DOCUMENT RESUME

ED 276 567 SE 047 541

AUTHOR Maletsky, Evan, Ed.; Yunker, Lee E., Ed.
TITLE NCTM Student Math Notes.
INSTITUTION National Council of Teachers of Mathematics, Inc., Reston, Va.
PUB DATE 86
NOTE 21p.
PUB TYPE Collected Works - Serials (022) -- Guides - Classroom Use - Materials (For Learner) (051)
JOURNAL CIT NCTM Student Math Notes; Jan-Nov 1986

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Enrichment Activities; *Geometric Concepts; Junior High School Students; *Mathematics Instruction; *Number Concepts; Prime Numbers; Probability; *Proof (Mathematics); Ratios (Mathematics); Secondary Education; *Secondary School Mathematics; Simulation

ABSTRACT

Five sets of activities for students are included in this document. Each is designed for use in junior high and secondary school mathematics instruction. The first Note concerns mathematics on postage stamps. Historical procedures and mathematicians, metric conversion, geometric ideas, and formulas are among the topics considered. Successful simulation is the focus of the second Note, with activities on planning a supermarket, bowling and random digits, and other suggestions for using a random digit table. The third set of activities concerns proof; geometric, algebraic, and numerical formulas are each considered. Golden rectangles and ratios are included in the fourth topic, with a physical model for generating golden rectangles, directions for constructing a golden rectangle and generating golden ratios via Logo, and comments on the divine proportion. Finally, the fifth Note concerns primes, and includes work with the Sieve of Eratosthenes, twin primes, and other problems. (MNS)

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
JANUARY 1986

Mathematics on Postage Stamps

Many postage stamps tell stories. This German stamp contains some facts about a German mathematician named Adam Riese. It was issued in 1959 on the 400th anniversary of his death. The symbols 30.3.1559 indicate the date of his death, 30 March 1559.

Riese wrote arithmetic books at a time when trade and commerce had begun requiring much computation. Up to this point, most calculating was done by *counter reckoning*. This was a mechanical process used by the ancient Romans where counters were moved on lines, much like an abacus. However, Riese gave strong support to a new form of calculating that used Hindu-Arabic numerals that could be written out with a quill pen. This was the beginning of the algorithmic form of computing we use today. The title of his book, *Rechnung auff der Linier und Federn,* written in 1522, can be translated as "Reckoning on Lines with a Pen."

In the old method of reckoning on lines, the only way a computation could be checked was to move the counters and begin again. The **X** and its four numbers shown on this stamp illustrate an algorithm or method Riese used to check calculations with numerals written with a pen. We know it today as the "method of casting out nines."

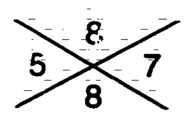
To find the remainder, or excess, when a number is divided by 9, cast 9s out of the number as shown in the computation below. This can be done by adding the number's digits. If their sum is *greater* than 9, as with the number 437 below whose digits add up to 14, add the digits of the new sum. In this case the new result, 5, is *less* than 9, and it is called the excess. If the sum is 9, cast it out and call the excess 0.

Study the process below to see how casting out 9s was used by Riese to check multiplication. In the check, multiply the two excess numbers of the multiplier and multiplicand: $5 \times 7 = 35$. Add the digits: $3 + 5 = 8$. This excess (8) must equal the excess of the product, or there is some error in the computation.

COMPUTATION

$437 \longrightarrow (4 + 3 + 7 = 14;\ 1 + 4 = 5)$ excess $= 5$
$\times 286 \longrightarrow (2 + 8 + 6 = 16;\ 1 + 6 = 7)$ excess $= 7$
2622
3496
874
$124982 \longrightarrow (1 + 2 + 4 + 9 + 8 + 2 = 26;\ 2 + 6 = 8)$ excess $= 8$

CHECK

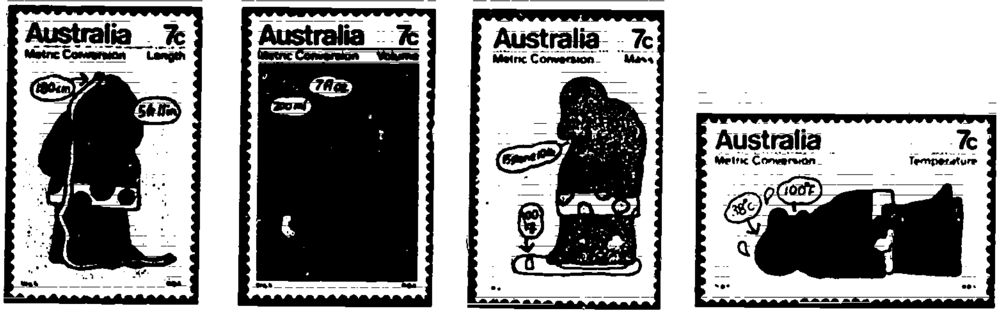


$(5 \times 7 = 35;\ 3 + 5 = 8)$
excess $= 8$

Find the product of 589 and 3417. Then check the answer using the method given by Riese.

Metric Conversion on Stamps

Australia issued this set of stamps to help educate its population on metric conversion. Use the information on them to help answer the questions below.



1. The stamp on **length** shows that a height of 180 cm is equivalent to 5 feet 11 inches. Mentally estimate the number of centimeters in a height of 5 feet. ________
2. The stamp on **volume** shows that 200 mL (milliliters) is equivalent to 7 fluid ounces. Using this relationship, how many fluid ounces more than a quart is a liter? ________
3. The stamp on **mass** shows that 100 kilograms is equivalent to 15 stone 10 pounds. If 1 kilogram is equivalent to 2.2 pounds, how many pounds are in one stone? ________
4. The stamp on **temperature** shows that 38°C is equivalent to 100°F. Explain why 19°C is not equivalent to 50°F. ________

Illusion or Reality?

These stamps from Brazil and Sweden show two interesting geometric surfaces.

5. One surface has two sides and the other has just one side. Which one is which?

____

6. One surface exists in the real world and is called a Möbius strip. The other is only an artist's optical illusion. Which one is which?

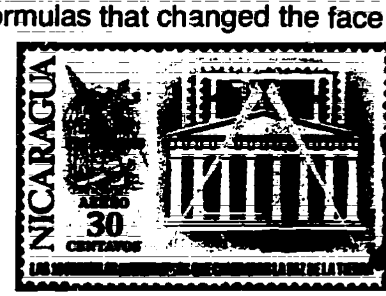
____

A

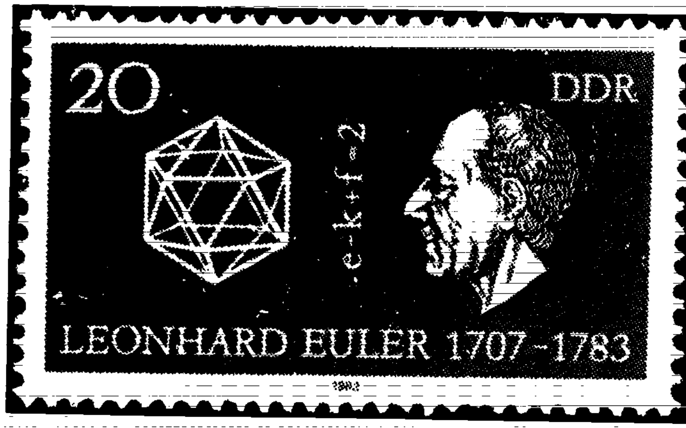
B

Formulas on Stamps

7. Nicaragua issued a series of stamps on the theme "Ten mathematical formulas that changed the face of the earth." This is one of them. Can you tell what mathematician it honors?
8. Use the formula given on the stamp to find the counting number c if $a = 24$ and $b = 7$.
9. Use the formula to find the counting numbers a and b if $c = 30$.



History on Stamps



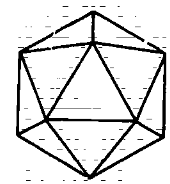
Here is a stamp honoring Switzerland's most famous mathematician, Leonhard Euler ("oiler"). It was issued in 1983 by East Germany to commemorate the bicentennial of Euler's death. The formula on the stamp contains the variables e, k, and f that correspond to the first letters of German expressions for vertices, edges, and faces of a simple closed polyhedron.

A more familiar form of this relationship using v for vertices, e for edges, and f for faces is

$$v - e + f = 2$$

It is interesting to note that although this stamp honors Euler, the relationship it depicts among vertices, edges, and faces of a simple closed polyhedron was known by Descartes a century earlier.

1. The geometric figure represented on the stamp is a regular icosahedron, a polyhedron with 20 congruent equilateral triangles as faces. Each of the 20 faces has three edges and three vertices. On the icosahedron, five faces come together at each vertex and two at each edge, as shown in this view of the icosahedron with only its front faces indicated. See if you can use some clever thinking to count v, e, and f for this solid. Verify your results using the formula $v - e + f = 2$.



Icosahedron

This Swiss stamp was issued in 1957 to commemorate the 250th anniversary of the birth of Leonhard Euler. He studied in every branch of mathematics then known, and his extensive writings are more numerous than those of any other mathematician.

Notice this formula on the stamp:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Those who know trigonometry will recognize that when $\theta = \pi$, we can get this surprising formula:

$$e^{i\pi} + 1 = 0$$

This formula connects five of the most important numbers in mathematics. Zero is the first whole number, and 1 is the first counting number. Both 0 and 1 are rational numbers. Both π and e are irrational numbers. All four of these are real numbers, but i is the imaginary unit, $\sqrt{-1}$, for complex numbers that can be written in the form $a + bi$, where a and b are real numbers.

2. The number e, sometimes called Euler's number, has the value 2.7182818284 . . . . It can be approximated using the infinite series

$$1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots$$

Use a calculator to evaluate the first ten terms in this series. See how the sum compares with the value given above.

Solutions.

Page 2: 1) 150 cm 2) 3 fl. oz. 3) 14 lb. 4) because 0° is not the freezing point on the Fahrenheit scale as it is on the Celsius scale

5 and 6) The surface on stamp A has just two sides, and the surface on stamp B has just one side. The surface on stamp B can be made by taking a strip of paper and giving it a half-twist before taping its ends together. The surface on stamp A is impossible to construct. It is an optical illusion.

7) Pythagoras 8) $c = 25$ 9) $a = 18$ and $b = 24$

Page 3: 1) Separately, the 20 triangles contain 60 vertices and 60 edges. Together on the surface of the icosahedron, there are 12 vertices (60 ÷ 5) and 30 edges (60 ÷ 2) as well as the 20 triangular faces. 12 − 30 + 20 = 2

2) 2.7182812 on an eight-digit calculator

Mathematicians on Stamps

Here are stamps honoring 5 mathematicians.

A


B


C


D


E


Bet you can't . . .

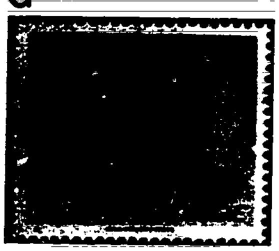
- Use the appropriate letter to match each stamp correctly, first by the mathematician's name, then his nationality, year of birth, and contribution.

Mathematician	Nationality	Year of Birth	Contribution
__ Einstein	__ Belgian	__ 430	__ One of the three greatest mathematicians of all time
__ Gauss	__ Chinese	__ 1548	__ Early supporter of the use of decimal fractions
__ Pascal	__ French	__ 1623	__ Father of probability
__ Stevin	__ German	__ 1777	__ $e = mc^2$
__ Tsu-Ching-Chih	__ American and German	__ 1879	__ Used 355/113 to approximate π

- Use the appropriate letter to match each stamp's mathematician with the statement that pertains to him. Read up on those who are unfamiliar to you.

F


Descartes

G


Ramanujan

H


Abel

I


Bolyai

J


Aristotle

__ Possessed amazing ability and insight in number theory

__ Developed a form of non-Euclidean geometry

__ Died at 27 never receiving the recognition deserved

__ Established the Cartesian coordinate system

__ Used models from mathematics to systematize logic

NCTM STUDENT MATH NOTES is published as part of the NEWS BULLETIN by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091. The five issues a year appear in September, November, January, March, and May. Pages may be reproduced for classroom use without permission.

Editor: Evan Maletsky, Department of Mathematics and Computer Science, Montclair State College, Upper Montclair, NJ 07043

Editorial Panel: Pamela Coffield, Brookstone School, 440 Bradley Park Drive, Columbus, GA 31995
Daniel T. Dolan, Office of Public Instruction, Helena, MT 59620
Margaret Kenney, Mathematics Institute, Boston College, Chestnut Hill, MA 02167 (Board liaison)
Lee E. Yunker, West Chicago Community High School, West Chicago, IL 60185

Printed in U.S.A

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
MARCH 1986

Successful Simulation: Approximate Answers to Real Questions

Planning a Supermarket

So you want to go into the supermarket business. You have found the perfect building, arranged the necessary financial backing, and made contacts for stocking and replenishing the store. Everything looks good. There's only one more question—checkout counters.

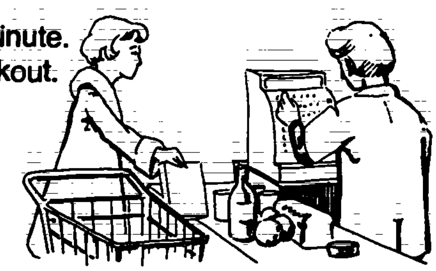
How many checkout counters should you have? If there are too many, some of the clerks will be standing around idle most of the time. If you have too few, the checkout lines will get pretty long and the waiting customers will be unhappy. You could make a guess and set up some counters. However, it's expensive to be wrong. A better way would be to *simulate* the situation—do a probability experiment based on assumptions about how the counters would operate. This is much less expensive than actually setting up the counters, and it allows you to try different combinations to see which works best.

Let's assume that a customer arrives at the checkout counter every three minutes, on the average. Suppose it takes the clerk three minutes to check each person out. We can throw a die to simulate the frequency of shoppers arriving at the counter: 1 or 2 means someone arrives; 3, 4, 5, or 6 means no one arrives. Let's look at a 30-minute work period during the day—one throw of the die for each minute. Here are the 30 rolls of the die:

	23212	25611	36153	24665	62513	44415
Customer:	A BCD	E FG	H	I	J K	L

Let's study these results. Remember, only a 1 or a 2 on a roll means a customer arrives. So in this simulation of 30 minutes, 12 customers arrived. Here is a minute-by-minute analysis of the first 7 minutes. The customers are identified by successive letters of the alphabet. At the end of the given minute—

Minute 1 Customer **A** arrives at the counter. The clerk has been idle for that minute.
2 Customer **A** is being checked out. No other customer arrives at checkout.
3 **B** arrives. **A** is still being checked out.
4 **C** arrives and **B** is waiting. **A** finishes.
5 **D** arrives and **C** is waiting. **B** is being checked out.
6 **E** arrives and **C** and **D** are waiting. **B** is still being checked out.
7 No new customer arrives. **C**, **D**, and **E** are waiting. **B** finishes.



Continue this analysis through the entire 30 minutes. Assume no customers arrive after 30 minutes but that the clerk continues working until all customers are checked out. Summarize the results by completing the table below. You will need to extend it to 37 on another sheet of paper.

End of minute	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Customer arrives	A	—	B	C	D	E	—	—	F	G																				
checking out	—	A	A	A	B	B	B	C	C	C																				
waiting	—	—	—	B	C	CD	CDE	DE	DE	DEF																				

Now complete the table on page 2 for each customer. Give the minute of arrival and final minute of checkout, along with the number of minutes spent waiting before checkout begins.

The editors wish to thank Albert Shulte, Oakland Schools, Pontiac, MI, for writing this issue.

Customer	A	B	C	D	E	F	G	H	I	J	K	L
Minute of arrival at checkout line	1	3	4	5	6	9	10					
Minute of completing checkout	4	7	10	13	16	19	22					
Minutes waited in line before checkout begins	0	1	3	5	7	7	9					

Find these values for this simulation:

1. Number of customers expected: ___
2. Number of customers actually arriving: ___
3. Total customer waiting time: ___
4. Average waiting time: ___
5. Total clerk idle time: ___
6. Total clerk overtime (beyond 30 minutes): ___

More on Checking Out

The simulated results of the checkout arrangement on the first page were not too good. Waiting that long would cause lots of customers to become impatient and unhappy. How could you improve the situation? You might try two checkout lines. The first customer goes to line 1. Other customers take the line that will serve them first (line 1 if there is a tie). Roll the die as before to check this new arrangement. You could continue to simulate situations, trying different things, until the results look reasonable to you.

Random Digits and Simulated Bowling

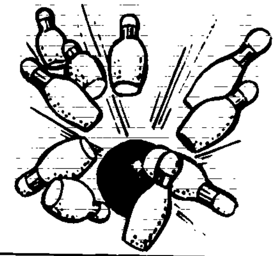
Simulation can be carried out using dice, spinners, coins, cards, or any other devices of chance. Computers are especially useful for running a simulation many times and accumulating results. A useful way to do simulations quickly without a computer is to use a *random digit* table. These tables are lists of digits (numbers 0–9) generated by a process that makes each digit equally likely to occur at each place in the table.

The random digit table below contains 1000 random digits arranged in blocks of five.

1000 Random Digits

91178	63914	64595	89172	04085	64395	93572	20383	93846	77476
62319	68719	48349	91172	37679	39944	33933	08402	25280	53118
44779	77020	52542	48064	06155	44303	43567	06228	97225	60371
14926	32769	89952	64944	76635	39828	66548	84706	45277	37556
56699	72694	41549	08682	15966	27395	33689	33134	64823	81367
67515	01008	85091	97881	76438	32122	22640	30774	34779	97267
85132	30076	35487	50049	71687	07368	85890	41601	91536	13926
58935	49219	66936	50428	15795	75141	91930	54603	90240	27426
48403	78402	47208	93252	20922	71642	95144	47828	69258	25782
51069	08648	46509	41073	16099	45829	10332	05131	87634	75321
29854	90646	81292	43092	25044	25646	12427	40867	44664	14145
43079	88271	34240	72522	31000	52807	37171	44166	37555	47017
41597	15888	23812	72748	23616	04615	53432	57647	98599	30483
58282	57337	14120	01638	72627	10735	95622	91801	45081	69629
99571	07315	82784	84503	68103	11650	79870	16818	34115	44615
44067	79637	88663	84441	75516	65473	65104	93890	55883	06048
56926	86889	33816	30444	49763	51862	45813	27612	71675	23907
88168	25823	38991	80801	72947	48877	14076	47106	34969	25742
49282	49247	75970	96903	35028	06911	03495	03737	50980	36127
07948	73543	26375	12311	82544	32452	57766	11974	89926	75343

How can we use a random digit table to simulate events? We can start anywhere and read digits, going in any direction. Just don't start in the same place each time.



Simulated Bowling Game

Let's use the random digit table to simulate a bowling game. Our game is much simpler than commercial simulation games.

First Ball		Second Ball			
		2-Pin Split		3 Split	
Digit	**Result**	**Digit**	**Result**	**Digit**	**Result**
1–3	Strike	1	Spare	1–3	Spare
4–5	2-pin split	2–8	Leave one pin	4–6	Leave 1 pin
6–7	9 pins down	9–0	Miss both pins	7–8	* Leave 2 pins
8	8 pins down			9	+ Leave 3 pins
9	7 pins down			0	Leave all pins
0	6 pins down				

* If there are fewer than 2 pins, result is a spare.
+ If there are fewer than 3 pins, those pins are left.

Here's how to score bowling:

1. There are 10 frames to a **game** or **line.**
2. You roll two balls for each frame, unless you knock all the pins down with the first ball (a **strike**).
3. Your score for a frame is the sum of the pins knocked down by the two balls, if you don't knock down all 10.
4. If you knock all 10 pins down with two balls (a **spare,** shown as ◪), your score is 10 pins plus the number knocked down with the next ball.
5. If you knock all 10 pins down with the first ball (a **strike,** shown as ⊠), your score is 10 pins plus the number knocked down with the next **two** balls.
6. A **split** (shown as 0) is when there is a big space between the remaining pins. Place in the circle the number of pins remaining after the second ball.
7. A **miss** is shown as —.

Here is how one person simulated a bowling game using the random digits 7 2 7 4 8 2 2 3 6 1 6 0 4 6 1 5 5, chosen in that order from the table.

Frame	1	2	3	4	5	6	7	8	9	10	
Digit(s)	7/2	7/4	8/2	2	3	6/1	6/0	4/0	1	5/5	
Bowling result	9 ◪ 19	9 — 28	8 ◪ 48	⊠ 77	⊠ 97	9 ◪ 116	9 — 125	8 (1) 134	⊠ 153	8 (1) 162	162

Now you try several.

Frame	1	2	3	4	5	6	7	8	9	10	
Digit(s)											
Bowling result											

	1	2	3	4	5	6	7	8	9	10	
Digit(s)											
Bowling result											

If you wish to, you can change the probabilities in the simulation to better reflect *your* actual bowling ability.

Using a Random Digit Table

Here are some suggestions for using a random digit table:

1. To simulate coin tossing: Let an odd digit represent a head, and let an even digit represent a tail; use 0 for a head, 1 for a tail; or use 0–4 for a head, 5–9 for a tail.
2. To simulate the role of a die: Let 1–6 represent faces on the die and ignore 7–9 and 0.
3. To simulate the sum of two dice: Use two digits (one for each die) just as in number 2; 46 gives a sum of 10. Sometimes more than two consecutive random digits will be needed from the table. Here is a case where a block of five digits is needed to secure a sum: 58901—you can use only the digits 5 and 1 for a sum of 6.
4. To simulate a 1–10 spinner: Let 0 represent a spin of 10; every other digit represents a spin of that number.
5. To simulate a 1–4 spinner: Represent spins this way:

   Digit 1 or 5 represents a spin of 1.
   Digit 2 or 6 represents a spin of 2.
   Digit 3 or 7 represents a spin of 3.
   Digit 4 or 8 represents a spin of 4.
   Ignore digits 9 and 0.

6. To select 10 people from 100 using random digits: Assign the people numbers 00–99. Read off the first two digits in a block and select the person represented by that number. Ignore repeats. For example, reading down the first two digits of the last column in the table, we select persons 77, 53, 60, 37, 81, 97, 13, 27, 25, 75.

Use Simulation to Answer These Problems

7. What is the probability that a .300 hitter in baseball who bats five times in a game will get exactly one hit?
8. Your airplane is scheduled to arrive at Major Airport at 1:15 P.M. Your connecting flight is due to depart at 2:00 P.M. Past records show that the probability of your flight arriving on or before 1:15 is .65, the probability of arriving 10 minutes late is .15, the probability of arriving 20 minutes late is .15, and the probability of arriving 30 minutes late is .05. Records also show that your connecting flight leaves on time 70% of the time, 10 minutes late 10% of the time, and 15 minutes late 20% of the time. Your luggage can be transferred if there is 30 minutes between your arrival and your departure. What is the probability that your luggage will be transferred to your connecting flight?

Bet you didn't know that...

- Simulations using probability are called *Monte Carlo* simulations. The term was invented during World War II by John Von Neumann at the Los Alamos Scientific Laboratory when he was working with Stanislaus Ulam on neutron diffusion problems.
- Monopoly is a simulation game of finance based on the streets, utilities, and railroads in Atlantic City, N.J.
- Pilots can log practice instrument approaches in *simulators*, and these simulated approaches count just as much as actually flying the approaches.
- Chess is a stylized simulation of ancient warfare.
- Computers and microcomputers have a function, RND ( ), that generates random digits for simulations or other uses.

Solutions. *Page 2:* 1) 10 2) 12 3) 68 minutes 4) 5 minutes, 40 seconds 5) 1 minute 6) 7 minutes
Page 4: 7) *P*(exactly one hit) ≈ .36. (A simulation of 100 games gave a result of .32.)
8) *P*(baggage will be transferred) = (.65) · (1) + (.15)(1) + (.15) · (.3) + (.05) · (.2) = .855. (A simulation of 100 trips resulted in baggage being transferred 86 times.)

NCTM STUDENT MATH NOTES is published as part of the **NEWS BULLETIN** by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091. The five issues a year appear in September, November, January, March, and May. Pages may be reproduced for classroom use without permission.

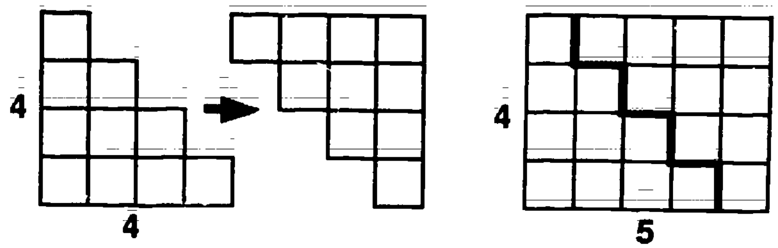
Printed in U.S.A.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
MAY 1986

Proof Without Words

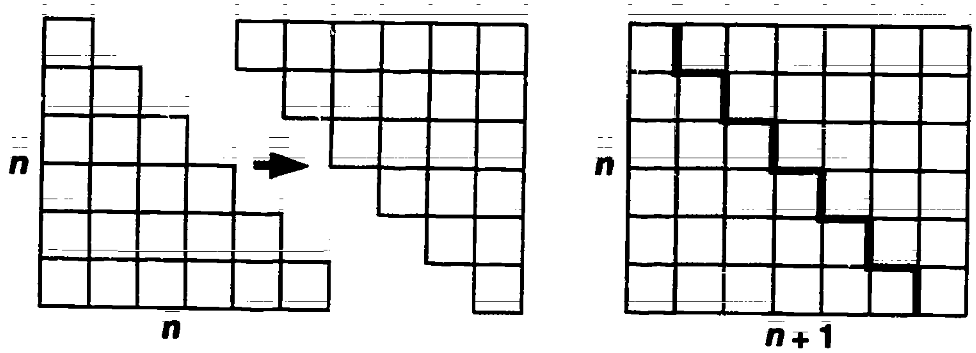
For some, the essence of mathematics is proof. But finding rigorous proofs is often tedious and difficult. For others, the beauty of mathematics lies in the patterns and relationships among numbers and figures that can be recognized from intuitive observations. In this issue we shall look at some patterns and from them draw some observations and derive some formulas.

For instance, look at how these two "staircases" of four steps each are combined to form a rectangle. The rectangle contains 4×5, or 20, small squares. So each staircase must contain $20 \div 2$, or 10, such squares. This is the same answer that we get when we count the squares directly:



The sum of the first four counting numbers $= 1 + 2 + 3 + 4 = \frac{4 \times 5}{2} = 10.$

Suppose we let each of the staircases below represent one with n steps and $1 + 2 + 3 + \ldots + n$ squares. If n were a large number, it would take a lot of time and work to count all the squares directly, but the sum could still be found in terms of n. Can you find the formula for the sum from the rectangle formed by the two staircases?



The sum of the first n counting numbers $= 1 + 2 + 3 + \ldots + n =$ ________.

Add the first 10 counting numbers. Check your formula for $n = 10$ to see if you get the same sum. Repeat the process for the first 20 counting numbers.

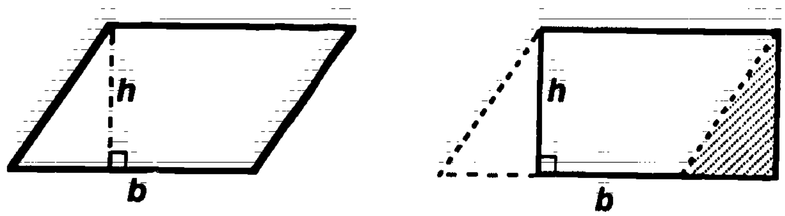
Geometry Formulas

Suppose you know that the area of a parallelogram can be found by multiplying the base and the height. How can you find the areas of these other figures?

1.

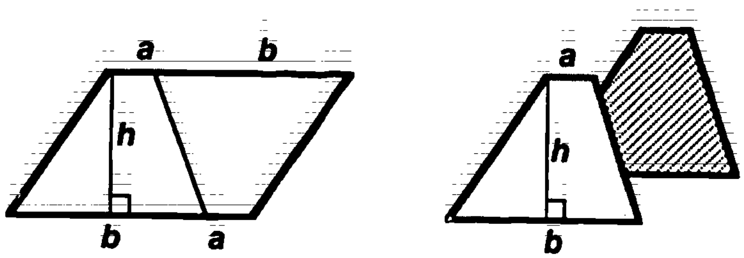
A = ______
(Triangle)

2.



A = ______
(Rectangle)

3.

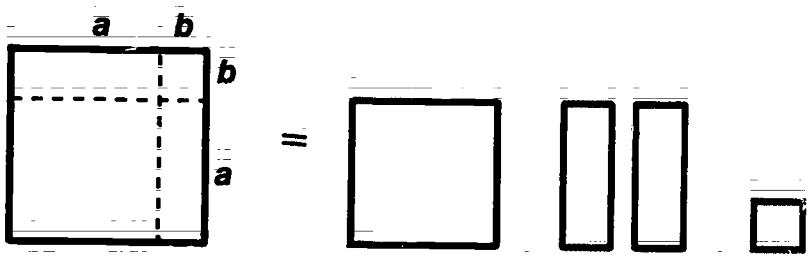


A = ______
(Trapezoid)

Algebra Formulas

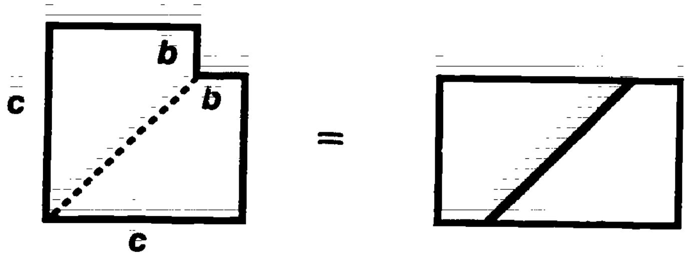
Think of areas in completing these formulas in algebra.

4.



$(a + b)^2$ = ______ + ______ + ______

5.



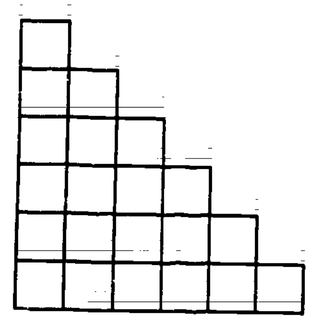
length______

width______

$c^2 - b^2$ = (__ + __)(__ − __)

Numerical Formulas

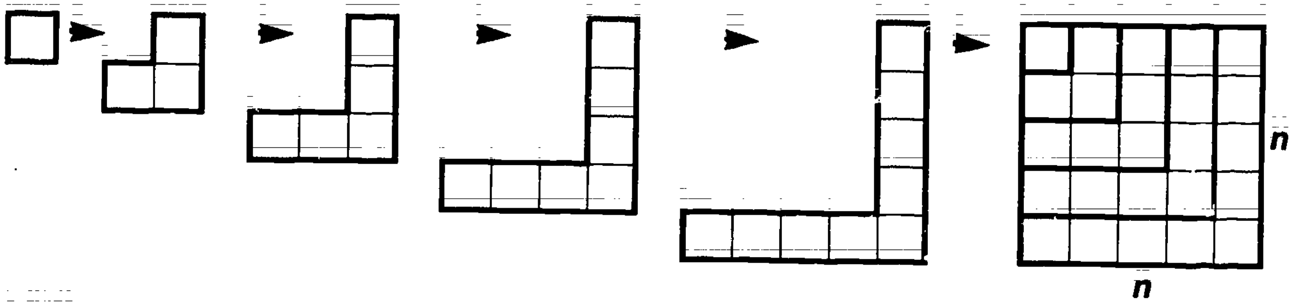
Numerical formulas can also be found from pictures. This staircase contains 6 steps but we can think of it as having n steps and containing $1 + 2 + 3 + \ldots + n$ small squares. When two such staircases are joined, as on the first page, a rectangle measuring n by $n + 1$ is formed. The rectangle contains $n(n + 1)$ squares. So each staircase contains $n(n + 1)/2$ squares. It follows that—



$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

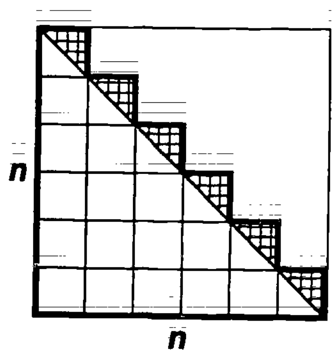
Here are some other proofs without words. Try to follow each argument. Then verify the formula you get for $n = 15$.

6.



Sum of the first n odd counting numbers $\quad 1 + 3 + 5 + \ldots + (2n - 1) = n^2$

7.

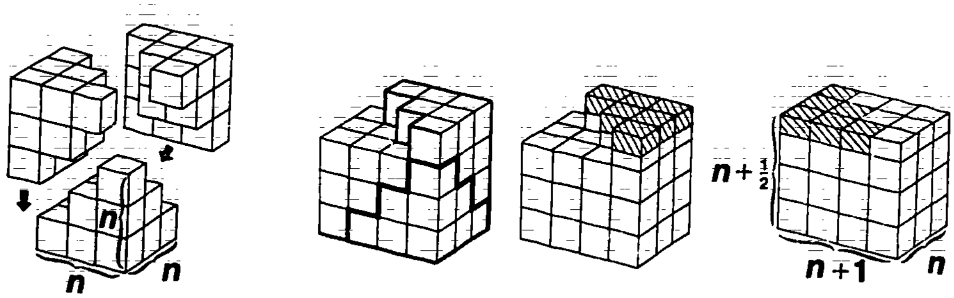


Sum of the first n counting numbers $\quad 1 + 2 + 3 + \ldots + n = \frac{n^2}{2} + \frac{n}{2}$

Three-dimensional figures can also be used in proofs, as illustrated here. Verify the formula for $n = 10$.

8.

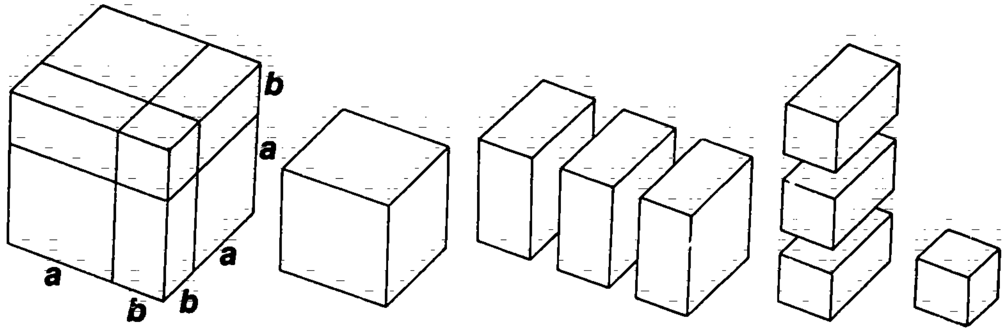
The ideas in this question are adapted from Man-Keung Siu (*Mathematics Magazine*, March 1984).



Sum of the squares of the first n counting numbers $\quad 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{3}n(n + 1)(n + \frac{1}{2})$

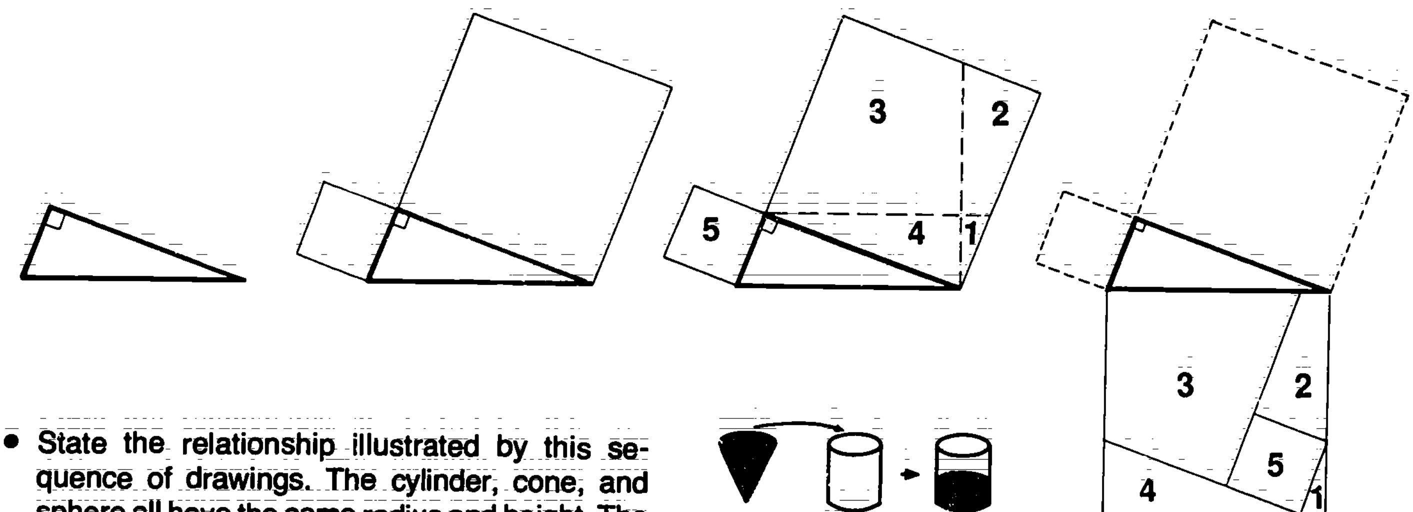
Bet you can't . . .

- Use the volumes of the prisms formed to complete this formula. Verify the results for $a = 5$ and $b = 2$.

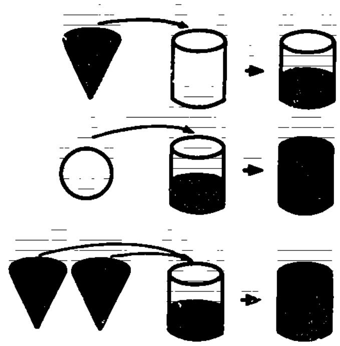


$$(a + b)^3 = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$

- Show how the formula for the area of a triangle can be found from the figure formed as follows: Cut out a piece of paper in the shape of a scalene acute triangle. Locate the foot of an altitude from one vertex by folding. Then fold all three vertices to that point.
- Describe the problem that might arise in the demonstration above if an obtuse triangle were used.
- Identify the famous theorem proved by these drawings. Will the method work for any right triangle?



- State the relationship illustrated by this sequence of drawings. The cylinder, cone, and sphere all have the same radius and height. The cylinder and cone have an open base, and the shading represents water being poured or displaced.



Solutions: 1) $A = 1/2bh$ 2) $A = bh$ 3) $A = 1/2h(a + b)$ 4) $(a + b)^2 = a^2 + 2ab + b^2$ 5) $c^2 - b^2 = (c + b)(c - b)$

Page 4: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Let b and h be the base and height of the starting triangle. The resulting rectangle has an area of $(1/2b)(1/2h)$, or $1/4bh$. But two make up the area of the triangle, so its area must be $1/2bh$. Pythagorean theorem (yes); $V_{cone} : V_{sphere} : V_{cylinder} = 1:2:3$

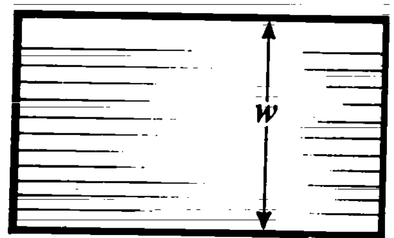
NCTM STUDENT MATH NOTES is published as part of the **NEWS BULLETIN** by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091. The five issues a year appear in September, November, January, March, and May. Pages may be reproduced for classroom use without permission.

Printed in U.S.A.

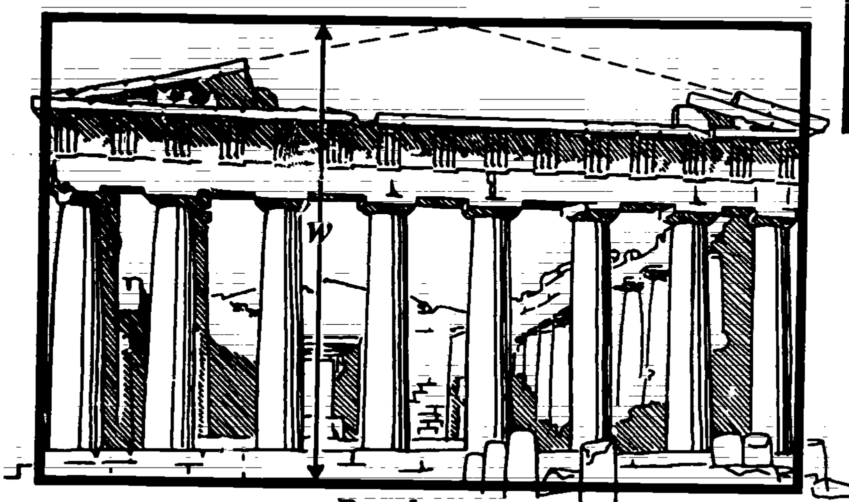
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
SEPTEMBER 1986

Golden Rectangles and Ratios

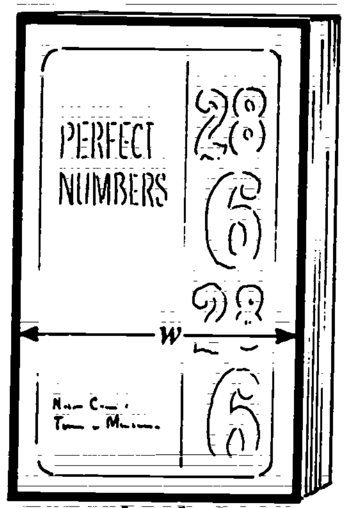
All the figures below have a rectangular shape. What else do they have in common? This particular rectangular shape, called the *golden rectangle,* is considered to be the most pleasing to the eye.



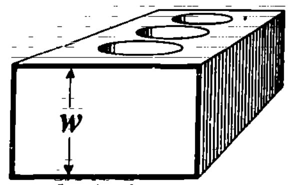
index card



Parthenon



paperback book



brick

Measure the length and width of each rectangle in millimeters. Express the ratio of the width to the length as a decimal rounded to two places.

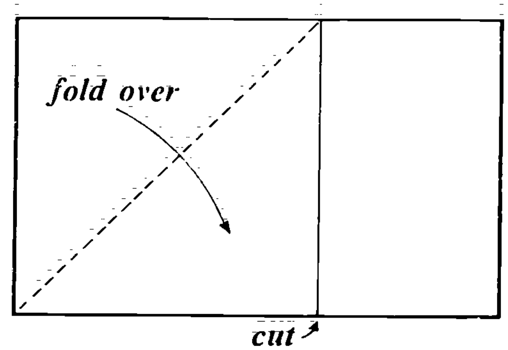
	Width (w)	Length (l)	w/l	Decimal Ratio
Parthenon	____	____	____	____
Index card	____	____	____	____
Paperback book	____	____	____	____
Brick	____	____	____	____

By expressing these ratios in decimal form, we can observe that each of them is approximately 0.6. The ratio 0.61803 . . . is called the *golden ratio*.

The editors wish to thank **Rick Billstein** and **Johnny W. Lott,** University of Montana, Missoula, MT 59812, for writing this issue of the *NCTM Student Math Notes.*

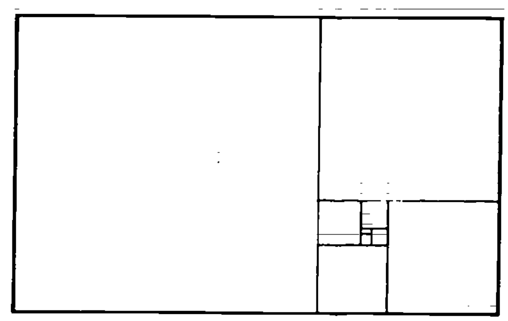
A Physical Model for Generating Golden Rectangles

Cut a sheet of paper to measure 25 cm by 15.5 cm. This rectangle closely approximates a golden rectangle. Fold over one corner of the rectangle as shown in the adjacent figure. Then cut off the square from the rectangle. The remaining rectangle has the same proportions as the original rectangle; hence it is also a golden rectangle.



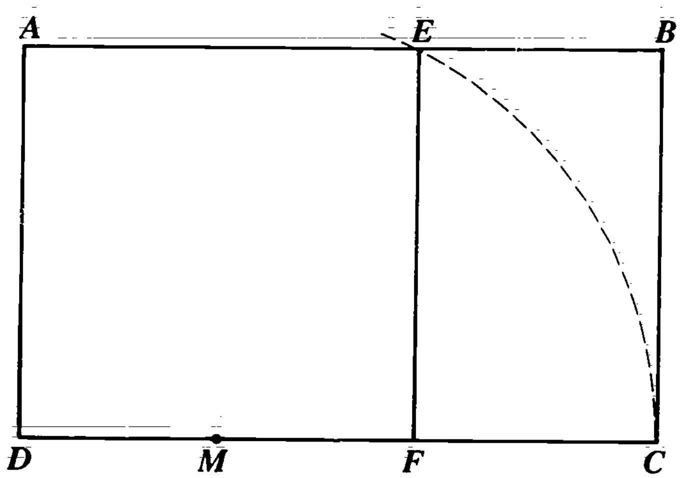
You can continue to generate golden rectangles by repeating this process. Each successive square has sides approximately 0.61803 . . . times the length of the sides of the preceding square.

Place the squares together to form the original golden rectangle as in the figure shown here. This representation of the golden rectangle is often referred to as "the rectangle with the whirling squares."



Constructing a Golden Rectangle

A golden rectangle can be constructed using a compass and straightedge. To accomplish this construction, follow the steps given below. The figure shows the appropriate lettering of the vertices.



1. Construct a square *AEFD*.
2. Bisect $\overline{DF}$. Label the midpoint *M*.
3. Extend $\overline{DF}$.
4. With center *M* and radius *ME*, draw an arc intersecting $\overline{DF}$ at *C*.
5. Construct a perpendicular to $\overline{DC}$ at *C*.
6. Extend $\overline{AE}$ to intersect the perpendicular at *B*.

The rectangle *ABCD* is a golden rectangle. It can be shown that *BCFE* is also a golden rectangle. To show that this statement is true, assume *MF* = 1 and find each of the following lengths:

(a) *FE* ____ (b) *BC* ____ (c) *ME* ____ (d) *MC* ____ (e) *FC* ____ (f) *DC* ____

Find the decimal values of these ratios: *BC/DC* ____ *FC/BC* ____ Are they equivalent?

Generating Golden Ratios via Logo

To generate a Logo procedure that draws successive golden rectangles using the idea of whirling squares, use Logo recursion. Recursion is the process of a procedure calling on a copy of itself. In mathematics, recursion can be thought of as a function that defines itself in terms of preceding terms.

First, write a procedure to draw a square of variable size that returns the turtle to its original position and heading.

```
TO SQUARE :SIZE
  REPEAT 4[FD :SIZE RT 90]
END
```

Next, add another square onto the square. However, you must first move to the upper-right-hand corner and give the turtle a heading of 90. You can do so by redrawing the first two sides of the square. Next, draw a square whose side is 0.61803 times the length of the original square. To generate the whirling squares, use recursion to repeat the process until you obtain a square that is as small as desired. This new procedure, used with the one above, will generate golden rectangles using whirling squares. (In Apple Logo, replace STOP with [STOP].)

```
TO GOLDEN.RECTANGLES :SIZE
  IF :SIZE < 1 STOP
  SQUARE :SIZE
  FD :SIZE RT 90 FD :SIZE
  GOLDEN.RECTANGLES :SIZE * 0.61803
  HIDETURTLE
END
```

An alternate procedure, called GOLDEN, for generating golden rectangles is given below. Note that it doesn't require the existence of the procedure SQUARE. Study it to determine how it works.

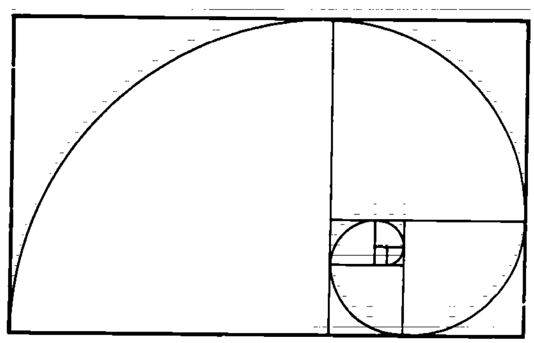
```
TO GOLDEN :SIZE
  IF :SIZE < 1 STOP
  REPEAT 2[FD :SIZE RT 90 FD :SIZE * 1.61803 RT 90]
  FD :SIZE RT 90 FD :SIZE
  GOLDEN * :SIZE * 0.61803
  HIDETURTLE
END
```

In the GOLDEN procedure, the first line names the procedure, the second line tells the procedure when to stop, and the third line draws the initial golden rectangle.

- What does the fourth line do?
- What does the fifth line do?

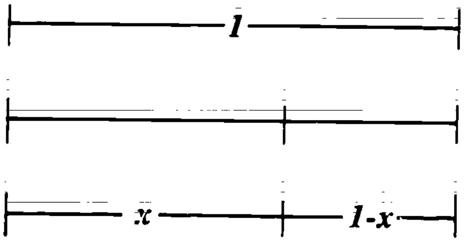
Draw the picture that would result if the fourth line were removed from the procedure.

Imagine drawing a 90-degree arc connecting the opposite corners of each square as shown here.



The resulting figure is called a *golden spiral*.

The Divine Proportion



The divine proportion was derived by the fifteenth-century mathematician Luca Pacioli. It is found by dividing a segment into two parts so that the length of the smaller part is to the length of the larger part as the length of the larger part is to the length of the entire segment. Divide a segment one unit long into two parts; label the longer segment x, as shown.

The segments are in the divine proportion if the following is true: $\frac{1-x}{x} = \frac{x}{1}$

Solve for x:

$$\begin{aligned} x^2 &= 1 - x \\ x^2 + x - 1 &= 0 \\ x &= \frac{-1+\sqrt{5}}{2} \text{ or } \frac{-1-\sqrt{5}}{2} \end{aligned}$$

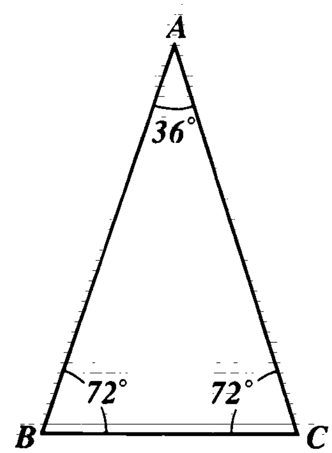
Since lengths cannot be negative, the negative root can be discarded. Use a calculator to evaluate the positive root to five or more decimal places. Are you surprised at the result? (It's our new friend, the golden ratio, 0.61803 . . . .) The algebraic expression allows us to compute the value of the golden ratio more precisely than the paper-folding geometry used earlier.

Did You Know That . . .

- in his art, Michelangelo used the golden ratio?
- the factorial function is an example of recursion?
- the golden ratio is the limit of the ratio of successive terms in the Fibonacci sequence, which is related to such natural phenomena as pinecones, sunflowers, pineapples, and seashells?
- the reciprocal of 0.61803 . . . equals 1.61803 . . . ? Does any other number differ from its reciprocal by 1?

Can You . . .

- write a Logo procedure to draw a golden spiral?
- use Euclidean tools to construct a regular pentagon?
- find the ratio of the measure of the side of a regular pentagon to the measure of its diagonal?
- show that BC/AB is a golden ratio in triangle ABC?



Solutions: *Page 2:* *a*) 2   *b*) 2   *c*) $\sqrt{5}$ *d*) $\sqrt{5}$
e) $\sqrt{5} - 1$ *f*) $\sqrt{5} + 1$;   0.61803 . . . ;   0.61803 . . . ;   yes

NCTM STUDENT MATH NOTES is published as part of the **NEWS BULLETIN** by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091. The five issues a year appear in September, November, January, March, and May. Pages may be reproduced for classroom use without permission.

Printed in U.S.A.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
NOVEMBER 1986

Primes

A *prime number* is a natural number that has exactly two factors, itself and 1. The pyramid below is called a *prime pyramid*. Each row in the pyramid begins with 1 and ends with the number that is the row number. In each row, the consecutive numbers from 1 to the row number are arranged so that the sum of any two adjacent numbers is a prime.

For example, look at row 5:

1) It must contain the numbers 1, 2, 3, 4, and 5.
2) It must begin with 1 and end with 5.
3) The sum of adjacent pairs must be a prime number.
4) $1 + 4 = \underline{5}$, $4 + 3 = \underline{7}$, $3 + 2 = \underline{5}$, and $2 + 5 = \underline{7}$.

*	Row 1
1 2	Row 2
1 2 3	Row 3
1 2 3 4	
1 4 3 2 5	
1 4 3 2 5 6	⋮
1 ___ ___ ___ ___ ___ 7	
1 ___ ___ ___ ___ ___ ___ 8	
1 2 3 4 7 6 5 8 9	
1 ___ ___ ___ ___ ___ ___ ___ ___ 10	
1 ___ ___ ___ ___ ___ ___ ___ ___ ___ 11	
1 ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ 12	
1 ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ 13	
1 ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ 14	
1 ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ 15	Row 15

- Supply the missing numbers in this prime pyramid.
- Can you extend the prime pyramid beyond row 15?
- What patterns do you see in your solutions?
- What is your solution strategy for completing the pyramid?

The editors wish to thank **Margaret Kenney**, Mathematics Institute, Boston College, Chestnut Hill, MA 02167, for writing this issue of the *NCTM Student Math Notes*.

Prime Start

Euclid (born about 300 B.C.), a Greek mathematician famous for his geometry, showed that the number of primes is infinite. Eratosthenes (born in 230 B.C.), also a Greek, was the chief librarian at the University of Alexandria in Egypt. He developed a method for finding primes that is still in use today.

Sieve of Eratosthenes

Start with a 10 × 10 array of the numbers 1 through 100. The number 1 is crossed out because it is not a prime. Why not? ______________________________

Circle 2, the first prime; then cross out the remaining multiples of 2 using a horizontal slash (—).

What visual pattern do the multiples of 2 make in the grid? ______________________

What's the next number that hasn't been crossed out? ______

Circle 3, the next prime; then cross out the remaining multiples of 3 using a vertical slash (|).

What visual pattern is made by the multiples of 3? ______________________

What is the next prime number that hasn't been crossed out? ______

Circle the 5 and cross out the remaining multiples of 5 using the diagonal slash (/).

In a similar manner, circle the next prime, 7, and cross out all its remaining multiples using the reverse diagonal slash (\ ).

~~1~~	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

At this point, except for 2, 3, 5, and 7, you have sifted out all the multiples of 2, 3, 5, and 7.

1. Are any multiples of 11, the next prime, not yet crossed out? ______

2. Why is this the case? ______________________________

____

The numbers that are not crossed out are the primes less than 100. Make a list of these primes.

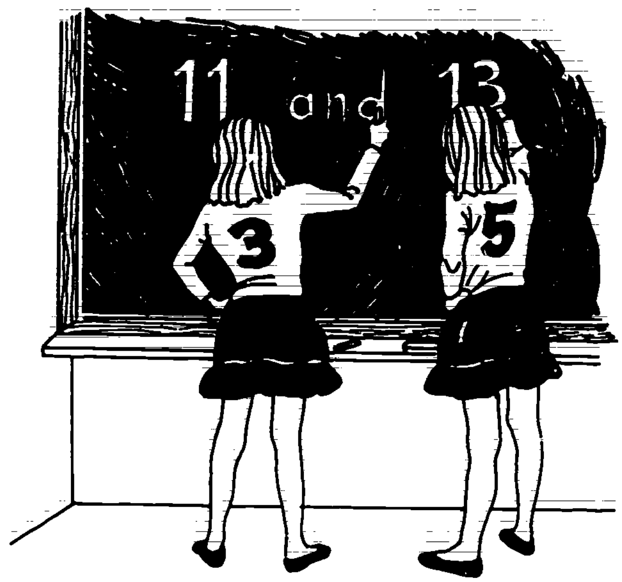
2, 3, 5, 7, 11, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____,

____, ____, ____, and ____.

Twin Primes

Several pairs of primes in the list of primes less than 100 have a difference of 2. For example, the pairs 3 and 5, 5 and 7, and 11 and 13 each have a difference of 2. These pairs are called *twin primes*. Complete the list of all twin primes less than 100. Also, find the sums and products of these twin primes.

Twin Primes	Sums	Products
3 and 5	____	____
5 and 7	____	____
11 and 13	____	____
____ and ____	____	____
____ and ____	____	____
____ and ____	____	____
____ and ____	____	____
____ and ____	____	____



1. Do you see a pattern in the column of sums? Can you *prove* a fact about the *sums* of twin primes?

2. Do you see a pattern in the column of products? Can you *prove* a fact about the *products* of twin primes?

3. Examine the primes larger than 5 in your list. What *different* digits appear in the units position of these primes? ____________ Will any prime larger than 100 have a different ending than the ones you have found? ______

The number 13 is a prime, and 31, the reverse of 13, is also a prime. 13 is called an *emirp* (*prime* spelled backward) because its reverse is a *different prime*. 31 is also an emirp. But 11 is not an emirp. Why not? ____________

4. List all the emirps less than 100. ____________

Prime Concerns

Over the centuries we have learned a great deal about prime numbers. But in all this time no one has discovered a simple formula that will produce all the primes starting with 2. Many attempts have been made, and no doubt will continue to be made, to find such a formula. One such attempt produced the following:

$$p_n = n^2 - n + 41$$

p_n is a prime number for $n = 1$ through $n = 40$. For example, $p_1 = 41$, $p_2 = 43$, and $p_3 = 47$. Use your calculator or write a computer program to find additional values of p_n for $n = 4, \ldots, 40$. What is p_{41}? ______ Is it prime or composite? Why? ____________

5. What are some other values of $n > 41$ for which p_n is a composite number? ____________

Is p_n prime for some values of $n > 41$? If so, list some of them. ____________

Can You . . .

- replace each blank with a "+" or "−" to get an equality relation?

  17 = 1 ____ 2 ____ 3 ____ 5 ____ 7 ____ 11 ____ 13 ____ 13
  19 = 1 ____ 2 ____ 3 ____ 5 ____ 7 ____ 11 ____ 13 ____ 17
  23 = 1 ____ 2 ____ 3 ____ 5 ____ 7 ____ 11 ____ 13 ____ 17 ____ 19 ____ 19
  29 = 1 ____ 2 ____ 3 ____ 5 ____ 7 ____ 11 ____ 13 ____ 17 ____ 19 ____ 23

- find some emirps that contain three, four, or more digits?
- in Eratosthenes' sieve, name the primes whose multiples *must* be crossed out to find all primes less than 200? 300? 500?
- write a computer program based on Eratosthenes' sieve to find all primes less than 1000?
- find a string of at least 1 million consecutive numbers that are all composite? For example, 24, 25, 26, 27, 28 is a string of five consecutive numbers that are all composite. Identify your string by naming the first number and the one-millionth number in the string. (*Hint:* Use factorials!)

Did You Know That . . .

- a conjecture states that the number of twin primes is infinite? No one has been able to prove or disprove this conjecture.
- J. P. Kulik spent twenty years, unassisted, computing a factor table of the numbers from 1 to 100 000 000? He completed his monumental work in 1867. It filled eight volumes, but volume 2 is now missing from the collection at the Vienna Royal Academy.
- a special kind of prime number, *Mersenne numbers*, are named after the French priest and amateur mathematician Marin Mersenne? In 1644, he published his (incomplete) list of primes that satisfied the rule $M_p = 2^p - 1$, where p is prime: $p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$. It took 304 years to resolve the errors in his short list.
- in 1978, the twenty-fifth Mersenne prime, $2^{21\,701} - 1$, was found by two eighteen-year-old students, Laura Nickel and Curt Noll, using a CYBER-174 computer at California State University at Hayward?
- the largest known Mersenne prime, $2^{216\,091} - 1$, consists of 65 050 digits and was discovered in 1985 in Houston, Texas, on a Cray X-MP supercomputer by scientists at Chevron Geosciences Company? This find will probably be declared the thirtieth Mersenne prime.
- Christian Goldbach, 1690–1764, conjectured that every even number greater than 4 is equal to the sum of two prime numbers? His conjecture remains unproved today.

Solutions: *Page 1:*

1 4 3 2 5 6 7
1 2 3 4 7 6 5 8
1 2 3 4 7 6 5 8 9 10
1 4 3 2 5 6 7 10 9 8 11
1 4 3 2 5 6 7 10 9 8 11 12
1 4 3 2 5 6 7 12 11 8 9 10 13
1 2 5 8 3 4 7 12 11 6 13 10 9 14
1 2 5 8 3 4 7 12 11 6 13 10 9 14 15

Page 2: 1) no; 2) You do not need to cross out multiples of 11 in addition to the ones already crossed out, since the first one to go would be 11 × 11, which is greater than 100. *Page 3:* 1) Except for 3 + 5, all are divisible by 12: $(6m - 1) + (6m + 1) = 12m$; 2) Each product is one less than the square of the average of the twin primes; 3) 1, 3, 7, 9; no; 4) 13, 17, 31, 37, 71, 73, 79, and 97; 5) $n = 42, 45, 50, 57, 66, 77, 82, 83, 85, 88, \ldots$

Page 4:
\+ − − + − + +
\+ − + + + + −
− + + − − + − + +
\+ − − + − + − + +

Emirps: 337, 733, 1471, 1741, . . .

$(10^6 + 1)! + 2, (10^6 + 1)! + 3, (10^6 + 1)! + 4, \ldots, (10^6 + 1)! + (10^6 + 1)$ is a string of 1 million consecutive numbers that are all composite.

NCTM STUDENT MATH NOTES is published as part of the **NEWS BULLETIN** by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091. The five issues a year appear in September, November, January, March, and May. Pages may be reproduced for classroom use without permission.

Printed in U.S.A.