus, and the role of feelings in learning mathematics. Topic groups considered exploratory problem solving in the mathematics classroom, epistemological fallacies, and recent Canadian research concerning teaching, gender, and mathematics. (MNS)

CMESG/GCEDM

1985 PROCEEDINGS

JUNE 7-11, 1985

UNIVERSITE LAVAL

EDITED BY C VERHILLE

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1985 ANNUAL MEETING

UNIVERSITE LAVAL

QUEBEC, QUEBEC

JUNE 7-11,1985

EDITED BY CHARLES VERHILLE

CMESG/GCEDM
1985 Meeting
PROCEEDINGS

Editors foreward

The proceedings for the 1985 CMESG/GCEDM meeting have been delayed for a long time. It was necessary to wait until a major contribution was received, otherwise the proceedings would have been most inadequate.

The proceedings, following the format of previous years, include the major lectures presented by Heinrich Bauersfeld and Henry Pollak followed by working group and topic group contributions in reduced format. This meeting represented our first effort to plan a joint speaker with the CMS - a group with whom we have many interests in common.

This represents our second meeting at Laval. The University in particular as well as Quebec City in general provide pleasant surroundings for such a gathering. We are especially appreciative to Claude Gaulin and Bernard Hodgson for making the local arrangements.

Charles Verhille
Editor

Canadian Mathematics Education Study Group
Groupe canadien d'étude en didactique des mathématiques

1985 Meeting

The ninth annual meeting of the Study Group was held at Laval University, June 7 to 11, 1985. Fifty mathematics educators and mathematicans met in plenary sessions and working groups. This year the conference was deliberately arranged to follow immediately on the CMS Summer Meeting and the first of the two guest lectures, by Henry Pollak (Bell Communications Research, was planned in collaboration with the CMS Education Committee. Dr. Pollak spoke "On the relations between the application of mathematics and the teaching of mathematics". He identified four different meanings commonly attached to the words "applied mathematics", and considered the implications of each for curriculum and for pedagogy. Also co-sponsored by CMS Education Committee was a session, led by Peter Taylor (Queen's), on "Exploratory problem solving in the mathematics classroom".

The second guest speaker was Heinrich Bauersfeld (IDM, Bielefeld) who made "Contributions to a fundamental theory of mathematics learning and teaching". Setting out to answer the question: How do we manage to retrieve what we require and adapt it to a new situation?, Professor Bauersfeld wove an intriguing account of constructivist theories.

Other lectures were given by Fernand Lemay (Laval), who presented a masterly sweep through the historical developments of analytic and synthetic geometry, and by Jacques Désautels (Laval), who applied the epistemological theories of Gaston Bachelard to the learning of science. Three accounts of specific researches on teaching, gender and mathematics were given by Roberta Mura (Laval), Gila Hanna (OISE) and Erika Kuendiger (Windsor).

The working groups at this conference focused on a positive view of students' errors, a group led by Stanley Erlwanger (Concordia) and Dieter Lunkenbein (Sherbrooke); on more advanced activities with LOGO, a group led by Joel Hillel (Concordia). A third group investigated the possibilities of symbolic manipulation software, led by Bernard Hodgson (Laval) and Eric Muller (Brock); the fourth tackled feelings and mathematics, led by Fran Rosamond (San Diego) and John Poland (Carleton).

This bald summary may indicate the scope of the conference but may not make clear the special characteristics of its style. Most conferences of comparable length offer participants many more lectures and paper presentations. The result, as everyone knows, is that participants at conventional conferences are selective in their attendance at sessions; no one can sit through continuous periods of being talked at. Participants at Study Group meetings, where ample time is allowed for cooperative work and discussion, tend to follow the whole programme. This generates more of a sense of common interest, a bridging of differences rather than an accentuation of them.

David Wheeler
Concordia University
Montreal

IN MEMORIAM DIETER LUNKENBEIN

The mathematics education community has been deeply shocked to hear bout the sudden death of our colleague Dieter Lunkenbein, on September 11, 1985, at 48 years of age.

Born and educated in Germany, he had come to Canada in 1968 to work as a research assistant for Dr. Zoltan P. Dienes at the Centre de Recherche en Psycho-mathématique in Sherbrooke. He subsequently got a Ph.D. in mathematics education at Laval University and he bacame a regular faculty member of Université de Sherbrooke, where he has displayed strong leadership in teacher education as well as in research and development in mathematics education.

In 1982 he was awarded the "Abel Gauthier Prize" by the Association Mathématique du Québec in recognition for his significant and exceptional contribution to mathematics education in Québec. At the Canadian level, he has been very active in the annual meetings of the Canadian Mathematics Education Study Group, particularly in working groups about teacher education and about the field of mathematics education, and as a leader of many groups on geometry education -- an area for which he was a recognized expert.

Dieter is the author of more than 70 scientific lectures or papers, including articles in Educational Studies in Mathematics, For the Learning of Mathematics, Bulletin de l'A.M.Q., etc. At the international level, he has been involved in many conferences and for about ten years he has been very active as a coopted member of the Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM), of which he was the President from 1982 to 1984.

For the mathematics education community, the death of Dieter Lunkenbein constitutes a great loss. Everyone will long remember his work and dedication to our field as well as his impressive human qualities.

LECTURE 1

CONTRIBUTIONS TO A FUNDAMENTAL THEORY OF MATHEMATICS LEARNING AND TEACHING

BY HEINRICH BAUERSFELD
UNIVERSITAT BIELEFELD

Contributions to a fundamental theory of mathematics learning and teaching

HEINRICH BAUERSFELD
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> "Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time. Collateral learning in the way of formation of enduring attitudes, of likes and dislikes, may be and often is much more important than the spelling lesson or lesson in geography that is learned. For these attitudes are what fundamentally count in the future." JOHN DEWEY (1938)

1. A theory gap in school practice

A few years ago the report of an outstanding piece of research appeared: it is D.HOPF's investigation on the teaching of mathematics in grade 7 of the Gymnasium[1)] (D.HOPF 1980). The study analyses data from 14 000 students, their teachers and parents, at 417 Gymnasien in the area of West Germany including West Berlin, and it is a representative sample. Detailed questionnaires were used in order to find out about the "social, cognitive, and motivational conditions under which learning outcomes and credits" are produced in mathematics lessons. In our view the most interesting results are:

* There is an overwhelming dominance of direct instruction, in particular the well-known game of teacher's questioning and student's response as well as teacher's monologues (lecturing) and similar types of instruction; and
* it is not possible to identify "any more general structure" in the extremely rich data base "which would indicate the existence of overall concepts for the orientation of method

and teaching". Clearly, this came out quite contrary to the researcher's expectation, that "at least some of the concepts which were under discussion in mathematics education for methods and teaching would appear more often than in single specific phases of the lessons only." (D.HOPF 1980, p. 192)
The lack of explicit theory in everyday school practice could prove to be a surface phenomenon: Perhaps teachers do not talk about theoretical backgrounds, but they may follow recipes for action rather consistently, which are based upon certain theoretical concepts. One might expect, therefore, that careful analyses could lead to reconstructions of a hidden though theory-based grammar of teacher's decisions.

Reviewing various well-known concepts of mathematics education, the researcher thought about such analyses, but "found no reason for establishing a search for interpretations which could be traced back to more general concepts." (D.HOPF 1980, p.191). That is to say, the researchers found continuities and regularities in the processes of the mathematics classroom - e.g. the preference for direct instruction - but they could not find any relation with the concepts that appear in the theoretical debates of the mathematics education community.

Now we can ask more generally: If not through theoretical reflection, how then do the often documented and criticized patterns of teaching and learning in mathematics classrooms come into being (see the "recitation game", HOETKER and AHLBRAND 1969)? On the one hand the available theories obviously do not cover the practitioner's needs; the theories do not have sufficient explanatory power. The hidden regularities of everyday classroom practice on the other hand function as if they arose from the subjective theories of the participants (teachers and students). So probably these hidden regularities are the outcomes of covert processes of optimization, that is, they may represent a bearable balance between the given actual, societal, institutional, and micro-sociological forces in the classroom (where bearable means: bearable for the participants). Provided this is an adequate description, then the hidden gensis of the regularities would explain the product´s tenacity and resistance against every reform.

The following remarks are grouped into three chapters. The main part, chapter 3, presents theoretical considerations from the many micro-analyses of teaching-learning situations in mathematics conducted by a research group at the IDM Bielefeld (BAUERSFELD, KRUMMHEUER, VOIGT). The thesis of the domain-specific orientation of a peron´s action leads to new views on (and descriptions of) abstraction/generalization, representation/embodiment of concepts, and learning.

The preceding chapter 2 can just as well be read after chapter 3, since the remarks on deficiencies and paradigms in theories of mathematics education may then be more understandable. It is meant as an introduction as presented here. The concluding chapter 4 relates the theoretical discussion to certain recent issues in problem solving. The application gives support to the thesis of the preceding chapter.

2. The paradigms of theories of mathematics education

The usual set of didactical questions: What is the nature of the subject? How is it learned? and How should we teach it? reproduces in itself disciplinary boundaries. Theories of mathematics education tend therefore to stress the relation either to the acting persons or to the subject matter of mathematics. Thus we receive psychological or mathematical-philosophical answers, such as student-centered "theories of learning" and teacher-centered "theories of instruction" or as subject-matter-centered theories of knowledge, of curriculum, of task analysis, of AI-simulations[2)] etc. Until very recently, linguistics, sociology, etc., were not disciplines to which the math ed community referred.

Both theoretical mainstreams use the stages metaphor when characterizing developmental aspects. Psychological approaches arrive at stages based on classes - or more precisely at progressive class-inclusions - of abilities (e.g. KRUTETSKII 1976), or of operations (e.g. PIAGET 1971), at levels of learning (e.g. VAN HIELE 1959) etc. Since mathematical abilities as well as the success of learning mathematics are described or measured through the quality of solving certain mathematical tasks, it becomes inevitable that the hierarchies of psychological constructs map subject-matter structures. They duplicate mathematical hierarchies, but do not create genuine psychological descriptions of the related actions. The subject-matter-centered theories on the other hand use mathematical structures directly for the modeling of stages. We can state therefore, that in both theoretical mainstreams the description of the field is dominated by mathematical means.[3]

But math educators will have to extend their fundamental theoretical questions, if at least a reasonable subset of classroom processes follows hidden regulations. The more since the regulations develop interactively rather than directly through the participant´s intentions, and with effects often inconsistent if not conflicting with the official aims. Then we will have to take into account not only that teachers and students enter and leave the classroom with certain individual dispositions, intentions, and expectations - which we do in order to draw inferences from the difference between the two cross-sections, but we will also have to ask what they make of it in a concrete situation, how they actually employ available states of knowledge, and when they activate and how they use schemata (and not only which ones, as is usually done). Cross-section analyses of input and output states with inferences about the process in between are no longer sufficient for an adequate understanding. If, as becomes evident, knowledge develops together with and as part of the knowledge, then this calls in question the process-product metaphor.

Furthermore the ongoing vivid interation in the classroom indeed leads to very personal (subjective) interpretations and constructions of meaning. But socially shared meanings and norms of content-processing are produced as well. And these are not just taken over like ready-made rules, rather they are constituted through the interaction, they become reality via the mutual processes of construction and negotiation. That is to say, we have to discriminate individual structures of potentially available knowledge from the interactional structuring of the actual actions. And on a social level we have to discriminate (so-called) objective subject-matter structures from the related meanings, norms, and claims for validity, as they are constituted in the course of the interaction in the classroom. This of course makes cause-effect analyses haphazard, because attributing cause to a single person´s action may become difficult.

There is a remarkable convergence in recent developments in mathematics and in cognitive science as well as psychology that supports the scepticism advocated here. In the view of cognitive psychologists the When and the How, as mentioned above, are mainly organized on metacognitive levels. The classical problem solving strategy from AI-developments - building up a hierarchy of operations or organizing control on a superordinate level - recurs here and has been the subject of intensive discussion in cognitive psychology recently (see ANN L. BROWN, J.C. CAMPIONE, and M.T.H.CHI in WEINERT 1984) under the name "metacognition". Investigations begin to focus on "dynamic learning situations" and on "interactive processes" (J.V. WERTSCH 1978, 1984; and A.L. BROWN in WEINERT 1984, p.101/102. One of the "many largely unsolved problems" in developing advanced intelligent computer systems for educational purposes that TIM O´SHEA has named is that "not enough thought has been given to represent inexpert reasoning". He has also pointed at the crucial role of "using natural language" (1984, p.266). Interestingly the attack comes from non-human information processing research, human understanding, learning, and reasoning in general and of mathematics in particular.[4)]

Even mathematics itself has been challenged from within the community, as by LAKATOS´ concept of "informal, quasi-empirical mathematics", an image of the discipline which he holds out against the counterpart of "authoritative, infallible, irrefutable mathematics"[5)] (1976, p.5). FREUDENTHAL has long since argued against the same enemy: "True mathematics is a meaningful activity in an open domain." (1983, p.39).

> "Why, come to think of it, do we have so few good ideas and theories about the mind? I propose the following answer to this question:
> 1. It may be the most difficult question Science ever asked.
> 2. It is made even harder because our first theories have led us in the wrong direction."
>
> MARVIN MINSKY (1982, p.35)

3. "Domains of Subjective Experience" and "Society of Mind"

In our research process the adoption of sociological methods and concepts has turned into a process of adapting the means to the end. Since we are interested in learning and teaching mathematics rather than in general social structures, as identified by sociologists across subject-matter, our analyses are focussed on the relations between the subject-matter aspects, as thematized by the participants, and the predominantly social nature of classroom processes and their conditions. This, we think, describes an important weakness in the dominating psychological and subject-matter-oriented theories.

Our micro-analyses of video-taped teaching-learning situations at different schools and with different ages have led to three related theoretical elements. GOTZ KRUMMHEUER has adapted GOFFMAN'S "frame analysis" in order to describe the participants'(teacher and students) definitions of situations - "frames" - and their stratified changes - "keyings" - in the flow of interactions. Complementary to these actual activities, my concept of "domains of subjective experience (DSE)" aims at the description of the sources and the organization of memory and of the related long-term effects called learning. These respresentations function as relatively stable dispositions and as the potential from which the individual's actual orientation and action is coined and formed. JORG VOIGT has investigated the hidden regulations of classroom procedure as they are constituted among the participants. He describes "patterns of social interaction" and their relation to "moves under duress" and to (DSE-rooted) individual "routines". (See KRUMMHEUER 1983 and 1984, BAUERSFELD 1980, 1983 and 1985, and VOIGT 1984 and 1985.)

In the following I shall restrict myself to discussing the main aspects of the DSE-metaphor. It should be noticed that we offer alternative interpretations but do not claim to describe "the" reality. The theoretical elements offer a well-founded perspective on classroom processes among other theoretical perspectives, with which it competes. The theses and their substained connections are the products of "abductions" (C.S. PEIRCE 1965, J.VOIGT 1984a). Thus a specific understanding of the genesis of theories as well as of theory itself is functional in our approach.

1. Thesis All subjective experience is domain-specific. Therefore all experiences of a person (subject) are organized in Domains of Subjective Experiences (DSE).[6)]

Whenever I have experiences, that is: I learn, actively and/or passively, this occurs in a concrete situation, something which I realize as context. Thus learning is situation specific, is learning-in-context. Learning is not limited to cognitive dimensions. Since I cannot switch off one or the other of my senses deliberately, all of my senses are involved, particularly the genetically older organs like the mid-brain (emotions) and the cerebellum (motor functioning). The stronger the accompanying emotions, the more distinct and richer are certain details and circumstances in the recollection. We therefore speak of the totality of experiences and learning.

Learning is also multidimensional: I learn how to do things, and along with that, though mostly indirectly, I learn about the when and the why. At all times I learn about myself and about others.

The specificity to situation, the totality, and the multidimensionality, give good reasons for the conjecture that all experiences of a person are stored in memory in disparate domains according to the related situations. Each DSE encloses all of the aspects and ascribed meanings which appeared to be relevant for the person who was acting within the situation. Encountering the same situation repeatedly contributes to the consolidation of the related DSE, but as well to its isolation from other DSE´s. When entering a specific known situation a person immediately ´knows´ very much, due to the activated DSE.

> An example: More than 25 years ago during teaching practice with student teachers in the country, I visited a little nongraded school of some twenty pupils ranging in age from 7 to 14. The teacher opened the first lesson with a series of spectacular actions. He called on the attention of the few 11 and 12 year olds and made the others work silently. Then he ostentatiously dropped a plate which burst into pieces. A defective teapot followed, and finally he broke a few wood sticks into pieces. His hand waved over the scene accompanied by the key question: "What is this?!" And a nice little girl answered: "It is the introduction to fractions!"

Apparently she had experienced this happening repeatedly in her earlier school years and she knew it would end up with naming and calculating with fractions. From that she gave a clear definition of the situation.

Under a phylogenetic perspective the immediate availability of an adequate DSE guarantees survival. The complex nature of the DSE´s enables the activation of a specific one just through a smell, a touch, a word, a picture, an action etc., and in such a way provides for the instantaneous identification of a dangerous situation for quick and appropriate (re-)actions, and for a certain coping with possible consequences. Obviously many of the students reactions in mathematics lessons are examples of such direct and prompt concatenation, ensuring survival in the classroom and saving unpleasant effort and reasoning.

The ideals of mathematizing, on the other hand, are clearly related to critical distance, to analytic decomposition and reflected construction, and to operations with symbols and models. These arts do not develop along the elicitation-reaction line. In order to overcome the troublesome phylogenetic conditions (which we cannot change nor deny), instructional situations should therefore give more attention to indirect learning on higher levels rather than to behavioral responses/evoking through invitations on the bottom level of direct action and reaction only.

2. Thesis The domains of subjective experiences (DSE) are stored in memory in a non-hierarchically ordered accumulation, following M.MINSKY´s idea of a "society of mind" (1982). In a given situation the DSE´s function in competition for activation, independently from each other, and this the more intensively they have been built up initially.

The model represents a powerful description for a functioning organisation of the isolated DSE´s. According to the flow of personal impressions and activities the "society of mind" is under continuous change and development. Permanent and lifelong new DSE´s are formed[7], older DSE´s are changing. The gradual fading away of DSE´s, not activated for a long time, diminishes the growing burden, the more, the lower, the emotional status and the frequency of activation of the DSE are.

Every activation produces change: Often activated DSE´s pass through many transformations: the meaning, the relations, and the importance of their elements may shift, the characteristics of the situations become less specific (they allow more variance, i.e. they generalize), and a hard core of routines, of easy meanings, and of preferred verbal or pictorial presentations is shaped. In an actual situation these well-developed DSE´s obtrude themselves through their smooth perspectives and therefore have the best chance to win the competition for reactivation. Thus success has stablizing and tracking effects, though not necessarily for optimal solutions, as an observer may note rather than relative personal optima. But since every situation is new in a certain sense, there is an opportunity for younger and less elaborated DSE´s ("soft state") as well as for easy and robust older DSE´s. There is no preference in principle in the activation game as the phenomenon of regression demonstrates: The relapse into certain pattern of understanding and action under stress, which are older and less adequate or less differentiated, but are functioning more quickly and more reliably, receives a simple explanation from within the "society of mind" model.

The model, by the way, leaves no room for an independent or superordinate authority in the "society", a "demon" or something similar, who selects and decidedly activates DSE´s. Clearly we can exercise a limited influence on our internal retrieval processes, but we are not in command of our memory as the many failures of mnemonics show. An idea suggests itself - or not.

Through microethnographical analyses a surprisingly high degree of separation between single DSE's has been demonstrated (LAWLER 1979, Bauersfeld 1982). Outcomes from quantitative-experimental research work gives support also. Recently E.FISCHBEIN et. al. (1985) have investigated the solving of verbal problems in multiplication and division with 623 Italian pupils in grades 5, 7, and 9. They focussed their attention on the role of "implicit, unconscious, and primitive intuitive models." Such models, so goes their hypothesis, might mediate "the identification of the operation needed to solve a problem" and thus "impose their own constraints on the search process." (FISCHBEIN et.al. 1985, p.4). The authors arrive at the unexpected profoundness of the expected effects, which they describe as "a fundamental dilemma" for the teacher:

> "The initial didactical models seem to become so deeply rooted in the learner's mind that they continue to exert an unconscious control over mental behavior even after the learner has acquired formal mathematical notions that are solid and correct."(p.16).

The authors identify two sources for the genesis of such personal (subjective) models. One is the direct relation to the concept and the operation as it was initially taught in school. As the other, they found a natural tendancy to produce subjective regularities and use them intuitively through continuous activities "beyond any formal rules one has learned" (p.15) and though they might be "formally meaningless and algorithmically incorrect" (p.14). This represents an example of the genesis of a DSE, pointing at the specifity of situation as well as at the totality and multidimensionality of subjective experience as stated above.

The rigid disparity of two DSE's which from a teacher's perspective should be extensively interrelated (as e.g. experiences with a special case and the general rule) characterizes not only the phase of initial development in the subject. Against the expectations of a natural growing together of separately-gained pieces of knowledge through repeated practice, the persistent subordination of knowledge to specific DSE´s remains effective and dominates the subject´s actions. The supposition that cognitive networks develop quasi-automatically through an adaptation to the logics of subject matters appears as an illusion. The "society of mind" model with its independently competing DSE´s allows a simple explanation for the persistence of disparate DSE´s for the "same" situation. This can happen even in cases where a DSE´s concepts and procedures are stored but not used though they are superior or more general in an observer´s view, because they do not cover the "same" problem under the subject´s perspectives. Even so-called general concepts stored in memory are inevitably related to the subject´s perception of the situation in which the concepts were built. And therefore ascertaining the "sameness" of two cases affords a comparing of elements from at least two different DSE´s (see thesis 4). Each activation from memory on the other side reinforces the activated DSE, but not an abstract relation to other DSE´s.

<u>3. Thesis</u> The activities of the subject and the related subjective constructions of meaning and sense, as these develop through social interaction, are the descisive fundamentals for the formation of DSE.

In mathematics education, in particular, the subjectively relevant activities are bound to the offered mediatization of the matter taught, to what is really done. Teacher and students act in relation to some matter meant, usually a mathematical structure as embodied or modelled by concrete action with physical means and signs. But neither the model, nor the teaching aids, nor the action, nor the signs are the matter meant by the teacher. What he/she tries to teach cannot be mapped, is not just visible, or readable, or otherwise easily decodable. There is access only via the subject´s active internal construction mingled with these activities. This is the beginning of a delicate process of negotiation about acceptance and rejection. That is why the production of meaning is intimately and interactively related to the subjective interpretation of both the subject's own actions as well as the teacher's and the peer's perceived actions in specific situations. Via these processes the (social) norms of mathematical action are also constituted in the classroom, covertly, regarding acceptability, validity, completeness, relevance, and so on.

The doctoral thesis of G.FELLER 1984) gives an idea of how important the activities with embodiments and physical means (teaching aids) are for the formation of mathematical experiences. She tested mathematical achievement at the end of grade 2 in Berlin in order to find out the extent to which the aims of the mathematics curriculum had been attained. As a by-product the author was "startled by the strong impact of the manner of representation". Her final assessment:

> "The outcomes indicate that the acquisition of each different type of representation requires the learner's explicit endeavour and connected rehearsal, an effort which is not less than is usually required for the learning of mathematical matter itself (like addition or subtraction)." (G.FELLER 1984, p.67).

In our terminology this would mean that, for many children, experiences with a new representation of subject matter, though perhaps well-known from other situations, lead to a new DSE, stored separately in memory and with weak if any relations to the older experiences.

A new DSE can also develop through the explicit connecting of elements from different older and available DSE's. The "Aha" insight, flashing up suddenly while acting within the horizon of an activated DSE and producing the idea of essentially "doing the same" as in another context (DSE), is the announcement of a birth, for the person as well as an observer. But the "Aha" alone does not produce by magic a fully developed network of relations here and now. It takes time and continual activities to elaborate the new DSE. An "Aha" insight, not elaborated after the first appearance, can fade away in the continuous flow and only light up again much later accompanied by the feeling that something like that was known already.

DSE´s disappear only (and slowly too) if they do not receive reactivation. Growing interrelations and even integration are not necessarily weakening effects. "The mind never subtracts" (M.MINSKY, 1981). As is the case with regression very old DSE´s can prevail in the competition for activation under stress against younger DSE´s where so-called "higher", "super-ordinate", "more sophisticated" knowledge is stored.

In the mathematics classroom students are often asked to identify common characteristics between two events or cases, which in the view of the teacher appear to be two models for the very same mathematical structure. This is the task of producing a generalizing abstraction from different embodiments upon request. In our view the student then has to compare elements which are rooted in two different DSE´s; in other words: which are incorporated in two different contexts. What can form the basis of the required comparing activities?
Usually the perspectives of the separate DSE´s themselves do not cover such operations, due to the specificity of actions, language and meaning. So where do the aims come from? Which kind of similarity or commonness do I have to search for? The adequate basis has to be a third DSE, the elements of which are the means for comparison and the possible aims. Comparing common characteristics by abstracting and neglecting other ones is a complex and highly constructive activity. Without an orientation, at least a diffuse image of the potential results and of the relevant characteristics, as well as an idea of the adequate means, there is no reasonable chance for the student´s success.

An example may demonstrate the difficulties. What do the following three situations have in common?

a) You plunge your hand into a paper bag three times and take out two eggs each time.
b) You see three blocks of houses with two houses in each block.
c) Three boys and two girls dance. How many different pairs are possible? (old-fashioned style: one girl one boy per pair!)

The question can also be put this way: For which more general issue can these three situations serve as models? Is it enough to answer - like fourth graders perhaps would do - "It´s always six!" or "All are three times two" or "It is multiplication!" or ...? What is the meaning of the concept "multiplication of natural number"? How may it be explained?

The critical step is the crossing of the borderlines of the three related DSE´s. The interesting commonness is not with the same twos, threes, and sixes in each situation. What are the conditions for seeing the well-known elements differently, to dissociate them from narrow concreteness, to attach another meaning, another relation, a more general relation, to them? Obviously, we can get hold of what we call a common structure only by means of a model, of a certain description; no matter how concrete or illustrative this model might be, provided that it can work for us as the more general model, which we can identify in (map onto) each of the three situations given.

For the above example a possible fourth model can be

d) Three parallel lines are cut by two other parallel lines. The first three lines can then represent the 1., 2., and 3. selection or house-per-block or boy. The second two lines represent the 1. and 2. egg per selection or block or girl. And the intersections (modes) stand for the six eggs or houses or pairs in total.

Clearly the learner either has to reconstruct from related help and hints or he/she has to construct such a model on his/her own. It should be clear, too, that this construction is not by nature an integrated part of any one of the three situations. It is not part of the experiences within the three related DSE's it is a new perspective.

From another point of view the geometrical configuration d) is nothing else than just another (specific) model for the multiplication of natural numbers. Under this view there are many more adequate models or descriptions, e.g. e) A table with three columns and two rows, including the three initial ones (more in H.RADATZ/W.SCHIPPER 1983, p.73).

From a developed understanding of the concept, each of the models can serve as description of "the" general structure of multiplication of natural number, at least potentially, and realizable through one-to-one coordination. Thinking about the available modes and possibilities for the representation and any structures at all, we might find that we cannot overcome the force of the use of models in communication. In principle there is no transgression. This brings us nearer to the relativity of so-called general concepts (see T.B.SEILER 1973). At this point, on the other hand, the common statement about "the best learning is learning by example" sounds somewhat tautological.

4. Thesis. In terms of memory there are no general or abstract - i.e. context-free - concepts, strategies, or procedures. The person can think (produce) relative generality in a given situation. But the products are not retrievable from memory in the same generality or abstractness, that is, they are not activatable independently of the related DSE's.

With advancing years the development of the "society of mind" leads to an accumulation of DSE's and also to a growing network of relations among their elements through even the relations are realized and retrievable only in specific domains. Their genesis is bound to the considerable constructive activities of the person as well as to the situations of practice and to the qualities of social interaction. The perspective of a certain DSE may become integrated into a new DSE, together with elements from other DSE's. In the perspectives of the new DSE the integrated older experiences may appear as subordinate and hierarchically lower elements. But in spite of that the older DSE still can compete for activation with the new DSE. R.LAWLER therefore speaks of a "structure of a mixed form, basically competitive but hierarchical at need" (1981, p.20), more precisely perhaps: hierarchically through special activation. General knowledge is available through special activation only, this is the meaning of thesis 4.

The disparity of the DSE's marks not only the phase of their initial formation but also the later phases when detailed or more general knowledge has been required, which of course is stored in different DSE's because of the differences in situation, as the investigations of FISCHBEIN et al. (1985) show. Microethnographical studies at preschool and early school ages substantiate the extent to which the ability for identifying two events as being "the same case" depends upon previous learning experiences and upon the subjective perception and definition of the actual situation. In several long-term studies R.LAWLER has documented and analyzed the encounters of his children with computers, arithmetic and geometry (1979, 1981, 1985). His early concept of "microworlds" is the cognitive shadow of the domains of subjective experience (DSE) as defined here (and elsewhere, BAUERSFELD 1982, 1983).

> LAWLER's daughter Miriam e.g. has solved tasks of the type 75 + 26 = ? according to the specificities of presentation in at least three different and for long incompatible microworlds.
>
> If the task appears as "75 cents plus 26 cents" Miriam calculates the solution via her activated "Money world", like: "That's three quarters, four and a penny, one-oh-one!" The presentation of "seventy-five plus twenty-six equals..." she solves in her "Serial world" like: "seven plus two,

nine, ninety-six, ninety-seven, ninety-eight, ninety-nine, hundred, one-oh-one!"

And if it is written as a vertical sum, Miriam adds up the columns and carries the tens (R.W.LAWLER 1981, H.BAUERSFELD 1983).

The "identical" arithmetical task, as a teacher would name it, is thus solved according to the activated special DSE using related but completely different procedures. For the child, obviously, the different presentations are perceived as different and independent tasks. The rigid disparity remains in effect even when all three representation are given consecutively. It is much later that through spectacular "Aha" events certain relations are produced.

The studies support the supposition that, in particular, the use of language is specific to the situation and hence to the activated DSE. In LAWLER's protocols Miriam uses the phonetic lly same words "six", "seventy", "plus", etc. across the di! ent situations, whilst her concrete actions indicate differer ecific meanings in correspondence with the different activated microworlds. For an observer therefore it is impossible to interpret an utterance without adequate reconstruction of the related subjective definition of the situation (DSE). Likewise it is impossible for Miriam to take a distancing and critical perspective against her specific procedures and interpretations from within the activated DSE. Evidently this is impossible in general - without having developed the distancing and critical perspective as an integrated habitual activity within the DSE. That is why a teacher's urging for comparing, for controlling, for looking closer etc. has no effect when these activities are not developed in relation to the activated narrow DSE.

There is by the way good reason for the development of disparate DSE's because of the strikingly different sensual characteristics of the concrete activities.

Miriam's "Money world" is built upon her intensive experiences with her pocket money, with buying and change. What mathematicians call the operations of addition and subtraction is here embedded in a world with its own specific sensuality: colour, and coinage etc. and with specific non-number names like penny, nickel, dime etc. (see H.RUMPF 1981).

In contrast to that her "Counting world" is ruled mainly by word sequences which obey certain rules of construction ("twenty, twenty-one,...") and which are produced through one-to-one procedures of speaking and touching the objects to be counted.

The Paper-sums world" is a medium of quite another type of sensuality: Writing symbols on paper using a pencil (with the typical fine-motoric muscle tensions), reading, and operating with the symbols (see H.BROGELMANN 1983).

So we can state that meaning is attached to a word through certain activities in a certain situation but a word has no definite meaning per se. This is true with speaking, hearing, reading, and writing. Likewise we interpret a word heard in a concrete situation within the range of the actually activated DSE. There is no other chance for understanding without additional effort, e.g. the activation of other DSE´s. In this sense even the so-called universal language of mathematics is not universally available (retrievable) for a person.
Theories become helpful models for realities when and insofar as they generate constructive orientation. So more interesting than the disparity of DSE´s and the unthinkable purity of context-free concepts, perhaps, are both the totality and the principle of multidimensionality of learning in social interaction:

<u>5. Thesis</u> Whenever we learn, all of the channels of human perception are involved; i.e. we learn with all senses, learning is total. And: simultaneously we learn on all dimensions and levels of human activities, at least potentially; i.e. learning is multidimensional.

A smell therefore can activate a certain DSE later on, as can a pattern of motion or a sophisticated metaphor. In a given situation we not only learn about the subject matter, directly and attentively, the what-to-do - e.g. the theme, facts and procedure (declarative and procedural knowledge) - we also learn, more covertly, about the how and the when to do it - e.g. orientations of action, strategies, the fit and the adequacy of situations - we also learn about the why to do it - e.g. sense, reasons, attached values - we learn about ourselves - e.g. anxiety and motivation, personal identity - and we learn about the others and how they see us - e.g. social norms, the person's social identify. The listing is far from complete. We also develop routines and pattern of habitual activities in all dimensions.

JOHN DEWEY already formulated this idea in 1938[7]:

"Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time. Collateral learning in the way of formation of enduring attitudes, of likes and dislikes, may be and often is much more important than the spelling lesson or lesson in geography that is learned. For these attitudes are what fundamentally count in the future."

The continuous flow of conscious production only marks the surface of a much deeper stream of experiences which form the orientation of a person's future actions. As DEWEY stated, the most important things are learned collaterally, across many activities and preconsciously, in a FREUDian sense. So what is learned beyond the official theme, this major and more powerful portion of learning appears as a core problem of classroom teaching. AUSUBEL's classical and often quoted words may now be read with a somewhat more differentiated understanding:

"If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Assertain this and teach him accordingly" (1968)

Notes

1) In West Germany (FRG) after four years in primary school about 25-40% of the 10-11 years old students enter a Gymnasium, where they normally pass grades 5-13 and end up with the Abitur, at age 19. The Abitur exam is the general pre-requisite for university entrance. The majority of the students enter grade 5 of Hauptschule, Realschule or Gesammtschule, the other types in the secondary school system.

2) These include not only direct simulations of mathematical content on the computer screen, but also simulations of the learning process, of the learner's previous knowledge and strategies, because all this information is processed in the form of mathematical or logical rules and with unambiguous ascriptions (meaning).

3) This, clearly, requires more detailed discussion, which cannot be done here. My interest is to point out the limitations which are carried by the unreflective use of my categories or descriptors. They seem to be "at hand" (like metonymies) for what we think we see. But we usually do not reflect upon their origin or their context, which leads to covert, narrow pursuit, and not to novel ideas. As operations in context, describing and interpreting are dependent on the qualities of these bases of the teaching-learning processes.

4) T.O'SHEA stated that often "the attempt to automate an activity forces a better understanding of the activity itself" (O'SHEA/SELF /1983, p.267). And he ends his diagnosis by saying: "...it is easier to let children try to learn BASIC than to develop learning environments which facilitate intellectual discoveries; it is easier to write programs which treat students uniformly than to write programs which try to take account of an individual student's interests, errors and aptitudes" (ibid., p.268).

5) Analysing the role of example and counterexample in "proofs and refutations" LAKATOS said: "...we may have two statements that are consistent in (a given language)L_1, but we switch to (a new language) L_2 in which they are inconsistent. Or we may

have two statements that are inconsistent in L_1, but we switch to L_2 in which they are consistent. As knowledge grows, languages change.

´Every period of creation is at the same time a period in which the language changes.´ (FELIX) The growth of language cannot be modelled in any given language." (I.LAKATOS 1976, p.93; brackets added from context, H.B.). LAKATOS identifies the change of language as "concept-stretching" (p. 93 f.). But "concept-stretching will refute <u>any</u> statement, and will leave no true statement whatsoever." (p.99) Indeed he denies the existence of "inelastic, exact concepts" as bases for rationality (p.102). There is no eternal truth, there is only "guessing" (p.76 f.) and "the incessant improvement of guesses" (p.5). D.SPALT (1985) discusses in detail the failure of LAKATOS´ solution to this fundamental problem: "mitigation" of concept-stretching (LAKATOS 1976, p.102 f.).

6) The notion of "subjective" experiences rather than "personal" experiences (which might be nearer to colloquial English) follows etymological considerations. The Latin origin, the verb "subjicere", means in the transitive sense that the person (the subject) actively subjugates something, makes it the person´s own through action. This of course describes the functioning of "subjective experiences". The active parts are at least the continuous constructions of meaning and the selecting and focussing in our changing definitions of the actual situation.

7) For this quotation I am indepted to HARRIET K. CUFFARO´s article in the Columbia Teachers College Record, summer 1984, p. 567, which interestingly critizes the present use of computers in schools.

The vigilant reader will find that chapter 4 as promised at the bottom of page 2 is missing here. The chapter will have to be added later on.

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LECTURE 2

ON THE RELATION BETWEEN THE APPLICATION OF MATHEMATICS AND THE TEACHING OF MATHEMATICS

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ON THE RELATIONSHIP BETWEEN APPLICATIONS OF MATHEMATICS AND THE TEACHING OF MATHEMATICS *

INTRODUCTION

Most mathematics educators believe in the importance of applications, but it is nevertheless very difficult to get applications into the curriculum. Why? One possible reason appears to be that there is no agreement on what is meant by applied mathematics. In the following we shall explore four different definitions, and their consequences both for the mathematics subject matter and for pedagogy.

1 THE DEFINITION OF APPLIED MATHEMATICS AND ITS VISUALIZATION

In discussions of applied mathematics, a large amount of unnecessary difficulty is sometimes created by differences in perception of the appropriate definition. These differences have come about quite naturally in recent years, since the variety of mathematics which has significant practical applications, the number of fields to which mathematics is applied, and the modes of applications have all undergone very rapid change. It is useful to think in terms of *four* different definitions.

* Dr. Pollak's lecture followed closely parts of the text of his paper "The interaction between mathematics and other school subjects", Volume 4, UNESCO. The appropriate parts of the text are reprinted here by permission of the author and UNESCO.

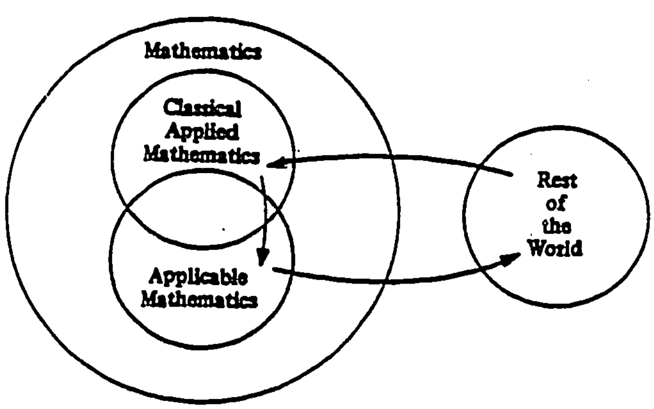
(1) *Applied mathematics means classical applied mathematics;* that is, the classical branches of analysis, including calculus, ordinary and partial differential equations, integral equations, the theory of functions as well as a number of related areas. It is sometimes convenient to annex those aspects of secondary mathematics which are essential prerequisites to calculus, in particular algebra, trigonometry and various versions of geometry. The fact that these branches of mathematics are the ones most applicable to classical physics is usually understood as part of this definition, but no actual connection with physical problems is implied.

(2) *Applied mathematics means all mathematics that has significant practical application.* This greatly enlarges the collection of mathematical disciplines included under (1). All the topics that have been considered world-wide for inclusion in the elementary and secondary school have significant practical applications – including sets and logic, functions, inequalities, linear algebra, modern algebra, probability, statistics and computing. Almost all the mathematics taught at the tertiary level (the undergraduate level at many universities) as well as much graduate mathematics are also included. In the views of many people, the most important areas of mathematics that are included in (2) but not in (1) are statistics, probability, linear algebra and computer science. There are many who feel that these topics are as important as classical analysis. Fields of potential applicability include more than physics, but, once again, only the mathematics itself is being considered.

(3) *Applied mathematics means beginning with a situation in some other field or in real life,* making a mathematical interpretation or model, doing mathematical work within that model, and applying the results to the original situation. Note that the other field is by no means restricted to lie in the physical sciences. In particular, applications in the biological sciences, the social sciences, and the management sciences have become extremely active. Many other areas of applications will also be considered.

(4) *Applied mathematics means what people who apply mathematics in their livelihood actually do.* This is like (3) but usually involves going around the loop between the rest of the world and the mathematics many times. An excellent example of the process involved in this definition of applied mathematics may be found in a report of the workings of the Oxford Seminar in the United Kingdom (Oxford, 1972).

A convenient aid in visualizing these four definitions is seen below:



In this picture the left-hand side shows mathematics as a whole, which contains two intersecting subsets we have called classical applied mathematics and applicable mathematics. Classical applied mathematics represents definition (1) and applicable mathematics, definition (2). Why doesn't (2) contain *all* of (1)? The overlap between these is great, but it is not true that all of classical applied mathematics is currently applicable mathematics. There is much work in the theory of ordinary and partial differential equations, for example, which is of great theoretical interest but has no applications which are visible at the moment. Such work is included in definition (1) as classical applied mathematics, since this contains all work in differential equations; on the other hand, if it is not currently applicable, it does not belong in definition (2).

The rest of the world includes all other disciplines of human endeavour as well as everyday life. An effort beginning in the rest of the world, going into mathematics and coming back again to the outside discipline belongs in definition (3). Definition (4) involves, as will be seen, going around the loop many times.

Other categorizations of applied mathematics have also been considered and can be examined in terms of the diagram. Typically, they involve a more detailed study of the process within mathematics itself than we shall undertake here. For example, applications of mathematics may consist of routine uses of mathematics, of mathematical reasoning as opposed to direct calculation, and of the building of models of various sizes going from small models through full mathematization of real situations to truly large-scale theories. Another very interesting way of slicing the pie may be found in Felix Browder (1976) "The relevance of mathematics". His categories consist of: (a) practical mathematics, that is mathematical practice in the common life of mankind in civilized societies; (b) technical mathematics, that is the use of mathematical techniques and concepts to formulate and solve problems in other intellectual disciplines; (c) mathematical research, that is the investigation of concepts, methods and problems of various mathematical disciplines including applied ones; and (d) mathematics as a universal pattern of knowledge, which means the science of significant form. His essay is highly recommended.

2 A DETAILED STUDY OF THE VARIOUS DEFINITIONS

2.1 The mathematics side of the diagram

The mathematical content of classical applied mathematics (definition (1)) and of applicable mathematics (definition (2)) have already been discussed. One recent trend has been the publication of books and articles showing the applicability of many of the mathematical disciplines which are not included in definition (1). To name just a few examples, Hans Freudenthal (1973) as well as M. Glaymann and Tamas Varga (1973) have written recent books on the applicability of probability; Tanur, Mosteller, Kruskal, Link, Pieters and Rising (1972) have edited a volume showing the great diversity of applications of statistics; R. H. Atkin (1974) in his book has included applications of topology, and Fred Roberts (1976) has devoted much space to applications of graphs and Markov chains. Journal articles are even more numerous; a few samples of particular interest follow — without the slightest pretence of coverage. Thus F. W. Sinden (1966) and Uwe Beck (1974) have shown some applications of topology; M. Dumont (1973) has discussed some uses of Boolean functions and J. H. Durran (1973) some applications of Markov chains. Recent applications of combinatorics and graph theory are examined, for example, by John Niman (1975), J. N. Kapur (1970) and W. F. Lunnon (1969).

A significant feature of applications of mathematics is that mathematical concepts and structures have important usefulness, not just mathematical technique. An interesting discussion of this point is given by H. G. Flegg (1974). Furthermore, since the relationship between mathematics and its applications is forever changing, there is a dynamic effect on mathematics itself. It has happened many times that areas of mathematics which were originally considered quite pure, and were developed with no thought of applications whatever, have turned out to be significantly useful. On the other hand, areas of mathematics which were invented only for application, with no thought of their possible contribution to core mathematics, have turned out to have an impact on pure mathematical disciplines. As an example of the former, the theory of entire functions has given notable insights in electrical communications; ideas of information theory, on the other hand, have been useful in such diverse fields as measure-preserving transformations and the theory of finite groups.

2.2 The rest of the world

Perhaps the outstanding feature of applications of mathematics in recent years is that the areas to which mathematics is applied have been increasing in number so rapidly. It is fair to say that no area of human endeavour is currently immune from quantitative reasoning or mathematical modelling. Besides the traditional physical sciences and engineering, the biological sciences, the social sciences, the management sciences, the humanities and everyday life are all arenas for interaction with mathematics. This is not meant to imply that mathematics is taking over all these other fields, but there *are* many interesting applications.

Perhaps the most extensive literature in recent years on applied mathematics from the point of view of the other disciplines has come in the biological sciences. An excellent overall survey appears in the book by J. Maynard Smith (1968). Books dealing with specific areas within the biological sciences include Victor Twersky (1967) on growth, decay and competition and R. M. May (1973) on the stability of ecosystems. Among the articles too numerous to summarize we note S. Karlin (1972a,b, the former jointly with M. Feldman) on genetics, S. P. Hastings (1975) on neurobiology, Arthur Engel (1971, 1975) and Beck (1975) on population models, W. D. Hamilton (1971) on the geometry of group behaviour, and several articles in "Computers in Higher Education" (1974) on the use of computers in biology. Not that new books and articles on mathematics in science have been lacking: We note particularly a little known volume by George Polya *Mathematical Methods in Science* (1963) as well as another portion of Victor Twersky (1967). Recent articles on mathematics in science include J. B. Griffiths (1976) on model building and mechanics, the conference report on "Modern Mathematics and the Teaching of Science" (1975), and the previously mentioned computer survey "Computers in Higher Education" (1974).

Another field which has recently flourished is the interaction of mathematics with the social sciences. Information on computers and statistics in the social sciences generally may be found in (Computers . . . , 1974) and (Teaching of statistics . . . , 1973); a fascinating and somewhat different viewpoint is represented in the article by H. R. Alker, Jr., "Computer simulations: Inelegant mathematics and worse social science?" (1974). The *Source book on Applications of Undergraduate Mathematics to the Social Sciences* (1977) contains descriptions of detailed mathematizations in many fields of the social sciences. To go on with specific fields, economics is extremely active for interactions with mathematics, although good expositions of the problems of model building in economics are not common. One nice example is "On the theory of interest" by David Gale (1973). Mathematical work in geography has also been quite popular in recent years, particularly in the United Kingdom. Again there are significant contributions in (Computers . . . , 1974) and (Source Book . . . , 1977), and an elementary treatment of weather forecasting in Durran (1973); see also King (1970). Mathematical psychology is represented by two recent survey articles by Anatol Rapoport (1976); *Source Book . . .* (1977) also contains extensive references to recent work. Besides their appearance in overall summaries, anthropology is represented by example in the book by L. Pospisil (1963) and the traditional mathematical theory of warfare by Arthur Engel in (1971). A magnificent example of mathematics applied to political science may be found in M. L. Balinski and H. P. Young (1975) "The quota method of apportionment". Mayer (1971) and Coxon (1970), for example, represent mathematical sociology.

The very large field of mathematical models in the management sciences including the entire area of operations research hardly needs description here. Sample articles of particular interest in recent years include those by F. J. Fay (1972), J. C. Herz (1973) and the delightful piece on mathematics applied to college presidency by J. G. Kemeny (1973). Mathematical models in medicine has been an increasingly active field; there is an excellent survey by J. S. Rustagi "Mathematical models in medicine" (1971). Mathematical linguistics has similarly become a major accepted field. Interesting particular articles appear, for example, as parts of Engel (1971) and *Source Book* . . . (1977), with Sankoff (1973) as another good source.

The penetration of mathematics into the humanities, including statistical and computer models, is a fairly recent event. Perhaps furthest advanced are mathematical analyses of art. We note, for example, A. V. Subnikov and V. A. Koptsik (1974) and a very valuable British summary of mathematical ideas and concepts in art by Beryl Fletcher (1976a). Mathematics applied to architecture is discussed by R. Fischler (1976) as well as in the summary work "Computers in Higher Education" (1974). Some examples of mathematical ideas in hobbies and handicrafts are given in Beryl Fletcher (1976b). Mathematical strategies for certain games such as NIM and the towers of Hanoi have long been familiar to, and enjoyed by, mathematicians. In recent years, there has been a great upswing in the discovery of optimal strategies for much more intricate games, and this has even provided one of the early applications of ideas from nonstandard analysis. We particularly note the work of E. R. Berlekamp and J. H. Conway, partly reported in Conway (1976). A nice example of optimal strategy for poker is given by W. H. Cutler (1975). Cryptanalysis has often been treated — see e.g. Sinkov (1968); for mathematics in sports see Klein (1972).

Besides the above-mentioned books and articles more or less devoted to specific areas of applications, there has been a trend in recent years towards the publication of excellent collections of articles and symposium reports which cover a broader spectrum. One of the earliest but still of great interest is the Utrecht colloquium "How to Teach Mathematics so as to be Useful" (Freudenthal, 1968). This was followed by the Echternach symposium "New Aspects of Mathematical Applications to School Level" (Echternach, 1973) and the Lyon seminar "Goals and Means Regarding Applied Mathematics in School Teaching" (Goals and Means . . . , 1974). Other noteworthy volumes of this kind include *Notes of Lectures on Mathematics in the Behavorial Sciences* edited by H. A. Selby (1973), *Topics in Behavorial Mathematics* by T. L. Saaty (1973), *A Source Book for Teachers and Students on Some Uses of Mathematics*, Max Bell (1967), *A Conference on the Applications of Undergraduate Mathematics* . . . (Knopp and Meyer, 1973) and *La Mathématique et ses Applications* by E. Galion (1972).

The preceding list well illustrates the current diversity of applications of mathematics in contrast with the historical monolith of applications to physics. It should not be assumed, however, that the arguments between those who stress the great variety of applications in recent years and those who feel that their total impact cannot compare to the 2000-year accumulation of success in mathematical physics have died down. In fact, this difference of instinctive value judgement underlies many of the arguments about mathematics education to which we will return later.

2.3 The model building process

When mathematics is actually applied to a situation in some other field, there are typically a number of distinguishable steps in the process. These consist of a recognition that a situation needs understanding, an attempt to formulate the situation in precise mathematical terms, mathematical work on the derived model, (frequently) numerical work to gain further insights into the results, and an evaluation of what has been learned in terms of the original external situation. This picture of the model building process has been widely accepted and there are many papers which elucidate the details from various points of view. Overall descriptions appear, for example, in the papers by M. S. Klamkin (1971), H. O. Pollak (1970) and P. L. Bhatnagar (1974). The same pattern, but applied specifically to operations research, appears in the paper by Gordon Raisbeck (1975) "Mathematicians in the practice of operations research"; its application to engineering may be found in A. C. Bajpai, L. R. Mustoe and D. Walker (1975), and again in the paper by H. G. Flegg (1974). M. E. Rayner (1973) in her paper "Mathematical applications in science" in the Echternach report describes in detail some of the difficulties in problem formulation. A quotation she gives from Eddington is particularly worth repeating, "The initial formulation of the problem is the most difficult part, as it is necessary to use one's brains all the time; afterwards, you can use mathematics instead". A proposal for better model building in mechanics is also given by J. B. Griffiths (1976). See also Wilder (1973).

The model building process has a number of interesting properties as well as pitfalls which we shall examine. A good model is one which is to some extent successful in explaining, or even predicting, external reality. If it fails to have this explanatory power then, no matter how satisfactory the mathematics itself, the model is not good applied mathematics and must be changed. This process can be quite painful for the mathematician but real progress in interdisciplinary efforts is often made through successive changes in the model. This is one of the reasons why definition (4) of applied mathematics involves going around the loop many times. Another phenomenon which sometimes happens is that a mathematical model predicts too much rather than too little. It may happen that phenomena observed in the other field are indeed explained satisfactorily, but that further logical implications of the model are not acceptable. For example, in the mathematics of communication a model of a signal which is of finite duration in time is very realistic. Similarly, a model of a communication signal using finite bandwidth comes up in many situations and gives quite satisfactory engineering results. Unfortunately, the two are contradictory and cannot be used at the same time in the same problem; models which do so unwittingly will lead to nonsense. On the other hand, attempts to understand this difficult situation fully have led to very interesting advances, see e.g. D. Slepian (1976).

Another feature of the model building process is that the purposes for which a mathematical model is created are also quite varied. In the physical sciences and engineering, the purpose is frequently very precise understanding which will in turn lead to action. In the social sciences, on the other hand, the purpose is often one of insight; you want to know whether a certain set of hypotheses could account for a particular observed phenomenon. It is often assumed, although not necessarily true, that these associations are in fact one-to-one correspondences. Physical models of why rivers meander, or why a rapidly slurped pice of spaghetti comes up and hits your nose, are not necessarily used for scientific decisions. On the other hand, mathematical models of shortest connecting networks and optimal pricing are often used for management action.

The overall picture of applications of mathematics would not be complete without a discussion of truly interdisciplinary activity. Much of the most exciting current work is in fact on the borderline between several fields, one of which being in the mathematical sciences. The above references will lead the reader to many examples of current interdisciplinary work.

3 EFFECTS OF APPLIED MATHEMATICS ON MATHEMATICS EDUCATION

3.1 Problems and problem solving in the schools

A framework for understanding the meaning of applied mathematics has now been established, and a number of ramifications of the various definitions have been examined. A look at effects of applied mathematics on education follows. It must be emphasized that many of the topics in this chapter represent ideas and experiments in various countries which cannot claim to be adopted on any large scale. Discussions at the Karlsruhe Congress did not bring forth any data which would substantiate broad use of applied mathematics in the schools.

Traditionally most of what was considered applied mathematics in the schools has been found under headings such as "word problems", "problem solving", etc. (This does not mean the "word problem" in the sense of modern algebra.) The meaning of such problem solving has been examined in a number of projects and articles in recent years. For example, the work of IOWO in the Netherlands is of particular importance. IOWO has also paid special attention to the differences in abstraction and precision between mathematical language and everyday language. The detailed meaning of problem solving is examined in papers by H. G. Flegg (1974), Beryl Fletcher (1976c) and H. O. Pollak (1969). Genuine applications of mathematics to other fields and to everyday life should ideally be in the character of definitions (3) and (4). It is often argued that a full presentation in the spirit of even definition (3) represents too large a project and takes too much time. In that case, the actual situation and numbers used in the word problem should at least be genuine extractions from an honest problem formulation. For example, estimates of crop yields and of times to complete a task should not be made to five significant figures, for this will never happen in real life. Too many plumbers in one room get in each other's way, and jobs are not always divisible. A current joint project of the National Council of Teachers of Mathematics and the Mathematical Association of America in the United States is producing a *Source Book* (1978) of hundreds of simple problems which are intended to be genuine in the above sense.

The opposite phenomenon is that the facts alleged in the statement of a problem are sometimes totally unreal. Problems which use wrong linguistics or impossible engineering or incorrect meteorology just to have some words from another discipline should be avoided. In this case, intent can nevertheless be important. Sometimes problems are clothed in a mantle of external vocabulary only for amusement, and the pretended application is not meant to be taken seriously. We shall call such problems whimsical problems. A strong argument in favor of such problems is made for example by Arthur Engel (1969) "Some examples are artificial, like fables. But just like fables, they have a moral, i.e., they facilitate insights into things that appear in the real world". For example, it can be quite effective to begin with an unsatisfactory oversimplification of a real situation, and to approach a genuine application in the sense of definition (4) through a series of increasingly realistic problems. Thus whimsical and unreal problems are not necessarily devoid of pedagogic value. However, if they are perceived as stupid, they may well be counterproductive. Similar discussions of real and unreal problems may be found in two particularly interesting papers by Margaret Brown (1972, 1973) and Mary Williams (1971). In particular, Mary Williams points out that the same difficulty of unreal models happens at a very advanced level as well as at the school level. See also section 1.1.5 of Chapter IV.

The increased awareness in many countries of the importance of teaching the applicability of mathematics has led to a number of very interesting attempts to collect real problems at various levels, and from various disciplines, and to make them available for teaching purposes. One collection at the school level (Source Book ... Secondary School, 1978) has already been mentioned. Other general collections have been made by Max Bell (1972), Ben Noble (1967),

D. A. Quadling (1975), and C. W. Sloyer (1974). Collections devoted to particular disciplines, mainly at the university level, include the series on statistics by example (Mosteller et al., 1973), the social sciences problem book (Source Book . . . Social Sciences, 1977) and the collection of mathematical models in biology (Thrall et al., 1967), although the realism of problems in the latter collection varies. Another text in the same spirit, although it is organized as an actual course in engineering concepts, is *The Man Made World* (1971). It can be expected that very interesting collections of real problems in the above spirit will also be appearing in China. One such example of which we are aware contains, among other things, a number of excellent geometric problems from industry and agriculture (Applications . . . , 1975).

3.2 Mathematical subject matter in the schools

The diversity of applicable mathematics (definition (2)) which has emerged in recent years has greatly complicated the task of designing curricula for elementary and secondary schools. The traditional goals of preparing students for either shopkeeping or calculus (associated with definition (1)) cease to be uniquely valid when so many more areas in the mathematical sciences are of undeniable importance to so many of the world's people. As the number of reasonable choices increases, so does the difficulty of designing a curriculum. It has been argued by many that, for example, probability, statistics and computer science are as important for applications as the calculus. School materials for applications to many different disciplines have become available in recent years. Collections of materials involving applications to many different fields may be found, for example, in *Crossing Subject Boundaries* (Schools Council, 1970) and the materials from the Minnesota School Mathematics Center (Rosenbloom, 1963). The Chelsea Centre for Science Education project, "Science Uses Mathematics" (Chelsea) contains interesting applications to science which can be used in an interdisciplinary way, although this is not always done. *Applied Mathematics in the High School* by Max Schiffer (1963) also gives excellent examples of the relationship of mathematics and scientific applications from the point of view of the schools. A collection of examples which turn the tables and use physics to do mathematics has been made by Uspenskii (1961).

A major work examining curricular goals and pedagogy in the framework of an application to economics may be found in Damerow, Elwitz, Keitel, and Zimmer (1974). Biological applications may be found in Gibbons and Blofield (1971), and applications to geography in the materials by IOWO, in *New Ways in Geography* by J. P. Cole and N. J. Beynon (1968) and also in B. Fletcher (1976c). Applications to geography are also featured in the *Travaux d'Orléans (Les Mathématiques dans l'Enseignement . . .*, 1975), which in fact contains many other fascinating applications to a variety of fields throughout the curriculum, including economics, technology and medicine. This work also features references to recent work on applications in France and interesting philosophy on the usefulness of mathematics. An interesting application to political science may be found in Steiner (1966); environmental applications occur in the work of IOWO and in the book by Fred Roberts (1976). As we look at applications organized from the mathematical point of view, a superb collection of applications of linear algebra may be found in T. J. Fletcher (1972), and of statistics and probability in the work of Arthur Engel, e.g. (1970, 1973) and in *The Teaching of Probability and Statistics* edited by Råde (1970). *Mathematics Applicable* by the Schools Council (1975) also motivates much secondary mathematics through examples; the volume entitled *Logarithmic/Exponential* is a particularly interesting sample.

This great diversity of possible applications of mathematics, and of elementary branches of mathematics with significant applicability, has made the curriculum design problem very difficult. For example, topic A deserves to precede topic B in the curriculum if topic A is socially more important at this particular time, or if topic A is a prerequisite to topic B at this particular time.

As technology and social goals change, so should the ordering of importance. As available tools for teaching change, so will the order of prerequisites. These orderings will differ also from country to country. These facts make it even more difficult than it has been in the past to export curricula from one part of the world to another. Since an imported curriculum incorporates problems, situations and values which make no sense in a new country, this was probably never desirable, but is is even more questionable now.

3.3 The possible effect of applications on pedagogy

An appreciation for the different forms of applications of mathematics should affect not only the curriculum materials of the schools but also the pedagogy. If you examine even relatively simple uses of mathematics, you find that it is necessary to understand when and how and why the mathematics works in order to apply it correctly. There are several reasons for this. One is that mathematics which has been understood will be remembered better. Another more fundamental reason is the danger that mathematics which has been memorized without understanding will be misapplied. It is necessary to know where a particular method or formula comes from, exactly what kind of problem it will handle, and when and how it works in order to be sure that it will apply to a new situation. Curriculum reform in many countries has emphasized the "why" of mathematics in recent years on the grounds that it is essential for proper teaching of mathematics. What we see is that "why" is just as important for interactions of mathematics with other disciplines as it is for mathematics itself. The natural desire of mathematics teachers to emphasize understanding as well as technique is reinforced, not contradicted, by applications.

The model building process as developed through definitions (3) and (4) of applied mathematics interacts with mathematical pedagogy in a still deeper sense. Model building requires an understanding of the situation outside mathematics and of the process of mathematization as well as of the mathematics itself. You cannot hope to mathematize a situation without understanding it. Here we have yet another way in which "applied" problems which do nothing more than mouth words from another discipline are likely to mislead the student. A great weakness of some courses with titles like "Methods of Applied Mathematics" is that no attempt is made to provide an opportunity for the student to understand the situation and the mathematization process. This point has been particularly emphasized by H. G. Flegg (1974) and is further substantiated, especially from the point of view of future employment, in R. R. McLone (1973). Some of the college-level collections of real problems mentioned previously, for example Noble (1967) and *Source Book ... Social Sciences* (1977), take particular pains towards the understanding of the situation in the real world.

Another pedagogic implication of the interaction between mathematics and other disciplines as it is described in definition (4) is that such interactions are clearly open-ended. Open-ended teaching of mathematics itself has long been recommended by mathematics educators in many countries, although adoption is rare. What does "open-ended" mean in this context? Besides the usual activities of solving problems and proving theorems, students should have the experience of finding their own problems to solve and their own theorems to prove. Such experience is an important factor in the mathematical development of the student. But exactly the same argument holds in the context of applications. It is very valuable for the student to have open-ended modelling experience, which besides its great pedagogic value is an accurate foretaste of mathematical applications in the real world. Experiments in open-ended discovery teaching of mathematical applications, many in the form of truly interdisciplinary materials, are under way in surprisingly many countries. An outstanding example is certainly China, where a major practical problem will be used for reference and inspiration throughout a course in calculus or linear algebra. There are many other examples of open-ended and truly interdisciplinary activities at the tertiary level, represented, for example, by the *Case Studies in Applied Mathematics* (1976), the books by T. J. Fletcher (1972), Maki and Thompson (1973) and Roberts (1976). At the elementary level, an outstanding example is provided by the USMES project in the United States (Lomon et al., 1975) in which students attack a series of action-oriented challenges by appropriate combinations of mathematics, science and social science. Truly open-ended applications are particularly difficult to introduce at the secondary level, and corresponding materials are very scarce.

3.4 Applications and teacher training

As mathematics teaching changes in the light of the increasing applicability of the subject, so should teacher training. Teachers should become familiar with the new fields of applicable mathematics, with the process of model building, and with the associated pedagogic emphases on understanding and open-endedness. There is a general tendency world-wide to reverse certain recent trends and to include more experiences involving applications in the training of prospective teachers. Perhaps the most exciting development in this direction is the pattern pioneered in the United Kingdom and now also spreading, for example, to Australia (Fensham and Davison, 1972), i.e. to make an internship in industry part of the training of a mathematics teacher. In this way, it is possible for the teacher to learn something of how the mathematical sciences are really applied. Practising teachers also sometimes help with the preparation of new interdisciplinary, open-ended materials (see e.g. Case Studies . . . , 1976). Especially in those countries in which there is currently an ample supply of teachers, those prepared in the broader mathematical sciences and familiar with applications of mathematics enjoy a stronger position in looking for employment. In other countries, applied mathematics in the sense of definition (1) has always been a strong component in teacher training, but experience with applications in the sense of definitions (3) and (4) has been missing. Once again, major industrial or agricultural experience has become part of teacher training in China.

3.5 Vocational education

A further educational effect of applications of mathematics is in vocational education. As the importance of the mathematical sciences increases for many disciplines, so does the need the workers and technicians in these disciplines to learn the most appropriate mathematical techniques. Noteworthy vocational materials in a variety of technical fields have been developed in a number of countries. For example, of the order of a dozen volumes of applications of mathematics in different technologies (clothing, carpentry, metal work, etc.) have been produced in Hungary. A different development in the same spirit is the increasing popularity of special curricula for technicians in computer science and data analysis. These have become particularly prevalent, for example, in the United States.

3.6 The implications of truly interdisciplinary teaching

Teaching which is truly multidisciplinary is very difficult to achieve at any level, but perhaps nearest to reality in the elementary school, where — in many countries — a single teacher normally handles most if not all subjects. The evidence for this may be found in the many multidisciplinary materials for the elementary school which have been mentioned. Such activities, when actually carried out in the schools, are especially satisfactory for students because they strengthen the relationship between school and real life. Students are not always satisfied with the promise of future gratification inherent in such statements as "you will find out why this is useful later on", and are pleased with the applicability of mathematics to problems in which they are interested. This is particularly stressed, for example, by IOWO and USMES (Education Development Center, 1974, 1975). However, if the time during the school day is apportioned according to disciplines, it is necessary that the time for multidisciplinary activities be contributed by the various disciplines involved. This implies, at a minimum, that multidisciplinary projects must state what responsibility they will take for specific topics in the several disciplines. Appropriate teacher training at the elementary level is very necessary. On the secondary level, the implications for the structure of the educational system are much more severe. If a single unit involves mathematics, science, social science and language arts all in a significant way, who is going to teach the material, who will contribute the time, how should the school be organized? These problems have not been solved, although team teaching is one possibility; see also Rao (1975). They are discussed particularly in section 3.7 of Chapter III and in the *Report of the Memphis Conference* (Education Development Center, 1974). At the university level, multidisciplinary educational activities may take the form, for example, of genuine model building courses discussed previously, or of team teaching by faculty from mathematics and from a field of application of sections in basic courses such as calculus, linear algebra, and statistics. An example of a master's programme with multidisciplinary experience is Hunter College (1974).

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JORKING GROUP A

SONS FROM RESEARCH

UT STUDENT'S ERRORS

UP LEADERS:

STANLEY ERLWANGER

DIETER LUNKENBEIN

1

LESSONS FROM RESEARCH ABOUT STUDENTS' ERRORS
by S.H. Erlwanger

PARTICIPANTS OF WORKING GROUP A

H. Bauersfeld, H. Bouazzaoui, V. Byers, B. Emmouna, H. Gerber, Hoffman, R. Kayler, D. Kirshner, A. Kramti, E. Kuendiger, D. I O. Mohammed, A. Powell, J. Vervoort, S. Erlwanger.

"Students' errors in mathematics learning have often been approached from a pathological point of view. In such an approach, the study of errors or error patterns is conceiv as the study of the symptoms of some disease for which a c has to be found or discovered. In other relatively recent studies, the phenomenon of errors in mathematics learning been approached from a more developmental, cognitive point view. In this latter approach, students' errors are seen more as signs of progress in learning, which may indicate incompleted process, a deviation from an expected developm or even a misconception but which essentially is a phenome of a cognitive process called learning. In this perspecti students' errors in mathematics are important indicators f the description of the learning process and its gradual de ment.

It is the intention of this working group to study and to the phenomena of error in mathematics learning from the la perspective.

1. by looking at some recent publications or research in this field;
2. by indicating the impact of particular results on the conception and description of models of the learning process;
3. by identifying some areas of research, where the descri approach could be of particular interest."

The Working Group consisted of a diverse set of individuals wit ests ranging from the elementary school to the university leve

1. Publications and Research

Several publications were on display for the group. Some o are listed in the Appendix. In addition, several copies of reports and examples of students work were made available t members. There was unfortunately, not a wide enough range as of research identified as of interest were in the articles display, especially two by Ginsberg which are discussed be-. The following points of interest emerged from the discussions.

or Analysis for Cognitive Purposes

as some discussion as well as general agreement on the idea of rrors and error analysis to study cognitive processes. The , "Cognitive Diagnosis of Children's Arithmetic" by Ginsburg cussed as a good example of one application of error analysis. felt that Ginsburg's classification of cognitive purpose and tive purpose was a useful way of considering error analysis.

the members here felt that such analysis could be useful for building while others were more interested in using such s for remediation. These differences reflected individual interest and experiences in the area of diagnosis.

the participants, H. Gerber, has aptly observed that "the got off to a ponderously slow beginning. Perhaps it was due heterogenious nature of the groups, or the variety of biases, tions, and concerns, that the first three hours were, to me at a waste of time. The meeting came alive at the start of the ession with your (Erlwanger) examples, confirming the theory should introduce a topic with a problem that interests the e." The point being made is an important one in that it re- the state of the art regarding error analysis in mathematics. icle by Ginsburg was a beginning attempt towards some sort of However, it became clearer over the three days that we were ily motivated by anecdotal examples and subsequent discussions d at the descriptive level and led to different interpretations viduals.

e remaining two days we attempted to follow a plan to discuss ral Errors, Conceptual Errors and finally errors in Problem .

cedural and Conceptual Errors

nction was made between these two types of errors by V. Byers. mer were errors in the steps of an algorithm and the latter were ed with concepts. The discussion on procedural errors led to lowing points:

e article, "Cognitive Analysis of Children's Mathematics Diffi-" by Russell and Ginsberg was introduced by Byers as an example strate procedural errors as specific defective procedures when ng a written algorithm. Members did not have enough time st the article. It was also felt that the paper summarized re- rather than described individual children. However, the results licate that fourth graders with learning difficulties made a number of errors than their peers.

(ii) Two examples of children's work were shown by Erlwange One example showed a boy who made procedural errors with an but could do the example mentally in his head. Moreover, th could handle fractions which were used in plans for his mode structions, but he made errors with fractions at school. Th example was of a boy who was able to use algorithms correctl as informal methods of working with percents for example. T were used to suggest that the so-called informal methods by both good and poor students, but it is only the latter wh often unable to use standard algorithm that are taught at sch

(iii) Two examples of conceptual errors were given. V. Byer an example by R. Davis on the zero product principle which sc school students used incorrectly to solve $(x-7)(x-8) = 3$ as x $x-8=3$.

A set of examples by Erlwanger of children's interview resp to questions about the equal sign was distributed. The examp illustrated how children gave their own (different) interpret of the equal sign in examples such as $2+3=5$, $3=3$ and $5+3=3+5$.

There was not enough time to consider these examples or any o

C. Errors in Problem Solving

There was no time to discuss this area at all.

D. Other aspects that were touched upon but not discussed at

1. Articles by Byers and Erlwanger. One article on "Content in Mathematics" raised the question whether errors should buted to either the content or the form or both. A second on "Memory and Mathematical Understanding" raised the ques What do good students in mathematics remember that poor on A related question is the role of memory in errors arising spurious genralizations.
2. The article by Russell and Ginsburg supported some of the o made by the group, namely: children with mathematics diff are not deficient in key informal mathematical concepts an but they have trouble recalling addition number facts and make procedural type errors.
3. The problem of how to minimise the occurrence of errors as how to use errors as they occur to assist students in l.ar mathematics was proposed by M. Hoffman.
4. Looking at errors from a broad context in which errors emer only one aspect of the totality of that student. (H. Bauers
5. The notion that errors are subject matter specific and refl content as well as its form. (Byers and Erlwanger)
6. The development approach in Geometry where errors are consi to be indicators of the level of development of children. (This raised the question that we speak of children's erro quently in subjects taught at school while we seldom think in subjects that are taught informally such as geometry.

discussions on errors remained at the descriptive level
lid not lead to any theory. (D. Kirshner)

rize, the working group demonstrated o... again that our know-
garding the learning of mathematics and the causes and nature
s is still incomplete, fragmentary and far from a theory.
the group met initially a great deal of time was spent in
evaluate each others views. It would probably have been ad-
to have focussed on introducing each aspect by means of
But it turns out that finding examples to cover different
g. elementary, secondary, college and university is quite
t.

l comments by participants:

Kuendiger

iated very much, that during discussions a variety of differ-
s came up, as to how an error can be defined and what role
in the learning process. Depending on the chosen conceptual
different aspects come to the fore.

er three different approaches, that partly are overlapping,
xclusive.

aley gave examples of a student, who could solve an addition
mentally in a non-school environment and could not do it
school, neither mentally nor by using the standard algorithm.
back, I think Heinrich's domains of learning are very suitable
ibe these difficulties: A cognitive structure is built up
main and by this is related to this domain and is not necessarily
er as a successful strategy into another domain.

situation the tasks of the teacher would be to recognise suit-
tegies a student posseses already, to enable the student to
this strategy into another domain and to demonstrate the
ship between strategies (standard algorithm - others).

another reason why I like the above mentioned example given
y: it demonstrates the relevance of the affective part of the
process. This affective aspect is - as to my opinion - one
st important characteristics of a domain, e.g. if a learning
nt is supportive in a way that a student ventures to think,
of cognitive structures from another domain is more likely.

ther aspect came in to the fore in Dieker's approach, that
evelopmental one. Taking geometrical concepts as an example
lopment of a cognitive structure could be described. In this
l framework an error can be looked upon as indicator of the
development. By choosing appropriate tasks the teacher is
support the development of the cognitive structure.

above, shortly described approaches complement one another, -
one, introduced by David Kirshner is - as to my opinion -
atible with the others. Frankly, I do not agree with David,
e his framework as too static: an error is defined as

deviation of a well defined norm system. Moreover the occurre
errors has to be avoided. If this is not possible, the teache
by intervening - has or will lead to the right way more or les
immediately.

This approach makes it easy to identify errors and to classify
but I think it is far from school learning or learning in gene

(b) D. Kirshner

"In this report, I focus on my own principal intervention in t
Error Analysis Working Group, concerning the relationship of
tence models to error analysis. The thesis consists of the f
components:

1. Data available on students' errors are (usually and appro
analysed through comparison to, or as deviatio... from, co
behaviours.

2. Error patterns are less uniform and "stable" than compete
patterns, both within and between subjects, because the c
of deviations from a procedure is (in principle and pract
broader than the procedure itself. Also, errors may pres
intermediate stage in the acquisition of competence. The
is therefore an 'end point' of a developmental process.

3. The greater stability of competence data permits, in prin
more systematic and rigorous analysis of competence patte
is possible, independently, of error or acquision pattern
dominant paradigms in the psychology of mathematical skil
Information Processing) do not exploit this potential, b
competence and error using the same tools and ascribing e
status to theories of error and theories of competence.
sult is that "in most (IP) analyses there has been consid
obscurity in the boundary between what is meant to be tru
subjects, and what is meant to be true of a particular su
(VanLehn, Brown, & Greeno, 1984, p.236)

4. More productive error analysis (i.e. more genralizable a
may have to attend the more rigorous modelling competenc
that case, error analysis may serve a new, subservient r
to the evaluation and verification of competence models.

(c) H. Gerber

The conference was an excellent one, well-organized and with
speakers. However, the session got off to a ponderously slo
Perhaps it was due to the heterogeneous nature of the groups
the variety of biases, expectations, and concerns, that the
three hours were, to me at least, a waste of time.

ating came alive at the start of the second session with your es, confirming the theory that we should introduce a topic problem that interests the audience. From that moment, and ne when the francophones began to speak, our session was first-

, the sessions opened a whole new aspect of teaching. I began erstand the problems, the terminology, and the present limita- on our understanding of errors. Moreover, I now have a biblio- on which to begin. The next time I see you, I intend to ipate in such a meeting in a more intelligent fashion.

nted an example. Let me remind you of the one I gave. The of the examination scores 22/30, 15/20, and 5/8, on tests calculated as (22+15+5)/(30+20+8)=42/58. My son thought that as the same as the old percentage average. In that case the of 60%, 70%, and 80% is (60+70+80)/(100+100+100)=210/300= 70%. This confusion led him to believe that if his cumu- average after 3 months was 60%, and he got 70% on his next ation his new average would be 65%. He was bright enough to error as soon as I pointed it out to him."

Erlwanger

s the second working group in five years that I have attended subject of errors. The first one in 1980 focussed on results n tests and examples of remediation. This time we tried to on the value of errors in cognitive analysis. I note that in se the groups got off to a very slow start. This is probably ction of the different biases and interests of individuals.

h the discussions did not go very far, I think they did reflect velopment in this area that ought to be pursued by furture groups - I hope in less than five years time.

suggest though that in future an attempt should be made to bers to contact each other before the conference. I think absolutely essential so that the group leaders can get some the interests of the participants and perhaps arrange that pants bring one or more examples of errors for discussion."

X

ticles

sburg, H.P., 1983, 'Cognitive Diagnosis of Children's Arithmetic', in J. C. Bergeron and N. Herscovics (eds): Proceedings of the Fifth Annual Meeting of the International Group for the Psychology of Mathematics Education, pp. 247-54.

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5. Matz, M., 1980, 'Towards a Computational Theory of Algebraic Competence, The Journal of Mathematical Behaviour, Vol.3, No pp. 93-165.

1. Byers, V. and Erlwanger, S.H., 1984, Content and Form in Mathem Educational Studies in Mathematics 15, 259-275.

2. Byers, V. and Erlwanger, S.H., 1985, 'Memory in Mathematical Understanding', Educational Studies in Mathematics 16, 259-2

3. Davis, R.B. et al., 1982, The Roles of "Understanding" in the Learning of Mathematics. Part II of the Final Report of the National Science Foundation, April 1982.

(b) Assortment of books.

WORKING GROUP B

LOGO ACTIVITIES FOR THE HIGH SCHOOL

GROUP LEADER:

JOEL HILLEL

Appendix

This appendix contains three examples of LOGO-
activities for the math classroom. The first
relates to the topic of Pattern and can be use
varying levels of sophistication through the
(elementary and secondary). The second activ
to the topic of Least Common Multiple and can
in late junior and intermediate level math cl
The third activity, relates to teaching about
and the circumference of a circle (intermedia

Report of Working Group 'B': LOGO

The Group spent most of its time in examining and evaluating several LOGO inspired investigative situations which had strong links to the math curriculum. This was a follow up to last year's group (Working Group A: LOGO and the math-curriculum) in which the consensus emerged that the availability of such explicit 'microworlds' represents the best strategy for having LOGO accepted and used by most teachers. It is an approach taking the path of 'minimal resistance' since it calls on no special programming expertise by the teacher, nor does it require a major perturbation of the existing classroom setup or the existing curriculum. This is not an argument against other possible implementations of LOGO in the school, including a more inclusive Papertian vision of a fully implemented LOGO curriculum. Rather, it is based on the pragmatic realisation that the acceptability of LOGO to most teachers will be based, rightly or wrongly, on their perception of its relevance to what is currently taught.

Aside from an emphasis on specific math content, last year's group employed other criteria which were intended to reflect the advantages of LOGO-based environments. These included: modifiability, extensibility, the possibility of users writing their own procedures and following several lines of inquiry, etc. (see last year's report). At the risk of an oversimplification, we can say that two general types of situations were examined during the three days. The first type comprised those situations created specifically to enhance a topic

within the existing math curriculum. The second type compr[...] uations whose underlying math concepts are not traditionally[...] but yet seem accessible to students because of the graphical [...] ties afforded by the computer.

Gary Flewelling of the Wellington County Board of Educat[...] duced many examples in which LOGO was used to generate "inve[...] situations" connected to topics in the math curriculum. The[...] included investigations involving fractions, vectors, motion [...] acceleration, trig functions and statistics (see the appendi[...] some examples). These were viewed by the group, which discus[...] they could be modified, or extended to allow the user more c[...]

Denis Therrien of Université Laval also demonstrated so[...] packages which dealt with number concepts such as divisors, [...] composite, odd/even numbers, etc.

A. Senteni of U.Q.A.M. demonstrated a non-turtle LOGO mi[...] that of variations on Conway's Game of Life (designed by B. [...] of L.C.S.I.). Here members of the group discussed briefly wh[...] this kind of situation is only for 'buffs' or whether such an[...] gation could be used to launch into some important math. conc[...] such as 'state', 'action on states', 'stability', finite and [...] 'orbits', etc.

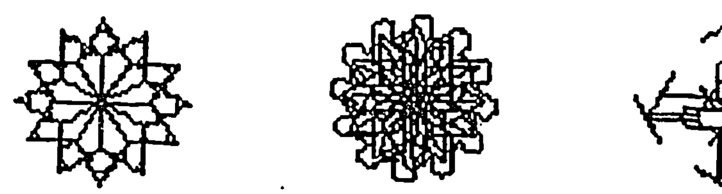
Finally, the possibility of using LOGO to investigate li[...] processes was discussed. Here the group thought out several [...] limiting behaviour which could be exhibited geometrically: l[...] shapes (e.g. circle as a limit of n-gons), limiting points an[...] of inspirals, numerical limits (e.g. ratio of perimeter to di[...] n-gon) and fractals using recursive procedures.

Members of the group:

- R. Blake (U.N.B.)
- G. Flewelling (Wellington County Board)
- J. Girard (U.Q.A.C.)
- J. Hillel (Concordia)
- B. Kastner (S.F.U.)
- H. Kayler (.U.Q.A.M.)
- T. Kieren (U. of Alberta)
- E. Lepage (U.Q.A.R.)
- A. Senteni (U.Q.A.M.)
- D. Therrien (Laval)
- C. Verhille (U.N.B.)
- D. Wheeler (Concordia)

DESIGNS FROM LETTER PATTERNS

(Using LOGO procedures)



Developed by

Gary Flewelling
Mathematics Consultant
Wellington County Board of Education

DESIGNS FROM LETTER PATTERNS

E

MATH
Letter Patterns
Properties of 2D
Designs

START UP INSTRUCTIONS

|. Load LOGO into your computer (see pin up card #1)
:. With Flewelling disk in drive, type
READ "LETTERS [RETURN]
. When the LETTERS file has been read in, type
LETTERS [RETURN]
OU WILL BE ASKED TO RESPOND TO ONE OR TWO INSTRUCTIONS.

you have responded to the instructions on the screen, lphabet keys you asked for will be activated.

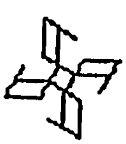
ch letter is typed in, it will appear in the upper portion of the screen.

dition, a larger version of the letter will appear at reen. (see below)

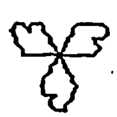
dditional letter begins to be drawn where the previous stops being drawn. This gives rise to a large number tter designs.

ery simple single-letter patterns will generate designs. elow)

AAAA



BBB



- More complicated designs result from using two or

ABABABABABABABAB

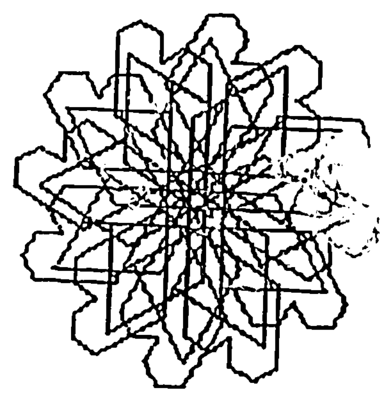
- If a key is hit in error and you wish to remove tha from your design, just hit the [←] key. If you wan several letters, hit the [←] key several times.
- If for any reason you want to blow up or shrink a d hit the [/] key.

You will be asked, "What scale factor?" If you wan its dimension, for example, respond by typing [2] [R Had you wanted to shrink it to half size you would by typing [.5] [RETURN]

To get back to original design size you must hit th respond with a scale factor [1] [RETURN].

Below is the 'ABABAB' design blown up using a scale

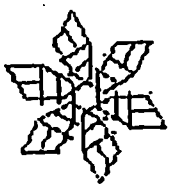
ABABABABABABABA



- Hit the [*] key to erase the screen and start over

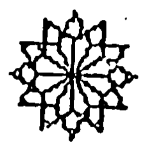
. 1

EMEMEMEMEMEME



e.g. 2

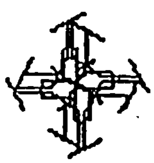
BOBBOBBOBBOBBOBBOBBOBBOBBOBBOBBOBBOBBOB



ter patterns need not be set out on just one line as above.
dimensional array of letter patterns can also be created.
n is achieved by pressing the [1] key at the end of each
e of letters in the array.

ing ART1ART1ART1ART1 for example, would give the letter
tern and design shown below.

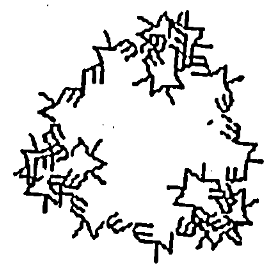
ART
ART
ART
ART



igns can be printed onto paper following the instructions
'pin up card #9'.

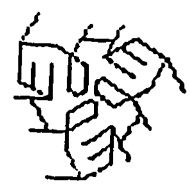
ew sample print outs are shown on the next page.

ZZHH
ZHHZ
HHZZ
HZZH
ZZHH
ZHHZ
HHZZ
HZZH
ZZHH
ZHHZ
HHZZ
HZZH
ZZHH



HELP
HELP
HELP

PIZZA
PIZZA
PIZZA
PIZZA

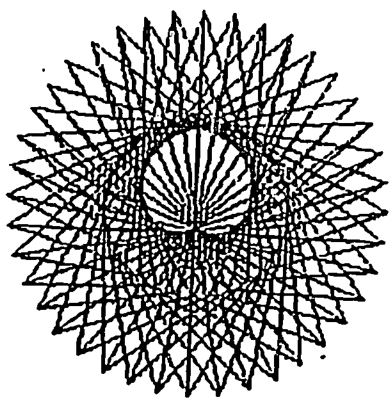


- See the supplement 'WHAT CAN I DO WITH THE PRET
for ideas on how to utilize these designs.

NOTE: Should something go wrong, for whatever rea
want to start over again, hold the [CRTL] and
down, together, and type in LETTERS [RETURN]

LCM

(Using LOGO procedures)



Developed by

Gary Flewelling
Mathematics Consultant
Wellington County Board of Education

LEAST COMMON MULTIPLE

LOGO
NONE

MATH
- Multiples
- Least Common Multiple
- Lowest Common Denomina
- Common Factors
- Coprime #'s
- Composite #'s
- Properties of 2D desig
  as gear ratios

START UP INSTRUCTIONS

1. Load LOGO (see pin up card #1)
2. With Flewelling disk in disk drive, type,
   READ "LCM [RETURN]
3. When LCM file is loaded, type,
   BEGIN [RETURN]

- You are first asked to type in the coordinates of the cen and radius for each of two circles.
I would suggest, in the beginning, typing,

0 0 [RETURN] and

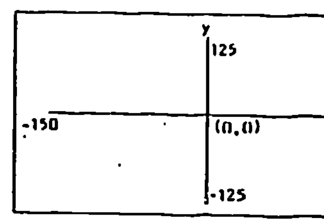
125 [RETURN]

for the first circle, and

0 0 [RETURN]

60 [RETURN]

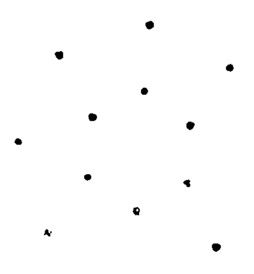
for the second circle.



(Keep the circles within the screen dimension shown above.

- You are then asked to input two natural numbers. Initiall should consider using one digit numbers. Had you typed, example, 8 [RETURN] and then 6 [RETURN] you would see, on screen, eight points on a large out circle and six points an inner circle. (fig 1)

fig 1



rocedures will cause the first point of circle one (C1) joined to the first point on circle two (C2), then the point on C1 to be joined to the second point on C2, Two coloured disks will appear on the points being . (fig 2 & fig 3)

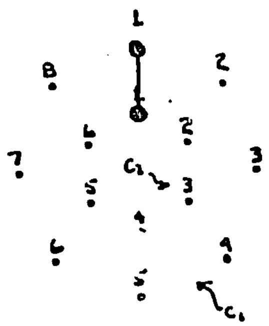
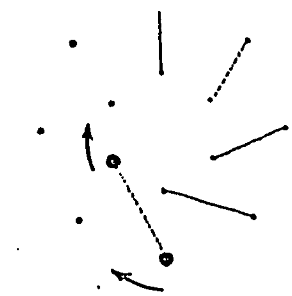


fig 3

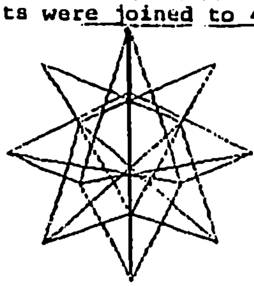


n control the action on the screen (type [S] [RETURN]) the procedure run continuously(type [R] [RETURN]).

[S] and [RETURN] will activate the [*] key. Each time] is pressed another pair of points will be joined. shows result of pressing [*] five times)

above example, it will be noticed that the design be complete (fig 4) until 24 pairs of points have been . In this time, the disk on C1 will have made 3 trips C1 and the disk on C2 will have made 4 trips around .e. 3 sets of 8 points were joined to 4 sets of 6).

fig 4



e action been run continuously, you would see the two run around their circular tracks, with the disk on C1 com- g 3 laps in the time that the disk on C2 completed four. touching 24 points)

ints loudly of the following

$$\frac{1}{6} + \frac{1}{8} = \frac{4\times1}{4\times6} + \frac{3\times1}{3\times8} = \frac{4}{24} + \frac{3}{24} = \text{etc.}$$

a gear with 8 teeth turning a gear with 6 teeth

4 is the "least common multiple" of 6 and 8. ncepts can be introduced with this package.

- Students should be able to predict screen behaviors an comes given any two inputs.

e.g. 1 C1: 8 points and C2: 5 points (fig 6)
e.g. 2 C1: 8 points and C2: 4 points (fig 7)
e.g. 3 C1: 8 points and C2: 8 points (fig 8)

fig 6

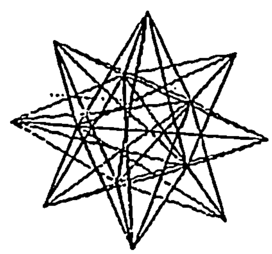
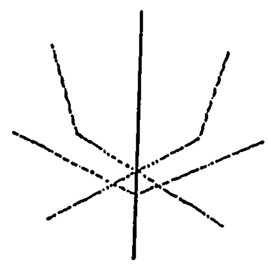


fig 7



- Natural numbers up to 100 can be entered (too large a n will result in an "out of memory" error).

- To print completed designs (fig 9-13) from to screen to follow these instructions.

1. Have Flewelling disk in disk drive and printer 'ON'.
2. Stop LCM procedures with [CTRL] and [G] keys held do together.
3. Press [P] key and the [RETURN] key.

(Figures 9-13 had both circles centred at (0,0), the ce of the screen.)

fig 9 C1: 4 C2: 50

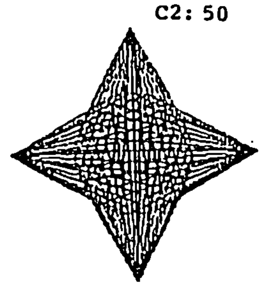


fig 10 C1: 67 C2: 2

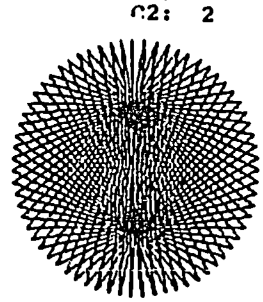


fig 11 C1: C2:

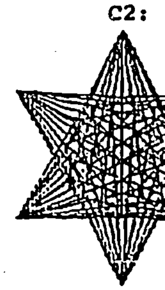


fig 12 C1: 68
C2: 11

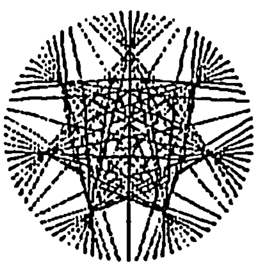


fig 13 C1:40
C2:30

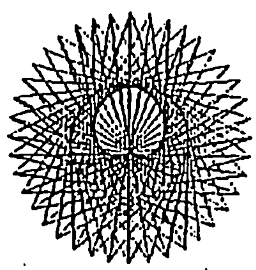


fig 14
C1: (0,50) r=60
C2: (0,-60) r=50
#'s 90 and 3

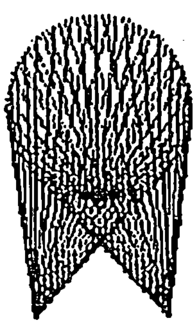


fig 15
C1: (0,50) r=75
C2: (0,-50) r=75
#'s 24 and 72

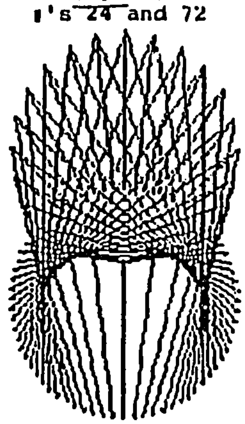
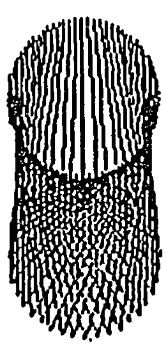


fig 16
C1: (0,50) r=50
C2: (0,-50) r=50
#'s 80 and 40



Things other than LCM's and gear ratios can be inves

Q1. How do successive segments vary in length? (coul measure each off screen and plot a graph, pair # v.s. length in mm)

Q2. Can you predict design characteristics given val inputs? (e.g. C1:16 and C2:12)

Q3. Given design, can you determine input values?

Q4. Are there characteristic differences in designs inputs:
a) are a multiple of the other (e.g. C1:24 and C
b) share a common factor (e.g. C1:24 and C2:8)
c) coprime (no common factors) (e.g. C1:7 and C2

Q5. Are there characteristic differences between des C1:a, C2:b and C1:b, C2:a?

NOTE 1: prolonged use of the [*] key to step out a des result in an "out of memory" error. At this p design can be completed by typing DESIGN [RETU

NOTE 2: The procedure is not self-stopping. You must [CRTL] and [G] keys down to stop the drawing a

NOTE 3: To enter two new numbers without changing the position of the two circles, type,

LCM [RETURN]

NOTE 4: To start with two new circles, type,

INFO [RETURN]

NOTE 5: Should anything go wrong, for whatever reason down the [CTRL] and [G] key then type, BEGIN

Make sure the Flewelling disk is in the disk

CIRCLE ACTIVITIES

TO, REPEAT, PD, RT, LT • [C]

MATH	
Activity 1	circles
Activity 2	circle designs dilatations
Activity 3	circular arcs designs
Activity 4	π circumference

(π AND THE CIRCUMFERENCE FORMULA)

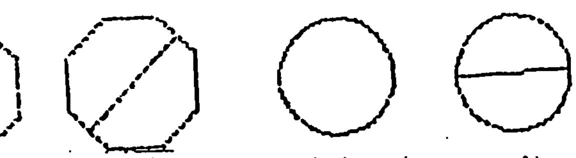
…cy does three things.

…aning to π ,
…e user a way of approximating π ,and
…e user a method for working out a circle's circumference.

… is a procedure from the PI file that draws regular … ist like the POLY1 procedure used in Activity 1. The …rence here is that once the polygon is drawn, the …es to the centre of the last side drawn, turns inwards … pulses the direction it is pointing in.

…now enter a command like FD 2 (or 1 or 5 etc.) and hit [RETURN] keys repeatedly until you get to the opposite … the polygon, you will have measured the polygon's width … count the number of steps taken x2 (or 1 or 5 etc.)

…gate how the width of a specific regular polygon …s to its perimeter.



…egular polygon (regardless of size) has its own peculiar …t (arrived at be dividing its parimeter by its width).

…lygon	Perimeter	Width	Constant P/W
…re			4
…gon			
…gon			
…N			
…gon			
…le			

…omputer to do arithmetic calculation, just enter …ter/width [RETURN])

…eans that the perimeter of a reguler polygon can be … simply by working out the answer to

width of polygon x polygon constant

…s a weird way of calculating a perimeter. Normally, …uld just take the length of one side and multiply by …mber of sides. And yet, it is a way of working out …ter thats worth getting used to!

…he regular polygon becomes a circle, you have no … but to use

width x circle constant !!

…e more familiar with the usual way of writing this formula.

Circumference of a circle = diameter x pi

WORKING GROUP C

IMPACT OF SYMBOLIC MANIPULATION SOFTWARE O… TEACHING OF CALCULUS

GROUP LEADERS:

BERNARD HODGSON

ERIC MULLER

ymbolic Manipulation Software on the teaching of Calculus

les Logiciels a Calculs Symboliques sur L'Enseignement du
ferentiel et Integral

Working Group C

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	Nöelange Boisclair	(Cégep Montmorency)
	Gila Hanna	(OISE)
	Charles Latour	(Cégep F.-X. Garneau)
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	Robert Sealy	(Mount Allison Univer
	Bernard Vanbrugghe	(Université de Moncto
	Edgar Williams	(Memorial University)

Acknowledgements: The leaders wish to thank Professors Dicke and Wainwright from the University of Waterloo. These three spent considerable time explaining the Maple System, its firs the introductory calculus course and the use of computers in ductory Linear Algebra course. The warm welcome to the Unive Waterloo and generous contribution of their time is much appr The group expresses its thanks to Gilbert Morin, an undergrad Universite Laval, for preparing documentation on the use of m

Report

this report the terms Symbolic Manipulation Software (SMS)
Computer Algebra Software (CAS) are assumed equivalent.
y refer to software which manipulates algebraic systems,
s rational arithmetic and can perform calculus operations.)

group started by spending three hours obtaining first hand
ce of the muMATH software in the Laval Mathematics Department's
puter laboratory. The group followed a set of instructions
d by Gilbert Morin - a mathematics undergraduate at the
te Laval (see Appendix 1).

arge number of shortcomings were found during this three hour
the most serious of these being that wrong and incomplete
were produced on the screen without comment. The general concerns
roup is that this particular software is not yet in a form
ntly consistent and correct to be used in or with a first year
The group is aware that such software as MAPLE and MACSYMA have
more widely used and tested and that they do not contain some
hortcomings of the muMATH. At present both MACSYMA and MAPLE
larger computer systems to operate. Nevertheless it is the opinion
rs in the field that both MAPLE and MACSYMA will be available on
micros very soon. The group therefore was looking ahead to times
ted and powerful (computer algebra) symbolic manipulation systems
readily available. Part A of Appendix 2, by Charles Latour, is a
arly good description of the experiences of an individual using
or the first time to solve a specific problem.

the end of the first session participants were asked to think
e impact of such systems on mathematics and to prepare a list of
concerns, etc., which could be studied and developed by the group.

following list was drawn up at the beginning of the second session:
not in order of importance)

Develop problems (examples) particularly suited to solution using symbolic manipulation software.

Develop guidelines for the use of SMS systems as a check to one's work.

Determine whether an SMS system permits the introduction of more advanced ideas at an earlier stage, i.e. order within curriculum when SMS system is used.

Discuss the use of such systems for non-university bound students.

5. Identify either

   a) "routine" parts of the curriculum which can be un
   by the SMS system and which have in themselves no
   towards achieving the aims of the course

   or

   b) isolate the important parts of the curriculum whi
   enhanced by, but not replaced by, the use of an S

6. Guidelines on how to use the SMS systems as a means f
   exploratory development of mathematics

7. Isolate those skills which are necessary for using th
   sensibly:-

   (a) Estimation
   (b) Sense of reasonableness
   (c) Knowledge of concepts
   (d) Are the procedures used in testing algorithms us
   testing solutions from an SMS package?

   (e) Use of graphical techniques as a check of reason

8. How much should one know about the algorithms and the
   language used in such packages? Do these algorithms
   languages give any insight into the the mathematics?

9. What properties should an SMS system have in order fo
   useful in education (as opposed to a pure research to
   capability to show intermediate steps etc.

The group then decided to isolate one topic within the differe
and integral calculus and to discuss the use of SMS systems in
of that concept. Without making any statement as to when or w
a calculus curriculum "limits" should be taught the group deci
at the possible impact of SMS systems on the teaching and lear

IS systems and the teaching of Limits

IS systems do not provide a rich environment for the teaching of
icept of limits. These systems can be used to simplify complicated
iic expressions but generally numerical procedures provide a
medium to motivate intuitive ideas of limit concepts in calculus,
s of the type $\frac{0}{0}$. A useful numerical software package would have
: screen displaying graphical values on one side and algebraic
entations on the other. The plotting of function values should be
: so that subsequent values appear one at a time. It should be
to enlarge any interval of values so that intervals which initially
y small could be enlarged to fill the whole graphical portion of the
Such software would be used to present simple cases in class and
llow students to explore many different functions which are normally
essible because either the student lacks the algebraic techniques,
computations are extremely tedious.

ce the concept of limit is understood SMS systems should be used
vate the laws of differentiation. Every effort should be made to
the derivative as a dynamic concept and not a numerical one. SMS
e allows quick access to more meaningful applications and to the
ction of differential equations which provide life to the derivative.

S systems and the teaching of Integration

en discussing integration techniques -- algebraic integration
res -- two disparate points of view are expressed:

) Too much time is spent on integration techniques both in class and student assignments. These techniques tend to dominate the use of the student's time and mastery of these techniques does not translate into a better understanding of integration. Some argue that we can now dispense completely with integration techniques as they are largely algebraic manipulations which shed no new light or insight on the concept of integration.

) Integration techniques are a necessary part of any calculus course. A student faced with a particular integral is forced to consider alternative procedures for solving it. There is therefore a certain openess or trial and error situation. It is one of the few areas where students apply the algebraic skills they have acquired in school mathematics.

up believes that the following points are sufficiently significant
ey can form the basis of further thought and study in the use of
tware in calculus courses.

When technology is available, course content, lecture pre
and student activities should shift to higher mental acti
Can calculus courses learn from the statistics experience
statistics courses spent many hours on simplification of
involving sums of squares and cross product expressions.
"good" for the students as they obtained experience using
notation and manipulation of indices to change the concep
definition to the efficiently calculable form. This is r
now and more time is spent on the statistical concept and
when and how to apply it. Is the calculus curriculum so
established that it no longer has any flexibility for cha
way to review the Calculus curriculum is to firstly isola
concepts which are essential to calculus and secondly to
curriculum with supporting activities restricted to those
the concepts and give a deeper understanding of calculus.
that SMS software will play a major role in such supporti
Many students presently complete a calculus course and ar
integral tables. They have a very limited experience of
techniques and many are unaware that the integral of the
functions do not have closed analytical forms. Hopefully
ware will change this situation and will place students i
experimental situation.

A reduced emphasis on algebraic manipulation in calculus c
have a major impact on school mathematics courses as much
algebra is directed towards preparation for calculus cours

It is clear that unviersity mathematics professors involve
year calculus and linear algebra courses have a lot to learn re
use of SMS software in these courses. It is imperative that th
are experimenting with the use of such software in their course
their findings. It is important that experimental use of such
well documented so that others can repeat these experiments in
settings. Either one of the leaders of this working group woul
receiving such information and to circulate it to interested in

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27 paper ... vided in five parts: Issues; The computer ... s teaching aid; Teaching mathematics th... ... programming; Diagnostic uses of the comput... ...; Bibliography.)

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A quick overview of different symbolic manipulation systems.

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A fresh discussion of some of the issues raised by the presence of computers and symbolic manipulation systems.

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informatique." Pour la science ... pp.90-98.)
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SIGSAM 85 SIGSAM Bulletin 18(4) and 19(1) (...
A special issue of the Bull... "Special Interest Group on Symb... braic Manipulation" (SIGSAM) ... Contains papers from the session ... mathematical systems and their e... curriculum" held at ICME-5, Ad... Sixty-two pages of interesting ... of the papers report on experim... high school or university.

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A short presentation of symbolic ... systems.

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Some comments on the general q... the computer, with its symbolic ... capability, put all mathematic... business ?

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STOUT 85 D.R. Stoutemeyer, "Using compu... math for learning by discovery ... BOURG 85], pp. 155-160.

Some nice suggestions of projects using computer algebra for math discovery.

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The papers presented by the participants to the ICMI symposium. A new edition of these supporting papers is to be published by the IREM of Strasbourg. Copies can be ordered from F. Pluvinage, Département de mathématiques, 7 rue René-Descartes, 67084 Strasbourg Cédex, France. The price is FF100.

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How to use the graphical capabilities of the computer to illustrate basic concepts of the calculus. See also, by the same author, "Continuous mathematics and discrete computing are complementary, not alternatives", Coll. Math. J. 15 (1984) 389-391 and "Visualizing calculus concepts using a computer", in [STRASBOURG 85] pp. 203-211.

4 Z. Usiskin, "Mathematics is getting easier." Math. Teacher 77 (1984) 82-83.
"Some skills are clearly necessary, but (...) too much else should be learned about mathematics to waste time in practicing obsolete skills. Mathematics is getting easier [with muMATH]. We will not be able to keep this secret from our students forever."

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muMATH is coming! muMATH is coming! A paper intended as a "distant early-warning signal" for the mathematical community.

H.S. Wilf, "Symbolic manipulation and algorithms in the curriculum of the first two years." In The Future of College Mathematics, A. Ralston and G.S. Young, ed., Springer-Verlag, 1983, pp. 27-40.
Expands on the issues raised in [WILF 82]. "It can be very unsettling to realize that what we previously thought was a very human ability (...) can actually be better done by "machines"." (Also contains a description of a second semester sophomore course introducing algorithms.)

WINKEL 84 B. Winkelmann, "The impact of the c[...] the teaching of analysis." Int. J. M[...] Sci. Techn. 15 (1984) 675-689.
A basic discussion of the ways th[...] capabilities (among others, the[...] capabilities) will influence the [...] calculus.

WINKEL 85 B. Winkelmann, "Some remarks on th[...] of elementary calculus in the comp[...] In [STRASBOURG 85] pp. 1-7 (Suppleme[...] "So if it seems possible to master [...] tial equations] at a more elemen[...] than hitherto was possible, they[...] regarded as the most appropriate [...] and goal even for the teaching of [...] calculus at schools and colleges."

YUN & STOUT 80 D.Y. Yun and D.R. Stoutemeyer, [...] mathematical computation." In Ency[...] Computer Science and Technology. J[...] al., eds. M. Dekker, 1980. vol[...] 235-310.
A general discussion of symbolic m[...] systems. Includes a guide to so[...] systems and a discussion of basic [...] alternatives for building up such[...] The last 30 pages are devoted to ap[...] algebra, nonscalar analysis, [...] analysis, celestial mechanics [...] relativity, high-energy physics.

Appendix 1

AN INTRODUCTION TO muMATH

olic mathematics package for micro-computers)

presented to the CMESG meeting

by

G. Morin

Université Laval

BASIC INSTRUCTIONS FOR THE USE OF muMATH SYMBOLIC

1- Insert the DOS 2.10 diskette in the left disk

2- Put the power on the video screen and on t
(right- side of the machine).

3- On the screen will appear: "ENTER NEW DATE:"
the "return" key (↵) in response; same th
"ENTER NEW TIME:"prompt.

4- Remove DOS 2.10 diskette from disk driv
"muMATH 1" disket in that drive and place
diskette in the right disk drive.

5- Type the word: MUSIMP on the keyboard, fo
"return" key (↵).

6- Press the key: [Caps Lock] (for the use of capit
it's important), then press: [Num Lock] .

7- Following the question mark, type: LOAD (MU
press the "return" key.

You are now in muMATH.

N.B. <u>In muMATH, always end a sentence by a
followed by a "return".</u>

A BRIEF SURVEY OF WHAT muMATH CAN DO

Name of file	What it does
ARITH.MUS....................	rational arithmetic
ALGEBRA.ARI..................	elementary algebra
EQN.ALG......................	equation simplificatio
SOLVE.EQN....................	equation solver
ARRAY.ARI....................	array operations
MATRIX.ARR...................	matrix operations
LINEQN.MAT...................	simultaneous linear equations
ABSVAL.ALG...................	absolute-value si
LOG.ALG......................	logarithmic simplific
TRG.ALG......................	trigonometric simplif
ATRG.TRG.....................	inverse trigonometri cation
HYPER.ALG....................	hyperbolic trigonome fication
DIF.ALG......................	symbolic different Taylor series
INT.DIF......................	symbolic integration

INT...................extended symbolic integration
......................limits of functions
IF. ....closed-form summation and products
......................first-order ordinary differential equations
ODE...................higher order ordinery differential equations
.ODE..................axtend first-order ODE methods
......................vector gabra
VEC...................vector calculus

want to see a demonstration of or f the above items, DS (<items's 1st nama>,<item's 2nd name>,B);

mple if you want to know how to differentiate with type: RDS (DIF,ALG,B); and wait for a few saconds. conds at most.)

ter each demonstration the following will appear: Break, Continue, DOS?

onsider "Break" or "DOS", just press "C" if you went inue with a different example or "A" if you want to he demonstration and do some of you own material the same punctuetion and orthograph as in the demons- of coursa).

he "system file" named MUMATH has been built to all the so-called "source files" above. When you pad LOAD (MUMATH); es indicated above, you thus have the memory all the tools offered by muMATH. But if t to see a demo, you need to type the RDS command

DEMONSTRATION OF muMATH

does exect retional arithmetic. Try these examples on board.

/3;

^(1/4);

/4) #E^(#I #PI/4) meening: $\sqrt{3}\, e^{i\pi/4}$

n assign an expression or a value to a "name", e.g.

? TOTO: Y+3*X;

@: Y+3X

Now to see that Y+3*X is really assigned to "TOTO"

? TOTO+Y;   N.B. "*" is the multiplication symbol wh often (but not in every case) be or replace by a "space".

@: 2Y+3x

Remember, you can do symbolic mathematics so it to handle variables who don't have values assigned

Here's the trigonometric expansion function, "TRGE

? TRGEXPD (SIN(2*X),-3);   ← these parameters tell th how to do the expansion.

@:2 COS X SIN X

? TRGEXPD (2*COS(X)*SIN(X),30);

@: SIN(2 X)

If you want to know more about trigomometry on mu RDS (TRG,ALG,B);

SOME USEFUL muMATH COMMANDS

To do:	Type:
- $\int f(x)dx$ (indefinite integral)	INT (F(X)
- $\int_a^b f(x)dx$ (definita integral)	DEFINT (F

N.B. b can be positive infinity "PINF" and a can be minus infinity "MINF"

- $\sum_{x=a}^{b} f(x)$	SIGMA (F(

where a and b can respectively be "MINF" and "

- $f'(x)$	DIF (F(X)
- $\frac{d^n f(x)}{dx^n}$	DIF (F(X

```
,y)                                DIF (F(X),X,N,Y,M);

Taylor expansion of f(x)          TAYLOR (F(X),X,A,N);
t A
oly. equ. P(x) = q(x)              SOLVE (P(X)==Q(X),X);

oly. equ. P(x) = 0                 SOLVE (P(X),X);

a system of n linear equations
spect to x1,x2,...,xn

          LINEQN([equ1,equ2,...,equN],[x1,...,xn]);
```

```
a differential equation.
le if you want to solve:
      (y'(x)+1)y'''(x) = (y''(x))²
DS (Y(X));
R: 'X;
((Y'+1)*Y'''==Y'''^2,Y);

uMATH  e=#E  (e.g. LN#E=1)
       π=#PI (e.g. SIN(#PI/2) = 1)
       i=#I  (e.g. #I^2 = 1)
1) is the 1st arbitrary constant of an expression
2) is the 2nd arbitrary constant of an expression

X from being "DIFVAR" and the dependency of  Y upon

     DIFVAR: FALSE;
     PUT ('Y,'X,FALSE);
```


Appendix 2

GCEDM 1985

Appendice au rapport du groupe

de travail C sur

les logiciels symboliques

Charles LATOUR
Département de mathém
CEGEP François-Xavier

PARTIE A

(Séance pratique tenue le 6 Juin 1985 sous la supe
Bernard R. Hodgson, Eric Muller et Gilbert Morin.)

Le document "An Introduction to muMATH" p
Gilbert Morin s'est révélé fort utile et tout à fai
Mentionnons cependant un petit oubli à la page 4: i
y lire au nota bene
(e.g. # I^2 = -1) au lieu de (e.g. # I^2 = 1).

Au cours de cette session nous avons choisi
les "démonstrations" suivantes:

1- LOG. ALG. sur les simplifications logarithmiq
n'y avons décelé rien d'inquiétant.

2- SIGMA. DIF. sur les sommations et produits.
nous avons pu faire une observati
peu surprenante. Pour la sommatio
nous avons obtenu une expression de

____ + ____ + ____ + ____ + A - A.

Les termes algébriques de la réponse se ressemblant tous, les A n'étant pas des expressions simples et si on y ajoute la difficulté de lire les réponses sous forme linéaire, il aura fallu un bon sens de l'observation pour se rendre compte de la possiblité de les soustreire. On a utilisé la fonction EXPAND pour faire disparaître les A et récupérer la réponse la plus réduite. On ne s'attendait pas à ne pas obtenir la meilleure réponse à l'intérieur d'une "démonstration". C'est d'ailleurs un problème constant dans l'emploi de ce logiciel de savoir si la réponse obtenue est la plus réduite possible. C'est sans doute un problème de grande taille pour l'étudiant qui apprend et qui ne possède donc par l'expérience requise pour évaluer la réponse.

ant les démonstrations j'ai suggéré de résoudre le blème suivent sur le mode autonome: de la formule

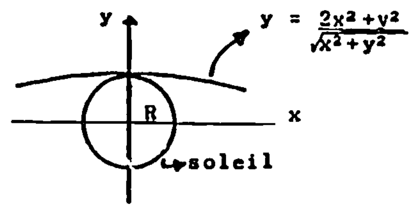
$$y = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

), trouver y'(x,y) et y"(x,y) puis calculer la

$$K. = \frac{y''}{[1+(y')^2]^{3/2}}$$

e se pose en relativité générale. La formule (1) uation en approximation du premier ordre de la

trajectoire d'un rayon lumineux rasant les bords (e.g. soleil). Le contexte physique indique q (1) devrait avoir l'allure suivante



Il s'agissait pour nous de confirmer l'allure de c par l'étude usuelle des dérivées première et s d'en calculer la courbure. Après quelques tâtonn avons procédé de la façon suivante à l'aide de mu

```
DEPENDS (Y(X));
DIFVAR: 'X;     [ligne peut-être superflue]
DIF (Y-(2*X^2+Y^2)/(X^2+Y^2)^(1/2),X);
       [peut-être inutile]
C: @;
SOLVE (C==0,y');
C1: (2*X^3+3*X*Y^2)/((X^2+Y^2)^(3/2)-Y^3);
[cette expression est Y'=Y'(X,Y): poser simp
ne fonctionne pas pour la suite]
DIF(DIF(Y-(2*X^2+Y^2)/(X^2+Y^2)^(1/2),X),X);
[je pense que DIF(C1,X); aurait été plus sim
facile pour la suite]
C:@
```

=0,Y");

ient

"(X,Y,Y')

$$\frac{3Y^2X^2(Y')^2-6Y^3XY'+3Y^4}{2Y^2X^2(Y^2+X^2)^{1/2}-Y^3X^2+Y^4(Y^2+X^2)^{1/2}+X^4(Y^2+X^2)^{1/2}-Y^5}$$

tenir Y" = Y"(X,Y), je réécris la formule précé-

n substituant C1 à Y' et utilise EXPAND]

(   );

ient Y" = Y"(X,Y) en 18 lignes!]

réécris l'expression pour Y"]

(C3/((C1^2+1)^(3/2)));

procure la courbure K = K(X,Y) en 3 1/4 pages!]

on de procéder évoquée ci-dessus est sans doute

ère mais elle est juste. Elle a été testée sur

$$y = x^3 , \quad K = \frac{6x}{(1+9x^4)^{3/2}}$$

$$x^{2/3} + y^{2/3} = a^{2/3} , \quad K = \frac{1}{3(axy)^{1/3}}$$

ent majeur résidq dans le fait de réécrire C1 et

g. Je suis raisonnablement sûr (et satisfait)

enu les bonnes expressions pour Y', Y" et K.

aussi testé le calcul de dérivées plus complexes

de la fonction $y = x^{1/x}$. On n'obtient pas la

plus simple, comme c'est le cas pour $y= x^4$ par

exemple. Il faut utiliser EXPAND.

Au niveau de l'intégration muMATH ne peut p

$\int\sqrt{x}\sqrt{1-x}\,dx$ directement mais il effectue très bien

qui est évidemment une forme équivalente. Mais c'

ai transformé $\sqrt{x}$ $\sqrt{1-x}$ en $\sqrt{x-x^2}$! Le logicie

d'obtenir $\sqrt{x-x^2}$ de $\sqrt{x}$ $\sqrt{1-x}$? Si non, l'étudi

s'appuyer sur le logiciel pour résoudre l'intégr

alors s'entraîner de façon traditionnelle à ma

expressions algébriques. Cette situation amène

plus générale suivante: "Etant donné qu'il est fré

doive transformer légèrement les intégrales pro

utiliser les tables d'intégration, dans quel

système muMATH permet-il de le faire?" "Quels m

ce logiciel pour écrire de façon différente une

algébrique ?"

(A ce propos, on a soulevé au cours de l

lendemain la pertinence d'utiliser un logicie

symbolique dans l'étude de l'intégration par

rationnelles pour éviter la longue "digression"

algébriques. Mon opinion à ce sujet est que je

que muMATH puisse convertir, par exemple, $\frac{2x+3}{x^3+x^2-}$

en $-\frac{3}{2x}+\frac{5}{3(x-1)}-\frac{1}{6(x+2)}$. Je n'ai pas eu le

tester muMATH à ce sujet.)

PARTIE B

ilisation de muMATH en classe ou en laboratoire)

atalier il fut surtout discuté de l'emploi d'un symbolique dans l'étude de la notion de limite.

opinion à ce sujet est que si l'on s'en tient à la ure" du type $\lim_{x \to 3} (x^2+4) = 13$ par exemple, alors le symbolique est à toute fin pratique inutile. Seul que est en jeu. Mais pour la définition de la ar une limite, le logiciel peut se révéler util.. le

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = f'(x_0)$$

$= x^4$ à $x_0 = 2$. L'étudiant peut demander le ment de $(2+\Delta x)^4-2^4$ puis le quotient par Δx (puisque Ce serait à explorer en laboratoire.

aut également penser à utiliser ce logiciel pour indéterminations par la règle de l'Hospital.

généralement, l'emploi de tels logiciels pose la fondamentale suivante: "Les étudiants perdent ils chose (si oui, quoi?) à ne plus s'entraîner à de façon traditionnelle?"

crois que oui parce que l'étude des mathématiques et ilisation comportera toujours du "calcul" sous une une autre.

lève de 3e année qui réussit une division s'exerce à calcul.

(b) L'élève de secondaire III qui exécute x^3-
à un calcul d'un cran plus abstrait.

(c) Plus tard celui ou celle qui montre que
pour f et g satisfaisant des conditio
s'exerce encore à un calcul plus abstrait.

(d) Lorsqu'on montre que dans un groupe,
inverse à droite est aussi élément invers
calcule encore à un niveau plus abstrait.

(e) Lorsqu'on démontre les trois théorèmes d'i
théorie des groupes, on calcule toujours
moins) à un niveau encore plus abstrait.

(f) Je soutiens que même en topologie, on "c
d'une façon particulière à un niveau plus

Je suis d'avis que les mathématiques rest lement une étude des formes de calcul (dans un avouons-le vague!)- la géométrie élémentaire exception quand elle n'est pas modelée par l'al ou la théorie des groupes.

En conséquence l'étudiant chez qui l'e calcul sous forme traditionnelle aura fait dé d'un effort accru au niveau de la conceptual mise en équation, de la "mathématisation" présentées pourra, selon moi, souffrir de niveau supérieur, où l'on ne peut plus reléguer la machine. En résumé, si dans un cheminement qui va de l'élémentaire à l'université, l'étu

eres étapes de son entraînement au calcul - étant
rtout "spectateur" au plan "calculatoire" durant ces
étapes, qui peuvent s'étendre jusqu'au collégial -
vraisemblable qu'il manifestera une faiblesse dans
on de toutes manipulations symboliques (lesquelles,
le, sont inévitables dans l'étude des sciences

uterai que les démonstrations mathématiques (de
les longues) offrent une occasion singulière de
longues chaînes de pensées ou d'idées en devant
d'un lien solide entre chacune. Je soutiens que
leté à "former des chaînes" est fondamentale à bien
dans l'exercice de la science. La mise en
t la conceptualisation, bien que très importantes,
pas un tel intérêt de ce point de vue. Quant à la
é d'acquérir ailleurs cette habileté (en jouant aux
exemple), je réponds qu'il est préférable que le
cquière la forme sur son vélo plutôt qu'en nageant
. (D'ailleurs, aux échecs, le lien entre chaque
n'est pas toujours aussi étroit et solide que dans
tration mathématique).

sion, je poursuivrai sûrement ma réflexion sur
de ce type de logiciel dans l'enseignement des
quee. Je le ferai autant par goût que par néces-

eulement il serait souhaitable que de tels logi-

ciels soient jumelés à des logiciels graphiques mais
des logiciels de traitement de textes mathématiques,
permettraient peut-être l'écriture "normale" des exp
algébriques dont le déchiffrement devient très
lorsqu'elles dépassent en longueur plus de deux ligne

Appendix 3

on au groupe de travail CMESG/GCEDM 1985
ENCE DES LOGICIELS A CALCULS SYMBO-
S SUR L'ENSEIGNEMENT DU CALCUL DIF-
TIEL ET INTEGRAL"

Noëlange Boisclair
Cégep Montmorency

COMMENTAIRES

'enseignement actuel des premiers cours de calcul: univer-
s et cégeps

oche axiomatique et formelle
livres de référence des étudiants)

prépondérance pour le calcul numérique à tendance acroba-
tique

i académique très vif mais, carence dans l'enseignement
rieur
s mathématiques du secondaire et le développement de la
nsée formelle s'harmonisent sur de courtes durées; ex-
ption peut-être pour la 12e année du High School)

entissage confié à l'étudiant responsable de sa formation
université lègue cette préoccupation au college ou au cégep;
collega ou le cégep lègue au High School ou à la poly-
alente;
là, l'instance responsable devient muette!)

EMERVEILLEMENT: efficacité et rendement s'approprient les
nommeurs du cours
tout étudiant écoute une certaine musique; tout étudiant
découvre les séries harmoniques; qui fait u . lien?)

COMMENTAIRES SUR L'ATELIER

Aucun participant n'étant spécialiste de l'incidence des l
calculs numériques dans l'enseignement du calcul différent
tégral, il en ressort que les nombreux thèmes soumis par l
nants représentent une ébauche intéressante qui nécessite
une classification suivant leur caractère pédagogique et l
té d'insertion dans une séquence d'apprentissage.

Toutefois, discuter du concept de LIMITE et/ou d'INTEGRATI
nous l'avons fait, m'apparaît une approche d'un dynamisme.
à court terme car, elle perd de vue la structure globale
A mon sens, la LIMITE est comme un architecte, elle crée,
entre autres, la DERIVATION et l'INTEGRATION et elle se v
posante inhérente.

De ceci, ma réaction aux discussions est que l'on a eu te
téger sa vision personnelle du calcul tout en manifestant
rêt àvouloir répondre aux nouvelles exigences scientifiqu
A mon avis, on a exploré des moyens de moderniser les cou
à vue le rythme traditionnel des concepts tels qu'enseign

Nonobstant cette remarque, cette concertation a eu un asp
dans le sens qu'elle a répondu à une nécessité de poser u
cussion qui se veut l'amorce d'une réflexion plus articul
congrès. Je partage l'idée qu'un tel débat mérite une dém
te et réfléchie.

SUGGESTION

Il me semblerait intéressant d'orienter le débat autour d
tincts qui rendraient pertinente l'utilisation des logici
gisse de MUMATH, de MACSYMA, de MAPLE ou d'autres.

Disons, en guise d'exemples, qu'un logiciel pourrait être
divers aspects, soient:

un outil pour alléger l'enseignement des notions reconnue
les cours préalables
v.g.: manipulation algébrique
domaine et image de fonctions élémentaire

un outil pour développer une représentation spaciale des
tiques
v.g.: graphiques statiques
graphiques dynamiques; mouvement des ε et
familles de courbes.

nt pour soutenir et/ou prolonger l'enseignement
v.g.: esquisces d'analyse
proposition de synthèse
interprétation des valeurs numériques

u'est-ce qu'on met la-dedans?... un peu de génie et..
créativité à l'épreuve.

NOELANGE BOISCLAIR
collège Montmorency

Appendix 4

The Canadian Mathematics Education Study Group

Laval 85 Meeting

A Personal Report from Working Group C

The impact of symbolic manipulation software on the teaching of calculus.

Edgar R. Williams

Memorial University of Newfoundland

I suspect that for some of us in Working Group C, our first two sessions co appropriately labelled as Learning Group C. We did try to come to some conclusio last session and overall, I can honestly say that, for me, the learning, the discuss product of Working Group C made it one of the most fruitful and interesting se have attended.

Without going into a lot of detail, I would like to summarize some of the concl drew from this session. In what follows, the abbreviation CAS will refer to Symbol tion Software Programs or simply Computer Algebra Software (CAS).

1. CAS has the potential to provide Professors with the opportunity to spend less classroom illustrating routine but time consuming computations and more tim and more exciting Mathematical concepts.

also has the potential to provide students with the opportunity to spend less time on
y and time consuming paper and pencil computations, as is normally required on
ments, and to spend more time doing real mathematics.

y put, CAS can be used as a tool to alleviate computational drudgery and allow more
ex examples to be introduced and studied.

can be used by both students and professors to check answers to assigned or com-
homework.

can be used to automate part of a task, for example, the computation of Taylor
when the task is to examine questions of convergence, etc.

examples can be done and done successfully when the computer takes over the chore
tine computation.

ceptional students, CAS may permit the introduction of Calculus and other areas of
much earlier than is possible at the moment.

the possibility of incorporating graphics capabilities into a CAS system, it may be
le to illustrate many concepts geometrically right before the students eyes in a very
ic and interactive way.

has the potential to improve student attitudes toward mathematics especially for
of average ability or below.

as the potential to permit us to re-establish the importance of creative thought and
m solving in the mathematics curriculum.

resent generation of CAS Systems were developed for the use of Scientists and
ers. However, with potential developments in Artificial Intelligence, the future
ial for improvement in CAS designed for educational purposes, seems enormous.

as the potential to provide opportunities for more individual attention to those stu-

dents who need it.

13. Successful mathematics students today appear to learn by being "pro
ple", i.e. after observing enough examples, a methodological technique
tunately, many (unsuccessful) Mathematics students never infer such
them correctly, or in some cases, never even realize such rules exist. CA
to convince weaker students that such rules exist and that even a du
programmed to carry them out.

14. CAS can be used to provide enrichment and motivation in the mathema

15. CAS has the potential to permit students to do exploratory mathemati
before possible.

16. CAS can be extended to include automatic drill, testing, and record
advantage to those of us who have better ways to spend our time.

17. What are we going to do when many, or most, or perhaps all of our st
to come to class with a relatively inexpensive hand held computer with
We must answer that question now. Otherwise, our students will answer

Appendix 5

Some thoughts on the "Impact of symbolic manipulation software on the teaching of calculus"

D.W. Alexander

at routines are unnecessary for understanding?

at routines are necessary for understanding?

w can the graphic capabilities and symbolic nipulation potential of computers be best used to hance learning (of calculus)?

w might the availability of symbolic manipulation ftware (and.graphics) effect priorities, order?

n these be used to promote understandings, en-endedness a la Pollack?

w does this relate the Whitehead's cycle: romance, ecision, and generalization?

ggested sequence:

. Graphical introduction to derivative: chords to tangent; "window" on screen; associate slope of tangent at a point; exploration - generalization for specific function, "derivative" (i.e. slope of tangent at any point).

. Symbolic manipulation code for derivative Maximum/minimum problems
  - approximation (graphically)
  - precision (using derivative code)

Should problems be limited to polynominals or w students "understand" derivatives of other func

Should equation solving capacity of symbolic ma be used?

Is there need to explore second derivatives or graphical capacity remove that need?

Could second derivative tests be introduced as confirming computer graphs? (reasonableness of

What other aspects of "curve sketching" techniq still appropriate assuming availability of grap packages?

Should inverse differentiation (differential eq problems be introduced?

3. Generalization: Explorations of derivativ given by symbolic manipulation to give y' rivative of sinx = cosx; derivative of c sinx; derivative of sinax, etc. Is this time to introduce limit ideas as a basis f

4. Other "Rules of Derivatives"
   - Derivative of a Sum
   - Product Rule for Derivatives
   - Derivative of Quotient
   - Chain Rule

Given the symbolic manipulator, how much of thi needed?

Could it be motivated by "need" to know how to results without the "black box"? By a desire t tand" how the derivatives are obtained?

his be "optional" and only done with some students?

mental issue: Do we desire to teach calculus as a
us" development with the need for "proofs" or is
l to use calculus in solving problems?
the latter, then 4. and perhaps 3. are
sary. (Is it only my conditioning that makes me
ous of this conclusion?)

WORKING GROUP D

THE ROLE OF FEELINGS
LEARNING MATHEMATI

GROUP LEADERS:

JOHN POLAND

FRAN ROSAMOND

MATHEMATICS AND FEELINGS

rticipants: Dorothy Buerk, Renee Caron, Claude Gaulin, son, Bill Higginson, John Poland, Pat Rogers, Fran Ralph Staal, Peter Taylor.

egan this working group by explaining that although lot of literature touching on the role of feelings in mathematics, there is almost nothing directly on it. important area to understand and we must rely strongly l examples shared in the group.

each participant introduced him or herself to the plaining his or her connection to this workshop. This going around the group to share was a key component of s of our working group. The following excerpts from the introductions indicate the wonderful collection of experiences in this group.

Mathematics is connected with feeling the power of looking at new and significant ideas. There is the thrill of invention, of being able to name, of making up new words.

There is the feeling of exploration and of uncovering new and exciting things. There is the eureka experience, the feeling of curiosity, of challenge, of aesthetic, and the philosophical side of uncovering real basic truths.

I would like to see how the enthusiasium of the teacher can influence students in the classroom.

I have team taught a math class with a poet. Half the class was spent in analyzing a piece of poetry. The other half was spent analyzing a math problem. I would like to explore the feelings that are common to poetry, music, math.

see that the beginner's view of math is far different rom the mathematician's vision and I would like to explore how to open up the latter vision to the beginners.

am interested in how the environment influences us. lso I would like to try to be specific about which eelings we pay attention to.

here are not many people at my school with whom I can iscuss these ideas. I feel isolated and would like his workshop to be the beginning of a support group.

here is no such thing as non-emotional motives. People seem less inhibited to express feelings about music. arge groups of mathematicians love music. Is there a omplimentarity here?

As a possible framework for our topic we i discussed the Perry development scheme. Several h scheme that are attached to this paper were kindly Dorothy Buerk. As these indicate, it seems that level we tend to teach to reinforce level 1 percepti students to evolve to level 4. Early levels collu of the world that what is correct is restricted to family, peers, school and is reflected in statement teacher last year didn't do it that way "

In this regard, Lars Jansson drew our attenti discussing the problems of beginning teachers (see We discussed but left unresolved whether emotions are more important at lower Perry levels than at h Does a change in pedagogy equate to a change in the of students perceptions? The experience of many in that a feeling of community and caring in the clas important role in Perry development. The role continued to be a theme on successive days.

Participants were asked to attend the To Problem-Solving by Peter Taylor that afternoon. keep careful track of their feelings during thi When we met on the second day, the sharing of these a great stimulus. Pat Rogers described how feelings and the need to be aware of them, experience of having them. This validation helped situations when negative or confusing feelings aros

One feeling for example, was Pat's anger with peers who were model students for the teacher durin Solving session. There also was an anger with the insisting on receiving his own answer from the s feeling of anger formed a block that kept Pat f involved participant in the Problem-Solving session the workshop described feelings such as anger, conf or competition. The owners of these negative feel being turned off or disengaged during the P session.

The sources of the negative feelings could, in traced to specific incidents. Male-female diff discussed in this context. It was noted that in across all subjects, research has shown that teac attention to the male students. Discussion shift press mathematics has in general. An argument was who were not particularily athletic could be accept they excelled in math. Many questions such as the raised. Does mathematics always assume one's wor related to an authoritative perception of mat unusually strong in the feeling of self-worth?

such as "cool", "controlled" have been used to thematics. These often convey a remoteness or ess on the part of the learner. We discussed, "How ed mathematics to suppress feelings?" In response to on the image of "cool" was pleasant and positive. d described his ability to focus attention on and thus distract himself during a painful illness. nd mentioned that she enjoyed mathematics as an because thoughts about mathematics could crowd out ut sex. Others commented that mathematics is a way eself from interaction with peers.

o concluded with many of us eagerly describing ics of the best teachers we have known. Fran lt it imperative that we also recognize and share our s. To this end we began the third day by spending two pairs, talking about "Why I am a good mathematics turning to the group, we spoke in turn about what we our partner say that struck us as important to good ome of the conversation follows.

th our students:

must make maximum effort to involve all and avoid eoccupation with just the bright students.

en students come in to office hours I go over the xt days lession with them. Then in class the next y they join the discussion because they have had a eview. This also helps me find their misconceptions advance.

deal with disappearing students, I have the class lit to small groups and then report back.

try to involve the students using modified Moore thod. There are weekly assignments leading to big sults. I play it by ear to give just the right ount of challenge and hints so the results become eirs.

want the students to learn to think wilder in the ture. We brainstorm in class. When a person ggests an idea, that person is the idea. Rejecting opinion is rejecting the person. In brainstorming, ideas are evaluated until all have been listed and a se of community has developed in the group.

o In the classroom:

Provide closure. At the end of every hour point to the positive accomplishment, if it is only the aski a good question. Look forward and backward in th

I use two overhead projectors. One is used prepared overheads and is a way for me to convey enthusiasm with the math and also look at the stude The other is used to write spontaneously on.

When discussing preassigned problems, keep po other interesting problems that come up. Let t become optinal homework problems.

I work with colleagues in team teaching. An Eng teacher jointly teaches my math class. We each half of a three hour class. The English prof discu what makes a poem work. Then I discuss what make math problem work. There is criticism of the wri experience as well as of the math problem-solving.

I relate what we are studying in math to other area math. Take a problem and approach it from seve areas in mathematics. Students sometimes re discussion of biographical, historical or cult aspects of the subject. They limit what they want profs to talk about. Its as if they feel we changed the ground rules on them.

I give marks for attendance. I assume progression or growth depends on attendance. If exam seems hard, I look at the marks of those attend regularily.

I give feedback periodically in the what looks li quiz but it is not for a grade. The students write what they feel is important. I give feedback.

o As a teacher:

There has to be harmony between being to egocentric or totally out-going. The teacher mu in charge of what is going on in the classroom. Al teacher must listen to what the students say and they say it In this way the teacher can hear mis The teacher can build on students' past experience

The teacher must hang on to a genuine egocentri Students don't want a teacher who disappears into background. Students want to hear what you are s because you've got something. Be yourself; that what you have the most of. Concentrating on o requires a detachment from yourself.

en I go into a professors office to ask advice, ually he or she has a special personal metaphor with ich to explain the concept. We should share our taphors in the classroom. Talk about mathematics as you are talking to a person while walking by a lake while on a stroll through the woods.

is important to be upfront about what we are doing. y that we expect our students to move through the rriculum by first being able to do problems of one two steps. Eventually they should be able to read on eir own and enjoy what mathematics has to offer. is can be written in a handout and said in class.

is important to build on students past experiences.

accept with good grace my own mistakes. The ideal urse is not one where teacher never makes mistakes.

ass must function as a support system. This must be ear to the student so there is no fear to opening-up. begin first class with lots of self-disclosure and mes in small groups. This lays the groundwork for scussion of feelings.

e of community was a dominant theme among our Community provides safety and belonging. This udents. This allows them to be in contact with to know themselves. We see the classroom as a (as in the '60's). We envisioned the superior high, high as of passing the mathematics.

ssed far too quickly and we have much to discuss. We lly interested in which emotions belong strictly to and cannot be avoided because of the nature of the ere or when do we see the "Ah ha" experience in our students? We also want to explore strategies for eaching that build community. Our main goal is to students.

Appendix to the Report of Working Group D

June, 1985

U. Laval

(R.A. Staal)

One of the by-products of working in this group was a he appreciation of the importance (and existence) of emotive asp learning in mathematics which have their source outside of th mathematics itself.

Within mathematical activity, there are numerous exampl what we might call "emotive" factors - while not strictly pa mathematics, they are inseparably connected with it, and ref essential nature of the total mathematical experience. A fe are: "Eureka!"; various forms of aesthetic satisfaction (pl the apparently complex and instructured to a simple, structu of a beautiful and ingenious proof...); the feeling of secu dealing with a "clean", well-defined structure with clear cri success; the excitement and suspense of exploration; the sen stimulation of mystery; the "down side", of frustration ("wh this work", "why couldn't I have seen that?" I just wasn't be a mathematician") etc. These examples are all pretty fam and come to mind rather easily.

At a less purely mathematical level, there are emotive arising from interactions of mathematics with other subjects at the seashore). These are hard to list in a systematic wa surface in our discussions.

e to the main point of this note: there are emotive aspects
sroom experience which have nothing especially to do with
per se - they apply to the classroom, rather than the subject
influence on the learning of whatever the subject might
ways adequately kept in mind. They have to do essentially
al-interpersonal matters, and include such things as:
on in the development of material, participation as a
member of a group, getting approval versus being put down,
dered important as a person.

to be emphasized that here we are concerned with the role
pects in the learning of mathematics, and have no intention
g the path in which concern for "the whole child" is expressed
phasis on the learning of a subject.

ls of the teacher is brought to the fore in this. Self-study,
ry materials, and computer-assisted-instruction (both of
time-to-time are touted as in the forefront of educational
eave this aspect of learning virtually untouched, unless,
used as a supplementary tool at the hands of a teacher. A
f our thesis, then, is that the teacher is uniquely important.
llowing description of four levels of teaching mathematics fits
omments into a broader scheme.

bject matter is presented, in logical form (Definition, Theorem,
mples are worked out, problems are assigned and solutions
and examinations are conducted.

Level II

As in Level I, but enriched by the addition of backgrou
(biographical and historical material included) , mathemat
and interconnections with other topics and subjects.

Level III

As in Level II, but in addition the students are broug
picture as participants in the mathematical activity. (The
fairly obvious - Socratic and similar approaches, the use of
exploratory assignments, etc.)

Level IV

As in Level III, but, in addition, the students are con
fully as persons, and the emotive aspects of the classroom en
are taken into account as part of the process of learning ma

The Dualistic View

Prepared For

Project MATCH Conference (Davidson College)

6/19/85

By Dorothy Buerk
Ithaca College

look more closely at the beliefs of those holding a Perry 2 view
cal knowledge. Students holding this view will have a number of
beliefs:

nswers are known by an authority for all mathematical questions.
re no unsolved problems, and no multiple answers. Right answers
sed on, not created by the authority.

s one right method to attain the right answer and while students
asked to find it for themselves, they know they are being asked
THE method to find THE answer.

ics is learned by memorization and hard work and by doing every
that is assigned, while following literally each instruction
e teacher (or the textbook) gives. We know how much practice
d.'

either good at mathematics or bad at mathematics. If you are
it you will catch on very quickly. Otherwise you will not.
s in contrast to the notion that one can come to understand over

s not act on a problem and one does not bring one's experience to
em. One brings the methods that have been taught for similar
. Even the authorities learn this way.

dent's role is to collect facts, not act on them, but to store
they are received. One does not use one's intuition.

e no gradations of truth - no gray areas.

ority (teacher, textbook, etc.) is responsible when a student
owledge.

education isn't necessary, since it "won't do me any good on my

To appear: Journal of Education Fall 1985, 16

Strategies to Enhance Learning

By Dorothy Buerk
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- Provide time to experience and clarify a problem (s
focusing on solution. Let each person think about the
anyone speaks. Respond to questions about interpreti
This would include providing background for application
student's field. Focus on resolution only after each p
problem (question) clearly.

- Include the historical perspective to help students bec
person-made quality of mathematics. Concepts as "simp
negative numbers were controversial and adopted with
and yet students are expected to accept them without que

- Acknowledge and encourage alternative methods a
approximation, guessing, estimation, partial solutions,
intuition.

- Answer questions with questions that both clarify
questions and that help the students realize their o
problem solvers and problem posers.

- Encourage students to share ideas, partial solutions,
interpretations of problems with each other. Establish
encourages collaboration and the pooling of ideas to
and/or new questions. Sharing authority in the classro
to the improvement of student learning.

- Encourage the asking of new questions and create an a
both teacher and student are free to wonder out loud.
see their teachers asking, thinking, puzzling, and conjec
in class.

- Make concerted attempts to avoid absolute language.

- Set as a goal the development of each student's internal
of confidence, and of control over the material. He
realize that mathematics can be learned by thinking
memorizing.

- Offer opportunities for students to reflect on paper ab
and feelings about mathematics. Often after acknowle
feelings and reactions a student can move on as if
burden. Writing out one's thoughts often brings a deeper
a new insight and with these come a new sense of confidenc

- Don't rush closure. It is important to continue to think
a problem, an idea, a question, and even a possible answe
leave resolution until the next or an even later class.

Our Expectations of our Students

Prepared For

roject MATCH Conference (Davidson College)
6/19/85

By Dorothy Buerk
Ithaca College

ar in talking with college mathematics faculty about their of their students, I have concluded that mathematics students to be:

al and able to think quantitatively and deductively.

do proofs, a skill which requires bringing together disparate knowledge and seeing the situation from several perspectives. in both "top down" and "bottom up" modes is often necessary to a proof.

see the relevance of applications to theory and theory to ions and, in addition, to understand the connections between .

use problem solving heuristics to approach non-traditional and to have the patience to try out several approaches - to h a difficult problem.

realize that one's intuitions are important and need to be d; that these intuitions can be misleading and need to be tested a theory or with evidence.

learn on their own and from each other; have the internal sense ssing necessary to do that.

make reasoned guesses, conjectures, and to estimate results in ess of inquiry.

ask good questions - especially new ones (problem posing).

ful of the power of mathematics, but still willing to experiment, out ideas that may not work.

write good definitions and to use them - to pull out relevant tion and to be complete.

Proposal for 1986 Working Group on Feelings and M

The 1985 Working Group on Feelings and Mathemat concept analysis to identify the meaning, role, workings of affect in mathematics instruction. propose to further develop the analysis of how affe are related to mathematics learning and teaching and theoretical framework to guide research in this area

While most research in mathematics education solving has focused on developing information-proce of purely cognitive systems, there has been conside recognition that affective dimensions are integra stimulate the cognitive. Emotions and belief syste the twelve major issues that Norman (1981) assert addressed in future research in cognitive scien (1985) urges careful attention to the language an instruction and says we badly need comprehensive an studies of affect in mathematics classes. In de implications from recent research on mathematics future research and policy, Good (1984) emphasize examine systematically how teacher belief systems belief systems in small-group and whole-class sett learning.

The immediate descriptors of affect are the signs such as flushed cheeks, muscle tension or ra McLeod (1984) has related to mathematics pro Mandler's theory that emotion results when an planned behavior is interrupted. Mandler's theory may need to be expanded to include emotions such relief and Ah! Ha! Eureka! described by participan Working Group. Emotion also is evoked by unconscio of present activity with past events. Recall of e memories is one way to raise level of awareness.

A cognitive interpretation of affective b include the influence of belief and value systems. the forms of intellectual and ethical development f Perry (1970) as a first model of how student belief learning are related. Work of Rosamond (1984), Bu Copes (1982) will demonstrate the relevance of Perr specific mathematics courses.

ve interpretation also will include elements of
decision making during problem-solving as described by
1985) Concerns such as that voiced by Brown (1984)
nt ability to generate mathematical questions and by
81) about making meaning and feeling the significance
ical ideas will be integrated into the framework.
discussed is the effect of emotion on memory and

r impetus to modification of learning behavior is the
evel of awareness With a low level of awareness a
react automatically to certain emotions while a higher
wareness allows the learner to choose appropriate
The Perry development scheme, problem-solving courses
enfeld's and tests such as that by Mason, Burton and
2) will be examined to ascertain the pitfalls and
efforts to influence level of awareness.

ognize that we need one-on-one research in laboratory
o help us describe and characterize those aspects of
impact on student learning behaior. Such "ideal"
can skew findings however, and for classroom
learning to be improved, methodology must be
o examine affect under conditions of large-group,
nstruction. The observations of Taylor(·) and
85) will be used to guide development of such

is of our proposed work then, would include the

X	SETTING	X	PHYSIOLOGICAL STATES	X	COGNITIVE INTERPRETATION
	specific or general math content		sensations		belief and value systems
	social content 1-1, small or large group		level of awareness		decision making strategies

hodology will be developed to test and elaborate the
nd thus lead the way to formulation of classroom
e scope of work outlined in these paragraphs is far
o be accomplished during the duration of one working
s indicates our direction however, and we expect to
rogress.

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e is happy to have anyone write for copies of these
Problem Solving book contains 6 problems he used
2 and 13 (good students) plus references and

TOPIC GROUP A

EXPLORING PROBLEM SOLV
IN THE MATHEMATICS CLA

BY: PETER TAYLOR

ion Unique?

an exercise in cooperative theorem discovery, formulation
hat may not be clear from the following review of the session
considerable time was spent playing with the formulation and
current result to make it satisfactory to me and to the

c was the uniqueness of factorization of natural numbers. I
viewing the notion of prime number and ensuring everyone was
the process by which a prime factorization is obtained. I
bout the amateur Canadian mathematician J.P. O'Reilly
hose hobby for many years was playing with large primes. In
vered by chance that if he multiplied the primes

$$p_0 = 2648552897$$

$$q_0 = 9133228103 \quad .$$

resulting number n_0 was divisible by 19. He realized
at if he factored the quotient $n_0/19$ he would get a second
of n_0. This he did, obtaining

$_0 = n_0 = 19.73.223.727.1481.2161.33613$.

$_0$ has been written as a product of primes in two different
primes in common. This was a revelation to O'Reilly because
t time generally supposed that prime factorizations were
order); indeed this was known to be true for reasonably
. O'Reilly's discovery received some attention from
s, and for many years, n_0 was the only number of this type
d the following definition is now standard.

An O'Reilly number is a number with at least two disjoint (no
mon) prime factorizations.

the class for another example of a number with two not
isjoint factorizations. After a moment they agreed that
e of n_0 had this property. They formulated:

f n is an O'Reilly number, then for any k, kn has 2
me factorizations.

one asked about the converse.

f n has two different prime factorizations, then n is, or
of, an O'Reilly number.

moments to find the simple proof of this, based on
mmon primes of the two factorizations.

point, one or two students declared some confusion: is it

not the case that all numbers have only one factorization? I
that while this was indeed the case for the numbers one met in
life, it can evidently(!) fail for large numbers. Indeed our
session was to discover just how widespread this failure might
young man, Ian by name, was not satisfied. He insisted that
divide either p_0 or q_0. That cannot be, I replied, they
prime. Someone had a calculator which took 10 digits and veri
indeed was not a divisor of p_0 or q_0. The youth became
angry. (I knew him to be one of the brighter and more active
the group.) O'Reilly must have made a mistake; 19 cannot div
n_0. I patiently explained that although I had not checked
such an error would surely have been noticed by now. He persi
was sure that factorization was unique. How do you know, I as
could not say. His fellows were embarrassed for him and asked
down. He did but he was upset.

Someone asked whether all O'Reilly numbers were as big as
there any smaller ones? I answered that although others have
they are all bigger than n_0. Indeed an American mathematicia
used a computer in 1952 to verify that all numbers less than
unique factorization; n_0 is the smallest O'Reilly number.

Of course, I continued, it is not pleasant to have number
unique factorization fails, and it is important to try to unde
it is about these numbers which gives them this property. The
theorems tell us that to understand such numbers, it is enough
understand O'Reilly numbers. The task I am proposing is to fi
theorems about O'Reilly numbers, which elucidate their propert

To start them off, I suggested

Theorem 3. An O'Reilly number cannot be even.

We spent some time finding and being careful about the pr
knew that this was to serve as a model for other proofs to com
seemed natural to start by contradiction. Suppose n is an e
number. Since n is even it has a factorization which contain
since it has two disjoint factorizations, it must have one tha
contain 2. Thus $n = p_1 \ldots p_k$ where the p_i are odd. But the
odd numbers is odd. So n is odd. Contradiction. Actually,
carefully at this proof, you will notice that it does not real
(should not?) proceed by contradiction, but can be done more e
directly. I will write subsequent proofs in this direct mode,
ones produced in class were always by contradiction.

I asked for another theorem of this nature. The one I go

Theorem 4. An O'Reilly number cannot end in 5.

The proof proceeds as above. An O'Reilly number n must
factorization that does not contain 5. The primes in this fac

in 5 (or they wouldn't be prime). So n is the product of ch don't end in 5, and so can't end in 5 either.

oof hinges on the fact that the product of two numbers which n 5 can't end in 5, and I asked how they could be sure of this. d that one just had to check the possibilities. The key point tiplcation has the property that if you know the last digit of , then you know the last digit of their product. So you draw digit" multiplication table for numbers not ending in 5.

pointed out that, because a number ending in 5 is odd, the only be constructed for the odd last digits 1, 3, 7 and 9.

	last digit of second number: 1	3	7	9
1	1	3	7	9
t 3	3	9	1	7
irst 7	7	1	9	3
er 9	9	7	3	1

Table of last digit of product of two odd numbers not ending in 5

d for more theorems. Someone put forward that an O'Reilly d not be prime and I called this Theorem 5. I asked for a onging to the same family as the previous two. It was remarked tate that O'Reilly numbers are not divisible by 2 or 5. What small primes?

An O'Reilly number is not a multiple of 3.

the class some time to think about this. Can they do for they did for 2 and 5? It was realized that 2 and 5 worked out y are the factors of 10 which is the base of our number scheme, redients of our proof were little facts about endings of this base. The required results were not available for 3.

t work in based 3? Let's try. The proof should begin as O'Reilly number n must have a factorization which does not If we write the prim[illegible] this factorization in base 3, then m will end in zero. [illegible] their product cannot end in is not divisible [illegible] that seems to do it.

d the fact that in base 3, the product of numbers not ending in and in zero. Is this true? Everyone said it was. Are you ked. After a moment, it was decided that you simply had to ast digit table.

	last digit of second number: 1	2
t digit first 1	1	2
ber 2	2	1

Base 3
Last digit table for product of two numbers not ending in zero

Since no zeros appear, the product of two numbers not ending 3) cannot end in zero.

We appeared to have an interesting "machine". What's ne It was suggested we should try 7 next. But the young man, Ia some trouble with earlier, who had been sitting scowling for while, said quietly, let's try 19.

<u>Theorem 7</u>. An O'Reilly number is not a multiple of 19.

Of course, I hastily explained to the class, we know thi be false. But in trying the above approach on it, we may, in find the proof, learn something about why 19 is different fro 5. So off we went.

An O'Reilly number must have at least one factorization not contain 19. Think of these prime factors in base 19. No end in zero, so the possible endings are 1, 2, 3, ..., 17, 18 treat the numbers 10, 11, ..., 18 as single "digits".) Can t two such numbers end in zero? I asked the class.

Someone said no, of course not, but someone else argued think of the base 19 representation of p_0 and q_0 above. T end in "digits" between 1 and 18. But their product n_0 must zero. Make up the table, someone said. I sketched out an 18 It's a big table, I said.

Ian had borrowed the 10 place calculator and was calcula final "digits" base 19 of p_0 and q_0. He did this by div 19 and taking the remainder. He got 17 and 18 respectively. multiplied them together and filled in that square of the tab The class was silent for a moment. I wonder what that means, means O'Reilly was wrong, said Ian immediately, and there was silence.

I think what it means, I said after a moment, is that I' one or both of these numbers p_0 or q_0. I'm sorry, they in my notes. Ian shook his head in dismay. Having tasted bl not about to be put off. Fill in the table, he said; you wo zero. It's a big table, I replied again.

Okay, I said, after a moment, suppose we fill in the tab we get no zero. What have we got? We know O'Reilly's exampl came the reply. No O'Reilly number is divisible by 19. Right where do we go from there? Do we do the same thing for other far can we get just by filling in larger and larger tables? any way of arguing directly that the table couldn't have a zer actually filling it in? Such an approach would be very power it might extend to a large family of primes. Suppose there's the table. Can you see anything wrong with that?

This was a large piece of direction I had given them, and

ay for awhile. Happily it was Ian who found the argument. It
to restore his equilibrium.

here is a zero in the base 19 table, say in the (h,k) position,
$h,k \leq 18$, then $hk = 19s$ for some s. Since h and k are
19, this gives us two different factorizations for the same
ance gives us a new O'Reilly number (possibly after cancelling
ctors). This new number is certainly smaller than n_0, and
ts the fact that n_0 was the smallest.

class was respectfully silent. Notice what's happened, I said.
illing in the base 19 table, we argued that it couldn't have a
t we used a piece of information we hadn't used before: the
y of n_0. How generally can you make this trick work?

yone felt game to try to tackle:

. There are no O'Reilly numbers.

ook a bit of trial and error to get the proof right. It turns
to generalize the n_0 argument there are really two important
s: that n_0 be the smallest O'Reilly number and that 19 be the
prime factor of n_0.

Theorem 8. Supposing the theorem false, let n be the smallest
number and let p be the smallest prime factor of n. Now n
a prime factorization that doesn't contain p, say $n = p_1 \cdots p_m$
$p_i > p$. Replace each p_i by its final "digit" r_i in base
at $t = r_1 \cdots r_m$. Since n is divisible by p, the last
of t (which is the same as the last "digit" of n) is zero (base
$t = pk$. This gives us two factorizations of t, which, after
of common primes, gives us a new O'Reilly number less than n_0
ach $r_i < p_i$). Contradiction.

the end, some of the class were a bit bewildered by what had been
ed. I pointed out that unique factorization was indeed a property
ntegers, and that that was in fact what Theorem 8 stated. What we
uced, in our explorations, was quite a reasonable proof of the
actorization result. Had anyone, I asked, seen a proof of the
actorization theorem before? One or two thought they had, but
en't sure.

I have given this exercise to four different groups: high school
, high school math teachers, university math seniors, and
ty math educators. In all groups there was some initial confusion
appearance of an example which appeared to contradict a firmly
ief. But if the example was properly dressed up with the right
al footnotes, I found my audience on the whole quite willing to
their disbelief" and enter actively into a search for theorems.

numbers p_0 and q_0 are chosen with care. I don't have any
o believe they are prime, but they have no factors ≤ 61. If you

multiply them out with a 10 digit hand calculator you get
which is also what you get if you multiply out the "small"
of n_0. Also if you do a "last 3 digit" analysis of the
factorizations you get 391 for the product of both sides.
factorizations are the same mod 1000.

The unique factorization result is usually (casually)
high school, and is proved in a first or second algebra co
university. [Nevertheless I had no trouble selling my exa
university students.] The usual proof uses the Euclidean
There is a standard proof similar in spirit to our "discov
Theorem 8, which Nathan Jacobsen [Basic Algebra I, Freeman
attributes to Zermelo. [I am grateful to John Poland for
It goes as follows: let n be the smallest number with tw
factorizations

$$p_1 \cdots p_m = n = q_1 \cdots q_k$$

and suppose $p_1 > q_1$. Then

$$(p_1 - q_1)(p_2 \cdots p_m) = q_1(q_2 \cdots q_k - p_2 \cdots p_m) .$$

By completing the factorization of both sides we get two p
factorizations of a number smaller than n, one of which
the other of which does not (since q_1 cannot divide $p_1 - q_1$
divide p_1).

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TOPIC GROUP C

STEMOLOGICAL FALLICIES ILL LEAD YOU NOWHERE

JACQUES DESAUTELS

EPISTEMOLOGICAL FALLACIES WILL LEAD YOU NOWHERE!

Titre arrogant pour une conférence si, d'une part, on tradu "fallacies" par "faussetés" et si, d'autre part, on imagine que invective ses auditeurs. Il perd cependant son impertinence s le sens d'illusion puisqu'il se transforme en lapalissade. Qu oserait affirmer que l'on peut aller "quelque part" dans le do tissage des mathématiques en se berçant d'illusions et d'illus ques au surplus. Mais cette lapalissade n'en est pas vraiment a pas le caractère premier, soit l'évidence liée à l'univocité connote. Pourtant, pour ceux d'entre nous qui ont réfléchi à cer de l'apprentissage des sciences, elle a acquis un sens évident, graduellement au jour par des travaux qui forment un véritable cherche que l'on reconnaît dans la littérature sous les étique tions pré-scientifiques, conceptions ou représentations spontan tre perd alors définitivement toute insolence ou prétention pui l'apprentissage des sciences et ne s'adresse donc pas, tout au aux didacticiens des mathématiques. Mais nos travaux peuvent-i certaine utilité?

C'est la question qui a orienté ma réflexion et je me prop succinctement avec vous des sujets suivants:

1) Quelques exemples de représentations spontanées.

2) M. Bachelard, ses obstacles et son profil épistémologiq

3) La droite, le point, le hasard.

1) QUELQUES EXEMPLES DE REPRÉSENTATIONS SPONTANÉES: La chaleur, le mouvement, etc.

Lorsqu'on demande à des enfants d'une dizaine d'années d'e l'extrémité A d'une tige de métal devient chaude alors que la s est située à l'extrémité B de celle-ci, on ne les prend pas au fournissent spontanément des explications. Celles-ci, bien que

eci en commun: il y a quelque chose qui se déplace du point B au
qui au demeurant est tout à fait logique. Mais qu'elle est la nature
e chose qui se déplace ainsi? Evidemment, c'est de la chaleur et
ne peut rien reprocher à l'explication. Si on poursuit le question-
'à leur demander ce que c'est la chaleur, on découvre que pour eux, il
substance plus ou moins volatile, qu'ils comparent à l'air, à la
n fluide quelconque. Ces explications ne correspondent
qui forment le champ de connaissance de la science moderne, bien que,
s cas, elles présentent des similarités étonnantes avec des théories
nt reonnnues [1] par les scientifiques, notamment la théorie du calo-
Cependant, ces explications enfantines,
e verrons ci-après, font obstacle à l'apprentissage des sciences et, à ce
aient être prises en considération dans l'élaboration de stratégies
.

cation du mouvement fournie par des élèves d'une dizaine d'années
autre exemple de représentation spontanée. Ceux-ci, à l'instar
ne peuvent concevoir qu'un objet puisse se mouvoir sans
on d'une force qui non seulement initie le mouvement mais le main-
tre part, si la vitesse d'un objet est constante, c'est que néces-
force agissante est constante, et plus celle-ci est grande, plus la
proportionnellement grande. Dans cette optique, un objet qui se
ande vitesse doit nécessairement être mu par une grande force.

d Driver [2] utilise l'expression "children's science" pour désigner
es explications que les enfants construisent spontanément pour ren-
es phénomènes avec lesquels ils interagissent, avant toute éducation
formelle, mais également pour souligner que ces explications forment
e conceptuelle dont on doit tenir compte en pédagogie des sciences,
parce qu'elle permet aux enfants de donner un sens à leurs ob-
uotidiennes. Or, jusqu'à tout récemment, on a négligé de le faire,
l suffisait de montrer la bonne solution pour que les élèves changent
ations. Les résultats de la recherche sont clairs [2], les élèves
t pas leurs explications premières et les réutilisent très volontiers
ontexte du problème qui leur est posé diffère de celui des problèmes
apitre dans un livre; ce qui d'ailleurs ne les empêche pas de réussir
Mais que devient la connaissance scolaire quelque temps après les

études? N'ayant pas été vraiment assimilée, elle est reléguée
lentement mais sûrement se transforme en vague souvenir – Ah!
le principe d'Archimède – l'eau qui monte dans la baignoire...
vraiment compris.

Le spectre des raisons qui peuvent être invoquées pour exp
nos enseignements respectifs [3] est varié: formation des maîtr
tique, stratégies pédagogiques, nature des disciplines, dévelop
des élèves, sont autant de facteurs à examiner afin d'éclairer
toutes ses dimensions. Or, parmi ceux-ci, je m'attarderai à la
gique intrinsèque du processus de la transformation de la conna
considère du point de vue historique ou du point de vue de l'ap
viduel, ce qui me permettra de spécifier en quoi les représenta
des élèves constituent des obstacles à leur apprentissage des s

2) Monsieur Bachelard, ses obstacles, son profil épistémologique

Il est étonnant de constater que la publication du petit l
Kuhn [4], *La Structure des Révolutions Scientifiques*, ait provo
chez les intellectuels de toutes les disciplines, alors que l'o
d'épistémologie historique de Gaston Bachelard continue à être
Dès ses premières publications [5], ce dernier, en interrogeant l
de la relativité et de la théorie quantique, posait les jalons d'
qui, à mon avis, est plus riche d'enseignement que l'oeuvre de
au regard de la compréhension de la nature du savoir scientifiq
formation, mais également du point de vue pédagogique, car s'il
mologue, Gaston Bachelard a d'abord été professeur de sciences.
pas s'étonner de trouver tout au long de l'oeuvre de Bachelard
pour l'enseignement scientifique; n'écrivait-il pas dès les ann

> *"Les professeurs de sciences imaginent que l'esprit commen*
> *une leçon, qu'on peut toujours refaire une culture nonchal*
> *redoublant une classe, qu'on peut faire comprendre une dém*
> *tion en la répétant point pour point."* (6)

Il ne saurait être question d'épuiser en quelques pages un
che; je me contenterai donc d'évoquer quelques-uns des concepts
cet auteur, qui permettent, à mon avis, de saisir en quoi les
spontanées constituent des obstacles à l'apprentissage.

helard[7] seule une philosophie dispersée des sciences peut rendre transformation historique du savoir scientifique, et c'est à partir de masse qu'il illustre cette idée. Il affirme que l'on peut dis- stades dans la transformation de cette notion correspondant à rants philosophiques, c'est-à-dire: le réalisme naïf, le réalisme- rationalisme, le rationalisme dialectique et le rationalisme complet. ade, la masse est conçue intuitivement comme une "appréciation comme gourmande de la réalité" [8]; au deuxième stade, la masse est iquement par l'opération de la balance et alors: "Peser c'est penser. peser" [9]. Ce n'est qu'au troisième stade que la notion prendra l'on peut parler ainsi, et sera rationnellement conçue comme s de notions et non plus seulement comme un élément primitif d'une médiate et directe." [10], et définie comme le rapport de deux s, la force et l'accélération. Cette belle assurance rationaliste u moment de la complexification de la notion de masse qui devient vitesse de l'objet en plus d'être transformable en énergie. Enfin, re et à la logique théorique et aux exigences empiriques, il a été accepter l'idée d'une masse négative.

iption de ces stades ne nous informe cependant pas quant au méca- able de cette transformation, et c'est pourquoi Bachelard a mis au ept de rupture épistémologique. Par exemple, le passage de la masse masse relative suppose l'abandon de certaines prémisses épistémolo- elles d'espace et de temps absolu, et d'en accepter d'autres dont itesse limite. Il y a donc une rupture qui rend ces notions incom- ce qui ne signifie pas pour autant que celles-ci ne soient pas ertains domaines spécifiques. D'une façon similaire, la théorie permet de définir en science la notion de chaleur exige e de considérer la chaleur comme une substance pour adopter le énergérique, beaucoup plus abstrait, puisque la chaleur est alors conçue ie cinétique moyenne des atomes ou molécules, telle que donnée par $= \frac{1}{2} mv^2$. Or, il s'agit d'une véritable rupture dans la mesure où aire de nier les impressions sensorielles à partir desquelles, tout on construit une certaine représentation de la chaleur, sans tre part, l'élimination de la notion de froid, qui n'a aucun sens xte des théories scientifiques. De même, l'enfant doit nier les ensorielles premières, qui le conduisent logiquement à croire au et à nier qu'un objet puisse se déplacer

sans l'action d'une force, pour accéder à la compréhension du pr C'est dans ce sens qu'il faut saisir le mot de Bachelard lorsqu'

> *"en fait, on connaît contre une connaissance antérieure, en des connaissances mal faites, en surmontant ce qui, dans l' même, fait obstacle à la spiritualisation."* [11],

d'où la notion d'obstacle épistémologique qui mériterait à elle commentaire. Je rappelle seulement que ces transformations de l intrinsèque à l'apprentissage des élèves sont l'équivalent d'une relle. Le rôle du pédagogue doit alors s'articuler aux exigence formations et on comprendra qu'il est alors nettement insuffisan version officielle des sciences, même si la présentation est log

3) Le point, la droite

Les notions de l'épistémologie bachelardienne nous ont aidé en quoi les représentations spontanées des élèves constituent de leur apprentissage des sciences. En effet, elles nous révèlent q transformation de ces connaissances exige la remise en question ment implicites qui forment la structure de base de la vision du laquelle les élèves règlent, avec un certain bonheur, leurs inte vers matériel. Il est dès lors illusoire de penser que ces chan s'opéreront au cours de quelques leçons bien faites. L'apprenti matiques pose-t-il des problèmes similaires?

Je ne me risquerais pas à affirmer que l'on retrouve exacte problèmes au niveau de cet apprentissage, compte tenu de la préc ture mathématique et ce, tant au plan des notions elles-mêmes que épistémologique. Cependant, il me semble qu'un certain nombre d géométrie euclédienne (la seule que je connaisse) présente des similaires à celles que j'ai évoquées ci-avant, au plan de leur des élèves.

Pour ces derniers, comme pour la plupart des gens, il n'y a de distinction entre la ligne et la droite. Celles-ci correspon physique observable qui manifestement a une longueur et une épai quant à lui, s'il est minuscule, n'est quand même pas infiniment là devant leurs yeux et bien visible. Et il est tout à fait con eux, un point qui se déplace dans l'espace engendre une ligne.

ace. Mais on sait que les définitions mathématiques du point et de
correspondent pas à ces représentations sensuelles. Il est fort
r les élèves de s'en détacher et de concevoir un point sans dimension,
à l'intersection de deux segments de droites qui n'ont qu'une lon-
d'épaisseur. Or, la compréhension des concepts de la géométrie eu-
essite un détachement par rapport à ces représentations concrètes
r à l'univers abstrait des constructions géométriques. Il s'agit là
saut qualitatif sans lequel on imagine mal comment les individus accé-
univers "étranges" des géométries à n dimensions où la référence au
titue un véritable obstacle épistémologique.

it-il pas ainsi une kyrielle d'obstacles épistémologiques à identi-
rt avec de nombreux concepts mathématiques au sujet desquels les
nstruit spontanément des représentations? Je pense, par exemple, aux
ants: l'infini, le hasard, la relation et, pourquoi pas, le nombre?
er des obstacles à l'apprentissage est une chose, créer les stratégies
pour les surmonter en est une autre.

s pas certain que mes propos aient été parfaitement clairs, ni
tout à fait pertinents par rapport aux problèmes rencontrés dans
e des mathématiques. Intuitivement, je pense que les concepts de
e bachelardienne ont un certain à-propos eu égard à vos préoccupa-
ticiens des mathématiques. La discussion qui suivra permettra, je
profondir ces questions.

Jacques Desautels
Professeur.

NOTES DE RÉFÉRENCES

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phylogenèse, bien que, dans certains de leurs aspects, il y ait
ments étonnants.

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TOPIC GROUP D

NT CANADIAN RESEARCH
ERNING TEACHING, GENDER
AND MATHEMATICS

ILA HANNA
RIKA KUENDIGER
OBERTA MURA

SEX DIFFERENCES IN THE MATHEMATICS ACHIEVE OF EIGHTH GRADERS IN ONTARIO

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In the past two decades researchers have shown considerable interest in tl between the sex and the mathematics achievement of children in the upper grad schools. Some have examined sex differences by comparing total test scores (Bac & Stanley, 1980, 1983; Maccoby & Jacklin, 1974), while others have focused on t students who answered a particular item correctly (Armstrong, 1980; Fennema, Wahlstrom & McLean, 1984). In a recent study by S.P. Marshall (1983) the anal comparison of the kinds of errors made by male and female students

Some of the studies done to date purport to have established that by age 13 difference in mathematical ability between the sexes, and that it is especially pr high-scoring exceptionally gifted students, with boys outnumbering girls 13 to 1 1983), while others have argued the opposite: that very little difference exists, if difference is detected it favours boys only slightly (Fennema & Carpenter 1981) *International Review on Gender and Mathematics* (Schildkamp-Kündiger. 1982 research carried out in nine countries, gender-related differences in achievemen vary considerably both within and among countries.

The purpose of this study is to assess the scope of sex-related differences ir achievement of Ontario Grade 8 students, making use of the pool of data collecte International Mathematics Study (SIMS).

Test Instruments and Data

For the SIMS study, a random sample of 130 schools was selected from a to schools after each school had been assigned to one of twenty-six strata based on t categories: (a) school size, (b) type of school (private, French, English Catholic, a rural or urban, and (d) geographical region of the province. (In Ontario, virtuall olds are enrolled in either a private or a public school.)

The present analysis does not use data for private schools (which are atten Grade 8 students). Since previous analyses (McLean, Raphael & Wahlstrom, 19 students in private schools had much higher rates of success, and since there wer boys as girls in the private-school stratum, it was decided to delete these data fro sample retained for this study consisted of all the Grade 8 students not attending whom data were available for both the pretest and the posttest: 3523 in total, 17' had developed 180 items for Grade 8, administered in five forms: a Core form of 40 Rotated forms of 35 items each; for technical reasons six items were not part of the 'he 174 Ontario items covered five broad topics: Arithmetic (58 items), Algebra (31 try (42 items), Probability and Statistics (17 items), and Measurement (26 items).

ents were administered both a pretest and a posttest. Each student responded to the to one of the Rotated forms A, B, C, or D on each occasion. Each student was given the n at both the pretest and the posttest, but not necessarily the same Rotated form; the were administered randomly on both occasions, each form to one quarter of the class. As of this method, there are variations in number of respondents among the four Rotated between the two occasions for the same Rotated form. In addition, the Core form yields ision of results, since it has about four times as many respondents as a Rotated form. arizes the pattern of responses to each of these test forms.

Table 1

Number of Respondents by Sex and Test Form

	Pretest		Posttest	
	Boys	Girls	Boys	Girls
	455	417	459	417
	427	470	465	444
	447	426	433	437
	444	437	416	452
otal)	1773	1750	1773	1750

number of respondents 3523.

ms were five-alternative multiple choice (one correct response and four distractors). se to each item was coded into one of three categories: correct, wrong, or item omitted. n, three percent values (correct, wrong and omitted) were calculated separately for boys with the student as the unit of analysis. (The percent correct of an item, for example, is ge of students who answered that item correctly.) Three mean percent values were then each topic by averaging the percent values for the individual items in that topic; these are ble 2.

Table 2

Mean Percent Values (and Standard Deviations) per Category of R by Sex and by Topic

		Pretest			
		Correct	Wrong	Omit	Correc
Arithmetic (58 items)	Boys:	49.5 (18.1)	48.0 (17.0)	2.5 (2.7)	54. (16.6
	Girls:	48.2 (20.0)	48.2 (18.4)	3.5 (3.9)	54. (17.8
Algebra (31 items)	Boys:	34.6 (15.9)	59.1 (16.4)	6.2 (3.6)	44. (15.7
	Girls:	33.5 (17.2)	57.8 (17.7)	8.5 (4.7)	44. (17.3
Geometry (42 items)	Boys:	36.4 (17.6)	56.9 (15.1)	6.7 (5.1)	45. (17.8
	Girls:	33.4 (17.2)	57.6 (14.5)	8.9 (6.5)	42. (18.3
Probability & Statistics (17 items)	Boys:	53.5 (19.6)	43.7 (18.7)	2.8 (1.9)	57. (18.8
	Girls:	52.9 (22.2)	42.9 (20.1)	4.2 (3.0)	56. (18.8
Measurement (26 items)	Boys:	45.9 (21.8)	51.1 (20.6)	3.1 (2.6)	53. (19.7
	Girls:	42.6 (22.8)	53.1 (21.2)	4.3 (3.4)	50. (21.

Note. Due to rounding error the figures for Correct, Wrong and Omit may not add

Results

For each topic the difference between boys and girls in the mean percent of omitted responses was analysed using the paired t-test with the item as the unit addition, a Wilcoxon matched-pairs test was performed to obtain the z-statistic a probability as well as information on the number of items with positive or negati between boys and girls.

in Correct Responses

vn in Table 3 no statistically significant differences were found between boys and girls on n for three of the topics (Arithmetic, Algebra, and Probability and Statistics). In l in Measurement, however, more boys gave correct responses on both occasions: in both areas the difference of about 3 percent is statistically significant at the .01 level.

retest as a whole, boys were more successful on 100 items and girls on 60: boys and girls t in Geometry and in Measurement, boys did better than girls on more than twice as This pattern of results was very much the same for the posttest.

Table 3

Differences Between Boys and Girls in Mean Percent Values by Topic

		Pretest			Posttest		
	df	Correct	Wrong	Omit	Correct	Wrong	Omit
	57	1.3	-0.2	-1.0*	0.7	0.4	-1.1*
	30	1.1	1.3	-2.3*	-0.5	1.8	-1.3*
	41	3.0*	-0.7	-2.2*	2.4*	-0.6	-1.7*
&	16	0.6	0.8	-1.4*	0.9	0.2	-1.0*
t	25	3.3*	-2.0	-1.2*	3.2*	-1.8	-1.2*

tive difference represents a higher mean percent for boys; a negative difference, n for girls.

in Wrong Responses

were no statistically significant differences between boys and girls at the .01 level on any the two sexes gave wrong responses with similar frequency.

pretest as a whole more boys gave wrong responses on 84 items while more girls did so on ms the percent of wrong response was the same for both sexes. In Arithmetic, Geometry s boys and girls gave wrong responses on approximately the same number of items. In wever, the rate of wrong responses was higher for boys on 20 items, while for girls it was ; this pattern was reversed in Measurement, with the girls giving more wrong responses and the boys on 9. The posttest results were very similar to those of the pretest in terms of tion of wrong responses.

Differences in Omitted Responses

As shown in Table 3, the differences between the sexes were negative, indi percent of omitted responses for girls was greater than that for boys on all the sul pretest and the posttest. Furthermore, the t-test paired comparisons showed tha between boys and girls were statistically significant at the .01 level.

In both the pretest and the posttest more girls than boys omitted responses. was higher for the boys only on 17 items (10% of the test), while it was higher for (70% of the test). The Wilcoxon analyses yielded z-statistics significant at the .01 topics, indicating that this trend was consistent from topic to topic.

A detailed examination of the omitted responses revealed that the percenta omitting items on the pretest ranged from 0 to 28 for girls and from 0 to 23 for boy 4.5 and 3.0, respectively. Although there was a decrease in these values for both t posttest (that is, fewer students omitted items), the gap between the sexes was ma posttest the range was 0 to 21 with a median of 3.0 for girls, while it was 0 to 17 wi boys.

Differences in Gains

The gains are based on the difference between the mean percent of correct topic on the posttest and on the pretest, for each group taken separately, and coul taken to represent the growth in mathematics achievement for the group. The re 4 would indicate that on average boys and girls improve at the same rate during no statistically significant differences (at the .01 level) between the two groups in in mean percent of correct responses by topic.

Girls showed greater gains on 93 items and boys on 63; girls and boys tied c In Measurement girls had greater gains on approximately the same number of it Arithmetic, Algebra and Geometry, taken together, girls had greater gains than many items.

Table 4

Gains in Mean Percent Values by Sex and Topic

	Boys	Girls
Arithmetic	5.4	6.0
Algebra	9.5	11.1
Geometry	8.6	9.1
Probability & Statistics	4.1	3.8
Measurement	7.5	7.6

Note. Differences between boys and girls not significant at the .01 level.

Discussion

s of this study may be summarized as follows:

percent of correct responses in two of the five topics (Geometry and ent) was slightly higher for boys than for girls. These differences, though not e statistically significant at the .01 level.

e differences between boys and girls in omitted responses. All the t-tests were t at the .01 level Girls had much higher omission rates on all topics. On average on ratio of boys to girls was 2:3.

ion of the gains indicated that instruction in Grade 8 had about the same on girls as on boys.

e findings assume educational significance when one bears in mind that the boys and the same randomly selected schools in approximately equal proportions and thus can atched on socio-economic level, on amount of formal training in mathematics, and on ng (ignoring possible differential treatment of the two sexes on the part of teachers). hus be generalized to students attending public schools in Ontario, and any sex d must be attributed to factors other than socio-economic level, formal training, or ng.

vable that the boys had had a certain amount of informal training through out-of-class rmally pursued by girls (following instructions for building models, reading charts and ike). Different informal training in mathematics could explain the differences in Geometry and in Measurement in particular.

to McLean, Raphael and Wahlstrom (1983), Ontario teachers reported that only about etry items had been taught at all, either before or during Grade 8. This would lend the idea that out-of-class activities contributed to the disparities in achievement es. On the other hand, the other topic which showed differences between the sexes, was among four topics in which most teachers reported covering about 80% of the s on the basis of the information available it is not possible to determine with any her out-of-class activities had an effect on the differences between the sexes in asurement.

es sometimes cited for sex differences in mathematical achievement, such as the mathematics as a male domain (Becker, 1982) or the presumed social conditioning and ations for boys and girls (Fennema, 1978), might explain why more girls omitted did boys. On the basis of the Grade 8 SIMS data no attempt could be made to determine of these factors, or indeed of informal training.

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<

ith differences between the sexes of 10 percentage

or more - -

em A Girls > Boys

em B to 0 Boys > Girls

em D - difference between sexes largest 20%

A. $\frac{3}{5} + \frac{2}{7}$ is equal to

A $\frac{21}{10}$

B $\frac{5}{12}$

C $\frac{10}{21}$

D $\frac{6}{35}$

E $\frac{31}{35}$

B. Which of the following is equal to a quarter of a million?

A 25 250

B 40 000

C $\frac{1}{4\ 000\ 000}$

D 250 000

E 2 500 000

C. There are 35 students in a $\frac{1}{5}$ of them come to school by another $\frac{2}{5}$ come by bicycle. many come to school by oth

A 7

B 14

C 21

D 28

E 35

D. The speed of sound is 340 m/s. How long will it take before the sound of a car horn reaches your ears if the car is 714 m away?

A 0.21 s

B 2.1 s

C 21 s

D 210 s

E None of these

E. 20 is what percent of 80?

A 4%

B 20%

C 25%

D 40%

E None of these

F. In a school election with three candidates, Joe received 120 votes. Mary received 50 votes, and George received 30 votes. What percent of the total number of votes did Joe receive?

A $\frac{6}{10}$%

B 40%

C 60%

D 80%

E 120%

G. One bell rings every 8 minutes, a second bell rings every 12 minutes. They both ring at exactly 12 o'clock. After how many minutes will they next ring together?

A 8

B 12

C 20

D 24

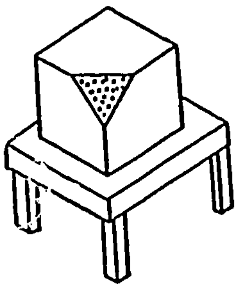
E 96

Figure 8

ture at
ountain
p of the
mperature
much warmer
he foot of

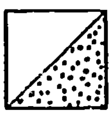
l, a boy
asts a
long. At
a nearby
45 units
hadow the
n, in the

J.

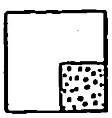


The figure above shows a wooden cube with one corner cut off and shaded. Which of the following drawings shows how this cube would look when viewed from directly above it?

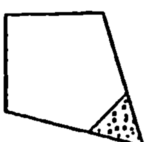
A



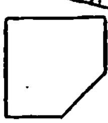
B



C



D



E

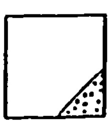
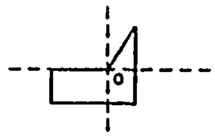


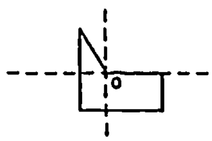
Figure 9

K.

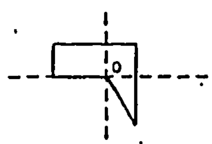


A half-turn (180°) about point O is applied to the figure above. Which of the figures below is the result?

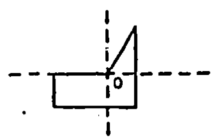
A



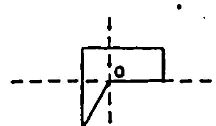
B



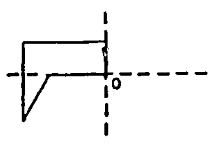
C



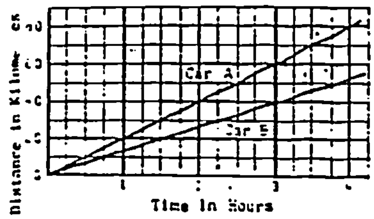
D



E



L.



How much longer does it take for car B to go 50 km than it does for car A to go 50 kilometres?

A 1 h 15 min

B 1 h 30 min

C 2 h

D 2 h 30 min

E 2 h 35 min

N. What is the capacity of a cubic container 10 cm by 10 cm by 10 cm?

A 1 L

B 10 L

C 100 L

D 1000 L

E 1000 cm

M. How many pieces of
each 20 m long, wo
required to constr
pipeline one kilom
length?

A 5

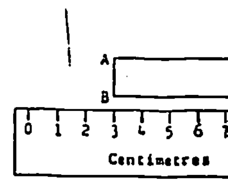
B 50

C 500

D 5000

E 50,000

O.



According to the scale s
the length of side BC of
rectangle ABCD (to the
NEAREST CENTIMETRE) is

A 5 cm

B 6 cm

C 7 cm

D 8 cm

E 9 cm

Figure 10

ger
f Windsor

PERCEPTIONS OF PRE-SERVICE STUDENT TEACHERS ON MATHEMATICAL ACHIEVEMENT AND ON TEACHING MATHEMATICS. RESULTS OF A PILOT STUDY

nce of teachers' expectations in the learning process
has been well recognized, since Rosenthal and
blished their book 'Pygmalion in the Classroom' in
attempts have been made to trace the channels by which
expectations and students' achievement are linked
in particular, this question has become of interest in
focusing on sex-related differences in students'
achievement and course-taking behavior.

ne for a moment a teacher has the following attitude:
ot as able as boys when it comes to mathematics and,
their future profession they are not going to need
as boys do. According to this attitude the teacher
expect the girls in his/her class to do very well
cs.
several possible ways in which this teacher's
might be communicated to the students. The teacher
display them when commenting on the poor work of a
r he/she might consciously or unconsciously use more
ys;e.g., praising a girl very much for correctly
easy question, asking mostly boys to solve really
roblems ,and attributing good mathematical achievement
a lot of effort and by boys to ability.

is not only a subject that more or less often gets
teachers, parents and students themselves but it is
ject that many people perceive as very difficult to

Let us assume for a moment a primary teacher who suc
passing high school math fairly well. The teacher
female because most primary teachers are female.
high school she did not take any math courses at th
level.It was not until she entered the pre-ser
training program that she had to deal with mathem
According to her personal experiences with mathemati
that it is a difficult subject to learn and that s
good grades because she worked very hard at it. Her
in relation to mathematics is low and she might even
that somehow men are the better mathematicians
appeared to her during high school that those studen
to have the least difficulty with mathematics and
the most self-confident in doing it were boys. Alth
not necessarily get the best grades, the boys did
need a lot of effort to grasp the main concepts.

During her time in preservice teacher training our
teacher pays particular attention to learn how to
mathematics,for, as she sees it, this is the mo
subject she is going to teach. She likes to coll
teaching ideas as possible. The more ready made t
better. She wants to be prepared for all possible s
she is going to use her entire stock when teaching
very hesitant about trying new things, in future y
not give her students much scope to bring math p
encounter outside school into the classroom. Ther
fear that she might not be able to solve unfamiliar

Of course the learning history of our teacher could
different. Let us assume for a moment that she is
teacher. She is one of the few female high school
have mathematics as a teachable subject. For her,
was always an enjoyable adventure. She is proud of
in this subject and found it easy to teach rig
beginning of her career as a teacher.

Her self-confidence in her math ability being well e
she is not afraid of challenging questions from her
the contrary she appreciates them as they demonstr
students are interested in math. She often uses the
as starting points for math investigations, of whic
does not know the results in advance.

examples are hypothetical. In fact very little is
out the mathematical learning history of teachers
ath at different grade levels, about the relationship of
ory to their perceptions on teaching mathematics, and
ir actual teaching. Moreover, the same is true for
entering a preservice program, that is for teachers to
seems to be reasonable to assume that the learning
rongly influences certain aspects of teaching and that
n investigating these variables.
5 a pilot study was carried out at the University of
The study focused - among other things - on answering
ing questions:

the personal learning history in mathematics of pre-
udent teachers?

ent are they in teaching mathematics?

ons do they give when the students they taught during
eaching did not make much progress in mathematics?

wers to the above stated questions depend on the sex of
nt teacher and/or on the division he/she has chosen to

Procedure

er training program of the University of Windsor is a
program and includes three divisions, these are the
nior (K - 6), the junior/intermediate (4 - 8) and the
te/senior (7 - 13) division. Students enrolled in the
nior program have to take the math education course
or their division, while for students in the other
math education is an optional course .Students enrolled
the junior/intermediate or the intermediate/senior
are grouped together for the analysis of the results
o as jun/int/senior division.

Relevant information was gathered via a questionna
The variable 'learning history in mathema
operationalized as follows: students were asked to ev
mathematical achievement during their schooldays and
reasons to their achievement. The questionnaire used
the latter variable was developed by the author in t
another research study (s. Schildkamp-Kuendiger 1980)

Moreover, the student teachers were asked to co
achievement in mathematics and their confidence in t
with that in other subjects. They were also asked to
related students'achievement differences they had
schools. To evaluate the reasons the student teache
the pupils they taught not makind satisfacory
mathematics, a questionnaire developed for the Second
International Mathematics Study was used.

The questionnaire was answered by students o
educational classes after they had been out for the
four practice teaching sessions. Students answered on
and anonymous basis.
Chi Square Tests were used to compare the responses
groups of students; e.g. male and female stude
enrolled in the primary/junior division.
In the graphs showing the results, arithmetic means
characterise the distributions.

Results

Overall 111 student teachers, enrolled in the pr
division, answered the questionnaire; 96 female and
teachers.
The corresponding numbers for the jun/int/senior div
overall 61 student teachers; 36 female and 25 male te

will be discussed for primary/junior student teachers
formation about their learning history in mathematics is
in graph 1. As a group primary/junior teachers remember
th achievement during their schooldays as average and
unt for it by a lot of reasons. The internal
tated are: math ability and learning effort; relevant
reasons are : math is difficult, good or poor teacher's
on and help by others.
nt sex-differences ($p < 0.05$) within the group are
one variable only, that is lack of help of others. This
is not considered as very relevant in general, but
udent teachers judge it as even less relevant than male

unior student teachers remember their math achievement
less good than their achievement in other school
(see grahp 2). This goes together with their
n of being comparatively less good in teaching this

ncouraging that they only sometimes encountered sex-
chievement differences in their pupils during practice
Moreover, the questionnaire reveals that, if sex-
differences had been observed, they did not show
icular subject like mathematics.
d female primary/junior teachers do not differ
ntly as to the variables considered in graph 2.

displays the reasons teachers perceive as relevant when
ls they taught during practice teaching did not make
ory progress in mathematics. Primary/junior student
mention two reasons the most: lack of ability of the
d lack of motivation. Lack of student ability is a
reason for the teacher; this is not his/her
ility. Motivating to learn on the other hand is
that falls in the duty of a teacher.
female teachers differ significantly in their evaluation
ts' misbehavior and lack of motivation; female teachers
se reasons as more important as their male colleagues.

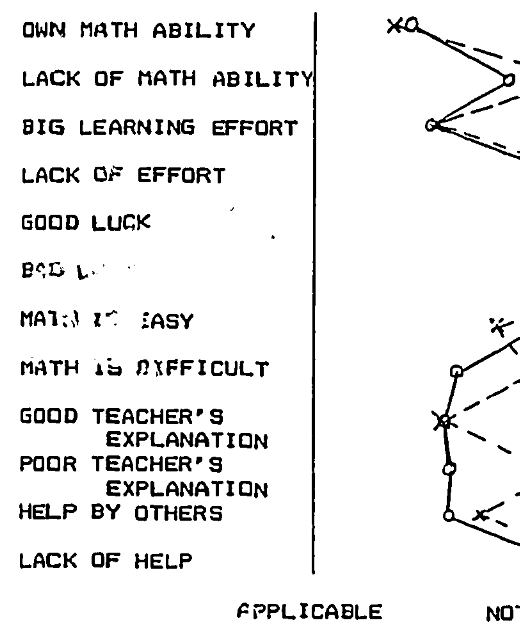
there are very few significant sex-related differences
ale and female primary/junior student teachers. This is
partly due to the fact that there are very few male
in this sample. It seems as if teaching in the
unior grades will stay mainly a female affair. Whether
le and female primary/junior student teachers can really
d upon as having the same charcteristics as to the
considered here has to be answered by subsequent

GRAPH 1

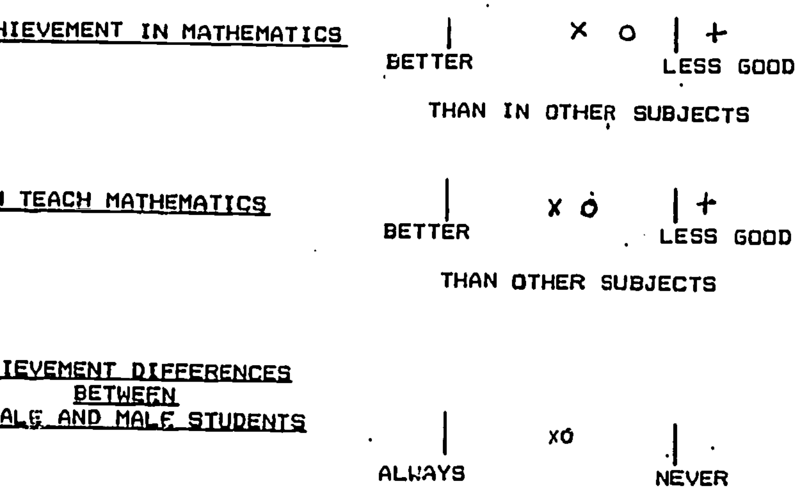
MATH ACHIEVEMENT DURING SCHOOLDAYS | X O
ABOVE
AVERA

CAUSAL ATTRIBUTION OF STUDENT TEACHE

MATHEMATICAL ACHIEVEMENT



O PRIMARY/JUNIOR STUDENT TEACHERS, N = 111
× JUN/INT/SENIOR STUDENT TEACHERS, N = 61
INDICATES SIGNIFICANT DIFFERENCES BETWEEN THESE TWO
+ ($p < 0.05$, CHI SQUARE TEST).

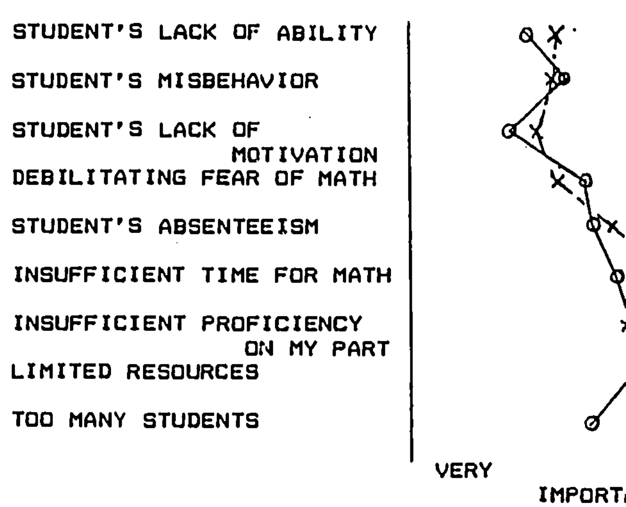


JUNIOR STUDENT TEACHERS, N = 111
SENIOR STUDENT TEACHERS, N = 61
S SIGNIFICANT DIFFERENCES BETWEEN THESE TWO GROUPS
5, CHI SQUARE TEST).

GRAPH 3

REASONS THAT STUDENT TEACHERS GIVE FOR PUPIL

MAKING SATISFACTORY PROGRESS IN MATHEMATIC



O PRIMARY/JUNIOR STUDENT TEACHERS, N = 111
X JUN/INT/SENIOR STUDENT TEACHERS, N = 61
+ INDICATES SIGNIFICANT DIFFERENCES BETWEEN THESE TWO
($p < 0.05$, CHI SQUARE TEST).

g history of jun/int/senior student teachers is quite from that of the primary/junior group (s. graph 1). ior student teachers remember their achievement during as above average. It is significantly higher than that ary/junior group. Moreover, the jun/int/senior group reasons for this achievement. Lack of ability, good luck, difficulty of math and poor teacher explanation d to significantly less as causes of achievement. At th is perceived as easier.

/senior group remembers its math achievement as about in other subjects and judges its ability in relation bject as average. For both variables the differences jun/int/senior teachers and the primary/junior significant.
do not differ as to the extent achievement differences and girls had been observed during practice teaching.

ms to indicate differences in the attribution pattern n/int/senior student teachers and primary/junior achers in the direction that the jun/int/senior ed fewer reasons to account for students not making progress in mathematics. Only for the reason t time for math' are the differences significant on , but there is a trend ($p < 0.07$) for the reasons: ack of motivation, limited resources, and too many

two groups of student teachers seem to differ in cts considered in this reseach.

differences between male and female primary/junior achers were rare; this is not the case for the or group. Although the whole group remembers its math during schooldays as above average, this is even more he female teachers ($p < 0.01$). Moreover female valuate their math ability and good teachers' as more relevant a reason for their achievement than achers ($p < 0.05$) ; whereas lack of effort is less a reason by female teachers ($p < 0.05$).

Finally there is another rather unexpected significant between male and female jun/int/senior student teachers comes to explaining why their pupils did not make sa progress female teachers more often perceive in proficiency on their part to be the reason. This is astonishing as ,according to their learning hi mathematics, they should be even more self-confident male teachers.

Summary

The results of this pilot study reveal considerable d between student teachers in the primary/junior division in the junior/intermediate/senior division. Student t the latter division have a much more positive learning mathematics than the primary/junior group.
Of course, it can be argued that the group of jun/ student teachers considered here would not have taken course, if they had not felt rather confident in this as they had a choice the primary/junior student teache have. The situation becomes more delicate , if this history is looked upon as having important impact on t of teaching. Afterall all these student teachers will t get. a teaching postion after finishing the program. A the primary/junior student teachers will start teaching less confidence in their ability to teach this subject teach other subjects.
It can be expected that - in doing the job - they will confident in teaching mathematics. But the hypothesi easily be turned down that they might gain this confi following a rather rigid teaching method that mini challenge of unexpected questions and problems.
Up to now the results of this pilot study indicate - a a trend - that the primary/junior student teachers nee upon more reasons than the other group to explain pupils fail to learn mathematics. Further information a student teachers think to be important to make math tea effective is available and will be analysed in the near

ırd to sex-differences the results indicate some
ı differences for jun/int/senior students. It seems as
student teachers only choose to teach mathematics if
very confident about their competence in the subject.
together with a readiness to explain failures in their
earning more often by personal insufficient proficiency
heir male colleagues. The question remains open what
ge these future female teachers are going to deliver
emale students.

out sex-related achievement differences at school have
ed that girls tend to have lower self-esteem in
to their math ability, even when they have the same
t as boys. It is worth investigating if there is a
at the teacher level in so far as the perception of
eaching proficiency comes into play.

R., Jacobson, L.: Pygmalion in the classroom. New York .

-Kuendiger, E.: Learnig the concept of a function. In:
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Roberta Mura

Université Laval

Mécanismes d'actualisation de la sous-représentation des femmes en mathém

présentation d'un projet en cours

Un sondage réalisé en 1981 sur l'état de la recherche concernant les différen
sexe en mathématique au Canada, avait indiqué qu'à peu près dans tout le pays, la
des filles aux cours de mathématique commence à décliner vers la fin du seco
qu'aucune recherche n'avait été effectuée pour tenter d'expliquer ce phénomène (M

Cette constatation m'a incitée à concevoir une première étude exploratoire
Comme il me semblait important d'étudier le phénomène dans sa globalité, j'
collaboration de collègues avec des compétences en sociologie et en psychologie; R
et Meredith Kimball, ont accepté de se joindre à moi et notre projet a obtenu une
Conseil de recherches en sciences humaines du Canada.

Au Québec, dans le secteur francophone, le phénomène de la sous-représentati
en mathématique s'amorce au passage du secondaire au collégial (Cegep) -- c'es
11ème à la 12ème année. D'après les statistiques fournies par le Ministère de
Québec, en 5ème secondaire (dernière année de l'école secondaire), même si
mathématique ne sont pas obligatoires, depuis plusieurs années, les filles représ
de la clientèle de ces cours. Au collégial par contre, à l'automne 1984, elles n'en
plus que 42%. Toujours d'après le Ministère de l'Education, la réussite des filles
comme au Cegep, est aussi bonne que celle des garçons, sinon meilleure.

out en étant conscientes que les racines des choix que les élèves font en entrant au Cegep
t remonter loin dans le passé, nous avons décidé d'aborder le problème en étudiant ce
u moment de sa formulation, c'est-à-dire vers la fin de la cinquième année du secondaire.
première phase de la cueillette de données a eu lieu de février à mai 1983 dans trois de
ses de mathématique de cinquième secondaire. Pendant cette période les élèves faisaient,
chéant, leur demande d'admission au Cegep. Les mêmes élèves ont ensuite été contacté/e/s
au un an plus tard.

s savions que le phénomène de la différenciation des choix scolaires selon le sexe était
nplexe et nous avons choisi d'en brosser un tableau global, plutôt que d'en étudier plus
il quelques aspects seulement. Dans cette perspective, nous avons opté pour l'emploi
né d'une variété de méthodes de cueillette des données: questionnaires aux élèves,
tions en classes, entrevues avec les élèves et avec leurs enseignant/e/s de mathématique.
s avons retenu un grand nombre de variables. Parmi les principales, on retrouve les
s:

occupation et la scolarité des parents,
écart entre l'image de soi et l'image d'une personne de science,
valeur intrinsèque et la valeur utilitaire attribuées à la mathématique,
attitude envers le succès en mathématique et en français,
confiance en ses capacités en mathématique,
s causes auxquelles les élèves attribuent leurs succès et échecs en mathématique et en
ançais,
s prévisions de réussite en mathématique,
s aspirations scolaires et professionnelles,

- la présence de modèles de rôles scientifiques dans le milieu de l'élève,
- les cours suivis et les notes obtenues,
- les motivations du choix scolaire telles qu'exprimées par les élèves,
- l'attitude du milieu de l'élève envers son choix scolaire,
- les interactions entre les élèves et leur enseignant/e de mathématique,
- la perception que l'enseignant/e a du potentiel de ses élèves en mathématiqu
  intérêt pour cette matière et de leur niveau de confiance,
- les prévisions de l'enseignant/e à l'égard de la réussite de ses élèves,
- les causes auxquelles les enseignant/e/s attribuent les succès et les échecs d

Dans le choix de ces variables, nous nous sommes en partie inspirées du modèl
Eccles (1985) -- modèle qui était déjà disponible avant le début de notre projet.

Toutes les variables ont été analysées en fonction du sexe et du choix scolaire
choix scolaire a été défini à partir de la demande d'admission au Cegep faite par
printemps 1984; nous avons ainsi distingué les élèves qui ont choisi u
scientifique de ceux et celles qui ont choisi une autre orientation. Tel que prév
groupe comprenait proportionnellement moins de filles que de garçons. Cette défi
scolaire a le désavantage d'élargir le champs d'étude de la mathématique aux scier
nous a semblé plus fiable qu'une définition basée sur les intentions de suivre
mathématique exprimées par les élèves, car dans la demande d'admission l'élè
programme auquel il, ou elle, veut s'inscrire sans préciser les cours particuli
suivis.

Je présente ici seulement quelques résultats préliminaires à titre d'exemple
personnes intéressées à se procurer le rapport final à la fin de 1985.

l'ensemble, nous avons trouvé plus de différences reliées au choix scolaire que de s reliées au sexe. Ainsi, l'écart entre l'image que les élèves ont d'eux-mêmes, ou êmes, et l'image qu'ils, ou elles, se font d'une personne de science est plus petit chez qui s'orientent vers les sciences que chez les autres. De même, le premier groupe une plus grande valeur intrinsèque et utilitaire à la mathématique et possède plus de en ses capacités dans cette matière. Parmi ces quatre variables, la dernière est la a donné lieu à une différence entre filles et garçons, ces derniers manifestent un plus au de confiance.

iste toutefois quelques exceptions. Par exemple, à propos des causes auxquelles les tribuent leurs succès et échecs en mathématique, nous avons trouvé des différences exe, mais non selon le choix scolaire: les filles attribuent très majoritairement leurs leurs efforts, tandis que les garçons sont partagés entre leurs efforts et leur habileté. ui est des explications de l'échec, la majorité des filles comme des garçons fait appel au 'effort, mais quelques filles invoquent aussi leur manque d'habileté ou la difficulté de la Les mêmes tendances se sont manifestées à propos des causes par lesquelles les nt/e/s expliquent les succès et échecs de leurs élèves. Nous n'avons pas trouvé de e analogue entre filles et garçons dans leur perception des causes de succès et d'échec en

autre différence importante entre filles et garçons est apparue dans leurs propres l'emploi et dans ce qu'elles, ou ils, prévoient pour le conjoint, ou la conjointe, lorsque t les enfants: garçons et filles s'accordent majoritairement pour dire que ce seront ces s qui assumeront les responsabilités majeures au niveau des tâches familiales et nt leur emploi à l'extérieur au temps partiel ou même le suspendront complètement.

L'influence de ce facteur sur le choix scolaire est liée à l'image des sciences com particulièrement exigeant, où il est difficile de poursuivre des études ou une carr partiel, ou de les reprendre après une interruption.

Enfin, un dernier exemple de différence entre filles et garçons touche leur com classe de mathématique: nous avons observé que les garçons participaient be vocalement que les filles, en répondant à 75% des questions de l'enseignant/e lors n'étaient pas adressées à un/e élève en particulier (les garçons constituaient 4 échantillon). Avant d'avancer des hypothèse sur le rôle de ce facteur dans les choix faudrait cependant effectuer des observations pour savoir si ce comportement ne se aussi dans des classes où l'on aborde des disciplines non scientifiques.

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